Nonlinear Problems- S2

Exercise 1 Consider a second-order autonomous system $\dot{x}(t) = f(x)$. For each of the following types of equilibrium points, classify whether the equilibrium point is stable, unstable, or asymptotically stable. Justify your answer using phase portraits.
(1) stable node (2) unstable node (3) stable focus(4) unstable focus (5) center (6) saddle

Exercise 2 For each of the following systems, use a quadratic Lyapunov function candidate to show that the origin is asymptotically stable. Then, investigate whether the origin is globally asymptotically stable.

(1) $$\begin{aligned}\dot{x}_1 &= -x_1 + x_2^2\\ \dot{x}_2 &= -x_2\end{aligned}$$

(2) $$\begin{aligned}\dot{x}_1 &= (x_1 - x_2)(x_1^2 + x_2^2 - 1)\\ \dot{x}_2 &= (x_1 + x_2)(x_1^2 + x_2^2 - 1)\end{aligned}$$

(3) $$\begin{aligned}\dot{x}_1 &= -x_1 + x_1^2 x_2\\ \dot{x}_2 &= -x_2 + x_1\end{aligned}$$

(4) $$\begin{aligned}\dot{x}_1 &= -x_1 - x_2\\ \dot{x}_2 &= x_1 - x_2^3\end{aligned}$$

Exercise 3 Using quadratic Lyapunov function, study stability of the origin of the system.

$$\begin{aligned}\dot{x}_1 &= x_1(k^2 - x_1^2 - x_2^2) + x_2(x_1^2 + x_2^2 + k^2)\\ \dot{x}_2 &= -x_1(k^2 + x_1^2 + x_2^2) + x_2(k^2 - x_1^2 - x_2^2)\end{aligned}$$

Exercise 4 Using the variable gradient method, find a Lyapunov function V(x) that shows asymptotic stability of the origin of the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -(x_1 + x_2) - \sin(x_1 + x_2)\end{aligned}$$

Exercise 5 Consider the second-order system

$$\begin{aligned}\dot{x}_1 &= \frac{-6x_1}{u^2} + 2x_2\\ \dot{x}_2 &= \frac{-2(x_1 + x_2)}{u^2}\end{aligned}$$

where $u = 1 + x_1^2$. Let $V(x) = x_1^2/(1 + x_1^2) + x_2^2$.

(a) Show that $V(x) > 0$ and $\dot{V}(x) < 0$ for all $x \in R^2 - \{0\}$.

Exercise 5 Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1 + g(x_3)\\ \dot{x}_2 &= -g(x_3)\\ \dot{x}_3 &= -ax_1 + bx_2 - cg(x_3)\end{aligned}$$

where a, b, and c are positive constants and $g(\cdot)$ satisfies

$$g(0) = 0 \text{ and } yg(y) > 0, \quad \forall\ 0 < |y| < k, \quad k > 0$$

(a) Show that the origin is an isolated equilibrium point.

(b) With $V(x) = \frac{1}{2}ax_1^2 + \frac{1}{2}bx_2^2 + \int_0^{x_3} g(y)\ dy$ as a Lyapunov function candidate, show that the origin is asymptotically stable.

(c) Suppose $yg(y) > 0\ \forall\ y \in R - \{0\}$. Is the origin globally asymptotically stable?

Exercise 6 Consider Lihnard's equation

$$\ddot{y} + h(y)\dot{y} + g(y) = 0$$

where g and h are continuously differentiable.

(a) Using $x_1 = y$ and $x_2 = \dot{y}$, write the state equation and find conditions on g and h to ensure that the origin is an isolated equilibrium point.

(b) Using $V(x) = \int_0^{x_1} g(y)\ dy + \frac{1}{2}x_2^2$ as a Lyapunov function candidate, find conditions on g and h to ensure that the origin is asymptotically stable.

(c) Repeat (b) using $V(x) = \frac{1}{2}\left[x_2 + \int_0^{x_1} h(y)\ dy\right]^2 + \int_0^{x_1} g(y)\ dy$.

Exercise 7 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -a\sin x_1 - kx_1 - dx_2 - cx_3\\ \dot{x}_3 &= -x_3 + x_2\end{aligned}$$

where all coefficients are positive and $k > a$. Using

$$V(x) = 2a\int_0^{x_1} \sin y\ dy + kx_1^2 + x_2^2 + px_3^2$$

with some $p > 0$, show that the origin is globally asymptotically stable.

Exercise 8 Show that the system

$$\begin{aligned}\dot{x}_1 &= \frac{1}{1+x_3} - x_1\\ \dot{x}_2 &= x_1 - 2x_2\\ \dot{x}_3 &= x_2 - 3x_3\end{aligned}$$

has a unique equilibrium point in the region $x_i \geq 0$, $i = 1,2,3$, and investigate stability of this point using linearization.

Exercise 9 Consider the system

$$\begin{aligned}\dot{x}_1 &= (x_1x_2 - 1)x_1^3 + (x_1x_2 - 1 + x_2^2)x_1\\ \dot{x}_2 &= -x_2\end{aligned}$$

(a) Show that $x = 0$ is the unique equilibrium point.

(b) Show, using linearization, that $x = 0$ is asymptotically stable.

Exercise 10 For each of the following systems, use linearization to show that the origin is asymptotically stable. Then, show that the origin is globally asymptotically stable.

(1) $$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= (x_1 + x_2)\sin x_1 - 3x_2\end{aligned}$$

(2) $$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_2\\ \dot{x}_2 &= -ax_1 - bx_2,\ a, b > 0\end{aligned}$$

Exercise 11 Consider the system

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + \alpha(t)x_2\\ \dot{x}_2 &= -\alpha(t)x_1 - x_2^3\end{aligned}$$

where $\alpha(t)$ is a continuous, bounded function. Show that the origin is globally uniformly asymptotically stable. Is it exponentially stable?

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Exercise 12 *A* pendulum with time-varying friction is represented by

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - g(t)x_2\end{aligned}$$

Suppose that $g(t)$ is continuously differentiable and satisfies

$$0 < a < \alpha \leq g(t) \leq \beta < \infty; \quad \dot{g}(t) \leq \gamma < 2$$

for all $t \geq 0$. Consider the Lyapunov function candidate

$$V(t,x) = \tfrac{1}{2}(a \sin x_1 + x_2)^2 + [1 + ag(t) - a^2](1 - \cos x_1)$$

(a) Show that $V(t,x)$ is positive definite and decrescent.

(b) Show that $\dot{V} \leq -(\alpha - a)x_2^2 - a(2-\gamma)(1-\cos x_1) + O(||x||^3)$, where $O(||x||^3)$ is a term bounded by $k||x||^3$ in some neighborhood of the origin.

(c) Show that the origin is uniformly asymptotically stable.

Exercise 13 Consider the system

$$\begin{aligned}\dot{x}_1 &= -2x_1 + g(t)x_2 \\ \dot{x}_2 &= g(t)x_1 - 2x_2\end{aligned}$$

where $g(t)$ is continuously differentiable and $|g(t)| \leq 1$ for all $t \geq 0$. Show that the origin is uniformly asymptotically stable.

Exercise 14 Consider the system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - (1 + b\cos t)x_2\end{aligned}$$

Find $b^* > 0$ such that the origin is exponentially stable for all $|b| < b^*$.

Exercise 15 Consider the linear system **x(k + 1) = Ax(k).** Show that the following statements are equivalent

(1) $x = 0$ is asymptotically stable.

(2) $|\lambda_i| < 1$ for all eigenvalues of A.

(3) Given any $Q = Q^T > 0$ there exists $P = P^T > 0$, which is the unique solution of the linear equation

$$A^T P A - P = -Q$$

Exercise 16

For the following systems, find the equilibrium points and determine their stability. Indicate whether the stability is asymptotic, and whether it is global.

(a) $\dot{x} = -x^3 + \sin^4 x$

(b) $\dot{x} = (5 - x)^5$

(c) $\ddot{x} + \dot{x}^5 + x^7 = x^2 \sin^8 x \cos^2 3x$

(d) $\ddot{x} + (x-1)^4 \dot{x}^7 + x^5 = x^3 \sin^3 x$

(e) $\ddot{x} + (x-1)^2 \dot{x}^7 + x = \sin(\pi x/2)$

Exercise 17

Show that given a constant matrix $\mathbf{M}$ and any time-varying vector $\mathbf{x}$, the time-derivative of the scalar $\mathbf{x}^T \mathbf{M} \mathbf{x}$ can be written

$$\frac{d}{dt} \mathbf{x}^T \mathbf{M} \mathbf{x} = \mathbf{x}^T (\mathbf{M} + \mathbf{M}^T) \dot{\mathbf{x}} = \dot{\mathbf{x}}^T (\mathbf{M} + \mathbf{M}^T) \mathbf{x}$$

and that, if $\mathbf{M}$ is symmetric, it can also be written

$$\frac{d}{dt} \mathbf{x}^T \mathbf{M} \mathbf{x} = 2\, \mathbf{x}^T \mathbf{M} \dot{\mathbf{x}} = 2\, \dot{\mathbf{x}}^T \mathbf{M} \mathbf{x}$$

Exercise 18

For different values of a in the following system, find the equilibrium points and study their stability.

$$\dot{x} = x^2 + a$$

Exercise 19

Find the equilibrium point of the following system and its stability completely.

$$\dot{x}_1 = x_2 + x_1(\beta^2 - x_1^2 - x_2^2)$$
$$\dot{x}_2 = -x_1 + x_2(\beta^2 - x_1^2 - x_2^2)$$

Exercise 20

Analyze the stability of the dynamics $\dot{v} + 2a|v|v + bv = c \qquad a > 0, b > 0$

Exercise 21

show that the linear time-varying system associated with the matrix

$$\mathbf{A}(t) = \begin{bmatrix} -1 & e^{t/2} \\ 0 & -1 \end{bmatrix}$$

is globally asymptotically stable.

Exercise 22

Determine whether the following systems have a stable equilibrium. Indicate whether the stability is asymptotic, and whether it is global.

$$(a) \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -10 & e^{3t} \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(b) \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2\sin t \\ 0 & -(t+1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$(c) \qquad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

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Exercise 23

Consider the system

$$\ddot{y} + (a + b\cos(y))\dot{y} + c\sin(y) = 0$$

a) This system can be viewed as a pendulum with a nonlinear damping coefficient. Using the total energy of the system as a Lyapunov candidate, show that (0,0) is a stable equilibrium point if $a \ge b \ge 0$.
b) Using Lasalle Theorem prove that (0, 0) is an *asymptotically* stable equilibrium point if $a \ge b \ge 0$.

Exercise 24

Consider the following forced system

$$\dot{x}_1 = \sin(x_2)\cos(x_1) + x_1^3 + u$$

$$\dot{x}_2 = -x_1 x_2 \cos(x_1)$$

a) Based on the Lyapunov function candidate $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$, suggest a control law $u(x_1, x_2)$ such that the system remains stable and $x_1(t) \to 0$ as $t \to \infty$.

b) Using MATLAB, show that your design satisfies the problem objectives.

Exercise 25

Consider the non-autonomous system:

$$a\,\ddot{y}(t) + p(t)\dot{y}(t) + e^{-t}y(t) = 0$$

a) If $a = 1$ Introduce a suitable time varying Lyapunov function and find some conditions of the function $p(t)$ that ensure the stability of the equilibrium 0.
b) Suppose $a = 0$ and $p(t) = \dfrac{e^{-t}}{6t\sin(t) - 2t}$ analyze the stability (and uniform stability) of the new differential equation.