

جزوه جبر جامع

سلام بر پاسبان

فرشته تکراری

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دانشگاه شهید بهشتی

رشته محاسبات نرم

حساب

٩٤, ٩, ١٨

□ A course in universal algebra

Bouris & Sankaranarayanan

ch I, II, IV (1-4), Select topic.

□ An introduction to lattice and other

Davey & Priestley

(General) Algebraic Structures.

Structures

Sub

Group

quotient

Ring

homomorphism

Vect space

Isomorphism Theories

Module

Lattice:

Def.

A lattice is an algebra structure consisted of
 a set L together with 2 binary operations
 (a function $L \times L \rightarrow L$)

$\wedge, \vee$ such that ...

(I) $\wedge, \vee$ are both commutative. ($x \vee y = y \vee x$ & $x \wedge y = y \wedge x$)

(II) $\wedge, \vee$ are both idempotent. ($x \vee x = x$; $x \wedge x = x$)

(III) $\wedge, \vee$ are both associative. $[(x \vee y) \vee z = x \vee (y \vee z)]$

$(x \vee y) \vee z = x \vee (y \vee z)$
 OFOGH
 SHARGH
 $\forall x, y, z \in L$

(IV) $\vee, \wedge$ satisfy absorption laws:

$$x \wedge (x \vee y) = x$$

$$x \vee (x \wedge y) = x$$

Example. Subgroups of a group form a Lattice G grb.

$$(\text{Sub}(G), \subseteq) \quad L = \text{Sub}(G)$$

$$\wedge: L \times L \rightarrow L$$

$$\vee: L \times L \rightarrow L$$

$$(H, K) \mapsto H \cap K$$

$$(H, K) \mapsto \langle H \cup K \rangle$$

Example. Ideals of a ring with $I \wedge J = I \cap J$

$$(\text{Id}(R), \subseteq)$$

$$I \vee J = \langle I \cup J \rangle$$

Example. Open subset of a topological space:

$$(\mathcal{O}(X), \subseteq) \quad U \wedge V = U \cap V \quad \text{کافیت بسته بود } \cap \text{ و } \cup \text{ نشان دهنده زیر}$$

$$U \vee V = U \cup V \quad \text{بهر خصوصیات شکلهای } \cap \text{ و } \cup \text{ برقرار است}$$

بسته بودن نیز به طور مستقیم از تعریف مجموعه های باز توپولوژی حاصل می شود.

Example. $\mathcal{P}(X)$ with $\cup, \cap$.

$\cup, \vee, \cap$ $\xrightarrow{\text{فصل}} \text{logical disjunction and conjunction}$

$\rightarrow$ proposition up to
ترتیب

$$\subseteq \subseteq \mathcal{P}(X) \quad \text{انگاشی و پائینانی و قی}$$

Def. A lattice is a poset $(P, \subseteq)$ in which

order $\forall x, y \in P, \sup \{x, y\}, \inf \{x, y\}$ exist.

Example. $L = \text{Sub Group } (G)$ such that G is group.

$$\wedge: L \times L \rightarrow L$$

$$(H, K) \mapsto H \cap K$$

$$\vee: L \times L \rightarrow L$$

$$(H, K) \mapsto \langle H \cup K \rangle$$

یادآوری: دسته‌ی گروه‌ها نسبت به زیرگروه، حاصل ضرب و خارج قسمت بسته است.
یعنی، زیردسته‌های مجزای گروه‌ها، حاصل ضرب گروه‌ها، و خارج قسمت گروه‌ها، خود گروه هستند.

تعریف: فرض کنیم X زیر مجموعه‌ی گروه G باشد. در این صورت، کوچکترین زیرگروه G را که شامل X باشد، زیرگروه تولیدشده توسط X می‌نامیم و آن را با $\langle X \rangle$ نشان می‌دهیم. اگر $G = \langle X \rangle$ ، می‌گوییم گروه G توسط X تولیدشده یا X مولد G است.

$$* \langle \emptyset \rangle = \{e\} = \langle e \rangle$$

قضیه: اگر $X \neq \emptyset$ زیر مجموعه‌ی گروه G باشد، آنگاه $\langle X \rangle$ (در اینجا دارای ضربی) شکل زیری حاصل ضرب بی‌تناهی به صورت $x_1, x_2, \dots, x_n$ است که در آن، برای هر n ،
 $x_i \in X$ یا $x_i \in X^{-1}$ یعنی:

$$\langle X \rangle = \{x_1 x_2 \dots x_n \mid n \in \mathbb{N}, x_i \in X \cup X^{-1}\}$$

$$\langle H \cup K \rangle = \{x_1 x_2 \dots x_n \mid n \in \mathbb{N}, x_i \in H \cup K \cup H^{-1} \cup K^{-1}\}$$

نتیجه: اگر $X \neq \emptyset$ زیر مجموعه‌ی گروه G باشد، آنگاه $\langle X \rangle = \{x_1^{n_1} x_2^{n_2} \dots x_k^{n_k} \mid x_i \in X, n_i \in \mathbb{Z}\}$

$$\mathbb{Z}[i] = \{m + ni \mid m, n \in \mathbb{Z}, i^2 = -1\} \quad \star \text{ گفته اعداد صحیح گاوسی}$$

$$\wedge: L \times L \rightarrow L$$

$$\vee: L \times L \rightarrow L$$

$$(H, K) \mapsto H \cap K$$

$$(H, K) \mapsto \langle H \cup K \rangle$$

۱) واضح است که هم جایی است.

$$\therefore H \wedge H = H \cap H = H$$

$$H \vee H = \langle H \cup H \rangle = \langle H \rangle = \{x_1, x_2, \dots, x_n \mid x_i \in H \cup H^{-1}\}$$

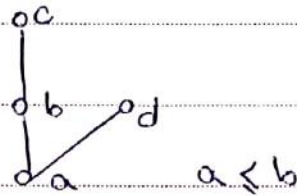
$$\therefore (H \wedge K) \wedge L = (H \cap K) \cap L = H \cap (K \cap L) = H \wedge (K \wedge L)$$

$$(H \vee K) \vee L = \langle H \cup K \rangle \vee L = \langle \langle H \cup K \rangle \cup L \rangle \quad ?$$

$$H = \{2\mathbb{Z}\}$$

$$K =$$

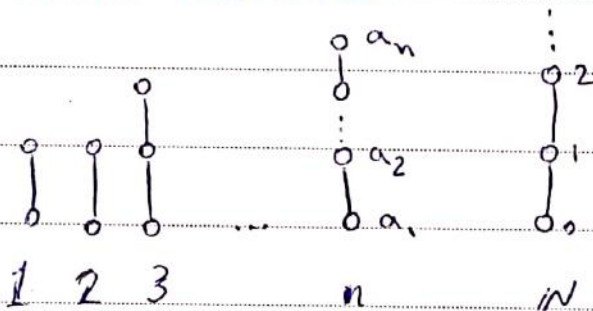
Hasse Diagram:



$$x \leq y \Leftrightarrow \sup \{x, y\} = y$$

$$x \leq y \Leftrightarrow \inf \{x, y\} = x$$

Some lattice:



نشان کنید X مجبوعه است. هر زیر مجموعه X به عبارتی
است از زیر مجموعه ای مثل τ از $P(X)$ که:

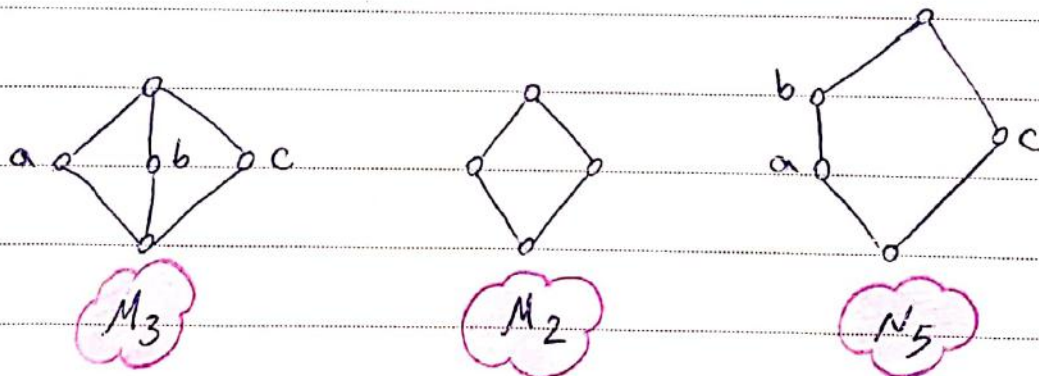
(i) $\phi, X \in \tau$

(ii) اگر $U \in \tau$ و V زیر مجموعه U باشد آنگاه

$U \cup \{V : V \in U\}$ در τ قرار دارد.

(iii) اگر $U, V \in \tau$ آنگاه $U \cap V \in \tau$

مجموعه های عضو τ باز هستند.


 $a \parallel b$

incomparable

Theorem. Algebraic Def lattice
 $\equiv$ order theoretic Def lattice.

نشان دهید

proof: $(\Rightarrow) L, \vee, \wedge \rightsquigarrow L$, define $\leq$ as: $alg \Rightarrow ord$
 $x \leq y \Leftrightarrow x \vee y = y$

$\Rightarrow x \leq x \quad \forall x$ $x \vee x = x$ $x \wedge x = x$
 II $\Rightarrow$ $x \leq y, y \leq x \Rightarrow x \vee y = y, y \vee x = x$
 III $\Rightarrow$ $x \leq y, y \leq z \Rightarrow (x \vee y) \vee z = z$
 IV $\Rightarrow$ $x \vee (y \vee z) = (x \vee y) \vee z$
 V $\Rightarrow$ $x \wedge (y \wedge z) = (x \wedge y) \wedge z$
 VI $\Rightarrow$ $x \leq y, y \leq z \Rightarrow x \leq z$
 VII $\Rightarrow$ $x \leq y, y \leq z \Rightarrow x \wedge z = x$
 VIII $\Rightarrow$ $x \leq y, y \leq z \Rightarrow x \vee z = z$
 IX $\Rightarrow$ $x \leq y, y \leq z \Rightarrow x \wedge (y \vee z) = x$
 X $\Rightarrow$ $x \leq y, y \leq z \Rightarrow x \vee (y \wedge z) = x \vee y$

$x \leq y, y \leq z \Rightarrow (x \vee y) \vee z = z$
 $x \vee y = y, y \vee z = z$
 $\Rightarrow x \vee (y \vee z) = z \Rightarrow x \leq z$

اینجا نشان دهم که $Poset$ است. inf و sup هم وجود دارد.

we claim $sup\{x, y\}^* = x \vee y$
 also $inf\{x, y\}^{**} = x \wedge y$

$x: x \vee (x \vee y) = (x \vee x) \vee y = x \vee y \Rightarrow x \leq x \vee y$

similarly $y \leq x \vee y$ $x \wedge y \leq x$ $x \wedge y \leq y$
 برای این مجموعه است.

$x \leq z, y \leq z \Rightarrow x \vee z = z, y \vee z = z$
 $\Rightarrow x \vee (y \vee z) = z \Rightarrow x \vee y \leq z$

$$** : x \vee (x \wedge y) = x \Rightarrow x \wedge y \leq x$$

$$\text{similarity } x \wedge y \leq y$$

این هر دو کران پایین است

$$\text{let } z \leq x, z \leq y \text{ then } z \vee x = x, z \vee y = y$$

$$x \leq y \Leftrightarrow x \vee y = y \stackrel{***}{\Leftrightarrow} x \wedge y = x$$

$$*** : x = x \wedge (x \vee y) = x \wedge y \quad (=>)$$

$$y = y \vee (x \wedge y) = y \vee x \quad (\Leftarrow)$$

$$z \wedge x = z, z \wedge y = z \Rightarrow (z \wedge y) \wedge x = z \wedge (y \wedge x) = z$$

$$\Rightarrow z \leq y \wedge x$$

($\Leftarrow$)

این هر دو کران پایین است

(P, $\leq$)

$\sup \{x, y\}$ and $\inf \{x, y\}$ exist.

$\Rightarrow P$, define $\vee, \wedge$

ord $\Rightarrow$ alg

$$x \vee y = \sup \{x, y\}$$

حاجه جایی دودنونی و شرکت پذیری واضح است.

$$x \wedge y = \inf \{x, y\}$$

قانون جابجایی (جذب)، ا ثابت می کنیم.

$$x \wedge (x \vee y) = \inf \{x, \sup \{x, y\}\} \stackrel{\text{daim}}{=} x$$

OFOGH
SHARGH

$$x \leq z, (x \leq x)$$

باید ثابت کنیم

$$z = \sup\{x, y\}$$

انگیزی

$$t \leq x, t \leq z \Rightarrow t \leq x$$



$$(L^0)^a \stackrel{?}{=} L$$

L lattice with alg def.

$$(P^a)^0 \stackrel{?}{=} P$$

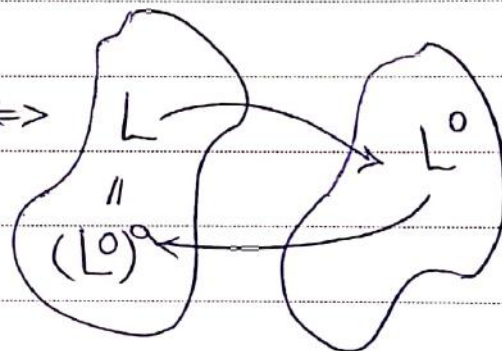
P lattice with ord def.

$$(L^0)^a = L, \leq$$

$$x \leq y \Leftrightarrow$$

$$x \vee y = y \Leftrightarrow$$

$$x \wedge y = x$$



$$\sup\{x, y\}$$

$$\inf\{x, y\}$$

$$\sup\{x, y\} \stackrel{?}{=} x \vee y$$

join, y است, join

$$\inf\{x, y\} \stackrel{?}{=} x \wedge y$$

sup, y است, join

$$(P^a)^0 = (P, \leq')$$

$$= (P, \leq)$$

$$\sup\{x, y\} \Leftrightarrow x \leq y \Rightarrow p$$

$$x \leq' y \Leftrightarrow x \leq y$$

Sublattice



$$L' \subseteq L, (L', \vee |_{L' \times L'} \wedge |_{L' \times L'}) \text{ است} \\ \equiv L' \subseteq L$$

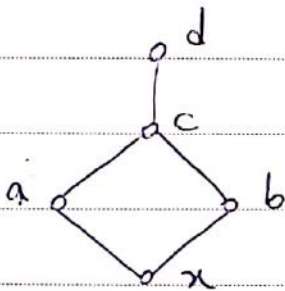
$$(x, y \in L' \Rightarrow x \wedge y, x \vee y \in L')$$

$$p' \subseteq p$$

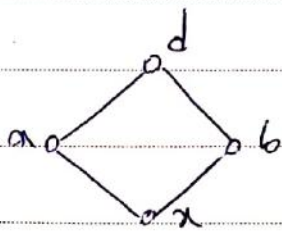
$$(x, y \in p' \Rightarrow \sup \{x, y\}, \inf \{x, y\} \in p')$$

نسبت به p

نشان بدهیم اینجاست $(p' \subseteq p)$ صدق است



صدق است



صدق است اما زیر صدق بالایی نیست

نیز

$$\sup \{a, b\}$$

که c است در شکل نیست

موضوع
۹۴، ۹، ۳.

Lattice homomorphism:

Def 1. A lattice homomorphism is a map between lattice $L_1, L_2, L, \xrightarrow{f} L_2$

such that $f(x \vee y) = f(x) \vee f(y)$ ①

$f(x \wedge y) = f(x) \wedge f(y)$ ②

$\forall x, y \in L_1$

Def 2. A poset homomorphism is a map between lattice $L_1, L_2, L, \xrightarrow{f} L_2$ such that

$x \leq_{L_1} y \Rightarrow f(x) \leq_{L_2} f(y)$ ③

(a monotone map, or order preserving)

monotone mapping

تکامل یکنوا

حافظ ترتیب

A lattice homo. is a poset homo. which preserves $\sup, \inf$, that is

حافظ عمل

$f(\sup_{L_1} \{x, y\}) = \sup_{L_2} \{f(x), f(y)\}$ ④

$f(\inf_{L_1} \{x, y\}) = \inf_{L_2} \{f(x), f(y)\}$ ⑤

حافظ ترتیب، تکامل یکنوا، حافظ عمل $\sup, \inf$

Remark. 1) If $L_1 \xrightarrow{f} L_2$ is a lattice homo. with def ① then is f a lattice

homo with def ②?

$$x \vee y = y$$

$$f(x \vee y) = f(y)$$

$$f(x) \vee f(y) = f(y)$$

$$\textcircled{1} \Leftrightarrow \textcircled{1'}$$

$$\textcircled{2} \Leftrightarrow \textcircled{2'}$$

$$\textcircled{1} + \textcircled{2} \Rightarrow \textcircled{3}$$

$$1 \Rightarrow 1'$$

$$\sup_{L_1} \{x, y\} = x \vee y$$

$$\sup_{L_2} \{f(x), f(y)\} = f(x \vee y)$$

2) If $f: L_1 \rightarrow L_2$ is a lattice homo with def 2. then is f a lattice homo with Def 1?

$$\text{yes, since } \textcircled{1'} \equiv \textcircled{1}$$

$$\textcircled{2'} \equiv \textcircled{2}$$

3) Does ③ imply ①, ②?

$$\textcircled{3} \Rightarrow \textcircled{1} \text{ or } \textcircled{2} \quad \textcircled{1} \text{ or } \textcircled{2} \Rightarrow \textcircled{3}$$

$$x \leq_L y \Leftrightarrow x \vee y = y \Leftrightarrow x \vee f(x) = x$$

$$x \leq_L y \Rightarrow x \vee y = y \Rightarrow f(x \vee y) = f(y)$$

$$\Rightarrow f(x) \vee f(y) = f(y)$$

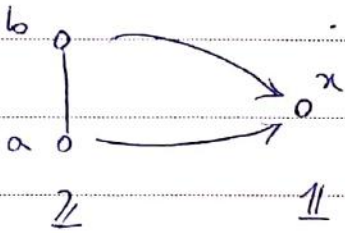
$$\Rightarrow f(x) \leq_{L_2} f(y)$$

$$f(x \vee y) = f(x) \vee f(y)$$

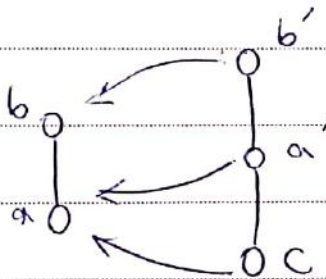
$$a) x \leq x \vee y \xrightarrow{\textcircled{3}} f(x) \leq f(x \vee y)$$

$$y \leq x \vee y \xrightarrow{\textcircled{3}} f(y) \leq f(x \vee y)$$

$$b) f(x) \leq z, f(y) \leq z \stackrel{?}{\Rightarrow} f(xvy) \leq z$$



$$\frac{f(a \vee b)}{x} = \frac{f(a) \vee f(b)}{x}$$



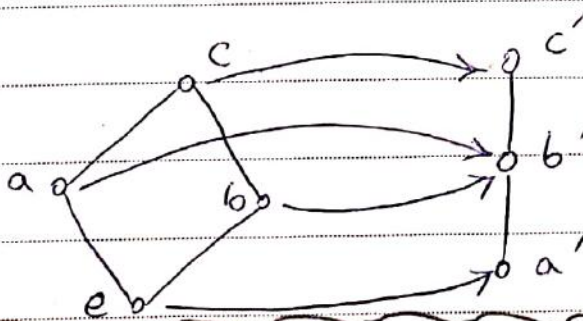
$$f(a' \vee c) = f(a') \vee f(c)$$

$$f(x) \leq z \leq f(xvy)$$

$$f(x) \vee f(y) = f(y)$$

Exersice.

f onto, f order-preserving $\Rightarrow f$ preserves $\vee, \wedge$.



$$a \wedge b = e$$

$$a' = f(e) = f(a \wedge b) \neq f(a) \wedge f(b)$$

$$e \leq_L a \rightarrow f(e) \leq_{L_2} f(a) \rightarrow a' \leq_{L_2} b' \quad \checkmark$$

$$e \leq_{L_1} b \rightarrow a' \leq_{L_2} b' \quad \checkmark$$

Isomorphism:

Def. lattice iso. lattice homo and bijection.
poset iso. poset homo and bijection and its inverse order-preserving.

Remark. If L_1, L_2 are lattices, and $f: L_1 \rightarrow L_2$ a map, then

- 1) f is a lattice iso.
 - 2) f is a poset iso.
- are equivalent

(1) $\stackrel{?}{\Rightarrow}$ (2)

$\vee, \wedge$ preserve $\Rightarrow f$ preserves $\wedge$
 f bijection $\Rightarrow f^{-1}$ bijection

f preserves $\vee, \wedge \stackrel{?}{\Rightarrow} f^{-1}$ preserves $\vee, \wedge$
 f bijection $\downarrow$
 $L_2 \rightarrow L_1$

$$f^{-1}(x' \vee y') = f^{-1}(x') \vee f^{-1}(y')$$

(2) $\stackrel{?}{\Rightarrow}$ (1)

$$f(x) \vee f(y) \stackrel{?}{\leq} f(x \vee y)$$

$$f(x) \leq z, f(y) \leq z \stackrel{?}{\Rightarrow} f(x \vee y) \leq z$$

$$f^{-1}(f(x)) \leq f^{-1}(z) \leq y \Rightarrow f(x \vee y) \leq z$$

Some kinds of lattices:

L a lattice

Def. (1) least element 0 : $0 \leq x \quad \forall x \in L$.

(2) greatest element 1 : $x \leq 1 \quad \forall x \in L$.

(3) Bounded lattice = A lattice which has least & greatest elements.

(4) Distributive lattice: (i) $x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z)$,
(ii) $x \wedge (y \vee z) \equiv (x \wedge y) \vee (x \wedge z)$

(5) Modular lattice.

Remark. (i) $\Leftrightarrow$ (ii)

(i) $\stackrel{?}{\Rightarrow}$ (ii)

$$x \wedge (y \vee z) \leq (x \vee y) \vee (x \wedge z)$$

$$(x \wedge y) \vee (x \wedge z) = ((x \wedge y) \vee x) \wedge ((x \wedge y) \vee z)$$

$$= x \wedge ((x \wedge y) \vee z) = x \wedge ((x \vee z) \wedge (y \vee z)) \stackrel{\text{جواب}}{=} x \wedge (y \vee z)$$

ملاحظہ

9F, 1V, 1F

A Course In Universal Algebra - Buris & Sankaranarayanan

ch I, II (12-14), IV (1-4) Selected topic.

Lattice: $L, \vee, \wedge: L \times L \rightarrow L$

s.t. 1, 2, 4

تقریباً
بہترین، شکرستیزی، جابجا، جوب

$:(L, \leq)$ s.t.

OFOGH
SHARGH

$\forall x, y \in L, \sup\{x, y\}, \inf\{x, y\}$ exist.

Distributive & modular lattice:

A lattice L is said to be distributive if:

$$i) x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \forall x, y, z \in L$$

$$ii) x \vee (x \wedge z) = (x \vee y) \wedge (x \vee z) \quad \forall x, y, z \in L$$

Remark. $(i) \Leftrightarrow (ii)$

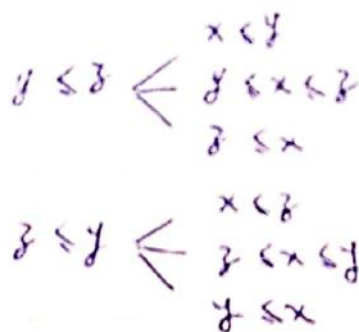
Example.

1

2

N

Every chain is a distributive lattice.

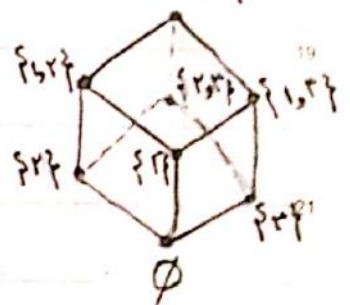


در زیر هر دو عنصری قابل تقایم اند.

کافیست برای اثبات این خاصیت
محاسبه کنیم.

Example. $(P(X), \subseteq)$ is a distributive lattice. $\{1, 2, 3\}$

Draw $(P(\{1, 2, 3\}), \subseteq)$.



Example. $(\text{sub}(G), \subseteq)$

$$H \wedge K = H \cap K$$

$$H \vee K = \langle H \cup K \rangle$$

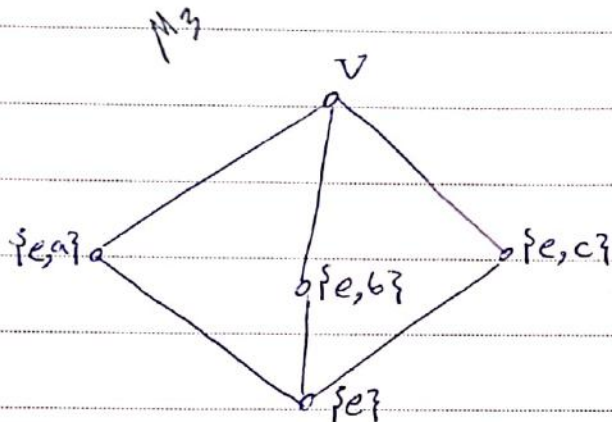
s.t. G is group.

! $\text{sub}(G)$ is not generally distributive.

$$V = \{e, a, b, c\}$$

نمودار

اجتماع دو زیر گروه، لزوماً گروه نیست.



$$\{e, a\} \vee (\{e, b\} \wedge \{e, c\})$$

$$= \{e, a\} \vee \{e\} = \{e, a\}$$

$$\neq (\{e, a\} \vee \{e, b\})$$

$$\wedge (\{e, a\} \vee \{e, c\})$$

$$= \{e, a, b, c\}$$

Example. $(O(X), \subseteq)$ is dist.

↓
open subsets

Theorem 1. A lattice L is non-distributive if and only if it has sublattice isomorphic to M_3 or to N_5 .
proof. Later on!

Def. A lattice L is called modular if $\forall x, y, z \in L$
 $x \leq y \Rightarrow x \vee (y \wedge z) = y \wedge (x \vee z)$

Remark. Distributive $\Rightarrow$ modular.

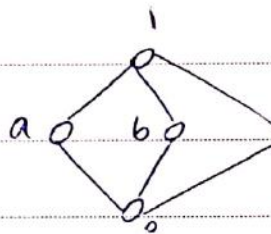
$\Leftarrow$

Proof. $(\Rightarrow) x \leq y$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \\ = y \wedge (x \vee z) \quad \checkmark$$

Remark. Every chain is modular.

$(\Leftarrow)$



M_3 is modular
but is not dist.

Example. N_5 is not modular.

$$a \leq b \Rightarrow a \vee (b \wedge c) \neq b \wedge (a \vee c) \\ 1 \wedge \quad \neq \quad b$$

Example $(\text{Sub}(G), \leq)$

$$H \leq K \quad H \vee (K \wedge N) \stackrel{?}{=} K \wedge (H \vee N)$$

$$H, K, N \leq G$$

$$\langle H \vee (K \wedge N) \rangle \supseteq K \wedge \langle H \vee N \rangle$$

$$\langle X \rangle = \{ x_1^{n_1} \dots x_k^{n_k} \mid x_i \in X, n_i \in \mathbb{Z}, k \in \mathbb{N} \}$$

این صحت دارد!

$$\text{مثال ۵:} \quad \langle H \vee N \rangle = \{ x_1 \dots x_k \mid x_i \in H \vee N, k \in \mathbb{N} \}$$

$$\begin{cases} x \in K \\ x \in \langle H \vee N \rangle \Rightarrow x = x_1 \dots x_k \quad x_i \in H \vee N \end{cases}$$

Example. $(\text{Sub}(M), \subseteq)$ is modular lattice.

sub modular → module

Remark. if $H, K \trianglelefteq G$ then $\langle H \cup K \rangle = HK$.

چون در مودول، گروه، این است، مودول راست!

$$H + (K \cap N) \stackrel{?}{=} K \cap (H + N)$$

$H \subseteq K$ what about $(N(G), \subseteq)$? مودول!

$$H \vee (K \wedge J) = K \wedge (H \vee J) \quad \text{Normal subgroup.}$$

$$H \cup (K \cap J) = K \cap (H \cup J), \quad x \in H \cup (K \cap J) \Rightarrow x \in H \text{ or } x \in K \cap J$$

Theorem 2. A lattice L is non-modular if and only if it has sublattice isomorphic to N_5 .

$$x \in H \text{ or } x \in K \cap J.$$

$$\Rightarrow x = k_j \in H \cup K \cap J \quad \text{Proof. } (\Leftarrow) \quad x \vee (y \wedge z) \geq y \wedge (x \vee z)$$

$$, k \in K, j \in J \quad (\Rightarrow): \text{non-modular}$$

$$\Rightarrow x \in K(H \cup J)$$

$$\exists x, y, z \in L$$

$$\text{برعکس } x \in K(H \cup J)$$

$$x \leq y, \quad x \vee (y \wedge z) \not\leq y \wedge (x \vee z)$$

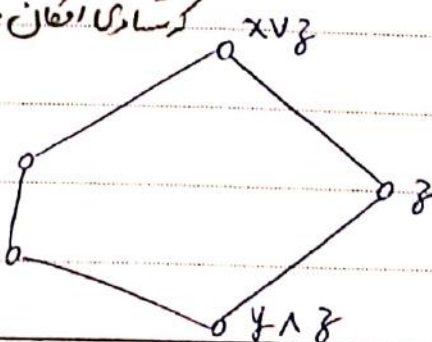
$$x = k_i \text{ s.t. } k \in K, i \in H \cup J \quad \text{کرتاری امکان پذیر نود!$$

$$\Rightarrow k \in K, i \in H \text{ or } i \in J$$

$$\Rightarrow k_i \in K \cap J \text{ or } k_i \in H$$

$$\Rightarrow x \in H \cup (K \cap J)$$

$$x \vee (y \wedge z)$$



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we show that the following is a sublattice of L .

کمیته نشان دهم بگو

$$y \wedge z \neq z \checkmark$$

$$\{y \wedge z, x \vee (y \wedge z), y \wedge (x \vee z), z, x \vee z\}$$

$$x \vee z \neq z \checkmark$$

مشکله است با ۸، ۷ و ۶ در شبکه L صدقند

$$y \wedge z \neq x \vee (y \wedge z) \checkmark$$

برای این منظور اول نشان می دهیم این عناصر

$$x \vee z \neq y \wedge (x \vee z) \checkmark$$

متمايزند و در نشان نسبت به

$$[x \vee (y \wedge z)] \vee z = x \vee z \checkmark$$

۸ و ۷ بسته است.

$$[x \vee (y \wedge z)] \wedge z = y \wedge z \checkmark$$

(با توجه به شکل ادغام)

$$[y \wedge (x \vee z)] \vee z = x \vee z \checkmark$$

$$[y \wedge (x \vee z)] \wedge z = y \wedge z \checkmark$$

proof Th. 1: $(\Leftarrow) \checkmark$

$(\Rightarrow)$: Let L do not have a sublattice isomorphic to N_5 .

Then by Th. 2, L is modular.

Since L is non-dist,

$$\exists x, y, z \in L$$

$$(x \vee y) \vee (x \vee z) \neq x \vee (y \vee z)$$

که در شبکه صدق نمی کند

$$d = (x \vee y) \vee (x \vee z) \vee (y \vee z) > e$$

$$(x \vee e) \vee d$$

$$(y \vee e) \vee d$$

$$(z \vee e) \vee d$$

$$e = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$$

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is a sublattice of $L(?)$

Def. A Complete Poset is a poset in which every subset has supremum and infimum.

$$\vee$$

$$\wedge$$

Def. Any complete poset is a lattice, called a complete lattice.

نوعی پوزیت است که هر زیرمجموعه آن بزرگترین و کوچکترین عضو دارد.
 $(\mathbb{R}_{-\infty}^{+\infty}, \leq)$

Example. $(P(X), \leq)$ is a complete lattice.

$$\bigvee A_i = \bigcup_{i \in I} A_i = \{x \in X \mid \exists i \in I, x \in A_i\}$$

$$\bigwedge A_i = \bigcap_{i \in I} A_i = \{x \in X \mid \forall i \in I, x \in A_i\}$$

Example. $([0, 1], \leq)$ is a complete lattice.

بزرگترین و کوچکترین عضو $\mathbb{R}$ است.

Example. $(Eq(X), \leq)$

↪ equivalence relations

Example. $(O(X), \leq)$

$$\bigwedge U_i = (\bigcap U_i)^0, \quad \bigvee U_i = \bigcup U_i$$

جواب
۹۴، ۷، ۶

Complete poset: A poset P in which $\forall x \subseteq P, \bigvee x, \bigwedge x$ exist.

Example. $(P(X), \subseteq)$ ✓

Example. $(Eq(X), \subseteq)$ (?)

Remark. $(P, \subseteq)$ a poset

$\forall x \subseteq P \quad \bigvee x \text{ exist} \iff \forall x \subseteq P \quad \bigwedge x \text{ exist.}$

$(\Rightarrow) \quad x \subseteq P \Rightarrow \mathcal{L}(x) \subseteq P \Rightarrow \bigvee \mathcal{L}(x) \text{ exist}$
چون:

$$\mathcal{L}(x) := \{t \in P : t \leq x, \forall x \in x\}$$

$$\mathcal{U}(x) = \{t \in P : x \leq t, \forall t \in \mathcal{U}(x)\}$$

$$\bigvee \mathcal{L}(x) = \bigwedge x$$

$$\forall t (t \in \mathcal{L}(x) \Rightarrow t \leq x, \forall x \in x)$$

$$x \in x \Rightarrow x \in \mathcal{U}(\mathcal{L}(x))$$

$$\mathcal{U}(\mathcal{L}(x)) = \{t \in P : \mathcal{L}(x) \leq t, \forall t \in \mathcal{U}(\mathcal{L}(x))\} \Rightarrow \bigvee \mathcal{L}(x) \leq x.$$

($\Leftarrow$)

$$\text{let } z \leq x, \forall x \in x$$

$$\Rightarrow z \in \mathcal{L}(x) \Rightarrow z \leq \bigvee \mathcal{L}(x).$$

Example. $(\mathcal{O}(X), \subseteq)$ Complete.
 $\downarrow$
 top. sp.

$$\bigcup U_i \in \mathcal{O}(X)$$

$$\text{if } U_i \in \mathcal{O}(X) \Rightarrow \bigvee U_i = \bigcup U_i$$

در این سیستم که اسی بودن و رابطه هم‌شکلی بودن و در رابطه ای هم‌ارزی است پس رابطه هم‌شکلی در آن که هم‌شکلی است پس

Example $(Eq(X), \subseteq)$ Complete. هم‌ارزی است.

$$\text{if } R_i \in Eq(X) \Rightarrow \bigcap R_i \in Eq(X)$$

$$\bigwedge R_i = \bigcap R_i$$

اگر R_1 و R_2 هم‌ارزی باشند آنگاه $R_1 \cap R_2$ نیز هم‌ارزی است اما

$R_1 \cup R_2$ هم‌ارزی نیست زیرا اگر $(x, y) \in R_1$ و $(y, z) \in R_2$ آنگاه (x, z) هم‌ارزی نیست

چون برای همین صورت join را بصورت دیگری تعریف می‌کنیم

what is $\vee$ in $Eq(X)$?

چون $R_2 \circ R_1$ قیاسی نیست پس برای استفاده به عنوان join نیست.

$$\begin{aligned} R_1 &\subseteq R_2 \circ R_1 \quad \checkmark \quad (x, y) \in R_2 \circ R_1 \Rightarrow (x, z) \in R_2 \Rightarrow (x, z) \in R_1 \\ R_2 &\subseteq R_2 \circ R_1 \quad \checkmark \end{aligned}$$

$$\begin{aligned} R_1 &\subseteq R_3 \\ R_2 &\subseteq R_3 \Rightarrow R_2 \circ R_1 \subseteq R_3 \quad \checkmark \end{aligned}$$

Lemma. If $R_i \in \text{Eq}_h(X)$, $i \in I$, then

$$VR_i = \langle UR_i \rangle = \bigcap S$$

generated by $\leftarrow$

$$S \in \text{Eq}_h(A)$$

$\text{Eq}_h \text{ rel.}$

$$UR_i \in S$$

$$= \bigcup_{\substack{n \in \mathbb{N} \\ i \in I}} (R_{i_1} \circ \dots \circ R_{i_n}) = R_1 \cup (R_1 \circ R_2) \cup (R_1 \circ R_2 \circ R_3) \cup \dots \\ \cup (R_1 \circ R_2 \circ R_1) \cup \dots$$

$$R_1 \vee R_2$$

$$= R_1 \cup (R_1 \circ R_2)$$

$$\cup R_2 \cup (R_2 \circ R_1)$$

$$\cup (R_1 \circ R_2 \circ R_1)$$

Exercise: $R_1 \vee R_2$

$$= R_1 \cup (R_1 \circ R_2)$$

$$\cup (R_1 \circ R_2 \circ R_1)$$

$$\cup (R_1 \circ R_2 \circ R_1 \circ R_2)$$

$\vdots$

$$X = \{1, 2, 3\}, T = \{(1, 2)\}$$

$$\langle T \rangle = T \cup T^{-1}$$

$$\Delta U = \{(x, x) : x \in X\}$$

$$\langle T \rangle^t \text{ Transitive closure}$$

$$T \subseteq X \times X$$

$$(x, y) \in \langle T \rangle \leftarrow \text{Eq. rel. generated}$$

$$\Leftrightarrow \exists z_1, z_2, \dots, z_n \in X$$

$$(x, z_1) \in T \text{ or } (z_1, x) \in T$$

! 2nd step

$$(z_1, z_2) \in T \text{ or } (z_2, z_1) \in T$$

! 1st step

⋮

$$(z_n, y) \in T \text{ or } (y, z_n) \in T$$

$$(1) \quad \bigcup (R_i \circ \dots \circ R_n) \in \text{Eq}(X) \quad \checkmark$$

$$(2) \quad R_i \subseteq R \quad \forall i \quad \checkmark$$

$$(3) \quad S \in \text{Eq}(X)$$

$$R_i \subseteq S \Rightarrow R \subseteq S \quad \checkmark$$

$$\text{because } R_i \subseteq S \Rightarrow R_i \circ \dots \circ R_n \subseteq S \circ \dots \circ S = S \\ \Rightarrow R \subseteq S$$

Remark. $R_i \subseteq S \iff R_i \subseteq S$

$$R \rightsquigarrow \frac{X}{R} = \{ [x]_R : x \in X \}$$

$$x \sim y \iff \boxed{\quad}$$

$$\iff$$

$$\exists C \in \boxed{\quad}, x, y \in C$$

Theorem. $(\Pi(X), \subseteq)$ all partitions on X is a complete lattice.

Exercise. $\leq$ on $\Pi(X)$? $\bigcup, \bigcap$ on $\Pi(X)$?

Algebraic Lattice:

Def. X is compact if $X = \bigcup_{i \in I} U_i$. Then $\exists U_1, \dots, U_n$
 $X = U_1 \cup \dots \cup U_n$. In $(O(X), \subseteq)$.

Def. Compact element of a lattice L :

$$x_0 \in L, \forall T \subseteq L$$

$$x_0 \leq \bigvee T \Rightarrow \exists T' \subset T, x_0 \leq \bigvee T' \\ \text{finite} \\ (y \subseteq UZ)$$

$$\forall Z \in P(X) \quad y \leq \bigvee Z \Rightarrow \exists Z' \subset Z, y \leq \bigvee Z'$$

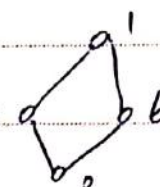
Example. In $(P(X), \subseteq)$, $Y \subseteq X$ is compact $\Leftrightarrow Y$
 is finite, $Y \in P(X)$

$$Y = \bigcup_{y \in Y} \{y\} \Rightarrow Y = \{y_1, y_2, \dots, y_n\}$$

Example. In $(\text{Sub}(G), \subseteq)$, $H \subseteq G$ is compact.
 $\Leftrightarrow H$ is finitely generated.

Example. In $([0, 1], \leq)$ (?)

Def. A Complete lattice L is called algebraic if
 every $x \in L$ is a supremum of a family
 of compact element.

Example.  $a = \bigvee \{a_1\}$

دفعہ ← Proof. حسب سبب

۹، ۷، ۱۱

$$\Rightarrow H \subseteq K, H \subseteq G, K \subseteq G \Rightarrow H \subseteq K \equiv x, y \in H \Rightarrow xy \in H$$

۳۔ H زیر گروہ G نہ کہ جس میں $xy \in H$ ۔ از قیہ $H \subseteq K$ جہاں H اسٹانڈرڈ ہے۔

۵۔ if $H \subseteq G, K \subseteq G$ then $H \subseteq K \Leftrightarrow H \subseteq K$.

Closure Operators $\equiv$ Closure Systems

→ in exercises!

Def. Let X be a set. Then a closure operator on X is a map. $C: \mathcal{P}(X) \rightarrow \mathcal{P}(X)$

such that:

(1) Extension : $A \subseteq C(A) \quad \forall A \subseteq X$

(2) Monotone : $A \subseteq B \Rightarrow C(A) \subseteq C(B) \quad \forall A, B \subseteq X$

(3) Idempotent : $C(C(A)) = C(A) \quad \forall A \subseteq X$

Def. $A \subseteq X$ is said to be C -Closed or Closed if

$$C(A) = A.$$

$$B \in L_C \Rightarrow B = C(B) \Rightarrow B \in \{C(A) \mid A \subseteq X\}$$

Notation. $L_C = \{A \subseteq X \mid C(A) = A\}$

کلاز بندی

$$B \in \{C(A) \mid A \subseteq X\} \Rightarrow \exists A \subseteq X \text{ st. } C(A) = B$$

$$= \{C(A) \mid A \subseteq X\} \checkmark$$

نہیں سیم $A \neq B$

Theorem. (i) For every closure operator C , $(L_C, \subseteq)$ is a complete lattice, $\forall A \subseteq L_C \exists \bigwedge A$

$$\bigwedge A \doteq A$$

(ii) Every complete lattice is isomorphic to

L_C , for some closure operator C .

نکته: تمام مجموعه های کامل از یک همگرایی کامل می شود.

Example. Kuratowski closure operator:

(X, τ) a top. sp. F closed $= F'$ open.

$\emptyset \in \tau \doteq \text{open}$,

$\bar{(\)} : P(X) \rightarrow P(X)$

$A \mapsto \bar{A} = \bigcap \{ F \subseteq X \mid F \text{ closed}, A \subseteq F \}$.

$A = \bar{A} \iff A$ closed in X .

Example. G group

$C : P(G) \rightarrow P(G) \quad C(A) = \langle A \rangle$

$A \mapsto \langle A \rangle$

$C(A) = A \iff \langle A \rangle = A$

$\iff A \leq G$

$L_C = \{ A \mid A \leq G \} = \text{Sub}(G)$

هر چه تا حالا از یک گروه، همگرایی برداریم

$A \subseteq \langle A \rangle$

$A \subseteq B \implies C(A) = \langle A \rangle \subseteq \langle B \rangle = C(B)$

$\langle \langle A \rangle \rangle = \langle A \rangle$

Proof (i): It is enough to show that L_C is closed under arbitrary intersection.

$$C(\bigcap_{i \in I} A_i) = \bigcap_{i \in I} A_i$$

$$A_1 \subseteq C(A_1) \rightarrow \text{we know } \bigcap_{i \in I} A_i \subseteq C(\bigcap_{i \in I} A_i)$$

$$A_2 \subseteq C(A_2)$$

$$A_1 \cap A_2 \subseteq C(A_1 \cap A_2)$$

:

It is enough to show that

$$\forall i \in I \quad C(\bigcap_{i \in I} A_i) \subseteq A_i$$

$$\text{we know } \forall i \quad \bigcap_{i \in I} A_i \subseteq A_i$$

$$\Rightarrow C(\bigcap_{i \in I} A_i) \subseteq \underbrace{C(A_i)}_{A_i} \quad \forall i$$

Exercise.

$$\bigvee_{i \in I} A_i = C(\bigvee_{i \in I} A_i) = \bigvee_{i \in I} A_i$$

$$A_i \subseteq \bigvee_{i \in I} A_i \Rightarrow C(A_i) \subseteq C(\bigvee_{i \in I} A_i)$$

$$C(A_i) = A_i$$

$$\bigvee_{i \in I} A_i \subseteq C(\bigvee_{i \in I} A_i)$$

Proof (ii): Let L be a complete lattice.

Define

$$C: P(L) \rightarrow P(L)$$

$$A \mapsto C(A) = \{x \in L : x \leq \bigvee A\}$$

$$i) A \subseteq C(A)$$

$$ii) A \subseteq B \Rightarrow C(A) \subseteq C(B)$$

$$\text{because } \bigvee A \leq \bigvee B$$

$$iii) C(C(A)) = C(A)$$

because

$$A \in C(A) \Rightarrow C(A) \in C(C(A))$$

$$\text{and if } z \in C(C(A))$$

$$\Rightarrow z \leq \bigvee C(A) \Rightarrow (\forall A \in C(A))$$

$$\text{Now } L_C \stackrel{\varphi}{=} L$$

$$\bigvee C(A) = \bigvee A$$

$$A \mapsto \bigvee A$$

$$C(A) = A$$

$$\downarrow x = \{y \in L \mid y \leq x\} \xleftarrow{\varphi^{-1}} x \in L$$

$$C(B) \stackrel{?}{=} B$$

$$A = \downarrow \bigvee A$$

$$x = \bigvee \downarrow x$$

$$\begin{array}{c} L \rightarrow L_C \\ \varphi = \varphi^{-1} \\ x \xrightarrow{\varphi} \downarrow x \end{array}$$

$$\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$$

$$\varphi(x \wedge y) = \varphi(x) \cap \varphi(y) \quad \checkmark$$

$$\downarrow(x \wedge y) \quad \downarrow x \quad \cap \quad \downarrow y$$

$$\downarrow(x \vee y) = \downarrow x \vee \downarrow y$$

$$\stackrel{\text{نفسه}}{=} C(\downarrow x \vee \downarrow y)$$

$$z \leq x \vee y$$

$$\vee (\downarrow x \cup \downarrow y)$$

$$\{t: t \leq x \text{ or } t \leq y\}$$

$$z \leq x \text{ or } z \leq y \Rightarrow z \leq x \vee y$$

and

$$\text{if } z \leq x \vee y \equiv \vee \{x, y\}$$

$$\leq \vee (\downarrow x \cup \downarrow y)$$

$$\text{because } \{x, y\} \subseteq (\downarrow x \cup \downarrow y)$$

Remark. $C(\{x\}) = \downarrow x = \{y \in L \mid y \leq x\}$

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \dots + \binom{n}{r} a^{n-r} b^r + \dots + b^n$$

بسته بودن که واضح است.

$$x \equiv x', y \equiv y' \Rightarrow (x, y) = (x', y') \text{ فرض کنیم}$$

$$x \equiv x', y \equiv y' \Rightarrow x \equiv x', y \equiv y'$$

$$\underbrace{x \dots x}_d \equiv \underbrace{x' \dots x'}_{d'} = \underbrace{(\varepsilon q_h + x) \dots (\varepsilon q_h + x)}_{\varepsilon q_h' + y}$$

$$\begin{aligned} x \equiv &= \frac{\varepsilon q_h' + y}{x} + (\varepsilon q_h' + y) x^{\varepsilon q_h' - 1 + y} \varepsilon q_h + \dots + \varepsilon q_h^{\varepsilon q_h' + y} \\ &\equiv \frac{\varepsilon q_h' + y}{x} = x^{\varepsilon q_h'} x^y \end{aligned}$$

$$\varepsilon = 2, \frac{y}{x} = 2, q_h' = 1, y = 2, x = 2 \text{ و } x \not\equiv x^{\varepsilon q_h' + y}$$

Universal Algebra:

A set with some operation on it.

A : a set.

operation on A ?

we studied binary operation that it is a map

$$f: A \times A \rightarrow A$$

such that:

$$I) \forall (x, y) \in A \times A, f(x, y) \in A$$

$$II) (x, y) = (x', y') \Rightarrow f(x, y) = f(x', y')$$

Example $\oplus_4: \mathbb{Z}_4 \times \mathbb{Z}_4 \rightarrow \mathbb{Z}_4$ is a binary operation.

but

نحوه بهمانند عمل دو تایی روی $\mathbb{Z}_4$ نیست

عمل یک تایی

unary operation:

$$f: A' \rightarrow A$$

تغییر در اعداد صحیح، عدد در اعداد گویا، متمم مجز در مجموعه...

عمل صفر تایی

nullary operation:

$$f: A^0 \rightarrow A$$

$$\{\emptyset\}$$

1 یک عضو هم باشد صفر، یک، و غیره خشتی و ...

3

تجربه نتایج از B بر A

$$A^B$$

$$A^2 = A^{\{1,2\}} \approx A \times A \rightarrow A$$

7 n -ary operation :

9

$$f: \underbrace{A \times \dots \times A}_{n \text{ بار}} \rightarrow A$$

a.b-1

11 سورنیم علمی ایجابی است. (از دو قضاچه و یا قضاچه نیست) ← مربوط به جبرناخانه

13

$$P(A) \xrightarrow{V} A$$

15 Hyper operation :

17

$$A^n \rightarrow P(A) \leftarrow \text{جبرناخانه}$$

Definition.

19 A universal algebra is a pair

$(A; (f_{n_f})_{f \in \Sigma})$ where A is a set and f_{n_f} is an n_f -ary operation on A and Σ is a set.

23 In this case, we say $\Sigma = (n_f)_{f \in \Sigma}$

is the type or signature of A .

Example. $(\mathbb{N}; +, \cdot, 1)$ is an algebra of type $(2, 2, 0)$.

$(\mathbb{Z}; +, -(), 0, 1)$ is an algebra of type $(2, 1, 0, 0)$.
عزلا تروی نوشته می شود.

نشان
جایگاه
۹۴, ۷, ۱۴

Universal algebra

$$f_{n_f}^A : A^{n_f} \rightarrow A$$

$$(A; (f_{n_f}^A)_{f \in \Sigma})$$

$$f \text{ type } (n_f)_{f \in \Sigma}$$

$$\lambda_A : A^{n_\lambda} \rightarrow A$$

$$(A, (\lambda_A)_{\lambda \in \Sigma})$$

$$f \text{ type } (n_f)_{f \in \Sigma}$$

Example.

Groupoid: $(G; *)$ of type (2)

Semigroup: $(S; *)$ of type (2) such that

$$*: S \times S \rightarrow S$$

$$\forall x, y, z \in S \quad x(* (x, y), z)$$

$$= * (x, * (y, z))$$

associativity identity
(equation)

quasi identity (منه شرطی شده وجودی ...)

$$* (x, y) = * (x, z) \Rightarrow y = z$$

Monoid: $(M; *, e)$ of type $(2, 0)$ s.t.

1) $*$ is associative.

$$2) * (x, e) = x = * (e, x) \quad \forall x \in M$$

$$e: \{\emptyset\} \rightarrow M$$

Group: $(G; *, (^{-1}), e)$ of type $(2, 1, 0)$

s.t.

$$1) * (x(x, y), z) = * (x, * (y, z))$$

$$2) * (x, e) = x = * (e, x)$$

$$3) * (x, x^{-1}) = e = * (x^{-1}, x)$$

تقریب کرده از $(G, *)$: Group Theory مثل جبر جابجایی

2) $\exists e \in G, \dots$

3) $\forall x \exists x^{-1}, \dots$

Ring: $(R; +, \cdot, -(), 0)$
type $(2, 2, 1, 0)$

1) $(R; +, -(), 0)$ is an abelian group.

2) $(R, \cdot)$ is a semigroup.

3) $x \cdot (y + z) = x \cdot y + x \cdot z$

$(y + z) \cdot x = y \cdot x + z \cdot x$

$\forall x, y, z \in R$

lattice: $(L; \vee, \wedge)$ of type $(2, 2)$

s.t.

1) $\dots$

2) $\dots$

3) $\dots$

4) $\dots$

Vector Space: over a Field F :

$(F; +, \cdot, -(), 0, 1)$ و $(F - \{0\}, \cdot, (\cdot)^{-1}, 1)$

یکایتمو شیم چون $(\cdot)^{-1}: F \rightarrow F$ دجوزدار.

Vector Space over a field F :

$$\begin{aligned} F \times V &\xrightarrow{\lambda} V \\ \lambda_a: V &\rightarrow V \\ a \in F \quad v &\mapsto \lambda(a, v) \end{aligned}$$

$(V; \oplus, -\odot, (\lambda_a)_{a \in F}, \theta)$ of type $(2, 1, (1)_F, 0)$ s.t.

1) $(V, \oplus, -\odot, \theta)$ is an abelian group.

$$2) \lambda_{a_1+a_2}(v) = \lambda_{a_1}(v) \oplus \lambda_{a_2}(v)$$

$$3) \lambda_{a_1 \cdot a_2}(v) = \lambda_{a_1}(\lambda_{a_2}(v))$$

$$\lambda_{a_1} \cdot a_2 = \lambda_{a_1} \circ \lambda_{a_2}$$

$$4) \lambda_a(v \oplus w) = \lambda_a(v) \oplus \lambda_a(w)$$

$$5) \lambda_1(v) = v$$

$$\lambda_1 = \text{id}_V$$

$$\lambda(a_1 + a_2, v) =$$

$$\lambda(a_1, v) \oplus \lambda(a_2, v)$$

$$\lambda(a_1 \cdot a_2, v) = \lambda(a_1, \lambda(a_2, v))$$

Module Over a ring R :

$$(M, +, -(\cdot), \odot, (\lambda_r)_{r \in R})$$

similar to Vec sp.

یون R نروا گیر، ریت پس نروا 5، Vec sp

نروا ریت

G -Set, G a group oid اگر گروه داشته باشد

X a set شرط دوم تکرار نیست!

$(X; (\lambda_x)_{x \in G})$ of type $(1)_G$ s.t.

$$1, \lambda_{x*y} = \lambda_x \circ \lambda_y$$

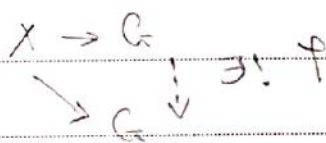
$$2, \lambda_e = id_X$$

R-ab group.

Automaton:

Q a set (of states) ایستگاه

A a set (of alphabet) الفبا



A^* : Free monoid generated by A .

If Q is an A^* -Set then

we say Q is a

deterministic automaton

$$A^* \times Q \rightarrow Q \quad (1)_{A^*}$$

Boolean algebra:

$$(B; \vee, \wedge, ()', 0, 1)$$

of type $(2, 2, 1, 0, 0)$

s.t.

1) $(B, \vee, \wedge, 0, 1)$ is a bounded distributive lattice.

$$\begin{cases} x \vee 1 = 1 \\ x \vee 0 = x \\ x \wedge 1 = x \\ x \wedge 0 = 0 \end{cases}$$

2) $x \vee x' = 1, x \wedge x' = 0$

Quasigroup:

$(Q; /, \cdot, 1)$ of type $(2, 2, 2)$ s.t.

1) $x \setminus (x \cdot y) = y$

2) $(x \cdot y) / y = x$

3) $x \cdot (x \setminus y) = y$

4) $(x / y) \cdot y = x$

Loop: A loop is an algebra

$(Q; /, \cdot, 1, 1)$ of type $(2, 2, 2, 2)$ s.t.

$(Q, /, \cdot, 1)$ is a quasigroup

and $x \cdot 1 = x = 1 \cdot x$

Heyting Algebra:

Brouwerian lattice
intuitionistic logic

$(H; \vee, \wedge, \rightarrow, 0, 1)$ of
type $(2, 2, 2, 0, 0)$ s.t.

1) $(H; \vee, \wedge, 0, 1)$ is
a bounded distributive
lattice.

$$2) (x \rightarrow x) = 1$$

$$(x \vee \neg y) \wedge y = (x \wedge y) \vee (\neg y \wedge y) = x \wedge y ? \quad 3) (x \rightarrow y) \wedge y = y$$

$$x \wedge (x \vee \neg y) = x ?$$

$$4) x \wedge (x \rightarrow y) = x \wedge y$$

$$x \vee \neg (y \wedge z) = x \vee \neg y \vee \neg z = (x \rightarrow y) \vee (x \rightarrow z) ? \quad 5) x \rightarrow (y \wedge z) =$$

$$(x \rightarrow y) \wedge (x \rightarrow z)$$

$$(x \vee \neg z) \wedge (y \vee \neg z) = \neg z \vee (x \wedge y) \\ = (x \wedge y) \rightarrow z ?$$

$$6) ((x \vee y) \rightarrow z)$$

$$= (x \rightarrow z) \wedge (y \rightarrow z)$$

Sub-universal algebra.

$$B \leq A \Leftrightarrow \begin{cases} B \subseteq A \\ \downarrow_A|_B = \downarrow_B \end{cases}$$

$$b_1, \dots, b_{n-1} \in B \Rightarrow \downarrow_A(b_1, \dots, b_{n-1}) \in B$$

تفريق بولین

۹۴، ۷، ۱۸
 جمله، سیم

Subalgebra. Let $(B, (\lambda_B)_{\lambda \in \mathbb{R}})$ and $(A, (\lambda_A)_{\lambda \in \mathbb{R}})$ be algebras of type $\tau = (\tau_\lambda)_{\lambda \in \mathbb{R}}$ we say that B is a subalgebra of A ; and write $B \leq A$, if:

$$1) B \subseteq A$$

$$2) \lambda_A|_{B^{n_\lambda}} = \lambda_B$$

Remark. A subset B of an algebra $(A; (\lambda_A)_{\lambda \in \mathbb{R}})$ can be an underlying set of a subalgebra iff B is closed under the operations on A ; that is

$$\forall \lambda \in \mathbb{R}, \forall b_1, \dots, b_{n_\lambda} \in B, \lambda_A(b_1, \dots, b_{n_\lambda}) \in B$$

$$\lambda_B(b_1, \dots, b_{n_\lambda}) = \lambda_A(b_1, \dots, b_{n_\lambda}) \in B$$

$$\lambda_B: B^{n_\lambda} \rightarrow B$$

Example. $(\mathbb{N}, +) \leq (\mathbb{Z}, +)$ عمل درستی است
 groupoid

$(\mathbb{N}, +) \leq (\mathbb{Z}, +)$ اتحاد است
 semigroup

اگر دو مجموعه برابر باشند
 در یک مجموعه حتماً اتحاد برابر است

$(\mathbb{N}_0, +, 0) \leq (\mathbb{Z}, +, 0)$
 monoid

$$- (\mathbb{Z} \times \{0\}, x) \leq (\mathbb{Z} \times \mathbb{Z}, x)$$

semigroup

$$s.t. (m, n) \times (m', n') = (mm', nn')$$

$$(1, 0) \in \mathbb{Z} \times \{0\}$$

عنصر

$$(1, 1) \in \mathbb{Z} \times \mathbb{Z}$$

عنصر

فرض کنید بنویسید!

But $\nless$ monoid because $(1, 0) \neq (1, 1)$.

$$\lambda_a^A : A^0 = \{x\} \rightarrow A$$

$x \mapsto a_x$

$$B \subseteq A$$

$$\lambda_a^A|_{B^0} = B^0 = \{0\} \rightarrow B$$

$$a_x \in B$$

is not monoid

$$(\mathbb{Z} \times \{0\}, x, (1, 1)) \nless (\mathbb{Z} \times \mathbb{Z}, x, (1, 1))$$

$(1, 0) \nless$

$$- \lambda (G, *) \leq (G', *')$$

subgroup

$$\Leftrightarrow G \subseteq G', *'|_{G \times G} = *, e' = e$$

$$- \lambda (G, *, ()^{-1}) \leq (G', *', ()^{-1})$$

subgroup

$$\Leftrightarrow G \subseteq G', *'|_{G \times G} = *, ()^{-1}|_{G \times G} = ()^{-1}$$

OFOGH SHARGH

$$\checkmark (G, *, (\cdot)^{-1}, e) \stackrel{\text{Subgroup}}{\leq} (G', *', (\cdot')^{-1}, e')$$

$$\Leftrightarrow G \leq G', *' \upharpoonright_{G \times G} = *, (\cdot')^{-1} \upharpoonright_G = (\cdot)^{-1}$$

$$, e = e' \in G$$

$$\times (G, *, e) \stackrel{\text{Subgroup}}{\leq} (G', *', e') \quad (\mathbb{N}, +) \leq (\mathbb{Z}, +)$$

$$\Leftrightarrow G \leq G', *' \upharpoonright_{G \times G} = *, e' = e$$

$$- (G, (\cdot)^{-1}, e) \stackrel{\text{Subgroup}}{\leq} (G', (\cdot')^{-1}, e')$$

$$\underbrace{(\{-1, 1, \cdot\}, -(\cdot), \cdot)}_x \leq \underbrace{(\mathbb{Z}, -(\cdot), \cdot)}_+$$

$$\text{Sub}(A) : \text{مجموعة جزئية من } A$$

Remark $\text{Sub}(A)$ is closed under intersection

اشترك جميعها $H_i \leq A$.

Sub(A) is not generally closed under $\cup$. اجتماع جزئية الزمرات ليست!

من الزمرات الجزئية $\subseteq A$
If $\{B_i : i \in I\}$ is an undirected family of subalgebras of A then $\bigcup_{i \in I} B_i \leq A$.

$$\forall B_i, B_j \exists B_k$$

$$B_i, B_j \leq B_k$$

$$1 \in \Omega, x_1, \dots, x_{n_1} \in \bigcup_{i \in I} B_i$$

$$\stackrel{(\S)}{\Rightarrow} 1(x_1, \dots, x_{n_1}) \in \bigcup_{i \in I} B_i$$

$$\exists B_1, \dots, B_{n_1}$$

$$x_1 \in B_1, \dots, x_{n_1} \in B_{n_1}$$

$$\xrightarrow[\text{induction}]{\text{by}} \exists B_K; B_1, \dots, B_{n_1} \subseteq B_K$$

$$\Rightarrow x_1, \dots, x_{n_1} \in B_K$$

$$\Rightarrow \lambda(x_1, \dots, x_{n_1}) \in B_K \subseteq \bigcup B_i$$

Remark. Every chain is an updirected.

Theorem. For every algebra A , $(\text{Sub}(A), \subseteq)$ is a complete lattice.

$$\bigwedge H_i = \bigcap H_i \leftarrow \text{proof}$$

Def. Let A be an algebra and $X \subseteq A$. Then we define the subalgebra generated by X to be the intersection of all subalgebra containing X .
اشهر زیر جبرهای شامل X

Notation. $\text{Sg}(X)$: subalgebra generated by X .

Remark. 1) $Sg(X)$ is the least subalgebra containing X . $\forall i \in I, X \subseteq A_i \Rightarrow \bigcap_{i \in I} A_i = Sg(X)$

1) $X \subseteq Sg(X) \checkmark$ $Sg(X) = \bigcap A_i \subseteq A_i$ ✓

2) $X \subseteq Y \Rightarrow Sg(X) \subseteq Sg(Y)$ $\leftarrow 2) \checkmark Sg(Sg(X)) = Sg(X)$ $X \subseteq A_i \Rightarrow \bigcap A_i = Sg(X)$

$Y \subseteq B_i \Rightarrow \bigcap B_i = Sg(Y)$

$X \subseteq Y \Rightarrow \bigcap$

3) $Sg(-) : P(A) \rightarrow P(A)$ is a closure operator ✓

$\exists C \in Sg(X) \text{ s.t. } C \notin Sg(Y) \rightarrow C \in Sg(X) \Rightarrow C \in \bigcap A_i \Rightarrow C = A_i \checkmark$

4) $L_{Sg(-)} = \{ X \subseteq A \mid Sg(X) = X \} = Sub(A)$

5) For every algebraic closure operator C ,

$L_C \cong Sub(A)$, for some algebra A .

6) A lattice L is algebraic iff $L \cong Sub(A)$ for some algebra A .

Def. A Closure operator

$C : P(X) \rightarrow P(X)$ is called algebraic if

$\forall A \subseteq X, C(A) = \bigcup \{ C(B) \mid B \subseteq A \text{ is finite} \}$

$\forall a \in L, \exists T, a = \bigvee T, T \subseteq K(L)$ compact element of L

$\equiv a = \bigvee \{ x \in K(L) \mid x \leq a \}$
 $\downarrow a \cap K(L)$

Algebraic lattice
: complete

Remark. If C is an algebraic closure operator, then L_C is an algebraic lattice. L_C is a complete lattice. مبداً

② Compact elements of L_C are exactly of the form $C(B)$ where B is finite. دوماً

($\Rightarrow$) Let $C(B)$ be a compact element of L_C . then by def (*),

$$C(B) = \bigvee \{ C(D) \mid D \subseteq B \} \xrightarrow{\text{finite}}$$

that is,

$$C(B) = \bigvee_{D \subseteq B} C(D)$$

$$D \subseteq B$$

But $C(B)$ is compact, so $\exists D_1, \dots, D_n \subseteq B$, $C(B)$

$$C(B) \leq C(D_1) \vee \dots \vee C(D_n)$$

$$\Rightarrow C(B) \leq C(D_1 \cup \dots \cup D_n)$$

$$\Rightarrow C(B) = C(\underbrace{D_1 \cup \dots \cup D_n}_D) \leq C(B)$$

($\Leftarrow$) $C(B)$, B finite, $C(B) \leq \bigvee_{i \in I} A_i$, $A_i \in L_C$, $B = \{b_1, \dots, b_k\}$, $\subseteq C(\bigvee_{i \in I} A_i)$

$$\Rightarrow \exists A_1, \dots, A_k, b_1 \in C(A_1), \dots, b_k \in C(A_k)$$

$$\Rightarrow C(B) \leq C(A_1 \cup \dots \cup A_k) = A_1 \vee \dots \vee A_k$$

③ Now, $\forall A \subseteq C$

$$A = C(A) = \bigcup \{C(B) : B \subseteq A\} \subseteq \bigvee \{C(B) : B \subseteq A\}$$

$$= \bigvee B_i$$

B_i Compact

$$B_i = C(B_i)$$

$$B_i \subseteq A$$

$$C(B_i) \subseteq C(A)$$

Theorem. Algebraic lattices are isomorphic to L_C ,
for some alg. closure operator.

For L alg. lattice Define

$$C : \mathcal{P}(L) \rightarrow \mathcal{P}(L)$$

$$C(A) = \{x \in L : x \leq \bigvee A\}$$

C is an algebraic closure operator.

$$C(A) \stackrel{?}{=} \bigcup \{C(B) : B \subseteq A\} \quad \supseteq \bigvee$$

$$x \in C(A)$$

$$\Rightarrow x \leq \bigvee A$$

$$\Rightarrow x = \bigvee \{x \in K(L) : z \leq x\}$$

$$\forall z \in K(L), z \leq x \Rightarrow z \leq \bigvee A$$

$$\Rightarrow \exists a_1, \dots, a_{k_2} \in A$$

$$z \leq a_1 \vee \dots \vee a_{k_2}$$

$$\Rightarrow x \leq \bigvee_z a_{1,z} \dots a_{k_2,z} \Rightarrow x \in C(B_2)$$

$$B_2 =$$

$$\{a_1, \dots, a_k\} \subseteq A$$

$$\Rightarrow x \in \bigcup \{C(B_2) : z \in K(L), x \geq z\}$$

$$\textcircled{5} \text{ } C \text{ alg. closure operator} \Rightarrow L_C \cong \text{Sub}(A)$$

$$\textcircled{6} \text{ } L \text{ alg. lattice} \Leftrightarrow L \cong \text{Sub}(A)$$

کتابت شده است، ⑤ و ⑥ درست اند.

Lemma ①. $Sg(-)$ is an algebraic closure operator.

Lemma ②. If C is an alg. closure operator then $L_C \cong \text{Sub}(A)$.

$$X \subseteq (A, (f_A)_{f \in \mathcal{F}})$$

$$Sg(X) \stackrel{?}{=} \bigcup_{n \in \mathbb{N}} E^n(X)$$

$$E^0(X) = (X)$$

$$E^1(X) = \{f_A(x_1, \dots, x_{n_f}) : x_i \in X, n_f \in \mathbb{N}, f \in \mathcal{F}\}$$

$$E^2(X) = \{f_A(y_1, \dots, y_{n_f}) : y_i \in E(X), n_f \in \mathbb{N}, f \in \mathcal{F}\}$$

$$\vdots$$

$$E^{n+1}(X) = E(E^n(X))$$

$a \in V, r.$

Lemma 1

$Sg(-)$ is an algebra closure operator.

proof: For $X \subseteq A$, note that $Sg(X) = \bigcup_{n \geq 0} E^n(X)$

$$E^0(X) = X$$

$$E^1(X) = \{ \lambda_A(x_1, \dots, x_{n_\lambda}) \cdot n_\lambda \in \mathcal{C}, x_i \in X, \lambda \in \Omega \}$$

$$E^{n+1}(X) = E^1(E^n(X))$$

$$X \subseteq E(X) \subseteq \dots$$

$$\text{now, } a \in Sg(X) \Rightarrow \exists n \geq 0, a \in E^n(X)$$

$$\Rightarrow \exists \lambda \in \Omega, a = \lambda_A(y_1, \dots, y_{n_\lambda})$$

$$, y_i \in E^{n-1}(X), n \geq 1$$

$$\Rightarrow a \in Sg(\{y_1, \dots, y_{n_\lambda}\})$$

Remember. $Sg(-): \mathcal{P}(A) \rightarrow \mathcal{P}(A)$

$$(A; (\lambda_A)_{\lambda \in \Omega}), \lambda_A: A^{n_\lambda} \rightarrow A$$

$$C(Y) = \bigcup_{Z \subseteq Y} C(Z)$$

Lemma 2

(i) For every algebra A , $\text{Sub}(A)$ is an algebraic lattice.

(ii) Every algebraic lattice L is isomorphic to $\text{Sub}(A)$, for some algebra A .

Proof. i) By Lemma 1 and $\text{Sub}(A) = L_{\text{sg}(-)'}'$
and since closed of each alg. closure
form an alg. lattice.

ii) Let L be an alg. lattice. Then $\exists$
an alg. closure c on L with

$L_c \cong L$. Take $A \doteq L$, and define:

$$\lambda_{F,b} : A^n \rightarrow A \text{ by } \lambda_{F,b}(x_1, \dots, x_n) \\ = \begin{cases} b & (x_1, \dots, x_n) = (a_1, \dots, a_n) \\ a_1 & \text{o.w.} \end{cases}$$

for $F = \{a_1, \dots, a_n\} \subseteq L$ ($n \in \mathbb{N}$)
 $b \in c(F)$

$$\{(F, b) : F \in \mathcal{P}_F(L), b \in c(F)\}$$

now, claim $L \cong \text{Sub}(A)$.

To show this, we prove

$$\forall X \subseteq A, \text{Sg}(X) = c(X) \quad (*)$$

Because then $L_{\text{sg}(-)} = L_c$

$$\Rightarrow \text{Sub}(A) = L$$

To prove $(*)$, it is enough to show

$$\forall F \subseteq A, \text{Sg}(F) = c(F)$$

Let $F \subseteq A$, $F = \{a_1, \dots, a_n\}$

we have $\text{Sg}(F) \subseteq c(F)$

because $\lambda_{F,b}(x_1, \dots, x_n) \begin{matrix} \nearrow b \\ \searrow a_i \end{matrix} \in c(F)$
 $\forall x_1, \dots, x_n \in E^m(F)$

$$\text{and } \text{Sg}(F) = \bigcup_{m \geq 0} E^m(F)$$

Conversely, $b \in c(F)$

$$\Rightarrow b = \lambda_{F,b}(a_1, \dots, a_n)$$

$$\in E^1(F) \subseteq \text{Sg}(F)$$

Remember. $(G, *) \xrightarrow{\varphi} (G', *)$

$$\varphi: G \rightarrow G' \quad \varphi \in \text{Hom}(G, G')$$

$$\varphi(x * y) = \varphi(x) *' \varphi(y)$$

$$\text{or } (\varphi \circ *) (x, y) = (*' \circ (\varphi \times \varphi)) (x, y)$$

$$\begin{array}{ccc} G \times G & \xrightarrow{*} & G \\ \varphi \times \varphi \downarrow & \curvearrowright & \downarrow \varphi \\ G' \times G' & \xrightarrow{*'} & G' \end{array}$$

Homomorphism. we mean a map $f: A \rightarrow B$

such that

$$\forall \lambda \in \Sigma$$

$$\begin{array}{ccc} A^{\eta_\lambda} & \xrightarrow{\lambda_A} & A \\ \downarrow f^{\eta_\lambda} & & \downarrow f \\ B^{\eta_\lambda} & \xrightarrow{\lambda_B} & B \end{array}$$

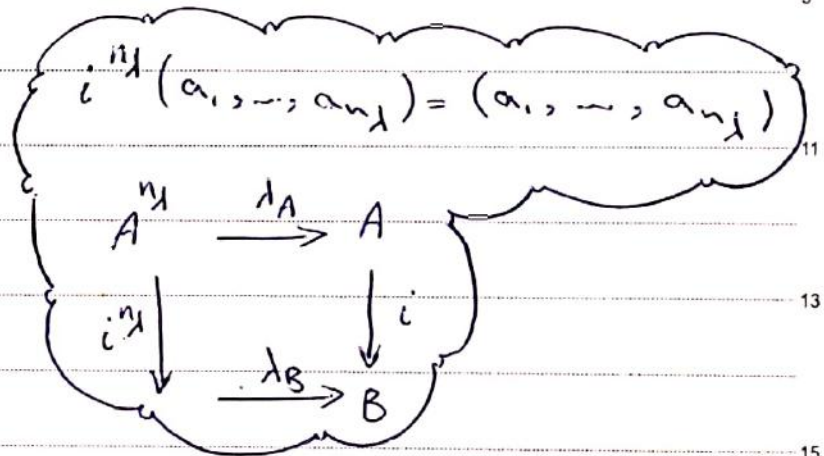
is commutative.

where $f^{n_1}(a_1, \dots, a_{n_1})$

$$= (f(a_1), \dots, f(a_{n_1}))$$

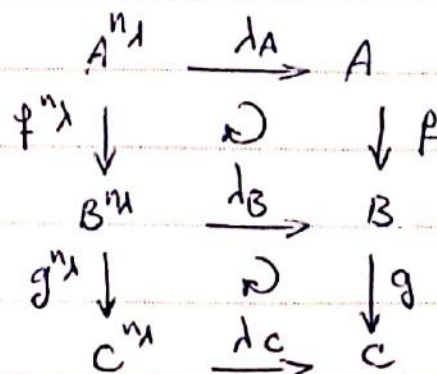
$$f \circ \text{id}_A = \text{id}_B \circ f^{n_1} \Leftrightarrow f(\text{id}(a_1, \dots, a_{n_1})) = \text{id}_B(f(a_1), \dots, f(a_{n_1}))$$

Remark. $1 \ A \leq B \Leftrightarrow i: A \rightarrow B$ is a homo.
 $a \mapsto a$



2/ $\text{id}: A \rightarrow A$ is a homo
 $a \mapsto a$

3/ Composition of homo is a homo.



4/ Algebras of the same type $\mathcal{A}$ form a category, denoted by $\text{Alg}(\mathcal{A})$.

5/ If $|B|=1$, then $\exists! A \xrightarrow{f} B$
homo.
(constant)

6/ we denote the set of all homo. from A to B by $\text{Homo}(A, B)$.
If $A=B$, $\text{Homo}(A, B) = \text{End}(A, A)$

نکته: همومورفیسم، ایندومورفیسم یک همومورفیسم است.

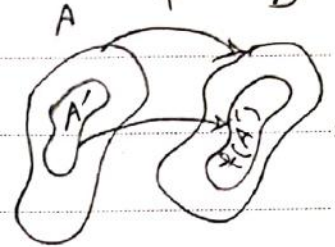
7/ If $|A|=0$ then $\exists!$ homo.
 $A \rightarrow B$

$$\begin{array}{ccc} A^{n_A} & \xrightarrow{\lambda_A} & A \\ \phi = \phi^{n_A} \downarrow & \curvearrowright & \downarrow \phi \\ B^{n_B} & \xrightarrow{\lambda_B} & B \end{array}$$

Lemma. If $A \xrightarrow{f} B$ is a homo, then f

$$(1) A' \leq A \Rightarrow \underline{f(A')} \leq B$$

$$\{f(x) : x \in A'\}$$



$$(2) B' \leq B \Rightarrow \underline{f^{-1}(B')} \leq A$$

$$\{a \in A \mid f(a) \in B'\}$$

Proof. exercise.

Exercise. If $A = \text{Sg}(X)$, $A \xrightarrow[f]{g} B$ are homo and

$$f|_X = g|_X \text{ then } f = g.$$

Product of Algebra:

The Cartesian product of two algebra A and B of type τ is an algebra of type τ with pointwise operations

$\forall l \in \tau$, define

$$\lambda_{A \times B}: (A \times B)^{n_l} \rightarrow A \times B$$

by

$$\lambda_{A \times B}((a_1, b_1), \dots, (a_{n_l}, b_{n_l})) = (\lambda_A(a_1, \dots, a_{n_l}), \lambda_B(b_1, \dots, b_{n_l}))$$

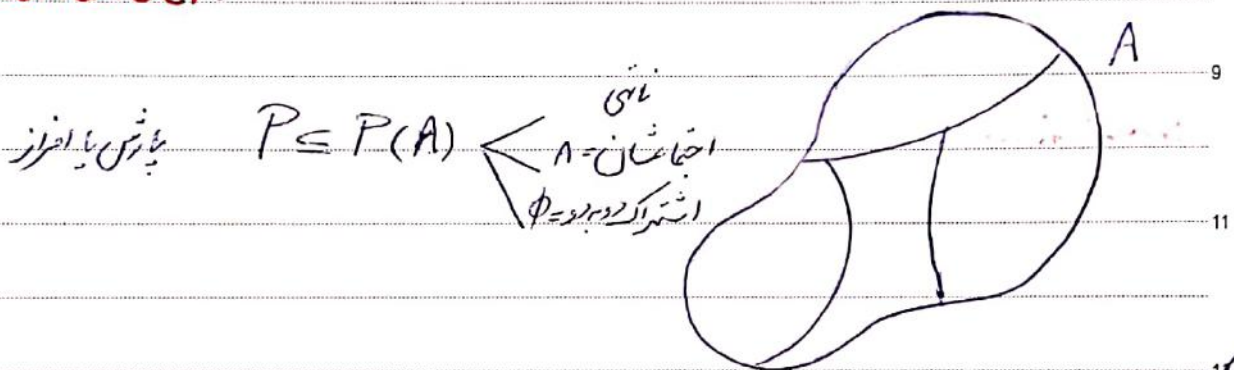
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and then $P_i: A \times B \rightarrow A$ are homo.

$$P_2: A \times B \rightarrow B$$

$$\begin{array}{ccc} (A \times B)^{\eta_1} & \xrightarrow{\lambda_{A \times B}} & A \times B \\ P_i^{\eta_1} \downarrow & & \downarrow P_i \\ A^{\eta_1} & \xrightarrow{\lambda_A} & A \end{array}$$

Remember.



$$[a]_R = \{x \in A \mid a R x\} \quad \text{s.t.} \quad R \subseteq A \times A \quad \leftarrow \begin{matrix} \text{ای} \\ \text{تای} \\ \text{تعی} \end{matrix}$$

$$\text{ای} \quad A \xrightarrow{f} B \approx A \cup \{y\} \quad \text{میزانی}$$

$$\{(x, y) \mid f(x) = f(y)\} = K_f \quad \text{میزانی}$$

$$\text{نقش:} \quad A \xrightarrow{\gamma_R} A/R = \{[a]_R \mid a \in R\} \quad \leftarrow R \subseteq A \times A$$

$$a \mapsto [a]$$

$$\begin{aligned}
 R &\xrightarrow{\quad} K_R = \{(x, y) \mid f(x) = f(y)\} \\
 \downarrow \gamma_R & \\
 Y_R & \\
 &= \{(x, y) \mid [x]_R = [y]_R\} \\
 &= \{(x, y) \mid (x, y) \in R\} = R
 \end{aligned}$$

$$\begin{aligned}
 A &\xrightarrow{f} B \quad f(a) \\
 \downarrow \gamma_{K_f} & \quad \uparrow \exists! \bar{f} \quad \checkmark \text{ یکتا} \\
 A/K_f & \quad [a] \quad \bar{f} \text{ onto} \Leftrightarrow f \text{ onto}
 \end{aligned}$$

جواب
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Def. By a Congruence on an algebra $(A; (\lambda_A)_{\lambda \in \Sigma})$ is an equivalence relation θ on A such that:

1, A/θ is an algebra of the same type of A .

2, $\gamma: A \rightarrow A/\theta$ is a homomorphism.
 $a \mapsto [a]$

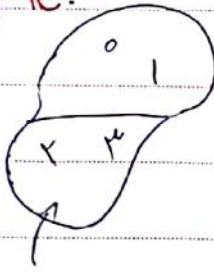
$$\begin{array}{ccc}
 A^{n_\lambda} & \xrightarrow{\gamma^{n_\lambda}} & (A/\theta)^{n_\lambda} \\
 \downarrow \lambda_A & & \downarrow \lambda_{A/\theta} \\
 A & \xrightarrow{\gamma} & A/\theta
 \end{array}$$

$$\lambda_{A/\theta}([a_1], \dots, [a_{n_\lambda}]) = [\lambda_A(a_1, \dots, a_{n_\lambda})]$$

$$\begin{aligned}
 \Leftrightarrow ([a_1] = [a'_1], \dots, [a_{n_\lambda}] = [a'_{n_\lambda}]) &\Rightarrow [\lambda_A(a_1, \dots, a_{n_\lambda})] \\
 &= [\lambda_A(a'_1, \dots, a'_{n_\lambda})]
 \end{aligned}$$

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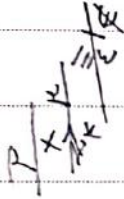
Example.



$$\mathbb{Z}_4 = \{ \overbrace{\{0, 1\}}^x, \overbrace{\{2, 3\}}^y \}$$

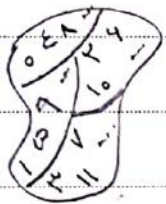
مجموعه خارجی نمی‌باشد. زیرا رابطه زیر برقرار نیست:

$$[a] +_{\mathbb{Z}_4} [b] = [a +_{\mathbb{Z}_4} b]$$



	x	y
[0] ← x	x	y
[2] ← y	y	x

	x	y
[0] ← x	x	y
[3] ← y	y	y



$$x = x'$$

$$y = y'$$

$$f(x, y) \neq f(x', y')$$

ف تابع نیست.

$$[3] = [2]$$

$$[3] = [2]$$

$$[2] \neq [0]$$

$$\begin{aligned} [1] &= [1] \\ [2] &= [2] \\ [3] &= [3] \\ [4] &= [4] \\ [5] &= [5] \\ [6] &= [6] \\ [7] &= [7] \\ [8] &= [8] \\ [9] &= [9] \\ [10] &= [10] \\ [11] &= [11] \\ [12] &= [12] \\ [13] &= [13] \\ [14] &= [14] \\ [15] &= [15] \\ [16] &= [16] \\ [17] &= [17] \\ [18] &= [18] \\ [19] &= [19] \\ [20] &= [20] \\ [21] &= [21] \\ [22] &= [22] \\ [23] &= [23] \\ [24] &= [24] \\ [25] &= [25] \\ [26] &= [26] \\ [27] &= [27] \\ [28] &= [28] \\ [29] &= [29] \\ [30] &= [30] \\ [31] &= [31] \\ [32] &= [32] \\ [33] &= [33] \\ [34] &= [34] \\ [35] &= [35] \\ [36] &= [36] \\ [37] &= [37] \\ [38] &= [38] \\ [39] &= [39] \\ [40] &= [40] \\ [41] &= [41] \\ [42] &= [42] \\ [43] &= [43] \\ [44] &= [44] \\ [45] &= [45] \\ [46] &= [46] \\ [47] &= [47] \\ [48] &= [48] \\ [49] &= [49] \\ [50] &= [50] \\ [51] &= [51] \\ [52] &= [52] \\ [53] &= [53] \\ [54] &= [54] \\ [55] &= [55] \\ [56] &= [56] \\ [57] &= [57] \\ [58] &= [58] \\ [59] &= [59] \\ [60] &= [60] \\ [61] &= [61] \\ [62] &= [62] \\ [63] &= [63] \\ [64] &= [64] \\ [65] &= [65] \\ [66] &= [66] \\ [67] &= [67] \\ [68] &= [68] \\ [69] &= [69] \\ [70] &= [70] \\ [71] &= [71] \\ [72] &= [72] \\ [73] &= [73] \\ [74] &= [74] \\ [75] &= [75] \\ [76] &= [76] \\ [77] &= [77] \\ [78] &= [78] \\ [79] &= [79] \\ [80] &= [80] \\ [81] &= [81] \\ [82] &= [82] \\ [83] &= [83] \\ [84] &= [84] \\ [85] &= [85] \\ [86] &= [86] \\ [87] &= [87] \\ [88] &= [88] \\ [89] &= [89] \\ [90] &= [90] \\ [91] &= [91] \\ [92] &= [92] \\ [93] &= [93] \\ [94] &= [94] \\ [95] &= [95] \\ [96] &= [96] \\ [97] &= [97] \\ [98] &= [98] \\ [99] &= [99] \end{aligned}$$

بر گروه زیر مجموعه از گروه جا نیست.

تعداد اعضای زیر گروه، تقود اعضای گروه و حاصل می‌گردد.

Remark.

1/ An equivalence relation θ on an algebra $\mathcal{A}$

note. $\equiv_n (n \in \mathbb{Z})$ are the only congruences on $(\mathbb{Z}, +, -, 0)$ or $(\mathbb{Z}_{\text{ring}}, +, \cdot, -, 0, 1)$

$$A/\nabla_A = \{A\}$$


همه چیز به یکباره عمل

$$A/\Delta_A = \{\{a\} : a \in A\}$$


Δ_A, ∇_A are the smallest cong. and greatest cong. on A , respectively.

عمل ضربی بر انتهای قسم یکبار است با هم ارزی

3, An equiv. relation θ is a cong. iff

$$(a_1, a'_1), \dots, (a_n, a'_n) \in \theta$$

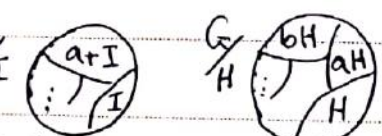
$$\Rightarrow (\lambda_A(a_1, \dots, a_n), \lambda_A(a'_1, \dots, a'_n)) \in \theta$$

$$\equiv \lambda_{A \times A}((a_1, a'_1), \dots, (a_n, a'_n)) \in \theta$$

that is; an equiv. rel. θ is cong. iff

$$\theta \leq A \times A.$$

در هر دو، خارج قسمت که باز گیرده نهال و در هر دو، با این حال که شکل من شوند



جواب
۹۴، ۱، ۲۷

منبع صوابی بنی و برزقن آورد

An equiv relation θ is a cong. on $A \iff$

$$\theta \leq A \times A$$

$$a_i \theta a'_i \Rightarrow \lambda(a_1, \dots, a_{n_1}) \theta \lambda(a'_1, \dots, a'_{n_1})$$

$$i = 1, \dots, n_1$$

Notation. $\text{Con}(A)$: the set of cong. on A .

Theorem. $\text{Con}(A)$ is an algebraic lattice.

proof: $\text{Con}(A) \leq \text{Eq}(A)$
sublattice

— The intersection of congruences is
a cong.

— The union of every chain of cong.
is a cong.

$$(a, b) \in \bigcup_{i \in I} R_i, (b, c) \in \bigcup_{i \in I} R_i$$

$$\Rightarrow \exists i (a, b) \in R_i, \exists j (b, c) \in R_j$$

$$R_i \leq R_j \quad \text{or} \quad R_j \leq R_i$$

$$\Downarrow (a, b), (b, c) \in R_j$$

$$\Downarrow (a, c) \in R_i$$

$$\Downarrow (a, c) \in R_j$$

$$\Downarrow (a, c) \in \bigcup_{i \in I} R_i$$

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So, $\text{Cong}(A)$ is an alg. closure system and it is an algebraic lattice.

$$\theta_1 \vee^{\text{Eq}} \theta_2 = \theta_1 \cup (\theta_1 \circ \theta_2) \cup (\theta_1 \circ \theta_2 \circ \theta_1) \cup \dots$$

$$\theta_1, \theta_2 \in \text{Cong}(A) \Rightarrow \theta_1 \vee \theta_2 \in \text{Con}(A)$$

$$\theta_1 \circ \theta_2 \stackrel{?}{\leq} A \times A$$

$$a_i (\theta_1 \circ \theta_2) a'_i, i=1, \dots, n$$

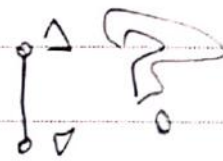
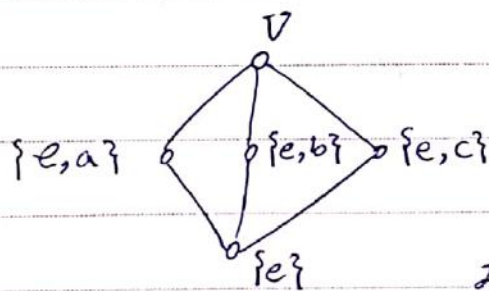
$$\exists c_i, a_i \theta_1 c_i, c_i \theta_2 a'_i$$

$$\lambda(a_1, \dots, a_n) \theta_1 \lambda(c_1, \dots, c_n)$$

$$\lambda(a_1, \dots, a_n) \theta_1 \lambda(a'_1, \dots, a'_n) \Rightarrow$$

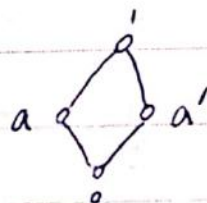
In fact, $\bigwedge_{\text{Con } A} \theta_i = \bigcap \theta_i, \bigvee_{\text{Con } A} \theta_i = \bigcup_{n \in \mathbb{N}} (\theta_i \circ \dots \circ \theta_i)$

Example.



In Groups, $\text{Con}(A) \cong N(A)$

Example.



$$\text{Con}(L) = ?$$

$$\frac{x \theta x' \quad y \theta y'}{(x \vee y) \theta (x' \vee y')} \quad (x \wedge y) \theta (x' \wedge y')$$

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Exercise.

برای هر شبکه θ ، $\text{Con}(L) = L$

اگر $a \theta a'$ و $a \theta a''$ چون $a \theta a'$ و $a \theta a''$ پس عناصر
در یک خانواده و Con خاصیت ترانزیتیست.

Remark.

i) If $f: A \rightarrow B$ is a homo, and $\theta \in \text{Con}(B)$ then $(f \times f)^{-1}(\theta) \in \text{Con}(A)$.

$$A \times A \xrightarrow{f \times f} B \times B \text{ homo (?)}$$

$$(a, a') \mapsto (f(a), f(a'))$$

$$\theta \subseteq B \times B \Rightarrow (f \times f)^{-1}(\theta) \subseteq A \times A$$

$$\theta \in \text{Eq}(B) \Rightarrow (f \times f)^{-1}(\theta) = \{(a, a') \in A \times A \mid (f(a), f(a')) \in \theta\} \in \text{Eq}(A)$$

ii) If $f: A \rightarrow B$ is an onto homo, and

f is 1-1 (or $\ker f \subseteq \theta$),

then $\theta \in \text{Con}(A) \Rightarrow (f \times f)(\theta) \in \text{Con}(B)$.

$$(f \times f)(\theta) = \{(f \times f)(a, a') \mid (a, a') \in \theta\} = \{(f(a), f(a'))\}$$

Exercise.

A homo f is 1-1 $\Leftrightarrow \ker f = \Delta$

$$f(a) = f(b) \Rightarrow (a, b) \in \ker f$$

$$\ker f = \Delta \Rightarrow (a, b) \in \ker f \Rightarrow f(a) = f(b) \Rightarrow a = b$$

$$\begin{aligned} \text{Cong.} & \Rightarrow (a, b) \in \ker f \Rightarrow f(a) = f(b) \\ & \Rightarrow (a, b) \in \ker f \Rightarrow f(a) = f(b) \\ & \Rightarrow (a, b) \in \ker f \Rightarrow f(a) = f(b) \end{aligned}$$

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iii) If $f: A \rightarrow B$ is a homo then $K_f \in \text{Con}(A)$.

$$K_f = \{ (a, a') \mid \underbrace{f(a) = f(a')}_{(f(a), f(a')) \in \Delta_B} \} = (f \times f)^{-1}(\Delta_B)$$



هم‌رشتی‌های A با f هم‌رشتی‌های B

$$K_f \in \text{Con}(A) \xleftrightarrow{\quad} f$$

$$\theta \in \text{Con}(A) \xleftrightarrow{\quad} \gamma: A \rightarrow A/\theta$$

$$a \mapsto [a]$$

homo.

Theorem. If $f: A \rightarrow B$ is an onto homo, then

$$\text{Con}(B) \cong \underbrace{[K_f, \nabla]}_{\substack{\text{Lattice} \\ \text{iso}}} \text{ in } \text{Con}(A).$$

$$\{ \theta \in \text{Con}(A) \mid K_f \subseteq \theta \subseteq \nabla \}$$

$$\gamma \mapsto (f^2)^{-1}(\gamma)$$

$$f^2(\theta) \subseteq \theta$$

$$K_f \subseteq (f^2)^{-1}(\gamma)$$

$$(a, a') \in K_f \Rightarrow f(a) = f(a') \Rightarrow (f(a), f(a')) \in \gamma \\ \Rightarrow (a, a') \in (f^2)^{-1}(\gamma)$$

اینست کابل خود

Exercise

$$\text{Con}(G) \subseteq N(G)$$

$$K_f \leftarrow \text{Ker } f$$

نقشه همگس، اشتراک دو نقطه می‌کند: برخلاف نقشه

همگس، دو همگس همگس، اجتماع دو نقطه می‌کند

Decomposition Theorem. If $f: A \rightarrow B$ is a homo.

then there exists a unique

homo. $h: A/K_f \rightarrow B$ such that

$$h \circ \gamma = f$$

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ & \searrow \gamma_{K_f} & \uparrow \exists! h \\ & & A/K_f \end{array}$$

$h'[a] = f(a) = h[a]$

شان می‌دهم h وجود دارد.
(خوش تعریف بودن تابع h)
هم بخش است.

Proof. Define

$$h[a] = f(a)$$

h is well-defined

(and 1-1) $h: A/K_f \rightarrow B$

$$[a] = [a']$$

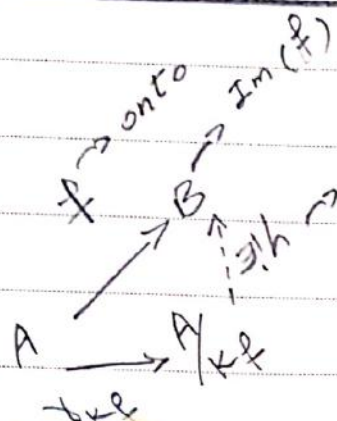
$$\Leftrightarrow (a, a') \in K_f$$

$$\Leftrightarrow f(a) = f(a')$$

h homo: $\forall \lambda \in \Omega$

$$\begin{aligned} h(\lambda_{A/K_f}([a_1], \dots, [a_{n_x}])) &= h([\lambda_A(a_1, \dots, a_{n_x})]) \\ &= h(\lambda_A(a_1, \dots, a_{n_x})) \end{aligned}$$

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$$\begin{aligned}
 h\left(\frac{1}{K_f}\right) &= f(1_A(a_1, \dots, a_n)) \\
 &= 1_B(f(a_1), \dots, f(a_n)) \\
 &= 1_B(h[a_1], \dots, h[a_n])
 \end{aligned}$$

Moreover, h onto $\Leftrightarrow f$ is onto

Corrollay.

(1st Iso Th) (Fundamental Homo Th)

If $f: A \rightarrow B$ is an onto homo, then

$$A/K_f \cong \text{Im } f$$

Exercise. (Generalized Decomposition Th.)

If $f: A \rightarrow B$ is a homo, and $g: A \rightarrow C$ is an onto homo with $K_g \subseteq K_f$,

then $\exists! h: C \rightarrow B$ homo such that $h \circ g = f$.

Moreover, h is 1-1 $\Leftrightarrow K_g = K_f$

and h is onto $\Leftrightarrow f$ is onto.

$$A \xrightarrow{f} B$$

$$\begin{array}{ccc}
 & & \nearrow \exists! h \\
 g \searrow & & \\
 & C &
 \end{array}$$

طبیعیات
۹۴، ۸، ۴

تبریز

First Isomorphism Theorem:

If $A \xrightarrow{f} B$ onto homo. then $A / \text{Ker } f \cong_{\text{iso}} \text{Im } f$

Second Isomorphism Theorem

If $A \xrightarrow{f} B$ is an onto homo and $\theta \in \text{Con}(A)$ with $\text{Ker } f \subseteq \theta$ then

$$A/\theta \cong_{\text{iso}} B/f^{-1}(\theta)$$

Third ISO Theorem:

If $\theta \in \text{Con}(A)$, $B, C \leq A$ with $B \leq C \leq A(B)$
Subalg

where $A(B) \doteq B^\theta = \bigcup_{b \in B} [b]_\theta = \{a \in A \mid \exists b \in B, (a, b) \in \theta\}$
then:

$$B / \theta \cap B^2 \cong C / \theta \cap C^2$$

Exercise: 1) Prove that for any group G , $N(G) \cong \text{Con}(G)$

2) Prove that for any ring R , $\text{Id}(R) \cong \text{Con}(R)$

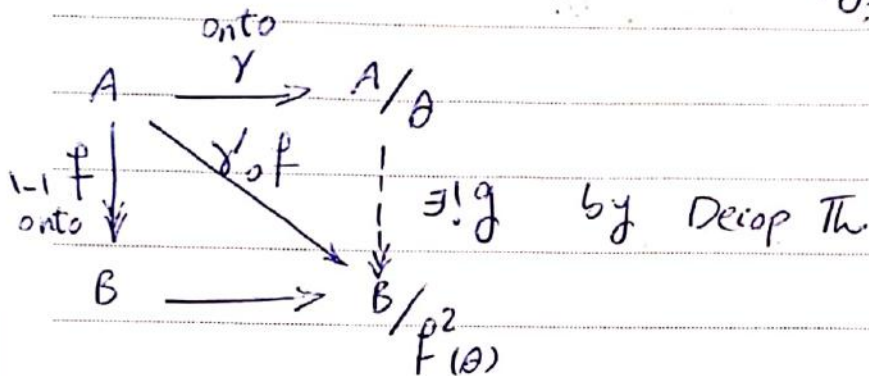
3) write the interpretation of the 1, 2, 3 ISO Th. in groups & rings.

Proof of the 2nd Iso th.

$$f^2_{f(\theta)} \in \text{Con}(B)$$

میدانیم که این دو هم جور و مان را میزنیم

فقط است پس



$$\Rightarrow a \in \text{Ker}(\gamma' \circ f) \iff \gamma'(f(a)) = \gamma'(f(a'))$$

$$\iff [f(a)]_{f^2_{f(\theta)}} = [f(a')]_{f^2_{f(\theta)}} \iff a \theta a'$$

← واضح است. خوش تائید و اراشان می دید

$\Rightarrow$

$$f(a) \quad f^2_{f(\theta)} \quad f(a')$$

$$\exists (a_1, a_2) \in \theta, f(a) = f(a_1), f(a') = f(a_2)$$

$$\Rightarrow (a, a_1) \in \text{Ker } f, (a', a_2) \in \text{Ker } f$$

because $\text{Ker } f = \theta$

$$\Rightarrow a \theta a_1, a' \theta a_2 \xRightarrow{a_1 \theta a_2} a \theta a'$$

Proof of the 3rd ISO Th.

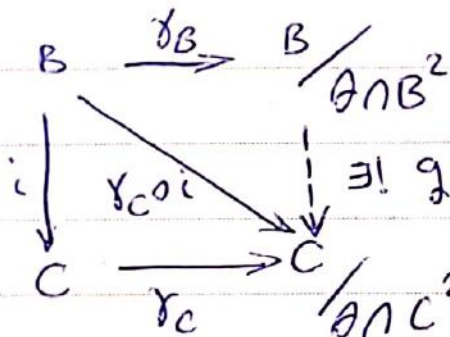
$$\theta \in \text{Con}(A)$$

از جبردارها به جبردار می‌کنیم. $B \times B$ زیرجبردار است پس زیرجبردار
مختص است.
باید هم ارزش نوشتن داریم

$$\Rightarrow \theta \cap B^2 \in \text{Con}(B) \quad \forall B$$

$$\theta \cap C^2 \in \text{Con}(C) \quad \forall C$$

حل یک هم‌رشتی پیدا می‌کنیم:



$$\theta \cap B^2 = \text{Ker}(\gamma_C \circ i)$$

g onto? $\gamma_C \circ i$ onto?

$$(b, b') \in B$$

$$b \in \text{Ker}(\gamma_C \circ i) \quad b'$$

$$\Rightarrow [b]_{\theta \cap C^2} = [b']_{\theta \cap C^2} \Leftrightarrow b (\theta \cap C^2) b'$$

$$\Leftrightarrow b \theta b', (b, b') \in C \times C = C^2$$

$$\Leftrightarrow (b, b') \in \theta \cap B^2$$

چون $B \leq C$ پس رابطه برابری پذیر است.

$$[x]_{\theta \cap C^2}, x \in C \xRightarrow{C \leq \theta(B)} \exists b \in B, (x, b) \in \theta$$

$$\Rightarrow_{B \leq C} [b]_{\theta \cap C^2} = [x]_{\theta \cap C^2} \Rightarrow (b, x) \in \theta \cap C^2$$

Corollary of second Iso. Th:

If $\theta, \varphi \in \text{Con}(A)$, $\varphi \subseteq \theta$, and suppose

$$\theta/\varphi = \{ ([a]_\varphi, [a']_\varphi) \mid (a, a') \in \theta \}$$

then $\theta/\varphi \in \text{Con}(A/\varphi)$

$$\text{and } A/\theta \cong \frac{A/\varphi}{\theta/\varphi}$$

Proof: Take $f: A \xrightarrow{\gamma_\varphi} A/\varphi$ and notice that $\text{Ker } f = \varphi \subseteq \theta$.

and by def. $\gamma_\varphi^2(\theta) = \theta/\varphi$

طبق تعریف هم هست.

Def. we say an algebra A is Congruence permutable if $\forall \theta_1, \theta_2 \in \text{Con}(A)$,

$$\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$$

Lemma. If A is Congruence permutable then

$$\theta_1 \vee \theta_2 = \theta_1 \circ \theta_2 \quad \forall \theta_1, \theta_2 \in \text{Con}(A)$$

$$\begin{aligned} \text{Proof: } \gamma_{\theta_1 \vee \theta_2} &= \theta_1 \vee (\theta_1 \circ \theta_2) \vee (\theta_1 \circ \theta_2 \circ \theta_1) \vee \dots \\ &\Rightarrow \theta_1 \vee \theta_2 = \theta_1 \circ \theta_2 \end{aligned}$$

۱/ این دو هم برابر هم است. ۲/ یک خبر است قدی را برای بنم.

$$a \theta_1 \theta_2 b \Rightarrow \exists x \in A, a \theta_2 x, x \theta_1 b$$

$$b \theta_1 \theta_2 c \Rightarrow \exists y \in A, b \theta_2 y, y \theta_1 c$$

$$\Rightarrow x(\theta_2 \circ \theta_1) y \Rightarrow x(\theta_1 \circ \theta_2) y$$

$$\Rightarrow \exists z, x \theta_2 z, z \theta_1 y$$

$$\Rightarrow a \theta_2 z, z \theta_1 c \Rightarrow a \theta_1 \circ \theta_2 c$$

Exercise: If A is Cong. permutable, $f: A \rightarrow B$ is an onto hmo, then $f^2(A) \in \text{Con}(B)$, $\forall \theta \in \text{Con}(A)$.

Theorem. If A is Cong. permutable then $\text{Con}(A)$ is modular.

Proof: By Lemma, $\theta_1 \vee \theta_2 = \theta_1 \circ \theta_2$. So

we prove

$$\theta_1 \leq \theta_2 \Rightarrow \theta_1 \circ (\theta_2 \cap \theta_3) = \theta_2 \cap (\theta_1 \circ \theta_3)$$

$$\forall \theta_1, \theta_2, \theta_3 \in \text{Con}(A)$$

$$x \theta_2 \cap (\theta_1 \circ \theta_3) y$$

$$\Rightarrow x \theta_2 y, x(\theta_1 \circ \theta_3) y$$

$$\Rightarrow x \theta_2 y, \exists z, x \theta_3 z, z \theta_1 y$$

$$\stackrel{\theta_1 \leq \theta_2}{\Rightarrow} z \theta_2 y$$

$$\Rightarrow x \theta_2 z, x \theta_3 z$$

$$\Rightarrow x(\theta_2 \cap \theta_3) z, z \theta_1 y \Rightarrow$$

$$\Rightarrow x(\theta, 0(\theta_2 \cap \theta_3))y.$$

Def. we say $\theta_1, \theta_2 \in \text{Con}(A)$ are factor congruence if:

$$1) \theta_1 \cap \theta_2 = \Delta$$

$$\Leftrightarrow \theta_1 \cap \theta_2 = \Delta$$

$$2) \theta_1 \cup \theta_2 = \nabla$$

$$\theta_1 \circ \theta_2 = \nabla$$

$$3) \theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$$

Theorem. If θ_1, θ_2 are fact cong. then

$A \stackrel{\text{alg}}{\cong}_{\text{iso}} A/\theta_1 \times A/\theta_2$ where $A/\theta_1 \times A/\theta_2$ is the algebra on cartesian product whose operations are

$$\forall \lambda \in \Omega$$

$$\lambda_{A/\theta_1 \times A/\theta_2} (([a_i]_{\theta_1}, [a'_i]_{\theta_2}), \dots, ([a_{n_\lambda}]_{\theta_1}, [a'_{n_\lambda}]_{\theta_2}))$$

$$= (\lambda_{A/\theta_1} ([a_i]_{\theta_1}, \dots, [a_{n_\lambda}]_{\theta_1}), \lambda_{A/\theta_2} ([a'_i]_{\theta_2}, \dots, [a'_{n_\lambda}]_{\theta_2}))$$

$$= ([\lambda_A(a_1, \dots, a_{n_\lambda})]_{\theta_1}, [\lambda_A(a'_1, \dots, a'_{n_\lambda})]_{\theta_2})$$

Define:

$$f: A \longrightarrow A/\theta_1 \times A/\theta_2$$

$$f(a) = ([a]_{\theta_1}, [a]_{\theta_2})$$

$$f(a) = f(a') \Rightarrow [a]_{\theta_1} = [a']_{\theta_1}, [a]_{\theta_2} = [a']_{\theta_2}$$

$$\Rightarrow a(\theta_1 \cap \theta_2) a' \Rightarrow a = a'$$

$$([a]_{\theta_1}, [a']_{\theta_2}) \in A/\theta_1 \times A/\theta_2$$

$$\Rightarrow (a, a') \in \nabla = \theta_2 \circ \theta_1 \Rightarrow a \theta_1 a'', a'' \theta_2 a'$$

$$\Rightarrow f(a'') = ([a]_{\theta_1}, [a']_{\theta_2})$$

$$[a'']_{\theta_1} = [a]_{\theta_1}, [a'']_{\theta_2} = [a']_{\theta_2}$$

$$A^n \xrightarrow{f^n} (A/\theta_1 \times A/\theta_2)^n$$

$$\begin{array}{ccc} \lambda_A \downarrow & \curvearrowright? & \downarrow \lambda_{A/\theta_1 \times A/\theta_2} \\ A & \xrightarrow{f} & A/\theta_1 \times A/\theta_2 \end{array}$$

f homo.

$$(a_1, a_2, \dots, a_{n_1}) \rightsquigarrow ([a_1]_{\theta_1}, [a_2]_{\theta_2}, \dots, [a_{n_1}]_{\theta_1}, [a_{n_1}]_{\theta_2})$$

$$\begin{aligned} & \downarrow \lambda_A \\ & \lambda_A(a_1, \dots, a_{n_1}) \rightsquigarrow ([\lambda_A(a_1, \dots, a_{n_1})]_{\theta_1}, [\lambda_A(a_1, \dots, a_{n_1})]_{\theta_2}) \\ & = f(\lambda_A(a_1, \dots, a_{n_1})) \end{aligned}$$

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حیثیہ دہلی

۹۴، ۸، ۹

در حقیقت قبل از این که اگر A_1, A_2 حیر باشند، آنگاه حاصل ضرب آنها هم حیر از سبیل نوع ۱

$$A_1, A_2 \in \text{Alg}(\mathcal{C}) \Rightarrow A_1 \times A_2 \in \text{Alg}(\mathcal{C})$$

$$\lambda_{A_1 \times A_2} ((a'_1, a'_2), \dots, (a'_n, a'_n))$$

$$= (\lambda_{A_1}(a'_1, \dots, a'_n), \lambda_{A_2}(a'_1, \dots, a'_n))$$

$$\begin{array}{ccc} (A_1 \times A_2)^{n_1} & \xrightarrow{P_1^{n_1}} & A_1^{n_1} \\ \downarrow \lambda_{A_1 \times A_2} & & \downarrow \lambda_{A_1} \\ A_1 \times A_2 & \xrightarrow{P_1} & A_1 \end{array}$$

$A_1 \times A_2$ is the product, then in $\text{Alg}(\mathcal{C})$.

$$P_1: A_1 \times A_2 \rightarrow A_1$$

$$(a_1, a_2) \rightsquigarrow a_1$$

$$\lambda_{A_1} P_1^{n_1} ((a'_1, a'_2), \dots, (a'_n, a'_n))$$

$$P_2: A_1 \times A_2 \rightarrow A_2$$

$$(a_1, a_2) \rightsquigarrow a_2$$

$$= \lambda_{A_1}(a'_1, \dots, a'_n)$$

are homo.

$$= P_1 \lambda_{A_1 \times A_2} ((a'_1, a'_2), \dots, (a'_n, a'_n))$$

Lemma.

A map $B \xrightarrow{h} A_1 \times A_2$ is homo iff

$$P_1 \circ h: B \rightarrow A_1$$

$$P_2 \circ h: B \rightarrow A_2$$

are homo.

Proof. $\leftarrow$

$$\begin{array}{ccccc}
 B^{n_1} & \xrightarrow{h^{n_1}} & (A_1 \times A_2)^{n_1} & \xrightarrow{p_1^{n_1}} & A_1^{n_1} \\
 \lambda_B \downarrow & & \downarrow \lambda_{A_1 \times A_2} & & \downarrow \lambda_{A_1} \\
 B & \xrightarrow{h} & A_1 \times A_2 & \xrightarrow{p_1} & A_1
 \end{array}$$

$$h(b_i) = (a_i^1, a_i^2) \quad \text{فرض}$$

$$h \lambda_B(b_1, \dots, b_{n_1}) = (a_1, a_2) \quad \text{فرض}$$

$$\begin{aligned}
 \lambda_{A_1} p_1^{n_1} h^{n_1}(b_1, \dots, b_{n_1}) &= p_1 h \lambda_B(b_1, \dots, b_{n_1}) \quad \text{فرض} \\
 \lambda_{A_1 \times A_2} h^{n_1}(b_1, \dots, b_{n_1}) &
 \end{aligned}$$

$$= \lambda_{A_1 \times A_2}(h(b_1), \dots, h(b_{n_1}))$$

$$= (\lambda_{A_1}(a_1^1, \dots, a_1^{n_1}), \lambda_{A_2}(a_2^1, \dots, a_2^{n_1}))$$

$$\stackrel{\text{فرض}}{=} (a_1, a_2)$$

از همین فرض داریم:

$$\lambda_{A_1} p_1^{n_1} h^{n_1}(b_1, \dots, b_{n_1}) = p_1 h \lambda_B(b_1, \dots, b_{n_1})$$

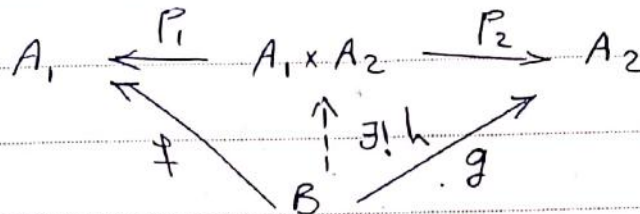
$$\underbrace{\lambda_{A_1} p_1^{n_1} h^{n_1}(b_1, \dots, b_{n_1})}_{(h(b_1), \dots, h(b_{n_1}))} = \underbrace{p_1 h \lambda_B(b_1, \dots, b_{n_1})}_{(a_1, a_2)}$$

$$\underbrace{(p_1 h(b_1), \dots, p_1 h(b_{n_1}))}_{(a_1^1, \dots, a_1^{n_1})} = a_1$$

$$\text{OFOGH} = \lambda_{A_1}(a_1^1, \dots, a_1^{n_1})$$

SHARGH

Theorem. If $A_1 \xleftarrow{f} B \xrightarrow{g} A_2$ are homo.
 then $\exists! B \xrightarrow{h} A_1 \times A_2$ homo.
 Such that $p_2 \circ h = g, p_1 \circ h = f$



Proof.

Define $h(b) = (f(b), g(b))$

then h is unique map with

$p_1 \circ h = f, p_2 \circ h = g$.

And, h is a homo. by lemma.

Remark. More generally, if $\{A_i : i \in I\}$, $A_i \in \text{Alg}(\mathcal{C})$
 then $\prod_{i \in I} A_i \in \text{Alg}(\mathcal{C})$ and the above lemma
 & th. in general form hold:

$$\prod_{i \in I} A_i = \{f: I \rightarrow \bigcup_{i \in I} A_i \mid f(i) \in A_i, \forall i \in I\}$$

$$\downarrow \prod_{i \in I} ((a_i)_{i \in I}, \dots, (a_i)_{i \in I})$$

$$= (\downarrow_{A_i} (a_i, \dots, a_i))_{i \in I}$$

$$A_1 \times A_2 = \{f: \{1, 2\} \rightarrow A_1 \cup A_2 \mid f(1) \in A_1, f(2) \in A_2\}$$

$$\lambda_{\prod A_i} (f_1, \dots, f_{n_1})$$

$$= f, f(i) = \lambda_{A_i} (f_1(i), \dots, f_{n_1}(i))$$

$$= (f(i))_{i \in I}$$

$$\begin{array}{ccc} \prod_{i \in I} A_i & \xrightarrow{p_i} & A_i \\ \exists! h \uparrow & \searrow h_i & \\ \prod_{i \in I} B_i & & \end{array}$$

Corollary. If $B_i, A_i, i \in I$ are in $\text{Alg}(\mathcal{C})$, and

$h_i: B_i \rightarrow A_i$ are homo.

then $\exists! h: \prod_{i \in I} B_i \rightarrow \prod_{i \in I} A_i$ homo. with

$$\begin{array}{ccc} \prod_{i \in I} A_i & \xrightarrow{p_i} & A_i \\ \exists! h \uparrow & \searrow h_i & \\ \prod_{i \in I} B_i & \xrightarrow{p'_i} & B_i \end{array} \quad \begin{array}{c} \uparrow k_i \\ h_i = k_i \circ p'_i \end{array}$$

$$p_i \circ h = k_i \circ p'_i \quad \forall i \in I$$

$$h(b) = (h_i(b))_{i \in I}$$

$$h((b_i)_{i \in I}) = (k_i \circ p'_i(b_i))_{i \in I}$$

$$= (k_i(b_i))_{i \in I}$$

$$\text{Ker } h = \{(b, b') \mid h_i(b) = h_i(b')\} = \bigcap_{i \in I} \text{Ker } h_i$$

$$\forall i \in I$$

$$(b, b') \in \text{Ker } h_i$$

Note.

$$\text{Ker}(h) = \bigcap_{i \in I} \text{Ker } h_i = \bigcap_{i \in I} \text{Ker}(p_i \circ h)$$

$$h: B \rightarrow \prod A_i$$

$$h: \prod B_i \rightarrow \prod A_i$$

$$\text{Ker}(h) = \bigcap_{i \in I} \text{Ker}(p_i \circ h) = \bigcap_{i \in I} \text{Ker}(p_i \circ h)$$

Exercise. h one-one $\iff \text{Ker } h = \Delta$ Thus for $h: B \rightarrow \prod A_i$ h one-one

$$\iff \bigcap_{i \in I} \text{Ker}(p_i \circ h) = \Delta$$

$$A \xrightarrow{f} B \quad \neq \text{1-1}$$

$$a_1 \neq a_2 \implies f(a_1) \neq f(a_2) \quad \text{if } f \text{ is 1-1}$$

$$B \xrightarrow{h} \prod A_i \iff$$

$$\forall b_1 \neq b_2, \exists i \in I, p_i \circ h(b_1) \neq p_i \circ h(b_2)$$

$$\iff \forall b_1 \neq b_2 \exists i \in I \quad h_i(b_1) \neq h_i(b_2)$$

$$\{h_i\} \text{ 1-1}$$

OFOGH
SHARGH

θ_1, θ_2 fact. cong.

$$\Leftrightarrow A \cong A/\theta_1 \times A/\theta_2$$

$\in \text{Con}(A)$

$$\bigcap_{i \in I} \text{Ker}(p_i \circ h) = \Delta \Leftrightarrow (p_i \circ h(b) = p_i \circ h(b'))$$

$i \in I$

$$\Rightarrow b = b'$$

$\forall i \in I$

Remark

$\text{Ker } p_1, \text{Ker } p_2$ are factor congruence.

$$i) \text{Ker } p_1 \cap \text{Ker } p_2 = \Delta \text{ (i.e., } \{p_1, p_2\}$$

خارجی و داخلی)

$$((a_1, a_2), (a'_1, a'_2)) \in \text{Ker } p_1 \cap \text{Ker } p_2$$

$$\Rightarrow a_1 = a'_1, a_2 = a'_2$$

$$\Rightarrow (a_1, a_2) = (a'_1, a'_2)$$

$$ii) \text{Ker } p_1 \circ \text{Ker } p_2 = \nabla_{A \times A_2}$$

$$((a_1, a_2), (a'_1, a'_2)) \in \nabla_{A \times A_2}$$

$$(a_1, a_2) \text{ Ker } p_2 (a'_1, a_2)$$

$$(a'_1, a_2) \text{ Ker } p_1 (a'_1, a'_2)$$

Def. A is called indecomposable if A cannot be written as a product of two non-trivial algebras.

$$A \cong A \times \{0\} \cong \{0\} \times A$$

Def. A is called simple if $\text{Con}(A) = \{\nabla, \Delta\}$.

Example. Every finite algebra A with $|A| = p$, p prim number, is indecomposable.

Lemma. A is indecomposable if the only fact. cong.

on A are ∇, Δ .

$$\nabla \cap \Delta = \Delta$$

θ_2, θ_1 is fact. cong. So $A \cong A/\theta_1 \times A/\theta_2$

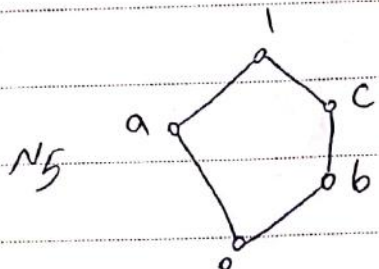
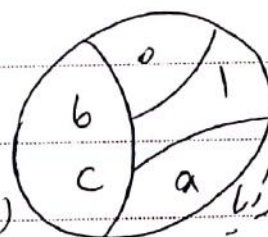
$$\nabla \cup \Delta = \nabla$$

$$\Delta \circ \nabla = \nabla \circ \Delta ?$$

note. Simple $\rightarrow$ indecomposable.

$\Leftarrow$
 N_5

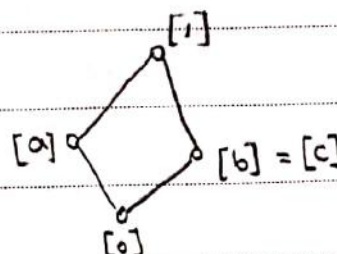
θ_1, θ_2 fact. cong. (N_5)



$$[b] = [c]$$

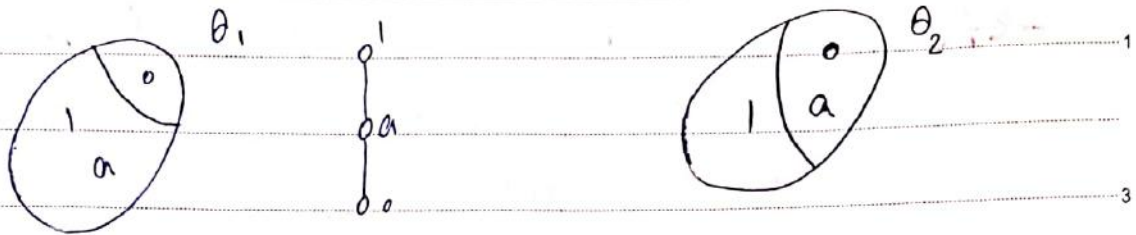
$$[b] = [c]$$

$$1: [b] = [c]$$



Example. N_4 indecomposable But not simple.

$\{0, 1, 2, 3\}$
 $\{0, 2\}$
 $\{0\}$
OFOGH
SHARGH



$(\text{Con}(A), \subseteq)$

$$\theta_1 \subseteq \theta_2$$

$$\theta_2 \subseteq \theta_1$$

$$\theta_1 \cap \theta_2 = \Delta$$

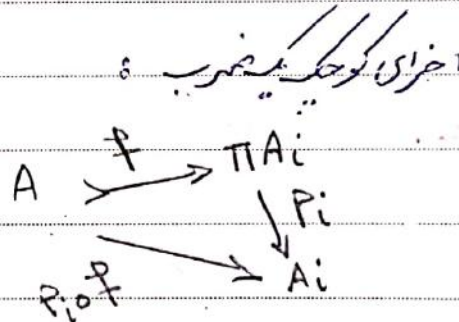
$$\theta_1 \circ \theta_2 \stackrel{(?)}{=} \nabla$$

حاصل می شود
 Δ, Δ, ∇

indecomposable

simple

irreducible



Def. we say an algebra A is a subdirect product of $\{A_i : i \in I\}$ if

(i) $\exists A \xrightarrow{f} \prod_{i \in I} A_i$ one-one homo

$$A \subseteq \text{Im } f \subseteq \prod_{i \in I} A_i$$

(ii) $\forall i \in I, p_i \circ f : A \rightarrow A_i$ is onto.
 homo.

In this case, we say $(f, \prod_{i \in I} A_i)$ is a subdirect representation of A .

Note.

$$I = \emptyset$$

$$A \text{ has S.R. } (\{f, \pi A_i\}_{i \in \emptyset})$$

 $\Leftrightarrow A$ is trivial (singleton)

$$\pi A_i = \{\emptyset\}$$

$$= \emptyset, \emptyset, \dots$$

Remark. $B \xrightarrow{h_i} A_i \Rightarrow B \xrightarrow{\exists! h} \pi A_i$
 $b \mapsto (h_i(b))$

$$\forall i \quad p_i \circ h = h_i$$

Lemma. $I \neq \emptyset$ $\theta_i \in \text{Con}(A)$, $\bigcap_{i \in I} \theta_i = \Delta$, then

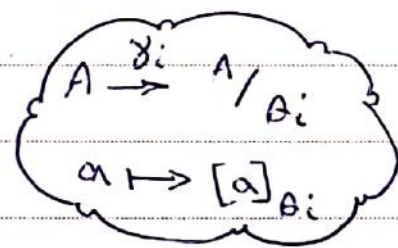
$$\gamma: A \rightarrow \prod_{i \in I} A/\theta_i \quad \text{is a S.R. of } A.$$

$$a \mapsto ([a]_{\theta_i})_{i \in I}$$

proof: $\text{Ker}(\gamma) = \bigcap_{i \in I} \text{Ker}(\gamma_i)$

$$= \bigcap_{i \in I} \theta_i = \Delta \quad \text{So } \gamma \text{ is one-one.}$$

$$\text{Also, } p_i \circ \gamma = \gamma_i \quad (\forall i \in I) \text{ onto.}$$



Def. we say A is subdirectly irreducible if
 for every S.R. $(f, \pi A_i)_{i \in I}$ of A ,
 $\exists i \in I$ $p_i \circ f$ is one-one.
 (that is, $\exists i \in I, A \stackrel{p_i \circ f}{\cong} A_i$)

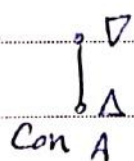
$$A \cong A_1 \times A_2 \Rightarrow A_1 \cong A \text{ or } A_2 \cong A$$

$$A \twoheadrightarrow A_1 \times A_2 \Rightarrow A \cong A_1 \text{ or } A \cong A_2$$

Remark. (1) we always can find a S.R. of A
 from a one-one hmo. $A \xrightarrow{f} \prod_{i \in I} A_i$
 as follows: $(f, \pi \underbrace{\text{Im}(p_i \circ f)}_{B_i})_{i \in I}$

(2) If A is S.I. then for every S.R.
 $(f, \pi A_i)_{i \in I}$ we have $\{p_i \circ f : i \in I\}$ separate
 elements of A .

A simple



$\text{Con } A$

A Indecomposable

$$\nexists \theta_1, \theta_2 \in \text{Con } A$$

$$\theta_1 \wedge \theta_2 = \Delta$$

$$\theta_1 \vee \theta_2 = \nabla$$

Fact $\text{Con}(A)$

غير بسيط
 قابل للتحلل

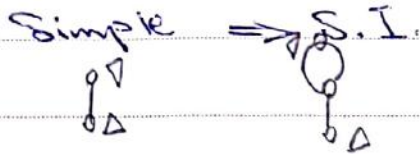
S.I Criterion:

$$\exists \theta_i \text{ monolith in } \text{Con}(A)$$

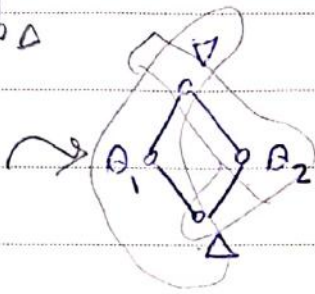
$$\theta_i = \bigcap (\text{Con}(A) = \{\Delta\})$$



$\text{Con}(A)$



Example.



is not S.I.

is not simple.

but it is indecomposable?

Example.

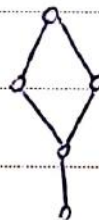
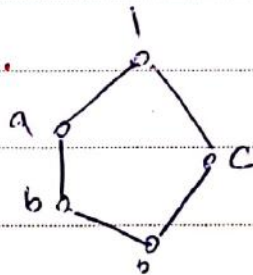


is S.I.

is simple.

is indecomposable.

example.



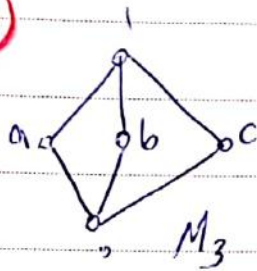
is S.I.

$Con(N_5)$?

is indecomposable

is not simple.

Example.



M_3



$\text{Con}(M_3)?$

Exercise. A finite abelian group G is S.I.

$$\iff |G| = p^n, p \text{ prime.}$$

• $\mathbb{Z}_{p^n}$ is an S.I. group.

• A vector space is S.I. $\iff$ it is S.I.
 $\iff$ it is trivial or is one-dimensional.

• Any two element algebra is S.I.

Remark.

simple $\implies$ S.I. $\implies$ indecomposable

$\nleftarrow$
 N_3

$\nleftarrow$
 $3I$

$\mathcal{A}_1, \mathcal{A}_2$ fact. cong.

$$\mathcal{A}_1 \cap \mathcal{A}_2 = \Delta$$

$$\bigcap \mathcal{A} = \mathcal{A} \neq \Delta$$

$$\mathcal{A} \neq \Delta$$

نمایش همبستگی جبرتی بل:

In Bool alg. A s.i.

$$\iff A \cong 2$$

$$B \xrightarrow{\text{s.r.}} \prod_x 2 = 2^X \cong P(X)$$

$\downarrow$

finite

B is atomic + complete

$$\Rightarrow B \cong P(X)$$

Proof of S.I. Criterion.

$(\Rightarrow)$ Let A be s.i. and

$$I = \{\theta \in \text{Con}(A) \mid \theta \neq \Delta\}$$

$$\text{Suppose } \bigcap_{\theta \in I} \theta = \Delta$$

then by lemma, $(\gamma: A \rightarrow \prod_{\theta \in I} A/\theta$

is an s.r. of A . But A is s.i., so

$$\exists \theta \in I, A \cong A/\theta.$$

$$\text{thus } \theta = \Delta. \times$$

$(\Leftarrow)$ Let $\bigcap_{\theta \in I} \theta \neq \Delta$ and $(f, \prod_{i \in I} A_i)$ be

an s.r. for A .

on the contrary, $\forall i \in I, p_i \circ f: A \rightarrow A_i$

is not one-one.

$$\text{now, } \Delta = \ker f = \bigcap_{i \in I} \ker(p_i \circ f)$$

$$\supseteq \bigcap_{\theta \in \text{Con}(A) - \{\Delta\}} \theta \neq \Delta$$

$$\theta \in \text{Con}(A) - \{\Delta\}$$

OF OGH
SHARGH

Theorem. Every finite algebra is isomorphic to a product of some indecomposable algebra.

Proof: If $|A| = 1$, then Q.E.D

Let $|A| > 1$, and the result is true for every B with $|B| < |A|$

If A is indecomposable then Q.E.D.

Let A be decomposable. So $\exists B, C$, non-trivial and $A \cong B \times C$. So

$$|B| < |A|, |C| < |A|.$$

By induction hypothesis

$\exists B_1, \dots, B_n, C_1, \dots, C_m$ indecomposable

$$\text{and } B \cong B_1 \times \dots \times B_n$$

$$, \quad C \cong C_1 \times \dots \times C_m$$

$$\text{Now } A \cong B_1 \times \dots \times B_n \times C_1 \times \dots \times C_m.$$

Theorem (Birkhoff) Every algebra has a S.R. of S.I. algebra

$$A \xrightarrow{\text{S.R.}} \prod_{i \in I} A_i \quad \leftarrow \text{S.I.}$$

Remark. A/θ is simple $\Leftrightarrow$

$$\text{Con}(A/\theta) = \{ \Delta_{A/\theta}, \nabla_{A/\theta} \}$$

$$\left\{ \begin{array}{l} \bigcap \theta_i = \Delta \\ A \xrightarrow{\text{S.R.}} \prod A/\theta_i \quad \leftarrow \text{S.I.} \end{array} \right.$$

$$\Leftrightarrow \{ \psi_{/\theta} : \psi \in \text{Con}(A), \theta \subseteq \psi \} = \{ \theta_{/\theta}, \nabla_{/\theta} \}$$

$$\{ ([a]_{\theta}, [a']_{\theta}) : a \psi a' \}$$

OFOGH
SHARGH

$\Leftrightarrow \theta$ is maximal congruence

(i.e. $\theta \neq \tau, \theta \leq \gamma \Rightarrow \theta = \gamma$)

Exercise.

A/θ is S.I. $\Leftrightarrow \theta$ is completely $\wedge$ -irreducible.

(i.e. $\theta = \bigcap_{i \in I} \gamma_i \Rightarrow \exists i, \theta = \gamma_i$)

$a \in L$ ← lattice

$\wedge$ -irr.

$a \leq c \wedge d \rightarrow a \leq c \wedge a \leq d$

$\Leftrightarrow \theta \subsetneq \bigcap \{ \gamma \in \text{Con}(A) \mid \theta \subsetneq \gamma \}$

pf: $\theta = \bigcap_{i \in I} \gamma_i \Rightarrow \theta \subsetneq \gamma_i$

Let $\theta \neq \gamma_i \forall i$

$\theta = \bigcap_{i \in I} \gamma_i \supsetneq \bigcap_{\substack{i \in I \\ \theta \subsetneq \gamma_i}} \gamma_i \neq \theta$

Proof of Birkhoff Th.

Take $\mathcal{I} = \{ \theta \in \text{Con}(A) \mid \theta \text{ is completely } \wedge\text{-irreducible} \}$

So A/θ is S.I. $\forall \theta \in \mathcal{I}$

We show $\bigcap_{\theta \in \mathcal{I}} \theta = \Delta_A$

on the contrary, let $(a, b) \in \bigcap \theta$ with $a \neq b$.

By Zorn's Lemma we find a cong. θ on A which is maximal with $(a, b) \notin \theta$.

$P = \{ \theta \in \text{Con}(A) \mid (a, b) \notin \theta \}$ now

by maximality of θ

$(a, b) \in \bigcap \{ \gamma \mid \theta \subsetneq \gamma \}$ so θ is

completely $\wedge$ -irr. (by exercise).

Thus $\theta \in I$, so $(a, b) \in \theta$. $\cdot \dot{x}$

$a \in \wedge, \vee$

حسب جدول

Variety (HSP closed)

Def. Let $\mathcal{K} \subseteq \text{Alg}(\mathcal{L})$

we write

$$A \in I(\mathcal{K}) \iff \exists B \in \mathcal{K}, A \cong B$$

$$A \in H(\mathcal{K}) \iff \exists B \in \mathcal{K}, \exists f: B \rightarrow A$$

onto
homo.

$$A \in S(\mathcal{K}) \iff \exists B \in \mathcal{K}, A \leq B.$$

($\cong$)

$$A \in P(\mathcal{K}) \iff \exists B_i \in \mathcal{K}, A = \prod_{i \in I} B_i$$

($\cong$) $i \in I$

$$A \in P_s(\mathcal{K}) \iff \exists B_i \in \mathcal{K}, A \xrightarrow{\text{S.R.}} \prod_{i \in I} B_i$$

Remark. For every $\mathcal{K} \subseteq \text{Alg}(\mathcal{L})$, and $Q \in \{I, S, P, P_s, H\}$

$$(1) \mathcal{K} \subseteq Q(\mathcal{K})$$

$$(2) \mathcal{K}_1 \subseteq \mathcal{K}_2 \implies Q(\mathcal{K}_1) \subseteq Q(\mathcal{K}_2)$$

$$(3) Q(Q(\mathcal{K})) = Q(\mathcal{K})$$

That is, all are closure operator on $\text{alg}(\mathcal{L})$

Notation. Q_1, Q_2 = composition Q_2 and Q_1 .

$$Q_1 \leq Q_2 \iff Q_1(\mathcal{K}) \subseteq Q_2(\mathcal{K})$$

$$\forall \mathcal{K} \in \text{Alg}(\mathcal{L})$$

Fact. $A \in \mathcal{K}, B \leq A \Rightarrow B \in \text{IQ}(\mathcal{K})$

$\forall Q \in \{I, S, H, P, P_S\}$

Lemma. $\text{IHS}, \text{ISP}, \text{IHP}, \text{IHP}_S$ are closure operator on $\text{Alg}(\mathcal{K})$.

$$\mathcal{K} \subseteq \text{HS}(\mathcal{K})$$

$$A \in \mathcal{K} \stackrel{?}{\Rightarrow} \exists B \in \mathcal{K}, \exists C \leq B, \exists f: C \rightarrow A \text{ onto homo.}$$

$$(1) \mathcal{K} \subseteq S(\mathcal{K}) \Rightarrow H(\mathcal{K}) \subseteq \text{HS}(\mathcal{K})$$

$$\mathcal{K} \subseteq H(\mathcal{K}) \subseteq \text{IHS}(\mathcal{K}) \checkmark$$

$$(2) \mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow S(\mathcal{K}_1) \subseteq S(\mathcal{K}_2)$$

$$\Rightarrow \text{IHS}(\mathcal{K}_1) \subseteq \text{IHS}(\mathcal{K}_2) \checkmark$$

$$(3) \text{IHS}(\text{IHS}(\mathcal{K})) \stackrel{?}{=} \text{IHS}(\mathcal{K})$$

$$E \in \text{IHS}(\text{IHS}(\mathcal{K})) \stackrel{?}{=} \text{IHS}(\mathcal{K})$$

بر کافیت شان دیم:

$$B \leq A \in \mathcal{K}$$

$$B \xrightarrow{f} C$$

$$D \leq C$$

$$D \xrightarrow{g} E$$

$$D \leq \frac{B}{\ker f} \cong C, \quad D \leq \frac{E}{\ker g}$$

$$\frac{D}{\ker g} \leq \frac{B/\ker f}{\ker g}$$

Lemma 2

(1) $SH \leq HS$

(4) $P_S H \leq H P_S$

(2) $PS \leq SP$

(5) $P_S P = P_S = P P_S$

(3) $PH \leq HP$

(6) $P_S S = SP = S P_S$

Pf: $\forall K \subseteq \text{Alg}(\mathcal{K})$

(1) $A \in SH(\mathcal{K})$

$$\Rightarrow \exists B \in \mathcal{K}, \exists B \xrightarrow{f} C, A \leq C$$

onto
homo.

$$\Rightarrow f^{-1}(A) \leq B \Rightarrow f^{-1}(A) \in S(\mathcal{K})$$

$$\Rightarrow f^{-1}(A) \xrightarrow{f} A \Rightarrow A \in HS(\mathcal{K})$$

onto

$$\text{Prop: } \mathcal{K} \subseteq S(\mathcal{K}) \subseteq HS(\mathcal{K})$$

$$H(\mathcal{K}) \subseteq H(HS(\mathcal{K})) = HS(\mathcal{K})$$

$$SH(\mathcal{K}) \subseteq SHS(\mathcal{K})$$

$$\underline{HS}H(\mathcal{K}) \subseteq HS HS(\mathcal{K}) = HS(\mathcal{K})$$

$$SH(\mathcal{K}) \subseteq HS(\mathcal{K})$$

(2) $A \in PS(\mathcal{K}) \Rightarrow \exists B_i \in \mathcal{K}, \exists C_i \leq B_i$

$$A = \prod C_i$$

$$\Rightarrow \prod C_i \leq \prod B_i$$

$$\overset{A}{\parallel}$$

$$\Rightarrow A \in SP(\mathcal{K})$$

(3) $PH \leq HP$

$$A \in PH(\mathcal{K}) \Rightarrow \exists B_i \in \mathcal{K},$$

$$B_i \xrightarrow{f_i} C_i, A = \prod C_i$$

$$\Rightarrow \prod B_i \xrightarrow{\prod f_i} \underbrace{\prod C_i}_A$$

$$\Rightarrow A \in HP(\mathcal{K})$$

عمل من بوشاد، بوشاد

(4) $PSH \leq HP_S$

$$A \in PSH(\mathcal{K}) \Rightarrow \exists B_i \in \mathcal{K}, B_i \xrightarrow{f_i} C_i$$

$$A \xrightarrow[h]{S.R.} \prod C_i$$

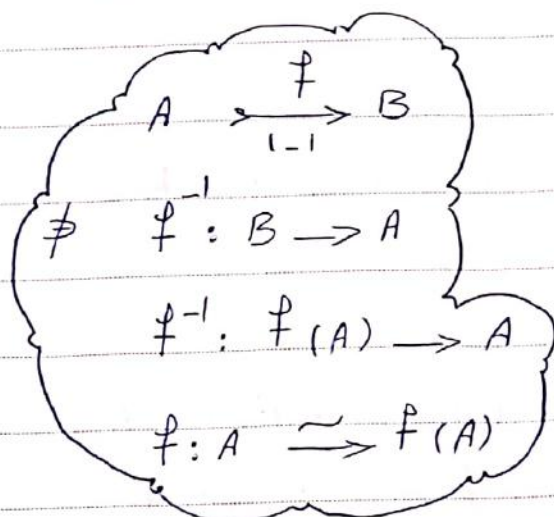
$$\underbrace{f_i(p_i h(A))}_{D_i} \leq B_i$$

$$D \doteq \prod D_i \xrightarrow{S.R.} \prod B_i \xrightarrow{\prod f_i} \prod C_i$$

$$\begin{array}{ccc} \prod D_i & \xrightarrow{\quad} & \prod B_i \\ & \searrow \checkmark & \downarrow p_i \\ & & B_i \end{array} \quad \begin{array}{l} \leftarrow \\ f_i(p_i h(A)) \\ x_i \in D_i \Rightarrow f_i(x_i) \in p_i h(A) \end{array}$$

$$(\prod f_i)(D) = h(A)$$

$$g(x) = h^{-1}((\prod f_i)(x)) \in A \quad \left(h|_{A, h(A)} \right)^{-1}$$



$$A \in H(\mathcal{K}) \Leftrightarrow \exists B \in \mathcal{K}, \exists f: B \xrightarrow{\text{onto}} A$$

homo

$$A \in S(\mathcal{K}) \Leftrightarrow \exists B \in \mathcal{K}, A \subseteq B$$

(3)

$$A \in P(\mathcal{K}) \Leftrightarrow \exists B_i \in \mathcal{K}, A = \prod B_i$$

(5)

Def. Let $\mathcal{K} \subseteq \text{Alg}(\mathcal{U})$.

(1) we say $\mathcal{K}$ is closed under an operator

Q if $Q(\mathcal{K}) \subseteq \mathcal{K}$.

specially, if $\mathcal{K}$ is closed under S , we say $\mathcal{K}$ is hereditary. (U.S.)

if $\mathcal{K}$ is closed under H , we say $\mathcal{K}$ is cohereditary.

if $\mathcal{K}$ is closed under P , we say $\mathcal{K}$ is productive.

(2) we say $\mathcal{K}$ is a variety if $\mathcal{K}$ is closed under S, H, P .

Theorem. $\mathcal{K}$ is a variety iff $\mathcal{K} = \text{HSP}(\mathcal{K})$

$$H(\mathcal{K}) = \mathcal{K}$$

$$S(\mathcal{K}) = \mathcal{K}$$

$$P(\mathcal{K}) = \mathcal{K}$$

$$\Leftrightarrow \text{HSP}(\mathcal{K}) = \mathcal{K}$$

Pf:

 $(\Rightarrow)$

$$HSP(K) = HS(K) = H(K) = K$$

 $(\Leftarrow)$

$$i) H(K) = H(HSP(K)) = HSP(K) = K$$

$$SH \subseteq HS \quad ii) S(K) = S(HSP(K)) \subseteq \underline{HSP}(K) \\ = HSP(K) = K$$

$$PH \subseteq HP \quad iii) P(K) = P(HSP(K)) \subseteq HSP(K)$$

$$PSSP \subseteq HSP(K) = HSP(K) = K$$

جواباً بر سؤال
۹۴، ۸، ۱۷

Free Algebra.

Def. An algebra $A \in \text{Alg}(\mathcal{U})$ is said to be free over set $X \subseteq A$ relative to a class $\mathcal{K} \subseteq \text{Alg}(\mathcal{U})$

if: 1) X generated A ; that is $\text{Sg}(X) = A$

2) For every function $h: X \rightarrow B, B \in \mathcal{K}$

there exists a homomorphism

$$\bar{h}: A \rightarrow B \text{ such that } \bar{h}|_X = h$$

$$X \xrightarrow{i} A = \text{Sg}(X)$$

$$\begin{array}{ccc} & \exists! \bar{h} & \\ & \downarrow & \\ h & \rightarrow & B \in \mathcal{K} \end{array}$$

note. $\bar{h}$ in the above definition is unique.

Proof: $\bar{h}, K: A \rightarrow B, K|_X = h, \bar{h}|_X = h$

$$\xrightarrow{A = \text{Sg}(X)} \bar{h} = K$$

Example. If K has trivial algebras, then every alg. A is free relative to K .

$$\begin{array}{ccc} \textcircled{X} & \xrightarrow{i} & A = \text{Sg}(X) \\ & \searrow h & \downarrow \exists! \bar{h} \\ & & B \in K \\ & & \text{free} \end{array}$$

$$h \upharpoonright_A (a_1, \dots, a_n) = \upharpoonright_B (h(a_1), \dots, h(a_n)) \quad \text{free}$$

Example. $N_0 = \mathbb{N} \cup \{0\} \in \text{Alg}((1))$

$(N, S), S: N_0 \rightarrow N, \quad \text{is free over } \{0\}.$
 $n \mapsto n+1$

relative to $\text{Alg}((1)).$

$$\{0\} \longrightarrow N_0 = \text{Sg}(0)$$

$$\begin{array}{ccc} h(0) = b_0 & \searrow h & \downarrow \exists! \bar{h} \\ & & B \in \text{Alg}((1)) \end{array}$$

$$\begin{cases} \bar{h}(0) = h(0) = b_0 \\ \bar{h}(n+1) = f(\bar{h}(n)). \end{cases} \quad \bar{h}(n) = f^n(b_0)$$

$$\bar{h}(S(n)) \stackrel{?}{=} f(\bar{h}(n))$$

$$\bar{h}(S(n)) = f^{n+1}(b_0) \quad f(\bar{h}(n)) = f^{n+1}(b_0)$$

Exercise. $(\mathbb{N}, +, 0)$ is free over $\{1\}$ relative to the class of all monoids.

Def. If A is free over X relative to $\text{Alg}(\tau)$ then we say A is absolutely free.

$$(p, q) \in F(X) \times F(X) \quad p \equiv q, p = q$$

اکا در از جبری خطا آزاد نیست
فاز جبری خطا آزاد $F(X)$

Lemma. If A is free over X_1 relative to K , and B is free over X_2 relative to K , and $|X_1| = |X_2|$ then, $A \cong B$.

Proof:

$$\begin{array}{ccc} X_1 & \xrightarrow{i_1} & A = \text{Sg}(X_1) \\ \downarrow \sigma \gg K & \searrow h = i_2 \circ K & \downarrow \exists \bar{h} \text{ homo.} \\ X_2 & \xrightarrow{i_2} & B = \text{Sg}(X_2) \end{array}$$

$$\begin{array}{ccc} X_1 & \xrightarrow{i_1} & A = \text{Sg}(X_1) \\ \uparrow \sigma \gg K' & \nearrow d = i_1 \circ K' & \uparrow \\ X_2 & \xrightarrow{\quad} & B = \text{Sg}(X_2) \end{array}$$

از ترکیب این دو تابع به تابع یکتا $\bar{\ell} \circ \bar{h}$ می‌رسیم که نشان می‌دهد $A \cong B$ است.

$$\begin{array}{ccc} X & \xrightarrow{i} & A \\ & \searrow i_! & \downarrow \text{id} \\ & & A \end{array} \quad \bar{\ell} \circ \bar{h} \Rightarrow \bar{\ell} \circ \bar{h} = \text{id}_A$$

Lemma. If A is free over X relative to K then A is free over X relative to $\text{sp}(K)$

proof: Let $C \in \text{sp}(K)$. Then $\exists B_i \in K, C \leq \prod_{i \in I} B_i$

Consider

$$h: X \rightarrow C$$

$$\begin{array}{ccccc} X & \xrightarrow{\quad} & A & & \\ h \downarrow & \text{Piojoh} & \searrow \exists! & & \\ C & \xrightarrow{j} \prod_{i \in I} B_i & \xrightarrow{P_i} & B_i \in K & \end{array}$$

از ترکیب این دو تابع به تابع یکتا $\bar{\ell} \circ \bar{h}$ می‌رسیم که نشان می‌دهد $A \cong B$ است.

$$\begin{array}{ccc} \prod_{i \in I} B_i & \xrightarrow{P_i} & B_i \\ \exists! K \uparrow & \nearrow \text{Piojoh} & \\ A & & \end{array}$$

$$K(a) = \overline{(P_i \circ j \circ h)(a)}$$

$$a \in A = \text{sg}(X) = \bigcup_{n \geq 0} E^n(X) \Rightarrow a \in E^{n+1}(X)$$

$$\exists \lambda \in K, a_1, \dots, a_n \in E^n(X), a = \lambda(a_1, \dots, a_n)$$

$$k(a) = k(\lambda(a_1, \dots, a_n)) = \lambda(k(a_1), \dots, k(a_n))$$

$\downarrow$ $\downarrow$
 $a_1 \in E(X)$

$$a = \lambda(x_1, \dots, x_n), x_i \in X \Rightarrow k(a) \in C$$

$$k(\lambda(x_1, \dots, x_n)) = \lambda(k(x_1), \dots, k(x_n)) \in C$$

$$k(x_i) = (\overline{p_i \circ j \circ h}(x_i))_{i \in I} \xrightarrow{x_i \in X} ((p_i \circ j \circ h)(x_i))_{i \in I} \in C$$

Lemma. If A is free over X relative to k then A is free over X relative to $H(k)$.

Proof:

Let $C \in H(k)$, and $h: X \rightarrow C$ a map.

Then $\exists B \in \mathcal{K} \exists \overline{f} \xrightarrow{\text{onto}} C$
 hom.

$$\begin{array}{ccc}
 X & \xrightarrow{i} & A \\
 h \downarrow & \searrow \overline{f \circ h} & \downarrow \exists! \overline{f' \circ h} \\
 C & \xleftarrow{f} & B \\
 & \searrow \overline{f'} &
 \end{array}$$

Now, $A \xrightarrow{\overline{f \circ h}} C$ and $(\overline{f \circ h})|_X = h$
 $f(\overline{f' \circ h}|_X) = f(\overline{f' \circ h}) = h$

now f is one-one and we claim $\text{Sg}(f(x))$ is free over $f(x)$ relative to K .

f is 1-1.

$$x_1 \neq x_2 \in X \Rightarrow \exists u, u(x_1) \neq u(x_2)$$

$$u: X \rightarrow B$$

$$B \in K, |B| \neq 1$$

$$x_1 \rightsquigarrow b_1$$

$$x_2 \rightsquigarrow b_2$$

$$\exists b_1 \neq b_2 \in B$$

$$x \neq x_1, x_2 \rightsquigarrow b_1$$

$$f(x) \rightarrow \text{Sg}_D(f(x))$$

$$\begin{array}{ccc} & \swarrow h & \searrow \exists! \bar{h}(?) \\ & & B \in K \end{array}$$

$$\text{now } |h(f(x))| \underset{\text{map}}{\leq} |f(x)| = |x| \text{ So}$$

$$\exists u \in M, \text{Sg}_B(h(f(x))) = B_u$$

Thus

$$\begin{array}{ccccc} f(x) & \hookrightarrow & \text{Sg}_D(f(x)) & \hookrightarrow & \prod_{u \in M} B_u \\ \downarrow h & \searrow \bar{p} & \downarrow p_u & & \downarrow p_u \\ h(f(x)) & \hookrightarrow & \text{Sg}_B(h(f(x))) & = & B_u \\ \downarrow & \searrow \bar{p} & & & \\ B & \leftarrow & B_u \end{array}$$

جبهه تدریس

۹۴، ۱، ۲۳

Remark. 1) In any variety $\mathcal{K}$ the free algebra over any set X relative to $\mathcal{K}$ belongs to $\mathcal{K}$.

2) Free algebras over X are unique up to isomorphism

$$\begin{array}{ccc} X & \xrightarrow{i_1} & F_1 \\ & \searrow i_2 & \downarrow \exists! h \exists! K \\ & & F_2 \end{array}$$

notation. Free algebra over X relative to $\mathcal{K}$: $F_{\mathcal{K}}(X)$

Absolutely free over X : $F(X)$

3) Every algebra is homomorphic image of a free algebra.

proof:

$$A \in \text{Alg}(\mathcal{L}) \Rightarrow |A| \text{ is a set}$$

underlying

$$|A| \xrightarrow{i} F(|A|)$$

set of the algebra $(A, (\lambda_A)_{\lambda \in \mathcal{L}})$

we show:

$$\begin{array}{ccc} & \downarrow \exists! h \\ \text{id} \searrow & & \downarrow \\ & A \in \text{Alg}(\mathcal{L}) \end{array}$$

A is a homo image of $F(|A|)$

$$\exists h, h \circ i = \text{id} \Rightarrow h \text{ onto homo.}$$

Exercise. If $K = S(K)$ then $A \in K$ is free in K over X iff every map $f: X \rightarrow B$, $B \in K$, can be uniquely extended to a homo $\bar{f}: A \rightarrow B$

$$\begin{array}{ccc} X & \hookrightarrow & A \\ f \downarrow & \nearrow \bar{f} & \\ B \in K & & \end{array}$$

Def. For any algebra A , and any set X with $x \in X$, we call the map $e_x: |A|^X \rightarrow A$ with $e_x(f) = f(x)$ ($e_x \in A^{|A|^X}$)

$$\downarrow$$

$$|A|^X = \{ f: X \rightarrow A \mid f \text{ map} \}$$

the evaluation map. $f(x) = e_x(f)$

Remark. (If A is non-trivial)

1) The evaluation map e_x gives a one-one map.

$$h: X \rightarrow A^{|A|^X} = \Pi A$$

$$\Pi A = \{ \alpha: |A|^X \rightarrow A \mid \alpha(f) \in A, \forall f \in |A|^X \}$$

defined by $h(x): |A|^X \rightarrow A$

$$h(x)(f) = e_x(f)$$

1-1:

$$h(x) = h(y) \Rightarrow h(x)(f) = h(y)(f) \quad \forall f \in |A|^X$$

$$\Rightarrow e_x(f) = e_y(f) \quad \forall f \in |A|^X$$

$$\Rightarrow f(x) = f(y) \quad \forall f \in |A|^X \Rightarrow x = y$$

Suppose $x \neq y$: برعکس فرض

Define $f: X \rightarrow |A|$ $a_1 \neq a_2 \in A$

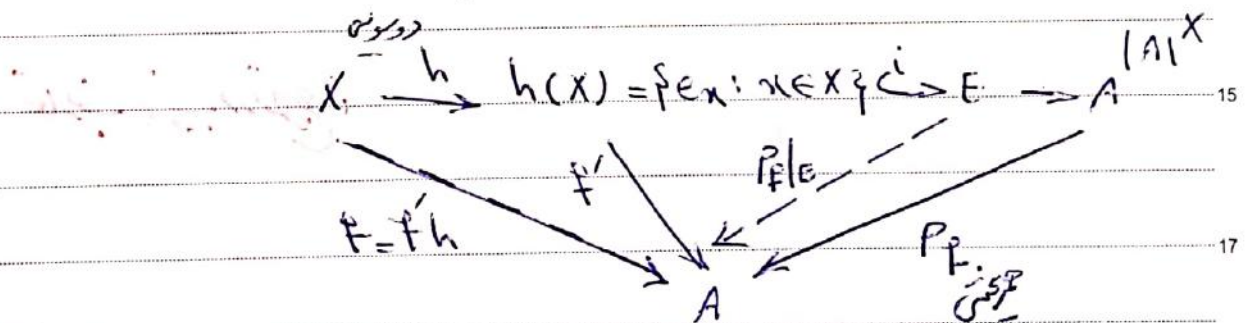
$$x \mapsto a_1$$

$$y \mapsto a_2$$

$$\Rightarrow f(x) \neq f(y) \cdot \dot{x}$$

Remark. Put $E = \text{Sg}(\{e_x : x \in X\}) \leq |A|^X$ Then

$$E \subseteq F_{|A|^X}(X)$$



نشان دهیم E روی $h(x)$ و $h(x)$ روی X است.

$$h(x) = e_x \mapsto e_x$$

$$\downarrow \quad \quad \downarrow p_f$$

$$f'(h(x)) = e_x(f) = f(x) = f'(h(x))$$

Def. Let $\tau = (n_i)_{i \in \Sigma}$ and $F(x)$ be the absolutely free algebra of type τ over the set $X = \{x_1, x_2, \dots, x_n\}$

Recall that

for $A \in \text{Alg}(\tau)$, $A^{|A|^n} \in \text{Alg}(\tau)$ with operations $\lambda_{A^{|A|^n}}(\alpha_1, \dots, \alpha_n) = |A|^n \rightarrow A$

$$\lambda_{A^{|A|^n}}(\alpha_1, \dots, \alpha_n)(a_1, \dots, a_n)$$

$$= \lambda_A(\alpha_1(a_1, \dots, a_n), \dots, \alpha_n(a_1, \dots, a_n))$$

$$|A|^n \xrightarrow{\prod_{i=1}^n \alpha_i} |A|^n \xrightarrow{\lambda_A} |A|$$

$\searrow \lambda$

تعريف ضرب در تنوع:

$$A \xleftarrow{P_1} A \times A \xrightarrow{P_2} A$$

$$\begin{array}{ccc} & \downarrow \{ (f, g) \} & \\ f \swarrow & & \searrow g \\ & B & \end{array}$$

$$b \mapsto (f(b), g(b))$$

$$\begin{array}{ccccc} A & \xleftarrow{P_1} & A \times A & \xrightarrow{P_2} & A \\ f \uparrow & & \uparrow P \times Q & & \uparrow g \\ B & \xleftarrow{Q_1} & B \times B & \xrightarrow{Q_2} & B \end{array}$$

$$\prod_{i=1}^{n_1} \alpha_i = (\alpha_i)_{i \in \{1, \dots, n_1\}}$$

$$|A|^{n_1} \xrightarrow{P_{n_1}} A$$

$\nwarrow$ $\nearrow \alpha_i$
 $|A|^n$

$$\prod_{i=1}^{n_1} \alpha_i(a_1, \dots, a_n) = (\alpha_i(a_1, \dots, a_n), \dots, \alpha_{n_1}(a_1, \dots, a_n))$$

now $x \hookrightarrow F(x) = S_g(x)$

$$h \searrow \exists! \varphi \text{ homo}$$

A^{h_1}

For every $p \in F(x)$ we call $P_A = \varphi(p): |A|^n \rightarrow A$
a polynomial (function) over A .

$$F[x] \xrightarrow{\varphi_\alpha} F[\alpha]$$

$\alpha \in F$

Example. $X = \{x_1, x_2\}$ $\alpha = (2)$ $\lambda = +$

$$p = x_1 + x_2 \in F(x)$$

$$\varphi(p) = P_A = (x_1 + x_2)_A: |A|^2 \rightarrow |A|$$

$$x_1, x_2 \in F(x)$$

$$\underbrace{\varphi(x_1 + x_2)}_p = \underbrace{\varphi(x_1) + \varphi(x_2)}_{A^{h_1}} \Rightarrow \varphi(p) = h(x_1) + h(x_2)$$

$$|A|^2 \rightarrow A$$

$$\begin{aligned}
 \psi(p)(a_1, a_2) &= h(x_1)(a_1, a_2) + h(x_2)(a_1, a_2) \\
 &= p_1(a_1, a_2) + p_2(a_1, a_2) \\
 &= a_1 + a_2
 \end{aligned}$$

Def. An Equation (identity) in a set X of variables is a pair $(p, q) \in F(X) \times F(X)$ we write it as $p \approx q$ or $p \equiv q$.

we say an algebra A satisfy $p \approx q$ if $p_A \approx q_A$ and write it as $A \models (p \approx q)$
 $p_A \approx q_A \iff \forall a_1, a_2, \dots, a_n \in A$
 $p_A(a_1, \dots, a_n) = q_A(a_1, \dots, a_n).$

حساب
۹، ۸، ۳

$$X = \{x_1, \dots, x_n\}$$

$$\begin{array}{ccc} X & \xrightarrow{i} & F X \\ & \searrow h & \downarrow \exists! \varphi \\ & & A^{|A|^n} = \prod A \end{array}$$

$$\varphi_i = h$$

$$h(x) = p_x \quad \{h$$

$$\varphi(p) = p_A: |A|^n \rightarrow A$$

$$e_{x_i}: |A|^n \rightarrow A$$

$$e_{x_i}(f) = f(x_i)$$

$$[e_{x_i}(a_1, \dots, a_n) = a_i]$$

$$(p, q) \in F X \times F X$$

$$p = q \quad \vdash \quad p \equiv q$$

$$A \models (p = q)$$

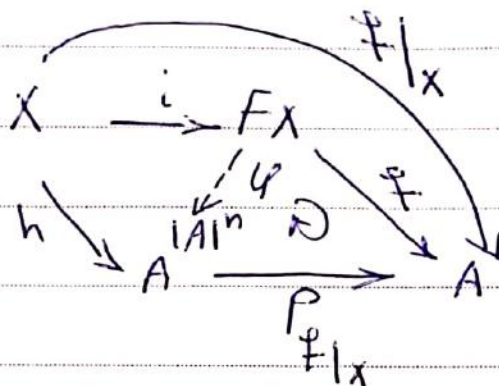
$$\Leftrightarrow$$

$$\text{for all } A, p, q$$

$$\begin{array}{c} p_A = q_A \\ \varphi(p) = \varphi(q) \end{array}$$

Lemma. For $A \in \text{Alg}(R)$, $A \models (p=q)$
 iff $\forall f: F_X \rightarrow A$,
 $f(p) = f(q)$.

Proof: ($\Rightarrow$) Let $f: F_X \rightarrow A$ be given.



$$\text{Now, } \varphi(p) = \varphi(q)$$

$$\text{So, } f(p) = f(q)$$

$$\text{hence } f(p) = f(q)$$

($\models$) By hypothesis,

$$\forall q \in |A|^X,$$

$$f: Fx \xrightarrow{\varphi} |A|^X \xrightarrow{p_g} A$$

$$f(p) = f(q)$$

thus $\varphi(p) = \varphi(q)$. because

$$f(p) = p_g \varphi(p), f(q) = p_g \varphi(q).$$

$$B \xrightarrow{k} A_1 \times A_2$$

$$k(b) = k(b') \iff p_1 k(b) = p_1 k(b')$$

$$p_2 k(b) = p_2 k(b')$$

Theorem. If $A \models (p=q)$ then,

$$(1) \forall B \leq A, B \models (p=q)$$

(2) Every homomorphic image of A satisfy $(p=q)$

$$(3) A_i \models (p=q) \forall i \in I \Rightarrow \prod_{i \in I} A_i \models (p=q)$$

Proof: ① Let $B \leq A$, and $f: Fx \rightarrow B$, then

$$Fx \xrightarrow{f} B \xrightarrow{i} A$$

$\searrow \quad \nearrow$
 iof

So, $(\text{iof})(p) = (\text{iof})(q)$, thus

$$f(p) = f(q)$$

② Let $A \xrightarrow{f} C$ onto homo and
 $Fx \xrightarrow{k} C$ homo.

f is onto, so $\exists f'_{C \rightarrow A}$ s.t. $f \circ f' = \text{id}_C$.

$$\begin{array}{ccc} Fx & \xrightarrow{k} & C \\ \uparrow i & \searrow f' \circ k & \downarrow f' \\ X & & A \end{array}$$

$$\begin{array}{ccc} X & \xrightarrow{i} & Fx \\ f' \circ k \downarrow & \searrow f & \downarrow k \\ A & \xleftarrow{f} & C \end{array}$$

now, $\bar{f}(p) = \bar{f}(q)$. we show
 $k(p) = k(q)$.

$$\bar{f}i = f' \circ k i \Rightarrow \bar{f} = f' \circ k$$

$$\text{so, } \bar{f}(p) = \bar{f}(q)$$

$$\Rightarrow f \bar{f}(p) = f \bar{f}(q)$$

$$\Rightarrow f f' k(p) = f f' k(q)$$

$$\Rightarrow k(p) = k(q).$$

③ $F_X \xrightarrow{f} \prod A_i$ be a homo.

thus $\forall i \in I$

$$F_X \xrightarrow{f} \prod A_i \xrightarrow{p_i} A_i$$

$$\searrow \quad \nearrow$$

$$p_i \circ f$$

$$\text{so } (p_i \circ f)(p) = (p_i \circ f)(q)$$

$$\text{thus, } f(p) = f(q).$$

Def. (1) we say $\mathcal{K} \subseteq \text{Alg}(\mathcal{C})$ satisfying $(p=q)$ if

$$A \models (p=q), \forall A \in \mathcal{K}$$

(2) Let $\Sigma \subseteq F_X \times F_X$. we say $A \models \Sigma$ if $\forall (p, q) \in \Sigma, A \models (p=q)$.

(3) Let $\Sigma \subseteq F_X \times F_X$ and $\mathcal{K} \subseteq \text{Alg}(\mathcal{C})$.

Then we say $\mathcal{K} \models \Sigma$ if

$$A \models (p=q) \quad \forall A \in \mathcal{K}, \forall (p, q) \in \Sigma$$

(4) For $\Sigma \subseteq F_X \times F_X$, we define

$$\text{Mod } \Sigma \doteq M(\Sigma)$$

$$= M(\Sigma, \mathcal{C})$$

$$= \{ A \in \text{Alg}(\mathcal{C}) : A \models \Sigma \}$$

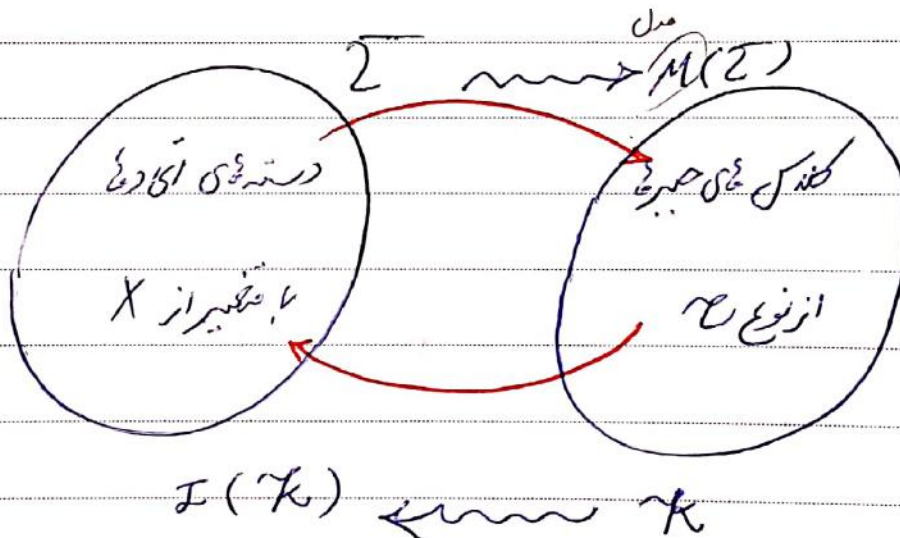
(5) For $\mathcal{K} \subseteq \text{Alg}(\mathcal{C})$, we define

$$I(\mathcal{K}) \stackrel{\text{مقادیر طرفین}}{=} Id(\mathcal{K}) \stackrel{\text{مقادیر}}{=} Id(\mathcal{K}, X)$$

$$= \{(p, q) \in FX \times FX \mid A \models (p = q)\} \\ \forall A \in \mathcal{K}$$

(6) we say $\mathcal{K} \subseteq Alg(\Sigma)$ is equational
if $\exists \Sigma \subseteq FX \times FX$ with
 $\mathcal{K} = mod(\Sigma)$

(7) we say $\Sigma \subseteq FX \times FX$ is model closed
if $\exists \mathcal{K} \subseteq Alg(\Sigma)$ with $I(\mathcal{K}) = \Sigma$



Theorem. 1) $\Sigma_1 \subseteq \Sigma_2 \Rightarrow M(\Sigma_2) \subseteq M(\Sigma_1)$

2) $\mathcal{K}_1 \subseteq \mathcal{K}_2 \Rightarrow I(\mathcal{K}_2) \subseteq I(\mathcal{K}_1)$

(3) $\Sigma \subseteq I(M(\Sigma))$

(4) $\mathcal{K} \subseteq M(I(\mathcal{K}))$

(5) $I(\mathcal{K}) = IMI(\mathcal{K})$

$$(6) M(\Sigma) = MIM(\Sigma)$$

$$\mathcal{K} = M(\Sigma)$$

$$\Leftrightarrow$$

$$HSP(\mathcal{K}) = \mathcal{K}$$

Proof:

$$(3): I(M(\Sigma))$$

$$= \{(p, q) \mid M(\Sigma) \models (p = q)\}$$

$$= \{(p, q) \mid A \models (p = q) \ \forall A \in M(\Sigma)\}$$

$$\{B \mid B \models \Sigma\}$$

$$(6): \Sigma \stackrel{(3)}{\subseteq} I(M(\Sigma))$$

$$\Rightarrow MIM(\Sigma) \stackrel{(1)}{\subseteq} M(\Sigma)$$

$$\stackrel{(4)}{\subseteq} MIM(\Sigma)$$

$$\Rightarrow M(\Sigma) = MIM(\Sigma)$$

$$(5): I(\mathcal{K}) \stackrel{(3)}{\subseteq} IMI(\mathcal{K})$$

$$\stackrel{(4)}{\subseteq} I(\mathcal{K})$$

مجموعه
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Th. (Birkhoff)

Variety $\equiv$ equational class

$\rightarrow$ we say $K \subseteq \text{Alg}(\mathcal{L})$ is eqn.

$K \subseteq \text{Alg}(\mathcal{L})$ iff $M(\Sigma) = K$.

equational class iff

$\exists X, \exists \Sigma \subseteq F_X \times F_X$

$K = M(\Sigma)$

$= \{ A \in \text{Alg}(\mathcal{L}) \mid A \models \Sigma \}$

Example. Group $\subseteq \text{Alg}(\mathcal{L}, 1, \cdot)$ is equational and

$\text{Grp} = M(\Sigma)$, where

$$\Sigma = \begin{cases} (x \cdot y) \cdot z = x \cdot (y \cdot z) \\ x \cdot e = x = e \cdot x \\ x \cdot x^{-1} = e = x^{-1} \cdot x \end{cases}$$

Do divisible groups form a variety?

Divisible Groups: $\forall n \in \mathbb{Z}, \forall a \in G \exists b \in G, nb = a$

Lemma 1. If $T = \{\theta \in \text{Con}(F_X) \mid F_X/\theta \in S(K)\}$ then

$$I_X(K) = \bigcap_{\theta \in T} \theta$$

$$\text{① } I_X(K) \subseteq \bigcap_{\theta \in T} \theta$$

Proof: Let $(p, q) \in I(K)$, so $A \models (p, q) \forall A \in K$

Take $\theta \in T$. then $F_X/\theta \in S(K)$.

now $F_X/\theta \models (p, q)$.

But for the normal epi.

$$\gamma: F_X \rightarrow F_X/\theta$$

we have $\gamma(p) = \gamma(q)$, thus $(p, q) \in \text{Ker } \gamma = \theta$

Take $(p, q) \in \bigcap_{\theta \in T} \theta$, and $A \in K$, and

$$f: F_X \rightarrow A$$

is a homo.

Thus $F_X/\text{Ker } f \cong \text{Im } f \leq A \in K$. Therefore

$\text{Ker } f \in T$.

Now $(p, q) \in \text{Ker } f$, i.e. $f(p) = f(q)$.

Lemma 2. If V is an infinite countable set, then

$$MI_V(K) \subseteq MI_X(K) \quad \forall X$$

Proof: Let X be a finite set, and consider

$$X \subseteq V.$$

Suppose $V \xrightarrow{f} X$ be a left inverse

of $i: X \rightarrow V$

$$f \circ i = \text{id}_X$$

$$\text{id}_X \Rightarrow \text{id}_X \neq f$$

$$V = \mathbb{N}$$

$$X = \{x_1, \dots, x_n\}$$

$$\cong \{1, 2, \dots, n\} \subseteq \mathbb{N} \subseteq V$$

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then we get

$$\begin{array}{ccc}
 V & \xrightarrow{i_2} & FV = \text{Sg}(V) \\
 \downarrow \bar{f} & \searrow i_1 \circ \bar{f} & \downarrow \exists! \bar{f} \text{ homo} \\
 X & \xrightarrow{i_1} & FX = \text{Sg}(X)
 \end{array}$$

we note that $\bar{f}$ is also onto (?)

$$x * y \in FX$$

$$x, y \in X$$

$$\bar{f}(v_1) = x, \bar{f}(v_2) = y$$

$$v_1, v_2 \in FV$$

$$\bar{f}(v_1 * v_2) = \bar{f}(v_1) * \bar{f}(v_2)$$

$$= \bar{f}i_2(v_1) * \bar{f}i_2(v_2)$$

$$= i_1 \bar{f}(v_1) * i_1 \bar{f}(v_2)$$

$$= x * y$$

Now, let $A \in \text{MI}_V(K)$ and $(p, q) \in I_X(K)$.

we show $A \models (p, q)$.

Consider a homo. $g: FX \rightarrow A$

then $g\bar{f}: FV \rightarrow A$ is a homo.

Also, since $\bar{f}$ is onto

$$\exists (\bar{p}, \bar{q}) \in FV \times FV, \bar{f}(\bar{p}) = p, \bar{f}(\bar{q}) = q$$

we claim $(\bar{p}, \bar{q}) \in I_V(K)$

Case 1: X finite

Suppose $X \xrightarrow{i} V$. Then $\exists f: V \xrightarrow{\text{onto}} X$

$f \circ i = \text{id}_X$. Hence

$$\begin{array}{ccc} V & \xrightarrow{i} & FV \\ f \downarrow & \uparrow i & \downarrow \exists! \bar{f} \\ X & \xrightarrow{i} & FX \end{array}$$

Let $A \in \text{MI}_V(\tilde{K})$, $(p, q) \in I_X(\tilde{K})$,

and $g: FX \rightarrow A$.

Claim $g(p) = g(q)$

$\bar{f}$ onto $\Rightarrow \exists (\bar{p}, \bar{q}) \in (FV)^2$

$\bar{f}(\bar{p}) = p$, $\bar{f}(\bar{q}) = q$.

(*) we show $(\bar{p}, \bar{q}) \in I_V(\tilde{K})$

Then $A \models (\bar{p}, \bar{q})$

So,

$$\underbrace{g(\bar{f}(\bar{p}))}_p = \underbrace{g(\bar{f}(\bar{q}))}_q$$

(*) : $(\bar{p}, \bar{q}) \in I_V(\tilde{K})$;

Let $h: FV \rightarrow B$, where $B \in \tilde{K}$.

then $X \xrightarrow{i} FX$ ~~shows~~

$$\begin{array}{ccccc} & & & \downarrow \exists! \bar{i} & \\ i \downarrow & & & & \\ V & \xrightarrow{i} & FV & \xrightarrow{h} & B \end{array}$$

claim: $(\bar{p}, \bar{q}) \in I_V(K)$.

Take $B \in \mathcal{R}$, $h: FV \rightarrow B$. then

$\exists ! \bar{i}: FX \rightarrow FV$ with $\bar{i}i_1 = i_2i$.

So, $FX \xrightarrow{\bar{i}} FV \xrightarrow{h} B$ is a homo.

and $h\bar{i}(p) = h\bar{i}(q)$ (since $B \models (p, q)$)

Thus $h(\bar{p}) = h(\bar{q})$

But $\bar{i}\bar{p} = \bar{p}oi = \bar{id} = id$

So, $h(\bar{p}) = h(\bar{q})$.

$$\begin{array}{ccc} V & \xrightarrow{i_2} & FV = \text{sg}(V) \\ \uparrow \downarrow \bar{p} & & \uparrow \downarrow \bar{q} \\ X & \xrightarrow{i_1} & FX = \text{sg}(X) \end{array}$$

$$A \models (\bar{p}, \bar{q})$$

$$\Rightarrow g\bar{p} = g\bar{q}$$

$$\Rightarrow g(p) = g(q)$$

$$\Rightarrow A \models (p, q)$$

now, let X be arbitrary. we have

$$(p, q) \in FX \times FX \Rightarrow \exists y \overset{\text{finite}}{\subset} X, (p, q) \in \text{sg}(y) \times \text{sg}(y)$$

$$FX = \text{sg}(X) = \bigcup \{ \text{sg}(y) \mid y \overset{\text{finite}}{\subset} X \}$$

$$p \in \text{sg}(y_1), q \in \text{sg}(y_2)$$

$$\Rightarrow p, q \in \text{sg}(y_1 \cup y_2) := \text{sg}(y)$$

But V is infinite countable.

$\exists U \subseteq V$, $|U| = |Y|$. then by the first case,

$$MI_V(K) \subseteq MI_U(K)$$

$$= MI_Y(K)$$

$$\xRightarrow{Y \subseteq X} MI_X(K).$$

proof Birkhoff Theorem:

Let K be equational then $\exists X$,

$$\Sigma \subseteq FX \times FX, K = M(\Sigma).$$

Then

Since we showed $K \models \Sigma$ implies $H(K) \models \Sigma$,

$$S(K) \models \Sigma, P(K) \models \Sigma.$$

$$\text{so } HSP(M(\Sigma)) \subseteq M(\Sigma)$$

$$\text{so } HSP(K) = K.$$

Conversely, we show if $K = HSP(K)$ then

$$K = MI_N(K).$$

Let $A \in MI_N(K)$. By lemma 2,

$$A \in MI_X(K), \forall X.$$

There is some X s.t. there is an onto homo.

$$F_X \xrightarrow{\bar{f}} A.$$

Since $A \in \text{MI}_X(\mathbb{K})$, $A \models (p, q) \quad \forall (p, q) \in I_{X,}(\mathbb{K})$

So, $\bar{f}(p) = \bar{f}(q)$, $\forall (p, q) \in I_{X,}(\mathbb{K})$.

thus $I_{X,}(\mathbb{K}) \subseteq \ker \bar{f}$.

Now

by Decomposition Th.

$$\begin{array}{ccc} F_X & \xrightarrow{\quad} & F_X / I_{X,}(\mathbb{K}) \\ & \searrow \bar{f} & \downarrow \exists! \bar{f} \text{ homo.} \\ & & A \end{array}$$

and $\bar{f}$ onto $\Rightarrow \bar{f}$ onto.

now consider

$$\begin{array}{ccc} F_X & \xrightarrow{\gamma} & F_X / I_{X,}(\mathbb{K}) \\ & \searrow \pi \gamma_\theta & \downarrow \exists! h \text{ (one-one)} \\ & & \prod_{\theta \in \mathcal{T}} F_X / \theta \end{array}$$

$$\text{Recall } \ker(\pi \gamma_\theta) = \bigcap_{\theta \in \mathcal{T}} \theta = I_{X,}(\mathbb{K})$$

By Decomposition Th, $\exists! h$ one-one.

$$\text{So, } F_X / I_{X,}(\mathbb{K}) \in \text{SPS}(\mathbb{K}) \subseteq \text{SSP}(\mathbb{K}) = \text{SP}(\mathbb{K}) \subseteq \mathbb{K}$$

$$\text{PS} \subseteq \text{SP}$$

Thus $A \in \text{HSP}(\mathbb{K}) = \mathbb{K}$.

جلبہ نوزدیم
۹۴، ۹، ۱۴

Def. A Boolean Algebra is an algebra

$(B; \vee, \wedge, (')', 0, 1)$ of type $(2, 2, 1, 0, 0)$

such that:

B1) $(B; \vee, \wedge)$ is a distributive lattice.

B2) $x \wedge 0 = 0, x \vee 1 = 1.$

B3) $x \wedge x' = 0, x \vee x' = 1.$

Note. The class $\mathbf{Bool}$ of Boolean algebras is a variety (because it is an equational class)

Remark.

B4) $a \wedge b = 0, a \vee b = 1 \Rightarrow b = a'$

B5) $(x')' = x$

B6) $(x \vee y)' = x' \wedge y', (x \wedge y)' = x' \vee y'$

Example. $(P(X), \cup, \cap, (')', \emptyset, X)$ is a Boolean alg. $(X \neq \emptyset)$

Example. $\prod_2^1$ is a Boolean alg.

Example. $2^X, \forall X$ is a Boolean alg.

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Example. $(\mathcal{P}, \vee, \wedge, \neg, F, T)$ is a Boolean alg.
 $\mathcal{P} \rightarrow$ equivalence of propositions.
 all propositions

Lemma: $\mathcal{P}(A) \cong 2^A$
 $\downarrow$
 as Boolean alg.

Def. Let B be a Boolean alg. $b \in B$.

Then $[a, b] \doteq \downarrow b = \{x \in B \mid x \leq b\}$ is a Boolean alg.

$$\vee_{[a, b]} = \vee_B$$

$$\wedge_{[a, b]} = \wedge_B$$

$$0_{[a, b]} = 0_B$$

$$1_{[a, b]} = b$$

$$x'_{[a, b]} = x'_B \wedge b$$



$$x \vee x'_{[a, b]} = x \vee (x'_B \wedge b)$$

$$= (x \vee x'_B) \wedge (x \vee b)$$

$$= 1 \wedge b$$

$$= b = 1_{[a, b]}$$

Lemma. Let B be a Boolean alg. $b \in B$ then

$$B \xrightarrow{f} \downarrow b$$

$$f(x) = x \wedge b$$

is a Boolean homo.

$$f \doteq - \wedge b \doteq \alpha_b$$

Exercise. $f: B_1 \rightarrow B_2$ is a Boo. homo.

$$\iff f \text{ preserve } \vee, \wedge, ()'. \quad f(x') = (f(x))'$$

$$\iff f \text{ preserve } \vee, \wedge, 0, 1.$$

$$\iff f \text{ preserve } \vee, ()', 0.$$

$$\iff f \text{ preserve } \wedge, ()', 1.$$

Th. Let B be a Boo. alg., $b \in B$. then $B \cong \downarrow b \times \downarrow b'$

$$\begin{array}{ccc} \downarrow b & \xleftarrow{\quad} \downarrow b \times \downarrow b' & \xrightarrow{\quad} \downarrow b' \\ & \nwarrow \alpha_b \quad \nearrow \alpha_{b'} & \\ & B & \end{array}$$

Proof. Consider $\alpha \equiv \alpha_b \times \alpha_{b'} : B \rightarrow \downarrow b \times \downarrow b'$
 $x \mapsto (x \wedge b, x \wedge b')$

we show that α is an iso.

α is 1-1:

$$x \wedge b = y \wedge b$$

$$x \wedge b' = y \wedge b'$$

we show that $x = y$.

$$x = x \wedge 1$$

$$= x \wedge (b \vee b')$$

$$= (x \wedge b) \vee (x \wedge b')$$

$$= (y \wedge b) \vee (y \wedge b')$$

$$= y \wedge (b \vee b') = y$$

α is onto:

$$(x, y) \in \downarrow b \times \downarrow b'$$

$$\Rightarrow z \doteq x \vee y, f(z) = (x, y)$$

$$(x \vee y) \wedge b \stackrel{?}{=} x$$

$$(x \wedge b) \vee (y \wedge b)$$

and

$$(x \vee y) \wedge b' \stackrel{?}{=} y$$

Corollary. 2 is the only (up to iso) indecomposable Boolean algebra.

Proof:

If $|B| > 2$, $b \neq 0, 1 \in B$ then

$B \cong \downarrow b \times \downarrow b'$ which is a nontrivial decomposable.

Stone Th.

Cor. Every finite Bool. alg. is isomorphic to a power of 2 , and hence is iso. to $P(X)$, for some X .

Def. A field of sets is a sub Bool. alg. of some $P(X)$.

Theorem. The only S.I. Bool. alg. is 2 , and so every Bool. alg. is isomorphic to a fielded

OF OGH
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of sets.

proof.

indecomposable $\Rightarrow$ S.I.
$\mathbb{Z}$ is S.I.since if $h: \mathbb{Z} \xrightarrow{\text{S.R.}} \prod B_i$ then $\exists i$
 $\mathbb{Z} \cong B_i$.

$$\mathbb{Z} \xrightarrow{h} \prod B_i$$

$$\begin{array}{ccc} & \searrow \text{P}_i \circ h & \swarrow \text{P}_i \\ & B_i & \end{array}$$

Let B be an S.I. Bool. alg. and $|B| > 2, b \in B, b \neq 0, 1$.then $B \cong \mathbb{Z}$ (by $P_{x/b} \circ h$)Now by Birkhoff every $BA, B \xrightarrow{\text{S.R.}} \prod_x \mathbb{Z} \cong P(x)$
that $B \leq P(x)$.that is, B is iso. to a field of sets.

Cor. $B_{\infty} = V(\mathbb{Z})$

$$= \text{HSP}\{\mathbb{Z}\} = \text{SP}\{\mathbb{Z}\}$$

$$= P_S(\{\mathbb{Z}\})$$

Def. A ring $(B; +, \cdot, -(), 0, 1)$ is called Boolean if

$x \cdot x = x \quad \forall x \in R$.

Remark. Any Boolean ring is commutative and $-x = x$ ($\forall x$)

$$(x+x)^r = x$$

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Th (stone): $Boo \cong Boo \text{ ring}$

Proof:

$$(B; \vee, \wedge, (')', 0, 1) \xrightarrow{\psi} (B; +, \cdot, -, 0, 1)$$

$$a + b = (a \wedge b') \vee (a' \wedge b)$$

$$a \cdot b = a \wedge b$$

$$-(a) = a$$

$$a \cdot (b + c) = a \cdot b + a \cdot c$$

$$(a + b) + c = a + (b + c)$$

$$\begin{aligned} a \cdot (b + c) &= a \wedge ((b \wedge c') \vee (b' \wedge c)) \\ &= (a \wedge (b \wedge c')) \vee (a \wedge (b' \wedge c)) \end{aligned}$$

$$(B; +, \cdot, -, 0, 1) \xrightarrow{\gamma} (B; \vee, \wedge, (')', 0, 1)$$

$$a \vee b = a + b + a \cdot b$$

$$a \wedge b = a \cdot b$$

$$a' = 1 + a$$

$$(a \vee b) \vee c = a \vee (b \vee c)$$

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$\begin{aligned} a \vee (1 + a) &= a + (1 + a) + a \cdot (1 + a) \\ &= 1 \end{aligned}$$

$$\psi \circ \psi \stackrel{?}{=} id$$

$$\psi \circ \psi \stackrel{?}{=} id$$

$$\rightarrow \psi \psi (B; v, \wedge, ()', 0, 1)$$

$$= (B; v, \wedge, ()', 0, 1)$$

$$\stackrel{?}{=} (B; v, \wedge, ()', 0, 1)$$

$$a \vee b = a + b + a \cdot b$$

$$= a \Delta b \Delta a \cdot b$$

$$= ((a \wedge b') \vee (a' \wedge b)) \Delta (a \wedge b)$$

$$= (* \wedge (a \wedge b')) \vee (*' \wedge (a \wedge b))$$

$$= * \vee (*' \wedge (a \wedge b))$$

$$= (* \vee *) \wedge (* \vee (a \wedge b))$$

$$= * \vee (a \wedge b)$$

$$= (a \wedge b') \vee (a' \wedge b) \vee (a \wedge b)$$

$$= a \vee ((a \wedge b') \vee (a' \wedge b)) \wedge$$

$$b \vee ((a \wedge b') \vee (a' \wedge b))$$

$$= (a \vee (a' \wedge b)) \wedge (b \vee (a \wedge b'))$$

$$= (a \vee a') \wedge (a \vee b) \wedge (b \vee a) \wedge (b \vee b')$$

$$= a \vee b$$

طبیعی
۹، ۹، ۱۴

Def. A subset I of a Bxl. alg. B is called ideal if:

$$1) 0 \in I$$

$$2) a, b \in I \Rightarrow a \vee b \in I$$

$$3) a \in I, x \leq b \Rightarrow ax \in I$$

$$n \in I$$

notation: $I \triangleleft B$

$$3') a \in I, x \leq a \Rightarrow x \in I$$

down closed

(ایمان بسته بودن)

Exercise. $\emptyset \neq I \triangleleft B$

$$\Leftrightarrow \forall a, b \in B (a \in I, b \in I \Leftrightarrow a \vee b \in I)$$

شماره نویسی

Remark. For B a B.A. $\theta \in \text{Con}(B)$. we have

$$(a, b) \in \theta \Leftrightarrow (a \vee b, a \wedge b) \in \theta$$

$$(a \vee b, b) \in \theta$$

$$(a \wedge b, b) \in \theta$$

$$(a \wedge (a \vee b), a \wedge (a \wedge b)) \in \theta$$

$$(a, a \wedge b) \in \theta$$

$$((a \vee b) \wedge b, (a \wedge b) \wedge b) \in \theta$$

$$(b, a \wedge b) \in \theta$$

Notation: $\text{Id}(B)$ the set of all ideals of B .

Th. For B a B.A.

$$\text{Con}(B) \cong \text{Id}(B)$$

↓
bijection

$$\theta \rightsquigarrow [\cdot]_{\theta} \doteq I_{\theta}$$

$$1) 0 \in I_{\theta}$$

$$2) \begin{matrix} a \theta_0 \\ b \theta_0 \end{matrix} \Rightarrow a \vee b \in \theta_0$$

$$3) a \theta_0, x \in B$$

$$x \wedge a \theta \underline{x \wedge_0}$$

$$\theta_I \leftarrow I$$

$$\text{where } x \theta_I y \Leftrightarrow (x \wedge y') \vee (x' \wedge y) \in I$$

برای هر دو نشان دهیم θ_I هم ارزی است. کافیست نشان دهیم θ_I عمل است.

$$x \theta_I y$$

$$x \vee z \theta_I y \vee z$$

$$x \wedge z \theta_I y \wedge z$$

$$x' \theta_I y' \quad \checkmark$$

$$((x \vee z) \wedge (y \vee z')) \vee ((x \vee z)' \wedge (y \vee z)) \stackrel{?}{\in} I$$

$$G_{I_0} \stackrel{?}{=} \theta$$

$$[0]_{\theta_I} = I_{\theta_I} \stackrel{?}{=} I$$

$$x \in [0]_{\theta_I} \Leftrightarrow x \theta_I 0$$

$$\Leftrightarrow x \in I$$

$$x \theta_{I_0} y \Rightarrow x \Delta y \in I_0 = [0]_{\theta}$$

$$\Rightarrow (x \Delta y) \theta 0$$

$$\Rightarrow x \vee (x \Delta y) \theta x$$

$$\Rightarrow x \vee y \theta x$$

$$\Rightarrow x \theta y$$

$$B/\theta \stackrel{?}{=} B/I_0, B/I \stackrel{?}{=} B/\theta_I$$

Def. A groupoid $(A, \cdot)$ such that

$$\forall a, b \in A \exists! c, d \in A$$

$$a \cdot c = b$$

$$d \cdot a = b$$

is called a quasi-group.

So, every element of a quasi-group appears just once in any row or column.

Cayley table of $(A, \cdot)$

$\cdot$	a	b	c
a	a	b	c
b	c	a	b
c	b	c	a

Latine Square

a	b	c
c	a	b
b	c	a

Th. A quasi-group is an algebra $(A; /, \cdot, \backslash)$ of type $(2, 2, 2)$ such that:

$$1) x \backslash (x \cdot y) = y$$

$$2) (x \cdot y) / y = x$$

$$3) x \cdot (x \backslash y) = y$$

$$4) (x / y) \cdot y = x$$

and conversely.

Proof: Take $a \backslash b$ to be the unique c with $a \cdot c = b$;

and a / b to be the unique d with $d \cdot b = a$;

then we get the equations (1) - (4).

$$(3) a \cdot (a \backslash b) = b \quad (\text{by Def})$$

$$(1) a \backslash (a \cdot b) \stackrel{?}{=} b$$

$$a / (a \cdot b) \stackrel{?}{=} b$$

$$\Rightarrow a \cdot (a \setminus a \cdot b) = a \cdot b$$

$$a \cdot b = a \cdot b$$

$$\Rightarrow a \setminus (a \cdot b) = b$$

جبهه سیت و سیم

۹۴، ۹، ۲۳

سینا آقایی خج

Congruence premutab: $\theta_1 \circ \theta_2 = \theta_2 \circ \theta_1$

Congruence distributive: $\theta_1 \wedge (\theta_2 \vee \theta_3)$

$$= (\theta_1 \wedge \theta_2) \vee (\theta_1 \wedge \theta_3)$$

Lemma. $K \models P = Q \Leftrightarrow F_K(X) \models P = Q$

V is variety of algebra with type τ .

$P(x_1, \dots, x_m, f_1, \dots, f_n)$

$Q(x_1, \dots, x_m, y_1, \dots, y_n)$ are terms.

$$\langle P(x_1, \dots, x_m, f_1, \dots, f_n), Q(x_1, \dots, x_m, y_1, \dots, y_n) \rangle \in \theta(y_1, \dots, y_n)$$

$$V \models P(x_1, \dots, x_n, f_1, \dots, f_n) = Q(x_1, \dots, x_n, f_1, \dots, f_n)$$

Proof:

$$\alpha: F_V(x_1, \dots, x_m, y_1, \dots, y_n) \rightarrow F_V(x_1, \dots, x_m, y)$$

$$\alpha(x_i) = x_i$$

$$\alpha(y_i) = y$$

$$\theta(y_1, \dots, y_n) \subseteq \ker \alpha$$

$$\alpha P(x_1, \dots, x_m, y_1, \dots, y_n) = \alpha q(x_1, \dots, x_m, y_1, \dots, y_n)$$

$$\Rightarrow P(\alpha x_1, \dots, \alpha x_m, \alpha y_1, \dots, \alpha y_n) = q(\alpha x_1, \dots, \alpha x_m, \alpha y_1, \dots, \alpha y_n)$$

$$\Rightarrow P(x_1, \dots, x_m, y, \dots, y) = q(x_1, \dots, x_m, y, \dots, y)$$

Theorem. Mal'cev: $\exists P(x, y, z)$ such that

$$V \models P(x, x, y) = y$$

$$V \models P(x, y, y) = x$$

iff V is congruence permutable.

$$P(x, y, z) = xy^{-1}z$$

$$P(x, x, y) = xx^{-1}y = y$$

$$P(x, y, y) = xy^{-1}y = x$$

$(\Rightarrow)$:

$$\langle x, z \rangle \in \theta(x, y) \circ \theta(y, z).$$

$$\Rightarrow \langle x, z \rangle \in \theta(y, z) \circ \theta(x, y)$$

$$\Rightarrow \exists P(x, y, z) \text{ s.t. } x \theta(y, z) P(x, y, z) \theta(x, y) z$$

$$\Rightarrow V \models P(x, y, y) = x$$

$$V \models P(x, x, y) = y$$

($\Leftarrow$):

$$A \in V, \varphi, \psi \in \text{Con } A$$

$$\langle a, b \rangle \in \varphi \circ \psi$$

$$\Rightarrow \exists c \quad a \varphi c \psi b$$

$$\Rightarrow b = p(c, c, b) \varphi p(a, c, b) \psi p(a, b, b) = a$$

$$\langle b, a \rangle \in \varphi \circ \psi \Rightarrow \langle a, b \rangle \in \psi \circ \varphi$$

Theorem. V is distributive iff $\exists M(x, y, z)$ such that:

$$V \models M(x, x, y) = M(x, y, x) = M(y, x, x) = x$$

Lattices are distributive because:

$$M(x, y, z) = (x \vee y) \wedge (x \vee z) \wedge (y \vee z)$$

Theorem (Pixley): $V \models m(x, y, x) = m(x, y, y)$
 $= m(y, y, x) = x$

Page 85-90.

پیمت

دسته جبری انود:

بر دسته جبری، خارج قسمت یک دسته جبری آزاد است.

تعریف: مجموعه $(A; *)$ باید $X \subseteq A$ آزاد است اگر:

۱) هر عضو $a \in A$ را بتوان به صورت $a = x_1 * \dots * x_n$ نوشت،
که در آن هر x_i در X است (یعنی، X دسته جبری A را تولید
می کند).

۲) این عبارت برای هر $a \in A$ یکتا باشد، یعنی، اگر $x_i, y_j \in X$ برای
 $1 \leq i \leq n, 1 \leq j \leq m$ ، آنگاه

$$a = x_1 * \dots * x_n = y_1 * \dots * y_m \Leftrightarrow m = n$$

$$\text{و } \forall i, x_i = y_i$$

* هیچ مجموعه متناهی $(A; *)$ نمی تواند پایه ای بگردد، زیرا متناهی بودن
اجاب می کند که برای هر $a \in A$ ، تعداد عضوی زیر متناهی باشد.

$$a, a * a, a * a * a, \dots$$

پس مثلاً $a * a * a * a = a * a$ که یکتایی تعریف را نقض می کند.

* برای هر $X \neq \emptyset$ ، مجموعه آزاد X^* نامتناهی است.

* مجموعه ایی آزاد: اصل بنده لا تعریف:

$$x_1 * x_2 * \dots * x_m = y_1 * y_2 * \dots * y_n \Leftrightarrow m = n \text{ و}$$

SHARGH

$$\{x_1, x_2, \dots, x_m\}$$

$$= \{y_1, y_2, \dots, y_n\}$$

تعریف: شماره آزاد: اصلاح بند ۲ تعریف: عبارت $a = x_1 * \dots * x_n$ باشد
حذف e های اضافی منحصر به فرد باشد یعنی

$$e * e * x * x * y * e * x = x * x * y * x$$

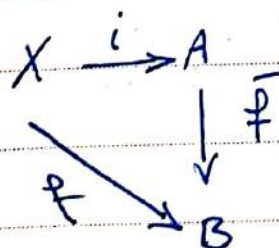
تعریف: گروه $(A; *, ^{-1}, e)$ بایایی X گروهی آزاد است اگر
۱) هر عضو $a \in A$ را بتوان به صورت $a = x_1 * \dots * x_n$ نوشت که
در آن هر x_i یا وارون آن در X است و
۲) این عبارت برای $a \neq e$ ، با شرط حذف e های اضافی و حذف عبارت
به صورت $x * x^{-1}$ و $x^{-1} * x$ منحصر به فرد باشد

تعریف: زیر مجموعه نهایی C از مجموعه آزاد X^* را یک کد، یا فرم روی X می نامیم، اگر شرایط
یکسانی برای هر $c_1, \dots, c_m, d_1, \dots, d_n \in C$
 $c_1 c_2 \dots c_m = d_1 d_2 \dots d_n \iff m = n \text{ و } \forall i, c_i = d_i$

* اگر مجموعه A روی هر دو زیر مجموعه X و Y آزاد باشد، آنگاه X و Y متمم هستند

* فرض کنید $(A; *)$ مجموعه آزاد بایایی $X \subseteq A$ است. نشان دهید برای هر مجموعه
 B و پراچ $X \xrightarrow{\bar{f}} B$ ، همپیکش منحصر به فرد $A \xrightarrow{\bar{f}} B$ وجود دارد طوری که
 $\bar{f}(x) = \bar{f}(x)$ ، $x \in X$ یعنی برای هر $x \in X$ ، $\bar{f}|_X = \bar{f}$

به عبارتش، نمودار (مثلث) زیر نقیض پذیر است.
این خاص شکلی است.



ولایت قضیه پرچون

نمونه کنیم $K \subseteq Alg(\mathcal{C})$ دسته‌ای از دستگاه‌های جبری از نوع $\mathcal{C}$ باشد.

I: دسته‌ی K نسبت به گیرگشت بسته باشد. یعنی هر جبری از نوع $\mathcal{C}$ که با عضوی از K گیرگشت است، متعلق به K باشد.

$$\{A \in Alg(\mathcal{C}) \mid (\exists B \in K) A \approx B\} = K$$

S: دسته‌ی K نسبت به زیر جبر بسته باشد. یعنی هر جبری از نوع $\mathcal{C}$ که زیر جبر عضوی از K است، متعلق به K باشد.

$$\{A \in Alg(\mathcal{C}) \mid (\exists B \in K) A \leq B\} = K$$

P_r: دسته‌ی K نسبت به ضرب بسته باشد. یعنی هر جبری از نوع $\mathcal{C}$ که حاصل ضرب اعضای از K است، متعلق به K باشد.

$$\{A \in Alg(\mathcal{C}) \mid (\exists \{B_i\}_{i \in I} \subseteq K) A \subseteq \prod_{i \in I} B_i\} = K$$

H: دسته‌ی K نسبت به خارج قسمت بسته باشد. یعنی هر جبری از نوع $\mathcal{C}$ که خارج قسمت عضوی از K است، متعلق به K باشد.

$$\{A \in Alg(\mathcal{C}) \mid (\exists B \in K) A \equiv B / \sim\} = K$$

(بسته بودن نسبت به نگاه‌های هم‌بخش)

تعریف: فرض کنیم $K \subseteq \text{Alg}(C)$ دسته‌ای از دستگاه‌های جبری از نوع C باشد.
 ۱) دسته‌ای K را وارثه می‌نامیم اگر K دارای ویژگی‌های H, P, S, I باشد.

۲) دسته‌ای K را معادله‌ای (یا اتحادی) می‌نامیم اگر مجموعه‌ای E از معادله‌ها وجود داشته باشد به طوری که K تشکیل از همه جبرهای از نوع C باشد
 طوری که بر معادله متعلق به E در هر عضو K اکتفا باشد. یعنی

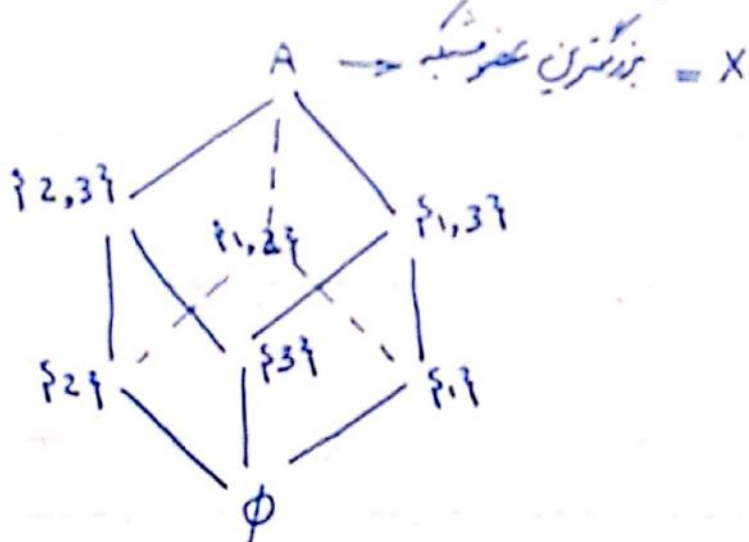
$$\{A \in \text{Alg}(C) \mid A \models E\} = K$$

نقشه: دسته‌ای $K \subseteq \text{Alg}(C)$ وارثه است اگر و تنها اگر دسته‌ای معادله‌ای باشد.

شکلی که از شبکه:

$$A = \{1, 2, 3\}$$

$(T(X), \subseteq)$ شبکه است.



$$S, T \in P(A) : S = \{1, 2\}, T = \{1, 3\}$$

$$S \cap T = \{1\}, S \cup T = \{1, 2, 3\} = A$$

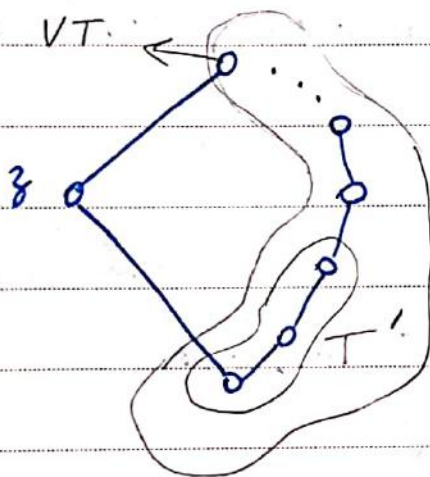
$(P(X), \subseteq)$ شبکه‌ای کامل است زیرا برای هر خانواده از $P(X)$

in f وجود است (خاصیت اشتراک بگیریم)

• $(P(X), \subseteq)$ شبکه‌ای حیرتی است. زیرا اولاً کامل است و دوماً هر عضو آن سوپریم خانواده‌ای از اعضا می‌باشد.

* مثال برای شبکه کامل که حیرتی نیست: k را بازه $[0, 1]$ روی اعداد حقیقی با ترتیب معمولی در نظر بگیریم. k^c را بگیریم پس k حیرتی نیست.

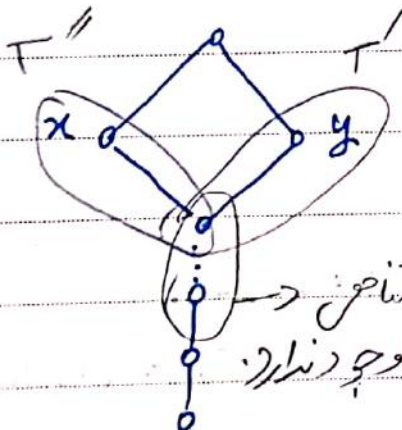
* مثال برای شبکه غیر حیرتی:



• شبکه نیست. بنا بر این سوپریم اعضا می‌باشد.

$$T' \subseteq T, \text{ و } z \leq VT'$$

در نتیجه لزوماً یک تقاطع می‌باشد.



* مثال برای شبکه نبودن:

x و y شبکه نیستند اما $x \vee y$ شبکه نیست.

بِسْمِ اللَّهِ الرَّحْمَنِ الرَّحِيمِ

اللَّهُمَّ صَلِّ عَلَى مُحَمَّدٍ

اللَّهُمَّ كُنْ لَوْلِيِّكَ الْحَبِيبِ بْنِ

الْحَسَنِ صَلِّ لَوْلَا تَكْ عَلَيْهِ وَعَلَى آبَائِهِ

فِي هَذِهِ السَّاعَةِ وَفِي كُلِّ سَاعَةٍ

وَلِيًّا وَحَافِظًا وَقَائِدًا وَنَاصِرًا وَدَلِيلًا

وَعَيْنًا حَتَّى تُسْكِنَهُ أَرْضَكَ طَوْعًا

وَتُمَتِّعَهُ فِيهَا طَوِيلًا وَهَبْ لَنَا

رَأْفَتَهُ وَرَحْمَتَهُ وَدُعَاءَهُ وَخَيْرَهُ

بِرَحْمَتِكَ يَا أَرْحَمَ الرَّاحِمِينَ

الْمَشْرُوعِي

اللَّهُمَّ عَجِّلْ لِي لِقَاءَ الْفَرَجِ