بسم الله الرحمن الرحیم

دانشگاه صنعتی خواجه نصیرالدین طوسی

محاسبات عددی

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1 Solutions of Equations in One Variable

In this chapter we consider

one of the most basic problems

of numerical approximation

the **root-finding problem**

This process involves

finding a **root**,

or solution,

of an equation

of the form $f(x) = 0$,

1 The Bisection Method

2 Fixed-Point Iteration

3 Newton's Method

The Secant Method

The Method of False Position

Newton's Method (or the *Newton-Raphson*) is based on $\boxed{\text{Taylor polynomials}}$

$$f(p) = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)),$$

Since $f(p) = 0$, this equation gives

$$0 = f(p_0) + (p - p_0)f'(p_0) + \frac{(p - p_0)^2}{2} f''(\xi(p)).$$

Newton's method is derived by assuming that since $|p - p_0|$ is small, the term involving $(p - p_0)^2$ is much smaller, so

$$0 \approx f(p_0) + (p - p_0)f'(p_0).$$

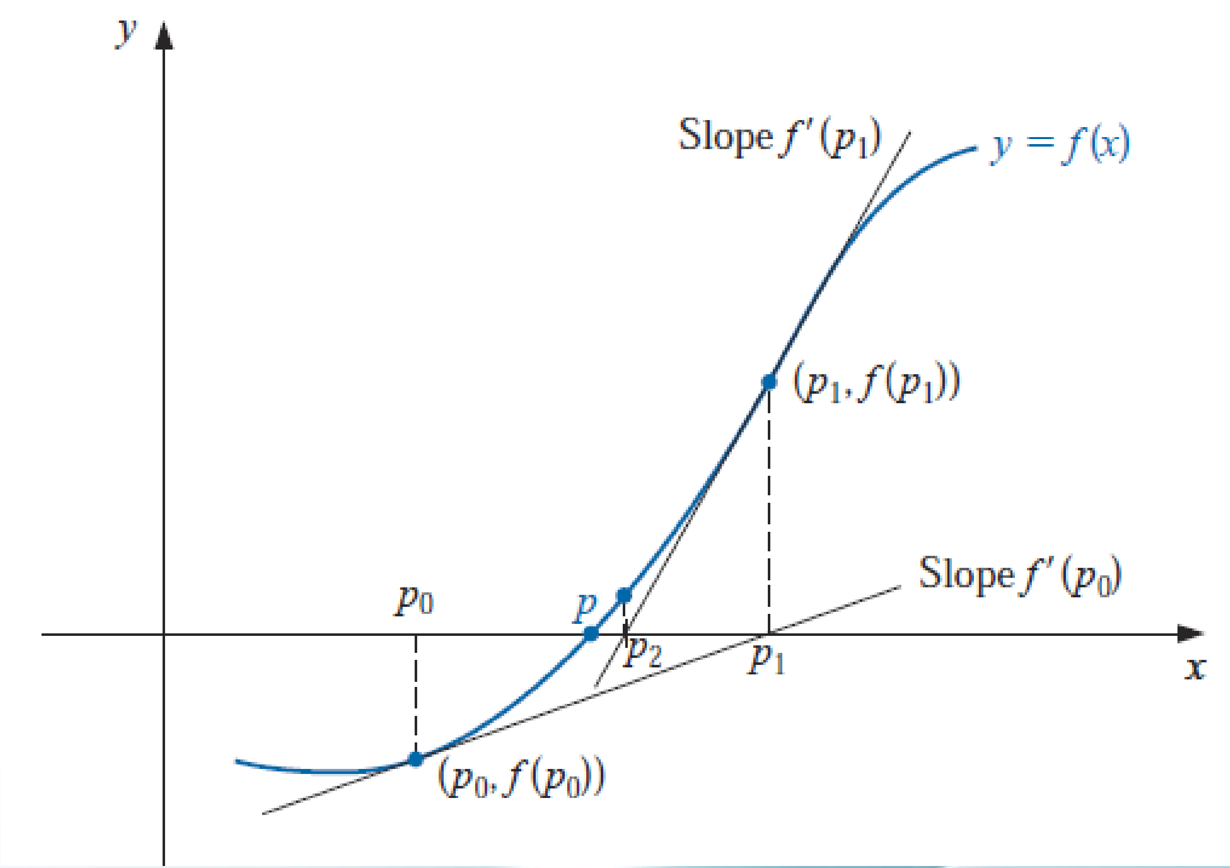
Solving for p gives

$$p \approx p_0 - \frac{f(p_0)}{f'(p_0)}$$

This sets the stage for Newton's method, which starts with an initial approximation p_0 and generates the sequence $\{p_n\}_{n=0}^{\infty}$, by

$$\boxed{p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1.}$$

Newton's Method



$$\boxed{p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1.}$$

Example 1

Consider the function $f(x) = \cos x - x = 0$. Approximate a root of f using (a) a fixed-point method, and (b) Newton's Method

Solution (a) $x = \cos x$, $[0, \pi/2]$.

Table 2.3 shows the results of fixed-point iteration with $p_0 = \pi/4$

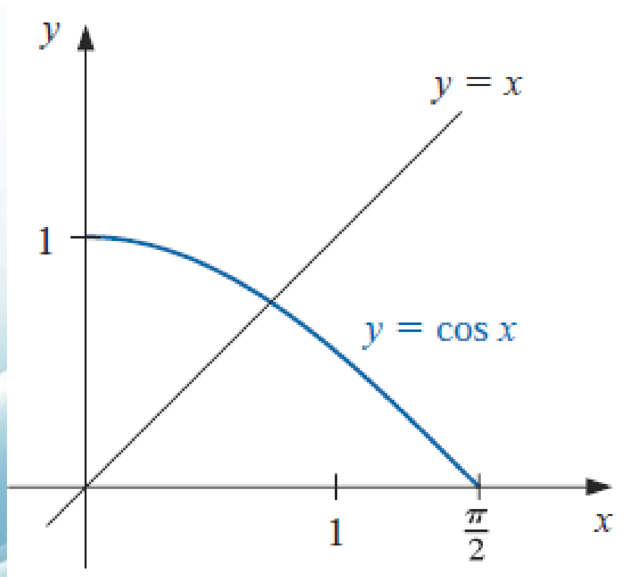


Table 2.3

n	p_n
0	0.7853981635
1	0.7071067810
2	0.7602445972
3	0.7246674808
4	0.7487198858
5	0.7325608446
6	0.7434642113
7	0.7361282565

(b) we need $f'(x) = -\sin x - 1$.

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f(p'_{n-1})}$$

$$= p_{n-1} - \frac{\cos p_{n-1} - p_{n-1}}{-\sin p_{n-1} - 1}$$

Table 2.4

n	p_n
0	0.7853981635
1	0.7395361337
2	0.7390851781
3	0.7390851332
4	0.7390851332

The Secant Method

$$p_n = p_{n-1} - \frac{f(p_{n-1})}{f'(p_{n-1})}, \quad \text{for } n \geq 1.$$

Newton's method is an extremely powerful technique

but it has a major weakness

the need to know the value of the derivative of f

at each approximation.

To circumvent the problem

we introduce a slight variation

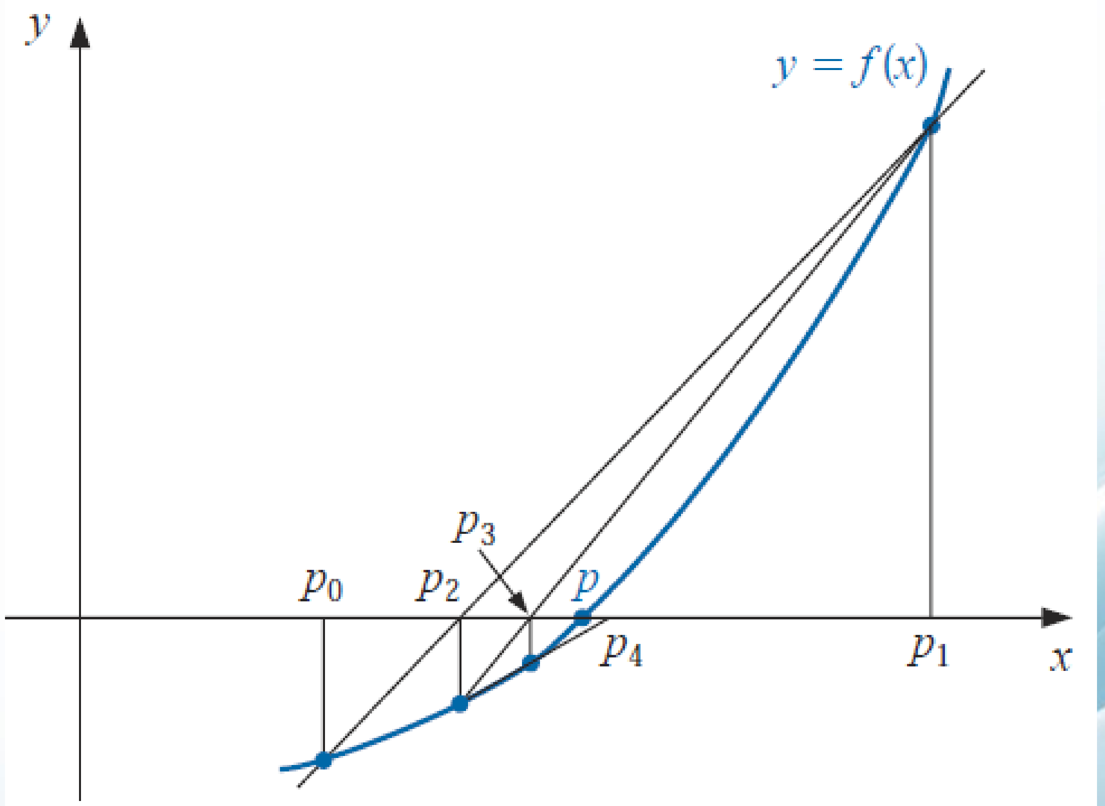
$$f'(p_{n-1}) = \lim_{x \to p_{n-1}} \frac{f(x) - f(p_{n-1})}{x - p_{n-1}}$$

If p_{n-2} is close to p_{n-1}, then

$$f'(p_{n-1}) \approx \frac{f(p_{n-2}) - f(p_{n-1})}{p_{n-2} - p_{n-1}} = \frac{f(p_{n-1}) - f(p_{n-2})}{p_{n-1} - p_{n-2}}.$$

$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$

The Secant Method



$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$

The Secant Method

Example 2

Use the Secant method to find a solution to $x = \cos x$, and compare the approximations with those given in Example 1 which applied Newton's method.

Solution For the Secant method we need two initial approximations

Suppose we use $p_0 = 0.5$ and $p_1 = \pi/4$

$$\boxed{p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.}$$

$$p_n = p_{n-1} - \frac{(p_{n-1} - p_{n-2})(\cos p_{n-1} - p_{n-1})}{(\cos p_{n-1} - p_{n-1}) - (\cos p_{n-2} - p_{n-2})}, \quad \text{for } n \geq 2$$

Table 2.5

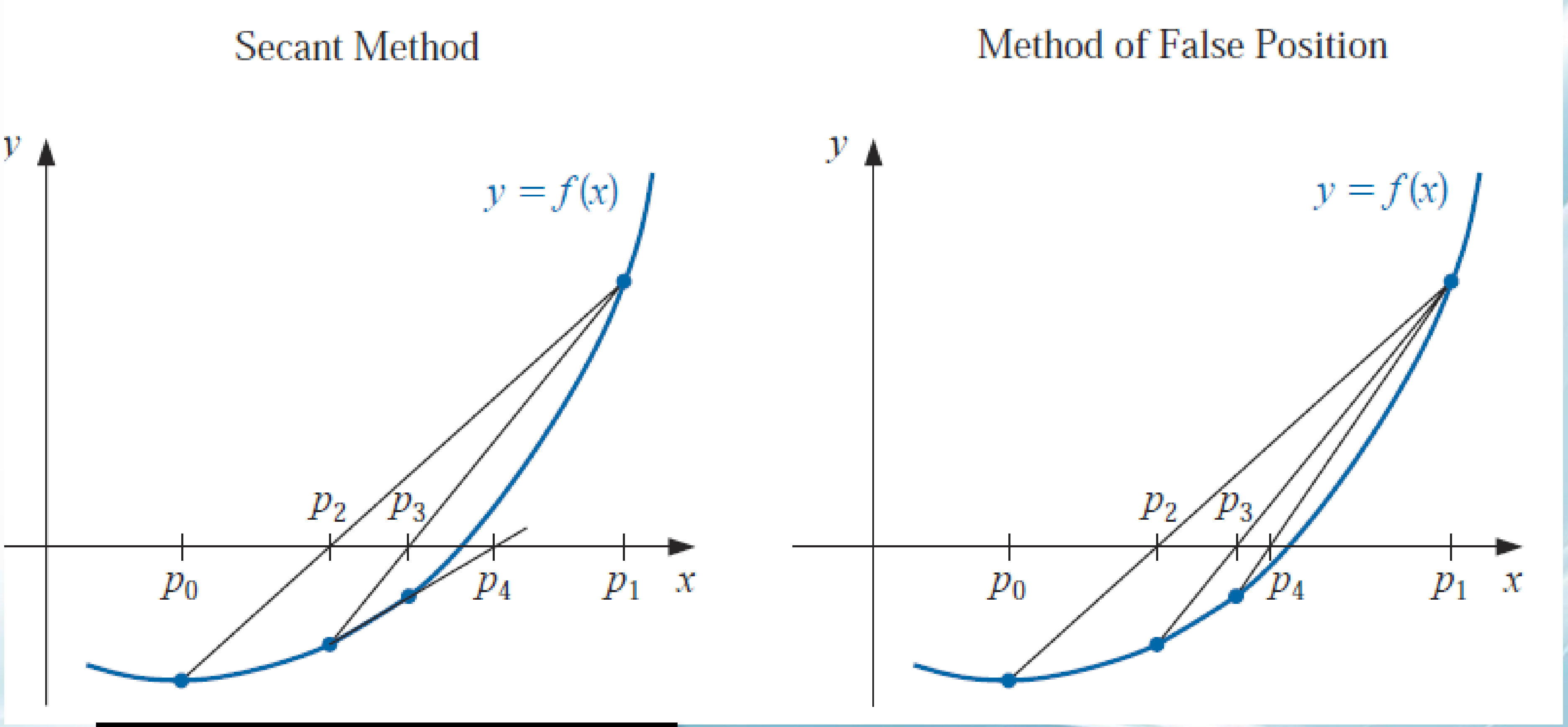
n	Secant p_n
0	0.5
1	0.7853981635
2	0.7363841388
3	0.7390581392
4	0.7390851493
5	0.7390851332

the convergence of the Secant method is much faster than functional iteration but slightly slower than Newton's method.

This is generally the case.

The Method of False Position

generates approximations in the same manner as the Secant method



$$p_n = p_{n-1} - \frac{f(p_{n-1})(p_{n-1} - p_{n-2})}{f(p_{n-1}) - f(p_{n-2})}.$$

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فصل۲)
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Example 3

Use the method of False Position to find a solution to $x = \cos x$, and compare the approximations with those given in Example 1 which applied fixed-point iteration and Newton's method, and to those found in Example 2 which applied the Secant method.

	False Position	Secant	Newton
n	p_n	p_n	p_n
0	0.5	0.5	0.7853981635
1	0.7853981635	0.7853981635	0.7395361337
2	0.7363841388	0.7363841388	0.7390851781
3	0.7390581392	0.7390581392	0.7390851332
4	0.7390848638	0.7390851493	0.7390851332
5	0.7390851305	0.7390851332	
6	0.7390851332		

Newton's Method

The Secant Method

The Method of False Position

EXERCISE SET 3

Use Newton's method to find solutions accurate to within 10^{-5} for the following problems.

a. $e^x + 2^{-x} + 2\cos x - 6 = 0$ for $1 \leq x \leq 2$

b. $\ln(x-1) + \cos(x-1) = 0$ for $1.3 \leq x \leq 2$

Use all three methods in this Section to find solutions to within 10^{-5} for the following problems.

Method of False Position

Secant method

Newton's method

a. $3xe^x = 0$ for $1 \leq x \leq 2$

b. $2x + 3\cos x - e^x = 0$ for $0 \leq x \leq 1$

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۱۴ فروردین ۹۹

با تشکر از توجه شما