2 SIGNALS AND SIGNAL SPACE

In this chapter we discuss certain basic signal concepts. Signals are processed by systems. We shall start with explaining the terms *signals* and *systems*.

Signals

A signal, as the term implies, is a set of information or data. Examples include a telephone or a television signal, the monthly sales figures of a corporation, and closing stock prices (e.g., in the United States, the Dow Jones averages). In all these examples, the signals are functions of the independent variable *time*. This is not always the case, however. When an electrical charge is distributed over a surface, for instance, the signal is the charge density, a function of *space* rather than time. In this book we deal almost exclusively with signals that are functions of time. The discussion, however, applies equally well to other independent variables.

Systems

Signals may be processed further by **systems**, which may modify them or extract additional information from them. For example, an antiaircraft missile launcher may want to know the future location of a hostile moving target, which is being tracked by radar. Since the radar signal gives the past location and velocity of the target, by properly processing the radar signal (the input), one can approximately estimate the future location of the target. Thus, a system is an entity that *processes* a set of signals (**inputs**) to yield another set of signals (**outputs**). A system may be made up of physical components, as in electrical, mechanical, or hydraulic systems (hardware realization), or it may be an algorithm that computes an output from an input signal (software realization).

2.1 SIZE OF A SIGNAL

Signal Energy

The size of any entity is a quantity that indicates its strength. Generally speaking, a signal varies with time. To set a standard quantity that measures signal strength, we normally view a signal $g(t)$ as a voltage across a one-ohm resistor. We define **signal energy** E_g of the signal $g(t)$ as the energy that the voltage $g(t)$ dissipates on the resistor. More formally, we define E_g

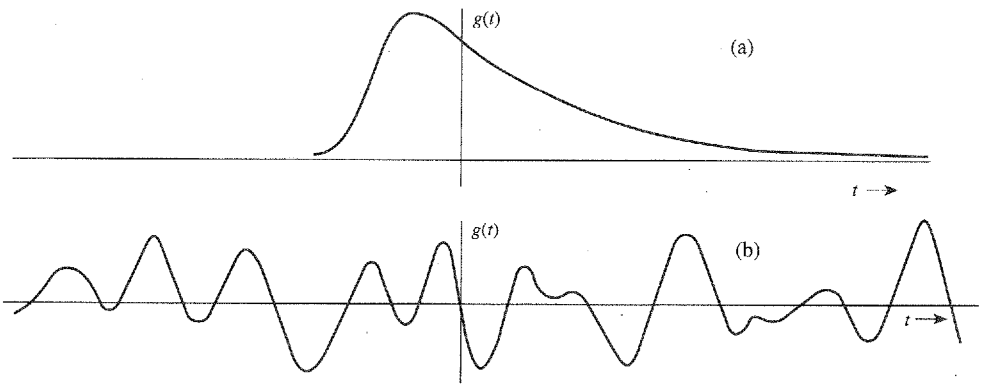


Figure 2.1 Examples of signals. (a) Signal with finite energy. (b) Signal with finite power.

(for a real signal) as

$$E_g = \int_{-\infty}^{\infty} g^2(t)\, dt \tag{2.1}$$

This definition can be generalized to a complex-valued signal $g(t)$ as

$$E_g = \int_{-\infty}^{\infty} |g(t)|^2\, dt \tag{2.2}$$

Signal Power

To be a meaningful measure of signal size, the signal energy must be finite. A necessary condition for energy to be finite is that the signal amplitude goes to zero as $|t|$ approaches infinity (Fig. 2.1a). Otherwise the integral in Eq. (2.1) will not converge.

If the amplitude of $g(t)$ does not go to zero as $|t|$ approaches infinity (Fig. 2.1b), the signal energy is infinite. A more meaningful measure of the signal size in such a case would be the time average of the energy (if it exists), which is the average power P_g defined (for a real signal) by

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} g^2(t)\, dt \tag{2.3}$$

We can generalize this definition for a complex signal $g(t)$ as

$$P_g = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2\, dt \tag{2.4}$$

Observe that the signal power P_g is the time average (mean) of the signal amplitude square, that is, the **mean square** value of $g(t)$. Indeed, the square root of P_g is the familiar **rms** (root mean square) value of $g(t)$.

The mean of an entity averaged over a large time interval approaching infinity exists if the entity either is periodic or has a statistical regularity. If such a condition is not satisfied, an average may not exist. For instance, a ramp signal $g(t) = t$ increases indefinitely as $|t| \to \infty$, and neither the energy, nor the power exists for this signal.

Units of Signal Energy and Power

The standard units of signal energy and power are the joule and the watt. However, in practice, it is often customary to use logarithmic scales to describe signal power. This notation saves the trouble of dealing with many decimal places when signal power is large or small. As a convention, a signal with average power of P watts can be said to have power of

$$[10 \cdot \log_{10} P]\,\text{dBw} \qquad \text{or} \qquad [30 + 10 \cdot \log_{10} P]\,\text{dBm}$$

For example, -30 dBm represents signal power of 10^{-6} W in normal decimal scale.

Example 2.1 Determine the suitable measures of the signals in Fig. 2.2.

The signal in Fig. 2.2a approaches 0 as $|t| \to \infty$. Therefore, the suitable measure for this signal is its energy E_g, given by

$$E_g = \int_{-\infty}^{\infty} g^2(t)\,dt = \int_{-1}^{0} (2)^2\,dt + \int_{0}^{\infty} 4e^{-t}\,dt = 4 + 4 = 8$$

The signal in Fig. 2.2b does not approach 0 as $|t| \to \infty$. However, it is periodic, and therefore its power exists. We can use Eq. (2.3) to determine its power. For periodic signals, we can simplify the procedure by observing that a periodic signal repeats regularly each period (2 seconds in this case). Therefore, averaging $g^2(t)$ over an infinitely large interval is equivalent to averaging it over one period (2 seconds in this case). Thus

$$P_g = \frac{1}{2}\int_{-1}^{1} g^2(t)\,dt = \frac{1}{2}\int_{-1}^{1} t^2\,dt = \frac{1}{3}$$

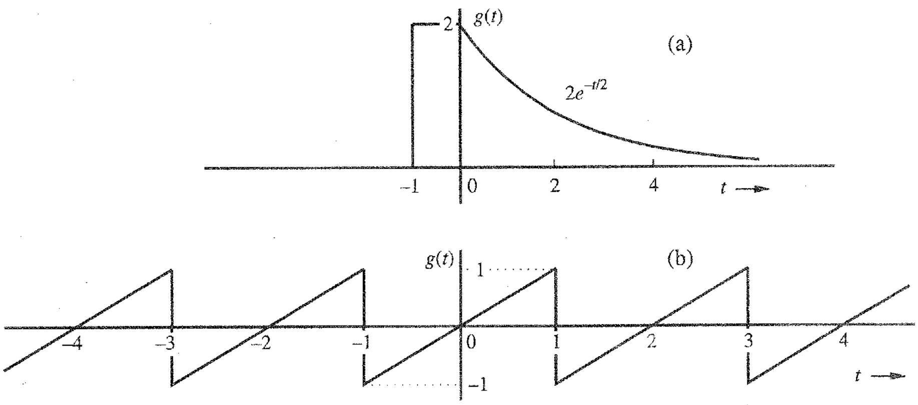


Figure 2.2 Signal for Example 2.1.

Recall that the signal power is the square of its rms value. Therefore, the rms value of this signal is $1/\sqrt{3}$.

2.2 CLASSIFICATION OF SIGNALS

There are various classes of signals. Here we shall consider only the following pairs of classes, which are suitable for the scope of this book.

1. Continuous time and discrete time signals
2. Analog and digital signals

3. Periodic and aperiodic signals
4. Energy and power signals
5. Deterministic and probabilistic signals

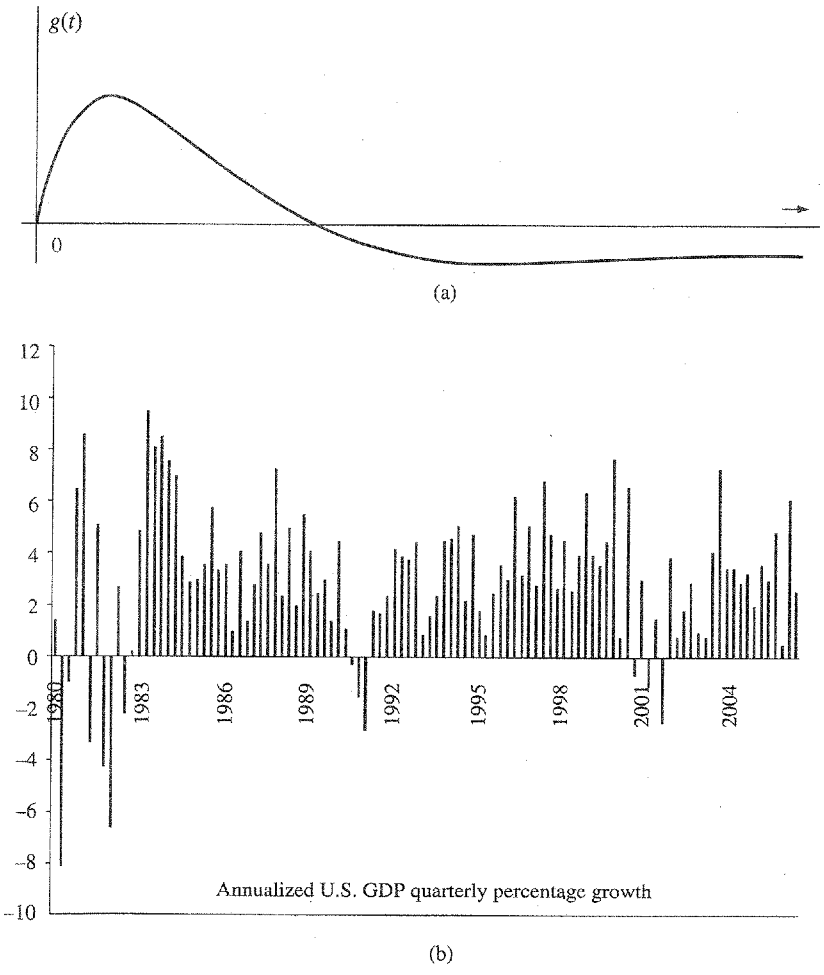
2.2.1 Continuous Time and Discrete Time Signals

A signal that is specified for every value of time t (Fig. 2.3a) is a **continuous time signal**, and a signal that is specified only at discrete points of $t = nT$ (Fig. 2.3b) is a **discrete time signal**. Audio and video recordings are continuous time signals, whereas the quarterly gross domestic product (GDP), monthly sales of a corporation, and stock market daily averages are discrete time signals.

2.2.2 Analog and Digital Signals

One should not confuse analog signals with continuous time signals. The two concepts are not the same. This is also true of the concepts of discrete time and digital. A signal whose amplitude can take on any value in a continuous range is an **analog signal**. This means that

Figure 2.3
(a) Continuous time signal.
(b) Discrete time signals.



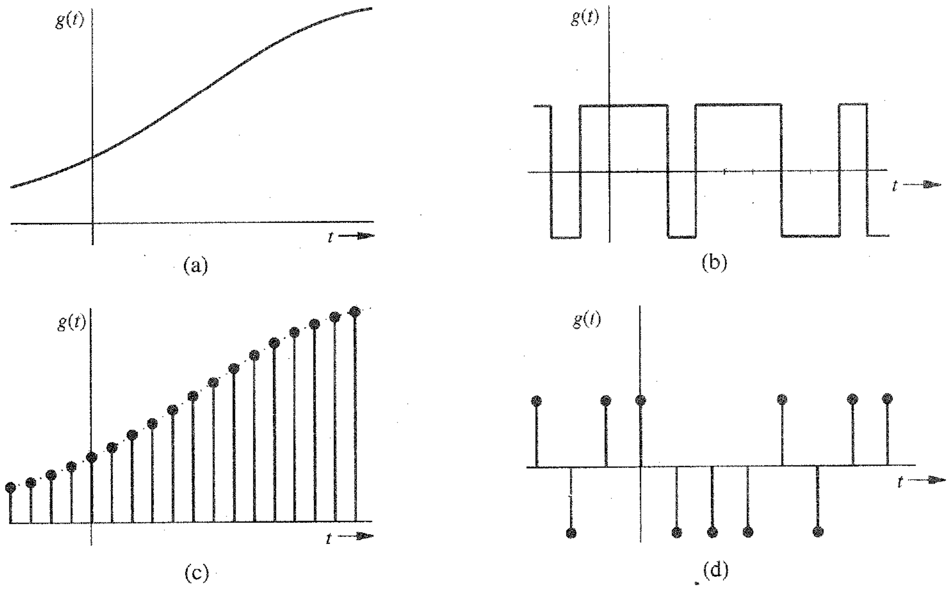


Figure 2.4 Examples of signals: (a) analog and continuous time, (b) digital and continuous time, (c) analog and discrete time, (d) digital and discrete time.

an analog signal amplitude can take on an (uncountably) infinite number of values. A **digital signal**, on the other hand, is one whose amplitude can take on only a finite number of values. Signals associated with a digital computer are digital because they take on only two values (binary signals). For a signal to qualify as digital, the number of values need not be restricted to two. It can be any finite number. A digital signal whose amplitudes can take on M values is an M-**ary** signal of which binary ($M = 2$) is a special case. The terms "continuous time" and "discrete time" qualify the nature of signal along the time (horizontal) axis. The terms "analog" and "digital," on the other hand, describe the nature of the signal amplitude (vertical) axis. Figure 2.4 shows examples of signals of various types. It is clear that analog is not necessarily continuous time, whereas digital need not be discrete time. Figure 2.4c shows an example of an analog but discrete time signal. An analog signal can be converted into a digital signal (via analog-to-digital, or A/D, conversion) through quantization (rounding off), as explained in Chapter 6.

2.2.3 Periodic and Aperiodic Signals

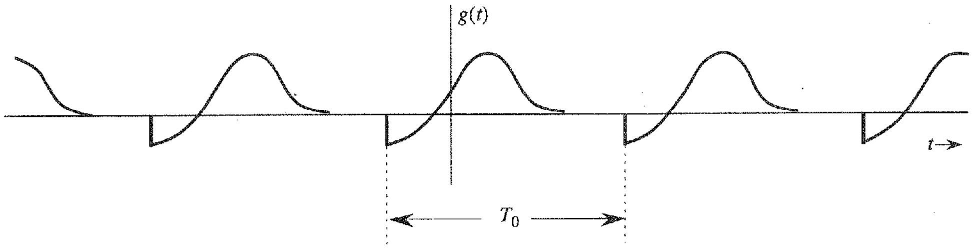
A signal $g(t)$ is said to be **periodic** if there exists a positive constant T_0 such that

$$g(t) = g(t + T_0) \quad \text{for all } t \tag{2.5}$$

The **smallest** value of T_0 that satisfies the periodicity condition of Eq. (2.5) is the **period** of $g(t)$. The signal in Fig. 2.2b is a periodic signal with period of 2. Naturally, a signal is **aperiodic** if it is not periodic. The signal in Fig. 2.2a is aperiodic.

By definition, a periodic signal $g(t)$ remains unchanged when time-shifted by one period. This means that a periodic signal must start at $t = -\infty$ because if it starts at some finite instant, say, $t = 0$, the time-shifted signal $g(t + T_0)$ will start at $t = -T_0$ and $g(t + T_0)$ would not be the same as $g(t)$. Therefore, a **periodic signal, by definition, must start from** $-\infty$ **and continue forever**, as shown in Fig. 2.5. Observe that a periodic signal shifted by an integral multiple of T_0 remains unchanged. Therefore, $g(t)$ may be considered to be a periodic signal

Figure 2.5 A periodic signal of period T_0.



with period mT_0, where m is any integer. However, by definition, the period is the smallest interval that satisfies periodicity condition of Eq. (2.5). Therefore, T_0 is the period.

2.2.4 Energy and Power Signals

A signal with finite energy is an **energy signal**, and a signal with finite power is a **power signal**. In other words, a signal $g(t)$ is an energy signal if

$$\int_{-\infty}^{\infty} |g(t)|^2 \, dt < \infty \tag{2.6}$$

Similarly, a signal with a finite and nonzero power (mean square value) is a power signal. In other words, a signal is a power signal if

$$0 < \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |g(t)|^2 \, dt < \infty \tag{2.7}$$

The signals in Fig. 2.2a and 2.2b are examples of energy and power signals, respectively. Observe that power is time average of the energy. Since the averaging is over an infinitely large interval, a signal with finite energy has zero power, and a signal with finite power has infinite energy. Therefore, a signal cannot be both an energy and a power signal. If it is one, it cannot be the other. On the other hand, some signals with infinite power are neither energy nor power signals. The ramp signal is one example.

Comments

Every signal observed in real life is an energy signal. A power signal, on the other hand, must have an infinite duration. Otherwise its power, which is its average energy (averaged over infinitely large interval) will not approach a (nonzero) limit. Obviously it is impossible to generate a true power signal in practice because such a signal would have infinite duration and infinite energy.

Also, because of periodic repetition, periodic signals for which the area under $|g(t)|^2$ over one period is finite are power signals; however, not all power signals are periodic.

2.2.5 Deterministic and Random Signals

A signal whose physical description is known completely, either in a mathematical form or a graphical form is a **deterministic signal**. A signal that is known only in terms of probabilistic description, such as mean value, mean square value, and distributions, rather than its full mathematical or graphical description is a **random signal**. Most of the noise signals encountered in practice are random signals. All message signals are random signals because, as will be shown

(a) (b)

Figure 2.6 A unit impulse and its approximation.

later, a signal, to convey information, must have some uncertainty (randomness) about it. The treatment of random signals will be discussed in later chapters.

2.3 UNIT IMPULSE SIGNAL

The unit impulse function $\delta(t)$ is one of the most important functions in the study of signals and systems. Its definition and application provide much convenience that is not permissible in pure mathematics.

The unit impulse function $\delta(t)$ was first defined by P. A. M. Dirac (hence often known as the "Dirac delta") as

$$\delta(t) = 0, \quad t \neq 0 \tag{2.8}$$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \tag{2.9}$$

We can visualize an impulse as a tall, narrow rectangular pulse of unit area, as shown in Fig. 2.6. The width of this rectangular pulse is a very small value ϵ; its height is a very large value $1/\epsilon$ in the limit as $\epsilon \to 0$. The unit impulse therefore can be regarded as a rectangular pulse with a width that has become infinitesimally small, a height that has become infinitely large, and an overall area that remains constant at unity.* Thus, $\delta(t) = 0$ everywhere except at $t = 0$, where it is, strictly speaking, undefined. For this reason, a unit impulse is graphically represented by the spearlike symbol in Fig. 2.6a.

Multiplication of a Function by an Impulse

Let us now consider what happens when we multiply the unit impulse $\delta(t)$ by a function $\phi(t)$ that is known to be continuous at $t = 0$. Since the impulse exists only at $t = 0$, and the value of $\phi(t)$ at $t = 0$ is $\phi(0)$, we obtain

$$\phi(t)\delta(t) = \phi(0)\delta(t) \tag{2.10a}$$

Similarly, if $\phi(t)$ is multiplied by an impulse $\delta(t - T)$ (an impulse located at $t = T$), then

$$\phi(t)\delta(t - T) = \phi(T)\delta(t - T) \tag{2.10b}$$

provided $\phi(t)$ is defined at $t = T$.

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* The impulse function can also be approximated by other pulses, such as a positive triangle, an exponential pulse, or a Gaussian pulse.

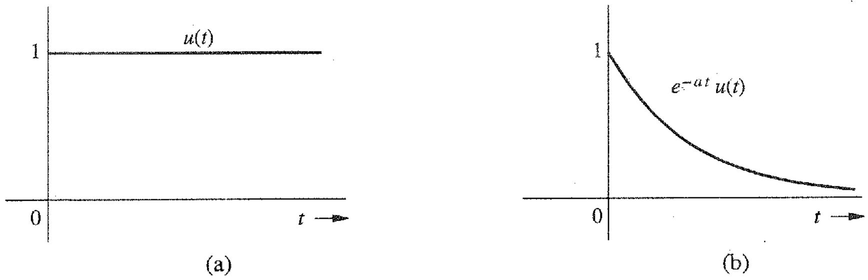


Figure 2.7 (a) Unit step function $u(t)$. (b) Causal exponential $e^{-at}u(t)$.

The Sampling Property of the Unit Impulse Function

From Eq. (2.10) it follows that

$$\int_{-\infty}^{\infty} \phi(t)\delta(t-T)\,dt = \phi(T)\int_{-\infty}^{\infty} \delta(t-T)\,dt = \phi(T) \tag{2.11a}$$

provided $\phi(t)$ is continuous at $t = T$. This result means that *the area under the product of a function with an impulse $\delta(t)$ is equal to the value of that function at the instant where the unit impulse is located.* This very important and useful property is known as the **sampling** (or **sifting**) **property** of the unit impulse.

Depending on the value of T and the integration limit, the impulse function may or may not be within the integration limit. Thus, it follows that

$$\int_{a}^{b} \phi(t)\delta(t-T)\,dt = \phi(T)\int_{a}^{b} \delta(t-T)\,dt = \begin{cases} \phi(T) & a \le T < b \\ 0 & T < a \le b, \text{ or } T \ge b > a \end{cases} \tag{2.11b}$$

The Unit Step Function $u(t)$

Another familiar and useful function is the **unit step function** $u(t)$, often encountered in circuit analysis and defined by Fig. 2.7a:

$$u(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases} \tag{2.12}$$

If we want a signal to start at $t = 0$ (so that it has a value of zero for $t < 0$), we need only multiply the signal by $u(t)$. A signal that starts after $t = 0$ is called a **causal signal**. In other words, $g(t)$ is a causal signal if

$$g(t) = 0 \qquad t < 0$$

The signal e^{-at} represents an exponential that starts at $t = -\infty$. If we want this signal to start at $t = 0$ (the causal form), it can be described as $e^{-at}u(t)$ (Fig. 2.7b). From Fig. 2.6b, we observe that the area from $-\infty$ to t under the limiting form of $\delta(t)$ is zero if $t < 0$ and unity if $t \ge 0$. Consequently,

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = \begin{cases} 0, & t < 0 \\ 1, & t \ge 0 \end{cases}$$
$$= u(t) \tag{2.13a}$$

From this result it follows that

$$\frac{du}{dt} = \delta(t) \tag{2.13b}$$

2.4 SIGNALS VERSUS VECTORS

There is a strong connection between signals and vectors. Signals that are defined for only a finite number of time instants (say N) can be written as vectors (of dimension N). Thus, consider a signal $g(t)$ defined over a closed time interval $[a, b]$. Let we pick N points uniformly on the time interval $[a, b]$ such that

$$t_1 = a, \quad t_2 = a + \epsilon, \quad t_3 = a + 2\epsilon, \quad t_N = a + (N-1)\epsilon = b, \qquad \epsilon = \frac{b-a}{N-1}$$

Then we can write a signal vector $\mathbf{g}$ as an N-dimensional vector

$$\mathbf{g} = \begin{bmatrix} g(t_1) & g(t_2) & \cdots & g(t_N) \end{bmatrix}$$

As the number of time instants N increases, the sampled signal vector $\mathbf{g}$ will grow. Eventually, as $N \to \infty$, the signal values will form a vector $\mathbf{g}$ of infinitely long dimension. Because $\epsilon \to 0$, the signal vector $\mathbf{g}$ will transform into the continuous-time signal $g(t)$ defined over the interval $[a, b]$. In other words,

$$\lim_{N\to\infty} \mathbf{g} = g(t) \quad t \in [a, b]$$

This relationship clearly shows that continuous time signals are straightforward generalizations of finite dimension vectors. Thus, basic definitions and operations in a vector space can be applied to continuous time signals as well. We now highlight this connection between the finite dimension vector space and the continuous time signal space.

We shall denote all vectors by boldface type. For example, $\mathbf{x}$ is a certain vector with magnitude or length $||\mathbf{x}||$. A vector has magnitude and direction. In a vector space, we can define the inner (dot or scalar) product of two real-valued vectors $\mathbf{g}$ and $\mathbf{x}$ as

$$< \mathbf{g}, \mathbf{x} >= ||\mathbf{g}|| \cdot ||\mathbf{x}|| \cos\theta \tag{2.14}$$

where θ is the angle between vectors $\mathbf{g}$ and $\mathbf{x}$. By using this definition, we can express $||\mathbf{x}||$, the length (norm) of a vector $\mathbf{x}$ as

$$||\mathbf{x}||^2 =< \mathbf{x}, \mathbf{x} > \tag{2.15}$$

This defines a normed vector space.

2.4.1 Component of a Vector along Another Vector

Consider two vectors $\mathbf{g}$ and $\mathbf{x}$, as shown in Fig. 2.8. Let the component of $\mathbf{g}$ along $\mathbf{x}$ be $c\mathbf{x}$. Geometrically the component of $\mathbf{g}$ along $\mathbf{x}$ is the projection of $\mathbf{g}$ on $\mathbf{x}$, and is obtained by drawing a perpendicular from the tip of $\mathbf{g}$ on the vector $\mathbf{x}$, as shown in Fig. 2.8. What is the mathematical significance of a component of a vector along another vector? As seen from

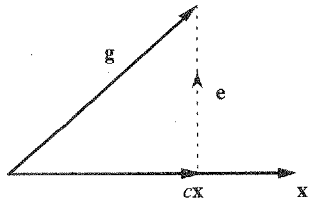


Figure 2.8 Component (projection) of a vector along another vector.

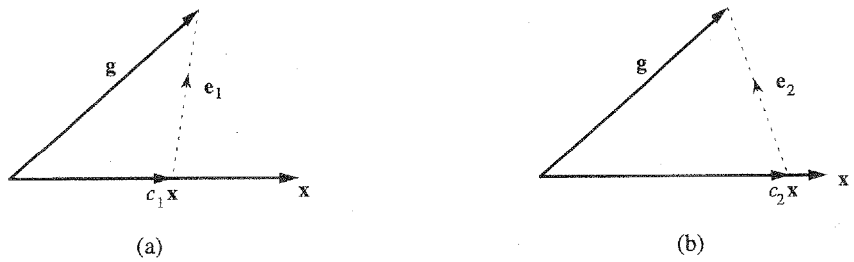


Figure 2.9 Approximations of a vector in terms of another vector.

Fig. 2.8, the vector **g** can be expressed in terms of vector **x** as

$$\mathbf{g} = c\mathbf{x} + \mathbf{e} \tag{2.16}$$

However, this does not describe a unique way to decompose **g** in terms of **x** and **e**. Figure 2.9 shows two of the infinite other possibilities. From Fig. 2.9a and b, we have

$$\mathbf{g} = c_1\mathbf{x} + \mathbf{e}_1 = c_2\mathbf{x} + \mathbf{e}_2 \tag{2.17}$$

The question is: Which is the "best" decomposition? The concept of optimality depends on what we wish to accomplish by decomposing **g** into two components.

In each of these three representations, **g** is given in terms of **x** plus another vector called the **error vector**. If our goal is to approximate **g** by $c\mathbf{x}$ (Fig. 2.8),

$$\mathbf{g} \simeq \hat{\mathbf{g}} = c\mathbf{x} \tag{2.18}$$

then the error in this approximation is the (difference) vector $\mathbf{e} = \mathbf{g} - c\mathbf{x}$. Similarly, the errors in approximations of Fig. 2.9a and b are $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. The approximation in Fig. 2.8 is unique because its error vector is the shortest (with the smallest magnitude or norm). We can now define mathematically the component (or projection) of a vector **g** along vector **x** to be $c\mathbf{x}$, where c is chosen to minimize the magnitude of the error vector $\mathbf{e} = \mathbf{g} - c\mathbf{x}$.

Geometrically, the magnitude of the component of **g** along **x** is $||\mathbf{g}|| \cos\theta$, which is also equal to $c||\mathbf{x}||$. Therefore

$$c||\mathbf{x}|| = ||\mathbf{g}|| \cos\theta$$

Based on the definition of inner product between two vectors, multiplying both sides by $||\mathbf{x}||$ yields

$$c||\mathbf{x}||^2 = ||\mathbf{g}||\,||\mathbf{x}|| \cos\theta = <\mathbf{g}, \mathbf{x}>$$

and

$$c = \frac{<\mathbf{g}, \mathbf{x}>}{<\mathbf{x}, \mathbf{x}>} = \frac{1}{||\mathbf{x}||^2} <\mathbf{g}, \mathbf{x}> \tag{2.19}$$

From Fig. 2.8, it is apparent that when **g** and **x** are perpendicular, or orthogonal, then **g** has a zero component along **x**; consequently, $c = 0$. Keeping an eye on Eq. (2.19), we therefore define **g** and **x** to be **orthogonal** if the inner (scalar or dot) product of the two vectors is zero, that is, if

$$< \mathbf{g}, \mathbf{x} >= 0 \tag{2.20}$$

2.4.2 Decomposition of a Signal and Signal Components

The concepts of vector component and orthogonality can be directly extended to continuous time signals. Consider the problem of approximating a real signal $g(t)$ in terms of another real signal $x(t)$ over an interval $[t_1, t_2]$:

$$g(t) \simeq cx(t) \qquad t_1 \leq t \leq t_2 \tag{2.21}$$

The error $e(t)$ in this approximation is

$$e(t) = \begin{cases} g(t) - cx(t) & t_1 \leq t \leq t_2 \\ 0 & \text{otherwise} \end{cases} \tag{2.22}$$

For "best approximation," we need to minimize the error signal, that is, minimize its norm. Minimum signal norm corresponds to minimum energy E_e over the interval $[t_1, t_2]$ given by

$$\begin{aligned} E_e &= \int_{t_1}^{t_2} e^2(t)\,dt \\ &= \int_{t_1}^{t_2} [g(t) - cx(t)]^2\,dt \end{aligned}$$

Note that the right-hand side is a definite integral with t as the dummy variable. Hence E_e is a function of the parameter c (not t), and E_e is minimum for some choice of c. To minimize E_e, a necessary condition is

$$\frac{dE_e}{dc} = 0 \tag{2.23}$$

or

$$\frac{d}{dc}\left[\int_{t_1}^{t_2} [g(t) - cx(t)]^2\,dt\right] = 0$$

Expanding the squared term inside the integral, we obtain

$$\frac{d}{dc}\left[\int_{t_1}^{t_2} g^2(t)\,dt\right] - \frac{d}{dc}\left[2c\int_{t_1}^{t_2} g(t)x(t)\,dt\right] + \frac{d}{dc}\left[c^2\int_{t_1}^{t_2} x^2(t)\,dt\right] = 0$$

from which we obtain

$$-2\int_{t_1}^{t_2} g(t)x(t)\,dt + 2c\int_{t_1}^{t_2} x^2(t)\,dt = 0$$

and

$$c = \frac{\int_{t_1}^{t_2} g(t)x(t)\,dt}{\int_{t_1}^{t_2} x^2(t)\,dt} = \frac{1}{E_x}\int_{t_1}^{t_2} g(t)x(t)\,dt \tag{2.24}$$

To summarize our discussion, if a signal $g(t)$ is approximated by another signal $x(t)$ as

$$g(t) \simeq cx(t)$$

then the optimum value of c that minimizes the energy of the error signal in this approximation is given by Eq. (2.24).

Taking our cue from vectors, we say that a signal $g(t)$ contains a component $cx(t)$, where c is given by Eq. (2.24). As in vector space, $cx(t)$ is the projection of $g(t)$ on $x(t)$. Consistent with the vector space terminolog, we say that if the component of a signal $g(t)$ of the form $x(t)$ is zero (i.e., $c = 0$), the signals $g(t)$ and $x(t)$ are orthogonal over the interval $[t_1, t_2]$. In other words, with respect to real-valued signals, two signals $x(t)$ and $g(t)$ are orthogonal when there is zero contribution from one signal to the other (i.e., $c = 0$). Thus, $x(t)$ and $g(t)$ are orthogonal if and only if

$$\int_{t_1}^{t_2} g(t)x(t)\,dt = 0 \tag{2.25}$$

Based on the illustrations of vectors in Fig. 2.9, we can say that two signals are orthogonal if and only if their inner product is zero. This relationship indicates that the integral of Eq. (2.25) is closely related to the concept of an inner product between vectors.

Indeed, the standard definition of the inner product of two N-dimensional vectors $\mathbf{g}$ and $\mathbf{x}$

$$< \mathbf{g}, \mathbf{x} >= \sum_{i=1}^{N} g_i x_i$$

is almost identical in form to the integration of Eq. (2.25). We therefore define the inner product of two (real-valued) signals $g(t)$ and $x(t)$, both defined over a time interval $[t_1, t_2]$, as

$$< g(t), x(t) >= \int_{t_1}^{t_2} g(t)x(t)\,dt \tag{2.26}$$

Recall from algebraic geometry that the square of a vector length $||\mathbf{x}||^2$ is equal to $< \mathbf{x}, \mathbf{x} >$. Keeping this concept in mind and continuing our analogy with vector analysis, we define the the norm of a signal $g(t)$ as

$$||g(t)|| = \sqrt{< g(t), g(t) >} \tag{2.27}$$

which is the square root of the signal energy in the time interval. It is therefore clear that the norm of a signal is analogous to the length of a finite dimensional vector. More generally, signals may not be merely defined over a continuous segment $[t_1, t_2]$.*

* Indeed, the signal space under consideration may be over a set of time segments represented simply by Θ. For such a more general space of signals, the inner product is defined as an integral over the time domain Θ. For

Example 2.2 For the square signal $g(t)$ shown in Fig. 2.10 find the component in $g(t)$ of the form of sin t. In other words, approximate $g(t)$ in terms of sin t:

$$g(t) \simeq c \sin t \qquad 0 \le t \le 2\pi$$

so that the energy of the error signal is minimum.

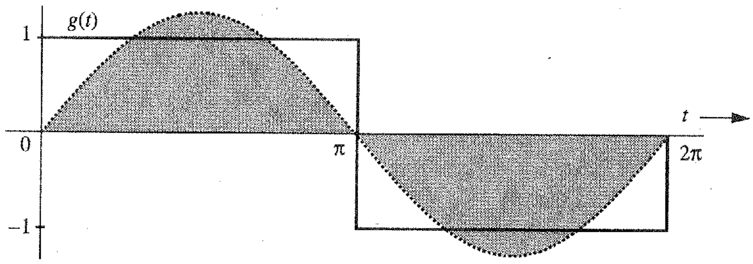


Figure 2.10 Approximation of square signal in terms of a single sinusoid.

In this case

$$x(t) = \sin t \qquad \text{and} \qquad E_x = \int_0^{2\pi} \sin^2(t)\, dt = \pi$$

From Eq. (2.24), we find

$$c = \frac{1}{\pi} \int_0^{2\pi} g(t) \sin t\, dt = \frac{1}{\pi} \left[\int_0^{\pi} \sin t\, dt + \int_{\pi}^{2\pi} (-\sin t)\, dt \right] = \frac{4}{\pi} \tag{2.29}$$

Therefore

$$g(t) \simeq \frac{4}{\pi} \sin t \tag{2.30}$$

represents the best approximation of $g(t)$ by the function sin t, which will minimize the error signal energy. This sinusoidal component of $g(t)$ is shown shaded in Fig. 2.10. As in vector space, we say that the square function $g(t)$ shown in Fig. 2.10 has a component of signal sin t with magnitude of $4/\pi$.

2.4.3 Complex Signal Space and Orthogonality

So far we have restricted our discussions to real functions of t. To generalize the results to complex functions of t, consider again the problem of approximating a function $g(t)$ by a

complex valued signals, the inner product is modified into

$$< g(t), x(t) >= \int_{\Theta} g(t) x^*(t)\, dt \tag{2.28}$$

Given the inner product definition, the signal norm $||g(t)|| = \sqrt{< g(t), g(t) >}$ and the signal space can be defined for any time domain signal.

function $x(t)$ over an interval $(t_1 \le t \le t_2)$

$$g(t) \simeq cx(t) \tag{2.31}$$

where $g(t)$ and $x(t)$ are complex functions of t. In general, both the coefficient c and the error

$$e(t) = g(t) - cx(t) \tag{2.32}$$

are complex. Recall that the energy E_x of the complex signal $x(t)$ over an interval $[t_1, t_2]$ is

$$E_x = \int_{t_1}^{t_2} |x(t)|^2 \, dt$$

For the best approximation, we need to choose c that minimizes E_e, the energy of the error signal $e(t)$ given by

$$E_e = \int_{t_1}^{t_2} |g(t) - cx(t)|^2 \, dt \tag{2.33}$$

Recall also that

$$|u + v|^2 = (u + v)(u^* + v^*) = |u|^2 + |v|^2 + u^*v + uv^* \tag{2.34}$$

Using this result, we can, after some manipulation, express the integral E_e in Eq. (2.33) as

$$E_e = \int_{t_1}^{t_2} |g(t)|^2 \, dt - \left| \frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} g(t)x^*(t) \, dt \right|^2 + \left| c\sqrt{E_x} - \frac{1}{\sqrt{E_x}} \int_{t_1}^{t_2} g(t)x^*(t) \, dt \right|^2$$

Since the first two terms on the right-hand side are independent of c, it is clear that E_e is minimized by choosing c such that the third term is zero. This yields the optimum coefficient

$$c = \frac{1}{E_x} \int_{t_1}^{t_2} g(t)x^*(t) \, dt \tag{2.35}$$

In light of the foregoing result, we need to redefine orthogonality for the complex case as follows: complex functions (signals) $x_1(t)$ and $x_2(t)$ are orthogonal over an interval $(t \le t_1 \le t_2)$ as long as

$$\int_{t_1}^{t_2} x_1(t)x_2^*(t) \, dt = 0 \quad \text{or} \quad \int_{t_1}^{t_2} x_1^*(t)x_2(t) \, dt = 0 \tag{2.36}$$

In fact, either equality suffices. This is a general definition of orthogonality, which reduces to Eq. (2.25) when the functions are real.

Similarly, the definition of inner product for complex signals over a time domain Θ can be modified:

$$< g(t), x(t) > = \int_{\{t: t \in \Theta\}} g(t)x^*(t) \, dt \tag{2.37}$$

Consequently, the norm of a signal $g(t)$ is simply

$$||g(t)|| = \left[\int_{\{t: t \in \Theta\}} |g(t)|^2 \, dt \right]^{1/2} \tag{2.38}$$

2.4.4 Energy of the Sum of Orthogonal Signals

We know that the geometric length (or magnitude) of the sum of two orthogonal vectors is equal to the sum of the magnitude squares of the two vectors. Thus, if vectors **x** and **y** are orthogonal, and if $\mathbf{z} = \mathbf{x} + \mathbf{y}$, then

$$||\mathbf{z}||^2 = ||\mathbf{x}||^2 + ||\mathbf{y}||^2$$

We have a similar result for signals. The energy of the sum of two orthogonal signals is equal to the sum of the energies of the two signals. Thus, if signals $x(t)$ and $y(t)$ are orthogonal over an interval $[t_1, t_2]$, and if $z(t) = x(t) + y(t)$, then

$$E_z = E_x + E_y \tag{2.39}$$

We now prove this result for complex signals of which real signals are a special case. From Eq. (2.34) it follows that

$$\begin{aligned}\int_{t_1}^{t_2} |x(t) + y(t)|^2\, dt &= \int_{t_1}^{t_2} |x(t)|^2\, dt + \int_{t_1}^{t_2} |y(t)|^2\, dt + \int_{t_1}^{t_2} x(t)y^*(t)\, dt + \int_{t_1}^{t_2} x^*(t)y(t)\, dt \\ &= \int_{t_1}^{t_2} |x(t)|^2\, dt + \int_{t_1}^{t_2} |y(t)|^2\, dt \end{aligned} \tag{2.40}$$

The last equality follows because, as a result of orthogonality, the two integrals of the cross products $x(t)y^*(t)$ and $x^*(t)y(t)$ are zero. This result can be extended to sum of any number of mutually orthogonal signals.

2.5 CORRELATION OF SIGNALS

By defining the inner product and the norm of signals, we paved the foundation for signal comparison. Here again, we can benefit by drawing parallels to the familiar vector space. Two vectors **g** and **x** are similar if **g** has a large component along **x**. In other words, if c in Eq. (2.19) is large, the vectors **g** and **x** are similar. We could consider c to be a quantitative measure of similarity between **g** and **x**. Such a measure, however, would be defective because it varies with the norms (or lengths) of **g** and **x**. To be fair, the amount of similarity between **g** and **x** should be independent of the lengths of **g** and **x**. If we double the length of **g**, for example, the amount of similarity between **g** and **x** should not change. From Eq. (2.19), however, we see that doubling **g** doubles the value of c (whereas doubling **x** halves the value of c). The similarity measure based on signal correlation is clearly faulty. Similarity between two vectors is indicated by the angle θ between the vectors. The smaller the θ, the larger the similarity, and vice versa. The amount of similarity can therefore be conveniently measured by cos θ. The larger the cos θ, the larger the similarity between the two vectors. Thus, a suitable measure would be $\rho = \cos\theta$, which is given by

$$\rho = \cos\theta = \frac{<\mathbf{g}, \mathbf{x}>}{||\mathbf{g}||\, ||\mathbf{x}||} \tag{2.41}$$

We can readily verify that this measure is independent of the lengths of **g** and **x**. This similarity measure ρ is known as the **correlation coefficient**. Observe that

$$-1 \le \rho \le 1 \tag{2.42}$$

Thus, the magnitude of ρ is never greater than unity. If the two vectors are aligned, the similarity is maximum ($\rho = 1$). Two vectors aligned in opposite directions have maximum dissimilarity ($\rho = -1$). If the two vectors are orthogonal, the similarity is zero.

We use the same argument in defining a similarity index (the correlation coefficient) for signals. For convenience, we shall consider the signals over the entire time interval from $-\infty$ to ∞. To establish a similarity index independent of energies (sizes) of $g(t)$ and $x(t)$, we must normalize c by normalizing the two signals to have unit energies. Thus, the appropriate similarity index ρ analogous to Eq. (2.41) is given by

$$\rho = \frac{1}{\sqrt{E_g E_x}} \int_{-\infty}^{\infty} g(t)x(t)\,dt \tag{2.43}$$

Observe that multiplying either $g(t)$ or $x(t)$ by any constant has no effect on this index. Thus, it is independent of the size (energies) of $g(t)$ and $x(t)$. Using the Cauchy-Schwarz inequality (proved in Appendix B),† one can show that the magnitude of ρ is never greater than 1:

$$-1 \leq \rho \leq 1 \tag{2.44}$$

2.5.1 Correlation Functions

We should revisit the application of correlation to signal detection in a radar unit, where a signal pulse is transmitted to detect a suspected target. By detecting the presence or absence of the reflected pulse, we confirm the presence or absence of the target. By measuring the time delay between the transmitted and received (reflected) pulse, we determine the distance of the target. Let the transmitted and the reflected pulses be denoted by $g(t)$ and $z(t)$, respectively. If we were to use Eq. (2.43) directly to measure the correlation coefficient ρ, we would obtain

$$\rho = \frac{1}{\sqrt{E_g E_z}} \int_{-\infty}^{\infty} z(t)g^*(t)\,dt = 0 \tag{2.45}$$

Thus, the correlation is zero because the pulses are disjoint (nonoverlapping in time). The integral in Eq. (2.45) will yield zero even when the pulses are identical but with relative time shift. To avoid this difficulty, we compare the received pulse $z(t)$ with the transmitted pulse $g(t)$ shifted by τ. If for some value of τ, there is a strong correlation, we not only detect the presence of the pulse but we also detect the relative time shift of $z(t)$ with respect to $g(t)$. For this reason, instead of using the integral on the right-hand side, we use the modified integral $\psi_{gz}(\tau)$, the **cross-correlation** function of two complex signals $g(t)$ and $z(t)$, defined by

$$\psi_{gz}(\tau) \equiv \int_{-\infty}^{\infty} z(t)g^*(t-\tau)\,dt = \int_{-\infty}^{\infty} z(t+\tau)g^*(t)\,dt \tag{2.46}$$

Therefore, $\psi_{gz}(\tau)$ is an indication of similarity (correlation) of $g(t)$ with $z(t)$ advanced (left-shifted) by τ seconds.

† The Cauchy-Schwarz inequality states that for two real energy signals $g(t)$ and $x(t)$, $\left(\int_{-\infty}^{\infty} g(t)x(t)\,dt\right)^2 \leq E_g E_x$ with equality if and only if $x(t) = Kg(t)$, where K is an arbitrary constant. There is similar inequality for complex signals.

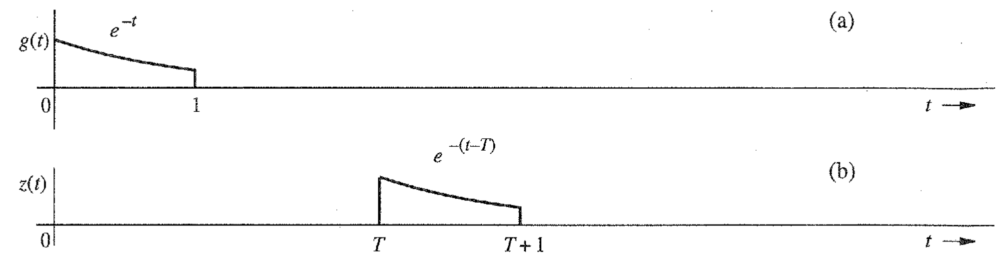


Figure 2.11 Physical explanation of the autocorrelation function.

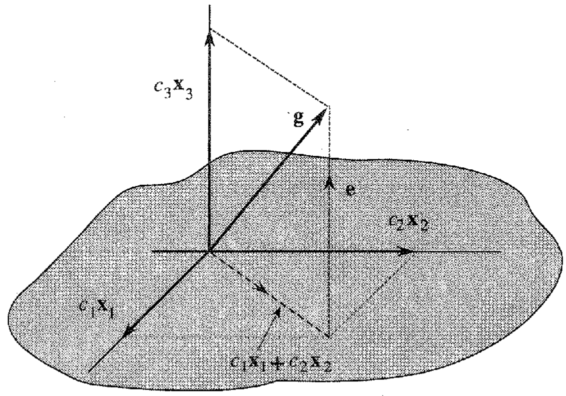


Figure 2.12 Representation of a vector in three-dimensional space.

2.5.2 Autocorrelation Function

As shown in Fig. 2.11, correlation of a signal with itself is called the **autocorrelation**. The autocorrelation function $\psi_g(\tau)$ of a real signal $g(t)$ is defined as

$$\psi_g(\tau) \equiv \int_{-\infty}^{\infty} g(t)g(t+\tau)\,dt \tag{2.47}$$

It measures the similarity of the signal $g(t)$ with its own displaced version. In Chapter 3, we shall show that the autocorrelation function provides valuable spectral information about the signal.

2.6 ORTHOGONAL SIGNAL SET

In this section we show a way of representing a signal as a sum of orthogonal set of signals. In effect, the signals in this orthogonal set form a basis for the specific signal space. Here again we can benefit from the insight gained from a similar problem in vectors. We know that a vector can be represented as a sum of orthogonal vectors, which form the coordinate system of a vector space. The problem in signals is analogous, and the results for signals are parallel to those for vectors. For this reason, let us review the case of vector representation.

2.6.1 Orthogonal Vector Space

Consider a multidimensional Cartesian vector space described by three mutually orthogonal vectors $\mathbf{x}_1$, $\mathbf{x}_2$, and $\mathbf{x}_3$, as shown in Fig. 2.12 for the special case of three-dimensional vector space. First, we shall seek to approximate a three-dimensional vector $\mathbf{g}$ in terms of two orthogonal vectors $\mathbf{x}_1$ and $\mathbf{x}_2$:

$$\mathbf{g} \simeq c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

The error $\mathbf{e}$ in this approximation is

$$\mathbf{e} = \mathbf{g} - (c_1\mathbf{x}_1 + c_2\mathbf{x}_2)$$

or equivalently,

$$\mathbf{g} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \mathbf{e}$$

In accordance with our earlier geometrical argument, it is clear from Fig. 2.12 that the length of error vector $\mathbf{e}$ is minimum when it is perpendicular to the ($\mathbf{x}_1$, $\mathbf{x}_2$) plane, and when $c_1\mathbf{x}_1$ and $c_2\mathbf{x}_2$ are the projections (components) of $\mathbf{g}$ on $\mathbf{x}_1$ and $\mathbf{x}_2$, respectively. Therefore, the constants c_1 and c_2 are given by formula in Eq. (2.19).

Now let us determine the best approximation to $\mathbf{g}$ in terms of all the three mutually orthogonal vectors $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$:

$$\mathbf{g} \simeq c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 \tag{2.48}$$

Figure 2.12 shows that a unique choice of c_1, c_2, and c_3 exists, for which (2.48) is no longer an approximation but an equality:

$$\mathbf{g} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3$$

In this case, $c_1\mathbf{x}_1, c_2\mathbf{x}_2$, and $c_3\mathbf{x}_3$ are the projections (components) of $\mathbf{g}$ on $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$, respectively. Note that the approximation error $\mathbf{e}$ is now zero when $\mathbf{g}$ is approximated in terms of three mutually orthogonal vectors: $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$. This is because $\mathbf{g}$ is a three-dimensional vector, and the vectors $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$ represent a *complete set* of orthogonal vectors in three-dimensional space. Completeness here means that it is impossible in this space to find any other vector $\mathbf{x}_4$, which is orthogonal to all the three vectors $\mathbf{x}_1, \mathbf{x}_2$, and $\mathbf{x}_3$. Any vector in this space can therefore be represented (with zero error) in terms of these three vectors. Such vectors are known as **basis** vectors, and the set of vector is known as a **complete orthogonal basis** of this vector space. If a set of vectors $\{\mathbf{x}_i\}$ is not complete, then the approximation error will generally not be zero. For example, in the three-dimensional case just discussed earlier, it is generally not possible to represent a vector $\mathbf{g}$ in terms of only two basis vectors without an error.

The choice of basis vectors is not unique. In fact, each set of basis vectors corresponds to a particular choice of coordinate system. Thus, a three-dimensional vector $\mathbf{g}$ may be represented in many different ways depending on the coordinate system used.

To summarize, if a set of vectors $\{\mathbf{x}_i\}$ is mutually orthogonal, that is, if

$$< \mathbf{x}_m, \mathbf{x}_n >= \begin{cases} 0 & m \neq n \\ |\mathbf{x}_m|^2 & m = n \end{cases}$$

and if this basis set is complete, a vector $\mathbf{g}$ in this space can be expressed as

$$\mathbf{g} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + c_3\mathbf{x}_3 \tag{2.49}$$

where the constants c_i are given by

$$c_i = \frac{< \mathbf{g}, \mathbf{x}_i >}{< \mathbf{x}_i, \mathbf{x}_i >} \tag{2.50a}$$

$$= \frac{1}{||\mathbf{x}_i||^2} < \mathbf{g}, \mathbf{x}_i > \qquad i = 1, 2, 3 \tag{2.50a}$$

2.6.2 Orthogonal Signal Space

We continue with our signal approximation problem, using clues and insights developed for vector approximation. As before, we define orthogonality of a signal set $x_1(t)$, $x_2(t)$, ... $x_N(t)$ over a time domain Θ (may be an interval $[t_1, t_2]$) as

$$\int_{t\in\Theta} x_m(t)x_n^*(t)\,dt = \begin{cases} 0 & m \neq n \\ E_n & m = n \end{cases} \tag{2.51}$$

If all signal energies are equal $E_n = 1$, then the set is *normalized* and is called an **orthonormal set**. An orthogonal set can always be normalized by dividing $x_n(t)$ by $\sqrt{E_n}$ for all n. Now, consider the problem of approximating a signal $g(t)$ over the Θ by a set of N mutually orthogonal signals $x_1(t), x_2(t), \ldots, x_N(t)$:

$$g(t) \simeq c_1x_1(t) + c_2x_2(t) + \cdots + c_Nx_N(t) \tag{2.52a}$$

$$= \sum_{n=1}^{N} c_nx_n(t) \qquad t \in \Theta \tag{2.52b}$$

It can be shown that E_e, the energy of the error signal $e(t)$ in this approximation, is minimized if we choose

$$c_n = \frac{\int_{t\in\Theta} g(t)x_n^*(t)\,dt}{\int_{t\in\Theta} |x_n(t)|^2\,dt}$$

$$= \frac{1}{E_n}\int_{\Theta} g(t)x_n^*(t)\,dt \qquad n = 1, 2, \ldots, N \tag{2.53}$$

Moreover, if the orthogonal set is **complete**, then the error energy $E_e \to 0$, and the representation in (2.52) is no longer an approximation, but an equality. More precisely, let the N-term approximation error be defined by

$$e_N(t) = g(t) - c_1x_1(t) + c_2x_2(t) + \cdots + c_Nx_N(t) = g(t) - \sum_{n=1}^{N} c_nx_n(t) \qquad t \in \Theta \tag{2.54}$$

If the orthogonal basis is **complete**, then the error signal energy converges to zero; that is,

$$\lim_{N\to\infty} \int_{t\in\Theta} |e_N(t)|^2 dt = 0 \tag{2.55}$$

In a strictly mathematical sense, however, a signal may not converge to zero even though its energy does. This is because a signal may be nonzero at some isolated points.* Still, for all practical purposes, signals are continuous for all t, and the equality (2.55) states that the error signal has zero energy as $N \to \infty$. Thus, for $N \to \infty$, the equality (2.52) can be loosely written as

$$\begin{aligned} g(t) &= c_1x_1(t) + c_2x_2(t) + \cdots + c_nx_n(t) + \cdots \\ &= \sum_{n=1}^{\infty} c_nx_n(t) \qquad t \in \Theta \end{aligned} \tag{2.56}$$

where the coefficients c_n are given by Eq. (2.53). Because the error signal energy approaches zero, it follows that the energy of $g(t)$ is now equal to the sum of the energies of its orthogonal components.

The series on the right-hand side of Eq. (2.56) is called the **generalized Fourier series** of $g(t)$ with respect to the set $\{x_n(t)\}$. When the set $\{x_n(t)\}$ is such that the error energy $E_N \to 0$ as $N \to \infty$ for every member of some particular signal class we say that the set $\{x_n(t)\}$ is complete on $\{t : \Theta\}$ for that class of $g(t)$, and the set $\{x_n(t)\}$ is called a set of **basis functions** or **basis signals**. In particular, the class of (finite) energy signals over Θ is denoted as $L^2\{\Theta\}$. Unless otherwise mentioned, in the future we shall consider only the class of energy signals.

2.6.3 Parseval's Theorem

Recall that the energy of the sum of orthogonal signals is equal to the sum of their energies. Therefore, the energy of the right-hand side of Eq. (2.56) is the sum of the energies of the individual orthogonal components. The energy of a component $c_nx_n(t)$ is $c_n^2E_n$. Equating the energies of the two sides of Eq. (2.56) yields

$$\begin{aligned} E_g &= c_1^2E_1 + c_2^2E_2 + c_3^2E_3 + \cdots \\ &= \sum_n c_n^2E_n \end{aligned} \tag{2.57}$$

This important result goes by the name of **Parseval's theorem**. Recall that the signal energy (area under the squared value of a signal) is analogous to the square of the length of a vector in the vector-signal analogy. In vector space we know that the square of the length of a vector is equal to the sum of the squares of the lengths of its orthogonal components. Parseval's theorem [Eq. (2.57)] is the statement of this fact as applied to signals.

2.7 THE EXPONENTIAL FOURIER SERIES

We noted earlier that orthogonal signal representation is NOT unique. While the traditional trigonometric Fourier series allows a good representation of all periodic signals, here we provide an orthogonal representation of periodic signals that is **equivalent** but has a simpler form.

* Known as a measure-zero set.

First of all, it is clear that the set of exponentials $e^{jn\omega_0 t}$ ($n = 0, \pm 1, \pm 2, \ldots$) is orthogonal over any interval of duration $T_0 = 2\pi/\omega_0$, that is,

$$\int_{T_0} e^{jm\omega_0 t}(e^{jn\omega_0 t})^* \, dt = \int_{T_0} e^{j(m-n)\omega_0 t} \, dt = \begin{cases} 0 & m \neq n \\ T_0 & m = n \end{cases} \tag{2.58}$$

Moreover, this set is a complete set.[1,2] From Eqs. (2.53) and (2.56), it follows that a signal $g(t)$ can be expressed over an interval of duration T_0 second(s) as an exponential Fourier series

$$\begin{aligned} g(t) &= \sum_{n=-\infty}^{\infty} D_n e^{jn\omega_0 t} \\ &= \sum_{n=-\infty}^{\infty} D_n e^{jn2\pi f_0 t} \end{aligned} \tag{2.59}$$

where [see Eq. (2.53)]

$$D_n = \frac{1}{T_0} \int_{T_0} g(t) e^{-jn2\pi f_0 t} \, dt \tag{2.60}$$

The exponential Fourier series in Eq. (2.59) consists of components of the form $e^{jn2\pi f_0 t}$ with n varying from $-\infty$ to ∞. It is periodic with period T_0.

Example 2.3 Find the exponential Fourier series for the signal in Fig. 2.13b.

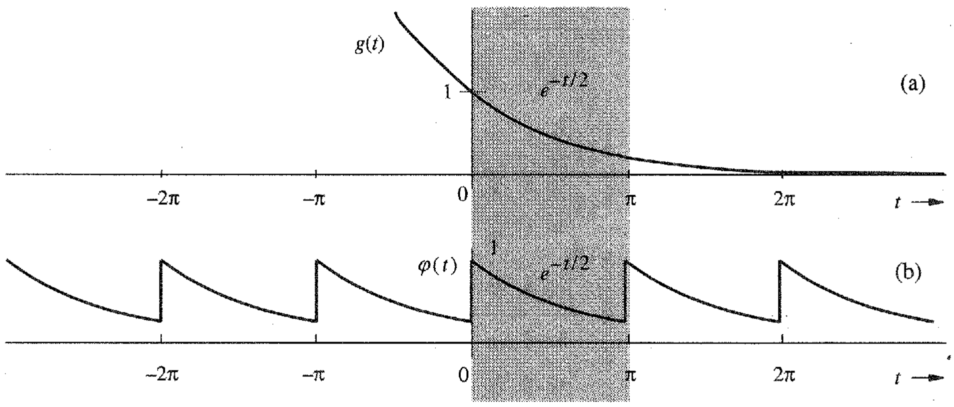


Figure 2.13 A periodic signal.

In this case, $T_0 = \pi$, $2\pi f_0 = 2\pi/T_0 = 2$, and

$$\varphi(t) = \sum_{n=-\infty}^{\infty} D_n e^{j2nt}$$

where

$$
\begin{aligned}
D_n &= \frac{1}{T_0}\int_{T_0} \varphi(t)e^{-j2nt}\,dt \\
&= \frac{1}{\pi}\int_0^{\pi} e^{-t/2}\,e^{-j2nt}\,dt \\
&= \frac{1}{\pi}\int_0^{\pi} e^{-(\frac{1}{2}+j2n)t}\,dt \\
&= \frac{-1}{\pi\left(\frac{1}{2}+j2n\right)} e^{-(\frac{1}{2}+j2n)t}\Big|_0^{\pi} \\
&= \frac{0.504}{1+j4n}
\end{aligned}
\tag{2.61}
$$

and

$$
\varphi(t) = 0.504 \sum_{n=-\infty}^{\infty} \frac{1}{1+j4n} e^{j2nt} \tag{2.62a}
$$

$$
\begin{aligned}
&= 0.504\left[1 + \frac{1}{1+j4}e^{j2t} + \frac{1}{1+j8}e^{j4t} + \frac{1}{1+j12}e^{j6t} + \cdots \right. \\
&\left. + \frac{1}{1-j4}e^{-j2t} + \frac{1}{1-j8}e^{-j4t} + \frac{1}{1-j12}e^{-j6t} + \cdots \right]
\end{aligned}
\tag{2.62b}
$$

Observe that the coefficients D_n are complex. Moreover, D_n and D_{-n} are conjugates, as expected.

Exponential Fourier Spectra

In exponential spectra, we plot coefficients D_n as a function of ω. But since D_n is complex in general, we need two plots: the real and the imaginary parts of D_n or the amplitude (magnitude) and the angle of D_n. We prefer the latter because of its close connection to the amplitudes and phases of corresponding components of the trigonometric Fourier series. We therefore plot $|D_n|$ versus ω and $\angle D_n$ versus ω. This requires that the coefficients D_n be expressed in polar form as $|D_n|e^{j\angle D_n}$.

For a real periodic signal, the twin coefficients D_n and D_{-n} are conjugates,

$$
|D_n| = |D_{-n}| \tag{2.63a}
$$

$$
\angle D_n = \theta_n \qquad \text{and} \qquad \angle D_{-n} = -\theta_n \tag{2.63b}
$$

Thus,

$$
D_n = |D_n|e^{j\theta_n} \qquad \text{and} \qquad D_{-n} = |D_n|e^{-j\theta_n} \tag{2.64}
$$

Note that $|D_n|$ are the amplitudes (magnitudes) and $\angle D_n$ are the angles of various exponential components. From Eq. (2.63) it follows that the amplitude spectrum ($|D_n|$ vs. f) is an even function of ω and the angle spectrum ($\angle D_n$ vs. f) is an odd function of f when $g(t)$ is a real signal.

For the series in Example 2.3, for instance,

$$D_0 = 0.504$$

$$D_1 = \frac{0.504}{1+j4} = 0.122e^{-j75.96^\circ} \Longrightarrow |D_1| = 0.122, \ \angle D_1 = -75.96^\circ$$

$$D_{-1} = \frac{0.504}{1-j4} = 0.122e^{j75.96^\circ} \Longrightarrow |D_{-1}| = 0.122, \ \angle D_{-1} = 75.96^\circ$$

and

$$D_2 = \frac{0.504}{1+j8} = 0.0625e^{-j82.87^\circ} \Longrightarrow |D_2| = 0.0625, \ \angle D_2 = -82.87^\circ$$

$$D_{-2} = \frac{0.504}{1-j8} = 0.0625e^{j82.87^\circ} \Longrightarrow |D_{-2}| = 0.0625, \ \angle D_{-2} = 82.87^\circ$$

and so on. Note that D_n and D_{-n} are conjugates, as expected [see Eq. (2.63b)].

Figure 2.14 shows the frequency spectra (amplitude and angle) of the exponential Fourier series for the periodic signal $\varphi(t)$ in Fig. 2.13b.

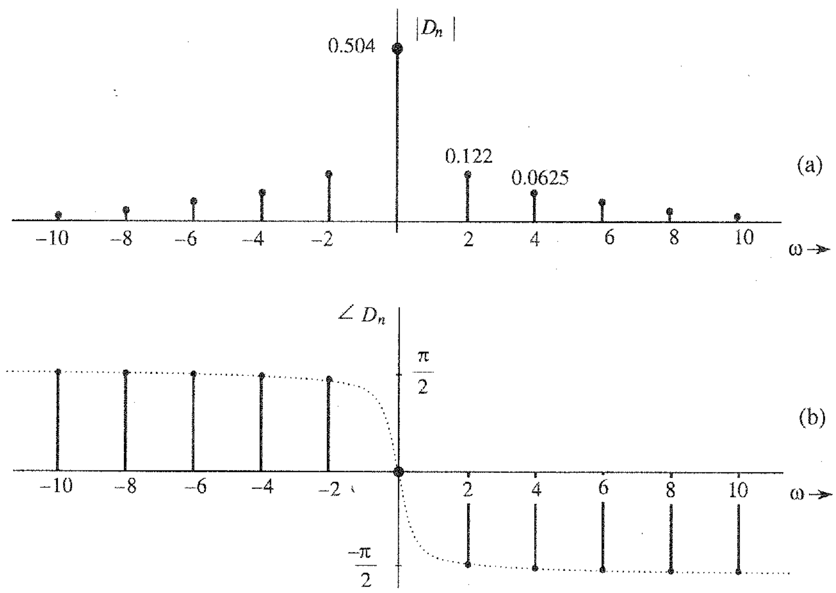
We notice some interesting features of these spectra. First, the spectra exist for positive as well as negative values of f (the frequency). Second, the amplitude spectrum is an even function of f and the angle spectrum is an odd function of f. Equations (2.63) show the symmetric characteristics of the amplitude and phase of D_n.

What Does Negative Frequency Mean?

The existence of the spectrum at negative frequencies is somewhat disturbing to some people because by definition, the frequency (number of repetitions per second) is a positive quantity. How do we interpret a negative frequency f_0? We can use a trigonometric identity to express a sinusoid of a negative frequency $-f_0$ by borrowing $\omega_0 = 2\pi f_0$, as

$$\cos(-\omega_0 t + \theta) = \cos(\omega_0 t - \theta)$$

Figure 2.14 Exponential Fourier spectra for the signal in Fig. 2.13a.



This clearly shows that the angular frequency of a sinusoid cos $(-\omega_0 t + \theta)$ is $|\omega_0|$, which is a positive quantity. The commonsense statement that a frequency must be positive comes from the traditional notion that frequency is associated with a real-valued sinusoid (such as a sine or a cosine). In reality, the concept of frequency for a real-valued sinusoid describes only the rate of the sinusoidal variation without addressing the direction of the variation. This is because real-valued sinusoidal signals do NOT contain information on the direction of its variation.

The concept of negative frequency is meaningful **only** when we are considering complex sinusoids for which the rate and the *direction* of variation are meaningful. Observe that

$$e^{\pm j\omega_0 t} = \cos\,\omega_0 t \pm j \sin\,\omega_0 t$$

This relationship clearly shows that either positive or negative ω leads to periodic variation of the same rate. However, the resulting complex signals are NOT the same. Because $|e^{\pm j\omega_0 t}| = 1$, both $e^{+j\omega_0 t}$ and $e^{-j\omega_0 t}$ are unit length complex variables that can be shown on the complex plane. We illustrate the two exponential sinusoids as unit length complex variables that vary with time t in Fig. 2.15. Thus, the rotation rate for both exponentials $e^{\pm j\omega_0 t}$ is $|\omega_0|$. It is clear that for positive frequency, the exponential sinusoid rotates counterclockwise while for negative frequency, the exponential sinusoid rotates clockwise. This illustrates the actual meaning of negative frequency.

There exists a good analogy between positive/negative frequency and positive/negative velocity. Just as people are reluctant to use *negative* velocity in describing a moving object, they are equally unwilling to accept the notion of "negative" frequency. However, once we understand that negative velocity simply refers to both the negative direction and the actual speed of a moving object, negative velocity makes perfect sense. Likewise, negative frequency does NOT describe the rate of periodic variation of a sine or a cosine. It describe the direction of rotation of a unit length exponential sinusoid and its rate of revolution.

Another way of looking at the situation is to say that *exponential spectra are a graphical representation of coefficients D_n as a function of f. Existence of the spectrum at $f = -nf_0$ merely indicates that an exponential component $e^{-jn2\pi f_0 t}$ exists in the series.* We know from Euler's identity

$$\cos\,(\omega t + \theta) = \frac{e^{j\theta}}{2} \exp\,(j\omega t) + \frac{e^{-j\theta}}{2} \exp\,(-j\omega t)$$

that a sinusoid of frequency $n\omega_0$ can be expressed in terms of a pair of exponentials $e^{jn\omega_0 t}$ and $e^{-jn\omega_0 t}$. That both sine and cosine consist of positive and negative frequency exponential sinusoidal components clearly indicates that we are NOT at all able to describe the *direction* of their periodic variations. Indeed, both sine and cosine functions of frequency ω_0 consist of two equal-size exponential sinusoids of frequency $\pm\omega_0$. Thus, the frequency of sine or cosine is the absolute value of its two component frequencies and denotes only the rate of the sinusoidal variations.

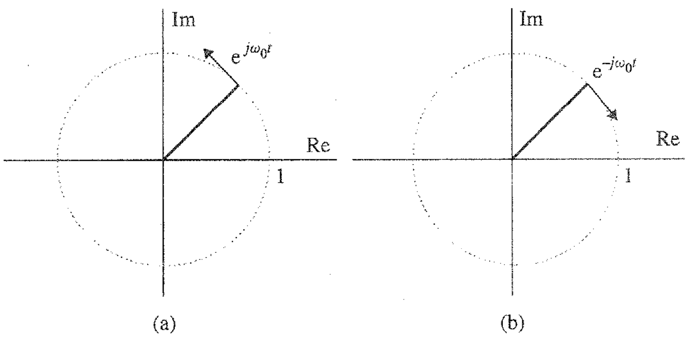


Figure 2.15 Unit length complex variable with positive frequency (rotating counter-clockwise) versus unit length complex variable with negative frequency (rotating clockwise).

Example 2.4 Find the exponential Fourier series for the periodic square wave $w(t)$ shown in Fig. 2.16.

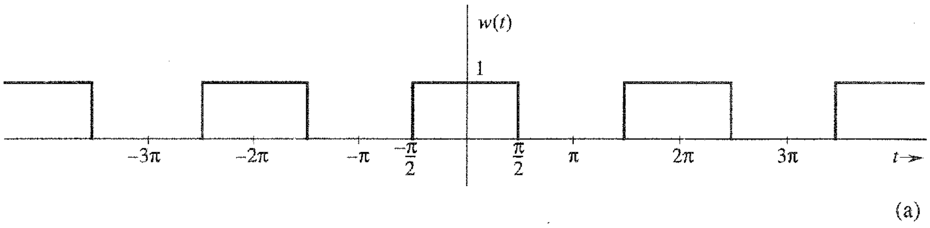


Figure 2.16 A square pulse periodic signal.

$$w(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn2\pi f_0 t}$$

where

$$D_0 = \frac{1}{T_0}\int_{T_0} w(t)\,dt = \frac{1}{2}$$

$$\begin{aligned} D_n &= \frac{1}{T_0}\int_{T_0} w(t)e^{-jn2\pi f_0 t}\,dt, \qquad n \neq 0 \\ &= \frac{1}{T_0}\int_{-T_0/4}^{T_0/4} e^{-jn2\pi f_0 t}\,dt \\ &= \frac{1}{-jn2\pi f_0 T_0}\left[e^{-jn2\pi f_0 T_0/4} - e^{jn2\pi f_0 T_0/4}\right] \\ &= \frac{2}{n2\pi f_0 T_0}\sin\left(\frac{n2\pi f_0 T_0}{4}\right) = \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

In this case D_n is real. Consequently, we can do without the phase or angle plot if we plot D_n vs. f instead of the amplitude spectrum ($|D_n|$ vs. f) as shown in Fig. 2.17.

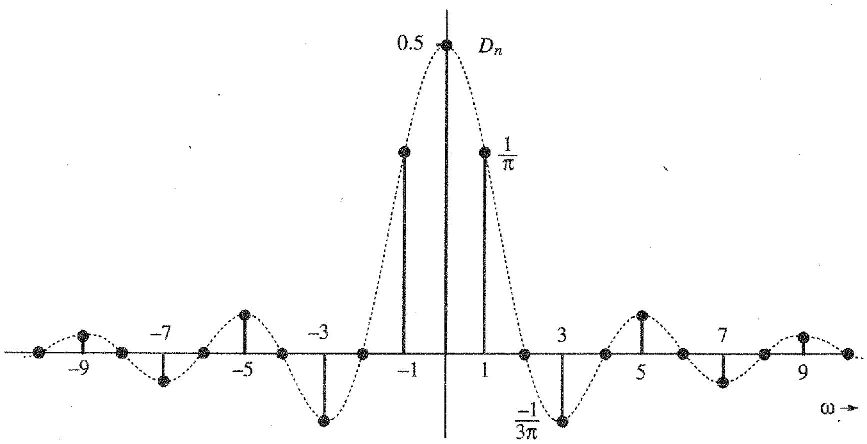


Figure 2.17 Exponential Fourier spectrum of the square pulse periodic signal.

Example 2.5 Find the exponential Fourier series and sketch the corresponding spectra for the impulse train $\delta_{T_0}(t)$ shown in Fig. 2.18a.

The exponential Fourier series is given by

$$\delta_{T_0}(t) = \sum_{n=-\infty}^{\infty} D_n e^{jn2\pi f_0 t} \qquad f_0 = \frac{1}{T_0} \tag{2.65}$$

where

$$D_n = \frac{1}{T_0}\int_{T_0} \delta_{T_0}(t) e^{-jn2\pi f_0 t}\, dt$$

Choosing the interval of integration $(\frac{-T_0}{2}, \frac{T_0}{2})$ and recognizing that over this interval $\delta_{T_0}(t) = \delta(t)$, were have

$$D_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \delta(t) e^{-jn2\pi f_0 t}\, dt$$

In this integral, the impulse is located at $t = 0$. From the sampling property of the impulse function, the integral on the right-hand side is the value of $e^{-jn2\pi f_0 t}$ at $t = 0$ (where the impulse is located). Therefore

$$D_n = \frac{1}{T_0} \tag{2.66}$$

and

$$\delta_{T_0}(t) = \frac{1}{T_0} \sum_{n=-\infty}^{\infty} e^{jn2\pi f_0 t} \qquad f_0 = \frac{1}{T_0} \tag{2.67}$$

Equation (2.67) shows that the exponential spectrum is uniform ($D_n = 1/T_0$) for all the frequencies, as shown in Fig. 2.18b. The spectrum, being real, requires only the amplitude plot. All phases are zero.

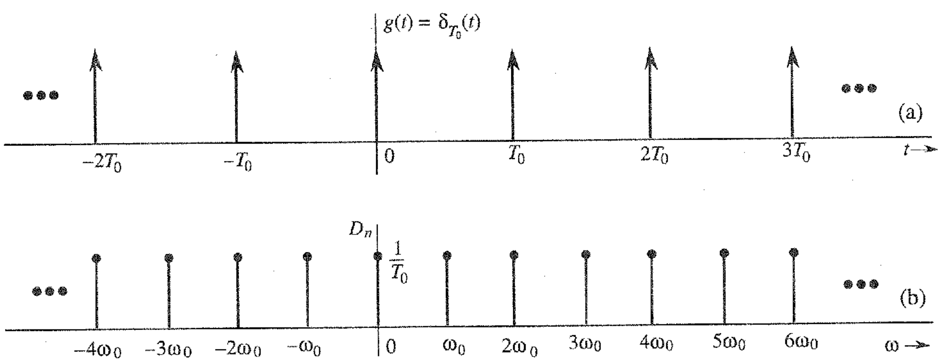


Figure 2.18 Impulse train and its exponential Fourier spectra.

Parseval's Theorem in the Fourier Series

A periodic signal $g(t)$ is a power signal, and every term in its Fourier series is also a power signal. The power P_g of $g(t)$ is equal to the power of its Fourier series. Because the Fourier series consists of terms that are mutually orthogonal over one period, the power of the Fourier series is equal to the sum of the powers of its Fourier components. This follows from Parseval's theorem.

Thus, for the exponential Fourier series

$$g(t) = D_0 + \sum_{n=-\infty,\, n\neq 0}^{\infty} D_n e^{jn\omega_0 t}$$

the power is given by (see Prob. 2.1-7)

$$P_g = \sum_{n=-\infty}^{\infty} |D_n|^2 \tag{2.68a}$$

For a real $g(t)$, $|D_{-n}| = |D_n|$. Therefore

$$P_g = D_0{}^2 + 2\sum_{n=1}^{\infty} |D_n|^2 \tag{2.68b}$$

Comment: Parseval's theorem occurs in many different forms, such as in Eqs. (2.57) and Eq. (2.68a). Yet another form is found in the next chapter for nonperiodic signals. Although these forms appear to be different, they all state the same principle: that is, the square of the length of a vector equals the sum of the squares of its orthogonal components. The first form [Eq. (2.57)] applies to energy signals, and the second [Eq. (2.68a)] applies to periodic signals represented by the exponential Fourier series.

Some Other Examples of Orthogonal Signal Sets

The signal representation by Fourier series shows that signals are vectors in every sense. Just as a vector can be represented as a sum of its components in a variety of ways, depending upon the choice of a coordinate system, a signal can be represented as a sum of its components in a variety of ways. Just as we have vector coordinate systems formed by mutually orthogonal vectors (rectangular, cylindrical, spherical, etc.), we also have signal coordinate systems, basis signals, formed by a variety of sets of mutually orthogonal signals. There exist a large number of orthogonal signal sets that can be used as basis signals for generalized Fourier series. Some well-known signal sets are trigonometric (sinusoid) functions, exponential functions, Walsh functions, Bessel functions, Legendre polynomials, Laguerre functions, Jacobi polynomials, Hermite polynomials, and Chebyshev polynomials. The functions that concern us most in this book are the exponential sets discussed next in the chapter.

2.8 MATLAB EXERCISES

In this section, we provide some basic MATLAB exercises to illustrate the process of signal generation, signal operations, and Fourier series analysis.

Basic Signals and Signal Graphing

Basic functions can be defined by using MATLAB's m-files. We gave three MATLAB programs that implement three basic functions when a time vector `t` is provided:

- `ustep.m` implements the unit step function $u(t)$
- `rect.m` implements the standard rectangular function $rect(t)$
- `triangl.m` implements standard triangle function $\Delta(t)$

```
% (file name:  ustep.m)
%  The unit step function is a function of time 't'.
%  Usage   y = ustep(t)
%
%  ustep(t) = 0     if t < 0
%  ustep(t) = 1,    if t >= 1
%
%  t - must be real-valued and can be a vector or a matrix
%
function y=ustep(t)
   y = (t>=0);
end
```

```
% (file name: rect.m)
%  The rectangular function is a function of time 't'.
%
%  Usage   y = rect(t)
%  t - must be real-valued and can be a vector or a matrix
%
%  rect(t) = 1,     if |t| < 0.5
%  rect(t) = 0,     if |t| > 0.5
%
function y=rect(t)
   y =(sign(t+0.5)-sign(t-0.5) >0);
end
```

```
% (file name: triangl.m)
%  The triangle function is a function of time 't'.
%
%  triangl(t) = 1-|t|, if  |t| < 1
%  triangl(t) = 0, if      |t| > 1
%
%  Usage  y = triangl(t)
%  t - must be real-valued and can be a vector or a matrix
%
```

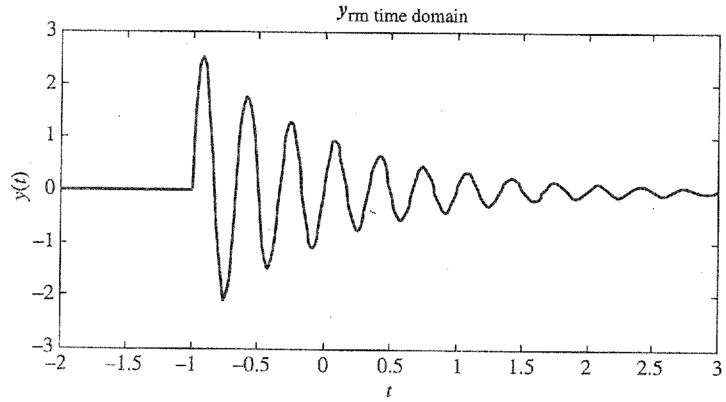


Figure 2.19 Graphing a signal.

```
function y=triangl(t)
    y = (1-abs(t)).*(t>=-1).*(t<1);
end
```

We now show how to use MATLAB to generate a simple signal plot through an example. `siggraf.m` is provided. In this example, we construct and plot a signal

$$y(t) = \exp(-t)\sin(6\pi t)u(t+1)$$

The resulting graph shown in Fig. 2.19.

```
% (file name: siggraf.m)
%  To graph a signal, the first step is to determine
%  the x-axis and the y-axis to plot
%  We can first decide the length of x-axis to plot
   t=[-2:0.01:3];        % "t" is from -2 to 3 in 0.01 increment
% Then evaluate the signal over the range of "t" to plot
   y=exp(-t).*sin(10*pi*t).*ustep(t+1);
   figure(1);  fig1=plot(t,y);      % plot t vs y in figure 1
   set(fig1,'Linewidth',2);         % choose a wider line-width
   xlabel('\it t');           % use italic 't' to label x-axis
   ylabel('\{\bf y\}(\{\it t})');      % use boldface 'y'
                                           to label y-axis
   title('\{\bf y\}\_\{\rm time domain\}');  % can use subscript
```

Periodic Signals and Signal Power

Periodic signals can be generated by first determining the signal values in one period before repeating the same signal vector multiple times.

In the following MATLAB program `PfuncEx.m`, we generate a periodic signal and observe its behavior over $2M$ periods. The period of this example is $T = 6$. The program also evaluates the average signal power which is stored as a variable `y_power` and signal energy in one period which is stored in variable `y_energyT`.

```
% (file name:  PfuncEx.m)
% This example generates a periodic signal, plots the signal
% and evaluates the average signal power in y_power and signal
% energy in 1 period T:     y_energyT
    echo off;clear;clf;
% To generate a periodic signal g_T(t),
% we can first decide the signal within the period of 'T' for g(t)
    Dt=0.002;    % Time interval (to sample the signal)
    T=6;         % period=T
    M=3;         % To generate 2M periods of the signal
    t=[0:Dt:T-Dt];  %"t" goes for one period [0, T] in Dt increment
% Then evaluate the signal over the range of "T"
    y=exp(-abs(t)/2).*sin(2*pi*t).*(ustep(t)-ustep(t-4));
% Multiple periods can now be generated.
    time=[];
    y_periodic=[];
for i=-M:M-1,
    time=[time i*T+t];
    y_periodic=[y_periodic y];
end
    figure(1); fy=plot(time,y_periodic);
    set(fy,'Linewidth',2);xlabel('{\it t}');
    echo on
%   Compute average power
    y_power=sum(y_periodic*y_periodic')*Dt/(max(time)-min(time))
%   Compute signal energy in 1 period T
    y_energyT=sum(y.*conj(y))*Dt
```

The program generates a periodic signal as shown in Fig. 2.20 and numerical answers:

```
y_power =
      0.0813

y_energyT =
      0.4878
```

Signal Correlation

The MATLAB program can implement directly the concept of signal correlation introduced in Section 2.5. In the next computer example, we provide a program, `sign_cor.m`, that evaluates the signal correlation coefficients between $x(t)$ and signals $g_1(t)$, $g_2(t)$, ... $g_5(t)$. The program first generates Fig. 2.21, which illustrates the six signals in the time domain.

```
% (file name: sign_cor.m)
clear
%   To generate 6 signals x(t), g_1(t), ...  g_5(t);
%   of this Example
%   we can first decide the signal within the period of 'T' for g(t)
```

Figure 2.20
Generating a periodic signal.

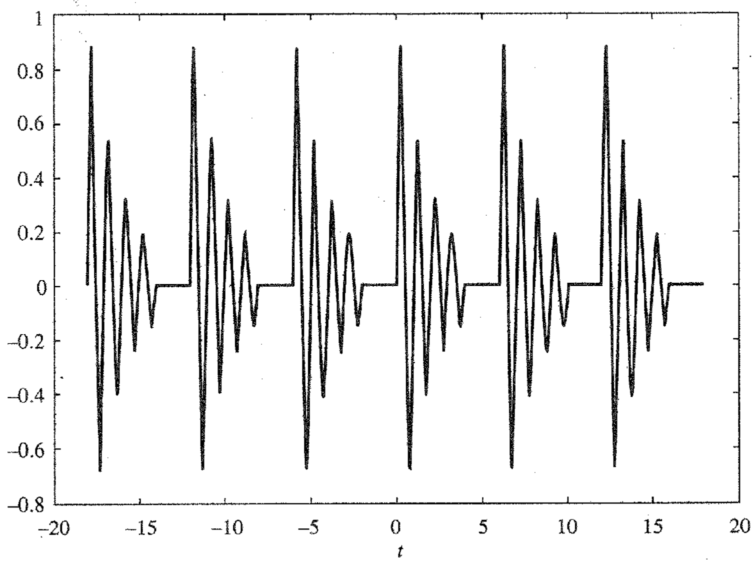
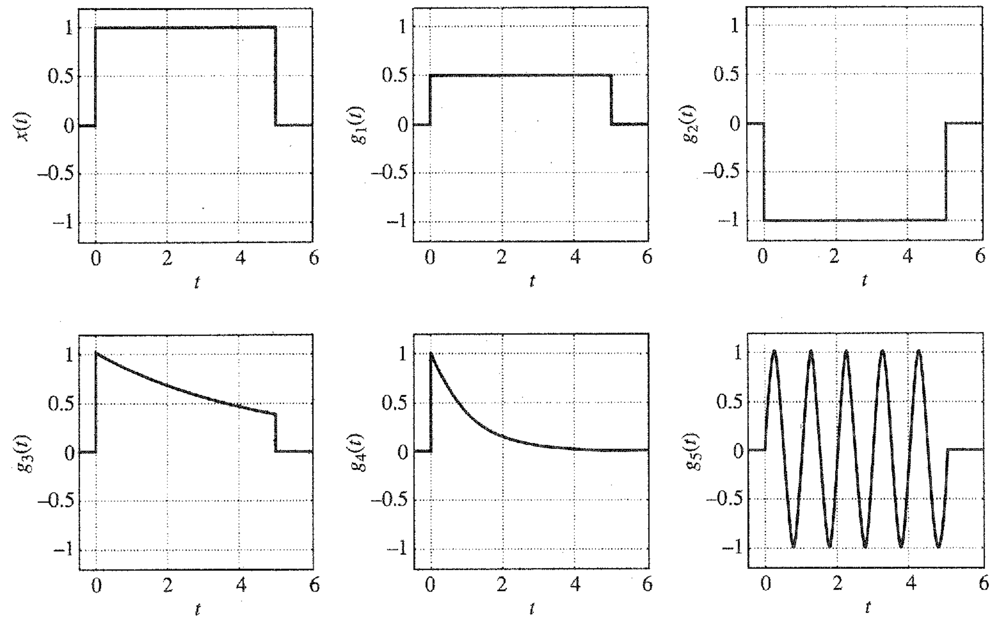


Figure 2.21
Six simple signals.




```
    Dt=0.01;      % time increment Dt
    T=6.0;        % time duration = T
    t=[-1:Dt:T];      %"t" goes between [-1, T] in Dt increment
% Then evaluate the signal over the range of "t" to plot
    x=ustep(t)-ustep(t-5);
    g1=0.5*(ustep(t)-ustep(t-5));
    g2=-(ustep(t)-ustep(t-5));
    g3=exp(-t/5).*(ustep(t)-ustep(t-5));
    g4=exp(-t).*(ustep(t)-ustep(t-5));
    g5=sin(2*pi*t).*(ustep(t)-ustep(t-5));
```

```
subplot(231); sig1=plot(t,x,'k');
xlabel('\it t'); ylabel('{\it x}({\it t})');    % Label axis
set(sig1,'Linewidth',2);                % change linewidth
axis([-.5 6 -1.2 1.2]); grid            % set plot range
subplot(232); sig2=plot(t,g1,'k');
xlabel('\it t'); ylabel('{\it g}_1({\it t})');
set(sig2,'Linewidth',2);
axis([-.5 6 -1.2 1.2]); grid
subplot(233); sig3=plot(t,g2,'k');
xlabel('\it t'); ylabel('{\it g}_2({\it t})');
set(sig3,'Linewidth',2);
axis([-.5 6 -1.2 1.2]); grid
subplot(234); sig4=plot(t,g3,'k');
xlabel('\it t'); ylabel('{\it g}_3({\it t})');
set(sig4,'Linewidth',2);
axis([-.5 6 -1.2 1.2]); grid
subplot(235); sig5=plot(t,g4,'k');
xlabel('\it t'); ylabel('{\it g}_4({\it t})');
set(sig5,'Linewidth',2);grid
axis([-.5 6 -1.2 1.2]);
subplot(236); sig6=plot(t,g5,'k');
xlabel('\it t'); ylabel('{\it g}_5({\it t})');
set(sig6,'Linewidth',2);grid
axis([-.5 6 -1.2 1.2]);

% Computing signal energies
E0=sum(x.*conj(x))*Dt;
E1=sum(g1.*conj(g1))*Dt;
E2=sum(g2.*conj(g2))*Dt;
E3=sum(g3.*conj(g3))*Dt;
E4=sum(g4.*conj(g4))*Dt;
E5=sum(g5.*conj(g5))*Dt;

c0=sum(x.*conj(x))*Dt/(sqrt(E0*E0))
c1=sum(x.*conj(g1))*Dt/(sqrt(E0*E1))
c2=sum(x.*conj(g2))*Dt/(sqrt(E0*E2))
c3=sum(x.*conj(g3))*Dt/(sqrt(E0*E3))
c4=sum(x.*conj(g4))*Dt/(sqrt(E0*E4))
c5=sum(x.*conj(g5))*Dt/(sqrt(E0*E5))
```

The six correlation coefficients are obtained from the program as

```
c0 =
     1

c1 =
     1

c2 =
    -1
```

```
c3 =
     0.9614

c4 =
     0.6282

c5 =
    8.6748e-17
```

Numerical Computation of Coefficients D_n

There are several ways to numerically compute the Fourier series coefficients D_n. We will use MATLAB to show how to use numerical integration in the evaluation of Fourier series.

To carry out a direct numerical integration of Eq. (2.60), the first step is to define the symbolic expression of the signal $g(t)$ under analysis. We use the triangle function $\Delta(t)$ in the following example.

```
% (funct_tri.m)
% A standard triangle function of base -1 to 1
    function y = funct_tri(t)
% Usage  y = func_tri(t)
%     t = input variable i
y=((t>-1)-(t>1)).*(1-abs(t));
```

Once the file funct_tri.m defines the function $y = g(t)$, we can directly carry out the necessary integration of Eq. (2.60) for a finite number of Fourier series coefficients $\{D_n, n = -N, \ldots, -1, 0, 1, \ldots, N\}$. We provide the following MATLAB program called FSexample.m to evaluate the Fourier series of $\Delta(t/2)$ with period $[a, b]$ $(a = -2, b = 2)$. In this example, $N = 11$ is selected. Executing this short program in MATLAB will generate Fig. 2.22 with both amplitude and angle of D_n.

```
% (file name:  FSexp_a.m)
%   This example shows how to numerically evaluate
%   the exponential Fourier series coefficients Dn
%   directly.
%   The user needs to define a symbolic function
%   g(t).  In this example, g(t)=funct_tri(t).
echo off; clear; clf;
    j=sqrt(-1); % Define j for complex algebra
    b=2; a=-2;  % Determine one signal period
    tol=1.e-5;  % Set integration error tolerance
    T=b-a;      % length of the period
    N=11;       % Number of FS coefficients
            % on each side of zero frequency
    Fi=[-N:N]*2*pi/T;    % Set frequency range
```

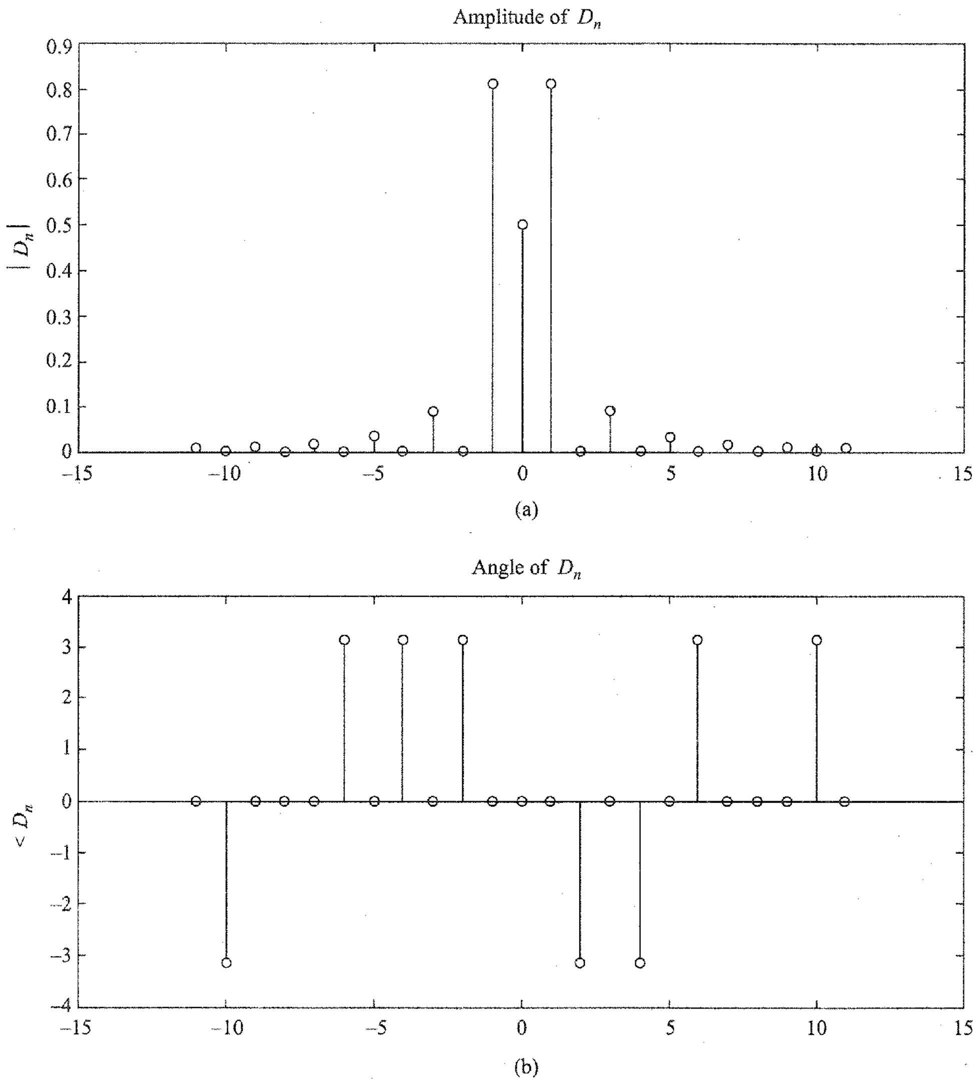


Figure 2.22 Exponential Fourier series coefficients of a repeated $\Delta(t/2)$ with period $T = 4$.

```
% now calculate D_0 and store it in D(N+1);
    Func= @(t) funct_tri(t/2);
    D(N+1)=1/T*quad(Func,a,b,tol);  % Using quad.m integration
for i=1:N
% Calculate Dn for n=1,...,N (stored in D(N+2) ... D(2N+1)
    Func= @(t) exp(-j*2*pi*t*i/T).*funct_tri(t/2);
    D(i+N+1)=quad(Func,a,b,tol);
% Calculate Dn for n=-N,...,-1 (stored in D(1) ... D(N)
    Func= @(t) exp(j*2*pi*t*(N+1-i)/T).*func_tri(t/2);
    D(i)= quad(Func,a,b,tol);
end
figure(1);
subplot(211);s1=stem([-N:N],abs(D));
set(s1,'Linewidth',2); ylabel('|{\it D}_{\it n}|');
```

```
title('Amplitude of {\it D}_{\it n}')
subplot(212);s2=stem([-N:N],angle(D));
set(s2,'Linewidth',2); ylabel('<{\it D}_{\it n}');
title('Angle of {\it D}_{\it n}');
```

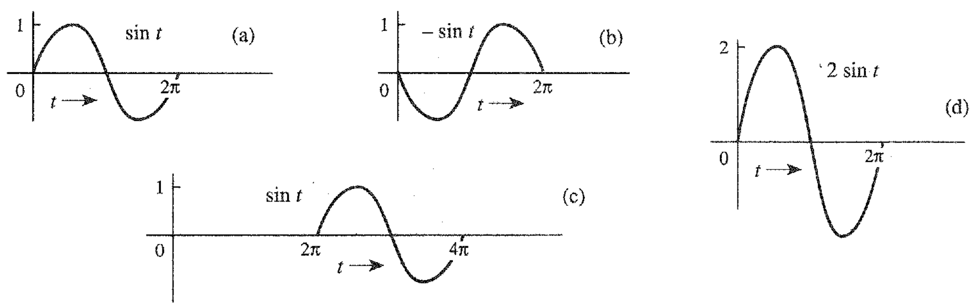
REFERENCES

1. P. L. Walker, *The Theory of Fourier Series and Integrals*, Wiley-Interscience, New York, 1986.
2. R. V. Churchill, and J. W. Brown, *Fourier Series and Boundary Value Problems*, 3rd ed., McGraw-Hill, New York, 1978.

PROBLEMS

2.1-1 Find the energies of the signals shown in Fig. P2.1-1. Comment on the effect on energy of sign change, time shift, or doubling of the signal. What is the effect on the energy if the signal is multiplied by k?

Figure P.2.1-1



2.1-2 **(a)** Find E_x and E_y, the energies of the signals $x(t)$ and $y(t)$ shown in Fig. P2.1-2a. Sketch the signals $x(t) + y(t)$ and $x(t) - y(t)$ and show that the energy of either of these two signals is equal to $E_x + E_y$. Repeat the procedure for signal pair in Fig. P2.1-2b.

(b) Now repeat the procedure for signal pair in Fig. P2.1-2c. Are the energies of the signals $x(t) + y(t)$ and $x(t) - y(t)$ identical in this case?

2.1-3 Find the power of a sinusoid $C \cos(\omega_0 t + \theta)$.

2.1-4 Show that if $\omega_1 = \omega_2$, the power of $g(t) = C_1 \cos(\omega_1 t + \theta_1) + C_2 \cos(\omega_2 t + \theta_2)$ is $[C_1{}^2 + C_2{}^2 + 2C_1C_2 \cos(\theta_1 - \theta_2)]/2$, which is not equal to $(C_1{}^2 + C_2{}^2)/2$.

2.1-5 Find the power of the periodic signal $g(t)$ shown in Fig. P2.1-5. Find also the powers and the rms values of **(a)** $-g(t)$ **(b)** $2g(t)$ **(c)** $cg(t)$. Comment.

2.1-6 Find the power and the rms value for the signals in **(a)** Fig. P2-1-6a; **(b)** Fig. 2.16; **(c)** Fig. P2-1-6b; **(d)** Fig. P2.7-4a; **(e)** Fig. P2.7-4c.

Figure P.2.1-2

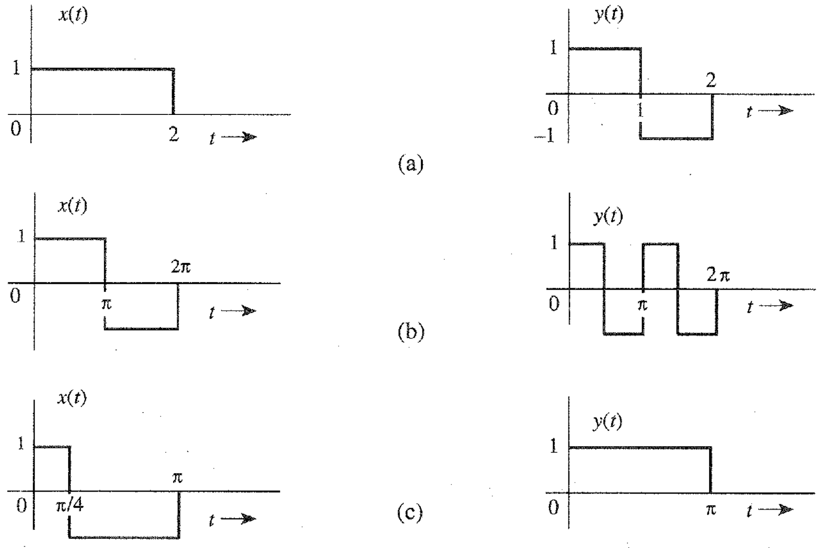


Figure P.2.1-5

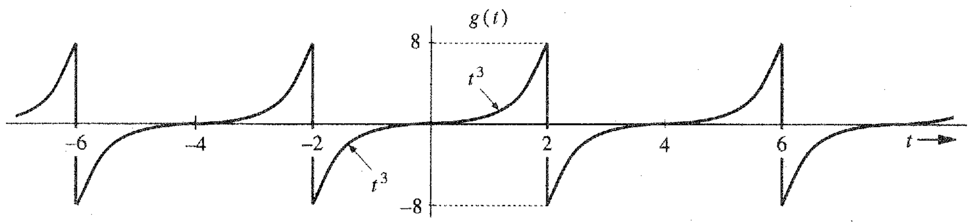
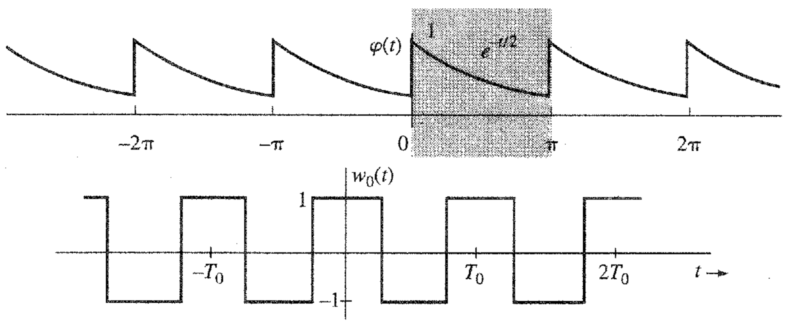


Figure P.2.1-6



2.1-7 Show that the power of a signal $g(t)$ given by

$$g(t) = \sum_{k=m}^{n} D_k e^{j\omega_k t} \qquad \omega_i \neq \omega_k \text{ for all } i \neq k$$

is (Parseval's theorem)

$$P_g = \sum_{k=m}^{n} |D_k|^2$$

2.1-8 Determine the power and the rms value for each of the following signals:

(a) $10\cos\left(100t + \frac{\pi}{3}\right)$ (d) $10\cos 5t \cos 10t$

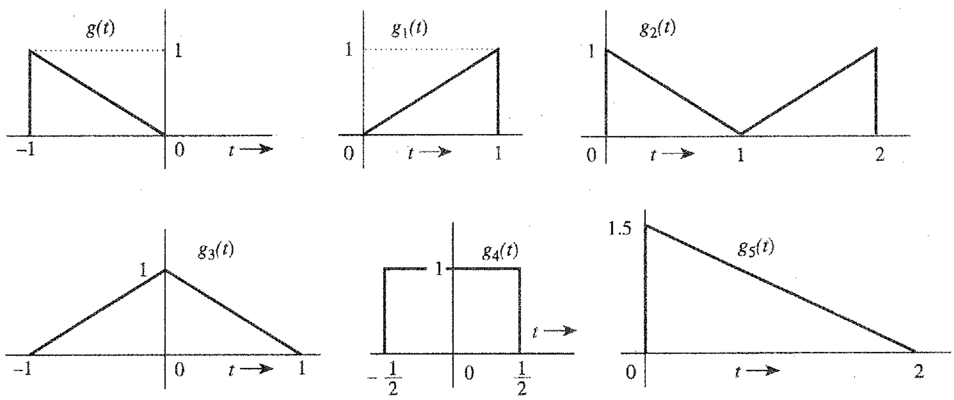
(b) $10\cos\left(100t + \frac{\pi}{3}\right) + 16\sin\left(150t + \frac{\pi}{5}\right)$ (e) $10\sin 5t \cos 10t$

(c) $(10 + 2\sin 3t)\cos 10t$ (f) $e^{j\alpha t}\cos\omega_0 t$

2.2-1 Show that an exponential e^{-at} starting at $-\infty$ is neither an energy nor a power signal for any real value of a. However, if a is imaginary, it is a power signal with power $P_g = 1$ regardless of the value of a.

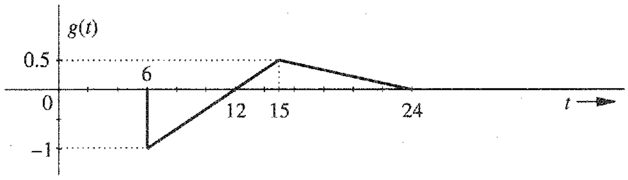
2.3-1 In Fig. P2.3-1, the signal $g_1(t) = g(-t)$. Express signals $g_2(t)$, $g_3(t)$, $g_4(t)$, and $g_5(t)$ in terms of signals $g(t)$, $g_1(t)$, and their time-shifted, time-scaled, or time-inverted versions. For instance, $g_2(t) = g(t-T) + g_1(t-T)$ for some suitable value of T. Similarly, both $g_3(t)$ and $g_4(t)$ can be expressed as $g(t-T) + g(t-T)$ for some suitable value of T. In addition, $g_5(t)$ can be expressed as $g(t)$ time-shifted, time-scaled, and then multiplied by a constant.

Figure P.2.3-1



2.3-2 For the signal $g(t)$ shown in Fig. P2.3-2, sketch the following signals: (a) $g(-t)$; (b) $g(t+6)$; (c) $g(3t)$; (d) $g(6-t)$.

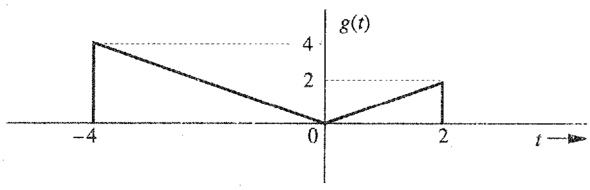
Figure P.2.3-2



2.3-3 For the signal $g(t)$ shown in Fig. P2.3-3, sketch (a) $g(t-4)$; (b) $g(t/1.5)$; (c) $g(2t-4)$; (d) $g(2-t)$.
Hint: Recall that replacing t with $t-T$ delays the signal by T. Thus, $g(2t-4)$ is $g(2t)$ with t replaced by $t-2$. Similarly, $g(2-t)$ is $g(-t)$ with t replaced by $t-2$.

2.3-4 For an energy signal $g(t)$ with energy E_g, show that the energy of any one of the signals $-g(t)$, $g(-t)$, and $g(t-T)$ is E_g. Show also that the energy of $g(at)$ as well as $g(at-b)$ is E_g/a. This shows that time inversion and time shifting do not affect signal energy. On the other

Figure P.2.3-3



hand, time compression of a signal by a factor a reduces the energy by the factor a. What is the effect on signal energy if the signal is **(a)** time-expanded by a factor a $(a > 1)$ and **(b)** multiplied by a constant a?

2.3-5 Simplify the following expressions:

(a) $\left(\frac{\tan t}{2t^2+1}\right)\delta(t)$

(b) $\left(\frac{j\omega-3}{\omega^2+9}\right)\delta(\omega)$

(c) $\left[e^{-t}\cos(3t-\pi/3)\right]\delta(t+\pi)$

(d) $\left(\frac{\sin \pi(t+2)}{t^2-4}\right)\delta(t-1)$

(e) $\left(\frac{\cos(\pi t)}{t+2}\right)\delta(2t+3)$

(f) $\left(\frac{\sin k\omega}{\omega}\right)\delta(\omega)$

Hint: Use Eq. (2.10b). For part **(f)** use L'Hospital's rule.

2.3-6 Evaluate the following integrals:

(a) $\int_{-\infty}^{\infty} g(\tau)\delta(t-\tau)\,d\tau$

(b) $\int_{-\infty}^{\infty} \delta(\tau)g(t-\tau)\,d\tau$

(c) $\int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}\,dt$

(d) $\int_{-\infty}^{1} \delta(t-2)\sin \pi t\,dt$

(e) $\int_{-2}^{\infty} \delta(3+t)e^{-t}\,dt$

(f) $\int_{-2}^{2} (t^3+4)\delta(1-t)\,dt$

(g) $\int_{-\infty}^{\infty} g(2-t)\delta(3-t)\,dt$

(h) $\int_{-\infty}^{\infty} e^{(x-1)}\cos\frac{\pi}{2}(x-5)\delta(2x-3)\,dx$

Hint: $\delta(x)$ is located at $x=0$. For example, $\delta(1-t)$ is located at $1-t=0$; that is, at $t=1$, and so on.

2.3-7 Prove that

$$\delta(at) = \frac{1}{|a|}\delta(t)$$

Hence show that

$$\delta(\omega) = \frac{1}{2\pi}\delta(f) \qquad \text{where} \quad \omega = 2\pi f$$

Hint: Show that

$$\int_{-\infty}^{\infty} \phi(t)\delta(at)\,dt = \frac{1}{|a|}\phi(0)$$

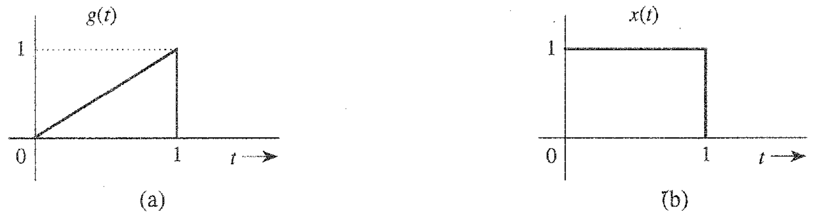
2.4-1 Derive Eq. (2.19) in an alternate way by observing that $\mathbf{e} = (\mathbf{g} - c\,\mathbf{x})$, and

$$|\mathbf{e}|^2 = (\mathbf{g} - c\,\mathbf{x}) \cdot (\mathbf{g} - c\,\mathbf{x}) = |\mathbf{g}|^2 + c^2|\mathbf{x}|^2 - 2c\mathbf{g}\cdot\mathbf{x}$$

To minimize $|\mathbf{e}|^2$, equate its derivative with respect to c to zero.

2.4-2 For the signals $g(t)$ and $x(t)$ shown in Fig. P2.4-2, find the component of the form $x(t)$ contained in $g(t)$. In other words, find the optimum value of c in the approximation $g(t) \approx cx(t)$ so that the error signal energy is minimum. What is the resulting error signal energy?

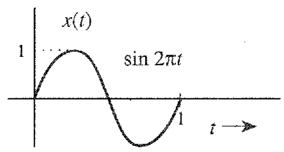
Figure P.2.4-2



2.4-3 For the signals $g(t)$ and $x(t)$ shown in Fig. P2.4-2, find the component of the form $g(t)$ contained in $x(t)$. In other words, find the optimum value of c in the approximation $x(t) \approx cg(t)$ so that the error signal energy is minimum. What is the resulting error signal energy?

2.4-4 Repeat Prob. 2.4-2 if $x(t)$ is a sinusoid pulse shown in Fig. P2.4-4.

Figure P.2.4-4



2.4-5 The Energies of the two energy signals $x(t)$ and $y(t)$ are E_x and E_y, respectively.

(a) If $x(t)$ and $y(t)$ are orthogonal, then show that the energy of the signal $x(t) + y(t)$ is identical to the energy of the signal $x(t) - y(t)$, and is given by $E_x + E_y$.

(b) if $x(t)$ and $y(t)$ are orthogonal, find the energies of signals $c_1x(t) + c_2y(t)$ and $c_1x(t) - c_2y(t)$.

(c) We define E_{xy}, the cross-energy of the two energy signals $x(t)$ and $y(t)$, as

$$E_{xy} = \int_{-\infty}^{\infty} x(t)y^*(t)\,dt$$

If $z(t) = x(t) \pm y(t)$, then show that

$$E_z = E_x + E_y \pm (E_{xy} + E_{yx})$$

2.4-6 Let $x_1(t)$ and $x_2(t)$ be two unit energy signals orthogonal over an interval from $t = t_1$ to t_2. Signals $x_1(t)$ and $x_2(t)$ are unit energy, orthogonal signals; we can represent them by two unit length, orthogonal vectors $(\mathbf{x}_1, \mathbf{x}_2)$. Consider a signal $g(t)$ where

$$g(t) = c_1x_1(t) + c_2x_2(t) \qquad t_1 \le t \le t_2$$

This signal can be represented as a vector $\mathbf{g}$ by a point (c_1, c_2) in the $x_1 - x_2$ plane.

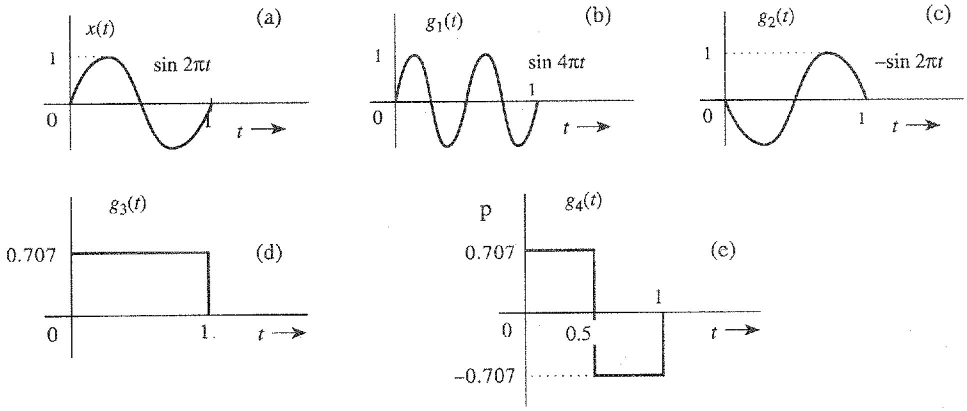
(a) Determine the vector representation of the following six signals in this two-dimensional vector space:

(i) $g_1(t) = 2x_1(t) - x_2(t)$	**(iv)** $g_4(t) = x_1(t) + 2x_2(t)$
(ii) $g_2(t) = -x_1(t) + 2x_2(t)$	**(v)** $g_5(t) = 2x_1(t) + x_2(t)$
(iii) $g_3(t) = -x_2(t)$	**(vi)** $g_6(t) = 3x_1(t)$

(b) Point out pairs of mutually orthogonal vectors among these six vectors. Verify that the pairs of signals corresponding to these orthogonal vectors are also orthogonal.

2.5-1 Find the correlation coefficient c_n of signal $x(t)$ and each of the four pulses $g_1(t)$, $g_2(t)$, $g_3(t)$, and $g_4(t)$ shown in Fig. P2.5-1. To provide maximum margin against the noise along the transmission path, which pair of pulses would you select for a binary communication?

Figure P.2.5-1



2.7-1 **(a)** Sketch the signal $g(t) = t^2$ and find the exponential Fourier series to represent $g(t)$ over the interval $(-1, 1)$. Sketch the Fourier series $\varphi(t)$ for all values of t.

(b) Verify Parseval's theorem [Eq. (2.68a)] for this case, given that

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

2.7-2 **(a)** Sketch the signal $g(t) = t$ and find the exponential Fourier series to represent $g(t)$ over the interval $(-\pi, \pi)$. Sketch the Fourier series $\varphi(t)$ for all values of t.

(b) Verify Parseval's theorem [Eq. (2.68a)] for this case, given that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

2.7-3 If a periodic signal satisfies certain symmetry conditions, the evaluation of the Fourier series coefficients is somewhat simplified.

(a) Show that if $g(t) = g(-t)$ (even symmetry), then the coefficients of the exponential Fourier series are real.

(b) Show that if $g(t) = -g(-t)$ (odd symmetry), the coefficients of the exponential Fourier series are imaginary.

(c) Show that in each case, the Fourier coefficients can be evaluated by integrating the periodic signal over the half-cycle only. This is because the entire information of one cycle is implicit in a half-cycle owing to symmetry.

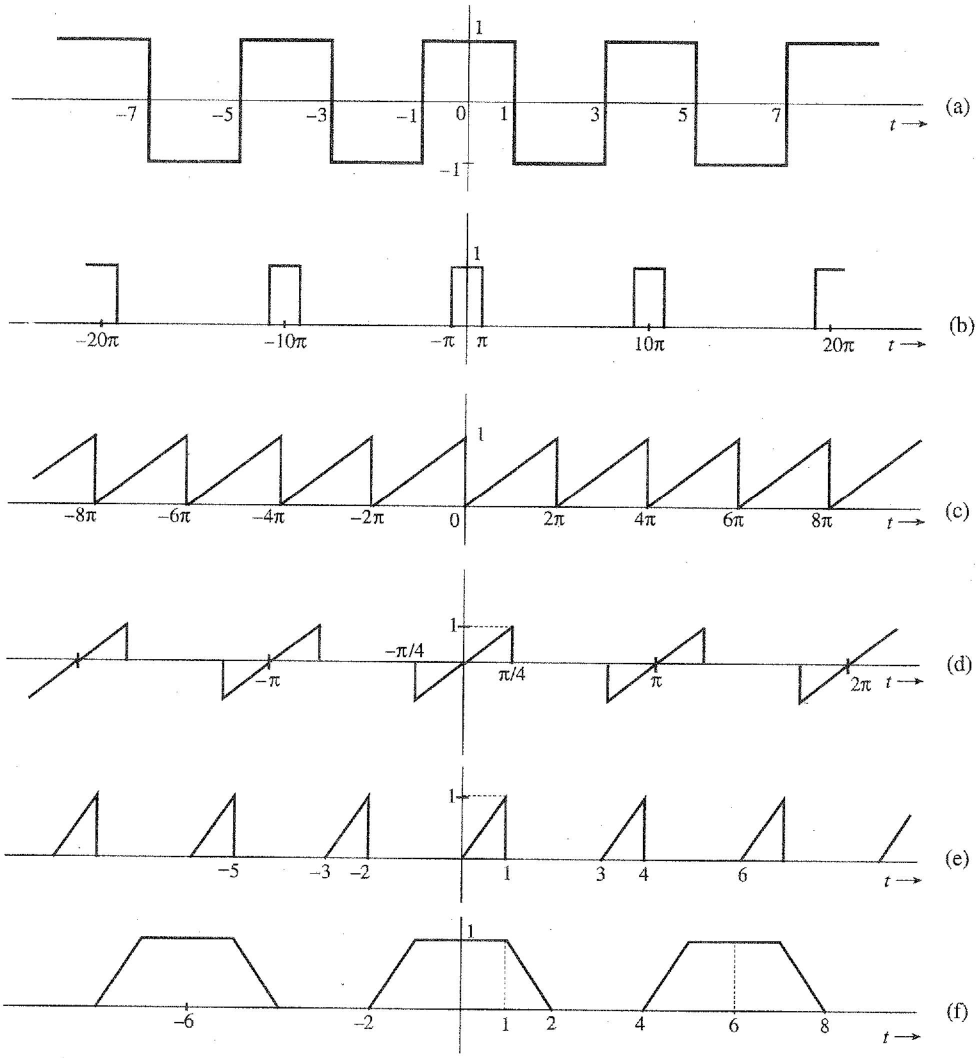
Hint: If $g_e(t)$ and $g_o(t)$ are even and odd functions, respectively, of t, then (assuming no impulse or its derivative at the origin),

$$\int_{-a}^{a} g_e(t)\,dt = \int_{0}^{2a} g_e(t)\,dt \quad \text{and} \quad \int_{-a}^{a} g_o(t)\,dt = 0$$

Also, the product of an even and an odd function is an odd function, the product of two odd functions is an even function, and the product of two even functions is an even function.

2.7-4 For each of the periodic signals shown in Fig. P2.7-4, find the exponential Fourier series and sketch the amplitude and phase spectra. Note any symmetric property.

Figure P.2.7-4



2.7-5 **(a)** Show that an arbitrary function $g(t)$ can be expressed as a sum of an even function $g_e(t)$ and an odd function $g_o(t)$:

$$g(t) = g_e(t) + g_o(t)$$

$$\textit{Hint}: \quad g(t) = \underbrace{\frac{1}{2}\Big[g(t) + g(-t)\Big]}_{g_e(t)} + \underbrace{\frac{1}{2}\Big[g(t) - g(-t)\Big]}_{g_o(t)}$$

(b) Determine the odd and even components of the following functions: **(i)** $u(t)$; **(ii)** $e^{-at}u(t)$; **(iii)** e^{jt}.

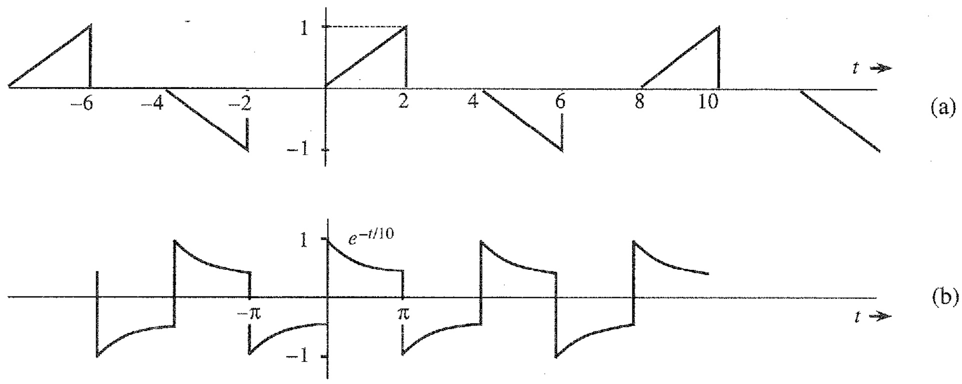
2.7-6 **(a)** If the two halves of one period of a periodic signal are of identical shape except that one is the negative of the other, the periodic signal is said to have a **half-wave symmetry**. If a periodic signal $g(t)$ with a period T_0 satisfies the half-wave symmetry condition, then

$$g\left(t - \frac{T_0}{2}\right) = -g(t)$$

In this case, show that all the even-numbered harmonics (coefficients) vanish.

(b) Use this result to find the Fourier series for the periodic signals in Fig. P2.7-6.

Figure P.2.7-6



2.8-1 A periodic signal $g(t)$ is expressed by the following Fourier series:

$$g(t) = 3\sin t + \cos\left(3t - \frac{2\pi}{3}\right) + 2\cos\left(8t + \frac{\pi}{3}\right)$$

(a) By applying Euler's identities on the signal $g(t)$ directly, write the exponential Fourier series for $g(t)$.

(b) By applying Euler's identities on the signal $g(t)$ directly, sketch the exponential Fourier series spectra.