

# Chapter 11: Rolling, Torque, and Angular Momentum

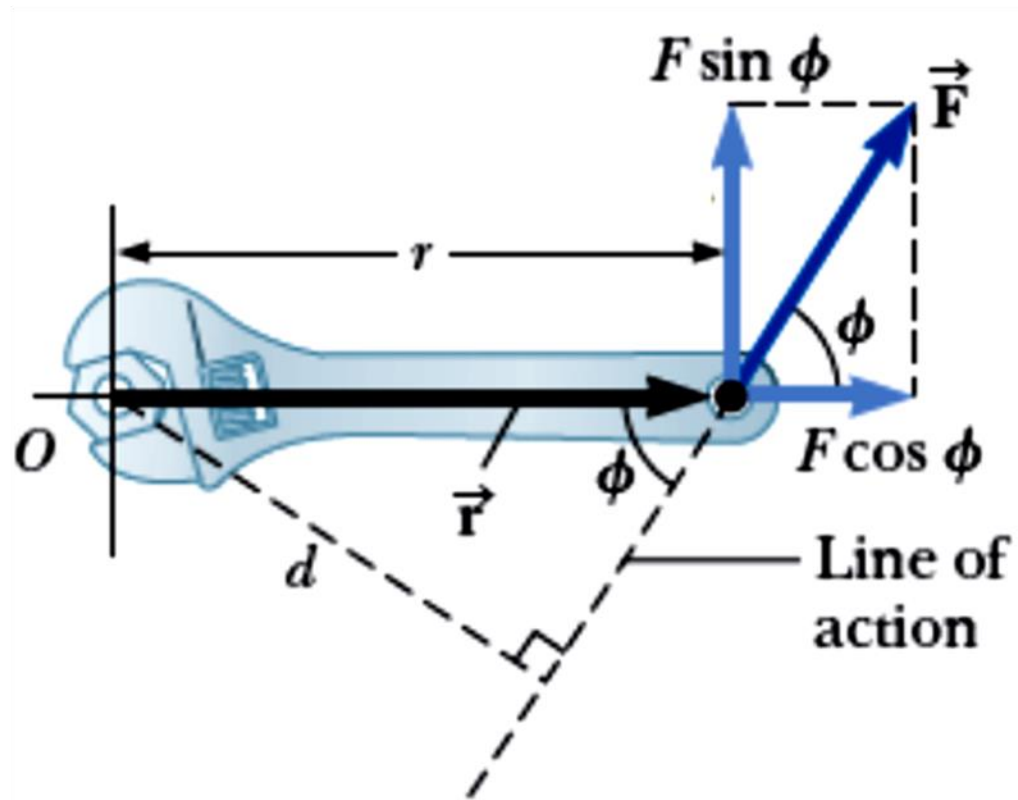
- ✓ **Rolling**
- ✓ **Torque**
- ✓ **Angular Momentum**
- ✓ **Newton's Second Law in Angular Form**
- ✓ **Conservation of Angular Momentum**

# Chapter 11: Rolling, Torque, and Angular Momentum

## Session 24:

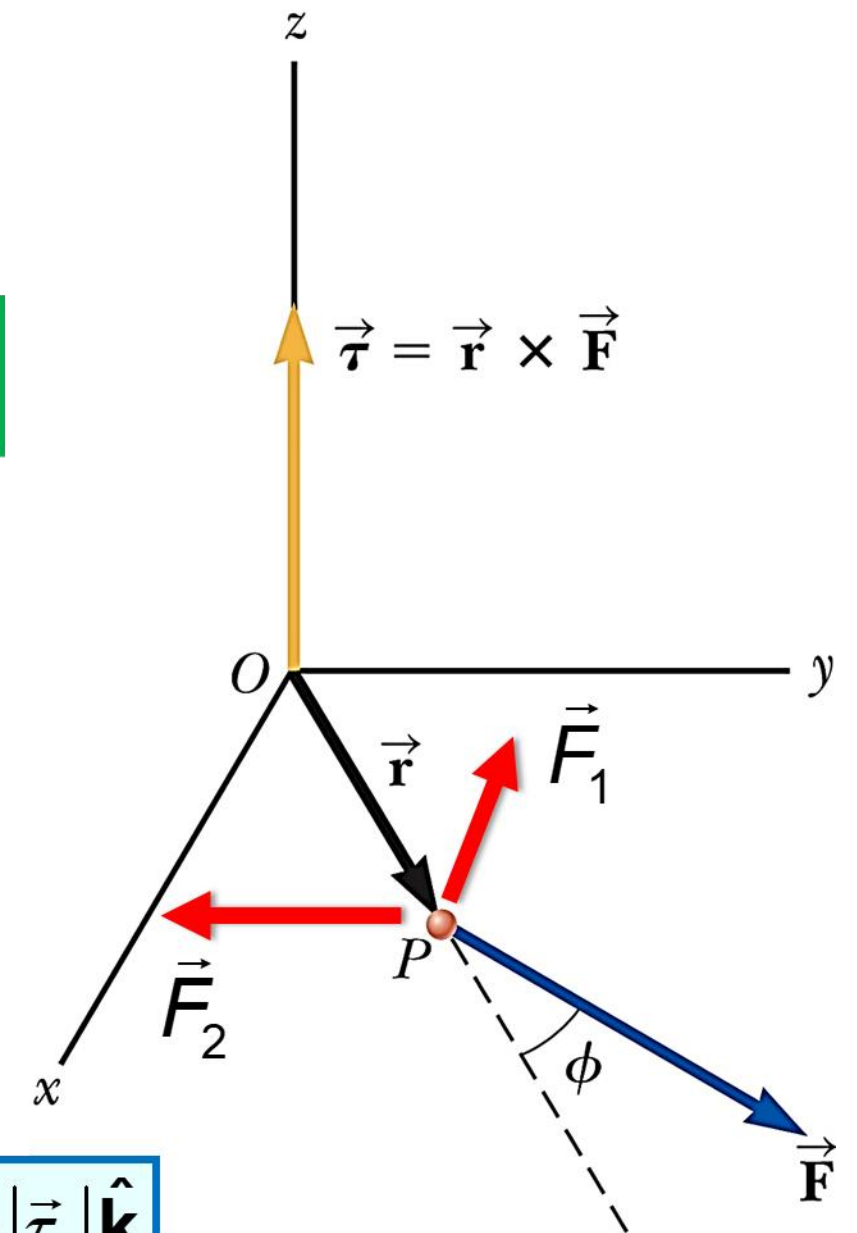
- ✓ **Torque**
- ✓ **Angular Momentum**
- ✓ **Examples**

# Torque



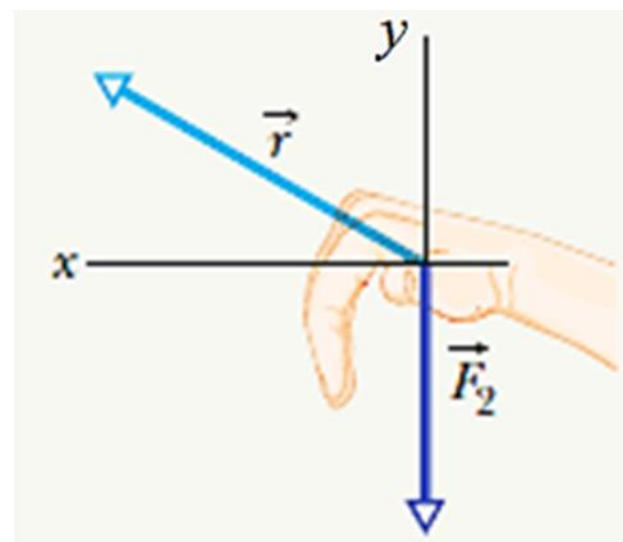
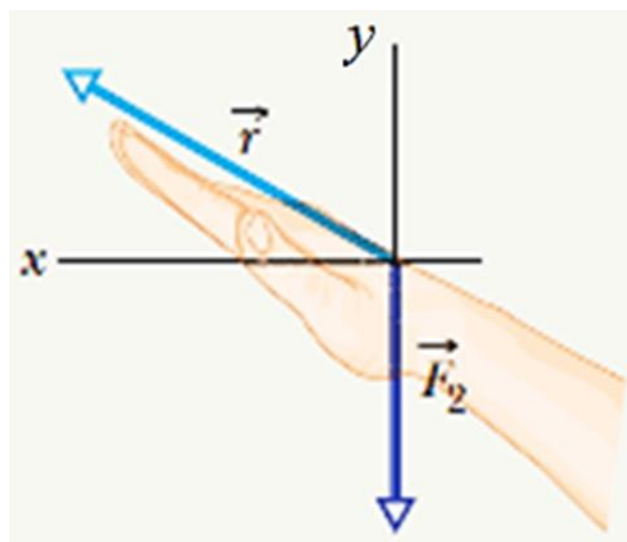
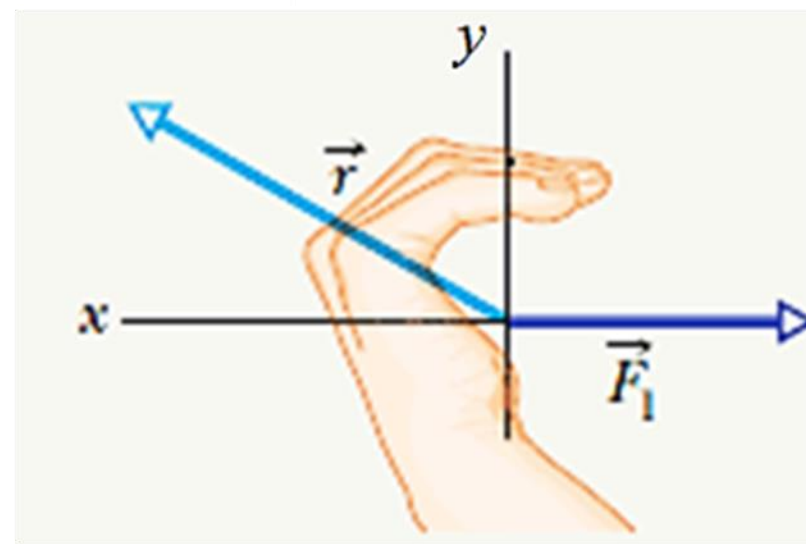
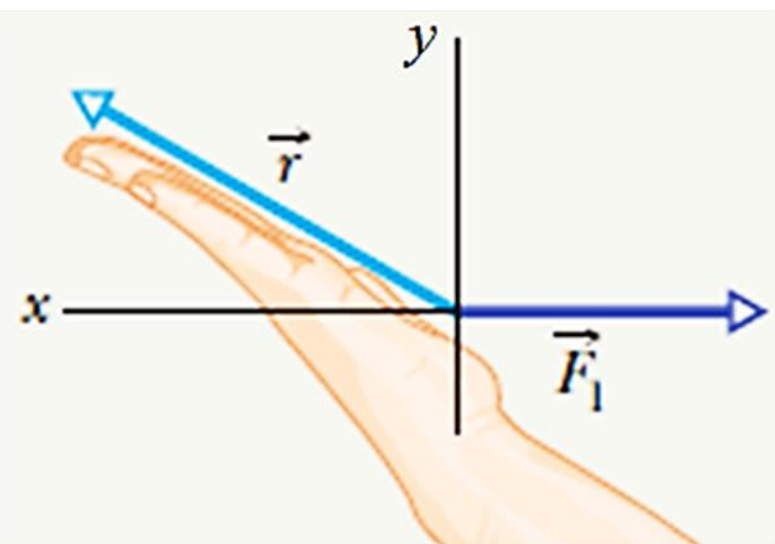
$$\tau = r F \sin \phi$$

$$\vec{\tau} = \vec{r} \times \vec{F}$$



$$\vec{\tau}_1 = |\vec{\tau}_1| \hat{\mathbf{k}}$$

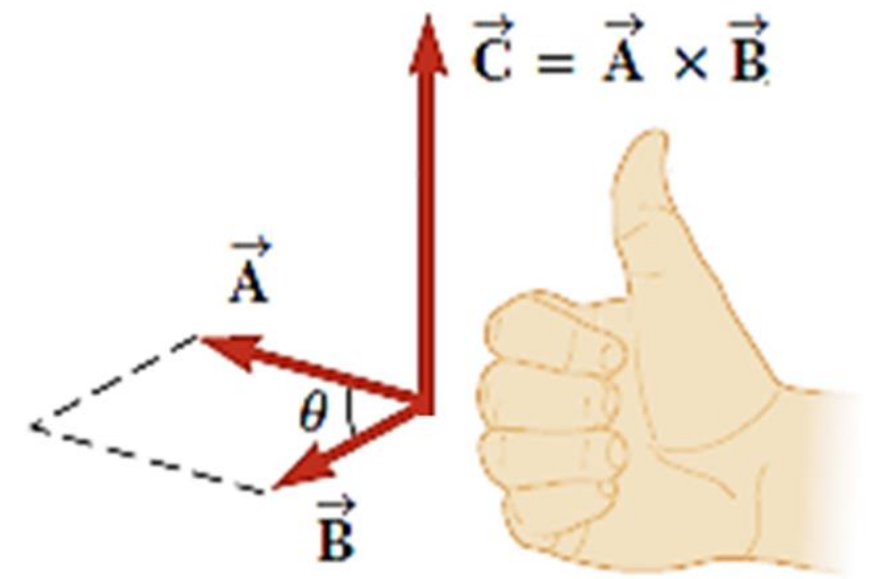
$$\vec{\tau}_2 = |\vec{\tau}_2| (-\hat{\mathbf{k}})$$



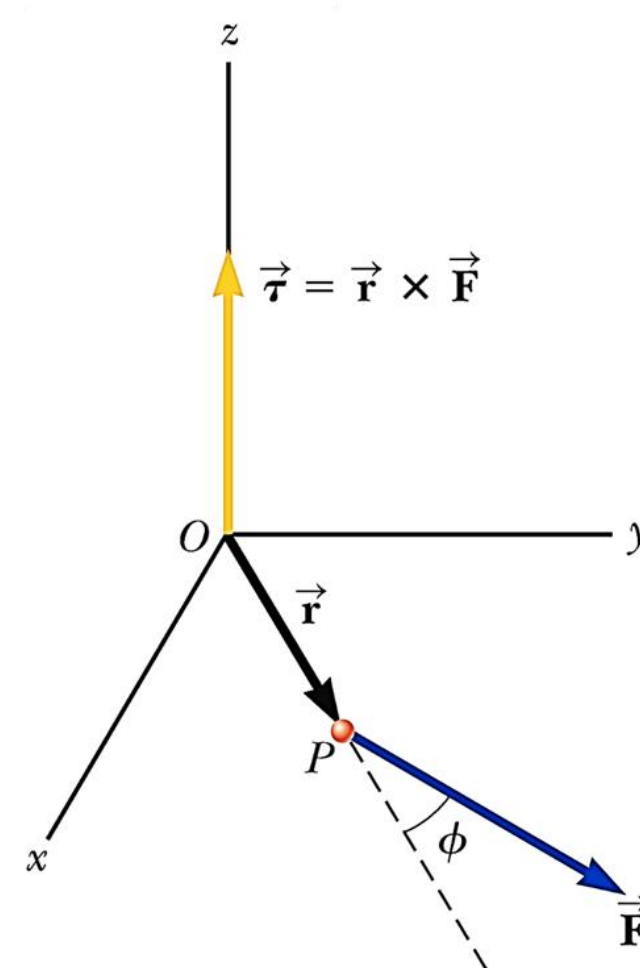
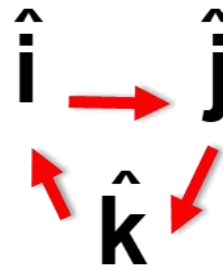
# Torque

$$\vec{C} = \vec{A} \times \vec{B}$$

$$|\vec{A} \times \vec{B}| = AB \sin \theta$$



$$\begin{aligned}\hat{i} \times \hat{i} &= \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0 \\ \hat{i} \times \hat{j} &= \hat{k} \\ \hat{j} \times \hat{k} &= \hat{i} \\ \hat{k} \times \hat{i} &= \hat{j}\end{aligned}$$



$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} \hat{i} - \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix} \hat{j} + \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \hat{k}$$

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \hat{i} - (A_x B_z - A_z B_x) \hat{j} + (A_x B_y - A_y B_x) \hat{k}$$

**Ex 5:** A force of  $\mathbf{F} = 2\mathbf{i} + 3\mathbf{j}$  N is applied to an object that is pivoted about a fixed axis aligned along the **z coordinate axis**. The force is applied at a point located at  $\mathbf{r} = 4\mathbf{i} + 5\mathbf{j}$  m. Find the torque applied to the object.

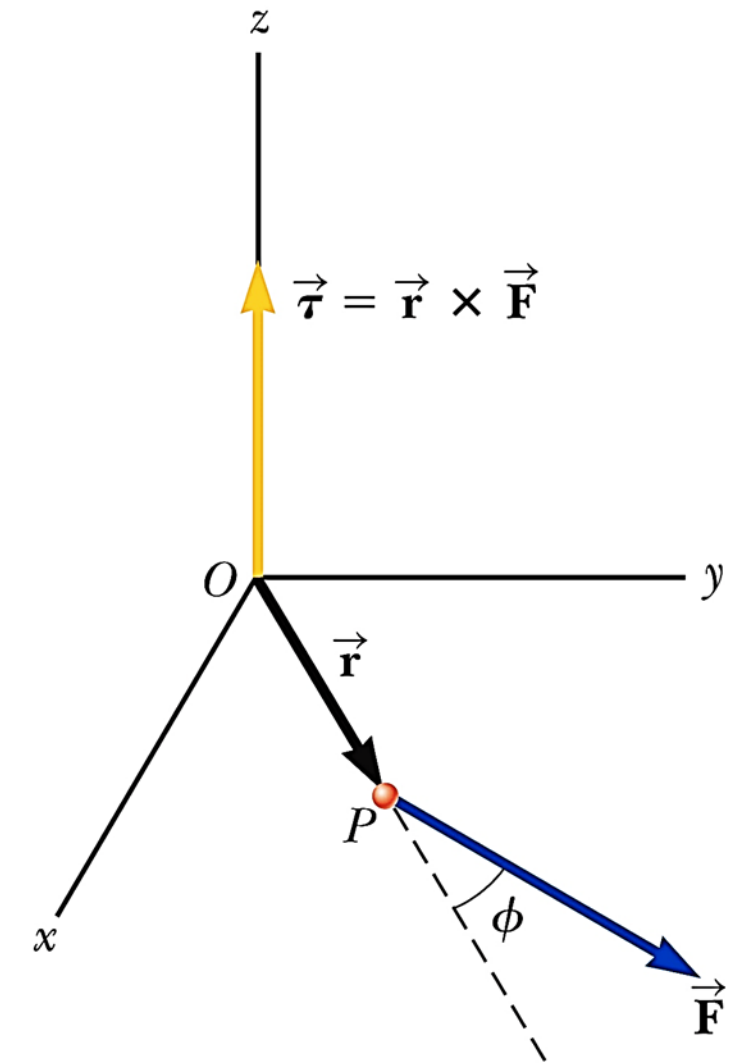
$$\vec{\tau} = \vec{r} \times \vec{F}$$

$$\vec{\tau} = (4\hat{\mathbf{i}} + 5\hat{\mathbf{j}}) \times (2\hat{\mathbf{i}} + 3\hat{\mathbf{j}})$$

$$\vec{\tau} = (4\hat{\mathbf{i}} \times 2\hat{\mathbf{i}}) + (4\hat{\mathbf{i}} \times 3\hat{\mathbf{j}}) + (5\hat{\mathbf{j}} \times 2\hat{\mathbf{i}}) + (5\hat{\mathbf{j}} \times 3\hat{\mathbf{j}})$$

$$\vec{\tau} = 0 + 12 \underbrace{(\hat{\mathbf{i}} \times \hat{\mathbf{j}})}_{\hat{\mathbf{k}}} + 10 \underbrace{(\hat{\mathbf{j}} \times \hat{\mathbf{i}})}_{-\hat{\mathbf{k}}} + 0$$

$$\vec{\tau} = 2\hat{\mathbf{k}} \text{ N.m}$$



**Ex 6: (Problem 11. 24 Halliday)**

In unit-vector notation, what is the torque about the origin on a jar of jalapeño peppers located at coordinates **(3 m, -2 m, 4 m)** due to (a) force  $\vec{F}_1 = 3\hat{i} - 4\hat{j} + 5\hat{k}$  (N) b) force  $\vec{F}_2 = -3\hat{i} - 4\hat{j} - 5\hat{k}$  (N) and (c) the vector sum of  $\vec{F}_1$  and  $\vec{F}_2$  ? (d) Repeat part (c) for the torque about the point with coordinates **(3 m, 2 m, 4 m)**.

$$\vec{\tau}_1 = \vec{r} \times \vec{F}_1 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 4 \\ 3 & -4 & 5 \end{vmatrix} = (-10 + 16)\hat{i} - (15 - 12)\hat{j} + (-12 + 6)\hat{k} = 6\hat{i} - 3\hat{j} - 6\hat{k}$$

$$\vec{\tau}_2 = \vec{r} \times \vec{F}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & -2 & 4 \\ -3 & -4 & -5 \end{vmatrix} = (10 + 16)\hat{i} - (-15 + 12)\hat{j} + (-12 - 6)\hat{k} = 26\hat{i} + 3\hat{j} - 18\hat{k}$$

$$\vec{\tau}_{net} = \vec{r} \times (\vec{F}_1 + \vec{F}_2) = (3\hat{i} - 2\hat{j} + 4\hat{k}) \times (-8\hat{j}) = 32\hat{i} - 24\hat{k}$$

$$\vec{r}' = \vec{r} - \vec{r}_0 = (3\hat{i} - 2\hat{j} + 4\hat{k}) - (3\hat{i} + 2\hat{j} + 4\hat{k}) = -4\hat{j}$$

$$\vec{\tau}_{net} = \vec{r}' \times (\vec{F}_1 + \vec{F}_2) = (-4\hat{j}) \times (-8\hat{j}) = 0$$

# Angular Momentum

$$\vec{\mathbf{F}}_{net} = \frac{d\vec{\mathbf{p}}}{dt}$$

$$\vec{\mathbf{p}} = m \vec{\mathbf{v}}$$

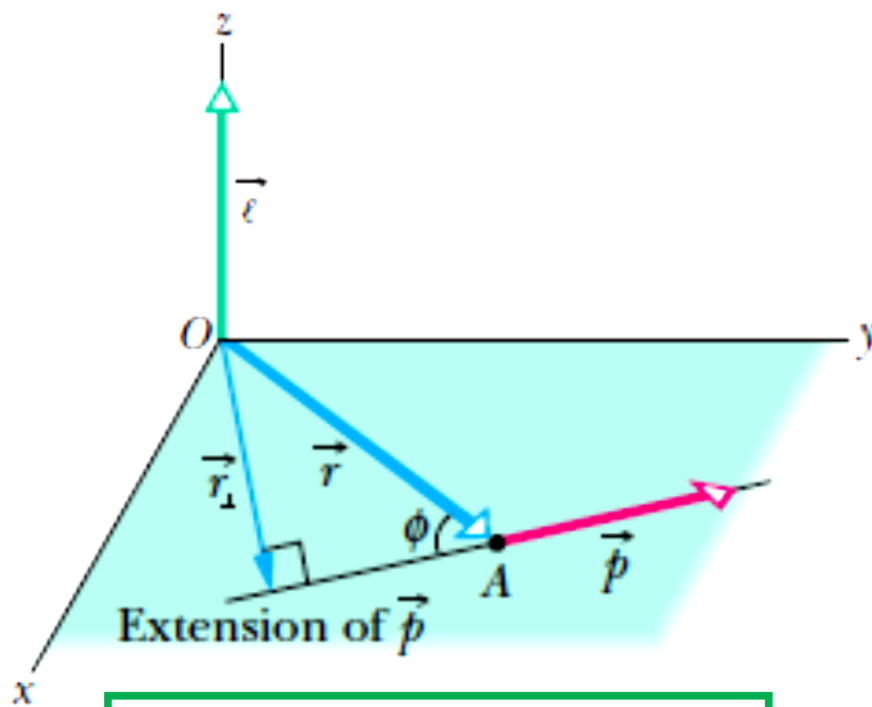
Linear Momentum (kg.m/s)

$$\vec{\tau}_{net} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}_{net}$$

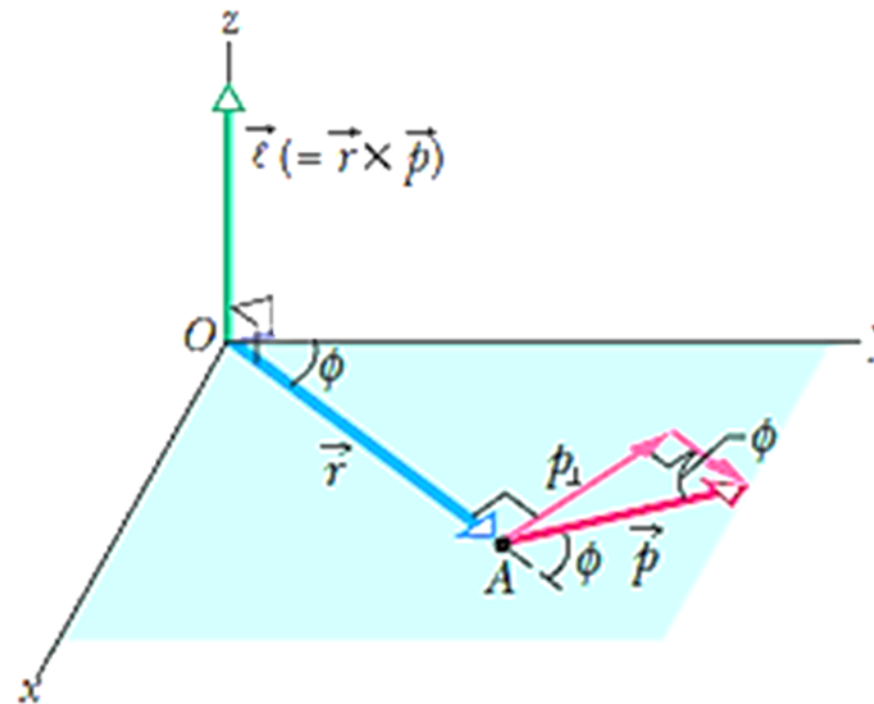
$$\vec{\mathbf{l}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}} = m(\vec{\mathbf{r}} \times \vec{\mathbf{v}})$$

Angular Momentum (kg.m<sup>2</sup>/s)

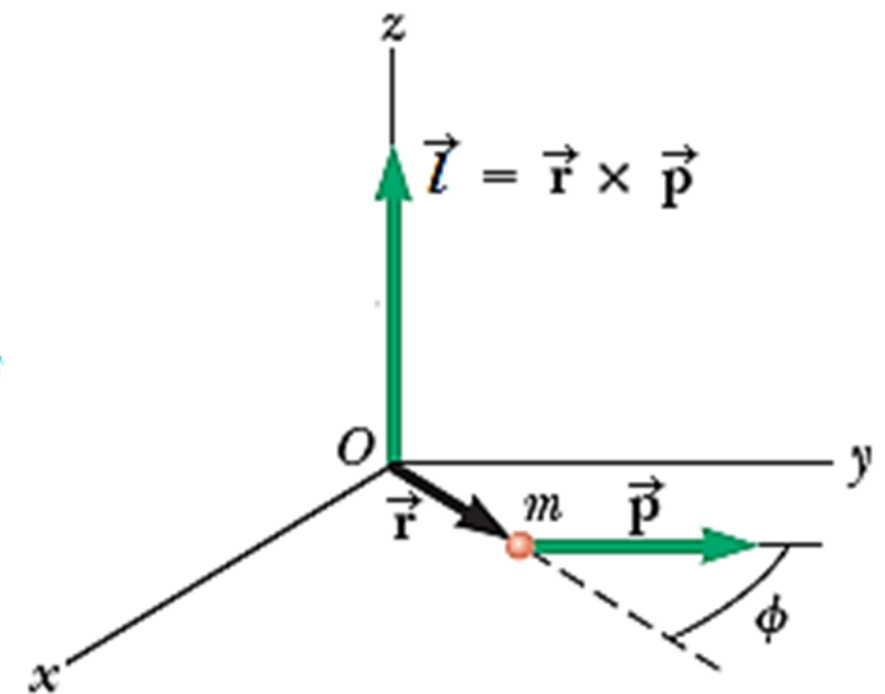
$$|\vec{\mathbf{l}}| = m r v \sin \phi$$



$$|\vec{\mathbf{l}}| = (r \sin \phi) m v = r_{\perp} p$$



$$|\vec{\mathbf{l}}| = r (m v \sin \phi) = r p_{\perp}$$



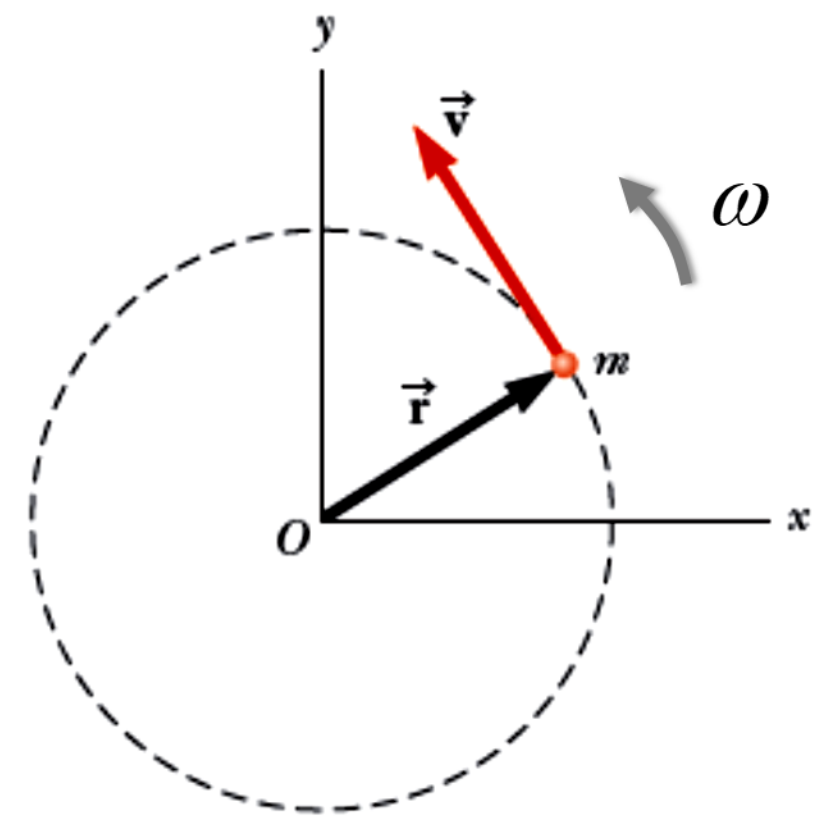
System of Particles:

$$\vec{\mathbf{L}}_{tot} = \vec{\mathbf{l}}_1 + \vec{\mathbf{l}}_2 + \dots + \vec{\mathbf{l}}_n = \sum_{i=1}^n \vec{\mathbf{l}}_i$$

**Ex 7:** Angular momentum of a particle in circular motion

$$\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v}) \quad \Rightarrow \quad \boxed{\vec{l} = mrv \hat{k}}$$

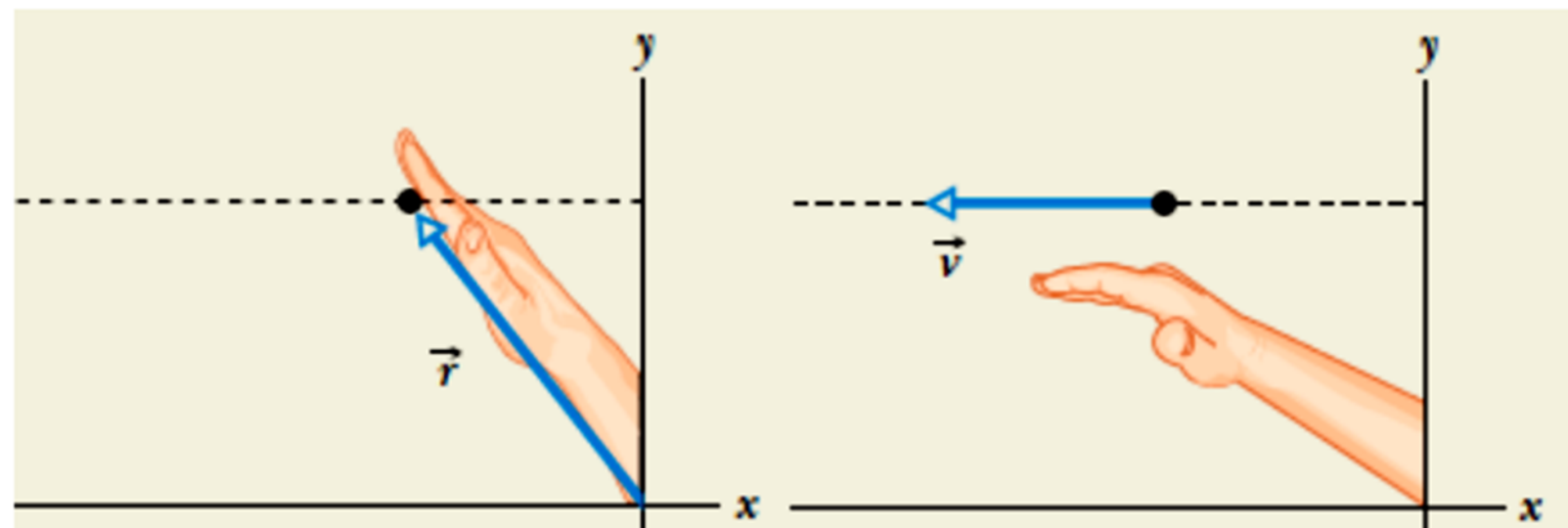
$$v = r\omega \quad \Rightarrow \quad \boxed{|\vec{l}| = mr^2\omega = I\omega}$$



**Ex 8:** The position vector of a particle of mass **2 kg** as a function of time is given by  $\mathbf{r} = -6t \mathbf{i} + 5 \mathbf{j}$ , where  $\mathbf{r}$  is in meters and  $t$  is in seconds. Determine the angular momentum of the particle about the origin as a function of time.

$$\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v}) \quad \Rightarrow \quad \boxed{\vec{l} = 2(-6t \hat{i} + 5 \hat{j}) \times (-6 \hat{i}) = 60 \hat{k} \text{ (kg.m}^2/\text{s)}}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d}{dt}(-6t \hat{i} + 5 \hat{j}) = -6 \hat{i}$$



**Ex 9:** Figure below shows an overhead view of two particles moving at constant momentum along horizontal paths. Particle 1, with momentum magnitude  $p_1 = 5 \text{ kg.m/s}$ , has position vector  $\mathbf{r}_1$  and will pass **2 m** from point O. Particle 2, with momentum magnitude  $p_2 = 2 \text{ kg.m/s}$ , has position vector  $\mathbf{r}_2$  and will pass **4 m** from point O. What are the magnitude and direction of the net angular momentum about point O of the two particle system?

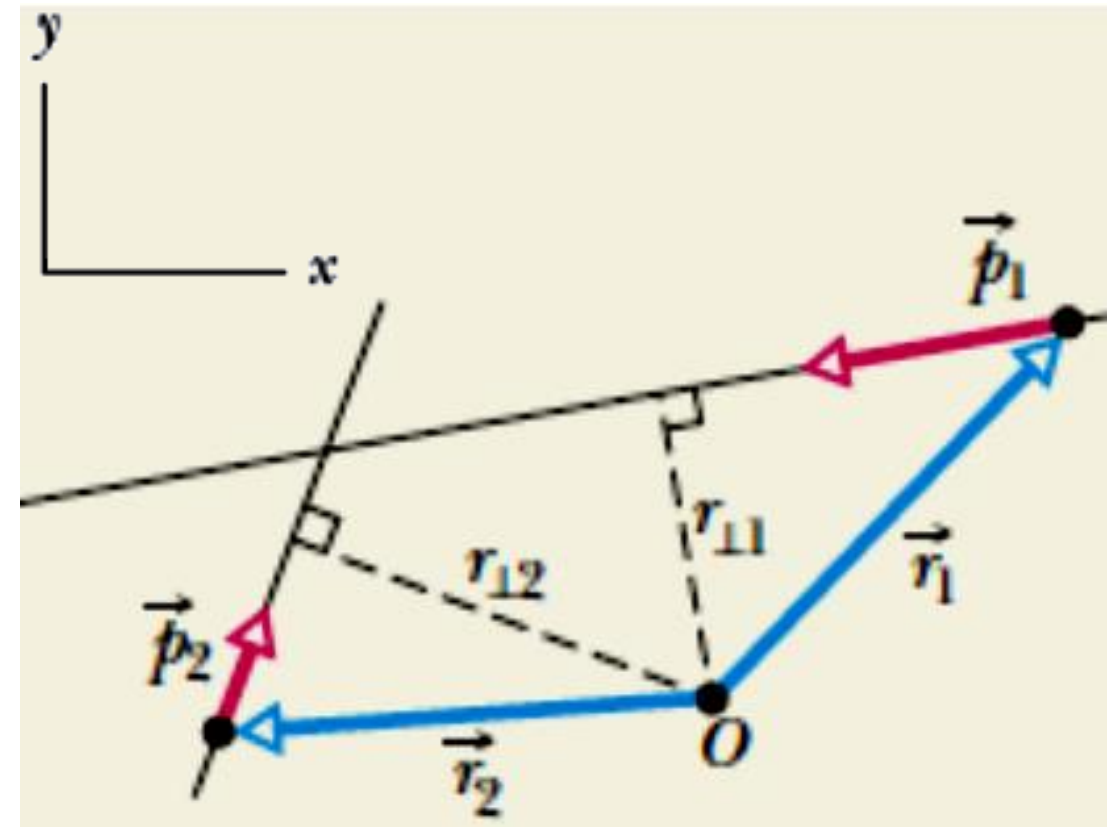
$$|\vec{l}| = r_{\perp} p$$

$$\left\{ \begin{array}{l} |\vec{l}_1| = r_{\perp 1} p_1 = (2)(5) = 10 \text{ kg.m}^2 / \text{s} \\ |\vec{l}_2| = r_{\perp 2} p_2 = (4)(2) = 8 \text{ kg.m}^2 / \text{s} \end{array} \right.$$

$$\left\{ \begin{array}{l} \vec{l}_1 = +10 \hat{\mathbf{k}} \\ \vec{l}_2 = -8 \hat{\mathbf{k}} \end{array} \right.$$

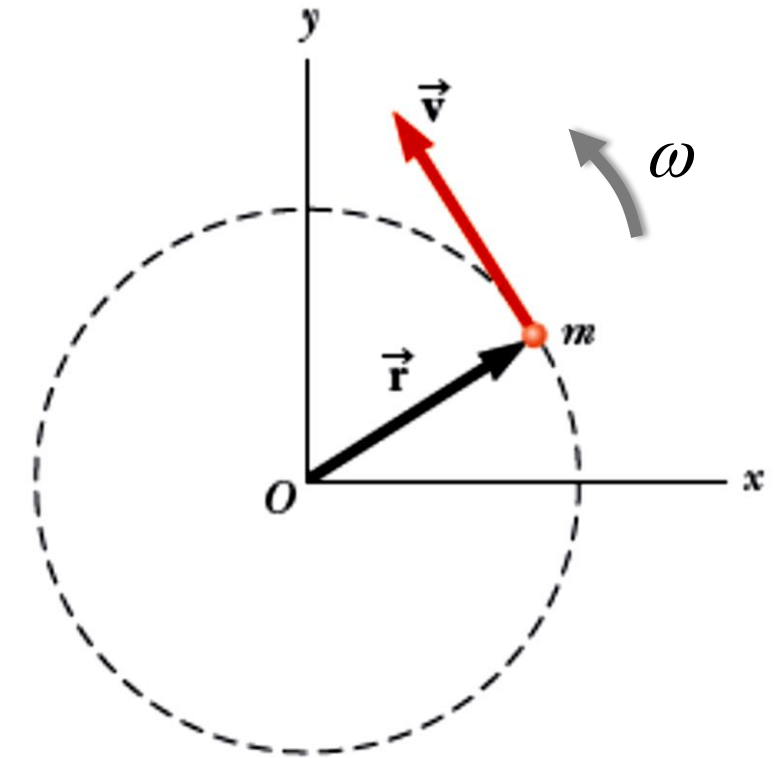


$$\vec{\mathbf{L}}_{tot} = \vec{l}_1 + \vec{l}_2 = +2 \hat{\mathbf{k}}$$



# Angular Momentum of a Rotating Rigid Object

$$\vec{l} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v}) \quad \Rightarrow \quad |\vec{l}| = mr^2\omega = I\omega$$

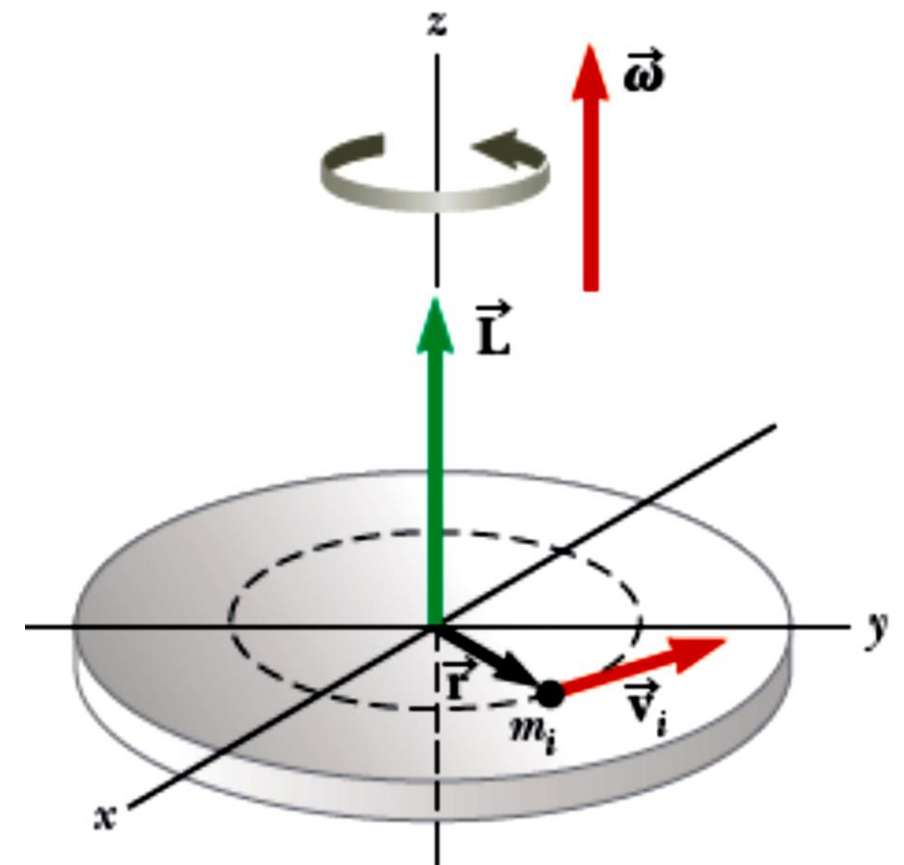


$$l_i = m_i r_i^2 \omega$$

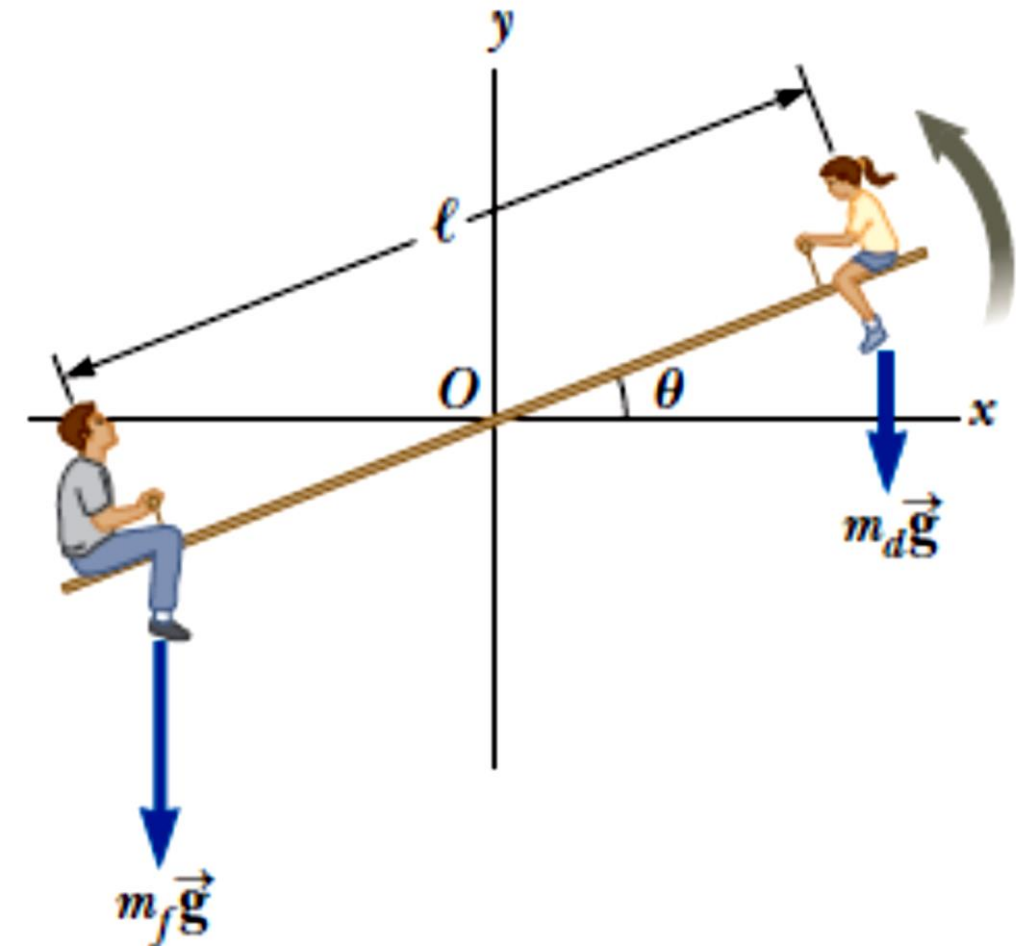
$$\mathbf{L} = \sum_i l_i = \sum_i m_i r_i^2 \omega = \left( \sum_i m_i r_i^2 \right) \omega$$

$$\mathbf{L} = I\omega$$

$$\vec{\mathbf{L}} = I\vec{\omega}$$



**Ex 10:** A father of mass  $m_f$  and his daughter of mass  $m_d$  sit on opposite ends of a seesaw at equal distances from the pivot at the center. The seesaw is modeled as a rigid rod of mass  $M$  and length  $l$ , and is pivoted without friction. At a given moment, the combination rotates in a vertical plane with an angular speed  $\omega$ . Find an expression for the magnitude of the system's angular momentum.



$$\mathbf{L} = I\omega$$

$$I = \frac{1}{12} M l^2 + m_f \left(\frac{l}{2}\right)^2 + m_d \left(\frac{l}{2}\right)^2 = \frac{l^2}{4} \left(\frac{M}{3} + m_f + m_d\right)$$

$$\mathbf{L} = \frac{l^2}{4} \left(\frac{M}{3} + m_f + m_d\right) \omega$$