# Photogrammetric Block Adjustment

## Chapter 4
## Additional Parameters & Self Calibration

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# Introduction

*The measured image or model coordinates are influenced by many source of errors. For the sake of simplicity, we consider them as two groups:*

- *Random errors*
- *Systematic errors*

*For example, the image coordinates should be corrected for both types:*

$$\bar{x} = x + v_x + \Delta x \qquad v_x, v_y : \text{Corrections for Random errors}$$
$$\bar{y} = y + v_y + \Delta y \qquad \Delta x, \Delta y : \text{Corrections for Systematic errors}$$

***Random errors** should be minimized through standard least squares adjustment procedure.*

***Systematic errors** should also be corrected one way or another. Nowadays, detection and elimination of the systematic errors in image or model coordinates is one of the main objectives of recent research work in aerial triangulation.*


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# Main sources of systematic errors

- **The photographic system**
  - *lens distortion*
    - *Radial distortion*
    - *Tangential distortion*
  - *Unflatness of film during exposure*
  - *Film deformation*
  - *Diapositives deformation*
- **Atmospheric refraction**
- **Instrumental errors**
- **Systematic errors of geodetic network**
- **Earth curvature**
  - *Correction of image coordinates*
  - *Correction of model coordinates for earth curvature*


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# Main sources of systematic errors

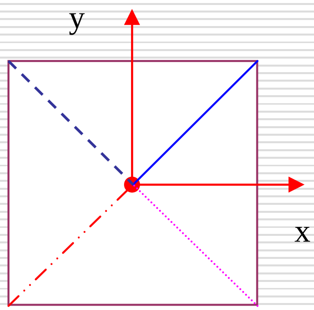
## The photographic system

### 1-Lens distortion:

*The lens distortion is usually measured along 4 semi-diagonals. The average of 4 curves is given as lens distortion. Types of Lens distortion:*

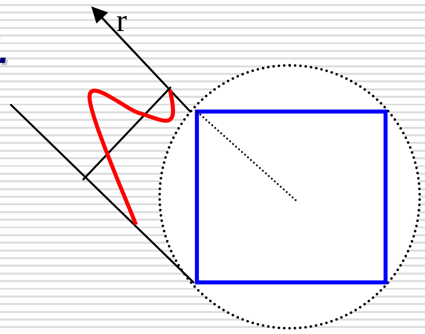
**a- Radial distortion**

*The magnitude of radial distortion is very small except at the corners.*




**b- Tangential distortion**

*Tangential distortion is neglected in most practical work.*







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# Main sources of systematic errors

## The photographic system

### 2- Unflatness of film during exposure:

Before each exposure the film must be flattened in the focal plan with the help of the vacuum plate.

### 3- Film deformation:

The film deformation depends to a great extend on the material from which the film base is made.

### 4- Diapositives deformation:

Preparation of diapositive is another phases, which can contribute to random and systematic errors.



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# Main sources of systematic errors

## Atmospheric Refraction

*Usually atmospheric refraction is corrected assuming standard atmosphere while the situation during photography can differ from one day to another. Turbulences and changes in temperature and pressure may add other errors. Atmospheric refraction cause radial distortion from the nadir point.*

## Instrumental errors

*Systematic errors of: mechanical and optical parts in Analogue instruments, electro mechanical and optical parts in Analytical instruments.*

## Systematic errors of geodetic network

*Systematic errors in geodetic network reflect on the result of block adjustment.*


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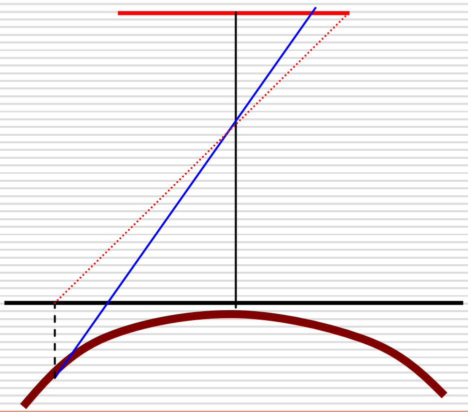
# Main sources of systematic errors

## Earth curvature

*The problem comes when the models are formed and the coordinates are measured with reference to a Cartesian coordinate system. One proper way to avoid this problem is to transform the geodetic coordinate into rectangular coordinate system.*

### *1- Correction of image coordinates*

*Correction of image coordinates is made by projecting the curved earth surface on a plane perpendicular to vertical line passing through the perspective center.*





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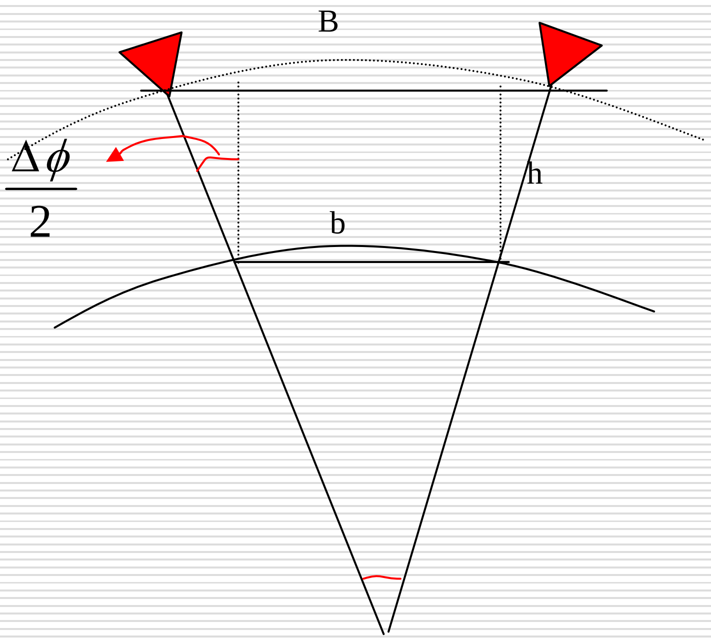
# Main sources of systematic errors

## Earth curvature

### 2- Correction of model coordinates for earth curvature

*In the model surface* $\Delta z = \dfrac{x^2 + y^2}{2R}$ *the middle is the highest point in the model. The convergence of the camera axis have to be corrected by reducing the base length* $\Delta x = h.\Delta\phi$ *where:*

$$\Delta\phi = \frac{B}{R+h} = \frac{B}{R}$$







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# Detectability of systematic errors

*The task of the additional parameters in the correction of systematic errors as far as. They can be detected from residuals at image or control points. The most important factors which affects the possibility of detecting the systematic errors are:*

*1- The type of overlap: (60% side overlap multi photographic coverage).*
*2- The number and location of tie or image points.*
*3- The number and distribution of control points.*

*Unfortunately not all-systematic errors can be detected from the standard configuration of photographic coverage.*


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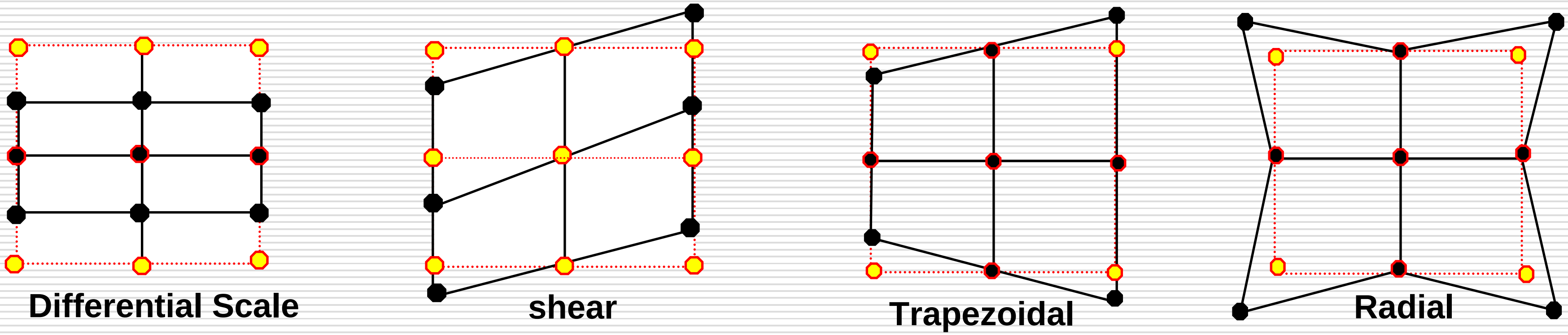
# The history of Investigations on propagation of systematic errors

*In **1972 Kubik** investigated the effects of four systematic image deformations on the results of bundle strip and block adjustment.*




*In practice the systematic errors are on average around 5μm or smaller while the maximum magnitude is about 10μm.*

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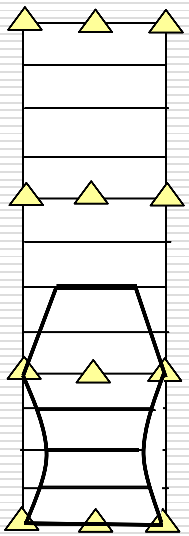
# The history of Investigations on propagation of systematic errors

## *Example of residual errors in a strip due to differential scale systematic error.*

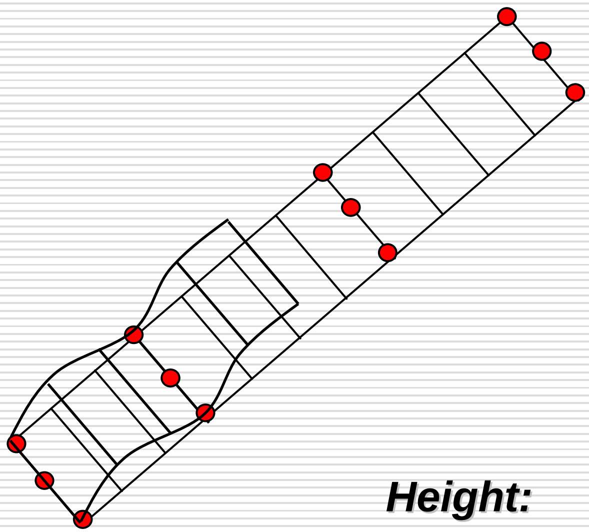


**Plan:**

$\sigma_x = 2.4\,\mu m \quad e_x \max = 3.6\,\mu m$

$\sigma_y = 7.9\,\mu m \quad e_y \max = 14.3\,\mu m$

$\tilde{\sigma}_0 = 2.7\,\mu m$



**Height:**

$e_z \max = 5.5\,\mu m \quad \sigma_x = 2.4\,\mu m$

$\tilde{\sigma}_0 = 2.7\,\mu m$



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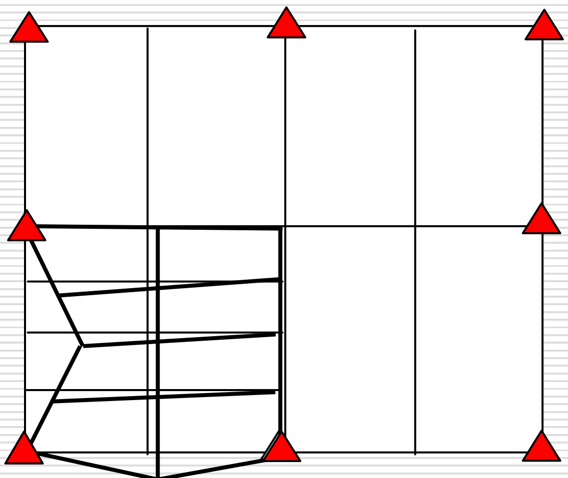
# The history of Investigations on propagation of systematic errors

## Example of residual error in a block due to systematic error pattern

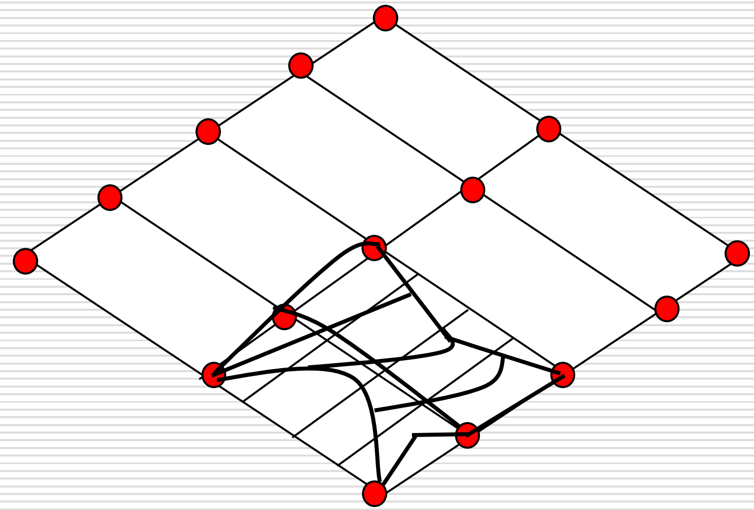


**Plan:**

$\sigma_x = 7.7\mu m$ $e_x \max = 20.0\mu m$

$\sigma_y = 13.2\mu m$ $e_y \max = 33.1\mu m$

$\tilde{\sigma}_0 = 3.3\mu m$



**Height:**

$\sigma_z = 8.2\mu m$ $e_z \max = 21.15\mu m$

$\tilde{\sigma}_0 = 3.3\mu m$



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# The history of Investigations on propagation of systematic errors

*Taken conclusions after the adjustment of different sizes of square blocks with different plan and height control distribution with using at Kubik.*

*1- The largest planimetric coordinate errors occur due to affine deformation. (scale & shear)*

*2- The largest height errors occur due to radial deformation.*

*3-* $\sigma_0 = \sqrt{\frac{[v_x . v_x] + [v_y . v_y]}{r}}$ *Computed from residual image errors after bundle adjustment can be significantly smaller than the actual mean square error of image coordinates introduced.*

*Consequently* $\sigma_0$ *, can not be used to indicate the magnitude of actual systematic errors.*


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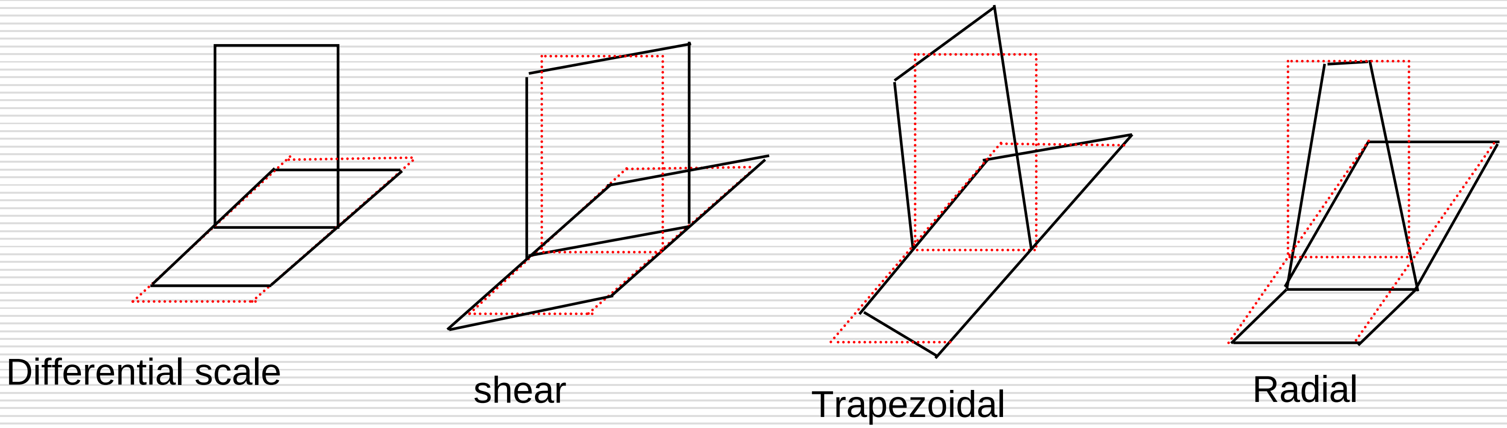
# The history of Investigations on propagation of systematic errors

*In **1973 Werner** (Stuttgart University) investigated the effect of the some four patterns of systematic image deformation on block adjusted by I.M.*




*Again similar conclusions were obtained namely:*

*1- Affine deformations give largest planimetric errors however they are less in magnitude than those obtained from bundle adjustment.*

*2- Radial deformations give largest height errors.*

*3- $\sigma_{op}, \sigma_{oh}$ Can not be used as a measure for magnitude of systematic errors.*

*4- The increase in control distribution reduced the magnitude of residual errors.*



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# Procedures for systematic errors correction

*The following are the three phases at which corrections for systematic errors may be applied:*

- *Before Adjustment*
- *During the A.T. Adjustment*
- *After A.T. Adjustment*



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# Procedures for systematic errors correction

**Before adjustment**

*It is obvious that if any systematic error is absolutely known (such as lens distortion), it should be corrected a prior of the adjustments*

- **System calibration**

  *A real calibration of the photogrammetric system can be gained by test-field calibration before and after each photogrammetric mission.*

- **Observation technique**

  *1- Special flight arrangement*

  *2- Good distribution of control point*

*The correction of image or model coordinates before A.T. adjustment is not entirely effective since instruments, cameras and materials can be calibrated to a limited accuracy thus leaving residual systematic errors.*

*However, in most practical projects, calibration test-fields are not available and the special flight arrangements and observation techniques are expensive and are not effective to correct all type of systematic errors.*



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# Procedures for systematic errors correction

## *After A.T. adjustment*

*One of the techniques was developed in France where systematic error corrections are applied after block adjustment. In this method, the residuals of the tie points are analyzed and systematic errors are reduced using second degree polynomials. Image coordinators are corrected and block adjustment is repeated. Several iterations may be required until no more significant systematic errors are detected.*

*Other technique:*

*Residual errors at control points are used to eliminate the systematic errors in the block. Usually an interpolation procedure is used such as the linear least squares interpolation method. This method is very efficient in case of rather dense control (Cadasteral survey).*



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# Procedures for systematic errors correction

## During the A.T. adjustment

*The most promising method is the so-called "Self-calibration" technique, which uses additional parameters in the process of block adjustment.*

*The advantages of this method:*

*1- No need for additional flight*

*2- No need for additional measurements*

*3- Highly accurate in bundle adjustment and etc.*

## Choice of additional parameters

*There are two schemes for the choice of additional parameters:*

*1- Chooses additional parameters in accordance with characteristics of image errors or according to physical sources. (such as D. Brown)*

*2- Chooses additional parameters in a general polynomial or choose the parameters according to the pattern of deformation. (such as E. Ebner)*


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# Additional parameters

## Based on Formulation of Physical Characteristics of Errors

**D. Brown** uses 29 parameters for correcting systematic errors in bundle adjustment, which adopts the following equations.

$$\Delta x = a_1 x + a_2 y + a_3 x^2 + a_4 xy + a_5 y^2 + a_6 x^2 y + a_7 xy^2 + \frac{x}{r}(c_1 x^2 + c_2 xy + c_3 y^2 + c_4 x^3 + c_5 x^2 y$$

$$+ c_6 xy^2 + c_7 y^3) + x(k_1 r^2 + k_2 r^4 + k_3 r^6) + p_1(y^2 + 3x^2) + 2p_2 xy + \delta x_p + (\frac{x}{c})\delta c$$

$$\Delta y = b_1 x + b_2 y + b_3 x^2 + b_4 xy + b_5 y^2 + b_6 x^2 y + b_7 xy^2 + \frac{y}{r}(c_1 x^2 + c_2 xy + c_3 y^2 + c_4 x^3 + c_5 x^2 y$$

$$+ c_6 xy^2 + c_7 y^3) + y(k_1 r^2 + k_2 r^4 + k_3 r^6) + 2p_1 xy + p_2(x^2 + 3y^2) + \delta y_p + (\frac{y}{c})\delta c$$

$c_1, c_2, \ldots, c_7$: **Coefficients indicating errors in film flatting**

$a_1, a_2, \ldots, a_7$
$b_1, b_2, \ldots, b_7$: **Coefficients expressing film deformation.**

$\Delta x, \Delta y$: **Additional corrections in the image coordinates**

$\delta x_p, \delta y_p, \delta c$: **Parameters for inner orientation elements.**

$p_1, p_2$: **Parameters for tangential lens distortion**

$x, y$: **Image coordinates**

$k_1, k_2, k_3$: **Coefficients for radial lens distortion**

$r$: **Radial distance of image points i.e.** $r^2 = x^2 + y^2$



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# Additional parameters

## Based on Formulation of Physical Characteristics of Errors

*The basic equations for the use of additional parameters for the use of additional parameters in the method of bundle adjustment are:*

$$x + \Delta x = -f\,\frac{a_1(X - X_0) + b_1(Y - Y_0) + c_1(Z - Z_0)}{a_3(X - X_0) + b_3(Y - Y_0) + c_3(Z - Z_0)}$$

$$y + \Delta y = -f\,\frac{a_2(X - X_0) + b_2(Y - Y_0) + c_2(Z - Z_0)}{a_3(X - X_0) + b_3(Y - Y_0) + c_3(Z - Z_0)}$$

$\Delta x, \Delta y$ *Represent the functions of additional parameters currently, more kinds of additional parameter functions have been introduced in photogrammetry. Each of the function may consist of 4 to 25 parameters.*



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# Additional parameters

## Based on General Polynomials

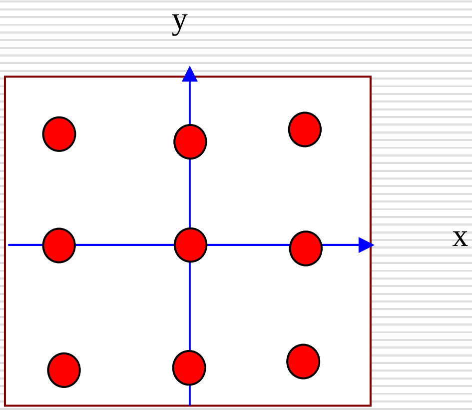
### Additional parameters in Image Space

*This method is to use a general polynomial or a trigonometrical function (spherical harmonic function) as the additional parameters.*

*The general forms are:*

$$\Delta x = \sum_{ij} a_{ij} x^i y^j$$

$$\Delta y = \sum_{ij} b_{ij} x^i y^j$$




$a_{ij}, b_{ij}$ *: The coefficients of the polynomials*

*In designing the parameters, we should try to secure their orthogonality so as to ensure that the coefficient matrix of the normal equations thus formed is in good condition.*



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# Additional parameters

## Based on General Polynomials

### Additional parameters in Image Space

Orthogonality should not only exit between the parameter themselves, but also between the parameters and the element of exterior orientation and the coordinates of the densification points.

No function can express all the effects of image deformations in a comprehensive way without excluding the unpredictable elements.

The choice of parameters depends on experience.

The displacement $\Delta x, \Delta y$ can be written as:

$$\Delta x = \begin{bmatrix} 1 & y & y^2 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$

$$\Delta y = \begin{bmatrix} 1 & y & y^2 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \begin{bmatrix} 1 \\ x \\ x^2 \end{bmatrix}$$



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# Additional parameters

## Based on General Polynomials

### Additional parameters in Image Space

Witch can be applied to compensate all the systematic errors of the 3*3 standard points.

we obtain:

$$\Delta x = a_{11} + a_{12}x + a_{21}y + a_{13}x^2 + a_{22}xy + a_{31}y^2 + a_{23}x^2y + a_{32}xy^2 + a_{33}x^2y^2$$

$$\Delta y = b_{11} + b_{12}x + b_{21}y + b_{13}x^2 + b_{22}xy + b_{31}y^2 + b_{23}x^2y + b_{32}xy^2 + b_{33}x^2y^2$$

Which have eighteen parameters,6 parameters for orientation and 12 parameters remaining, Thus the above equations can be reduced. (Such as H. Ebner method)



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# Additional parameters

## Based on General Polynomials

### Additional parameters in Image Space

### H. Ebner method

*H. Ebner method for bundle adjustment recommended the use of 12 parameters for bundle adjustment depending on the 9 standard image points*

**Ebner equations:**

$$\Delta x = b_1 x + b_2 y - b_3(2x^2 - 4\frac{b^2}{3}) + b_4 xy + b_5(y^2 + 2\frac{b^2}{3}) + b_7(y^2 - 2\frac{b^2}{3})x$$
$$+ b_9(x^2 - 2\frac{b^2}{3})y + b_{11}(x^2 - 2\frac{b^2}{3})(y^2 - 2\frac{b^2}{3})$$

$$\Delta y = -b_1 y + b_2 x + b_3 xy - b_4(2y^2 - 4\frac{b^2}{3}) + b_6(x^2 - 2\frac{b^2}{3}) + b_8(x^2 - 2\frac{b^2}{3})y$$
$$+ b_{10}(y^2 - 2\frac{b^2}{3})x + b_{12}(x^2 - 2\frac{b^2}{3})(y^2 - 2\frac{b^2}{3})$$

*If the nine points observed on the photo are located at the standard position that is the interval is equal to the length of the photo base line b, and the ground is horizontal with photographing approximately vertically downward.*



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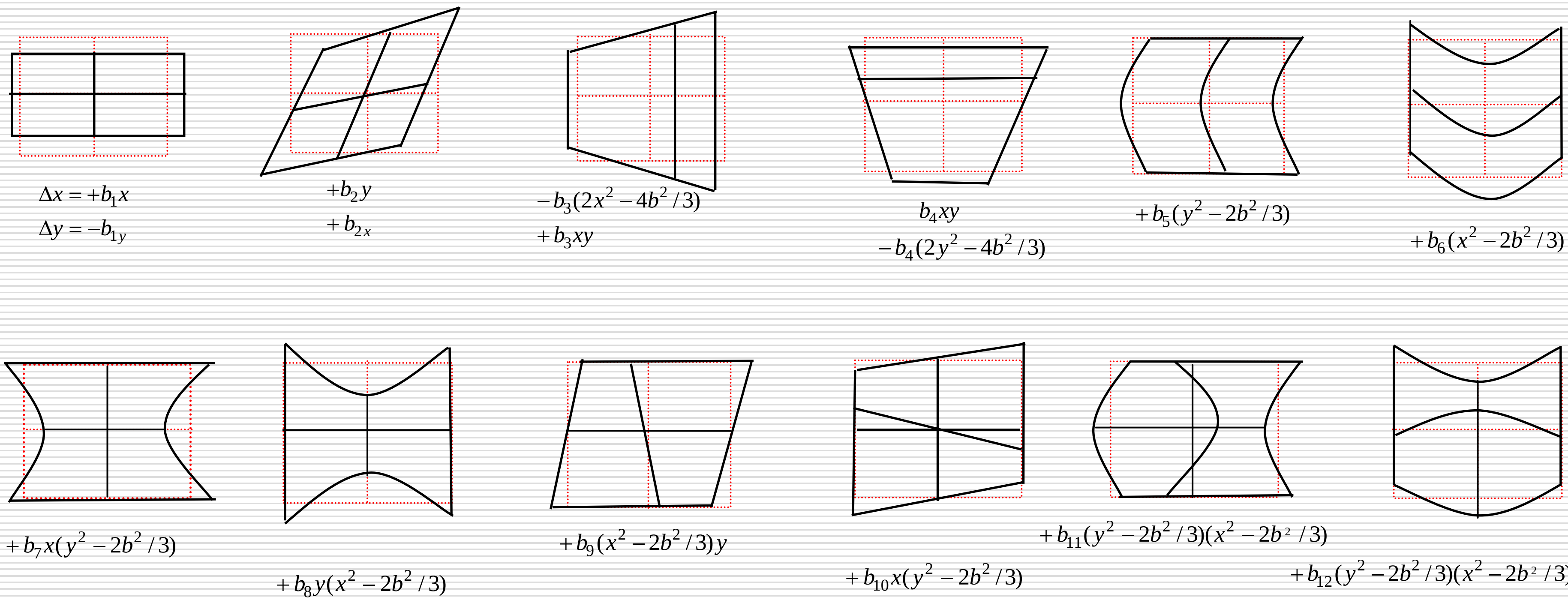
# Additional parameters

## Based on General Polynomials

### Additional parameters in Image Space

### H. Ebner method

*Then we can prove that there is a strict orthogonality between the additional parameters themselves and between the additional parameters and the unknowns of orientation as well. The error models are shown in figure as follows:*



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# Additional parameters

## Based on General Polynomials

### Additional parameters in Image Space

### Grun method

Grun (1978) derived the following 44 parameters which correspond to the additional parameters in consideration of the orthogonality at the 5×5=25 standard points

$$\Delta x = a_{12}x + a_{21}y + a_{22}xy + a_{31}l - b_{23}\frac{10}{7}k + a_{14}xp + a_{23}yk + a_{32}xl + a_{41}yq + a_{15}r + a_{24}xyp + a_{33}kl$$
$$+ a_{42}xyq + a_{51}s + a_{25}yr + a_{34}xlp + a_{43}ykq + a_{52}xs + a_{35}lr + a_{44}xypq + a_{53}ks + a_{45}yqr + a_{54}xps + a_{55}rs$$

$$\Delta y = -a_{12}y + a_{21}x - a_{22}\frac{10}{7}l + b_{13}k + b_{22}xy + b_{14}xp + b_{23}yk + b_{32}xl + b_{41}yq + b_{15}r + b_{24}xyp + b_{33}kl$$
$$+ b_{42}xyq + b_{51}s + b_{25}yr + b_{34}xlp + b_{43}ykq + b_{52}xs + b_{35}lr + b_{44}xypq + b_{53}ks + b_{45}yqr + b_{54}xps + b_{55}rs$$

$$k = x^2 - \frac{b^2}{2} \qquad l = y^2 - \frac{b^2}{2} \qquad p = x^2 - \frac{17}{20}b^2$$

$$q = y^2 - \frac{17}{20}b^2 \qquad r = x^2(x^2 - \frac{31}{28}b^2) + \frac{9}{70}b^4 \qquad s = y^2(y^2 - \frac{31}{28}b^2) + \frac{9}{70}b^4$$



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# Additional parameters

## Based on General Polynomials

### Additional parameters in Image Space

### Faig and EL-Hakim method

*The additional parameters expressed by spherical harmonic functions.*

$$\Delta x = a_1 x + a_2 y + q\frac{x}{r}$$

$$\Delta y = -a_1 y + a_2 x + q\frac{y}{r}$$

$$q = a_3 r\cos\lambda + a_4 r\sin\lambda + a_5 r^2 + a_6 r^2\cos 2\lambda + a_7 r^2\sin 2\lambda + a_8 r^3\cos\lambda$$

$$+ a_9 r^3\sin\lambda + a_{10}\, r^3\cos 3\lambda + a_{11} r^3\sin 3\lambda \qquad r = \sqrt{x^2 + y^2} \qquad \lambda = arctg(y/x)$$



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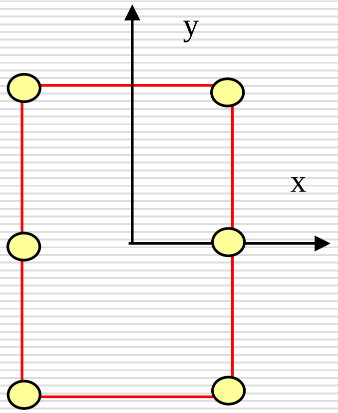
# Additional parameters

## Based on General Polynomials

### Additional parameters in Model Space

Fill illustration, let us as an example the parameters introduced into the block adjustment by independent models in the Ebner method.

The photogrammetric independent model consists of six standard image points.




The interval along the x direction of the model is about the half of that along the y direction. The terms containing $X^2$ are eliminated from the polynomial. For the model coordinates X, Y and Z, $3 \times 6=18$ terms are required. Besides three terms should be added to compensate the systematic errors of the coordinates $X_0$ , $Y_0$ and $Z_0$ of the perspective centers. Since seven of these twenty-one terms can be brought into consideration by space similarity transformation of each model, therefore only fourteen terms remain.



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# Additional parameters

## Based on General Polynomials

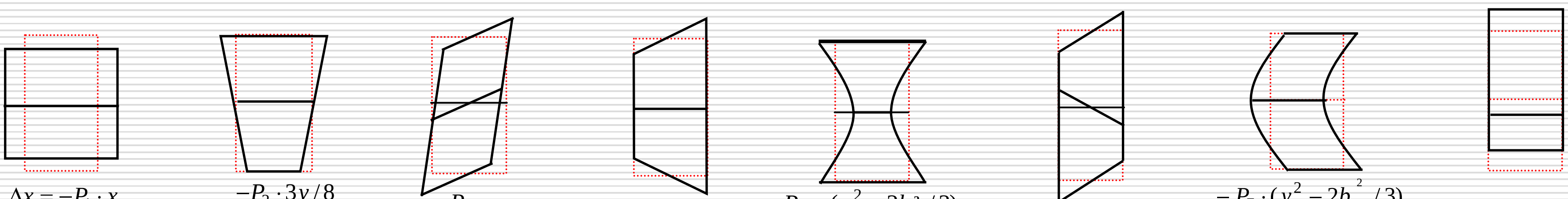
### Additional parameters in Model Space

### Ebner method

Ebner method for independent models adjustment according to thee 6 standard points.

- 8 parameters ( $P_1$ to $P_8$ ) into planimetric block adjustment.
- 6 parameters ( $h_1$ to $h_8$ ) into height adjustment.

The forms of the correction terms and their influences on model points are shown.



$\Delta x = -P_1 \cdot x$, $\Delta y = P_1 \cdot 3y/8$

$-P_2 \cdot 3y/8$, $-P_2 \cdot x$

$-P_3 \cdot xy$

$-P_4 \cdot xy$

$-P_5 \cdot x(y^2 - 2b^2/3)$

$-P_6 \cdot x(y^2 - 2b^2/3)$

$-P_7 \cdot (y^2 - 2b^2/3)$

$-P_8 \cdot (y^2 - 2b^2/3)$

$\Delta z = h_1 \cdot xy$

$-h_2 \cdot x(y^2 - 2b^2/3)$

$-h_3 \cdot (y^2 - 2b^2/3)$

$\Delta x_{Pz} = h_4 \cdot x/b$

$\Delta y_{Pz} = h_5 \cdot x/b$

$\Delta z_{Pz} = h_6 \cdot x/b$

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# Stability of the normal equation system

*Two general principles in applying additional parameters:*

*(1) The number of parameters should be as small as feasible to avoid over-parameterization and to keep the additional computational effort small.*

*(2) The parameters are to be chosen such that their, correlations with the other unknowns are negligible, otherwise the normal equation matrix becomes ill-conditional or singular.*

*Since for most photogrammetric projects the same stable aerial camera is used to photograph the whole block in one flight mission under stable atmospheric conditions. The parameters can be assumed to be the same for all photographs.*


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# Introduction

*This leads to the so-called block invariant approach, which is most favorable for computation economy. If however different cameras are used or if other conditions change (reflying of a strip) then a block variant approach has to be applied, where the parameters are valid for only a group of photographs. In the extreme case (when using nonmetric camera, as in close-range photogrammetry) a photo variant approach is needed.*

*In order to minimize the chances of having an ill-conditional system Ebner proposed the following:*



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# Stability of the normal equation system

**(1) Using parameters in terms which orthogonal to each other and with respect to orientation parameters.**

*-The orthogonality of the additional parameters:*

*(i) Leads to well conditioned normal equations*

*(ii) Allows the statistical checking of each individual correction term separately from the normal equations of the adjustment. Insignificant terms can then be eliminated.*



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# Stability of the normal equation system

***(2) The additional parameters are treated as observations beside being unknowns.***

*The additional parameters are relatively small and they vary from project to project with regard to sign and magnitude with a theoretical mean value of zero.*

*This technique is achieved by adding to the standard set of observation equations a new set of equations:*

$$V_{\overline{P}} + \overline{P} = 0$$

*Where $\overline{P}$ is the vector of additional parameters.*



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# Stability of the normal equation system

## (3) The operational problems

*The user of self-calibrating adjustment program should not be burdened with the selection of the additional parameters and the critical evaluation of their computed values from the adjustment. This task should be automized as far as possible.*

### (i) Choice of parameter groups

*It may not be necessary to use all of parameters for each block to be adjusted. Therefore, the additional burden of the program user should be limited to an additional input determining. This depends on chancing one of the project parameters such as camera, film type or measuring instruments.*


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# Stability of the normal equation system

## (3) The operational problems

### (ii) Statistical check of computed correction terms

*In introducing additional parameters, we may start with a large group of parameters. Then, by the use of statistical tests, we can eliminate those, which can not be determined with sufficient accuracy, and those which can not be separated with certainly or we may first take a small group of basic additional parameters base on experience. Then, having these as a reliable basis, we can add some more parameters as we see proper and check their significants tests.*

*When the additional parameters are orthogonal or almost orthogonal, we can use t-distribution in mathematical statistics to conduct significant test for all the parameters obtained one by one. The t-distribution is based upon available defined by the following equation:*

$$t = \frac{\xi}{\eta}$$



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# Stability of the normal equation system

where:

$$t = \frac{\xi}{\eta}$$

$\xi$ is standard normal N (0,1), $\eta = \sqrt{\frac{X^2}{v}}$ , $v$ is the degree of freedom of variable $X^2$.

Now the statistical hypothesis is: $H_0 : E(a_i) = 0$ , with $a_i$ being the estimate value of the ith additional parameters. Take

$$\xi = \frac{a_i - E(a_i)}{\sigma_0 \sqrt{q_{ii}}} \sim N(0,1) \qquad \eta = \sqrt{\frac{(n-u)s_0^2}{\sigma_0^2} / (n-u)} \sim \sqrt{\frac{X^2}{v}} \qquad , v = n-u$$

Where $\sigma_0^2$ is the variance of unit weight, $s_0^2 = \frac{V^T PV}{n-u}$ : Square of the mean square error of unit weight obtained from adjustment computations, and its expectation is $\sigma_0^2$, $q_{ii}$ is taken from the corresponding diagonal element of the cofactor matrix Q of the unknowns in the adjustment. Since we assume $E(\hat{a}_i) = 0$, the statistical variable of the t-distribution is obtained as follows:

$$t = \frac{\xi}{\eta} = \frac{\hat{a}_i}{\sigma_0 \sqrt{q_{ij}}} \sqrt{\frac{\sigma_0^2}{s_0^2}} = \frac{\hat{a}_i}{s_0 \sqrt{q_{ii}}}$$



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# Stability of the normal equation system

*When the significant level is given then we can find the critical value to from the t-distribution table. If $t < t_0$ then the null hypothesis is accepted, which means that the parameter is not significant and can be eliminated in the next iterative adjustment.*

*When there is a strong correlation between the additional parameters one-dimensional t-tests may lead to erroneous conclusions. The parameters of this group be put together and the multi-dimensional tests are used. In this case, the null hypothesis is:*

$$H_0 : E(\hat{a}) = 0$$

$\hat{a}^T = (\hat{a}_{i+1} ... \hat{a}_{i+k})$ *are the K parameters put together for the test. Take*

$$\xi' = \frac{[\hat{a} - E(\hat{a})]^T Q_{\hat{a}\hat{a}}^{-1} [\hat{a} - E(\hat{a})]}{\sigma_0^2} / k \sim X^2 / \nu \qquad , \nu = k$$

$$\eta' = \frac{(n-u)s_0^2}{\sigma_0^2} / (n-u) \sim X^2 / \nu \qquad , \nu = n - u$$



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# Stability of the normal equation system

*Thus the statistical variable of distribution F is obtained as follows:*

$$F = \frac{\xi'}{\eta'} = \frac{\hat{a}^T Q_{\hat{a}\hat{a}}^{-1} \hat{a}}{k s_0^2}$$

*From the two degree of freedom. (i.e. k and (n-u) ) and the assumed significance level α, we can consult the F distribution table. If $H_0$ is true then all the parameters $a_{i+1} \ldots a_{i+k}$ are eliminated.*



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# Stability of the normal equation system

## Adjustment algorithm schemes

*In performing block adjustment, the additional parameters can be regarded as constants, or as unknowns, or as observed values depending upon different situations.*

$$v = Ax + Bz - l \qquad Q_{bb}$$

- *l: Vector of observed values*
- *v: Vector of correction*
- *x: Vector of unknowns*
- *z: Vector of additional parameters*
- *B: Coefficient matrix of additional parameters*
- $Q_{bb}$ *: Weight matrix of observed values.*



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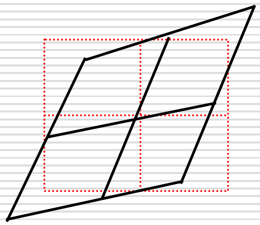
# Example of self-calibrating bundle adjustment

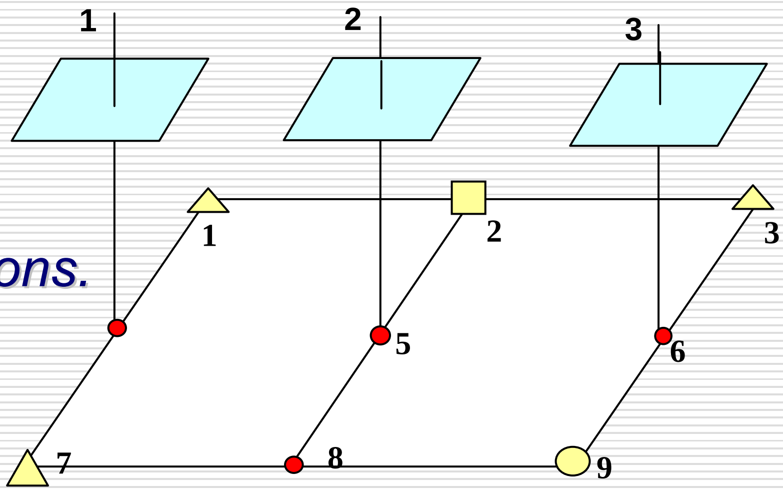
*A block of three bundles with control distribution as shown in figure has to be adjusted with necessary additional parameters for correcting systematic errors in image coordinates for differential scale and shear deformations.*



$$\therefore \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \begin{bmatrix} d \\ s \end{bmatrix}$$

$$\Delta x = +d \cdot x \qquad +s \cdot y$$

$$\Delta y = -d \cdot y \qquad +s \cdot x$$




*The following are considered as observations.*

*(1) The image coordinates*

*(2) The control points*

*(3) The additional parameters*



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# Example of self-calibrating bundle adjustment

**The unknowns**

- **(1) Orientation parameters:** 3 bundles x 6=18
- **(2) Adjusted coordinates:** 9 points x 3=27
- **(3) Additional parameters:** 2 parameters=2

Total number of unknowns =18+27+2=47

**Number of observation equation**

- **(1) Photogrammetric observation**
  - Image coordinates: (6+9+6) x 2= 42
- **(2) Terrestrial observation**
  - 3 full control x 3 = 9
  - 1 plan control x 2 = 2
  - 1 height control x 1 = 1
- **(3) Additional parameters as observations:** 2

Total number of observation equation = 56

**Degree of freedom = 56 – 47 = 9**



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# Example of self-calibrating bundle adjustment

## Type of observation equations

### 1- Original non-linear equations:

$$v_{x_{ij}} + x_{ij} + d \cdot x_{ij} + s \cdot y_{ij} = -c\left\{\frac{a_1^j(X_i - X_0^j) + a_4^j(Y_i - Y_0^j) + a_7^j(Z_i - Z_0^j)}{a_3^j(X_i - X_0^j) + a_6^j(Y_i - Y_0^j) + a_9^j(Z_i - Z_0^j)}\right\}$$

$$v_{x_{ij}} + y_{ij} - d.y_{ij} + sx_{ij} = -c\left\{\frac{a_2^j(X_i - X_0^j) + a_5^j(Y_i - Y_0^j) + a_8^j(Z_i - Z_0^j)}{a_3^j(X_i - X_0^j) + a_6^j(Y_i - Y_0^j) + a_9^j(Z_i - Z_0^j)}\right\}$$

### Linearized equations:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} \end{bmatrix}_{ij} \begin{bmatrix} \Delta\omega \\ \Delta\phi \\ \Delta\kappa \\ \Delta X_0 \\ \Delta Y_0 \\ \Delta Z_0 \end{bmatrix}_j + \begin{bmatrix} -a_{14} & -a_{15} & -a_{16} \\ -a_{24} & -a_{25} & -a_{26} \end{bmatrix}_{ij} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix}_j + \begin{bmatrix} x & y \\ -y & x \end{bmatrix}_{ij} \begin{bmatrix} d \\ s \end{bmatrix}_{(2,1)} = \begin{bmatrix} r_x \\ r_y \end{bmatrix}_{ij}$$

$$A_{ij} \cdot \Delta P_j + B_{ij} \cdot \Delta C_j + S_{ij} \cdot \overline{P} = R_{ij}$$



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# Example of self-calibrating bundle adjustment

## Type of observation equations

**2-** Observation equations for terrestrial observation

$$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \\ \Delta Z \end{bmatrix} = \begin{bmatrix} \bar{X} - E \\ \bar{Y} - N \\ \bar{Z} - H \end{bmatrix}$$

For **full control**

$$\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \Delta X \\ \Delta Y \end{bmatrix} = \begin{bmatrix} \bar{X} - E \\ \bar{Y} - N \end{bmatrix}$$

For **plan control**

$$\begin{bmatrix} -1 \end{bmatrix} \begin{bmatrix} \Delta Z \end{bmatrix} = \begin{bmatrix} \bar{Z} - H \end{bmatrix}$$

For **height control**

**3-** Observation equation for additional parameters

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} d \\ s \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



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# Example of self-calibrating bundle adjustment

## Weight matrix

*In most bundle adjustment programs the image coordinates are usually in millimetres, while terrestrial coordinates are in meters.*

**1- In general unit weight is given to the observed image coordinates.**

$$\sigma_0 = \sigma_x = \sigma_y = 6\mu m = 6\times10^{-6} mm \qquad W_{image\ \ coords} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

**2- Weight for terrestrial coordinates**

**-For example:**

$$\sigma_E = \sigma_N = 18cm = 18\times10^{-2} m \ , \quad \sigma_H = 12cm = 12\times10^{-2} m$$

$$P_E = P_N = \frac{6^2\times10^{-6}(mm)^2}{18^2\times10^{-4}(m)2} = \frac{1}{9}\times10^{-2}(\frac{mm}{m})^2 \qquad P_H = \frac{6^2\times10^{-6}(mm)^2}{12^2\times10^{-4}(m)^2} = \frac{1}{4}\times10^{-2}(\frac{mm}{m})^2$$

$$W_{Terresterl\ \ coords} = \begin{bmatrix} 1/9 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/4 \end{bmatrix}\times10^{-2}$$



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# Example of self-calibrating bundle adjustment

## Weight matrix

### 3- Weights for additional parameters

*Practice indicated the maximum systematic errors are of a magnitude of 5 -10 μm at the corners of the photograph. The mean value=0 and the standard deviation of systematic errors in coordinates Δx, Δy are about five microns.*

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \begin{bmatrix} d \\ s \end{bmatrix} \qquad \begin{bmatrix} \sigma_{\Delta x}^2 & \sigma_{\Delta x,\Delta y} \\ \sigma_{\Delta x,\Delta y} & \sigma_{\Delta y}^2 \end{bmatrix} = \begin{bmatrix} x & y \\ -y & x \end{bmatrix} \begin{bmatrix} \sigma_d^2 & \sigma_{d,s} \\ \sigma_{d,s} & \sigma_s^2 \end{bmatrix} \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$

*If we assume* $\sigma_d = \sigma_s, \sigma_{ds} = 0$

$$\begin{bmatrix} \sigma_{\Delta x}^2 & \sigma_{\Delta x,\Delta y} \\ \sigma_{\Delta x,\Delta y} & \sigma_{\Delta y}^2 \end{bmatrix} = \sigma_d^2 \cdot \begin{bmatrix} x^2 + y^2 & 0 \\ 0 & x^2 + y^2 \end{bmatrix}$$



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# Example of self-calibrating bundle adjustment

## Weight matrix

### 3- Weights for additional parameters

Assuming $\sigma_{\Delta x} = 2\mu m$ and (x = y = 100mm)

$$\sigma_d^2 = \frac{\sigma_{\Delta x}^2}{x^2 + y^2} = \frac{2^2 \times 10^{-6} (mm)^2}{(100)^2 + (100)^2 \quad (mm)^2} = 2 \times 10^{-10}$$

$$P_d = P_s = \frac{6^2 \times 10^{-6} (mm)^2}{2 \times 10^{-10}} = 18 \times 10^{+4} (mm)^2$$

$$W_{additional \quad parameters} = \begin{bmatrix} 18 & 0 \\ 0 & 18 \end{bmatrix} \times 10^{+4}$$


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# Example of self-calibrating bundle adjustment

## The observation equation matrix

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_{11}$ | | | $B_{11}$ | | | | | | | | | $S_{11}$ | $R_{1,1}$ |
| $A_{21}$ | | | | $B_{21}$ | | | | | | | | $S_{21}$ | $R_{2,1}$ |
| $A_{41}$ | | | | | | | $B_{41}$ | | | | | $S_{41}$ | $R_{4,1}$ |
| $A_{51}$ | | | | | | | | $B_{51}$ | | | | $S_{51}$ | $R_{5,1}$ |
| $A_{71}$ | | | | | | | | | | $B_{71}$ | | $S_{71}$ | $R_{7,1}$ |
| $A_{81}$ | | | | | | | | | | | $B_{81}$ | $S_{81}$ | $R_{8,1}$ |
| | $A_{12}$ | | $B_{12}$ | | | | | | | | | $S_{12}$ | $R_{1,2}$ |
| | $A_{22}$ | | | $B_{22}$ | | | | | | | | $S_{22}$ | $R_{2,2}$ |
| | $A_{32}$ | | | | $B_{32}$ | | | | | | | $S_{32}$ | $R_{3,2}$ |
| | $A_{42}$ | | | | | $B_{42}$ | | | | | | $S_{42}$ | $R_{4,2}$ |
| | $A_{52}$ | | | | | | $B_{52}$ | | | | | $S_{52}$ | $R_{5,2}$ |
| | $A_{62}$ | | | | | | | $B_{62}$ | | | | $S_{62}$ | $R_{6,2}$ |
| | $A_{72}$ | | | | | | | | $B_{72}$ | | | $S_{72}$ | $R_{7,2}$ |
| | $A_{82}$ | | | | | | | | | $B_{82}$ | | $S_{82}$ | $R_{8,2}$ |
| | $A_{92}$ | | | | | | | | | | $B_{92}$ | $S_{92}$ | $R_{9,2}$ |
| | | $A_{23}$ | | $B_{23}$ | | | | | | | | $S_{23}$ | $R_{2,3}$ |
| | | $A_{33}$ | | | $B_{33}$ | | | | | | | $S_{33}$ | $R_{3,3}$ |
| | | $A_{53}$ | | | | | $B_{53}$ | | | | | $S_{53}$ | $R_{5,3}$ |
| | | $A_{63}$ | | | | | | $B_{63}$ | | | | $S_{63}$ | $R_{6,3}$ |
| | | $A_{83}$ | | | | | | | | $B_{83}$ | | $S_{83}$ | $R_{8,3}$ |
| | | $A_{93}$ | | | | | | | | | $B_{93}$ | $S_{93}$ | $R_{9,3}$ |
| | | | -1 | | | | | | | | | | $\bar{X}_1 - E_1$ |
| | | | -1 | | | | | | | | | | $\bar{Y}_1 - N_1$ |
| | | | -1 | | | | | | | | | | $\bar{Z}_1 - H_1$ |
| | | | | | -1 | | | | | | | | $\bar{X}_3 - E_3$ |
| | | | | | -1 | | | | | | | | $\bar{Y}_3 - N_3$ |
| | | | | | -1 | | | | | | | | $\bar{Z}_3 - H_3$ |
| | | | | | | | | | -1 | | | | $\bar{X}_7 - E_7$ |
| | | | | | | | | | -1 | | | | $\bar{Y}_7 - N_7$ |
| | | | | | | | | | -1 | | | | $\bar{Z}_7 - H_7$ |
| | | | | -1 | | | | | | | | | $\bar{X}_2 - E_2$ |
| | | | | -1 | | | | | | | | | $\bar{Y}_2 - N_2$ |
| | | | | | | | | | | | -1 | | $\bar{Z}_9 - H_9$ |
| | | | | | | | | | | | | -1 | 0 |
| | | | | | | | | | | | | -1 | 0 |

| |
|---|
| $\Delta P_1$ |
| $\Delta P_2$ |
| $\Delta P_3$ |
| $\Delta C_1$ |
| $\Delta C_2$ |
| $\Delta C_3$ |
| $\Delta C_4$ |
| $\Delta C_5$ |
| $\Delta C_6$ |
| $\Delta C_7$ |
| $\Delta C_8$ |
| $\Delta C_9$ |
| d |
| s |

# Example of self-calibrating bundle adjustment

## The Normal equation in matrix notation

| | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\Sigma A_{i,1}^T \cdot A_{i,1}$ $i=1,2,4,5,7,8$ | | | | | | | | | | | | $\Sigma A_1^T S_1$ | $\Sigma A_1^T \cdot R_1$ |
| | $\Sigma A_{i,2}^T \cdot A_{i,2}$ $i=1,2,3,4,5,6,7,8$ | | | | | $N_{21}^T$ | | | | | | $\Sigma A_2^T S_2$ | $\Sigma A_2^T \cdot R_2$ |
| | | $\Sigma A_{i,3}^T \cdot A_{i,3}$ $i=2,3,5,6,8,9$ | | | | | | | | | | $\Sigma A_3^T S_3$ | $\Sigma A_3^T \cdot R_3$ |
| $B_{1,1}^T \cdot A_{1,1}$ | $B_{1,2}^T \cdot A_{1,2}$ | | $\sum_{i=1,2} B_{1,j}^T B_{1,j}$ $+G_F$ | | | | | | | | | $\Sigma B_1^T S_1$ | $\sum_{i=1,2} B_{1,j}^T R_{1,j}$ $+F_1$ |
| $B_{2,1}^T \cdot A_{2,1}$ | $B_{2,2}^T \cdot A_{2,2}$ | $B_{2,3}^T \cdot A_{2,3}$ | | $\sum_{i=1,2,3} B_{2,j}^T B_{2,j}$ $+G_F$ | | | | | | | | $\Sigma B_2^T S_2$ | $\sum_{i=1,2,3} B_{2,j}^T R$ $+F_2$ |
| | $B_{3,2}^T \cdot A_{3,2}$ | $B_{3,3}^T \cdot A_{3,3}$ | | | $\sum_{i=2,3} B_{3,j}^T B_{3,j}$ $+G_F$ | | | | | | | $\Sigma B_3^T S_3$ | $\sum_{i=2,3} B_{3,j}^T R_{3,j}$ $+F_3$ |
| $B_{4,1}^T \cdot A_{4,1}$ | $B_{4,2}^T \cdot A_{4,2}$ | | | | | $\Sigma B_{4,j}^T B_{4,j}$ $i=1,2$ | | | | | | $\Sigma B_4^T S_4$ | $\Sigma B_{4,j}^T R_{4,j}$ $i=1,2$ |
| $B_{5,1}^T \cdot A_{5,1}$ | $B_{5,2}^T \cdot A_{5,2}$ | $B_{5,3}^T \cdot A_{5,3}$ | | | | | $\Sigma B_{5,i}^T B_{5,i}$ $i=1,2,3$ | | | | | $\Sigma B_5^T S_5$ | $\Sigma B_{5,i}^T R_{5,i}$ $i=1,2,3$ |
| | $B_{6,2}^T \cdot A_{6,2}$ | $B_{6,3}^T \cdot A_{6,3}$ | | | | | | $\Sigma B_{6,j}^T B_{6,j}$ $i=2,3$ | | | | $\Sigma B_6^T S_6$ | $\Sigma B_{6,j}^T R_{6,j}$ $i=2,3$ |
| $B_{7,1}^T \cdot A_{7,1}$ | $B_{7,2}^T \cdot A_{7,2}$ | | | | | | | | $\sum_{i=1,2} B_{7,j}^T B_{7,j}$ $+G_F$ | | | $\Sigma B_7^T S_7$ | $\sum_{i=1,2} B_{7,j}^T R_{7,j}$ $+F_7$ |
| $B_{8,1}^T \cdot A_{8,1}$ | $B_{8,2}^T \cdot A_{8,2}$ | $B_{8,3}^T \cdot A_{8,3}$ | | | | | | | | $\Sigma B_{8,j}^T B_{8,j}$ $i=1,2,3$ | | $\Sigma B_8^T S_8$ | $\Sigma B_{8,j}^T R_{8,j}$ $i=1,2,3$ |
| | $B_{9,2}^T \cdot A_{9,2}$ | $B_{9,3}^T \cdot A_{9,3}$ | | | | | | | | | $\sum_{i=2,3} B_{9,j}^T B_{9,j}$ $+G_H$ | $\Sigma B_9^T S_9$ | $\sum_{i=2,3} B_{9,j}^T R_{9,j}$ $+H_9$ |
| $\Sigma S_1^T A_1$ | $\Sigma S_2^T A_2$ | $\Sigma S_3^T A_3$ | $\Sigma S_1^T B_1$ | $\Sigma S_2^T B_2$ | $\Sigma S_3^T B_3$ | $\Sigma S_4^T B_4$ | $\Sigma S_5^T B_5$ | $\Sigma S_6^T B_6$ | $\Sigma S_7^T B_7$ | $\Sigma S_8^T B_8$ | $\Sigma S_{91}^T B_9$ | $\sum S^T S$ $+G_{AP}$ | $\Sigma S \cdot R$ |