CHAPTER 2, PART A

2.1 Given

$$\left[S_{ij}\right]=\begin{bmatrix}1&0&2\\0&1&2\\3&0&3\end{bmatrix} \text{ and } \left[a_i\right]=\begin{bmatrix}1\\2\\3\end{bmatrix}$$

Evaluate (a) S_{ii}, (b) $S_{ij}S_{ij}$, (c) $S_{ji}S_{ji}$, (d) $S_{jk}S_{kj}$ (e) $a_m a_m$, (f) $S_{mn}a_m a_n$, (g) $S_{nm}a_m a_n$

Ans. (a) $S_{ii}=S_{11}+S_{22}+S_{33}=1+1+3=5$.

(b) $S_{ij}S_{ij}=S_{11}^2+S_{12}^2+S_{13}^2+S_{21}^2+S_{22}^2+S_{23}^2+S_{31}^2+S_{32}^2+S_{33}^2=$

$1+0+4+0+1+4+9+0+9=28$.

(c) $S_{ji}S_{ji}=S_{ij}S_{ij}=28$.

(d) $S_{jk}S_{kj}=S_{1k}S_{k1}+S_{2k}S_{k2}+S_{3k}S_{k3}$

$=S_{11}S_{11}+S_{12}S_{21}+S_{13}S_{31}+S_{21}S_{12}+S_{22}S_{22}+S_{23}S_{32}+S_{31}S_{13}+S_{32}S_{23}+S_{33}S_{33}$

$=(1)(1)+(0)(0)+(2)(3)+(0)(0)+(1)(1)+(2)(0)+(3)(2)+(0)(2)+(3)(3)=23$.

(e) $a_m a_m=a_1^2+a_2^2+a_3^2=1+4+9=14$.

(f) $S_{mn}a_m a_n=S_{1n}a_1a_n+S_{2n}a_2a_n+S_{3n}a_3a_n=$

$S_{11}a_1a_1+S_{12}a_1a_2+S_{13}a_1a_3+S_{21}a_2a_1+S_{22}a_2a_2+S_{23}a_2a_3+S_{31}a_3a_1+S_{32}a_3a_2+S_{33}a_3a_3$

$=(1)(1)(1)+(0)(1)(2)+(2)(1)(3)+(0)(2)(1)+(1)(2)(2)+(2)(2)(3)+(3)(3)(1)$

$+(0)(3)(2)+(3)(3)(3)=1+0+6+0+4+12+9+0+27=59$.

(g) $S_{nm}a_m a_n=S_{mn}a_m a_n=59$.

2.2 Determine which of these equations have an identical meaning with $a_i=Q_{ij}a'_j$.

(a) $a_p=Q_{pm}a'_m$, (b) $a_p=Q_{qp}a'_q$, (c) $a_m=a'_n Q_{mn}$.

Ans. (a) and (c)

2.3 Given the following matrices

$$\left[a_i\right]=\begin{bmatrix}1\\0\\2\end{bmatrix},\left[B_{ij}\right]=\begin{bmatrix}2&3&0\\0&5&1\\0&2&1\end{bmatrix}$$

Demonstrate the equivalence of the subscripted equations and corresponding matrix equations in the following two problems.

(a) $b_i=B_{ij}a_j$ and $[b]=[B][a]$, (b) $s=B_{ij}a_ia_j$ and $s=[a]^{\mathrm{T}}[B][a]$

Ans. (a)

$b_i=B_{ij}a_j\rightarrow b_1=B_{1j}a_j=B_{11}a_1+B_{12}a_2+B_{13}a_3=(2)(1)+(3)(0)+(0)(2)=2$

$b_2=B_{2j}a_j=B_{21}a_1+B_{22}a_2+B_{23}a_3=2,\quad b_3=B_{3j}a_j=B_{31}a_1+B_{32}a_2+B_{33}a_3=2$.

$[b]=[B][a]=\begin{bmatrix}2&3&0\\0&5&1\\0&2&1\end{bmatrix}\begin{bmatrix}1\\0\\2\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}$. Thus, $b_i=B_{ij}a_j$ gives the same results as $[b]=[B][a]$

(b)

$s=B_{ij}a_ia_j=B_{11}a_1a_1+B_{12}a_1a_2+B_{13}a_1a_3++B_{21}a_2a_1+B_{22}a_2a_2+B_{23}a_2a_3$

$+B_{31}a_3a_1+B_{32}a_3a_2+B_{33}a_3a_3=(2)(1)(1)+(3)(1)(0)+(0)(1)(2)+(0)(0)(1)$

$+(5)(0)(0)+(1)(0)(2)+(0)(2)(1)+(2)(2)(0)+(1)(2)(2)=2+4=6.$

and $s=[a]^{\mathrm{T}}[B][a]=[1\quad 0\quad 2]\begin{bmatrix}2&3&0\\0&5&1\\0&2&1\end{bmatrix}\begin{bmatrix}1\\0\\2\end{bmatrix}=[1\quad 0\quad 2]\begin{bmatrix}2\\2\\2\end{bmatrix}=2+4=6$.

2.4 Write in indicial notation the matrix equation (a) $[A]=[B][C]$, (b) $[D]=[B]^{\mathrm{T}}[C]$ and (c) $[E]=[B]^{\mathrm{T}}[C][F]$.

Ans. (a) $[A]=[B][C]\rightarrow A_{ij}=B_{im}C_{mj}$, (b) $[D]=[B]^{\mathrm{T}}[C]\rightarrow A_{ij}=B_{mi}C_{mj}$.

(c) $[E]=[B]^{\mathrm{T}}[C][F]\rightarrow E_{ij}=B_{mi}C_{mk}F_{kj}$.

2.5 Write in indicial notation the equation (a) $s=A_1^2+A_2^2+A_3^2$ and (b) $\dfrac{\partial^2\phi}{\partial x_1^2}+\dfrac{\partial^2\phi}{\partial x_2^2}+\dfrac{\partial^2\phi}{\partial x_3^2}=0$.

Ans. (a) $s=A_1^2+A_2^2+A_3^2=A_iA_i$. (b) $\dfrac{\partial^2\phi}{\partial x_1^2}+\dfrac{\partial^2\phi}{\partial x_2^2}+\dfrac{\partial^2\phi}{\partial x_3^2}=0\rightarrow\dfrac{\partial^2\phi}{\partial x_i\partial x_i}=0$.

2.6 Given that $S_{ij}=a_ia_j$ and $S'_{ij}=a'_ia'_j$, where $a'_i=Q_{mi}a_m$ and $a'_j=Q_{nj}a_n$, and $Q_{ik}Q_{jk}=\delta_{ij}$. Show that $S'_{ii}=S_{ii}$.

Ans. $S'_{ij}=Q_{mi}a_mQ_{nj}a_n=Q_{mi}Q_{nj}a_ma_n\rightarrow S'_{ii}=Q_{mi}Q_{ni}a_ma_n=\delta_{mn}a_ma_n=a_ma_m=S_{mm}=S_{ii}$.

2.7 Write $a_i=\dfrac{\partial v_i}{\partial t}+v_j\dfrac{\partial v_i}{\partial x_j}$ in long form.

Ans.

$$i=1\rightarrow a_1=\frac{\partial v_1}{\partial t}+v_j\frac{\partial v_1}{\partial x_j}=\frac{\partial v_1}{\partial t}+v_1\frac{\partial v_1}{\partial x_1}+v_2\frac{\partial v_1}{\partial x_2}+v_3\frac{\partial v_1}{\partial x_3}.$$

$$i=2\rightarrow a_2=\frac{\partial v_2}{\partial t}+v_j\frac{\partial v_2}{\partial x_j}=\frac{\partial v_2}{\partial t}+v_1\frac{\partial v_2}{\partial x_1}+v_2\frac{\partial v_2}{\partial x_2}+v_3\frac{\partial v_2}{\partial x_3}.$$

$$i=3\rightarrow a_3=\frac{\partial v_3}{\partial t}+v_j\frac{\partial v_3}{\partial x_j}=\frac{\partial v_3}{\partial t}+v_1\frac{\partial v_3}{\partial x_1}+v_2\frac{\partial v_3}{\partial x_2}+v_3\frac{\partial v_3}{\partial x_3}.$$

2.8 Given that $T_{ij} = 2\mu E_{ij} + \lambda E_{kk}\delta_{ij}$, show that

(a) $T_{ij}E_{ij} = 2\mu E_{ij}E_{ij} + \lambda\left(E_{kk}\right)^2$ and (b) $T_{ij}T_{ij} = 4\mu^2 E_{ij}E_{ij} + \left(E_{kk}\right)^2(4\mu\lambda + 3\lambda^2)$

Ans. (a)

$$T_{ij}E_{ij} = (2\mu E_{ij} + \lambda E_{kk}\delta_{ij})E_{ij} = 2\mu E_{ij}E_{ij} + \lambda E_{kk}\delta_{ij}E_{ij} = 2\mu E_{ij}E_{ij} + \lambda E_{kk}E_{ii} = 2\mu E_{ij}E_{ij} + \lambda(E_{kk})^2$$

(b)

$$T_{ij}T_{ij} = (2\mu E_{ij} + \lambda E_{kk}\delta_{ij})(2\mu E_{ij} + \lambda E_{kk}\delta_{ij}) = 4\mu^2 E_{ij}E_{ij} + 2\mu\lambda E_{ij}E_{kk}\delta_{ij} + 2\mu\lambda E_{kk}\delta_{ij}E_{ij}$$

$$+\lambda^2\left(E_{kk}\right)^2\delta_{ij}\delta_{ij} = 4\mu^2 E_{ij}E_{ij} + 2\mu\lambda E_{ii}E_{kk} + 2\mu\lambda E_{kk}E_{ii} + \lambda^2\left(E_{kk}\right)^2\delta_{ii}$$

$$= 4\mu^2 E_{ij}E_{ij} + \left(E_{kk}\right)^2(4\mu\lambda + 3\lambda^2).$$

2.9 Given that $a_i = T_{ij}b_j$, and $a'_i = T'_{ij}b'_j$, where $a_i = Q_{im}a'_m$ and $T_{ij} = Q_{im}Q_{jn}T'_{mn}$.

(a) Show that $Q_{im}T'_{mn}b'_n = Q_{im}Q_{jn}T'_{mn}b_j$ and (b) if $Q_{ik}Q_{im} = \delta_{km}$, then $T'_{kn}(b'_n - Q_{jn}b_j) = 0$.

Ans. (a) Since $a_i = Q_{im}a'_m$ and $T_{ij} = Q_{im}Q_{jn}T'_{mn}$, therefore, $a_i = T_{ij}b_j \rightarrow$.

$Q_{im}a'_m = Q_{im}Q_{jn}T'_{mn}b_j$ (1), Now, $a'_i = T'_{ij}b'_j \rightarrow a'_m = T'_{mj}b'_j = T'_{mn}b'_n$, therefore, Eq. (1) becomes

$Q_{im}T'_{mn}b'_n = Q_{im}Q_{jn}T'_{mn}b_j$. (2)

(b) To remove Q_{im} from Eq. (2), we make use of $Q_{ik}Q_{im} = \delta_{km}$ by multiplying the above equation, Eq.(2) with Q_{ik}. That is,

$$Q_{ik}Q_{im}T'_{mn}b'_n = Q_{ik}Q_{im}Q_{jn}T'_{mn}b_j \rightarrow \delta_{km}T'_{mn}b'_n = \delta_{km}Q_{jn}T'_{mn}b_j \rightarrow T'_{kn}b'_n = Q_{jn}T'_{kn}b_j$$

$$\rightarrow T'_{kn}(b'_n - Q_{jn}b_j) = 0.$$

2.10 Given $[a_i] = \begin{bmatrix}1\\2\\0\end{bmatrix}$ and $[b_i] = \begin{bmatrix}0\\2\\3\end{bmatrix}$ Evaluate $[d_i]$, if $d_k = \varepsilon_{ijk}a_ib_j$ and show that this result is the same as $d_k = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{e}_k$.

Ans. $d_k = \varepsilon_{ijk}a_ib_j \rightarrow$

$$d_1 = \varepsilon_{ij1}a_ib_j = \varepsilon_{231}a_2b_3 + \varepsilon_{321}a_3b_2 = a_2b_3 - a_3b_2 = (2)(3) - (0)(2) = 6$$

$$d_2 = \varepsilon_{ij2}a_ib_j = \varepsilon_{312}a_3b_1 + \varepsilon_{132}a_1b_3 = a_3b_1 - a_1b_3 = (0)(0) - (1)(3) = -3$$

$$d_3 = \varepsilon_{ij3}a_ib_j = \varepsilon_{123}a_1b_2 + \varepsilon_{213}a_2b_1 = a_1b_2 - a_2b_1 = (1)(2) - (2)(0) = 2$$

Next, $(\mathbf{a}\times\mathbf{b}) = (\mathbf{e}_1 + 2\mathbf{e}_2)\times(2\mathbf{e}_2 + 3\mathbf{e}_3) = 6\mathbf{e}_1 - 3\mathbf{e}_2 + 2\mathbf{e}_3$.

$d_1 = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{e}_1 = 6,\quad d_2 = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{e}_2 = -3,\quad d_3 = (\mathbf{a}\times\mathbf{b})\cdot\mathbf{e}_3 = 2$.

2.11 (a) If $\varepsilon_{ijk}T_{ij} = 0$, show that $T_{ij} = T_{ji}$, and (b) show that $\delta_{ij}\varepsilon_{ijk} = 0$

Ans. (a) *for* $k=1,\ \varepsilon_{ij1}T_{ij}=0\rightarrow\varepsilon_{231}T_{23}+\varepsilon_{321}T_{32}=0\rightarrow T_{23}-T_{32}\rightarrow T_{23}=T_{32}$.

for $k=2,\ \varepsilon_{ij2}T_{ij}=0\rightarrow\varepsilon_{312}T_{31}+\varepsilon_{132}T_{13}=0\rightarrow T_{31}-T_{13}\rightarrow T_{31}=T_{13}$.

for $k=3,\ \varepsilon_{ij3}T_{ij}=0\rightarrow\varepsilon_{123}T_{12}+\varepsilon_{213}T_{21}=0\rightarrow T_{12}-T_{21}\rightarrow T_{12}=T_{21}$.

(b) $\delta_{ij}\varepsilon_{ijk}=\delta_{11}\varepsilon_{11k}+\delta_{22}\varepsilon_{22k}+\delta_{33}\varepsilon_{33k}=(1)(0)+(1)(0)+(1)(0)=0$.

2.12 Verify the following equation: $\varepsilon_{ijm}\varepsilon_{klm}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$.

(Hint): there are 6 cases to be considered (i) $i=j$, (2) $i=k$, (3) $i=l$, (4) $j=k$, (5) $j=l$, and (6) $k=l$.

Ans. There are 4 free indices in the equation. Therefore, there are the following 6 cases to consider: (i) $i=j$, (2) $i=k$, (3) $i=l$, (4) $j=k$, (5) $j=l$, and (6) $k=l$. We consider each case below where we use LS for left side, RS for right side and repeated indices with parenthesis are not sum:

(1) For $i=j$, LS=$\varepsilon_{(i)(i)m}\varepsilon_{klm}=0$, $RS=\delta_{(i)k}\delta_{(i)l}-\delta_{(i)l}\delta_{(i)k}=0$.

(2) For $i=k$, LS=$\varepsilon_{(i)j1}\varepsilon_{(i)l1}+\varepsilon_{(i)j2}\varepsilon_{(i)l2}+\varepsilon_{(i)j3}\varepsilon_{(i)l3}$, $RS=\delta_{(i)(i)}\delta_{jl}-\delta_{(i)l}\delta_{j(i)}$

$$\text{LS=RS}=\begin{cases}0 & \text{if } j\neq l\\ 0 & \text{if } j=l=i\\ 1 & \text{if } j=l\neq i\end{cases}.$$

(3) For $i=l$, LS=$\varepsilon_{(i)jm}\varepsilon_{k(i)m}$, $RS=\delta_{(i)k}\delta_{j(i)}-\delta_{(i)(i)}\delta_{jk}$

$$\text{LS=RS}=\begin{cases}0 & \text{if } j\neq k\\ 0 & \text{if } j=k=i\\ -1 & \text{if } j=k\neq i\end{cases}$$

(4) For $j=k$, LS=$\varepsilon_{i(j)m}\varepsilon_{(j)lm}$, $RS=\delta_{i(j)}\delta_{(j)l}-\delta_{il}\delta_{(j)(j)}$

$$\text{LS=RS}=\begin{cases}0 & \text{if } i\neq l\\ 0 & \text{if } i=l=j\\ -1 & \text{if } i=l\neq j\end{cases}$$

(5) For $j=l$, LS=$\varepsilon_{i(j)m}\varepsilon_{k(j)m}$, $RS=\delta_{ik}\delta_{(j)(j)}-\delta_{i(j)}\delta_{(j)k}$

$$\text{LS=RS}=\begin{cases}0 & \text{if } i\neq k\\ 0 & \text{if } i=k=j\\ 1 & \text{if } i=k\neq j\end{cases}$$

(6) For $k=l$, LS=$\varepsilon_{ijm}\varepsilon_{(k)(k)m}$=0, $RS=\delta_{i(k)}\delta_{j(k)}-\delta_{i(k)}\delta_{j(k)}=0$

2.13 Use the identity $\varepsilon_{ijm}\varepsilon_{klm}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$ as a short cut to obtain the following results: (a) $\varepsilon_{ilm}\varepsilon_{jlm}=2\delta_{ij}$ and (b) $\varepsilon_{ijk}\varepsilon_{ijk}=6$.

Ans. (a) $\varepsilon_{ilm}\varepsilon_{jlm}=\delta_{ij}\delta_{ll}-\delta_{il}\delta_{lj}=3\delta_{ij}-\delta_{ij}=2\delta_{ij}$.

(b) $\varepsilon_{ijk}\varepsilon_{ijk}=\delta_{ii}\delta_{jj}-\delta_{ij}\delta_{ji}=(3)(3)-\delta_{ii}=9-3=6$.

2.14 Use the identity $\varepsilon_{ijm}\varepsilon_{klm}=\delta_{ik}\delta_{jl}-\delta_{il}\delta_{jk}$ to show that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

Ans. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = a_m\mathbf{e}_m \times (\varepsilon_{ijk}b_jc_k\mathbf{e}_i) = \varepsilon_{ijk}a_mb_jc_k(\mathbf{e}_m\times\mathbf{e}_i)$

$= \varepsilon_{ijk}a_mb_jc_k(\varepsilon_{nmi}\mathbf{e}_n) = \varepsilon_{ijk}\varepsilon_{nmi}a_mb_jc_k\mathbf{e}_n = \varepsilon_{jki}\varepsilon_{nmi}a_mb_jc_k\mathbf{e}_n$

$= (\delta_{jn}\delta_{km} - \delta_{jm}\delta_{kn})a_mb_jc_k\mathbf{e}_n = \delta_{jn}\delta_{km}a_mb_jc_k\mathbf{e}_n - \delta_{jm}\delta_{kn}a_mb_jc_k\mathbf{e}_n$

$= a_kb_nc_k\mathbf{e}_n - a_jb_jc_n\mathbf{e}_n = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$.

2.15 (a) Show that if $T_{ij} = -T_{ji}$, $T_{ij}a_ia_j = 0$ and (b) if $T_{ij} = -T_{ji}$, and $S_{ij} = S_{ji}$, then $T_{ij}S_{ij} = 0$

Ans. Since $T_{ij}a_ia_j = T_{ji}a_ja_i$ (switching the original dummy index i to j and the original index j to i), therefore $T_{ij}a_ia_j = T_{ji}a_ja_i = -T_{ij}a_ja_i = -T_{ij}a_ia_j \to 2T_{ij}a_ia_j = 0 \to T_{ij}a_ia_j = 0$.

(b) $T_{ij}S_{ij} = T_{ji}S_{ji}$ (switching the original dummy index i to j and the original index j to i), therefore, $T_{ij}S_{ij} = T_{ji}S_{ji} = -T_{ij}S_{ji} = -T_{ij}S_{ij} \to 2T_{ij}S_{ij} = 0 \to T_{ij}S_{ij} = 0$.

2.16 Let $T_{ij} = \left(S_{ij} + S_{ji}\right)/2$ and $R_{ij} = \left(S_{ij} - S_{ji}\right)/2$, show that $T_{ij} = T_{ji}$, $R_{ij} = -R_{ji}$, and $S_{ij} = T_{ij} + \mathrm{R}_{ij}$.

Ans. $T_{ij} = \left(S_{ij} + S_{ji}\right)/2 \to T_{ji} = \left(S_{ji} + S_{ij}\right)/2 = T_{ij}$.

$R_{ij} = \left(S_{ij} - S_{ji}\right)/2 \to R_{ji} = \left(S_{ji} - S_{ij}\right)/2 = -\left(S_{ij} - S_{ji}\right)/2 = -R_{ij}$.

$T_{ij} + R_{ij} = \left(S_{ij} + S_{ji}\right)/2 + \left(S_{ij} - S_{ji}\right)/2 = S_{ij}$.

2.17 Let $f(x_1, x_2, x_3)$ be a function of $x_1, x_2,$ and x_3 and $v_i(x_1, x_2, x_3)$ be three functions of $x_1, x_2,$ and x_3 . Express the total differential df and dv_i in indicial notation.

Ans. $df = \dfrac{\partial f}{\partial x_1}dx_1 + \dfrac{\partial f}{\partial x_2}dx_2 + \dfrac{\partial f}{\partial x_3}dx_3 = \dfrac{\partial f}{\partial x_i}dx_i$.

$dv_i = \dfrac{\partial v_i}{\partial x_1}dx_1 + \dfrac{\partial v_i}{\partial x_2}dx_2 + \dfrac{\partial v_i}{\partial x_3}dx_3 = \dfrac{\partial v_i}{\partial x_m}dx_m$.

2.18 Let $\left|A_{ij}\right|$ denote that determinant of the matrix $\left[A_{ij}\right]$. Show that $\left|A_{ij}\right| = \varepsilon_{ijk}A_{i1}A_{j2}A_{k3}$

Ans. $\varepsilon_{ijk}A_{i1}A_{j2}A_{k3} = \varepsilon_{1jk}A_{11}A_{j2}A_{k3} + \varepsilon_{2jk}A_{21}A_{j2}A_{k3} + \varepsilon_{3jk}A_{31}A_{j2}A_{k3}$

$= \varepsilon_{123}A_{11}A_{22}A_{33} + \varepsilon_{132}A_{11}A_{32}A_{23} + \varepsilon_{231}A_{21}A_{32}A_{13} + \varepsilon_{213}A_{21}A_{12}A_{33} + \varepsilon_{312}A_{31}A_{12}A_{23} + \varepsilon_{321}A_{31}A_{22}A_{13}$

$= A_{11}A_{22}A_{33} - A_{11}A_{32}A_{23} + A_{21}A_{32}A_{13} - A_{21}A_{12}A_{33} + A_{31}A_{12}A_{23} - A_{31}A_{22}A_{13}$

$$= \begin{vmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{vmatrix}$$

CHAPTER 2, PART B

2.19 A transformation $\mathbf{T}$ operate on any vector $\mathbf{a}$ to give $\mathbf{Ta} = \mathbf{a}/|\mathbf{a}|$, where $|\mathbf{a}|$ is the magnitude of $\mathbf{a}$. Show that $\mathbf{T}$ is not a linear transformation.

Ans. Since $\mathbf{Ta} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$ for any $\mathbf{a}$, therefore $\mathbf{T}(\mathbf{a}+\mathbf{b}) = \dfrac{\mathbf{a}+\mathbf{b}}{|\mathbf{a}+\mathbf{b}|}$. Now $\mathbf{Ta}+\mathbf{Tb} = \dfrac{\mathbf{a}}{|\mathbf{a}|}+\dfrac{\mathbf{b}}{|\mathbf{b}|}$ therefore $\mathbf{T}(\mathbf{a}+\mathbf{b}) \neq \mathbf{Ta}+\mathbf{Tb}$ and $\mathbf{T}$ is not a linear transformation.

2.20 (a) A tensor $\mathbf{T}$ transforms every vector $\mathbf{a}$ into a vector $\mathbf{Ta} = \mathbf{m}\times\mathbf{a}$ where $\mathbf{m}$ is a specified vector. Show that $\mathbf{T}$ is a linear transformation and (b) If $\mathbf{m} = \mathbf{e}_1+\mathbf{e}_2$, find the matrix of the tensor $\mathbf{T}$.

Ans. (a) $\mathbf{T}(\alpha\mathbf{a}+\beta\mathbf{b}) = \mathbf{m}\times(\alpha\mathbf{a}+\beta\mathbf{b}) = \mathbf{m}\times\alpha\mathbf{a}+\mathbf{m}\times\beta\mathbf{b} = \alpha\mathbf{m}\times\mathbf{a}+\beta\mathbf{m}\times\mathbf{b} = \alpha\mathbf{Ta}+\beta\mathbf{Tb}.$ Thus, the given $\mathbf{T}$ is a linear transformation.

(b) $\mathbf{Te}_1 = \mathbf{m}\times\mathbf{e}_1 = (\mathbf{e}_1+\mathbf{e}_2)\times\mathbf{e}_1 = -\mathbf{e}_3$, $\quad \mathbf{Te}_2 = \mathbf{m}\times\mathbf{e}_2 = (\mathbf{e}_1+\mathbf{e}_2)\times\mathbf{e}_2 = \mathbf{e}_3$,

$\mathbf{Te}_3 = \mathbf{m}\times\mathbf{e}_3 = (\mathbf{e}_1+\mathbf{e}_2)\times\mathbf{e}_3 = -\mathbf{e}_2+\mathbf{e}_1$. Thus,

$$[\mathbf{T}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

2.21 A tensor $\mathbf{T}$ transforms the base vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ such that $\mathbf{Te}_1 = \mathbf{e}_1+\mathbf{e}_2$ and $\mathbf{Te}_2 = \mathbf{e}_1-\mathbf{e}_2$. If $\mathbf{a} = 2\mathbf{e}_1+3\mathbf{e}_2$ and $\mathbf{b} = 3\mathbf{e}_1+2\mathbf{e}_2$, use the linear property of $\mathbf{T}$ to find (a) $\mathbf{Ta}$, (b) $\mathbf{Tb}$, and (c) $\mathbf{T}(\mathbf{a}+\mathbf{b})$.

Ans.

(a) $\mathbf{Ta} = \mathbf{T}(2\mathbf{e}_1+3\mathbf{e}_2) = 2\mathbf{Te}_1+3\mathbf{Te}_2 = 2(\mathbf{e}_1+\mathbf{e}_2)+3(\mathbf{e}_1-\mathbf{e}_2) = 5\mathbf{e}_1-\mathbf{e}_2$.

(b) $\mathbf{Tb} = \mathbf{T}(3\mathbf{e}_1+2\mathbf{e}_2) = 3\mathbf{Te}_1+2\mathbf{Te}_2 = 3(\mathbf{e}_1+\mathbf{e}_2)+2(\mathbf{e}_1-\mathbf{e}_2) = 5\mathbf{e}_1+\mathbf{e}_2$.

(c) $\mathbf{T}(\mathbf{a}+\mathbf{b}) = \mathbf{Ta}+\mathbf{Tb} = (5\mathbf{e}_1-\mathbf{e}_2)+(5\mathbf{e}_1+\mathbf{e}_2) = 10\mathbf{e}_1$.

2.22 Obtain the matrix for the tensor $\mathbf{T}$ which transforms the base vectors as follows: $\mathbf{Te}_1 = 2\mathbf{e}_1+\mathbf{e}_3$, $\mathbf{Te}_2 = \mathbf{e}_2+3\mathbf{e}_3$, $\mathbf{Te}_3 = -\mathbf{e}_1+3\mathbf{e}_2$.

Ans. $[\mathbf{T}] = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 3 & 0 \end{bmatrix}$.

2.23 Find the matrix of the tensor $\mathbf{T}$ which transforms any vector $\mathbf{a}$ into a vector $\mathbf{b} = \mathbf{m}(\mathbf{a}\cdot\mathbf{n})$ where $\mathbf{m} = \left(\sqrt{2}/2\right)(\mathbf{e}_1+\mathbf{e}_2)$ and $\mathbf{n} = \left(\sqrt{2}/2\right)(-\mathbf{e}_1+\mathbf{e}_3)$.

Ans. $\mathbf{Te}_1 = \mathbf{m}(\mathbf{e}_1\cdot\mathbf{n}) = n_1\mathbf{m} = \left(-\sqrt{2}/2\right)\left[\left(\sqrt{2}/2\right)(\mathbf{e}_1+\mathbf{e}_2)\right] = -(\mathbf{e}_1+\mathbf{e}_2)/2$.

$\mathbf{Te}_2 = \mathbf{m}(\mathbf{e}_2 \cdot \mathbf{n}) = n_2\mathbf{m} = 0\mathbf{m} = \mathbf{0}$.

$\mathbf{Te}_3 = \mathbf{m}(\mathbf{e}_3 \cdot \mathbf{n}) = n_3\mathbf{m} = (\sqrt{2}/2)\left[(\sqrt{2}/2)(\mathbf{e}_1 + \mathbf{e}_2)\right] = (\mathbf{e}_1 + \mathbf{e}_2)/2$.

Thus, $[\mathbf{T}] = \begin{bmatrix} -1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$.

2.24 (a) A tensor $\mathbf{T}$ transforms every vector into its mirror image with respect to the plane whose normal is $\mathbf{e}_2$. Find the matrix of $\mathbf{T}$. (b) Do part (a) if the plane has a normal in the $\mathbf{e}_3$ direction.

Ans. (a) $\mathbf{Te}_1 = \mathbf{e}_1,\ \mathbf{Te}_2 = -\mathbf{e}_2,\quad \mathbf{Te}_3 = \mathbf{e}_3$, thus, $[\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $\mathbf{Te}_1 = \mathbf{e}_1,\ \mathbf{Te}_2 = \mathbf{e}_2,\quad \mathbf{Te}_3 = -\mathbf{e}_3$, thus, $[\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$.

2.25 (a) Let $\mathbf{R}$ correspond to a right-hand rotation of angle θ about the x_1-axis. Find the matrix of $\mathbf{R}$. (b) do part (a) if the rotation is about the x_2-axis. The coordinates are right-handed.

Ans.(a) $\mathbf{Re}_1 = \mathbf{e}_1,\ \mathbf{Re}_2 = 0\mathbf{e}_1 + \cos\theta\mathbf{e}_2 + \sin\theta\mathbf{e}_3,\quad \mathbf{Re}_3 = 0\mathbf{e}_1 - \sin\theta\mathbf{e}_2 + \cos\theta\mathbf{e}_3$. Thus,

$[\mathbf{R}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$.

(b) $\mathbf{Re}_1 = -\sin\theta\mathbf{e}_3 + \cos\theta\mathbf{e}_1,\ \mathbf{Re}_2 = \mathbf{e}_2,\quad \mathbf{Re}_3 = \cos\theta\mathbf{e}_3 + \sin\theta\mathbf{e}_1$. Thus,

$[\mathbf{R}] = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$.

2.26 Consider a plane of reflection which passes through the origin. Let $\mathbf{n}$ be a unit normal vector to the plane and let $\mathbf{r}$ be the position vector for a point in space. (a) Show that the reflected vector for $\mathbf{r}$ is given by $\mathbf{Tr} = \mathbf{r} - 2(\mathbf{r}\cdot\mathbf{n})\mathbf{n}$, where $\mathbf{T}$ is the transformation that corresponds to the reflection. (b) Let $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$, find the matrix of $\mathbf{T}$. (c) Use this linear transformation to find the mirror image of the vector $\mathbf{a} = \mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$.

Ans. (a) Let the vector $\mathbf{r}$ be decomposed into two vectors $\mathbf{r}_n$ and $\mathbf{r}_t$, where $\mathbf{r}_n$ is in the direction of $\mathbf{n}$ and $\mathbf{r}_t$ is in a direction perpendicular to $\mathbf{n}$. That is, $\mathbf{r}_n$ is normal to the plane of reflection and $\mathbf{r}_t$ is on the plane of reflection and $\mathbf{r} = \mathbf{r}_t + \mathbf{r}_n$. In the reflection given by $\mathbf{T}$, we have,

$\mathbf{Tr}_n = -\mathbf{r}_n$ and $\mathbf{Tr}_t = \mathbf{r}_t$, so that $\mathbf{Tr} = \mathbf{Tr}_t + \mathbf{Tr}_n = \mathbf{r}_t - \mathbf{r}_n = (\mathbf{r} - \mathbf{r}_n) - \mathbf{r}_n = \mathbf{r} - 2\mathbf{r}_n = \mathbf{r} - 2(\mathbf{r}\cdot\mathbf{n})\mathbf{n}$.

(b) $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3} \rightarrow \mathbf{e}_1\cdot\mathbf{n} = \mathbf{e}_2\cdot\mathbf{n} = \mathbf{e}_3\cdot\mathbf{n} = 1/\sqrt{3}$.

$\mathbf{Te}_1 = \mathbf{e}_1 - 2(\mathbf{e}_1 \cdot \mathbf{n})\mathbf{n} = \mathbf{e}_1 - 2\left(1/\sqrt{3}\right)\left[(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}\right] = \left(\mathbf{e}_1 - 2\mathbf{e}_2 - 2\mathbf{e}_3\right)/3$.

$\mathbf{Te}_2 = \mathbf{e}_2 - 2(\mathbf{e}_2 \cdot \mathbf{n})\mathbf{n} = \mathbf{e}_2 - 2\left(1/\sqrt{3}\right)\left[(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}\right] = \left(-2\mathbf{e}_1 + \mathbf{e}_2 - 2\mathbf{e}_3\right)/3$.

$\mathbf{Te}_3 = \mathbf{e}_3 - 2(\mathbf{e}_3 \cdot \mathbf{n})\mathbf{n} = \mathbf{e}_3 - 2\left(1/\sqrt{3}\right)\left[(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}\right] = \left(-2\mathbf{e}_1 - 2\mathbf{e}_2 + \mathbf{e}_3\right)/3$.

$$[\mathbf{T}] = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}.$$

(c) $[\mathbf{T}][\mathbf{a}] = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ -1 \end{bmatrix} \rightarrow \mathbf{Ta} = -\left(3\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3\right)$.

2.27 Knowing that the reflected vector for $\mathbf{r}$ is given by $\mathbf{Tr} = \mathbf{r} - 2(\mathbf{r} \cdot \mathbf{n})\mathbf{n}$ (see the previous problem), where $\mathbf{T}$ is the transformation that corresponds to the reflection and $\mathbf{n}$ is the normal to the mirror, show that in dyadic notation, the reflection tensor is given by $\mathbf{T} = \mathbf{I} - 2\mathbf{nn}$ and find the matrix of $\mathbf{T}$ if the normal of the mirror is given by $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$,

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Ans. From the definition of dyadic product, we have ,

$\mathbf{Tr} = \mathbf{r} - 2(\mathbf{r} \cdot \mathbf{n})\mathbf{n} = \mathbf{r} - 2(\mathbf{nn})\mathbf{r} = (\mathbf{Ir} - 2(\mathbf{nn})\mathbf{r}) = (\mathbf{I} - 2\mathbf{nn})\mathbf{r} \rightarrow \mathbf{T} = \mathbf{I} - 2\mathbf{nn}$.

For $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3} \rightarrow [2\mathbf{nn}] = \frac{2}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} = \frac{2}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

$$\rightarrow [\mathbf{T}] = [\mathbf{I}] - [2\mathbf{nn}] = \frac{1}{3}\begin{bmatrix} 1 & -2 & -2 \\ -2 & 1 & -2 \\ -2 & -2 & 1 \end{bmatrix}.$$

2.28 A rotation tensor $\mathbf{R}$ is defined by the relation $\mathbf{Re}_1 = \mathbf{e}_2$, $\mathbf{Re}_2 = \mathbf{e}_3$, $\mathbf{Re}_3 = \mathbf{e}_1$ (a) Find the matrix of $\mathbf{R}$ and verify that $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$ and $\det \mathbf{R} = 1$ and (b) find a unit vector in the direction of the axis of rotation that could have been used to effect this particular rotation.

Ans. (a) $[\mathbf{R}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \rightarrow [\mathbf{R}]^{\mathrm{T}}[\mathbf{R}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\det[\mathbf{R}] = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = 1$.

(b) Let the axis of rotation be $\mathbf{n} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$, then

$$\mathbf{Rn} = \mathbf{n} \rightarrow [\mathbf{R} - \mathbf{I}][\mathbf{n}] = [\mathbf{0}] \rightarrow \begin{bmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow -\alpha_1 + \alpha_3 = 0, \quad \alpha_1 - \alpha_2 = 0, \quad \alpha_2 - \alpha_3 = 0 \ .$$

Thus, $\alpha_1 = \alpha_2 = \alpha_3$, so that a unit vector in the direction of the axis of rotation is

$\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$.

2.29 A rigid body undergoes a right hand rotation of angle θ about an axis which is in the direction of the unit vector $\mathbf{m}$. Let the origin of the coordinates be on the axis of rotation and $\mathbf{r}$ be the position vector for a typical point in the body. (a) show that the rotated vector of $\mathbf{r}$ is given by: $\mathbf{Rr}=(1-\cos\theta)(\mathbf{m}\cdot\mathbf{r})\mathbf{m}+\cos\theta\mathbf{r}+\sin\theta(\mathbf{m}\times\mathbf{r})$, where $\mathbf{R}$ is the rotation tensor. (b) Let $\mathbf{m}=(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}$, find the matrix for $\mathbf{R}$.

Ans. (a) Let the vector $\mathbf{r}$ be decomposed into two vectors $\mathbf{r}_m$ and $\mathbf{r}_p$, where $\mathbf{r}_m$ is in the direction of $\mathbf{m}$ and $\mathbf{r}_p$ is in a direction perpendicular to $\mathbf{m}$, that is, $\mathbf{r}=\mathbf{r}_p+\mathbf{r}_m$. Let $\mathbf{p}\equiv\mathbf{r}_p/|\mathbf{r}_p|$ be the unit vector in the direction of $\mathbf{r}_p$, and let $\mathbf{q}\equiv\mathbf{m}\times\mathbf{p}$. Then, $(\mathbf{m},\mathbf{p},\mathbf{q})$ forms an orthonormal set of vectors which rotates an angle of θ about the unit vector $\mathbf{m}$. Thus,

$\mathbf{Rr}_m=\mathbf{r}_m$ and $\mathbf{Rr}_p=|\mathbf{r}_p|(\cos\theta\mathbf{p}+\sin\theta\mathbf{q})$. From $\mathbf{r}=\mathbf{r}_p+\mathbf{r}_m$, we have,

$$\mathbf{Rr}=\mathbf{Rr}_p+\mathbf{Rr}_m=|\mathbf{r}_p|(\cos\theta\mathbf{p}+\sin\theta\mathbf{q})+\mathbf{r}_m=\left\{\cos\theta|\mathbf{r}_p|\mathbf{p}+\sin\theta|\mathbf{r}_p|(\mathbf{m}\times\mathbf{p})\right\}+\mathbf{r}_m$$

$$=\left\{\cos\theta\mathbf{r}_p+\sin\theta(\mathbf{m}\times\mathbf{r}_p)\right\}+\mathbf{r}_m=\left\{\cos\theta(\mathbf{r}-\mathbf{r}_m)+\sin\theta(\mathbf{m}\times(\mathbf{r}-\mathbf{r}_m))\right\}+\mathbf{r}_m$$

$$=\mathbf{r}\cos\theta+\mathbf{r}_m(1-\cos\theta)+\sin\theta\mathbf{m}\times(\mathbf{r}-\mathbf{r}_m)=\mathbf{r}\cos\theta+\mathbf{r}_m(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{r}$$

We note that $\mathbf{r}_m=(\mathbf{r}\cdot\mathbf{m})\mathbf{m}$, so that $\mathbf{Rr}=\mathbf{r}\cos\theta+(\mathbf{r}\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{r}$.

(b) Use the result of (a), that is, $\mathbf{Rr}=\mathbf{r}\cos\theta+(\mathbf{r}\cdot\mathbf{m})(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{r}$, we have,

$\mathbf{Re}_1=\mathbf{e}_1\cos\theta+(\mathbf{e}_1\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{e}_1$,

$\mathbf{Re}_2=\mathbf{e}_2\cos\theta+(\mathbf{e}_2\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{e}_2$,

$\mathbf{Re}_3=\mathbf{e}_3\cos\theta+(\mathbf{e}_3\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{e}_3$.

Now, $\mathbf{m}=(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}$, therefore, $\mathbf{m}\cdot\mathbf{e}_1=\mathbf{m}\cdot\mathbf{e}_2=\mathbf{m}\cdot\mathbf{e}_3=1/\sqrt{3}$

$\mathbf{m}\times\mathbf{e}_1=(1/\sqrt{3})(-\mathbf{e}_3+\mathbf{e}_2)$, $\mathbf{m}\times\mathbf{e}_2=(1/\sqrt{3})(\mathbf{e}_3-\mathbf{e}_1)$, $\mathbf{m}\times\mathbf{e}_3=(1/\sqrt{3})(-\mathbf{e}_2+\mathbf{e}_1)$. Thus,

$$\mathbf{Re}_1=\mathbf{e}_1\cos\theta+(\mathbf{e}_1\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{e}_1$$

$$=\mathbf{e}_1\cos\theta+(1/3)(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)(1-\cos\theta)+\sin\theta(1/\sqrt{3})(-\mathbf{e}_3+\mathbf{e}_2)$$

$$=(1/3)\{1+2\cos\theta\}\mathbf{e}_1+\mathbf{e}_2\left\{(1/3)(1-\cos\theta)+\sin\theta(1/\sqrt{3})\right\}+\mathbf{e}_3\left\{(1/3)(1-\cos\theta)-\sin\theta(1/\sqrt{3})\right\}$$

$$\mathbf{Re}_2=\mathbf{e}_2\cos\theta+(\mathbf{e}_2\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{e}_2$$

$$=\mathbf{e}_2\cos\theta+(1/3)(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)(1-\cos\theta)+\sin\theta(1/\sqrt{3})(\mathbf{e}_3-\mathbf{e}_1)$$

$$=\left\{(1/3)(1-\cos\theta)-(1/\sqrt{3})\sin\theta\right\}\mathbf{e}_1+(1/3)(1+2\cos\theta)\mathbf{e}_2+\left\{(1/3)(1-\cos\theta)+\sin\theta(1/\sqrt{3})\right\}\mathbf{e}_3$$

$$\mathbf{Re}_3=\mathbf{e}_3\cos\theta+(\mathbf{e}_3\cdot\mathbf{m})\mathbf{m}(1-\cos\theta)+\sin\theta\mathbf{m}\times\mathbf{e}_3$$

$$=\mathbf{e}_3\cos\theta+(1/3)(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)(1-\cos\theta)+\sin\theta(1/\sqrt{3})(-\mathbf{e}_2+\mathbf{e}_1)$$

$$=\left\{(1/3)(1-\cos\theta)+(1/\sqrt{3})\sin\theta\right\}\mathbf{e}_1+\left\{(1/3)(1-\cos\theta)-\sin\theta(1/\sqrt{3})\right\}\mathbf{e}_2+(1/3)(1+2\cos\theta)\mathbf{e}_3$$

Thus,

$$[\mathbf{T}]=\frac{1}{3}\begin{bmatrix} 1+2\cos\theta & (1-\cos\theta)-\sqrt{3}\sin\theta & (1-\cos\theta)+\sqrt{3}\sin\theta \\ (1-\cos\theta)+\sqrt{3}\sin\theta & (1+2\cos\theta) & (1-\cos\theta)-\sqrt{3}\sin\theta \\ (1-\cos\theta)-\sqrt{3}\sin\theta & (1-\cos\theta)+\sqrt{3}\sin\theta & (1+2\cos\theta) \end{bmatrix}.$$

2.30 For the rotation about an arbitrary axis $\mathbf{m}$ by an angle θ, (a) show that the rotation tensor is given by $\mathbf{R} = (1-\cos\theta)(\mathbf{mm})+\cos\theta\mathbf{I}+\sin\theta\mathbf{E}$, where $\mathbf{mm}$ denotes that dyadic product of $\mathbf{m}$ and $\mathbf{E}$ is the antisymmetric tensor whose dual vector (or axial vector) is $\mathbf{m}$, (b) find the $\mathbf{R}^{\mathrm{A}}$, the antisymmetric part of $\mathbf{R}$ and (c) show that the dual vector for $\mathbf{R}^{\mathrm{A}}$ is given by $(\sin\theta)\mathbf{m}$. Hint, $\mathbf{Rr}=(1-\cos\theta)(\mathbf{m}\cdot\mathbf{r})\mathbf{m}+\cos\theta\mathbf{r}+\sin\theta(\mathbf{m}\times\mathbf{r})$ (see previous problem).

Ans. (a) We have, from the previous problem, $\mathbf{Rr}=(1-\cos\theta)(\mathbf{m}\cdot\mathbf{r})\mathbf{m}+\cos\theta\mathbf{r}+\sin\theta(\mathbf{m}\times\mathbf{r})$.
Now, by the definition of dyadic product, we have $(\mathbf{m}\cdot\mathbf{r})\mathbf{m} = (\mathbf{mm})\mathbf{r}$, and by the definition of dual vector we have, $\mathbf{m}\times\mathbf{r} = \mathbf{Er}$, thus $\mathbf{Rr}=(1-\cos\theta)(\mathbf{mm})\mathbf{r}+\cos\theta\mathbf{r}+\sin\theta\mathbf{Er}$
$=\{(1-\cos\theta)(\mathbf{mm})+\cos\theta\mathbf{I}+\sin\theta\mathbf{E}\}\mathbf{r}$, from which, $\mathbf{R}=(1-\cos\theta)(\mathbf{mm})+\cos\theta\mathbf{I}+\sin\theta\mathbf{E}$.

(b) $\mathbf{R}^{\mathrm{A}}=(\mathbf{R}-\mathbf{R}^{\mathrm{T}})/2\rightarrow$

$2\mathbf{R}^{\mathrm{A}}=\{(1-\cos\theta)(\mathbf{mm})+\cos\theta\mathbf{I}+\sin\theta\mathbf{E}\}-\{(1-\cos\theta)(\mathbf{mm})^{\mathrm{T}}+\cos\theta\mathbf{I}+\sin\theta\mathbf{E}^{\mathrm{T}}\}$. Now

$[\mathbf{mm}]=[m_im_j]=[m_jm_i]=[\mathbf{mm}]^{\mathrm{T}}$, and the tensor $\mathbf{E}$, being antisymmetric, $\mathbf{E}=-\mathbf{E}^{\mathrm{T}}$, therefore,

$2\mathbf{R}^{\mathrm{A}}=2\sin\theta\mathbf{E}$, that is, $\mathbf{R}^{\mathrm{A}}=\sin\theta\mathbf{E}$.

(c) dual vector of $\mathbf{R}^{\mathrm{A}}=(\sin\theta)(\text{dual vector of }\mathbf{E})=\sin\theta\mathbf{m}$.

2.31 (a) Given a mirror whose normal is in the direction of $\mathbf{e}_2$. Find the matrix of the tensor $\mathbf{S}$ which first transforms every vector into its mirror image and then transforms them by a 45° right-hand rotation about the $\mathbf{e}_1$-axis. (b) Find the matrix of the tensor $\mathbf{T}$ which first transforms every vector by a 45° right-hand rotation about the $\mathbf{e}_1$-axis, and then transforms them by a reflection with respect to the mirror (whose normal is $\mathbf{e}_2$). (c) Consider the vector $\mathbf{a}=(\mathbf{e}_1+2\mathbf{e}_2+3\mathbf{e}_3)$, find the transformed vector by using the transformation $\mathbf{S}$.
(d) For the same vector $\mathbf{a}=(\mathbf{e}_1+2\mathbf{e}_2+3\mathbf{e}_3)$, find the transformed vector by using the transformation $\mathbf{T}$.

Ans. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ correspond to the reflection and the rotation respectively. We have

$$\mathbf{T}_1\mathbf{e}_1=\mathbf{e}_1,\ \mathbf{T}_1\mathbf{e}_2=-\mathbf{e}_2,\ \mathbf{T}_1\mathbf{e}_3=\mathbf{e}_3\rightarrow[\mathbf{T}_1]=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}.$$

$$\mathbf{T}_2\mathbf{e}_1=\mathbf{e}_1,\ \mathbf{T}_2\mathbf{e}_2=\frac{1}{\sqrt{2}}(\mathbf{e}_2+\mathbf{e}_3),\ \mathbf{T}_2\mathbf{e}_3=\frac{1}{\sqrt{2}}(-\mathbf{e}_2+\mathbf{e}_3)\rightarrow[\mathbf{T}_2]=\begin{bmatrix}1&0&0\\0&1/\sqrt{2}&-1/\sqrt{2}\\0&1/\sqrt{2}&1/\sqrt{2}\end{bmatrix}.$$

(a) $[\mathbf{S}]=[\mathbf{T}_2][\mathbf{T}_1]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & -1/\sqrt{2} & -1/\sqrt{2}\\ 0 & -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$.

(b) $[\mathbf{T}]=[\mathbf{T}_1][\mathbf{T}_2]=\begin{bmatrix}1 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1/\sqrt{2} & -1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & -1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}$.

(c) $[\mathbf{b}]=[\mathbf{S}][\mathbf{a}]=\begin{bmatrix}1 & 0 & 0\\ 0 & -1/\sqrt{2} & -1/\sqrt{2}\\ 0 & -1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}=\begin{bmatrix}1\\ -5/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix}$.

(d) $[\mathbf{c}]=[\mathbf{T}][\mathbf{a}]=\begin{bmatrix}1 & 0 & 0\\ 0 & -1/\sqrt{2} & 1/\sqrt{2}\\ 0 & 1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}=\begin{bmatrix}1\\ 1/\sqrt{2}\\ 5/\sqrt{2}\end{bmatrix}$.

2.32 Let $\mathbf{R}$ correspond to a right-hand rotation of angle θ about the x_3-axis (a) find the matrix of $\mathbf{R}^2$. (b) Show that $\mathbf{R}^2$ corresponds to a rotation of angle 2θ about the same axis (c) Find the matrix of $\mathbf{R}^n$ for any integer n.

Ans. (a) $[\mathbf{R}]=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$.

$$\rightarrow\left[\mathbf{R}^2\right]=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\cos^2\theta-\sin^2\theta & -2\sin\theta\cos\theta & 0\\ 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta & 0\\ 0 & 0 & 1\end{bmatrix}.$$

(b)

$$\left[\mathbf{R}^2\right]=\begin{bmatrix}\cos^2\theta-\sin^2\theta & -2\sin\theta\cos\theta & 0\\ 2\sin\theta\cos\theta & \cos^2\theta-\sin^2\theta & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}\cos 2\theta & -\sin 2\theta & 0\\ \sin 2\theta & \cos 2\theta & 0\\ 0 & 0 & 1\end{bmatrix}.$$

Thus, $\mathbf{R}^2$ corresponds to a rotation of angle 2θ about the same axis

(c) $\left[\mathbf{R}^n\right]=\begin{bmatrix}\cos n\theta & -\sin n\theta & 0\\ \sin n\theta & \cos n\theta & 0\\ 0 & 0 & 1\end{bmatrix}$.

2.33 Rigid body rotations that are small can be described by an orthogonal transformation $\mathbf{R}=\mathbf{I}+\varepsilon\mathbf{R}^*$ where $\varepsilon\rightarrow 0$ as the rotation angle approaches zero. Consider two successive small rotations $\mathbf{R}_1$ and $\mathbf{R}_2$, show that the final result does not depend on the order of rotations.

Ans. $\mathbf{R}_2\mathbf{R}_1 = \left(\mathbf{I}+\varepsilon\mathbf{R}_2^*\right)\left(\mathbf{I}+\varepsilon\mathbf{R}_1^*\right) = \mathbf{I}+\varepsilon\mathbf{R}_2^*+\varepsilon\mathbf{R}_1^*+\varepsilon^2\mathbf{R}_2^*\mathbf{R}_1^* = \mathbf{I}+\varepsilon\left(\mathbf{R}_2^*+\mathbf{R}_1^*\right)+\varepsilon^2\mathbf{R}_2^*\mathbf{R}_1^*$.
As $\varepsilon \to 0$, $\mathbf{R}_2\mathbf{R}_1 \approx \mathbf{I}+\varepsilon\left(\mathbf{R}_2^*+\mathbf{R}_1^*\right) = \mathbf{R}_1\mathbf{R}_2$.

2.34 Let $\mathbf{T}$ and $\mathbf{S}$ be any two tensors. Show that (a) $\mathbf{T}^{\mathrm{T}}$ is a tensor, (b) $\mathbf{T}^{\mathrm{T}}+\mathbf{S}^{\mathrm{T}} = (\mathbf{T}+\mathbf{S})^{\mathrm{T}}$ and (c) $(\mathbf{TS})^{\mathrm{T}} = \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}$.

Ans. Let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three arbitrary vectors and α β be any two scalars, then

(a) $\mathbf{a}\cdot\mathbf{T}^{\mathrm{T}}(\alpha\mathbf{b}+\beta\mathbf{c}) = (\alpha\mathbf{b}+\beta\mathbf{c})\cdot\mathbf{Ta} = \alpha\mathbf{b}\cdot\mathbf{Ta}+\beta\mathbf{c}\cdot\mathbf{Ta} = \alpha\mathbf{a}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{b}+\beta\mathbf{a}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{c}$
$= \mathbf{a}\cdot\left(\alpha\mathbf{T}^{\mathrm{T}}\mathbf{b}+\beta\mathbf{T}^{\mathrm{T}}\mathbf{c}\right) \to \mathbf{T}^{\mathrm{T}}(\alpha\mathbf{b}+\beta\mathbf{c}) = \left(\alpha\mathbf{T}^{\mathrm{T}}\mathbf{b}+\beta\mathbf{T}^{\mathrm{T}}\mathbf{c}\right)$. Thus, $\mathbf{T}^{\mathrm{T}}$ is a linear transformation, i.e., tensor.

(b) $\mathbf{a}\cdot(\mathbf{T}+\mathbf{S})^{\mathrm{T}}\mathbf{b} = \mathbf{b}\cdot(\mathbf{T}+\mathbf{S})\mathbf{a} = \mathbf{b}\cdot\mathbf{Ta}+\mathbf{b}\cdot\mathbf{Sa} = \mathbf{a}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{b}+\mathbf{a}\cdot\mathbf{S}^{\mathrm{T}}\mathbf{b}$
$= \mathbf{a}\cdot(\mathbf{T}^{\mathrm{T}}+\mathbf{S}^{\mathrm{T}})\mathbf{b} \to (\mathbf{T}+\mathbf{S})^{\mathrm{T}} = \mathbf{T}^{\mathrm{T}}+\mathbf{S}^{\mathrm{T}}$.

(c) $\mathbf{a}\cdot(\mathbf{TS})^{\mathrm{T}}\mathbf{b} = \mathbf{b}\cdot(\mathbf{TS})\mathbf{a} = \mathbf{b}\cdot\mathbf{T}(\mathbf{Sa}) = (\mathbf{Sa})\cdot\mathbf{T}^{\mathrm{T}}\mathbf{b} = \mathbf{a}\cdot\mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}\mathbf{b} \to (\mathbf{TS})^{\mathrm{T}} = \mathbf{S}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}}$.

2.35 For arbitrary tensors $\mathbf{T}$ and $\mathbf{S}$, without relying on the component form, prove that (a) $(\mathbf{T}^{-1})^{\mathrm{T}} = (\mathbf{T}^{\mathrm{T}})^{-1}$ and (b) $(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\mathbf{T}^{-1}$

Ans. (a) $\mathbf{TT}^{-1} = \mathbf{I} \to (\mathbf{TT}^{-1})^{\mathrm{T}} = \mathbf{I} \to (\mathbf{T}^{-1})^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} = \mathbf{I} \to (\mathbf{T}^{-1})^{\mathrm{T}} = (\mathbf{T}^{\mathrm{T}})^{-1}$.
(b) $(\mathbf{TS})(\mathbf{S}^{-1}\mathbf{T}^{-1}) = \mathbf{T}(\mathbf{SS}^{-1})\mathbf{T}^{-1} = \mathbf{TT}^{-1} = \mathbf{I}$, thus, $(\mathbf{TS})^{-1} = \mathbf{S}^{-1}\mathbf{T}^{-1}$.

2.36 Let $\{\mathbf{e}_i\}$ and $\{\mathbf{e}_i'\}$ be two Rectangular Cartesian base vectors. (a) Show that if $\mathbf{e}_i' = Q_{mi}\mathbf{e}_m$, then $\mathbf{e}_i = Q_{im}\mathbf{e}_m'$ and (b) verify $Q_{mi}Q_{mj} = \delta_{ij} = Q_{im}Q_{jm}$.

Ans. (a) $\mathbf{e}_i' = Q_{mi}\mathbf{e}_m \to \mathbf{e}_i'\cdot\mathbf{e}_j = Q_{mi}\mathbf{e}_m\cdot\mathbf{e}_j = Q_{mi}\delta_{mj} = Q_{ji} \to \mathbf{e}_j = Q_{jm}\mathbf{e}_m' \to \mathbf{e}_i = Q_{im}\mathbf{e}_m'$.
(b) We have, $\mathbf{e}_i'\cdot\mathbf{e}_j' = \delta_{ij} = \mathbf{e}_i\cdot\mathbf{e}_j$, thus,
$\delta_{ij} = \mathbf{e}_i'\cdot\mathbf{e}_j' = Q_{mi}\mathbf{e}_m\cdot Q_{nj}\mathbf{e}_n = Q_{mi}Q_{nj}\mathbf{e}_m\cdot\mathbf{e}_n = Q_{mi}Q_{nj}\delta_{mn} = Q_{mi}Q_{mj}$. And
$\delta_{ij} = \mathbf{e}_i\cdot\mathbf{e}_j = Q_{im}\mathbf{e}_m\cdot Q_{jn}\mathbf{e}_n = Q_{im}Q_{jn}\mathbf{e}_m\cdot\mathbf{e}_n = Q_{im}Q_{jn}\delta_{\mathrm{mn}} = Q_{im}Q_{jm}$.

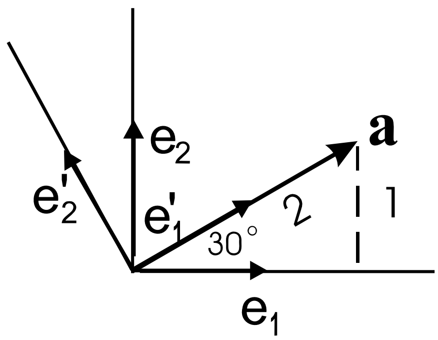
2.37 The basis $\{\mathbf{e}_i'\}$ is obtained by a 30° counterclockwise rotation of the $\{\mathbf{e}_i\}$ basis about the $\mathbf{e}_3$ axis. (a) Find the transformation matrix $[\mathbf{Q}]$ relating the two sets of basis, (b) by using the vector transformation law, find the components of $\mathbf{a} = \sqrt{3}\mathbf{e}_1+\mathbf{e}_2$ in the primed basis, i.e., find a_i' and (c) do part (b) geometrically.

Ans. (a) $\mathbf{e}_1' = \cos 30^{\circ}\mathbf{e}_1+\sin 30^{\circ}\mathbf{e}_2$, $\mathbf{e}_2' = -\sin 30^{\circ}\mathbf{e}_1+\cos 30^{\circ}\mathbf{e}_2$, $\mathbf{e}_3' = \mathbf{e}_3$. Thus,

$$[\mathbf{Q}]_{\mathbf{e}_i} = \begin{bmatrix} \cos 30^{\circ} & -\sin 30^{\circ} & 0 \\ \sin 30^{\circ} & \cos 30^{\circ} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) $[\mathbf{a}]_{\mathbf{e}_i'} = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{a}]_{\mathbf{e}_i} \rightarrow \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & 1/2 & 0 \\ -1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \rightarrow \mathbf{a} = 2\mathbf{e}_1'$

(c) Clearly $\mathbf{a} = \sqrt{3}\mathbf{e}_1 + \mathbf{e}_2$ is a vector in the same direction as $\mathbf{e}_1'$ and has a length of 2. See figure below



2.38 Do the previous problem with the $\{\mathbf{e}_i'\}$ basis obtained by a 30^{o} clockwise rotation of the $\{\mathbf{e}_i\}$ basis about the $\mathbf{e}_3$ axis.

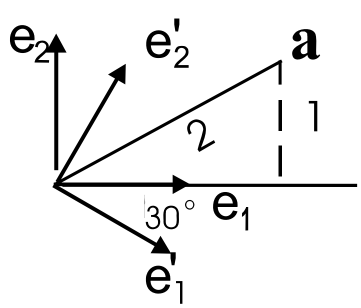
Ans.

(a) $\mathbf{e}_1' = \cos 30^{\mathrm{o}}\mathbf{e}_1 - \sin 30^{\mathrm{o}}\mathbf{e}_2$, $\mathbf{e}_2' = \sin 30^{\mathrm{o}}\mathbf{e}_1 + \cos 30^{\mathrm{o}}\mathbf{e}_2$, $\mathbf{e}_3' = \mathbf{e}_3$. Thus,

$$[\mathbf{Q}]_{\mathbf{e}_i} = \begin{bmatrix} \cos 30^{\mathrm{o}} & \sin 30^{\mathrm{o}} & 0 \\ -\sin 30^{\mathrm{o}} & \cos 30^{\mathrm{o}} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b) $[\mathbf{a}]_{\mathbf{e}_i'} = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{a}]_{\mathbf{e}_i} \rightarrow \begin{bmatrix} a_1' \\ a_2' \\ a_3' \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 0 \\ 1/2 & \sqrt{3}/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{3} \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ \sqrt{3} \\ 0 \end{bmatrix} \rightarrow \mathbf{a} = \mathbf{e}_1' + \sqrt{3}\mathbf{e}_2'$

(c) See figure below



2.39 The matrix of a tensor $\mathbf{T}$ with respect to the basis $\{\mathbf{e}_i\}$ is

$$[\mathbf{T}] = \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix}$$

Find T'_{11}, T'_{12} and T'_{31} with respect to a right-handed basis $\{\mathbf{e}'_i\}$ where $\mathbf{e}'_1$ is in the direction of $-\mathbf{e}_2 + 2\mathbf{e}_3$ and $\mathbf{e}'_2$ is in the direction of $\mathbf{e}_1$.

Ans. The basis $\{\mathbf{e}'_i\}$ is given by:

$\mathbf{e}'_1 = (-\mathbf{e}_2 + 2\mathbf{e}_3)/\sqrt{5}, \ \ \mathbf{e}'_2 = \mathbf{e}_1, \ \ \ \mathbf{e}'_3 = \mathbf{e}'_1 \times \mathbf{e}'_2 = (2\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{5}$.

$$T'_{11} = \mathbf{e}'_1 \cdot \mathbf{T}\mathbf{e}'_1 = \begin{bmatrix} 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = 4/5 .$$

$$T'_{12} = \mathbf{e}'_1 \cdot \mathbf{T}\mathbf{e}'_2 = \begin{bmatrix} 0 & -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -15/\sqrt{5} .$$

$$T'_{31} = \mathbf{e}'_3 \cdot \mathbf{T}\mathbf{e}'_1 = \begin{bmatrix} 0 & 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} = 2/5 .$$

2.40 (a) For the tensor of the previous problem, find $\left[T'_{ij}\right]$, i.e., $[\mathbf{T}]_{\mathbf{e}'_i}$ if $\{\mathbf{e}'_i\}$ is obtained by a 90^{o} right hand rotation about the $\mathbf{e}_3$ axis and (b) obtain T'_{ii} and the determinant $\left|T'_{ij}\right|$ and compare them with T_{ii} and $\left|T_{ij}\right|$.

Ans. (a) $\mathbf{e}'_1 = \mathbf{e}_2, \ \ \mathbf{e}'_2 = -\mathbf{e}_1, \ \ \ \mathbf{e}'_3 = \mathbf{e}_3 \rightarrow [\mathbf{Q}] = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

$$\left[T'_{ij}\right] = [\mathbf{T}]' = [\mathbf{Q}]^{\text{T}} [\mathbf{T}][\mathbf{Q}] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 & -5 \\ 5 & 0 & 0 \\ -5 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -5 & 0 \\ -5 & 1 & 5 \\ 0 & 5 & 1 \end{bmatrix}$$

(b) $T'_{ii} = T'_{11} + T'_{22} + T'_{33} = 0 + 1 + 1 = 2, \ \ \ \left|T'_{ij}\right| = -25$.

$T_{ii} = T_{11} + T_{22} + T_{33} = 1 + 0 + 1 = 2, \ \ \ \left|T_{ij}\right| = -25$.

2.41 The dot product of two vectors $\mathbf{a} = a_i\mathbf{e}_i$ and $\mathbf{b} = b_i\mathbf{e}_i$ is equal to $a_i b_i$. Show that the dot product is a scalar invariant with respect to orthogonal transformations of coordinates.

Ans. From $a'_i = Q_{mi} a_m$ and $b'_i = Q_{mi} b_m$, we have,

$a'_i b'_i = Q_{mi} a_m Q_{ni} b_n = Q_{mi} Q_{ni} a_m b_n = \delta_{mn} a_m b_n = a_m b_m = a_i b_i$.

2.42 If T_{ij} are the components of a tensor (a) show that $T_{ij}T_{ij}$ is a scalar invariant with respect to orthogonal transformations of coordinates, (b) evaluate $T_{ij}T_{ij}$ with respect to the basis $\{\mathbf{e}_i\}$ for $[\mathbf{T}] = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}_{\mathbf{e}_i}$, (c) find $[\mathbf{T}]'$, if $\mathbf{e}_i' = \mathbf{Q}\mathbf{e}_i$, where $[\mathbf{Q}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{\mathbf{e}_i}$ and

(d) verify for the above $[\mathbf{T}]$ and $[\mathbf{T}]'$ that $T_{ij}'T_{ij}' = T_{ij}T_{ij}$.

Ans. (a) Since T_{ij} are the components of a tensor, $T_{ij}' = Q_{mi}Q_{nj}T_{mn}$. Thus,

$T_{ij}'T_{ij}' = Q_{mi}Q_{nj}T_{mn}(Q_{pi}Q_{qj}T_{pq}) = (Q_{mi}Q_{pi})(Q_{nj}Q_{qj})T_{mn}T_{pq} = \delta_{mp}\delta_{nq}T_{mn}T_{pq} = T_{mn}T_{mn}$

(b) $T_{ij}T_{ij} = T_{11}^2 + T_{12}^2 + T_{13}^2 + T_{21}^2 + T_{22}^2 + T_{23}^2 + T_{31}^2 + T_{32}^2 + T_{33}^2 = 1+1+4+25+1+4+9 = 45$.

(c) $[\mathbf{T}]' = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{T}][\mathbf{Q}] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 5 \\ 1 & 2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 2 & 5 & 1 \\ 2 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 5 & 1 \\ 2 & 3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $T_{ij}'T_{ij}' = 4+25+1+4+9+1+1 = 45$.

2.43 Let $[\mathbf{T}]$ and $[\mathbf{T}]'$ be two matrices of the same tensor $\mathbf{T}$, show that $\det[\mathbf{T}] = \det[\mathbf{T}]'$.

Ans. $[\mathbf{T}]' = [\mathbf{Q}]^{\mathrm{T}}[\mathbf{T}][\mathbf{Q}] \rightarrow \det[\mathbf{T}]' = \det[\mathbf{Q}]^{\mathrm{T}}\det[\mathbf{Q}]\det[\mathbf{T}] = (\pm 1)(\pm 1)\det[\mathbf{T}] = \det[\mathbf{T}]$.

2.44 (a) If the components of a third order tensor are R_{ijk}, show that R_{iik} are components of a vector, (b) if the components of a fourth order tensor are R_{ijkl}, show that R_{iikl} are components of a second order tensor and (c) what are components of $R_{iik\ldots}$, if $R_{ijk\ldots}$ are components of a tensor of n^{th} order?

Ans. (a) Since R_{ijk} are components of a third order tensor, therefore,

$R_{ijk}' = Q_{mi}Q_{nj}Q_{pk}R_{mnp} \rightarrow R_{iik}' = Q_{mi}Q_{ni}Q_{pk}R_{mnp} = \delta_{mn}Q_{pk}R_{mnp} = Q_{pk}R_{nnp}$, therefore, R_{iik} are components of a vector.

(b) Consider a 4^{th} order tensor R_{ijkl}, we have,

$R_{ijkl}' = Q_{mi}Q_{nj}Q_{pk}Q_{ql}R_{mnpq} \rightarrow R_{iikl}' = Q_{mi}Q_{ni}Q_{pk}Q_{ql}R_{mnpq} = \delta_{mn}Q_{pk}Q_{ql}R_{mnpq} = Q_{pk}Q_{ql}R_{nnpq}$, therefore, R_{iikl} are components of a second order tensor.

(c) $R_{iik\ldots}$ are components of a tensor of the $(n-2)^{th}$ order.

2.45 The components of an arbitrary vector $\mathbf{a}$ and an arbitrary second tensor $\mathbf{T}$ are related by a triply subscripted quantity R_{ijk} in the manner $a_i = R_{ijk}T_{jk}$ for any rectangular Cartesian basis $\{\mathbf{e}_i\}$. Prove that R_{ijk} are the components of a third-order tensor.

Ans. Since $a_i = R_{ijk}T_{jk}$ is true for any basis, therefore, $a_i' = R_{ijk}'T_{jk}'$; Since $\mathbf{a}$ is a vector, therefore, $a_i' = Q_{mi}a_m$ and since $\mathbf{T}$ is a second order tensor, therefore, $T_{ij}' = Q_{mi}Q_{nj}T_{mn}$. Thus, $a_i' = Q_{mi}a_m \to R_{ijk}'T_{jk}' = Q_{mi}(R_{mjk}T_{jk})$. Multiply the last equation with Q_{si} and noting that $Q_{si}Q_{mi} = \delta_{sm}$, we have,

$Q_{si}R_{ijk}'T_{jk}' = Q_{si}Q_{mi}(R_{mjk}T_{jk}) \to Q_{si}R_{ijk}'T_{jk}' = \delta_{sm}R_{mjk}T_{jk} \to Q_{si}R_{ijk}'T_{jk}' = R_{sjk}T_{jk}$

$\to Q_{si}R_{ijk}'Q_{mj}Q_{nk}T_{mn} = R_{sjk}T_{jk} \to Q_{si}R_{ijk}'Q_{mj}Q_{nk}T_{mn} = R_{smn}T_{mn}$. Thus,

$\left(R_{smn} - Q_{si}Q_{mj}Q_{nk}R_{ijk}'\right)T_{mn} = 0$. Since this last equation is to be true for all T_{mn} , therefore, $R_{smn} = Q_{si}Q_{mj}Q_{nk}R_{ijk}'$, which is the transformation law for components of a third order tensor.

2.46 For any vector $\mathbf{a}$ and any tensor $\mathbf{T}$, show that (a) $\mathbf{a}\cdot\mathbf{T}^{\mathrm{A}}\mathbf{a} = 0$ and (b) $\mathbf{a}\cdot\mathbf{T}\mathbf{a} = \mathbf{a}\cdot\mathbf{T}^{\mathrm{S}}\mathbf{a}$, where $\mathbf{T}^{\mathrm{A}}$ and $\mathbf{T}^{\mathrm{S}}$ are antisymmetric and symmetric part of $\mathbf{T}$ respectively.

Ans. (a) $\mathbf{T}^{\mathrm{A}}$ is antisymmetric, therefore, $(\mathbf{T}^{\mathrm{A}})^{\mathrm{T}} = -\mathbf{T}^{\mathrm{A}}$, thus,

$\mathbf{a}\cdot\mathbf{T}^{\mathrm{A}}\mathbf{a} = \mathbf{a}\cdot(\mathbf{T}^{\mathrm{A}})^{\mathrm{T}}\mathbf{a} = -\mathbf{a}\cdot\mathbf{T}^{\mathrm{A}}\mathbf{a} \to 2\mathbf{a}\cdot\mathbf{T}^{\mathrm{A}}\mathbf{a} = 0 \to \mathbf{a}\cdot\mathbf{T}^{\mathrm{A}}\mathbf{a} = 0$.

(b) Since $\mathbf{T} = \mathbf{T}^{\mathrm{S}} + \mathbf{T}^{\mathrm{A}}$, therefore, $\mathbf{a}\cdot\mathbf{T}\mathbf{a} = \mathbf{a}\cdot(\mathbf{T}^{\mathrm{S}} + \mathbf{T}^{\mathrm{A}})\mathbf{a} = \mathbf{a}\cdot\mathbf{T}^{\mathrm{S}}\mathbf{a} + \mathbf{a}\cdot\mathbf{T}^{\mathrm{A}}\mathbf{a} = \mathbf{a}\cdot\mathbf{T}^{\mathrm{S}}\mathbf{a}$.

2.47 Any tensor can be decomposed into a symmetric part and an antisymmetric part, that is $\mathbf{T} = \mathbf{T}^{\mathrm{S}} + \mathbf{T}^{\mathrm{A}}$. Prove that the decomposition is unique. (Hint, assume that it is not true and show contradiction).

Ans. Suppose that the decomposition is not unique, then ,we have,

$\mathbf{T} = \mathbf{T}^{\mathrm{S}} + \mathbf{T}^{\mathrm{A}} = \mathbf{S}^{\mathrm{S}} + \mathbf{S}^{\mathrm{A}} \to (\mathbf{T}^{\mathrm{S}} - \mathbf{S}^{\mathrm{S}}) + (\mathbf{T}^{\mathrm{A}} - \mathbf{S}^{\mathrm{A}}) = \mathbf{0}$. Let $\mathbf{a}$ be any arbitrary vector, we have,

$\mathbf{a}\cdot(\mathbf{T}^{\mathrm{S}} - \mathbf{S}^{\mathrm{S}})\mathbf{a} + \mathbf{a}\cdot(\mathbf{T}^{\mathrm{A}} - \mathbf{S}^{\mathrm{A}})\mathbf{a} = 0 \to \mathbf{a}\cdot\mathbf{T}^{\mathrm{S}}\mathbf{a} - \mathbf{a}\cdot\mathbf{S}^{\mathrm{S}}\mathbf{a} + \mathbf{a}\cdot\mathbf{T}^{\mathrm{A}}\mathbf{a} - \mathbf{a}\cdot\mathbf{S}^{\mathrm{A}}\mathbf{a} = 0$.

But $\mathbf{a}\cdot\mathbf{T}^{\mathrm{A}}\mathbf{a} = \mathbf{a}\cdot\mathbf{S}^{\mathrm{A}}\mathbf{a} = 0$ (see the previous problem). Therefore,

$\mathbf{a}\cdot\mathbf{T}^{\mathrm{S}}\mathbf{a} - \mathbf{a}\cdot\mathbf{S}^{\mathrm{S}}\mathbf{a} = 0 \to \mathbf{a}\cdot(\mathbf{T}^{\mathrm{S}} - \mathbf{S}^{\mathrm{S}})\mathbf{a} = 0 \to \mathbf{T}^{\mathrm{S}} - \mathbf{S}^{\mathrm{S}} = 0 \to \mathbf{T}^{\mathrm{S}} = \mathbf{S}^{\mathrm{S}}$. It also follows from $(\mathbf{T}^{\mathrm{S}} - \mathbf{S}^{\mathrm{S}}) + (\mathbf{T}^{\mathrm{A}} - \mathbf{S}^{\mathrm{A}}) = \mathbf{0}$ that $\mathbf{T}^{\mathrm{A}} = \mathbf{S}^{\mathrm{A}}$. Thus , the decomposition is unique.

2.48 Given that a tensor $\mathbf{T}$ has the matrix $[\mathbf{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$, (a) find the symmetric part and the anti-symmetric part of $\mathbf{T}$ and (b) find the dual vector (or axial vector) of the anti-symmetric part of $\mathbf{T}$.

Ans. (a) $\left[\mathbf{T}^{\mathrm{S}}\right] = \dfrac{1}{2}\left\{[\mathbf{T}] + [\mathbf{T}]^{\mathrm{T}}\right\} = \dfrac{1}{2}\left\{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}\right\} = \dfrac{1}{2}\begin{bmatrix} 2 & 6 & 10 \\ 6 & 10 & 14 \\ 10 & 14 & 18 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{bmatrix}$.

$$\left[\mathbf{T}^{A}\right]=\frac{1}{2}\left\{[\mathbf{T}]-[\mathbf{T}]^{T}\right\}=\frac{1}{2}\left\{\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}-\begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix}\right\}=\frac{1}{2}\begin{bmatrix}0&-2&-4\\2&0&-2\\4&2&18\end{bmatrix}=\begin{bmatrix}0&-1&-2\\1&0&-1\\2&1&0\end{bmatrix}.$$

(b) $\mathbf{t}^{A}=-(T_{23}^{A}\mathbf{e}_1+T_{31}^{A}\mathbf{e}_2+T_{12}^{A}\mathbf{e}_3)=-(-1\mathbf{e}_1+2\mathbf{e}_2-1\mathbf{e}_3)=\mathbf{e}_1-2\mathbf{e}_2+\mathbf{e}_3$.

2.49 Prove that the only possible real eigenvalues of an orthogonal tensor $\mathbf{Q}$ are $\lambda=\pm 1$. Explain the direction of the eigenvectors corresponding to them for a proper orthogonal(rotation) tensor and for an improper orthogonal (reflection) tensor.

Ans. Since $\mathbf{Q}$ is orthogonal, therefore, for any vector $\mathbf{n}$, we have, $\mathbf{Qn}\cdot\mathbf{Qn}=\mathbf{n}\cdot\mathbf{n}$. Let $\mathbf{n}$ be an eigenvector, then $\mathbf{Qn}=\lambda\mathbf{n}$, so that $\mathbf{Qn}\cdot\mathbf{Qn}=\mathbf{n}\cdot\mathbf{n}\rightarrow$

$\lambda^2(\mathbf{n}\cdot\mathbf{n})=(\mathbf{n}\cdot\mathbf{n})\rightarrow(\lambda^2-1)(\mathbf{n}\cdot\mathbf{n})=0\rightarrow\lambda^2-1=0\rightarrow\lambda=\pm 1$.

The eigenvalue $\lambda=1$ ($\mathbf{Qn}=\mathbf{n}$) corresponds to an eigenvector parallel to the axis of rotation for a proper orthogonal tensor (rotation tensor); Or, it corresponds to an eigenvector parallel to the plane of reflection for an improper orthogonal tensor (reflection tensor). The eigenvalue $\lambda=-1$, ($\mathbf{Qn}=-\mathbf{n}$) corresponds to an eigenvector perpendicular to the axis of rotation for an 180^{o} rotation; or, it corresponds to an eigenvector perpendicular to the plane of reflection.

2.50 Given the improper orthogonal tensor $[\mathbf{Q}]=\frac{1}{3}\begin{bmatrix}1&-2&-2\\-2&1&-2\\-2&-2&1\end{bmatrix}$. (a) Verify that $\det[\mathbf{Q}]=-1$.

(b) Verify that the eigenvalues are $\lambda=1$ and -1 (c) Find the normal to the plane of reflection (i.e., eigenvectors corresponding to $\lambda=-1$) and (d) find the eigenvectors corresponding $\lambda=1$ (vectors parallel to the plane of reflection).

Ans. (a) $\det[\mathbf{Q}]=(1/3)^3(1-8-8-4-4-4)=(-27)/27=-1$.

(b) $I_1=3/3=1,\quad I_2=(1/3)^2\{(1-4)+(1-4)+(1-4)\}=-1,\quad I_3=-1\rightarrow$

$\lambda^3-\lambda^2-\lambda+1=0\rightarrow(\lambda-1)(\lambda^2-1)=0\rightarrow\lambda=1,1,-1$

(c) For $\lambda=-1$,

$\left(\frac{1}{3}+1\right)\alpha_1-\frac{2}{3}\alpha_2-\frac{2}{3}\alpha_3=0,\quad-\frac{2}{3}\alpha_1+\left(\frac{1}{3}+1\right)\alpha_2-\frac{2}{3}\alpha_3=0,\quad-\frac{2}{3}\alpha_1-\frac{2}{3}\alpha_2+\left(\frac{1}{3}+1\right)\alpha_3=0$. That is, $2\alpha_1-\alpha_2-\alpha_3=0,\quad-\alpha_1+2\alpha_2-\alpha_3=0,\quad-\alpha_1-\alpha_2+2\alpha_3=0$, thus, $\alpha_1=\alpha_2=\alpha_3$, therefore, $\mathbf{n}=\pm(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}$, this is the normal to the plane of reflection.

(d) For $\lambda=1$,

$\left(\frac{1}{3}-1\right)\alpha_1-\frac{2}{3}\alpha_2-\frac{2}{3}\alpha_3=0,-\frac{2}{3}\alpha_1+\left(\frac{1}{3}-1\right)\alpha_2-\frac{2}{3}\alpha_3=0,-\frac{2}{3}\alpha_1-\frac{2}{3}\alpha_2+\left(\frac{1}{3}-1\right)\alpha_3=0$

All three equations lead to $\alpha_1+\alpha_2+\alpha_3=0\rightarrow\alpha_3=-\alpha_1-\alpha_2$. Thus,

$\mathbf{n}=\frac{1}{\sqrt{\alpha_1^2+\alpha_2^2+\alpha_3^2}}[\alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2-(\alpha_1+\alpha_2)\mathbf{e}_3]$, e.g., $\mathbf{n}=\frac{1}{\sqrt{6}}(\mathbf{e}_1+\mathbf{e}_2-2\mathbf{e}_3)$ etc. these vectors are all perpendicular to $\mathbf{n}=\pm(\mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3)/\sqrt{3}$ and thus parallel to the plane of reflection.

2.51 Given that tensors $\mathbf{R}$ and $\mathbf{S}$ have the same eigenvector $\mathbf{n}$ and corresponding eigenvalue r_1 and s_1 respectively. Find an eigenvalue and the corresponding eigenvector for the tensor $\mathbf{T}=\mathbf{RS}$.

Ans. We have, $\mathbf{Rn}=r_1\mathbf{n}$ and $\mathbf{Sn}=s_1\mathbf{n}$, thus, $\mathbf{Tn}=\mathbf{RSn}=\mathbf{R}s_1\mathbf{n}=s_1\mathbf{Rn}=r_1s_1\mathbf{n}$. Thus, an eigenvalue for $\mathbf{T}=\mathbf{RS}$ is r_1s_1 with eigenvector $\mathbf{n}$.

2.52 Show that if $\mathbf{n}$ is a real eigenvector of an antisymmetric tensor $\mathbf{T}$, then the corresponding eigenvalue vanishes.

Ans. $\mathbf{Tn}=\lambda\mathbf{n}\rightarrow\mathbf{n}\cdot\mathbf{Tn}=\lambda(\mathbf{n}\cdot\mathbf{n})$. Now, from the definition of transpose, we have $\mathbf{n}\cdot\mathbf{Tn}=\mathbf{n}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{n}$. But, since $\mathbf{T}$ is antisymmetric, i.e., $\mathbf{T}^{\mathrm{T}}=-\mathbf{T}$, therefore, $\mathbf{n}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{n}=-\mathbf{n}\cdot\mathbf{Tn}$. Thus, $\mathbf{n}\cdot\mathbf{Tn}=-\mathbf{n}\cdot\mathbf{Tn}\rightarrow 2\mathbf{n}\cdot\mathbf{Tn}=0\rightarrow\mathbf{n}\cdot\mathbf{Tn}=0$. Thus, $\lambda(\mathbf{n}\cdot\mathbf{n})=0\rightarrow\lambda=0$.

2.53 (a) Show that $\mathbf{a}$ is an eigenvector for the dyadic product $\mathbf{ab}$ of vectors $\mathbf{a}$ and $\mathbf{b}$ with eigenvalue $\mathbf{a}\cdot\mathbf{b}$, (b) find the first principal scalar invariant of the dyadic product $\mathbf{ab}$ and (c) show that the second and the third principal scalar invariants of the dyadic product $\mathbf{ab}$ vanish, and that zero is a double eigenvalue of $\mathbf{ab}$.

Ans. (a) From the definition of dyadic product, we have, $(\mathbf{ab})\mathbf{a}=\mathbf{a}(\mathbf{b}\cdot\mathbf{a})$, thus $\mathbf{a}$ is an eigenvector for the dyadic product $\mathbf{ab}$ with eigenvalue $\mathbf{a}\cdot\mathbf{b}$.

(b) Let $\mathbf{T}\equiv\mathbf{ab}$, then $T_{ij}=a_ib_j$ and the first scalar invariant of $\mathbf{ab}$ is $T_{ii}=a_ib_i=\mathbf{a}\cdot\mathbf{b}$.

(c) $I_2=\begin{vmatrix}a_1b_1 & a_1b_2\\ a_2b_1 & a_2b_2\end{vmatrix}+\begin{vmatrix}a_2b_2 & a_2b_3\\ a_3b_2 & a_3b_3\end{vmatrix}+\begin{vmatrix}a_1b_1 & a_1b_3\\ a_3b_1 & a_3b_3\end{vmatrix}=0+0+0=0$.

$$I_3=\begin{vmatrix}a_1b_1 & a_1b_2 & a_1b_3\\ a_2b_1 & a_2b_2 & a_2b_3\\ a_3b_1 & a_3b_2 & a_3b_3\end{vmatrix}=a_1a_2a_3\begin{vmatrix}b_1 & b_2 & b_3\\ b_1 & b_2 & b_3\\ b_1 & b_2 & b_3\end{vmatrix}=0.$$

Thus, the characteristic equation is

$\lambda^3-I_1\lambda^2=0\rightarrow(\lambda-I_1)\lambda^2=0\rightarrow\lambda_1=I_1,\ \ \lambda_2=\lambda_3=0$.

2.54 For any rotation tensor, a set of basis $\{\mathbf{e}_i'\}$ may be chosen with $\mathbf{e}_3'$ along the axis of rotation so that $\mathbf{Re}_1'=\cos\theta\mathbf{e}_1'+\sin\theta\mathbf{e}_2',\quad \mathbf{Re}_2'=-\sin\theta\mathbf{e}_1'+\cos\theta\mathbf{e}_2',\quad \mathbf{Re}_3'=\mathbf{e}_3'$, where θ is the angle of right hand rotation. (a) Find the antisymmetric part of $\mathbf{R}$ with respect to the basis $\{\mathbf{e}_i'\}$, i.e., find $[\mathbf{R}^{\mathrm{A}}]_{\mathbf{e}_i'}$.

(b) Show that the dual vector of $\mathbf{R}^{\mathrm{A}}$ is given by $\mathbf{t}^{\mathrm{A}}=\sin\theta\mathbf{e}_3'$ and (c) show that the first scalar invariant of $\mathbf{R}$ is given by $1+2\cos\theta$. That is, for any given rotation tensor $\mathbf{R}$, its axis of rotation and the angle of rotation can be obtained from the dual vector of $\mathbf{R}^{\mathrm{A}}$ and the first scalar invariant of $\mathbf{R}$.

Ans. (a) From $\mathbf{Re}_1'=\cos\theta\mathbf{e}_1'+\sin\theta\mathbf{e}_2',\quad \mathbf{Re}_2'=-\sin\theta\mathbf{e}_1'+\cos\theta\mathbf{e}_2',\quad \mathbf{Re}_3'=\mathbf{e}_3'$, we have,

$$[\mathbf{R}]_{\mathbf{e}_i'}=\begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}_{\mathbf{e}_i'}\rightarrow[\mathbf{R}^{\mathrm{A}}]_{\mathbf{e}_i'}=\begin{bmatrix}0 & -\sin\theta & 0\\ \sin\theta & 0 & 0\\ 0 & 0 & 0\end{bmatrix}_{\mathbf{e}_i'}$$

(b) the dual vector (or axial vector) of $\mathbf{R}^{\mathrm{A}}$ is given by

$\mathbf{t}^A = -(T'_{23}\mathbf{e}'_1 + T'_{31}\mathbf{e}'_2 + T'_{12}\mathbf{e}'_3) = -(0\mathbf{e}'_1 + 0\mathbf{e}'_2 - \sin\theta\mathbf{e}'_3) = \sin\theta\mathbf{e}'_3$.

(c) The first scalar invariant of $\mathbf{R}$ is $I_1 = \cos\theta + \cos\theta + 1 = 1 + 2\cos\theta$.

2.55 The rotation of a rigid body is described by $\mathbf{Re}_1 = \mathbf{e}_2, \quad \mathbf{Re}_2 = \mathbf{e}_3, \quad \mathbf{Re}_3 = \mathbf{e}_1$. Find the axis of rotation and the angle of rotation. Use the result of the previous problem.

Ans From the result of the previous problem, we have, the dual vector of $\mathbf{R}^A$ is given by $\mathbf{t}^A = \sin\theta\mathbf{e}'_3$, where $\mathbf{e}'_3$ is in the direction of axis of rotation and θ is the angle of rotation. Thus, we can obtain the direction of axis of rotation and the angle of rotation θ by obtaining the dual vector of $\mathbf{R}^A$. From $\mathbf{Re}_1 = \mathbf{e}_2, \quad \mathbf{Re}_2 = \mathbf{e}_3, \quad \mathbf{Re}_3 = \mathbf{e}_1$, we have,

$$[\mathbf{R}] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to [\mathbf{R}^A] = \frac{1}{2}\begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \to \mathbf{t}^A = \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3).$$ Thus ,

$$\mathbf{t}^A = \frac{\sqrt{3}}{2}\frac{(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)}{\sqrt{3}} = \frac{\sqrt{3}}{2}\mathbf{e}'_3,$$ where $\mathbf{e}'_3 = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$ is in the direction of the axis of rotation and the angle of rotation is given by $\sin\theta = \sqrt{3}/2$, which gives $\theta = 60^o$ or 120^o. On the other hand, the first scalar invariant of $\mathbf{R}$ is 0. Thus, from the result in (c) of the previous problem, we have, $I_1 = 1 + 2\cos\theta = 0$, so that $\cos\theta = -1/2$ which gives $\theta = 120^o$ or 240^o. We therefore conclude that $\theta = 120^o$.

2.56 Given the tensor $[\mathbf{Q}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. (a) Show that the given tensor is a rotation tensor. (b) Verify that the eigenvalues are $\lambda = 1$ and -1. (c) Find the direction for the axis of rotation (i.e., eigenvectors corresponding to $\lambda = 1$). (d) Find the eigenvectors corresponding $\lambda = -1$ and (e) obtain the angle of rotation using the formula $I_1 = 1 + 2\cos\theta$ (see Prob. 2.54), where I_1 is the first scalar invariant of the rotation tensor.

Ans. (a) $\det[\mathbf{Q}] = +1$, and $[\mathbf{Q}][\mathbf{Q}]^T = [\mathbf{I}]$ therefore it is a rotation tensor.

(b) The principal scalar invariants are: $I_1 = -1, \quad I_2 = -1, \quad I_3 = 1 \to$ characteristic equation is

$\lambda^3 + \lambda^2 - \lambda - 1 = (\lambda + 1)(\lambda^2 - 1) = 0 \to$ the eigenvalues are: $\lambda = -1, 1, 1$.

(c) For $\lambda = 1$, clearly, the eigenvector are: $\mathbf{n} = \pm\mathbf{e}_3$, which gives the axis of rotation.

(d) For $\lambda = -1$, with eigenvector $\mathbf{n} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3$, we have

$0\alpha_1 = 0, \quad 0\alpha_2 = 0, \quad 2\alpha_3 = 0$. Thus, $\alpha_1 =$ arbitrary, $\alpha_2 =$ arbitrary, $\alpha_3 = 0$. The eigenvectors are:

$\mathbf{n} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2, \quad \alpha_1^2 + \alpha_2^2 = 1$. That is, all vectors perpendicular to the axis of rotation are eigenvectors.

(e) The first scalar invariant of $\mathbf{Q}$ is $I_1 = -1$. Thus, $1 + 2\cos\theta = -1 \to \cos\theta = -1 \to \theta = \pi$. (We note that for this problem, the antisymmetric part of $\mathbf{Q} = \mathbf{0}$, so that $\mathbf{t}^A = \mathbf{0} = \sin\theta\,\mathbf{n}$, of which $\theta = \pi$ is a solution).

2.57 Let $\mathbf{F}$ be an arbitrary tensor. (a) Show that $\mathbf{F}^T\mathbf{F}$ and $\mathbf{FF}^T$ are both symmetric tensors. (b) If $\mathbf{F} = \mathbf{QU} = \mathbf{VQ}$, where $\mathbf{Q}$ is orthogonal and $\mathbf{U}$ and $\mathbf{V}$ are symmetric, show that $\mathbf{U}^2 = \mathbf{F}^T\mathbf{F}$ and $\mathbf{V}^2 = \mathbf{FF}^T$ (c) If λ and $\mathbf{n}$ are eigenvalue and the corresponding eigenvector for $\mathbf{U}$, find the eigenvalue and eigenvector for $\mathbf{V}$. [note corrections for text]

Ans. (a) $(\mathbf{F}^T\mathbf{F})^T = \mathbf{F}^T(\mathbf{F}^T)^T = \mathbf{F}^T\mathbf{F}$, thus $\mathbf{F}^T\mathbf{F}$ is symmetric. Also $(\mathbf{FF}^T)^T = (\mathbf{F}^T)^T\mathbf{F}^T = \mathbf{FF}^T$, therefore, $\mathbf{FF}^T$ is also symmetric.

(b) $\mathbf{F} = \mathbf{QU} \rightarrow \mathbf{F}^T = \mathbf{U}^T\mathbf{Q}^T \rightarrow \mathbf{F}^T\mathbf{F} = \mathbf{U}^T\mathbf{Q}^T\mathbf{QU} = \mathbf{U}^T\mathbf{U} \rightarrow \mathbf{F}^T\mathbf{F} = \mathbf{U}^2$.

$\mathbf{F} = \mathbf{VQ} \rightarrow \mathbf{F}^T = \mathbf{Q}^T\mathbf{V}^T \rightarrow \mathbf{FF}^T = \mathbf{VQQ}^T\mathbf{V}^T = \mathbf{VV}^T \rightarrow \mathbf{FF}^T = \mathbf{V}^2$.

(c) Since $\mathbf{F} = \mathbf{QU} = \mathbf{VQ}$, and $\mathbf{Un} = \lambda\mathbf{n}$, therefore, $\mathbf{VQn} = \mathbf{QUn} = \mathbf{Q}(\lambda\mathbf{n}) \rightarrow \mathbf{V}(\mathbf{Qn}) = \lambda(\mathbf{Qn})$, therefore, $\mathbf{Qn}$ is an eigenvector for $\mathbf{V}$ with the eigenvalue λ.

2.58 Verify that the second principal scalar invariant of a tensor $\mathbf{T}$ can be written: $I_2 = \left(T_{ii}T_{jj} - T_{ij}T_{ji}\right)/2$.

Ans. $T_{ii}T_{jj} = (T_{11} + T_{22} + T_{33})^2 = T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{11}T_{22} + 2T_{22}T_{33} + 2T_{33}T_{11}$.

$T_{ij}T_{ji} = T_{1j}T_{j1} + T_{2j}T_{j2} + T_{3j}T_{j3} = T_{11}^2 + T_{12}T_{21} + T_{13}T_{31} + T_{21}T_{12} + T_{22}^2 + T_{23}T_{32} + T_{31}T_{13} + T_{32}T_{23} + T_{33}^2$.

Thus, $T_{ii}T_{jj} - T_{ij}T_{ji} = (T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{11}T_{22} + 2T_{22}T_{33} + 2T_{33}T_{11})$

$-(T_{11}^2 + T_{22}^2 + T_{33}^2 + 2T_{12}T_{21} + 2T_{13}T_{31} + 2T_{23}T_{32}) = 2(T_{11}T_{22} - T_{12}T_{21} + T_{22}T_{33} - T_{23}T_{32} + T_{33}T_{11} - T_{13}T_{31})$.

Thus,

$\left(T_{ii}T_{jj} - T_{ij}T_{ji}\right)/2 = (T_{11}T_{22} - T_{12}T_{21} + T_{22}T_{33} - T_{23}T_{32} + T_{33}T_{11} - T_{13}T_{31})$

$$= \begin{vmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{vmatrix} + \begin{vmatrix} T_{22} & T_{23} \\ T_{32} & T_{33} \end{vmatrix} + \begin{vmatrix} T_{11} & T_{13} \\ T_{31} & T_{33} \end{vmatrix} = I_2.$$

2.59 A tensor has a matrix $[\mathbf{T}]$ given below. (a) Write the characteristic equation and find the principal values and their corresponding principal directions. (b) Find the principal scalar invariants. (c) If $\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3$ are the principal directions, write $[\mathbf{T}]_{\mathbf{n}_i}$. (d) Could the following matrix $[\mathbf{S}]$ represent the same tensor $\mathbf{T}$ with respect to some basis.

$$[\mathbf{T}] = \begin{bmatrix} 5 & 4 & 0 \\ 4 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad [\mathbf{S}] = \begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Ans.

(a) The characteristic equation is:

$$\begin{vmatrix} 5-\lambda & 4 & 0 \\ 4 & -1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = 0 \rightarrow (3-\lambda)\left[(5-\lambda)(-1-\lambda) - 16\right] = (3-\lambda)(\lambda^2 - 4\lambda - 21) = (3-\lambda)(\lambda+3)(\lambda-7) = 0$$

Thus, $\lambda_1 = 3, \lambda_2 = -3, \lambda_3 = 7$.

For $\lambda_1 = 3$, clearly, $\mathbf{n}_1 = \pm\mathbf{e}_3$.

For $\lambda_2 = -3$

$(5+3)\alpha_1 + 4\alpha_2 = 0, \quad 4\alpha_1 + (-1+3)\alpha_2 = 0, \quad (3+3)\alpha_3 = 0, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$

$\to 8\alpha_1 + 4\alpha_2 = 0, \quad 4\alpha_1 + 2\alpha_2 = 0, \quad 6\alpha_3 = 0 \to \alpha_2 = -2\alpha_1, \; \alpha_3 = 0. \to \mathbf{n}_2 = \pm(\mathbf{e}_1 - 2\mathbf{e}_2)/\sqrt{5}.$

For $\lambda_3 = 7$

$(5-7)\alpha_1 + 4\alpha_2 = 0, \quad 4\alpha_1 + (-1-7)\alpha_2 = 0, \quad (3-7)\alpha_3 = 0, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$

$\to -2\alpha_1 + 4\alpha_2 = 0, \quad 4\alpha_1 - 8\alpha_2 = 0, \quad -4\alpha_3 = 0 \to \alpha_1 = 2\alpha_2, \quad \alpha_3 = 0. \to \mathbf{n}_3 = \pm(2\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{5}.$

(b) The principal scalar invariants are:

$I_1 = 5 - 1 + 3 = 7, \quad I_2 = (-5-16) + (-3-0) + (15-0) = -9, \quad I_3 = -15 - 48 = -63$. We note that

$\lambda^3 - 7\lambda^2 - 9\lambda + 63 = 0 \to (\lambda - 7)\lambda^2 - 9(\lambda - 7) = 0 \to (\lambda - 7)(\lambda^2 - 9) = 0$, same as obtained in (a)

(c) $[\mathbf{T}]_{\mathbf{n}_i} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 7 \end{bmatrix}_{\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}}$. (d) $\det \begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -3 \neq -63$, therefore, the answer is NO. Or, clearly one of the eigenvalue for $[\mathbf{S}]$ is -1, which is not an eigenvalue for $[\mathbf{T}]$, therefore the answer is NO.

2.60 Do the previous problem for the following matrix:

$$[\mathbf{T}] = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 4 & 0 \end{bmatrix}$$

Ans. (a) The characteristic equation is:

$$\begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & 0-\lambda & 4 \\ 0 & 4 & 0-\lambda \end{vmatrix} = 0 \to (3-\lambda)(\lambda^2 - 16) = (3-\lambda)(\lambda - 4)(\lambda + 4) = 0$$

Thus, $\lambda_1 = 3, \quad \lambda_2 = 4, \quad \lambda_3 = -4$.

For $\lambda_1 = 3$, clearly, $\mathbf{n}_1 = \pm\mathbf{e}_1$, because $\mathbf{T}\mathbf{e}_1 = 3\mathbf{e}_1$.

For $\lambda_2 = 4$

$(3-4)\alpha_1 = 0, \quad (0-4)\alpha_2 + 4\alpha_3 = 0, \quad 4\alpha_2 + (0-4)\alpha_3 = 0, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$

$-\alpha_1 = 0, \quad -4\alpha_2 + 4\alpha_3 = 0, \quad 4\alpha_2 - 4\alpha_3 = 0 \to \alpha_1 = 0, \quad \alpha_2 = \alpha_3, \to \mathbf{n}_2 = \pm(\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{2}.$

For $\lambda_3 = -4$

$(3+4)\alpha_1 = 0, \quad (0+4)\alpha_2 + 4\alpha_3 = 0, \quad 4\alpha_2 + (0+4)\alpha_3 = 0, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$

$\to 7\alpha_1 = 0, \quad 4\alpha_2 + 4\alpha_3 = 0, \quad 4\alpha_2 + 4\alpha_3 = 0 \to \alpha_1 = 0, \quad \alpha_2 = -\alpha_3, \to \mathbf{n}_3 = \pm(\mathbf{e}_2 - \mathbf{e}_3)/\sqrt{2}$

(b)

$I_1 = 3, \quad I_2 = (0-0) + (0-16) + (0-0) = -16, \quad I_3 = -48$.

$\lambda^3 - 3\lambda^2 - 16\lambda + 48 = 0 \to (\lambda - 3)\lambda^2 - 16(\lambda - 3) = 0 \to (\lambda - 3)(\lambda^2 - 16) = 0$, same as in (a) .

(c) $[\mathbf{T}]_{\mathbf{n}_i} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}_{\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}}$

(d) $\det[\mathbf{S}] = \det\begin{bmatrix} 7 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = -(7-4) = -3 \neq -48$, therefore, the answer is NO.

Or, clearly one of the eigenvalue for $[\mathbf{S}]$ is -1, which is not an eigenvalue for $[\mathbf{T}]$, therefore the answer is NO.

2.61 A tensor $\mathbf{T}$ has a matrix given below. Find the principal values and three mutually perpendicular principal directions: $[\mathbf{T}] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Ans. The characteristic equation is:

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0 \rightarrow (2-\lambda)\left[(1-\lambda)^2 - 1\right] = (2-\lambda)(-2\lambda+\lambda^2) = -\lambda(2-\lambda)^2 = 0.$$

Thus, $\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = 2$. That is, there is a double root $\lambda_2 = \lambda_3 = 2$.

For $\lambda_1 = 0$,

$$(1-0)\alpha_1 + \alpha_2 = 0, \quad \alpha_1 + (1-0)\alpha_2 = 0, \quad (2-0)\alpha_3 = 0, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$$

$$\rightarrow \alpha_1 + \alpha_2 = 0, \quad 2\alpha_3 = 0 \rightarrow \alpha_1 = -\alpha_2, \quad \alpha_3 = 0, \rightarrow \mathbf{n}_1 = \pm(\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}.$$

For $\lambda_2 = \lambda_3 = 2$, one eigenvector is clearly $\mathbf{n}_3$. There are infinitely many others all lie on the plane whose normal is $\mathbf{n}_1 = \pm(\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$. In fact, we have,

$$(1-2)\alpha_1 + \alpha_2 = 0, \quad \alpha_1 + (1-2)\alpha_2 = 0, \quad (2-2)\alpha_3 = 0, \quad \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$$

$$\rightarrow -\alpha_1 + \alpha_2 = 0, \quad 0\alpha_3 = 0 \rightarrow \alpha_1 = \alpha_2 = \alpha, \quad \alpha_3 = \sqrt{1-2\alpha^2} \rightarrow \mathbf{n} = \pm(\alpha\mathbf{e}_1 + \alpha\mathbf{e}_2 + \alpha_3\mathbf{e}_3),$$

which include the case where $\alpha = 0, \quad \alpha_3 = \pm 1 \rightarrow \mathbf{n} = \pm\mathbf{e}_3$.

CHAPTER 2, PART C

2.62 Prove the identity $\frac{d}{dt}(\mathbf{T}+\mathbf{S}) = \frac{d\mathbf{T}}{dt} + \frac{d\mathbf{S}}{dt}$, using the definition of derivative of a tensor.

Ans.

$$\frac{d}{dt}(\mathbf{T}+\mathbf{S}) = \lim_{\Delta t\to 0}\frac{\{\mathbf{T}(t+\Delta t)+\mathbf{S}(t+\Delta t)\} - \{\mathbf{T}(t)+\mathbf{S}(t)\}}{\Delta t} = \lim_{\Delta t\to 0}\frac{\{\mathbf{T}(t+\Delta t)-\mathbf{T}(t)\} + \{\mathbf{S}(t+\Delta t)-\mathbf{S}(t)\}}{\Delta t} -$$

$$\lim_{\Delta t\to 0}\frac{\{\mathbf{T}(t+\Delta t)-\mathbf{T}(t)\}}{\Delta t} + \lim_{\Delta t\to 0}\frac{\{\{\mathbf{S}(t+\Delta t)-\mathbf{S}(t)\}}{\Delta t} = \frac{d\mathbf{T}}{dt} + \frac{d\mathbf{S}}{dt}.$$

2.63 Prove the identity $\frac{d}{dt}(\mathbf{TS}) = \mathbf{T}\frac{d\mathbf{S}}{dt} + \frac{d\mathbf{T}}{dt}\mathbf{S}$ using the definition of derivative of a tensor.

Ans. $\frac{d}{dt}(\mathbf{TS}) = \lim_{\Delta t\to 0} \frac{\mathbf{T}(t+\Delta t)\mathbf{S}(t+\Delta t) - \mathbf{T}(t)\mathbf{S}(t)}{\Delta t}$

$= \lim_{\Delta t\to 0} \frac{\mathbf{T}(t+\Delta t)\mathbf{S}(t+\Delta t) - \mathbf{T}(t+\Delta t)\mathbf{S}(t) + \mathbf{T}(t+\Delta t)\mathbf{S}(t) - \mathbf{T}(t)\mathbf{S}(t)}{\Delta t}$

$= \lim_{\Delta t\to 0} \frac{\mathbf{T}(t+\Delta t)\{\mathbf{S}(t+\Delta t) - \mathbf{S}(t)\}}{\Delta t} + \lim_{\Delta t\to 0} \frac{\{\mathbf{T}(t+\Delta t) - \mathbf{T}(t)\}\mathbf{S}(t)}{\Delta t}$

$= \mathbf{T}(t) \lim_{\Delta t\to 0} \frac{\{\mathbf{S}(t+\Delta t) - \mathbf{S}(t)\}}{\Delta t} + \lim_{\Delta t\to 0} \frac{\{\mathbf{T}(t+\Delta t) - \mathbf{T}(t)\}}{\Delta t}\mathbf{S}(t) = \mathbf{T}\frac{d\mathbf{S}}{dt} + \frac{d\mathbf{T}}{dt}\mathbf{S}.$

2.64 Prove that $\frac{d\mathbf{T}^{\mathrm{T}}}{dt} = \left(\frac{d\mathbf{T}}{dt}\right)^{\mathrm{T}}$ by differentiating the definition $\mathbf{a}\cdot\mathbf{Tb} = \mathbf{b}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{a}$, where $\mathbf{a}$ and $\mathbf{b}$ are constant arbitrary vectors.

Ans. $\mathbf{a}\cdot\mathbf{Tb} = \mathbf{b}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{a} \rightarrow \mathbf{a}\cdot(d\mathbf{T}/dt)\mathbf{b} = \mathbf{b}\cdot(d\mathbf{T}^{\mathrm{T}}/dt)\mathbf{a}$. Now, the definition of transpose also gives $\mathbf{a}\cdot(d\mathbf{T}/dt)\mathbf{b} = \mathbf{b}\cdot(d\mathbf{T}/dt)^{\mathrm{T}}\mathbf{a}$. Thus, $\mathbf{b}\cdot(d\mathbf{T}/dt)^{\mathrm{T}}\mathbf{a} = \mathbf{b}\cdot(d\mathbf{T}^{\mathrm{T}}/dt)\mathbf{a}$.

Since $\mathbf{a}$ and $\mathbf{b}$ arbitrary vectors, therefore, $\left(\frac{d\mathbf{T}}{dt}\right)^{\mathrm{T}} = \left(\frac{d\mathbf{T}^{\mathrm{T}}}{dt}\right)$.

2.65 Consider the scalar field $\phi = x_1^2 + 3x_1x_2 + 2x_3$. (a) Find the unit vector normal to the surface of constant ϕ at the origin (0,0,0) and at $(1,0,1)$. (b) what is the maximum value of the directional derivative of ϕ at the origin? At (1,0,1)? (c) Evaluate $d\phi/dr$ at the origin if $d\mathbf{r} = ds(\mathbf{e}_1 + \mathbf{e}_3)$.

Ans. (a) $\nabla\phi = (2x_1 + 3x_2)\mathbf{e}_1 + 3x_1\mathbf{e}_2 + 2\mathbf{e}_3$,

at $(0,0,0)$, $\nabla\phi = 2\mathbf{e}_3 \rightarrow \mathbf{n} = \mathbf{e}_3$, at $(1,0,1)$, $\nabla\phi = 2\mathbf{e}_1 + 3\mathbf{e}_2 + 2\mathbf{e}_3, \rightarrow \mathbf{n} = (2\mathbf{e}_1 + 3\mathbf{e}_2 + 2\mathbf{e}_3)/\sqrt{17}$.

(b) At $(0,0,0)$, $(d\phi/dr)_{\max} = |\nabla\phi| = 2$ in the direction of $\mathbf{n}=\mathbf{e}_3$.

At $(1,0,1)$, $(d\phi/dr)_{\max} = |\nabla\phi| = \sqrt{17}$.

(c) At $(0,0,0)$, $d\phi/dr = (\nabla\phi)_{\mathrm{o}}\cdot d\mathbf{r}/dr = 2\mathbf{e}_3\cdot(\mathbf{e}_1 + \mathbf{e}_3)/\sqrt{2} = \sqrt{2}$.

2.66 Consider the ellipsoidal surface defined by the equation $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Find the unit vector normal to the surface at a given point (x,y,z).

Ans. Let $f(x,y,z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1$, then

$\frac{\partial f}{\partial x} = \frac{2x}{a^2}$, $\frac{\partial f}{\partial y} = \frac{2y}{b^2}$, $\frac{\partial f}{\partial z} = \frac{2z}{c^2} \rightarrow \nabla f = \frac{2x}{a^2}\mathbf{e}_1 + \frac{2y}{b^2}\mathbf{e}_2 + \frac{2z}{c^2}\mathbf{e}_3$, thus,

$$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \left[\left(\frac{2x}{a^2}\right)^2 + \left(\frac{2y}{b^2}\right)^2 + \left(\frac{2z}{c^2}\right)^2\right]^{-1/2} \left(\frac{2x}{a^2}\mathbf{e}_1 + \frac{2y}{b^2}\mathbf{e}_2 + \frac{2z}{c^2}\mathbf{e}_3\right).$$

2.67 Consider the temperature field given by: $\Theta = 3x_1x_2$. (a) Find the heat flux at the point $A(1,1,1)$, if $\mathbf{q} = -k\nabla\Theta$. (b) Find the heat flux at the same point if $\mathbf{q} = -\mathbf{K}\nabla\Theta$, where

$$[\mathbf{K}] = \begin{bmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 3k \end{bmatrix}$$

Ans. $\Theta = 3x_1x_2 \to \nabla\Theta = 3(x_2\mathbf{e}_1 + x_1\mathbf{e}_2) \to (\nabla\Theta)_A = 3(\mathbf{e}_1 + \mathbf{e}_2)$.

(a) $\mathbf{q} = -k\nabla\Theta = -3k(\mathbf{e}_1 + \mathbf{e}_2)$.

(b) $[\mathbf{q}] = -[\mathbf{K}\nabla\Theta] = -\begin{bmatrix} k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 3k \end{bmatrix}\begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = -\begin{bmatrix} 3k \\ 6k \\ 0 \end{bmatrix} \to \mathbf{q} = -(3k\mathbf{e}_1 + 6k\mathbf{e}_2)$.

2.68 Let $\phi(x_1,x_2,x_3)$ and $\psi(x_1,x_2,x_3)$ be scalar fields, and let $\mathbf{v}(x_1,x_2,x_3)$ and $\mathbf{w}(x_1,x_2,x_3)$ be vector fields. By writing the subscripted components form, verify the following identities.

(a) $\nabla(\phi+\psi) = \nabla\phi + \nabla\psi$, sample solution: $[\nabla(\phi+\psi)]_i = \dfrac{\partial(\phi+\psi)}{\partial x_i} = \dfrac{\partial\phi}{\partial x_i} + \dfrac{\partial\psi}{\partial x_i} = \nabla\phi + \nabla\psi$.

(b) $\text{div}(\mathbf{v}+\mathbf{w}) = \text{div}\mathbf{v} + \text{div}\mathbf{w}$, (c) $\text{div}(\phi\mathbf{v}) = (\nabla\phi)\cdot\mathbf{v} + \phi(\text{div}\mathbf{v})$ and (d) $\text{div}(\text{curl}\mathbf{v}) = 0$.

Ans. (b) $\text{div}(\mathbf{v}+\mathbf{w}) = \dfrac{\partial(v_i + w_i)}{\partial x_i} = \dfrac{\partial v_i}{\partial x_i} + \dfrac{\partial w_i}{\partial x_i} = \text{div}\mathbf{v} + \text{div}\mathbf{w}$.

(c) $\text{div}(\phi\mathbf{v}) = \dfrac{\partial(\phi v_i)}{\partial x_i} = \phi\dfrac{\partial v_i}{\partial x_i} + \dfrac{\partial\phi}{\partial x_i}v_i = \phi(\text{div}\mathbf{v}) + (\nabla\phi)\cdot\mathbf{v}$.

(d) $\text{curl}\mathbf{v} = -\varepsilon_{ijk}\dfrac{\partial v_j}{\partial x_k}\mathbf{e}_i = \varepsilon_{ijk}\dfrac{\partial v_k}{\partial x_j}\mathbf{e}_i \to \text{div}(\text{curl}\mathbf{v}) = \dfrac{\partial}{\partial x_i}\varepsilon_{ijk}\dfrac{\partial v_k}{\partial x_j} = \varepsilon_{ijk}\dfrac{\partial}{\partial x_i}\dfrac{\partial v_k}{\partial x_j}$.

By changing the dummy indices, ($i \to j, j \to i$) we have, $\varepsilon_{ijk}\dfrac{\partial}{\partial x_i}\dfrac{\partial v_k}{\partial x_j} = \varepsilon_{jik}\dfrac{\partial}{\partial x_j}\dfrac{\partial v_k}{\partial x_i}$. Thus,

$\varepsilon_{ijk}\dfrac{\partial}{\partial x_i}\dfrac{\partial v_k}{\partial x_j} = -\varepsilon_{ijk}\dfrac{\partial}{\partial x_j}\dfrac{\partial v_k}{\partial x_i} \to 2\varepsilon_{ijk}\dfrac{\partial}{\partial x_i}\dfrac{\partial v_k}{\partial x_j} = 0 \to \dfrac{\partial}{\partial x_i}\left(\varepsilon_{ijk}\dfrac{\partial v_k}{\partial x_j}\right) = 0$. Thus, $\text{div}(\text{curl}\mathbf{v}) = 0$.

2.69 Consider the vector field $\mathbf{v} = x_1^2\mathbf{e}_1 + x_3^2\mathbf{e}_2 + x_2^2\mathbf{e}_3$. For the point $(1,1,0)$, find (a) $\nabla\mathbf{v}$, (b) $(\nabla\mathbf{v})\mathbf{v}$, (c) $\text{div}\,\mathbf{v}$ and $\text{curl}\,\mathbf{v}$ and (d) the differential $d\mathbf{v}$ for $d\mathbf{r} = ds(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)/\sqrt{3}$.

Ans.(a) $[\nabla\mathbf{v}] = \begin{bmatrix} 2x_1 & 0 & 0 \\ 0 & 0 & 2x_3 \\ 0 & 2x_2 & 0 \end{bmatrix} \to [\nabla\mathbf{v}]_{(1,1,0)} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$.

(b) $[(\nabla\mathbf{v})\mathbf{v}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \to (\nabla\mathbf{v})\mathbf{v} = 2\mathbf{e}_1$.

(c) $\text{div}\,\mathbf{v} = 2x_1 + 0 + 0 = 2x_1 \to$ at (1,1,0), $\text{div}\,\mathbf{v} = 2$.

$$\text{curl}\mathbf{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}\right)\mathbf{e}_1 + \left(\frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}\right)\mathbf{e}_2 + \left(\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_3 = 2(x_2 - x_3)\mathbf{e}_1 .$$

At $(1,1,0)$, $\text{curl}\mathbf{v} = 2(1-0)\mathbf{e}_1 = 2\mathbf{e}_1$.

(d)

$$\text{At}(1,1,0),\ d\mathbf{v} = (\nabla\mathbf{v})d\mathbf{r} \rightarrow [d\mathbf{v}] = [\nabla\mathbf{v}][d\mathbf{r}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\begin{bmatrix} ds/\sqrt{3} \\ ds/\sqrt{3} \\ ds/\sqrt{3} \end{bmatrix} = \begin{bmatrix} 2ds/\sqrt{3} \\ 0 \\ 2ds/\sqrt{3} \end{bmatrix}.$$

$\rightarrow d\mathbf{v} = 2ds(\mathbf{e}_1 + \mathbf{e}_3)/\sqrt{3}$

CHAPTER 2, PART D

2.70 Calculate $\text{div}\,\mathbf{u}$ for the following vector field in cylindrical coordinates:
(a) $u_r = u_\theta = 0,\quad u_z = A + Br^2$. (b) $u_r = \sin\theta / r,\ u_\theta = u_z = 0$, and
(c) $u_r = r^2 \sin\theta / 2,\quad u_\theta = r^2 \cos\theta / 2,\quad u_z = 0$.

Ans.(a) $u_r = u_\theta = 0,\quad u_z = A + Br^2 \rightarrow \text{div}\,\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = 0+0+0+0 = 0$.

(b) $u_r = \sin\theta / r,\ u_\theta = u_z = 0 \rightarrow \text{div}\,\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = -\sin\theta / r^2 + 0 + \sin\theta / r^2 + 0 = 0$

(c) $u_r = r^2 \sin\theta / 2,\quad u_\theta = r^2 \cos\theta / 2,\quad u_z = 0$

$\rightarrow \text{div}\,\mathbf{u} = \frac{\partial u_r}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r} + \frac{\partial u_z}{\partial z} = r\sin\theta - r\sin\theta/2 + r\sin\theta/2 + 0 = r\sin\theta$.

2.71 Calculate $\nabla\mathbf{u}$ for the following vector field in cylindrical coordinate:
$u_r = A/r,\quad u_\theta = Br,\quad u_z = 0$.

$$\textit{Ans.}\ [\nabla\mathbf{u}] = \begin{bmatrix} \frac{\partial u_r}{\partial r} & \frac{1}{r}\left(\frac{\partial u_r}{\partial \theta} - u_\theta\right) & \frac{\partial u_r}{\partial z} \\ \frac{\partial u_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right) & \frac{\partial u_\theta}{\partial z} \\ \frac{\partial u_z}{\partial r} & \frac{1}{r}\frac{\partial u_z}{\partial \theta} & \frac{\partial u_z}{\partial z} \end{bmatrix} = \begin{bmatrix} -\frac{A}{r^2} & -B & 0 \\ B & \frac{A}{r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

2.72 Calculate $\text{div}\,\mathbf{u}$ for the following vector field in spherical coordinates
$u_r = Ar + B/r^2,\quad u_\theta = u_\phi = 0$

Ans. $\rightarrow \text{div}\,\mathbf{u} = \frac{1}{r^2}\frac{\partial (r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial (u_\theta \sin\theta)}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi}$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\left(Ar+\frac{B}{r^2}\right)\right\}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(Ar^3+B\right)=3A.$$

2.73 Calculate $\nabla\mathbf{u}$ for the following vector field in spherical coordinates: $u_r=Ar+B/r^2,\quad u_\theta=u_\phi=0$.

Ans.

$$[\nabla\mathbf{u}]=\begin{bmatrix}\frac{\partial u_r}{\partial r} & \left(\frac{1}{r}\frac{\partial u_r}{\partial\theta}-\frac{u_\theta}{r}\right) & \left(\frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\phi}-\frac{u_\phi}{r}\right)\\ \frac{\partial u_\theta}{\partial r} & \left(\frac{1}{r}\frac{\partial u_\theta}{\partial\theta}+\frac{u_r}{r}\right) & \left(\frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\phi}-\frac{u_\phi\cot\theta}{r}\right)\\ \frac{\partial u_\phi}{\partial r} & \frac{1}{r}\frac{\partial u_\phi}{\partial\theta} & \left(\frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial\phi}+\frac{u_r}{r}+\frac{u_\theta\cot\theta}{r}\right)\end{bmatrix}$$

$$=\begin{bmatrix}\partial u_r/\partial r & 0 & 0\\ 0 & u_r/r & 0\\ 0 & 0 & u_r/r\end{bmatrix}=\begin{bmatrix}A-2B/r^3 & 0 & 0\\ 0 & A+B/r^3 & 0\\ 0 & 0 & A+B/r^3\end{bmatrix}.$$

2.74 From the definition of the Laplacian of a vector, $\nabla^2\mathbf{v}=\nabla(\text{div }\mathbf{v})-\text{curl}\,\text{curl }\mathbf{v}$, derive the following results in cylindrical coordinates:

$$\left(\nabla^2\mathbf{v}\right)_r=\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}\right)\text{ and}$$

$$\left(\nabla^2\mathbf{v}\right)_\theta=\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}.$$

Ans. Let $\mathbf{v}(r)$ be a vector field. The Laplacian of $\mathbf{v}$ is $\nabla^2\mathbf{v}=\nabla(\text{div }\mathbf{v})-\text{curl}\,\text{curl }\mathbf{v}$. Now,

$$\text{div }\mathbf{v}=\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\text{, so that}$$

$$\nabla(\text{div }\mathbf{v})=\frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\right)\mathbf{e}_r+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\right)\mathbf{e}_\theta$$

$$+\frac{\partial}{\partial z}\left(\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\right)\mathbf{e}_z=\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}+\frac{\partial^2 v_z}{\partial r\partial z}\right)\mathbf{e}_r$$

$$+\left(\frac{1}{r}\frac{\partial^2 v_r}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{1}{r}\frac{\partial^2 v_z}{\partial\theta\partial z}\right)\mathbf{e}_\theta+\left(\frac{\partial^2 v_r}{\partial z\partial r}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial z\partial\theta}+\frac{1}{r}\frac{\partial v_r}{\partial z}+\frac{\partial^2 v_z}{\partial z^2}\right)\mathbf{e}_z.$$

Next,

$$\text{curl }\mathbf{v}=\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}-\frac{\partial v_\theta}{\partial z}\right)\mathbf{e}_r+\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta+\left(\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\mathbf{e}_z\text{, so that}$$

$$(\text{curl}\,\text{curl}\,\mathbf{v})_r = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)-\frac{\partial}{\partial z}\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)$$

$$=\left(\frac{1}{r}\frac{\partial^2 v_\theta}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}\right)-\left(\frac{\partial^2 v_r}{\partial z^2}-\frac{\partial^2 v_z}{\partial z\partial r}\right),$$

$$(\text{curl}\,\text{curl}\,\mathbf{v})_\theta = \frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}-\frac{\partial v_\theta}{\partial z}\right)-\frac{\partial}{\partial r}\left(\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)$$

$$(\text{curl}\,\text{curl}\,\mathbf{v})_z = \frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)+\frac{1}{r}\left(\frac{\partial v_r}{\partial z}-\frac{\partial v_z}{\partial r}\right)-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}-\frac{\partial v_\theta}{\partial z}\right)$$

Thus,

$$\left(\nabla^2\mathbf{v}\right)_r = \left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}+\frac{\partial^2 v_z}{\partial r\partial z}\right)$$

$$-\left(\frac{1}{r}\frac{\partial^2 v_\theta}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}\right)+\left(\frac{\partial^2 v_r}{\partial z^2}-\frac{\partial^2 v_z}{\partial z\partial r}\right)=\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\partial^2 v_r}{\partial z^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{v_r}{r^2}\right).$$

$$\left(\nabla^2\mathbf{v}\right)_\theta = \left(\frac{1}{r}\frac{\partial^2 v_r}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{1}{r}\frac{\partial^2 v_z}{\partial\theta\partial z}\right)$$

$$-\frac{\partial}{\partial z}\left(\frac{1}{r}\frac{\partial v_z}{\partial\theta}-\frac{\partial v_\theta}{\partial z}\right)+\frac{\partial}{\partial r}\left(\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)=\left(\frac{1}{r}\frac{\partial^2 v_r}{\partial\theta\partial r}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{1}{r}\frac{\partial^2 v_z}{\partial\theta\partial z}\right)$$

$$+\left(-\frac{1}{r}\frac{\partial^2 v_z}{\partial z\partial\theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right)+\left(\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}-\frac{v_\theta}{r^2}-\frac{1}{r}\frac{\partial^2 v_r}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}\right)$$

$$=\frac{\partial^2 v_\theta}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial\theta^2}+\frac{\partial^2 v_\theta}{\partial z^2}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}.$$

2.75 From the definition of the Laplacian of a vector, $\nabla^2\mathbf{v}=\nabla(\text{div}\,\mathbf{v})-\text{curlcurl}\,\mathbf{v}$, derive the following result in spherical coordinates:

$$\left(\nabla^2\mathbf{v}\right)_r = \left(\frac{1}{r^2}\frac{\partial^2 r^2 v_r}{\partial r^2}-\frac{2}{r^3}\frac{\partial r^2 v_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}-\frac{2}{r^2\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)$$

Ans.

From $\frac{1}{r^2}\frac{\partial\left(r^2 v_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(v_\theta\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}$, we have,

$$\nabla(\text{div}\mathbf{v})=\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\mathbf{e}_r+\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\mathbf{e}_\theta$$

$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r^2}\frac{\partial r^2 v_r}{\partial r}++\frac{1}{r\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\mathbf{e}_\phi,\text{ that is}$$

$$\nabla(\text{div}\mathbf{v})=\left(\frac{1}{r^2}\frac{\partial^2\left(r^2 v_r\right)}{\partial r^2}-\frac{2}{r^3}\frac{\partial\left(r^2 v_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial^2 v_\theta\sin\theta}{\partial r\partial\theta}-\frac{1}{r^2\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial^2 v_\phi}{\partial r\partial\phi}-\frac{1}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\mathbf{e}_r$$

$$+\left(\frac{1}{r^3}\frac{\partial^2\left(r^2v_r\right)}{\partial\theta\partial r}+\frac{1}{r^2\sin\theta}\frac{\partial^2\left(v_\theta\sin\theta\right)}{\partial\theta^2}+\frac{1}{r^2\sin\theta}\frac{\partial^2 v_\theta\sin\theta}{\partial\theta^2}-\frac{1}{r^2}\left(\frac{\cos\theta}{\sin^2\theta}\right)\frac{\partial v_\theta\sin\theta}{\partial\theta}+\frac{1}{r}\frac{\partial}{\partial\theta}\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)\mathbf{e}_\theta$$

$$+\left(\frac{1}{r^3\sin\theta}\frac{\partial}{\partial\phi}\frac{\partial\left(r^2v_r\right)}{\partial r}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\left(v_\theta\sin\theta\right)}{\partial\phi\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_\phi}{\partial^2\phi}\right)\mathbf{e}_\phi$$. Also,

$$\text{curl }\mathbf{v}=\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi\sin\theta}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right)\mathbf{e}_r+\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{1}{r}\frac{\partial rv_\phi}{\partial r}\right)\mathbf{e}_\theta+\left(\frac{1}{r}\frac{\partial rv_\theta}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\mathbf{e}_\phi$$

so that

$$\text{curl}\,\text{curl }\mathbf{v}=\frac{1}{r\sin\theta}\left\{\frac{\partial}{\partial\theta}\left(\frac{1}{r}\frac{\partial rv_\theta}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\sin\theta-\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{1}{r}\frac{\partial rv_\phi}{\partial r}\right)\right\}\mathbf{e}_r$$

$$+\left\{\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi\sin\theta}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right)-\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{1}{r}\frac{\partial rv_\theta}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)\right\}\mathbf{e}_\theta$$

$$+\left\{\frac{1}{r}\frac{\partial}{\partial r}r\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{1}{r}\frac{\partial rv_\phi}{\partial r}\right)-\frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi\sin\theta}{\partial\theta}-\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}\right)\right\}\mathbf{e}_\phi$$

i.e.,

$\text{curl}\,\text{curl }\mathbf{v}=$

$$\left\{\frac{1}{r^2}\left(\frac{\partial^2 rv_\theta}{\partial\theta\partial r}-\frac{\partial^2 v_r}{\partial\theta^2}\right)+\frac{\cot\theta}{r}\left(\frac{1}{r}\frac{\partial rv_\theta}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)-\left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}-\frac{1}{r^2\sin\theta}\frac{\partial^2 rv_\phi}{\partial\phi\partial r}\right)\right\}\mathbf{e}_r$$

$$+\left\{\frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2 v_\phi\sin\theta}{\partial\phi\partial\theta}-\frac{\partial^2 v_\theta}{\partial\phi^2}\right)-\frac{1}{r^2}\left(\frac{\partial rv_\theta}{\partial r}-\frac{\partial v_r}{\partial\theta}\right)-\left(\frac{1}{r}\frac{\partial^2 rv_\theta}{\partial r^2}-\frac{1}{r^2}\frac{\partial rv_\theta}{\partial r}-\frac{1}{r}\frac{\partial^2 v_r}{\partial r\partial\theta}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}\right)\right\}\mathbf{e}_\theta$$

$$+\left\{\begin{array}{l}\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial r\partial\phi}-\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{1}{r}\frac{\partial^2 rv_\phi}{\partial r^2}+\frac{1}{r^2}\frac{\partial rv_\phi}{\partial r}\right)+\frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{\partial rv_\phi}{\partial r}\right)\\ -\frac{1}{r^2}\frac{1}{\sin\theta}\left(-v_\phi\sin\theta+\sin\theta\frac{\partial^2 v_\phi}{\partial\theta^2}-\frac{\partial^2 v_\theta}{\partial\theta\partial\phi}\right)+\frac{\cos\theta}{r^2\sin^2\theta}\left(\frac{\partial v_\phi\sin\theta}{\partial\theta}-\frac{\partial v_\theta}{\partial\phi}\right)\end{array}\right\}\mathbf{e}_\phi$$

Thus, $\nabla^2\mathbf{v}=\nabla\left(\text{div}\mathbf{v}\right)-\text{curl}\,\text{curl }\mathbf{v}$ gives:

$$\left(\nabla^2\mathbf{v}\right)_r=\left(\frac{1}{r^2}\frac{\partial^2\left(r^2v_r\right)}{\partial r^2}-\frac{2}{r^3}\frac{\partial\left(r^2v_r\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial^2\left(v_\theta\sin\theta\right)}{\partial r\partial\theta}-\frac{1}{r^2\sin\theta}\frac{\partial\left(v_\theta\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial^2 v_\phi}{\partial r\partial\phi}-\frac{1}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)$$

$$-\left\{\frac{1}{r^2}\left(\frac{\partial^2 rv_\theta}{\partial\theta\partial r}-\frac{\partial^2 v_r}{\partial\theta^2}\right)+\frac{\cot\theta}{r}\left(\frac{1}{r}\frac{\partial rv_\theta}{\partial r}-\frac{1}{r}\frac{\partial v_r}{\partial\theta}\right)-\left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}-\frac{1}{r^2\sin\theta}\frac{\partial^2 rv_\phi}{\partial\phi\partial r}\right)\right\}$$

i.e.,

$$\left(\nabla^2\mathbf{v}\right)_r=\left(\frac{1}{r^2}\frac{\partial^2 r^2v_r}{\partial r^2}-\frac{2}{r^3}\frac{\partial r^2v_r}{\partial r}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial v_r}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}-\frac{2}{r^2\sin\theta}\frac{\partial v_\theta\sin\theta}{\partial\theta}-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)$$

2.76 From the equation $(\text{div}\mathbf{T})\cdot\mathbf{a} = \text{div}(\mathbf{T}^{\text{T}}\mathbf{a}) - \text{tr}(\mathbf{T}^{\text{T}}\nabla\mathbf{a})$ [See Eq. 2.29.3)] verify that in polar coordinates, the θ-component of the vector $(\text{div}\mathbf{T})$ is: $(\text{div}\mathbf{T})_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{T_{r\theta} + T_{\theta r}}{r}$.

Ans. $(\text{div}\mathbf{T})\cdot\mathbf{a} = \text{div}(\mathbf{T}^{\text{T}}\mathbf{a}) - \text{tr}(\mathbf{T}^{\text{T}}\nabla\mathbf{a}) \to (\text{div}\mathbf{T})\cdot\mathbf{e}_\theta = \text{div}(\mathbf{T}^{\text{T}}\mathbf{e}_\theta) - \text{tr}(\mathbf{T}^{\text{T}}\nabla\mathbf{e}_\theta)$

Now,

$$\mathbf{Te}_{\text{r}} = T_{rr}\mathbf{e}_{\text{r}} + T_{\theta r}\mathbf{e}_\theta, \mathbf{Te}_\theta = T_{r\theta}\mathbf{e}_{\text{r}} + T_{\theta\theta}\mathbf{e}_\theta \to \mathbf{e}_{\text{r}}\cdot\mathbf{T}^{\text{T}}\mathbf{e}_\theta = \mathbf{e}_\theta\cdot\mathbf{Te}_{\text{r}} = T_{\theta r},\ \ \mathbf{e}_\theta\cdot\mathbf{T}^{\text{T}}\mathbf{e}_\theta = \mathbf{e}_\theta\cdot\mathbf{Te}_\theta = T_{\theta\theta}$$

i.e., $\mathbf{T}^{\text{T}}\mathbf{e}_\theta = T_{\theta r}\mathbf{e}_{\text{r}} + T_{\theta\theta}\mathbf{e}_\theta \to \text{div}(\mathbf{T}^{\text{T}}\mathbf{e}_\theta) = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{T_{\theta r}}{r}$. Also

$$\mathbf{e}_\theta = 0\mathbf{e}_{\text{r}} + 1\mathbf{e}_\theta \to [\nabla\mathbf{e}_\theta] = \begin{bmatrix} 0 & -1/r \\ 0 & 0 \end{bmatrix} \to \left[\mathbf{T}^{\text{T}}\nabla\mathbf{e}_\theta\right] = \begin{bmatrix} T_{rr} & T_{\theta r} \\ T_{r\theta} & T_{\theta\theta} \end{bmatrix}\begin{bmatrix} 0 & -1/r \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -T_{rr}/r \\ 0 & -T_{r\theta}/r \end{bmatrix}$$

Thus,

$$(\text{div}\mathbf{T})_\theta = \text{div}(\mathbf{T}^{\text{T}}\mathbf{e}_\theta) - \text{tr}(\mathbf{T}^{\text{T}}\nabla\mathbf{e}_\theta) = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{T_{\theta r}}{r} - (0 - T_{r\theta}/r) = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{T_{\theta r} + T_{r\theta}}{r}.$$

2.77 Calculate $\text{div}\mathbf{T}$ for the following tensor field in cylindrical coordinates;

$$T_{rr} = A + \frac{B}{r^2},\quad T_{\theta\theta} = A - \frac{B}{r^2},\quad T_{zz} = \text{constant},\quad T_{r\theta} = T_{\theta r} = T_{rz} = T_{zr} = T_{\theta z} = T_{z\theta} = 0$$

Ans. $(\text{div}\mathbf{T})_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial\theta} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{\partial T_{rz}}{\partial z} = -\frac{2B}{r^3} + \frac{2B}{r^3} = 0$.

$$(\text{div}\mathbf{T})_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{T_{r\theta} + T_{\theta r}}{r} + \frac{\partial T_{\theta z}}{\partial z} = 0.$$

$$(\text{div}\mathbf{T})_z = \frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial\theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r} = 0.$$

2.78 Calculate $\text{div}\mathbf{T}$ for the following tensor field in cylindrical coordinates;

$$T_{rr} = \frac{Az}{R^3} - \frac{3Br^2z}{R^5},\quad T_{\theta\theta} = \frac{Az}{R^3},\quad T_{zz} = -\left(\frac{Az}{R^3} + \frac{3Bz^3}{R^5}\right),\quad T_{rz} = T_{zr} = -\left(\frac{Ar}{R^3} + \frac{3Brz^2}{R^5}\right)$$

$$T_{r\theta} = T_{\theta r} = T_{\theta z} = T_{z\theta} = 0,\ \ R^2 = r^2 + z^2.$$

Ans.

$$(\text{div}\mathbf{T})_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial\theta} + \frac{T_{rr} - T_{\theta\theta}}{r} + \frac{\partial T_{rz}}{\partial z} = \frac{\partial}{\partial r}\left(\frac{Az}{R^3} - \frac{3Br^2z}{R^5}\right) - \frac{3Brz}{R^5} - \frac{\partial}{\partial z}\left(\frac{Ar}{R^3} + \frac{3Brz^2}{R^5}\right)$$

$$= \left(Az\frac{\partial}{\partial r}\frac{1}{R^3} - 3Br^2z\frac{\partial}{\partial r}\frac{1}{R^5} - \frac{3Bz}{R^5}\frac{\partial}{\partial r}r^2\right) - \frac{3Brz}{R^5} - \left(Ar\frac{\partial}{\partial z}\frac{1}{R^3} + \frac{3Br}{R^5}\frac{\partial}{\partial z}z^2 + 3Brz^2\frac{\partial}{\partial z}\frac{1}{R^5}\right)$$

$$= \left(-\frac{3Az}{R^4}\frac{\partial R}{\partial r} + \frac{15Br^2z}{R^6}\frac{\partial R}{\partial r} - \frac{6Brz}{R^5}\right) - \frac{3Brz}{R^5} - \left(-\frac{3Ar}{R^4}\frac{\partial R}{\partial z} + \frac{6Bzr}{R^5} - \frac{15Brz^2}{R^6}\frac{\partial R}{\partial z}\right)$$

$$= \left(-\frac{3Arz}{R^5} + \frac{15Br^3z}{R^7} - \frac{6Brz}{R^5}\right) - \frac{3Brz}{R^5} + \left(\frac{3Arz}{R^5} - \frac{6Bzr}{R^5} + \frac{15Brz^3}{R^7}\right)$$

$$=B\left(\frac{15r^3z}{R^7}-\frac{15rz}{R^5}+\frac{15rz^3}{R^7}\right)=B\left(\frac{15rz}{R^7}\left(r^2+z^2\right)-\frac{15rz}{R^5}\right)=\left(\frac{15rz}{R^5}-\frac{15rz}{R^5}\right)=0.$$

$$(\text{div}\mathbf{T})_\theta=\frac{\partial T_{\theta r}}{\partial r}+\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta}+\frac{T_{r\theta}+T_{\theta r}}{r}+\frac{\partial T_{\theta z}}{\partial z}=0+0+0+0=0$$

$$(\text{div}\mathbf{T})_z=\frac{\partial T_{zr}}{\partial r}+\frac{1}{r}\frac{\partial T_{z\theta}}{\partial\theta}+\frac{\partial T_{zz}}{\partial z}+\frac{T_{zr}}{r}=-\frac{\partial}{\partial r}\left(\frac{A}{R^3}+\frac{3Brz^2}{R^5}\right)-\frac{\partial}{\partial z}\left(\frac{Az}{R^3}+\frac{3Bz^3}{R^5}\right)-\left(\frac{A}{R^3}+\frac{3Bz^2}{R^5}\right)$$

$$=-\left(\frac{A}{R^3}-\frac{3Ar^2}{R^5}+\frac{3Bz^2}{R^5}-\frac{15Br^2z^2}{R^7}\right)-\left(\frac{A}{R^3}-\frac{3Az^2}{R^5}+\frac{9Bz^2}{R^5}-\frac{15Bz^4}{R^7}\right)-\left(\frac{A}{R^3}+\frac{3Bz^2}{R^5}\right)$$

$$=\left(-\frac{3A}{R^3}+\frac{3A}{R^5}\left(r^2+z^2\right)-\frac{15Bz^2}{R^5}+\frac{15Bz^2}{R^7}\left(r^2+z^2\right)\right)==\left(-\frac{3A}{R^3}+\frac{3A}{R^3}-\frac{15Bz^2}{R^5}+\frac{15Bz^2}{R^5}\right)=0.$$

2.79 Calculate $\text{div}\,\mathbf{T}$ for the following tensor field in spherical coordinates;

$$T_{rr}=A-\frac{2B}{r^3},\quad T_{\theta\theta}=T_{\phi\phi}=A+\frac{B}{r^3},\quad T_{r\theta}=T_{\theta r}=T_{\theta\phi}=T_{\phi\theta}=T_{r\phi}=T_{\phi r}=0$$

Ans. $(\text{div}\mathbf{T})_r=\frac{1}{r^2}\frac{\partial\left(r^2T_{rr}\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(T_{r\theta}\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi}-\frac{T_{\theta\theta}+T_{\phi\phi}}{r}$

$$=\frac{1}{r^2}\frac{\partial\left(r^2T_{rr}\right)}{\partial r}-\frac{T_{\theta\theta}+T_{\phi\phi}}{r}=\frac{1}{r^2}\frac{\partial}{\partial r}\left(Ar^2-\frac{2B}{r}\right)-2\left(\frac{A}{r}+\frac{B}{r^4}\right)$$

$$=\frac{1}{r^2}\left(2Ar+\frac{2B}{r^2}\right)-2\left(\frac{A}{r}+\frac{B}{r^4}\right)=\left(\frac{2A}{r}+\frac{2B}{r^4}\right)-2\left(\frac{A}{r}+\frac{B}{r^4}\right)=0.$$

$$(\text{div}\mathbf{T})_\theta=\frac{1}{r^3}\frac{\partial\left(r^3T_{\theta r}\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(T_{\theta\theta}\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi}+\frac{T_{r\theta}-T_{\theta r}-T_{\phi\phi}\cot\theta}{r}$$

$$=\frac{T_{\theta\theta}\cot\theta}{r}+\frac{-T_{\phi\phi}\cot\theta}{r}=0.$$

$$(\text{div}\mathbf{T})_\phi=\frac{1}{r^3}\frac{\partial\left(r^3T_{\phi r}\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(T_{\phi\theta}\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi}+\frac{T_{r\phi}-T_{\phi r}+T_{\theta\phi}\cot\theta}{r}=0.$$

2.80 From the equation $(\text{div}\mathbf{T})\cdot\mathbf{a}=\text{div}(\mathbf{T}^{\mathrm{T}}\mathbf{a})-\text{tr}(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{a})$ [See Eq. 2.29.3)] verify that in spherical coordinates, the θ-component of the vector $(\text{div}\mathbf{T})$ is:

$$(\text{div}\mathbf{T})_\theta=\frac{1}{r^3}\frac{\partial(r^3T_{\theta r})}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial(T_{\theta\theta}\sin\theta)}{\partial\theta}++\frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi}+\frac{T_{r\theta}-T_{\theta r}-T_{\phi\phi}\cot\theta}{r}.$$

Ans. $(\text{div}\mathbf{T})\cdot\mathbf{a}=\text{div}(\mathbf{T}^{\mathrm{T}}\mathbf{a})-\text{tr}(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{a})\rightarrow(\text{div}\mathbf{T})\cdot\mathbf{e}_\theta=\text{div}(\mathbf{T}^{\mathrm{T}}\mathbf{e}_\theta)-\text{tr}(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{e}_\theta)$. Now,

$\mathbf{T}^{\mathrm{T}}\mathbf{e}_\theta = T_{\theta r}\mathbf{e}_\mathrm{r} + T_{\theta\theta}\mathbf{e}_\theta + T_{\theta\phi}\mathbf{e}_\phi \rightarrow \mathrm{div}(\mathbf{T}^{\mathrm{T}}\mathbf{e}_\theta) = \frac{1}{r^2}\frac{\partial(r^2 T_{\theta r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{\theta\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi}$. Also,

$$\mathbf{e}_\theta = 0\mathbf{e}_\mathrm{r} + 1\mathbf{e}_\theta + 0\mathbf{e}_\phi \rightarrow [\nabla\mathbf{e}_\theta] = \begin{bmatrix} 0 & -1/r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \cot\theta/r \end{bmatrix}$$

$$\rightarrow \left[\mathbf{T}^{\mathrm{T}}\nabla\mathbf{e}_\theta\right] = \begin{bmatrix} T_{rr} & T_{\theta r} & T_{\phi r} \\ T_{r\theta} & T_{\theta\theta} & T_{\phi\theta} \\ T_{r\phi} & T_{\theta\phi} & T_{\phi\phi} \end{bmatrix}\begin{bmatrix} 0 & -1/r & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \cot\theta/r \end{bmatrix} = \begin{bmatrix} 0 & -T_{rr}/r & T_{\phi r}\cot\theta/r \\ 0 & -T_{r\theta}/r & T_{\phi\theta}\cot\theta/r \\ 0 & -T_{r\phi}/r & T_{\phi\phi}\cot\theta/r \end{bmatrix}$$

$\rightarrow \mathrm{tr}\left[\mathbf{T}^{\mathrm{T}}\nabla\mathbf{e}_\theta\right] = -\frac{T_{r\theta}}{r} + \frac{T_{\phi\phi}\cot\theta}{r}$. Thus,

$$(\mathrm{div}\mathbf{T})_\theta = \mathrm{div}(\mathbf{T}^{\mathrm{T}}\mathbf{e}_\theta) - \mathrm{tr}(\mathbf{T}^{\mathrm{T}}\nabla\mathbf{e}_\theta)$$

$$= \frac{1}{r^2}\frac{\partial(r^2 T_{\theta r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(T_{\theta\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi} + \frac{T_{r\theta}}{r} - \frac{T_{\phi\phi}\cot\theta}{r}$$

$$= \frac{1}{r^3}\frac{\partial(r^3 T_{\theta r})}{\partial r} - \frac{T_{\theta r}}{r} + \frac{1}{r\sin\theta}\frac{\partial(T_{\theta\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi} + \frac{T_{r\theta}}{r} - \frac{T_{\phi\phi}\cot\theta}{r}.$$

CHAPTER 3

3.1 Consider the motion: $x_1 = (1+kt)X_1/(1+kt_o),\ \ x_2 = X_2,\ \ x_3 = X_3$.

(a) Show that reference time is $t = t_o$. (b) Find the velocity field in spatial coordinates. (c) Show that the velocity field is identical to that of the following motion:
$x_1 = (1+kt)X_1,\ \ x_2 = X_2,\ \ x_3 = X_3$.

Ans. (a) At $t = t_o$, $x_1 = X_1,\ \ x_2 = X_2,\ \ x_3 = X_3$. Thus, $t = t_o$ is the reference time.
(b) In material description, $v_1 = kX_1/(1+kt_o),\ \ v_2 = v_3 = 0$. Now, from $x_1 = (1+kt)X_1/(1+kt_o)$, $\rightarrow X_1 = (1+kt_o)x_1/(1+kt)$, therefore, $\rightarrow v_1 = kX_1 = kx_1/(1+kt),\ \ v_2 = v_3 = 0$.
(c) For $x_1 = (1+kt)X_1,\ \ x_2 = X_2,\ \ x_3 = X_3,\ \ \rightarrow v_1 = kX_1,\ \ v_2 = v_3 = 0$
$\rightarrow v_1 = kx_1/(1+kt),\ \ v_2 = v_3 = 0$, which are the same as the velocity components in (b).

3.2 Consider the motion: $x_1 = \alpha t + X_1,\ \ x_2 = X_2,\ \ x_3 = X_3$, where the material coordinates X_i designate the position of a particle at $t = 0$. (a) Determine the velocity and acceleration of a particle in both a material and a spatial description. (b) If the temperature field in spatical description is given by $\theta = Ax_1$, what is its material description? Find the material derivative of θ, using both descriptions of the temperature. (c) Do part (b) if the temperature field is $\theta = Bx_2$

Ans. (a) Material description: $v_1 = Dx_1/Dt = (\partial x_1/\partial t)_{X_i-\text{fixed}} = \alpha,\ \ v_2 = v_3 = 0$,
$a_1 = Dv_1/Dt = (\partial v_1/\partial t)_{X_i-\text{fixed}} = 0,\ \ a_2 = a_3 = 0$.
Spatial description: The same as above $v_1 = \alpha,\ \ v_2 = v_3 = 0,\ \ \ a_1 = a_2 = a_3 = 0$..
(b) The material description of θ is $\theta = A(\alpha t + X_1)$.
Using the material description: $\theta = A(\alpha t + X_1) \rightarrow D\theta/Dt = (\partial/\partial t)\left[A(\alpha t + X_1)\right] = A\alpha$.
Using the spatical description: $\theta = Ax_1 \rightarrow$

$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + v_1\frac{\partial\theta}{\partial x_1} + v_2\frac{\partial\theta}{\partial x_2} + v_3\frac{\partial\theta}{\partial x_3} = 0 + \alpha(A) + (0)(0) + (0)(0) = A\alpha .$$

(c) Using the material description: $\theta = BX_2 \rightarrow D\theta/Dt = (\partial/\partial t)(BX_2) = 0$.
Using the spatical description: $\theta = Bx_2 \rightarrow$

$$\frac{D\theta}{Dt} = \frac{\partial\theta}{\partial t} + v_1\frac{\partial\theta}{\partial x_1} + v_2\frac{\partial\theta}{\partial x_2} + v_3\frac{\partial\theta}{\partial x_3} = 0 + \alpha(0) + (0)(B) + (0)(0) = 0 .$$

3.3 Consider the motion
$x_1 = X_1,\ \ x_2 = \beta X_1^2 t^2 + X_2,\ \ x_3 = X_3$, where X_i are the material coordinates. (a) at $t = 0$, the corners of a unit square are at $A(0,0,0), B(0,1,0), C(1,1,0)$ and $D(1,0,0)$. Determine the position of $ABCD$ at $t = 1$ and sketch the new shape of the square. (b) Find the velocity $\mathbf{v}$ and the acceleration in a material description and (c) Find the spatial velocity field.

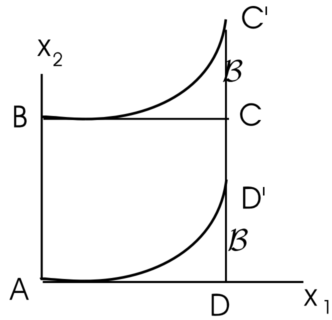
Ans. For the material line *AB*, $(X_1,X_2,X_3)=(0,X_2,0)$; at $t=1$, $(x_1,x_2,x_3)=(0,X_2,0)$

For the material line *BC*, $(X_1,X_2,X_3)=(X_1,1,0)$; at $t=1$, $(x_1,x_2,x_3)=\left(X_1,\beta X_1^2+1,0\right)$

For the material line *AD*, $(X_1,X_2,X_3)=(X_1,0,0)$; at $t=1$, $(x_1,x_2,x_3)=\left(X_1,\beta X_1^2,0\right)$

For the material line *CD*, $(X_1,X_2,X_3)=(1,X_2,0)$; at $t=1$, $(x_1,x_2,x_3)=(1,\beta+X_2,0)$

The shape of the material square at $t=1$ is shown in the figure.



(b) $v_i=\left(\dfrac{\partial x_i}{\partial t}\right)_{X_i-\text{fixed}}, a_i=\left(\dfrac{\partial v_i}{\partial t}\right)_{X_i-\text{fixed}} \rightarrow v_1=v_3=0,\ v_2=2\beta X_1^2 t;\ a_1=a_3=0,\ a_2=2\beta X_1^2$

(c) Since $x_1=X_1$, in spatial descrip. $v_1=v_3=0,\ v_2=2\beta x_1^2 t;\ a_1=a_3=0,\ a_2=2\beta x_1^2$

3.4 Consider the motion: $x_1=\beta X_2^2 t^2+X_1,\ \ x_2=kX_2 t+X_2,\ \ x_3=X_3$

(a) At $t=0$, the corners of a unit square are at $A(0,0,0), B(0,1,0), C(1,1,0)$ and $D(1,0,0)$. Sketch the deformed shape of the square at $t=2$. (b) Obtain the spatial description of the velocity field. (c) Obtain the spatial description of the acceleration field.

Ans. (a)

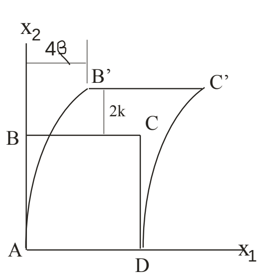
For material line *AB*, $(X_1,X_2,X_3)=(0,X_2,0)$; at $t=2$, $(x_1,x_2,x_3)=\left(4\beta X_2^2,2kX_2+X_2,0\right)$.

For material line *BC*, $(X_1,X_2,X_3)=(X_1,1,0)$; at $t=2$, $(x_1,x_2,x_3)=(4\beta+X_1,2k+1,0)$.

For material line *AD*, $(X_1,X_2,X_3)=(X_1,0,0)$; at $t=2$, $(x_1,x_2,x_3)=(X_1,0,0)$.

For mat. line *CD*, $(X_1,X_2,X_3)=(1,X_2,0)$; at $t=2$, $(x_1,x_2,x_3)=\left(4\beta X_2^2+1,2kX_2+X_2,0\right)$.

The shape of the material square at $t=2$ is shown in the figure.



b) $v_i=\left(\frac{\partial x_i}{\partial t}\right)_{X_i-\text{fixed}}, a_i=\left(\frac{\partial v_i}{\partial t}\right)_{X_i-\text{fixed}} \rightarrow, \ v_1=2\beta X_2^2 t,\ v_2=kX_2,\ v_3=0;\ a_1=2\beta X_2^2,\ a_2=a_3=0$.

(c) $x_2=(kt+1)X_2 \rightarrow,\ \ v_1=\frac{2\beta x_2^2 t}{(1+kt)^2},\ v_2=\frac{kx_2}{(1+kt)},\ v_3=0;\ a_1=\frac{2\beta x_2^2}{(1+kt)^2},\ \ a_2=a_3=0$.

3.5 Consider the motion: $x_1=k\left(s+X_1\right)t+X_1,\ \ x_2=X_2,\ \ x_3=X_3$.

(a) For this motion, repeat part (a) of the previous problem. (b) Find the velocity and acceleration as a function of time of a particle that is initially at the orgin. (c) Find the velocity and acceleration as a function of time of the particles that are passing through the origin.

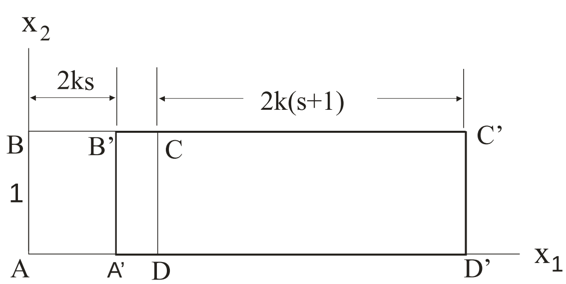
Ans. a) For material line $AB, \left(X_1,X_2,X_3\right)=\left(0,X_2,0\right)$; at $t=2$, $\left(x_1,x_2,x_3\right)=\left(2ks,X_2,0\right)$.

For material line BC, $\left(X_1,X_2,X_3\right)=\left(X_1,1,0\right)$; at $t=2$, $\left(x_1,x_2,x_3\right)=\left(2ks+2kX_1+X_1,1,0\right)$.

For material line AD, $\left(X_1,X_2,X_3\right)=\left(X_1,0,0\right)$; at $t=2$, $\left(x_1,x_2,x_3\right)=\left(2ks+2kX_1+X_1,0,0\right)$.

For material line CD, $\left(X_1,X_2,X_3\right)=\left(1,X_2,0\right)$; at $t=2$, $\left(x_1,x_2,x_3\right)=\left(2ks+2k+1,X_2,0\right)$.

The shape of the material square at $t=2$ is shown in the figure.



(b) $v_i=\left(\frac{\partial x_i}{\partial t}\right)_{X_i-\text{fixed}}$ and $a_i=\left(\frac{\partial v_i}{\partial t}\right)_{X_i-\text{fixed}} \rightarrow,\ v_1=k\left(s+X_1\right),\ \ v_2=0,\ \ v_3=0;\ \ a_1=a_2=a_3=0$.

Thus, for the particle $\left(X_1,X_2,X_3\right)=\left(0,0,0\right),\ v_1=ks,\ \ v_2=0,\ \ v_3=0$ and $a_1=0,\ \ a_2=0,\ \ a_3=0$

(c) $x_1=k\left(s+X_1\right)t+X_1 \rightarrow x_1=kst+\left(kt+1\right)X_1 \rightarrow X_1=(x_1-kst)/(1+kt)$,

thus, in spatial descriptions,

$$v_1=k\left\{s+\frac{x_1-kst}{(1+kt)}\right\}=\frac{k\left(s+x_1\right)}{(1+kt)},\ \ v_2=0,\ \ v_3=0 \text{ and } a_1=0,\ \ a_2=0,\ \ a_3=0.$$

At the position $\left(x_1,x_2,x_3\right)=\left(0,0,0\right)$, $v_1=ks/(1+kt),\ \ v_2=0,\ \ v_3=0$ and $a_1=0,\ \ a_2=0,\ \ a_3=0$.

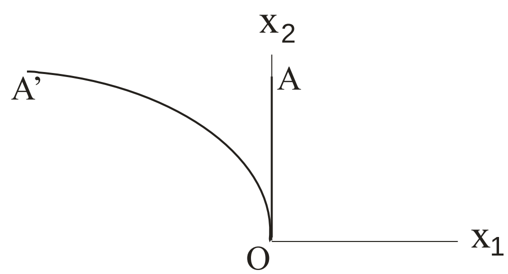
3.6 The position at time t of a particle initially at $\left(X_1,X_2,X_3\right)$ is given by

$$x_1=X_1-2\beta X_2^2 t^2,\ \ x_2=X_2-kX_3 t,\ \ x_3=X_3, \text{ where } \beta=1 \text{ and } k=1.$$

(a) Sketch the deformed shape, at time $t=1$ of the material line OA which was a straight line at $t=0$ with the point O at $\left(0,0,0\right)$ and the point A at $\left(0,1,0\right)$. (b) Find the velocity at $t=2$, of the particle which was at $(1,3,1)$ at $t=0$. (c) Find the velocity of the particle which is at $(1,3,1)$ at $t=2$.

Ans. With $\beta = 1$ and $k = 1$, $x_1 = X_1 - 2X_2^2 t^2,\ x_2 = X_2 - X_3 t,\ x_3 = X_3$

For the material line $OA: (X_1, X_2, X_3) = (0, X_2, 0)$: at $t = 1$, $x_1 = -2X_2^2,\ x_2 = X_2,\ x_3 = 0$. Thus, the deformed shape of the material line at $t = 1$ is a parabola given in the figure shown.



(b) $v_1 = Dx_1/Dt = -4X_2^2 t,\ v_2 = Dx_2/Dt = -X_3,\ v_3 = Dx_3/Dt = 0$

For the particle $(X_1, X_2, X_3) = (1,3,1)$, at $t = 2$, $v_1 = -4(3)^2(2) = -72,\ v_2 = -1,\ v_3 = 0.$

(c) The particle, which is at $(x_1, x_2, x_3) = (1,3,1)$ at $t = 2$, has the material coordinates given by the following equations: $1 = X_1 - 8X_2^2,\ 3 = X_2 - 2X_3,\ 1 = X_3 \to X_1 = 201,\ X_2 = 5,\ X_3 = 1$

$\to v_1 = -4X_2^2 t = -4(5)^2(2) = -200,\ v_2 = -X_3 = -1,\ v_3 = 0.$

3.7 The position at time t of a particle initially at (X_1, X_2, X_3) is given by:

$x_1 = X_1 + k(X_1 + X_2)t,\ x_2 = X_2 + k(X_1 + X_2)t,\ x_3 = X_3,$

(a) Find the velocity at $t = 2$, of the particle which was at $(1,1,0)$ at the reference time $t = 0$.

(b) Find the velocity of the particle which is at $(1,1,0)$ at $t = 2$.

Ans. (a) $v_1 = Dx_1/Dt = k(X_1 + X_2),\ v_2 = Dx_2/Dt = k(X_1 + X_2),\ v_3 = Dx_3/Dt = 0.$

For the particle $(X_1, X_2, X_3) = (1,1,0)$, at $t = 2$, $v_1 = k(1+1) = 2k,\ v_2 = k(1+1) = 2k,\ v_3 = 0$

(b) The particle, which is at $(x_1, x_2, x_3) = (1,1,0)$ at $t = 2$, has the material coordinates given by the following equations: $1 = X_1 + 2k(X_1 + X_2),\ 1 = X_2 + 2k(X_1 + X_2),\ 0 = X_3.$

$\to X_1 = \frac{1}{1+4k},\ X_2 = \frac{1}{1+4k},\ X_3 = 0,\quad \to v_1 = v_2 = k(X_1 + X_2) = \frac{2k}{1+4k},\ v_3 = 0$

3.8 The position at time t of a particle initially at (X_1, X_2, X_3) is given by

$$x_1 = X_1 + \beta X_2^2 t^2,\ x_2 = X_2 + kX_2 t,\ x_3 = X_3, \text{ where } \beta = 1 \text{ and } k = 1.$$

(a) for the particle which was initially at (1,1,0), what are its positions in the following instant of time: $t = 0, t = 1, t = 2$. (b) Find the initial position for a particle which is at (1,3,2) at $t = 2$. (c) Find the acceleration at $t = 2$ of the particle which was initially at (1,3,2) and (d) find the acceleration of a particle which is at (1,3,2) at $t = 2$.

Ans. With $\beta = 1$ and $k = 1$, $x_1 = X_1 + X_2^2 t^2,\ x_2 = X_2 + X_2 t,\ x_3 = X_3$

(a) $t=0\to(x_1,x_2,x_3)=(X_1,X_2,X_3)=(1,1,0)$,

$t=1\to(x_1,x_2,x_3)=(X_1+X_2^2,X_2+X_2,X_3)=(2,2,0)$

$t=2\to(x_1,x_2,x_3)=(X_1+4X_2^2,X_2+2X_2,X_3)=(5,3,0)$

(b) $x_1=X_1+X_2^2t^2\;\; x_2=X_2+X_2t,\;\; x_3=X_3$, at $t=2\to 1=X_1+4X_2^2,\;\; 3=3X_2,\;\; 2=X_3$

$\to X_1=-3,\;\; X_2=1,\;\; X_3=2$.

(c) $x_1=X_1+X_2^2t^2\;\; x_2=X_2+X_2t,\;\; x_3=X_3\to v_1=2X_2^2t,\;\; v_2=X_2,\;\; v_3=0$.

$\to a_1=2X_2^2,\;\; a_2=0,\;\; a_3=0$. For $(X_1,X_2,X_3)=(1,3,2),\to a_1=2(3)^2=18,\;\; a_2=a_3=0$ at any time.

(d) The initial position of this particle was obtained in (b), i.e., $\to X_1=-3,\;\; X_2=1,\;\; X_3=2$.

Thus, $\to a_1=2X_2^2=2(1)^2=2,\;\; a_2=0,\;\; a_3=0$.

3.9 (a) Show that the velocity field $v_i=kx_i/(1+kt)$ corresponds to the motion $x_i=X_i(1+kt)$ and (b) find the acceleration of this motion in material description.

Ans. (a) From $x_i=X_i(1+kt)$ and $X_i=x_i/(1+kt)\to v_i=kX_i=kx_i/(1+kt)$**.**

(b) $v_i=kX_i\;\;\to a_i=0$, or

$$a_i=\left(\frac{\partial v_i}{\partial t}\right)_{x_i-fixed}+v_j\frac{\partial v_i}{\partial x_j}=-\frac{k^2x_i}{(1+kt)^2}+\frac{kx_j}{(1+kt)}\frac{k\delta_{ij}}{(1+kt)}=-\frac{k^2x_i}{(1+kt)}+\frac{kx_i}{(1+kt)}\frac{k}{(1+kt)}=0.$$

3.10 Given the two dimensional velocity field: $v_x=-2y,\;\; v_y=2x$. (a) Obtain the acceleration field and (b) obtain the pathline equation.

Ans. (a) $a_x=\dfrac{\partial v_x}{\partial t}+v_x\dfrac{\partial v_x}{\partial x}+v_y\dfrac{\partial v_x}{\partial y}=0+(-2y)(0)+(2x)(-2)=-4x$,

$a_y=\dfrac{\partial v_y}{\partial t}+v_x\dfrac{\partial v_y}{\partial x}+v_y\dfrac{\partial v_y}{\partial y}=0+(-2y)(2)+(2x)(0)=-4y$, i.e., $\mathbf{a}=-4x\mathbf{e}_x-4y\mathbf{e}_y$

(b) $\dfrac{dx}{dt}=-2y$ and $\dfrac{dy}{dt}=2x\to\dfrac{dy}{dx}=-\dfrac{x}{y}\to xdx+ydy=0$, $\to x^2+y^2=\text{constant}=X^2+Y^2$,

Or, $\dfrac{dx}{dt}=-2y$ and $\dfrac{dy}{dt}=2x\to\dfrac{d^2x}{dt}=-2\dfrac{dy}{dt}=-2(2x)\to\dfrac{d^2x}{dt}+4x=0$

$\to x=A\sin 2t+B\cos 2t$ and $y=-A\cos 2t+B\sin 2t$, where $A=-Y,B=X$.

3.11 Given the two dimensional velocity field: $v_x=kx,\;\; v_y=-ky$. (a) Obtain the acceleration field and (b) obtain the pathline equation.

Ans. (a) $a_x=\dfrac{\partial v_x}{\partial t}+v_x\dfrac{\partial v_x}{\partial x}+v_y\dfrac{\partial v_x}{\partial y}=0+(kx)(k)+(-ky)(0)=k^2x$

$$a_y = \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = 0 + (kx)(0) + (-ky)(-k) = k^2 y$$, That is, $\mathbf{a} = k^2\left(x\mathbf{e}_x + y\mathbf{e}_y\right)$

(b) $\frac{dx}{dt} = kx \rightarrow \int_X^x \frac{dx}{x} = \int_0^t kdt \rightarrow \ln x - \ln X = kt \rightarrow \ln\frac{x}{X} = kt \rightarrow x = Xe^{kt}$.

Similarly derivation gives $\rightarrow y = Ye^{-kt}$. Or, $xy = XY$ where (X,Y) are material coordinates.

3.12 Given the two dimensional velocity field: $v_x = k(x^2 - y^2),\ v_y = -2kxy$. Obtain the acceleration field.

Ans. $a_x = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} = 0 + k\left(x^2 - y^2\right)(2kx) + (-2kxy)(-2ky) = 2xk^2(x^2+y^2)$.

$$a_y = \frac{\partial v_y}{\partial t} + v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} = 0 + k\left(x^2 - y^2\right)(-2ky) - 2kxy(-2kx) = 2yk^2(x^2+y^2).$$

That is, $\mathbf{a} = 2k^2\left(x^2 + y^2\right)\left(x\mathbf{e}_x + y\mathbf{e}_y\right)$

3.13 In a spatial description, the equation to evaluate the acceleration $\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla\mathbf{v})\mathbf{v}$ is nonlinear. That is, if we consider two velocity fields $\mathbf{v}^{A}$and $\mathbf{v}^{B}$, then $\mathbf{a}^{A} + \mathbf{a}^{B} \neq \mathbf{a}^{A+B}$, where $\mathbf{a}^{A}$and $\mathbf{a}^{B}$ denote respectively the acceleration fields corresponding to the velocity fields $\mathbf{v}^{A}$and $\mathbf{v}^{B}$ each existing alone, $\mathbf{a}^{A+B}$ denotes the acceleration field corresponding to the combined velocity field $\mathbf{v}^{A} + \mathbf{v}^{B}$. Verify this inequality for the velocity fields:

$$\mathbf{v}^{A} = -2x_2\mathbf{e}_1 + 2x_1\mathbf{e}_2,\quad \mathbf{v}^{B} = 2x_2\mathbf{e}_1 - 2x_1\mathbf{e}_2$$

Ans. From $\frac{D\mathbf{v}}{Dt} = \frac{\partial \mathbf{v}}{\partial t} + (\nabla\mathbf{v})\mathbf{v}$

$$\left[\mathbf{a}^{A}\right] = \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}0 & -2\\2 & 0\end{bmatrix}\begin{bmatrix}-2x_2\\2x_1\end{bmatrix} = \begin{bmatrix}-4x_1\\-4x_2\end{bmatrix},\quad \left[\mathbf{a}^{B}\right] = \begin{bmatrix}0\\0\end{bmatrix} + \begin{bmatrix}0 & 2\\-2 & 0\end{bmatrix}\begin{bmatrix}2x_2\\-2x_1\end{bmatrix} = \begin{bmatrix}-4x_1\\-4x_2\end{bmatrix}$$

$\rightarrow \mathbf{a}^{A} = -4x_1\mathbf{e}_1 - 4x_2\mathbf{e}_2,\ \mathbf{a}^{B} = -4x_1\mathbf{e}_1 - 4x_2\mathbf{e}_2$

$\rightarrow \mathbf{a}^{A} + \mathbf{a}^{B} = -8x_1\mathbf{e}_1 - 8x_2\mathbf{e}_2$.

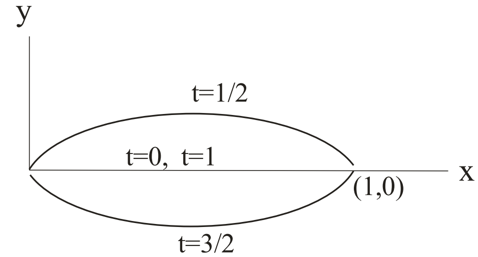
On the other hand, $\mathbf{v}^{A} + \mathbf{v}^{B} = \mathbf{0}$, so that $\mathbf{a}^{A+B} = 0$. Thus, $\mathbf{a}^{A} + \mathbf{a}^{B} \neq \mathbf{a}^{A+B}$

3.14 Consider the motion: $x_1 = X_1,\ x_2 = X_2 + (\sin\pi t)(\sin\pi X_1),\ x_3 = X_3$

(a) At $t = 0$, a material filament coincides with the straight line that extends from $(0,0,0)$ to $(1,0,0)$. Sketch the deformed shape of this filament at $t = 1/2,\ t = 1$ and $t = 3/2$.

(b) Find the velocity and acceleration in a material and a spatial description.

Ans. (a) Since $x_1 = X_1$ and $x_3 = X_3$, therefore there is no motion of the particles in the x_1 and x_3 directions. Every particle moves only up and down in the x_2 direction.
When $t = 1/2 \rightarrow x_2 = X_2 + \sin \pi X_1$, $t = 1 \rightarrow x_2 = X_2$, $t = 3/2 \rightarrow x_2 = X_2 - \sin \pi X_1$
The deformed shapes of the material at three different times are shown in the figure.



(b) $v_1 = 0$, $v_2 = \pi(\cos \pi t)(\sin \pi X_1)$, $v_3 = 0$, $a_1 = 0$, $a_2 = -\pi^2(\sin \pi t)(\sin \pi X_1)$, $a_3 = 0$
Since $x_1 = X_1$, the spatial descriptions are of the same form as above except that X_1 is replaced with x_1.

3.15 Consider the following velocity and temperature fields:

$$\mathbf{v} = \alpha(x_1\mathbf{e}_1 + x_2\mathbf{e}_2)/(x_1^2 + x_2^2), \quad \Theta = k(x_1^2 + x_2^2)$$

(a) Write the above fields in polar coordinates and discuss the general nature of the given velocity field and temperature field (e.g.,what do the flow and the isotherms look like?) (b) At the point $A(1,1,0)$, determine the acceleration and the material derivative of the temperature field.

Ans. (a) In polar coordinates, $x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = r\mathbf{e}_r$, where $r^2 = x_1^2 + x_2^2$ and $\mathbf{e}_r$ is the unit vector in the r direction, so that $\mathbf{v} = \frac{\alpha}{r}\mathbf{e}_r$, $\Theta = kr^2$. Thus, the given velocity field is that of a two dimensional source flow from the origin, the flow is purely radial with radial velocity inversely proportional to the radial distance from the origin. With $\Theta = kr^2$, the isotherms are circles.

(b) From $v_r = \frac{\alpha}{r}$ and $v_\theta = 0$, and Eq. (3.4.12)

$$a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} = 0 + \left(\frac{\alpha}{r}\right)\left(-\frac{\alpha}{r^2}\right) + 0 + 0 = -\frac{\alpha^2}{r^3}.$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = 0.$$

That is, $\mathbf{a} = -\alpha^2/r^3\mathbf{e}_r$. At the point $A(1,1,0)$, $r = \sqrt{2}$, $\mathbf{a} = -\alpha^2/(\sqrt{2})^3\mathbf{e}_r = -\alpha^2\sqrt{2}/4\mathbf{e}_r$.

$$\frac{D\Theta}{Dt} = \frac{\partial \Theta}{\partial t} + v_r\frac{\partial \Theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial \Theta}{\partial \theta} = 0 + \left(\frac{\alpha}{r}\right)(2kr) = 2\alpha k.$$

3.16 Do the previous problem for the following velocity and temperature fields:

$$\mathbf{v} = \frac{\alpha\left(-x_2\mathbf{e}_1 + x_1\mathbf{e}_2\right)}{x_1^2 + x_2^2}, \quad \Theta = k\left(x_1^2 + x_2^2\right)$$

Ans. With $x_1 = r\cos\theta,\ \ x_2 = r\sin\theta$ and $x_1^2 + x_2^2 = r^2$, we have

$$\mathbf{v} = \frac{\alpha\left(-x_2\mathbf{e}_1 + x_1\mathbf{e}_2\right)}{x_1^2 + x_2^2} = \frac{\alpha r\left(-\sin\theta\mathbf{e}_1 + \cos\theta\mathbf{e}_2\right)}{r^2} = \frac{\alpha}{r}\mathbf{e}_\theta \text{ and } \Theta = kr^2$$

Particles move in concentric circles with their speed inversely proportional to r. Isotherms are circles.

(b) With $v_r = 0,\ \ v_\theta = \dfrac{\alpha}{r}$, we have, from Eq.(3.4.12).

$$a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} = -\left(\frac{\alpha}{r}\right)^2\frac{1}{r} = -\frac{\alpha^2}{r^3}, \qquad a_\theta = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_r v_\theta}{r} = 0$$

i.e., $\mathbf{a} = -\alpha^2 / r^3\mathbf{e}_r$. At the point A, $r = \sqrt{2}$, therefore, $\mathbf{a} = -\sqrt{2}\alpha^2 / 4\mathbf{e}_r$

$$\frac{D\Theta}{Dt} = \frac{\partial\Theta}{\partial t} + \mathbf{v}\cdot\nabla\Theta = 0 + \left(\frac{\alpha}{r}\mathbf{e}_\theta\right)\cdot 2kr\mathbf{e}_r = 0.$$

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3.17 Consider : $\mathbf{x} = \mathbf{X} + X_1k\mathbf{e}_1$. let $d\mathbf{X}^{(1)} = \left(dS_1 / \sqrt{2}\right)\left(\mathbf{e}_1 + \mathbf{e}_2\right)$ & $d\mathbf{X}^{(2)} = \left(dS_2 / \sqrt{2}\right)\left(-\mathbf{e}_1 + \mathbf{e}_2\right)$ be differential material elements in the undeformed configuration. (a) Find the deformed elements $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$. (b) Evaluate the stretches of these elements ds_1 / dS_1 and ds_2 / dS_2 and the change in the angle between them. (c) Do part (b) for $k = 1$ and $k = 10^{-2}$ and (d) compare the results of part (c) to that predicted by the small strain tensor $\mathbf{E}$.

Ans. (a) $x_1 = X_1 + kX_1,\ \ x_2 = X_2,\ \ x_3 = X_3 \rightarrow [\mathbf{F}] = \begin{bmatrix} 1+k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \ d\mathbf{x} = \mathbf{F}d\mathbf{X} \rightarrow$

$$\left[d\mathbf{x}^{(1)}\right] = \left(\frac{dS_1}{\sqrt{2}}\right)\begin{bmatrix} 1+k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \rightarrow d\mathbf{x}^{(1)} = \left(\frac{dS_1}{\sqrt{2}}\right)\left[(1+k)\mathbf{e}_1 + \mathbf{e}_2\right].$$

$$\left[d\mathbf{x}^{(2)}\right] = \left(\frac{dS_2}{\sqrt{2}}\right)\begin{bmatrix} 1+k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \rightarrow d\mathbf{x}^{(2)} = \left(\frac{dS_2}{\sqrt{2}}\right)\left[-(1+k)\mathbf{e}_1 + \mathbf{e}_2\right].$$

(b) $\dfrac{ds_1}{dS_1} = \dfrac{ds_2}{dS_2} = \left(\dfrac{1}{\sqrt{2}}\right)\sqrt{(1+k)^2 + 1}$.

Let γ be the decrease in angle (from 90°), then $(\pi/2) - \gamma$ is the angle between the two deformed differential elements. Thus,

$$\cos\left(\frac{\pi}{2}-\gamma\right)=\frac{d\mathbf{x}^{(1)}\cdot d\mathbf{x}^{(2)}}{ds_1 ds_2}=\frac{1}{ds_1 ds_2}\left(\frac{dS_1}{\sqrt{2}}\right)\left(\frac{dS_2}{\sqrt{2}}\right)\left[-(1+k)^2+1\right]=\frac{-(1+k)^2+1}{(1+k)^2+1}\rightarrow$$

$$\sin\gamma=\frac{-(1+k)^2+1}{(1+k)^2+1}.$$

(c) For $k=1$, $\frac{ds_1}{dS_1}=\frac{ds_2}{dS_2}=\sqrt{\frac{5}{2}}$, $\sin\gamma=-\frac{3}{5}$.

For $k=10^{-2}$, $\frac{ds_1}{dS_1}=\frac{ds_2}{dS_2}=\left(\frac{1}{\sqrt{2}}\right)\sqrt{(1+k)^2+1}\approx\left(\frac{1}{\sqrt{2}}\right)\sqrt{2+2k}=\sqrt{1+k}=\sqrt{1.01}=1.005$.

$\sin\gamma=\frac{-(1+k)^2+1}{(1+k)^2+1}\approx\frac{-2k}{2+2k}=\frac{-k}{1+k}=\frac{-0.01}{1.01}\rightarrow\gamma=-0.0099$ *radian* ($-$ sign indicates increase in angle).

(d) $\mathbf{u}=\mathbf{x}-\mathbf{X}=kX_1\mathbf{e}_1\rightarrow u_1=kX_1,\ u_2=u_3=0,\ \rightarrow[\nabla\mathbf{u}]=\begin{bmatrix}k&0&0\\0&0&0\\0&0&0\end{bmatrix}\rightarrow[\mathbf{E}]=\begin{bmatrix}k&0&0\\0&0&0\\0&0&0\end{bmatrix}$,

$$\mathbf{e}_1'=\frac{1}{\sqrt{2}}(\mathbf{e}_1+\mathbf{e}_2)\rightarrow E_{11}'=\frac{1}{2}\begin{bmatrix}1&1&0\end{bmatrix}\begin{bmatrix}k&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&1&0\end{bmatrix}\begin{bmatrix}k\\0\\0\end{bmatrix}=\frac{k}{2},$$

$E_{11}'=\frac{ds-dS}{dS}=\frac{k}{2}\rightarrow\frac{ds}{dS}=1+\frac{k}{2}=1.005$, same as the result of part (c).

Also with

$$\mathbf{e}_2'=\frac{1}{\sqrt{2}}(-\mathbf{e}_1+\mathbf{e}_2)\rightarrow E_{12}'=\frac{1}{2}\begin{bmatrix}1&1&0\end{bmatrix}\begin{bmatrix}k&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}-1\\1\\0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1&1&0\end{bmatrix}\begin{bmatrix}-k\\0\\0\end{bmatrix}=-\frac{k}{2}\rightarrow 2E_{12}'=-k$$

Thus, the decrease in angle $=-k$, or the increase in angle is $0.01\approx0.0099$.

3.18 Consider the motion: $\mathbf{x}=\mathbf{X}+\mathbf{AX}$, where $\mathbf{A}$ is a small constant tensor (i.e., whose components are small in magnitude and independent of X_i). Show that the infinitesimal strain tensor is given by $\mathbf{E}=(\mathbf{A}+\mathbf{A}^{\mathrm{T}})/2$.

Ans. $\mathbf{u}=\mathbf{x}-\mathbf{X}=\mathbf{AX}\rightarrow\nabla\mathbf{u}=\nabla(\mathbf{AX})$. Since $\mathbf{A}$ is a constant, therefore,

$\nabla\mathbf{u}=\nabla(\mathbf{AX})=\mathbf{A}(\nabla\mathbf{X})$. Now, $[\nabla\mathbf{X}]=\left[\partial X_i/\partial X_j\right]=\left[\delta_{ij}\right]=[\mathbf{I}]\rightarrow\nabla\mathbf{u}=\mathbf{A}\rightarrow\mathbf{E}=(\mathbf{A}+\mathbf{A}^{\mathrm{T}})/2$

3.19 At time t, the position of a particle, initially at (X_1,X_2,X_3) is defined by:

$x_1=X_1+kX_3,\ x_2=X_2+kX_2,\ x_3=X_3,\ \ k=10^{-5}$. (a) Find the components of the strain tensor and (b) find the unit elongation of an element initially in the direction of $\mathbf{e}_1+\mathbf{e}_2$.

Ans. (a) $u_1 = x_1 - X_1 = kX_3,\ \ u_2 = x_2 - X_2 = kX_2,\ \ u_3 = x_3 - X_3 = 0$

$$\rightarrow [\nabla \mathbf{u}] = \begin{bmatrix} 0 & 0 & k \\ 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow [\mathbf{E}] = \frac{[\nabla \mathbf{u}] + [\nabla \mathbf{u}]^{\mathrm{T}}}{2} = \begin{bmatrix} 0 & 0 & k/2 \\ 0 & k & 0 \\ k/2 & 0 & 0 \end{bmatrix}$$

(b) Let $\mathbf{e}_1' = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2) \rightarrow E_{11}' = \mathbf{e}_1' \cdot \mathbf{E}\mathbf{e}_1' \rightarrow E_{11}' = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & k/2 \\ 0 & k & 0 \\ k/2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{k}{2} = \frac{10^{-5}}{2}$

3.20 Consider the displacements: $u_1 = k(2X_1^2 + X_1X_2),\ u_2 = kX_2^2,\ \ u_3 = 0,\ k = 10^{-4}$. (a) Find the unit elongations and the change of angles for two material elements $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$ that emanate from a particle designated by $\mathbf{X} = \mathbf{e}_1 + \mathbf{e}_2$. (b) Sketch deformed positions of these two elements.

Ans. (a) $[\nabla \mathbf{u}] = \begin{bmatrix} 4kX_1 + kX_2 & kX_1 & 0 \\ 0 & 2kX_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

At $(X_1, X_2, X_3) = (1,1,0)$, $[\nabla \mathbf{u}] = \begin{bmatrix} 5k & k & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow [\mathbf{E}] = \begin{bmatrix} 5k & k/2 & 0 \\ k/2 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Unit elong. for $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$ is $E_{11} = 5k = 5\times 10^{-4}$, unit elong. for $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$ is $E_{22} = 2k = 2\times 10^{-4}$.

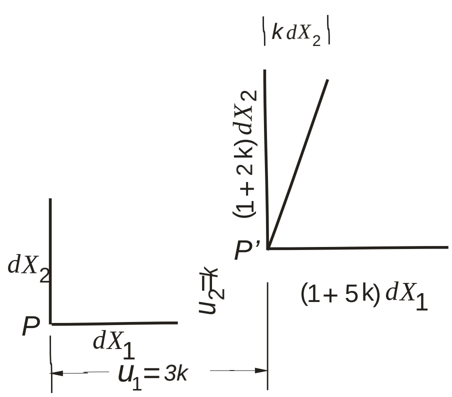
Decrease in angle between them is $2E_{12} = k = 10^{-4}\,radian$.

(b) For $d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1$, $d\mathbf{x}^{(1)} = d\mathbf{X}^{(1)} + (\nabla \mathbf{u})d\mathbf{X}^{(1)} = dX_1\mathbf{e}_1 + 5kdX_1\mathbf{e}_1 = (1+5k)dX_1\mathbf{e}_1$,

For $d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2$,

$d\mathbf{x}^{(2)} = d\mathbf{X}^{(2)} + (\nabla \mathbf{u})d\mathbf{X}^{(2)} = dX_2\mathbf{e}_2 + (kdX_2\mathbf{e}_1 + 2kdX_2\mathbf{e}_2) = kdX_2\mathbf{e}_1 + (1+2k)dX_2\mathbf{e}_2$

The deformed positions of these two elements are shown below:



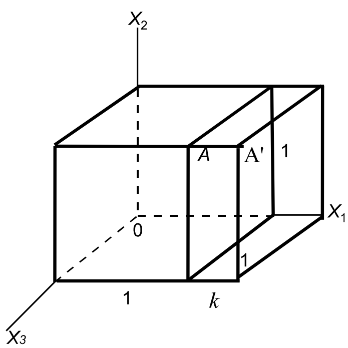
3.21 Given displacement field: $u_1 = kX_1,\ u_2 = u_3 = 0,\ k = 10^{-4}$. Determine the increase in length for the diagonal element OA of the unit cube (see figure below) in the direction of $\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3$ (a) by using the strain tensor and (b) by geometry.

Ans. (a) $[\mathbf{u}] = \begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\mathbf{E}]$. Let $\mathbf{e}_1' = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3)$, then the unit elongation in the $\mathbf{e}_1'$-direction is $E_{11}' = \mathbf{e}_1' \cdot \mathbf{E}\mathbf{e}_1' = \frac{1}{3}[1 \quad 1 \quad 1]\begin{bmatrix} k & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \frac{k}{3} = \frac{10^{-4}}{3}$.

(b) From the given displacement field, we see that the unit cube becomes longer in the x_1 direction by an amount of k, while the other two sides remain the same. The diagonal OA becomes OA', (see Figure), where $OA = \sqrt{3}$ and

$OA' = \sqrt{(1+k)^2 + 1 + 1} = \sqrt{3 + 2k + k^2} = \sqrt{3(1 + 2k/3 + k^2/3)}$

$\rightarrow OA' - OA = \sqrt{3}(1 + 2k/3 + k^2/3)^{1/2} - \sqrt{3}$.



Using binomial theorem, $\left(1 + 2k/3 + k^2/3\right)^{1/2} = 1 + (1/2)(2k/3) + ... \approx 1 + k/3$

Thus, $OA' - OA = \sqrt{3}(1 + k/3) - \sqrt{3} = \sqrt{3}k/3 \rightarrow (OA' - OA)/OA = k/3$, same as that obtained in part (a).

3.22 With reference to a rectangular Cartesian coordinate system, the state of strain at a point is given by the matrix $[\mathbf{E}] = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix} \times 10^{-4}$. (a) What is the unit elongation in the direction of $2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$? (b) What is the change in angle between two perpendicular lines (in the undeformed state) emanating from the point and in the directions of $2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$ and $3\mathbf{e}_1 - 6\mathbf{e}_3$?

Ans. Let $\mathbf{e}_1' = (2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3)/3$, the unit elongation in this direction is:

$$E'_{11} = \mathbf{e}'_1 \cdot \mathbf{E}\mathbf{e}'_1 = \frac{1}{9}\begin{bmatrix} 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}\times 10^{-4} = \frac{58}{9}\times 10^{-4}.$$

Let $\mathbf{e}'_2 = \frac{1}{\sqrt{45}}\left(3\mathbf{e}_1 - 6\mathbf{e}_3\right)$, then the decrease in angle between the two elements is:

$$2E'_{12} = 2\mathbf{e}'_1 \cdot \mathbf{E}\mathbf{e}'_2 = \frac{2}{3\sqrt{45}}\begin{bmatrix} 2 & 2 & 1 \end{bmatrix}\begin{bmatrix} 5 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}\times 10^{-4} = \frac{32}{\sqrt{45}}\times 10^{-4}\,rad.$$

3.23 For the strain tensor given in the previous problem, (a) find the unit elongation in the direction of $3\mathbf{e}_1 - 4\mathbf{e}_2$ and (b) find the change in angle between two elements in the dir. of $3\mathbf{e}_1 - 4\mathbf{e}_3$ and $4\mathbf{e}_1 + 3\mathbf{e}_3$.

Ans. (a) Let $\mathbf{e}'_1 = \frac{1}{5}\left(3\mathbf{e}_1 - 4\mathbf{e}_2\right)$**,** the unit elongation in this direction is:

$$E'_{11} = \mathbf{e}'_1 \cdot \mathbf{E}\mathbf{e}'_1 = \left(\frac{1}{5}\right)^2\begin{bmatrix} 3 & -4 & 0 \end{bmatrix}\begin{bmatrix} 5 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ -4 \\ 0 \end{bmatrix}\times 10^{-4} = \frac{37}{25}\times 10^{-4} = 1.48\times 10^{-4}$$

(b) Let $\mathbf{e}''_1 = \frac{1}{5}\left(3\mathbf{e}_1 - 4\mathbf{e}_3\right)$ and $\mathbf{e}''_2 = \frac{1}{5}\left(4\mathbf{e}_1 + 3\mathbf{e}_3\right)$, then the decrease in angle between these two elements is:

$$2E'''_{12} = 2\mathbf{e}''_1 \cdot \mathbf{E}\mathbf{e}''_2 = 2\left(\frac{1}{5}\right)^2\begin{bmatrix} 3 & 0 & -4 \end{bmatrix}\begin{bmatrix} 5 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}\times 10^{-4} = \frac{72}{25}\times 10^{-4} = 2.88\times 10^{-4}\,rad.$$

3.24 (a) Determine the principal scalar invariants for the strain tensor given below at the left and (b) show that the matrix given below at the right can not represent the same state of strain.

$$\left[\mathbf{E}\right] = \begin{bmatrix} 5 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 2 \end{bmatrix}\times 10^{-4}, \quad \begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix}\times 10^{-4}$$

Ans. (a) $I_1 = \left(5+4+2\right)\times 10^{-4} = 11\times 10^{-4}$,

$$I_2 = \begin{vmatrix} 5 & 3 \\ 3 & 4 \end{vmatrix}\times 10^{-8} + \begin{vmatrix} 4 & -1 \\ -1 & 2 \end{vmatrix}\times 10^{-8} + \begin{vmatrix} 5 & 0 \\ 0 & 2 \end{vmatrix}\times 10^{-8} = 28\times 10^{-8}$$

$$I_3 = \begin{vmatrix} 5 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 2 \end{vmatrix}\times 10^{-12} = 17\times 10^{-12}$$

(b) For $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 2 \end{bmatrix} \times 10^{-4}$, $I_3 = 36 \times 10^{-12}$**,** which is different from the I_3 in (a), therefore, the two matrices can not represent the same tensor.

3.25 Calculate the principal scalar invariants for the following two tensors. What can you say about the results?

$$\left[\mathbf{T}^{(1)}\right] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and} \left[\mathbf{T}^{(2)}\right] = \begin{bmatrix} 0 & -\tau & 0 \\ -\tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Ans. For $\left[\mathbf{T}^{(1)}\right] = \begin{bmatrix} 0 & \tau & 0 \\ \tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\{\mathbf{e}_i\}}$, $I_1 = 0,\ I_2 = -\tau^2,\ I_3 = 0$.

For $\left[\mathbf{T}^{(2)}\right] = \begin{bmatrix} 0 & -\tau & 0 \\ -\tau & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{\{\mathbf{e}_i\}}$ $I_1 = 0,\ I_2 = -\tau^2,\ I_3 = 0$

We see that these two tensors have the same principal scalar invariants. This result demonstrates that two different tensors can have the same three principal scalar invariants and therefore the same eigenvalues (in fact, $\lambda_1 = \tau, \lambda_2 = -\tau, \lambda_3 = 0$). However, corresponding to the same eigenvalue τ, the eigenvector for $\mathbf{T}^{(1)}$ is $(\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2}$, whereas the eigenvector for $\mathbf{T}^{(2)}$ is $(\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$. We see from this example that having the same principal scalar invariants is a necessary but not sufficient condition for the two tensors to be the same.

3.26 For the displacement field: $u_1 = kX_1^2,\ u_2 = kX_2X_3,\ u_3 = k\left(2X_1X_3 + X_1^2\right),\ k = 10^{-6}$, find the maximum unit elongation for an element that is initially at $(1,0,0)$.

Ans. $[\nabla\mathbf{u}] = \begin{bmatrix} 2kX_1 & 0 & 0 \\ 0 & kX_3 & kX_2 \\ k\left(2X_3 + 2X_1\right) & 0 & 2kX_1 \end{bmatrix}$, thus, for $(X_1, X_2, X_3) = (1,0,0)$,

$[\nabla\mathbf{u}] = \begin{bmatrix} 2k & 0 & 0 \\ 0 & 0 & 0 \\ 2k & 0 & 2k \end{bmatrix} \to [\mathbf{E}] = \begin{bmatrix} 2k & 0 & k \\ 0 & 0 & 0 \\ k & 0 & 2k \end{bmatrix}$, the characteristic equation for this tensor is:

$$\begin{vmatrix} 2k-\lambda & 0 & k \\ 0 & 0-\lambda & 0 \\ k & 0 & 2k-\lambda \end{vmatrix} = 0 \to (-\lambda)\left[(2k-\lambda)^2 - k^2\right] = 0 \to \lambda_1 = 0,\ \lambda_2 = 3k,\ \lambda_3 = k.$$

Thus, the maximum unit elongation at $(1,0,0)$ is $\lambda_2 = 3k = 3\times 10^{-6}$.

3.27 Given the matrix of an infinitesimal strain tensor as

$$[\mathbf{E}] = \begin{bmatrix} k_1 X_2 & 0 & 0 \\ 0 & -k_2 X_2 & 0 \\ 0 & 0 & -k_2 X_2 \end{bmatrix}.$$

(a) Find the location of the particle that does not undergo any volume change.
(b) What should the relation between k_1 and k_2 be so that no element changes its volume?

Ans. (a) $\dfrac{\Delta(dV)}{dV} = E_{11} + E_{22} + E_{33} = (k_1 - 2k_2)X_2 = 0$. Thus, the particles which were on the plane $X_2 = 0$ do not suffer any change of volume.
(b) If $(k_1 - 2k_2) = 0, ie., k_1 = 2k_2$, then no element changes its volume.

3.28 The displacement components for a body are:
$u_1 = k(X_1^2 + X_2),\ \ u_2 = k(4X_3^2 - X_1),\ \ u_3 = 0,\ \ \ k=10^{-4}$.
(a) Find the strain tensor. (b) Find the change of length per unit length for an element which was at $(1,2,1)$ and in the direction of $\mathbf{e}_1 + \mathbf{e}_2$. (c) What is the maximum unit elongation at the same point $(1,2,1)$? (d) What is the change of volume for the unit cube with a corner at the origin and with three of its edges along the positive coordinate axes?

Ans. (a) $[\nabla\mathbf{u}] = \begin{bmatrix} 2kX_1 & k & 0 \\ -k & 0 & 8kX_3 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow [\mathbf{E}] = \begin{bmatrix} 2kX_1 & 0 & 0 \\ 0 & 0 & 4kX_3 \\ 0 & 4kX_3 & 0 \end{bmatrix}$

(b) At $(1,2,1)$, $[\mathbf{E}] = \begin{bmatrix} 2k & 0 & 0 \\ 0 & 0 & 4k \\ 0 & 4k & 0 \end{bmatrix}$,

for $\mathbf{e}_1^{'} = \dfrac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$, $\ E_{11}^{'} = \mathbf{e}_1^{'} \cdot \mathbf{E}\mathbf{e}_1^{'} = \dfrac{1}{2}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2k & 0 & 0 \\ 0 & 0 & 4k \\ 0 & 4k & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = k$

(c) The characteristic equation is $\begin{vmatrix} 2k-\lambda & 0 & 0 \\ 0 & -\lambda & 4k \\ 0 & 4k & -\lambda \end{vmatrix} = 0 \rightarrow (2k-\lambda)\left[\lambda^2 - (4k)^2\right] = 0$

$\rightarrow \lambda_1 = 2k,\ \ \lambda_2 = 4k,\ \ \lambda_3 = -4k$. The maximum elongation is $4k$.
(d) Change of volume per unit volume $= E_{ii} = 2kX_1$, which is a function of X_1. Thus,

$$\Delta V = \int 2kX_1 dV = 2\int_o^1 kX_1(1)dX_1 = kX_1^2\Big|_o^1 = k.$$

3.29 For any motion, the mass of a particle (material volume) remains a constant (conservation of mass principle). Consider the mass to be the product of its volume and its mass density and show that (a) for infinitesimal deformation $\rho(1+E_{kk})=\rho_o$ where ρ_o denote the initial density and ρ, the current density. (b) Use the smallness of E_{kk} to show that the current density is given by $\rho=\rho_o(1-E_{kk})$.

Ans. (a) $\rho_o dV_o=\rho dV \rightarrow \rho_o=\frac{\rho dV}{dV_o}=\rho\frac{dV_o+\Delta dV}{dV_o}=\rho\left(1+\frac{\Delta dV}{dV_o}\right)$,

For small deformation, $\frac{\Delta dV}{dV_o}=E_{kk} \rightarrow \rho_o=\rho\left(1+E_{kk}\right)$.

(b) From bionomial theorem, for small E_{kk}, $\left(1+E_{kk}\right)^{-1}\approx 1-E_{kk}$, thus,

$\rho=\rho_o\left(1+E_{kk}\right)^{-1}=\rho_o\left(1-E_{kk}\right)$.

3.30 True or false: At any point in a body, there always exist two mutually perpendicular material elements which do not suffer any change of angle in an arbitrary small deformation of the body. Give reason(s).

Ans. True. The strain tensor **E** is a real symmetric tensor, for which there always exists three principal directions, with respect to which, the matrix of **E** is diagonal. That is, the non-diagonal elements, which give one-half of the change of angle between the elements which were along the principal directions, are zero.

3.31 Given the following strain components at a point in a continuum:

$E_{11}=E_{12}=E_{22}=k,\quad E_{33}=3k,\quad E_{13}=E_{23}=0,\quad k=10^{-6}$

Does there exist a material element at the point which decreases in length under the deformation? Explain your answer.

Ans.

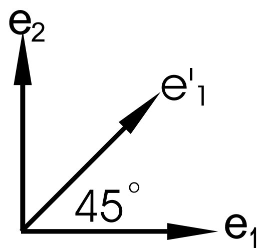
$$[\mathbf{E}]=\begin{bmatrix}k&k&0\\k&k&0\\0&0&3k\end{bmatrix}\rightarrow\begin{vmatrix}k-\lambda&k&0\\k&k-\lambda&0\\0&0&3k-\lambda\end{vmatrix}=0,\quad\rightarrow(3k-\lambda)\left[(k-\lambda)^2-k^2\right]=0$$

$$\rightarrow(3k-\lambda)\left(-2\lambda k+\lambda^2\right)=0\rightarrow\lambda_1=3k,\quad\lambda_2=0,\quad\lambda_3=2k.$$

Thus, the minimum unit elongation is 0. Therefore, there does not exist any element at the point which has a negative unit elongation (i.e., decreases in length).

3.32 The unit elongation at a certain point on the surface of a body are measured experimentally by means of strain gages that are arranged 45^o apart (called the 45^o strain rosette) in the direction of $\mathbf{e}_1,\frac{1}{\sqrt{2}}(\mathbf{e}_1+\mathbf{e}_2)$ and $\mathbf{e}_2$. If these unit elongation are designated by a,b,c respectively, what are the strain components E_{11},E_{22} and E_{12}?

Ans.



With $\mathbf{e}_1' = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$, we have,

$$E_{11}' = \mathbf{e}_1' \cdot \mathbf{E}\mathbf{e}_1' = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} E_{11} & E_{12} & E_{13} \\ E_{21} & E_{22} & E_{23} \\ E_{31} & E_{32} & E_{33} \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}(E_{11} + E_{12} + E_{21} + E_{22}), \quad \text{with } E_{12} = E_{21},$$

$E_{11}' = \frac{1}{2}(E_{11} + 2E_{12} + E_{22}) \to E_{12} = E_{11}' - \frac{(E_{11} + E_{22})}{2}$. Thus, the strain components are:

$E_{11} = a, \ E_{22} = c, \ E_{12} = b - \frac{(a+c)}{2}$.

3.33 (a) Do the previous problem, if the measured strains are 200×10^{-6}, 50×10^{-6} and 100×10^{-6} in the direction $\mathbf{e}_1, \mathbf{e}_1'$ and $\mathbf{e}_2$ respectively. (b) Find the principal directions, assuming $E_{31} = E_{32} = E_{33} = 0$. (c) How will the result of part b be altered if $E_{33} \neq 0$.

Ans. (a) With $E_{11} = 200 \times 10^{-6}$, $E_{11}' = 50 \times 10^{-6}$ and $E_{22} = 100 \times 10^{-6}$, we have, from the results of the previous problem, $E_{12} = E_{11}' - \frac{E_{11} + E_{22}}{2} = \left(50 - \frac{200 + 100}{2}\right) \times 10^{-6} = -100 \times 10^{-6}$

(b) $\begin{vmatrix} E_{11} - \lambda & E_{12} & 0 \\ E_{12} & E_{22} - \lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \to \lambda\left[(E_{11} - \lambda)(E_{22} - \lambda) - E_{12}^2\right] = 0$

$\to \lambda\left[+\lambda^2 - \lambda(E_{11} + E_{22}) + \left(E_{11}E_{22} - E_{12}^2\right)\right] = 0$,

$\lambda_{1,2} = \frac{(E_{11} + E_{22}) \pm \sqrt{(E_{11} - E_{22})^2 + 4E_{12}^2}}{2}$, $\lambda_3 = 0$,

thus,

$$\lambda_{1,2} = \left[\frac{(200 + 100) \pm \sqrt{(200 - 100)^2 + 4(-100)^2}}{2}\right] \times 10^{-6} = \begin{matrix} 261.8 \times 10^{-6} \\ 38.2 \times 10^{-6} \end{matrix}, \ \lambda_3 = 0$$

The principal direction for λ_3 is $\mathbf{e}_3$. The principal directions corresponding to the other two eigenvalues lie on the plane of $\mathbf{e}_1$ and $\mathbf{e}_2$. Let

$\mathbf{n} = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 \equiv \cos\theta\mathbf{e}_1 + \sin\theta\mathbf{e}_2$, then $(\mathrm{E}_{11} - \lambda)\alpha_1 + E_{12}\alpha_2 = 0$,

$$\rightarrow \frac{\alpha_2}{\alpha_1} = \tan\theta = \frac{(\lambda - E_{11})}{E_{12}},$$

For $\lambda_1 = 261.8\times10^{-6}$, $\tan\theta = \frac{\lambda_1 - E_{11}}{E_{12}} = \frac{261.8-200}{-100} = \frac{61.8}{-100} = -0.618 \rightarrow \theta = -31.7^{o}$,

Or, $\mathbf{n} = 0.851\mathbf{e}_1 - 0.525\mathbf{e}_2$

For $\lambda_2 = 38.2\times10^{-6}$, $\tan\theta = \frac{\lambda_2 - E_{11}}{E_{12}} = \frac{38.2-200}{-100} = 1.618 \rightarrow \theta = 58.3^{o}$

Or, $\mathbf{n} = 0.525\mathbf{e}_1 + 0.851\mathbf{e}_2$.

(c) If $E_{33} \neq 0$, then the principal strain corresponding to the direction $\mathbf{e}_3$ is E_{33} instead of zero. Nothing else changes.

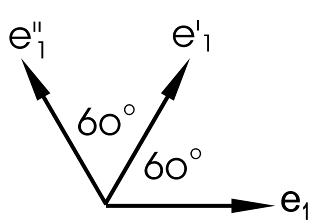
3.34 Repeat the previous problem with $E_{11} = E'_{11} = E_{22} = 1000\times10^{-6}$.

Ans. (a) From the results of Problem 3.32, $E_{12} = E'_{11} - \frac{E_{11}+E_{22}}{2} = \left(1000 - \frac{2000}{2}\right)\times10^{-6} = 0$,

(b) and (c) $[\mathbf{E}] = \begin{bmatrix} 10^{-3} & 0 & 0 \\ 0 & 10^{-3} & 0 \\ 0 & 0 & E_{33} \end{bmatrix}$, the principal strains are 10^{-3} in any directions lying on the plane of $\mathbf{e}_1$ and $\mathbf{e}_2$ and the principal strain E_{33} is in $\mathbf{e}_3$ direction.

3.35 The unit elongation at a certain point on the surface of a body are measured experimentally by means of strain gages that are arranged 60^{o} apart (called the 60^{o} strain rosette) in the direction of $\mathbf{e}_1, \frac{1}{2}\left(\mathbf{e}_1 + \sqrt{3}\mathbf{e}_2\right)$ and $\frac{1}{2}\left(-\mathbf{e}_1 + \sqrt{3}\mathbf{e}_2\right)$. If these unit elongation are designated by a,b,c respectively, what are the strain components E_{11}, E_{22} and E_{12}?

Ans.



With $\mathbf{e}'_1 = (\mathbf{e}_1 + \sqrt{3}\mathbf{e}_2)/2$, $\mathbf{e}''_1 = (-\mathbf{e}_1 + \sqrt{3}\mathbf{e}_2)/2$, we have,

$$E_{11}^{'}=\mathbf{e}_1^{'}\cdot\mathbf{E}\mathbf{e}_1^{'}=\frac{1}{4}\begin{bmatrix}1 & \sqrt{3} & 0\end{bmatrix}\begin{bmatrix}E_{11} & E_{12} & E_{13}\\ E_{21} & E_{22} & E_{23}\\ E_{31} & E_{32} & E_{33}\end{bmatrix}\begin{bmatrix}1\\ \sqrt{3}\\ 0\end{bmatrix}=\frac{1}{4}\left(E_{11}+2\sqrt{3}E_{12}+3E_{22}\right)\quad \text{(i)}$$

$$E_{11}^{''}=\mathbf{e}_1^{''}\cdot\mathbf{E}\mathbf{e}_1^{''}=\frac{1}{4}\begin{bmatrix}-1 & \sqrt{3} & 0\end{bmatrix}\begin{bmatrix}E_{11} & E_{12} & E_{13}\\ E_{21} & E_{22} & E_{23}\\ E_{31} & E_{32} & E_{33}\end{bmatrix}\begin{bmatrix}-1\\ \sqrt{3}\\ 0\end{bmatrix}=\frac{1}{4}\left(E_{11}-2\sqrt{3}E_{12}+3E_{22}\right)\quad \text{(ii)}$$

(i) & (ii), $\rightarrow E_{22}=\frac{1}{3}\left[2E_{11}^{'}+2E_{11}^{''}-E_{11}\right]=\frac{1}{3}\left[2b+2c-a\right]$, $E_{12}=\frac{E_{11}^{'}-E_{11}^{''}}{\sqrt{3}}=\frac{b-c}{\sqrt{3}}$, $E_{11}=a$.

3.36 If the 60^{o} strain rosette measurements give $a=2\times10^{-6}$, $\mathrm{b}=1\times10^{-6}$, $\mathrm{c}=1.5\times10^{-6}$, obtain E_{11},E_{12} and E_{22}. Use the formulas obtained in the previous problem.

Ans. Using the formulas drived in the previous problem, we have,

$$E_{22}=\frac{1}{3}\left[2b+2c-a\right]=\frac{1}{3}\left[(2)(1)+(2)(1.5)-2\right]\times10^{-6}=1\times10^{-6},$$

$$E_{12}=\frac{b-c}{\sqrt{3}}=-\frac{1}{2\sqrt{3}}\times10^{-6},\quad E_{11}=2\times10^{-6}.$$

3.37 Repeat the previous problem for the case $a=\mathrm{b}=\mathrm{c}=2000\times10^{-6}$.

Ans. $E_{22}=\frac{1}{3}\left[2b+2c-a\right]=\frac{1}{3}\left[(2)(2000)+(2)(2000)-2000\right]\times10^{-6}=2\times10^{-3}$,

$$E_{12}=\frac{b-c}{\sqrt{3}}=0,\quad E_{11}=2\times10^{-3}$$

3.38 For the velocity field: $\mathbf{v}=kx_2^2\mathbf{e}_1$, (a) find the rate of deformation and spin tensors. (b) Find the rate of extension of a material element $d\mathbf{x}=ds\mathbf{n}$ where $\mathbf{n}=\left(\mathbf{e}_1+\mathbf{e}_2\right)/\sqrt{2}$ at $\mathbf{x}=5\mathbf{e}_1+3\mathbf{e}_2$.

Ans. $v_1=kx_2^2,\ v_2=v_3=0$,

$$\rightarrow\left[\nabla\mathbf{v}\right]=\begin{bmatrix}0 & 2kx_2 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\rightarrow\left[\mathbf{D}\right]=\left[\nabla\mathbf{v}\right]^{\mathrm{S}}=\begin{bmatrix}0 & kx_2 & 0\\ kx_2 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad\left[\mathbf{W}\right]=\left[\nabla\mathbf{v}\right]^{\mathrm{A}}=\begin{bmatrix}0 & kx_2 & 0\\ -kx_2 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

(b) At the position $\mathbf{x}=5\mathbf{e}_1+3\mathbf{e}_2$,

$$\left[\mathbf{D}\right]=\begin{bmatrix}0 & 3k & 0\\ 3k & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad\left[\mathbf{W}\right]=\begin{bmatrix}0 & 3k & 0\\ -3k & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

For the element $d\mathbf{x} = ds\mathbf{n}$ with $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2}$, the rate of extension is:

$$D_{(n)(n)} = \mathbf{n}\cdot\mathbf{D}\mathbf{n} = \frac{1}{2}\begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}0 & 3k & 0\\ 3k & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = 3k\,.$$

3.39 For the velocity field: $\mathbf{v} = \alpha\left(\dfrac{t+k}{1+x_1}\right)\mathbf{e}_1$, find the rates of extension for the following material elements: $d\mathbf{x}^{(1)} = ds_1\mathbf{e}_1$ and $d\mathbf{x}^{(2)} = \left(ds_2/\sqrt{2}\right)\left(\mathbf{e}_1 + \mathbf{e}_2\right)$ at the origin at time $t = 1$.

Ans. $v_1 = \alpha\left(\dfrac{t+k}{1+x_1}\right),\ v_2 = v_3 = 0 \rightarrow [\nabla\mathbf{v}] = \begin{bmatrix}-\alpha(t+k)/(1+x_1)^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} = [\mathbf{D}]$.

At $t = 1$ and at $(x_1, x_2, x_3) = (0,0,0)$, $[\mathbf{D}] = \begin{bmatrix}-\alpha(1+k) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$.

Rate of extension for $d\mathbf{x}^{(1)} = ds_1\mathbf{e}_1$ is $D_{11} = -\alpha(1+k)$**;** for $d\mathbf{x}^{(2)} = \left(ds_2/\sqrt{2}\right)\left(\mathbf{e}_1 + \mathbf{e}_2\right)$, it is:

$$D_{11}^{'} = \frac{1}{2}\begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}-\alpha(1+k) & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = -\frac{1}{2}\alpha(1+k)$$

3.40 For the velocity field $\mathbf{v} = (\cos t)(\sin \pi x_1)\mathbf{e}_2$ (a) find the rate of deformation and spin tensors, and (b) find the rate of extension at $t = 0$ for the following elements at the origin: $d\mathbf{x}^{(1)} = ds_1\mathbf{e}_1$, $d\mathbf{x}^{(2)} = ds_2\mathbf{e}_2$ and $d\mathbf{x}^{(3)} = \left(ds_3/\sqrt{2}\right)\left(\mathbf{e}_1 + \mathbf{e}_2\right)$.

Ans. (a) With $v_1 = 0,\ v_2 = (\cos t)(\sin \pi x_1),\ v_3 = 0$,

$$[\nabla\mathbf{v}] = \begin{bmatrix}0 & 0 & 0\\ \pi\cos t\cos\pi x_1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} \rightarrow [\mathbf{D}] = \begin{bmatrix}0 & (\pi\cos t\cos\pi x_1)/2 & 0\\ (\pi\cos t\cos\pi x_1)/2 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},$$

$$[\mathbf{W}] = \begin{bmatrix}0 & -(\pi\cos t\cos\pi x_1)/2 & 0\\ (\pi\cos t\cos\pi x_1)/2 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

(b) At $t = 0$ and $(x_1, x_2, x_3) = (0,0,0)$, $[\mathbf{D}] = \begin{bmatrix}0 & \pi/2 & 0\\ \pi/2 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$.

For $d\mathbf{x}^{(1)} = ds_1\mathbf{e}_1$, rate of extension is D_{11}=0, for $d\mathbf{x}^{(2)} = ds_2\mathbf{e}_2$, $D_{22} = 0$ and

for $d\mathbf{x}^{(3)} = \left(ds_3 / \sqrt{2}\right)\left(\mathbf{e}_1 + \mathbf{e}_2\right)$, $D_{11}^{'} = \frac{1}{2}[1 \quad 1 \quad 0]\begin{bmatrix} 0 & \pi/2 & 0 \\ \pi/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{\pi}{2}$

3.41 Show that the following velocity components correspond to a rigid body motion.

$$v_1 = x_2 - x_3, \quad v_2 = -x_1 + x_3, \quad v_3 = x_1 - x_2$$

Ans. $[\nabla\mathbf{v}] = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \rightarrow [\mathbf{D}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Therefore, the velocity field is a rigid body motion..

3.42 Given the velocity field $\mathbf{v} = \frac{1}{r}\mathbf{e}_r$, (a) find the rate of deformation tensor and the spin tensor and (b) find the rate of extension of a radial material line element.

Ans. With $v_r = \frac{1}{r}, \ v_\theta = v_z = 0$, we have, using Eq. (2.34.5)

$$[\nabla\mathbf{v}] = \begin{bmatrix} \frac{\partial v_r}{\partial r} & \frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) & \frac{\partial v_r}{\partial z} \\ \frac{\partial v_\theta}{\partial r} & \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) & \frac{\partial v_\theta}{\partial z} \\ \frac{\partial v_z}{\partial r} & \frac{1}{r}\frac{\partial v_z}{\partial \theta} & \frac{\partial v_z}{\partial z} \end{bmatrix} = \begin{bmatrix} -\frac{1}{r^2} & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\mathbf{D}], \quad [\mathbf{W}] = [\mathbf{0}].$$

(b) The rate of extension for a radial element is $D_{rr} = -\frac{1}{r^2}$.

3.43 Given the two-dimensional velocity field in polar coordinates:

$$v_r = 0, \quad v_\theta = 2r + \frac{4}{r}$$

(a) Find the acceleration at $r = 2$ and (b) find the rate of deformation tensor at $r = 2$.

Ans. (a) Using Eq. (3.4.12), $a_r = \frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) = -\frac{(v_\theta)^2}{r} = -\frac{1}{r}\left(2r + \frac{4}{r}\right)^2$,

$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) = 0$. At $r = 2$, $a_r = -(6)^2/2 = -18$, $a_\theta = 0$.

(b) Eq. (2.34.5) $\rightarrow [\nabla \mathbf{v}] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - \mathrm{v}_\theta\right) \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{\mathrm{v}_\theta}{r} \\ \dfrac{\partial v_\theta}{\partial r} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\left(2 + \dfrac{4}{r^2}\right) \\ \left(2 - \dfrac{4}{r^2}\right) & 0 \end{bmatrix}$.

$[\mathbf{D}] = [\nabla \mathbf{v}]^S = \begin{bmatrix} 0 & -4/r^2 \\ -4/r^2 & 0 \end{bmatrix}$, at $r = 2$, $[\mathbf{D}] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$.

3.44 Given the velocity field in spherical coordinates:

$$v_r = 0,\ \ v_\theta = 0,\ \ v_\phi = \left(Ar + \frac{B}{r^2}\right)\sin\theta$$

(a) Determine the acceration field and (b) find the rate of deformation tensor.

Ans. (a) From Eq. (3.4.16),

$$a_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial v_r}{\partial \theta} - v_\theta\right) + \frac{v_\phi}{r\sin\theta}\left(\frac{\partial v_r}{\partial \phi} - v_\phi \sin\theta\right) = -\frac{v_\phi^2}{r} = -\frac{1}{r}\left(Ar + \frac{B}{r^2}\right)^2 \sin^2\theta$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right) + \frac{v_\phi}{r\sin\theta}\left(\frac{\partial v_\theta}{\partial \phi} - v_\phi \cos\theta\right) = -\frac{v_\phi^2}{r}\cot\theta = -\frac{\cos\theta\sin\theta}{r}\left(Ar + \frac{B}{r^2}\right)^2$$

$$a_\phi = \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\left(\frac{\partial v_\phi}{\partial \phi} + v_r \sin\theta + v_\theta \cos\theta\right) = 0$$

(b) Eq. (2.35.25) $\rightarrow$

$$[\nabla \mathbf{v}] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \left(\dfrac{1}{r}\dfrac{\partial v_r}{\partial \theta} - \dfrac{v_\theta}{r}\right) & \left(\dfrac{1}{r\sin\theta}\dfrac{\partial v_r}{\partial \phi} - \dfrac{v_\phi}{r}\right) \\ \dfrac{\partial v_\theta}{\partial r} & \left(\dfrac{1}{r}\dfrac{\partial v_\theta}{\partial \theta} + \dfrac{v_r}{r}\right) & \left(\dfrac{1}{r\sin\theta}\dfrac{\partial v_\theta}{\partial \phi} - \dfrac{v_\phi \cot\theta}{r}\right) \\ \dfrac{\partial v_\phi}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_\phi}{\partial \theta} & \left(\dfrac{1}{r\sin\theta}\dfrac{\partial v_\phi}{\partial \phi} + \dfrac{v_r}{r} + \dfrac{v_\theta \cot\theta}{r}\right) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\dfrac{v_\phi}{r} \\ 0 & 0 & -\dfrac{v_\phi \cot\theta}{r} \\ \dfrac{\partial v_\phi}{\partial r}, & \dfrac{1}{r}\dfrac{\partial v_\phi}{\partial \theta} & 0 \end{bmatrix}, \text{ thus}$$

the nonzero components of rate of deformation tensor are:

$$D_{r\phi} = \frac{1}{2}\left(-\frac{v_\phi}{r} + \frac{\partial v_\phi}{\partial r}\right) = -\frac{3B}{2r^3}\sin\theta\,,$$

$$D_{\theta\phi} = \frac{1}{2}\left(-\frac{v_\phi \cot\theta}{r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \theta}\right) = \frac{1}{2}\left[-\left(A + \frac{B}{r^3}\right) + \left(A + \frac{B}{r^3}\right)\right]\cos\theta = 0\,.$$

3.45 A motion is said to be irrotational if the spin tensor vanishes. Show that the following velocity field is irrotational:

$$\mathbf{v} = \frac{-x_2\mathbf{e}_2 + x_1\mathbf{e}_2}{r^2},\ \ r^2 = x_1^2 + x_2^2$$

Ans. $v_1 = -\frac{x_2}{r^2},\ v_2 = \frac{x_1}{r^2},\ r^2 = x_1^2 + x_2^2,\ \to [\nabla\mathbf{v}] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{2x_2}{r^3}\frac{\partial r}{\partial x_1} & -\frac{1}{r^2} + \frac{2x_2}{r^3}\frac{\partial r}{\partial x_2} \\ \frac{1}{r^2} - \frac{2x_1}{r^3}\frac{\partial r}{\partial x_1} & -\frac{2x_1}{r^3}\frac{\partial r}{\partial x_2} \end{bmatrix},$

$r^2 = x_1^2 + x_2^2 \to 2r\frac{\partial r}{\partial x_1} = 2x_1 \to \frac{\partial r}{\partial x_1} = \frac{x_1}{r}$, also, $\frac{\partial r}{\partial x_2} = \frac{x_2}{r}$,

$$[\nabla\mathbf{v}] = \begin{bmatrix} \frac{2x_1x_2}{r^4} & \frac{x_2^2 - x_1^2}{r^4} \\ \frac{x_2^2 - x_1^2}{r^4} & -\frac{2x_1x_2}{r^4} \end{bmatrix} = [\nabla\mathbf{v}]^S \to [\mathbf{W}] = 0.$$

3.46 Let $d\mathbf{x}^{(1)} = ds_1\mathbf{n}$ and $d\mathbf{x}^{(2)} = ds_2\mathbf{m}$ be two material elements that emanate from a particle P which at present has a rate of deformation $\mathbf{D}$. (a) Consdier $(D/Dt)(d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)})$ to show that

$$\left[\frac{1}{ds_1}\frac{D(ds_1)}{Dt} + \frac{1}{ds_2}\frac{D(ds_2)}{Dt}\right]\cos\theta - \sin\theta\frac{D\theta}{Dt} = 2\mathbf{m}\cdot\mathbf{Dn}$$

where θ is the angle between $\mathbf{m}$ and $\mathbf{n}$.

(b) Consider the case of $d\mathbf{x}^{(1)} = d\mathbf{x}^{(2)}$, what does the above formula reduce to?

(c) Consider the case where $\theta = \frac{\pi}{2}$, i.e., $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ are perpendicular to each other, where does the above formula reduces to?

Ans.

$$\frac{D}{Dt}\left(d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)}\right) = \left(\frac{D}{Dt}d\mathbf{x}^{(1)}\right)\cdot d\mathbf{x}^{(2)} + \left(d\mathbf{x}^{(1)} \cdot \frac{Dd\mathbf{x}^{(2)}}{Dt}\right) = (\nabla\mathbf{v})d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} + d\mathbf{x}^{(1)} \cdot (\nabla\mathbf{v})d\mathbf{x}^{(2)}$$

$$= d\mathbf{x}^{(1)} \cdot (\nabla\mathbf{v})^{\mathrm{T}} d\mathbf{x}^{(2)} + d\mathbf{x}^{(1)} \cdot (\nabla\mathbf{v})d\mathbf{x}^{(2)} = d\mathbf{x}^{(1)} \cdot \left\{(\nabla\mathbf{v})^{\mathrm{T}} + (\nabla\mathbf{v})\right\}d\mathbf{x}^{(2)} = 2d\mathbf{x}^{(1)} \cdot \mathbf{D}d\mathbf{x}^{(2)}.$$

With $d\mathbf{x}^{(1)} = ds_1\mathbf{n}$ and $d\mathbf{x}^{(2)} = ds_2\mathbf{m}$**,** the above formula give,

$\frac{D}{Dt}(ds_1ds_2\mathbf{n}\cdot\mathbf{m}) = 2ds_1ds_2(\mathbf{n}\cdot\mathbf{Dm}) \to \frac{D}{Dt}(ds_1ds_2\cos\theta) = 2ds_1ds_2(\mathbf{n}\cdot\mathbf{Dm})$. Thus,

$$\frac{Dds_1}{Dt}(ds_2\cos\theta) + \frac{Dds_2}{Dt}(ds_1\cos\theta) + \frac{D\cos\theta}{Dt}(ds_1ds_2) = 2ds_1ds_2(\mathbf{n}\cdot\mathbf{Dm}),$$

$$\to \left\{\frac{1}{ds_1}\frac{D(ds_1)}{Dt} + \frac{1}{ds_2}\frac{D(ds_2)}{Dt}\right\}\cos\theta - \sin\theta\frac{D\theta}{Dt} = 2(\mathbf{n}\cdot\mathbf{Dm}) = 2(\mathbf{m}\cdot\mathbf{Dn}).$$

(b) For**,** $d\mathbf{x}^{(1)} = d\mathbf{x}^{(2)} = ds\mathbf{n}$ the above formula $\to \left\{\frac{1}{ds}\frac{D(ds)}{Dt}\right\} = (\mathbf{n}\cdot\mathbf{Dn}) = D_{(n)(n)}$, no sum on n.

(c) For $d\mathbf{x}^{(1)}$ perpendicular to $d\mathbf{x}^{(2)}$**,** $\theta = 90^{\circ}$**,** we have,

$$-\frac{D\theta}{Dt} = 2(\mathbf{n}\cdot\mathbf{Dm}) = 2D_{nm}.$$

3.47 Let $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ and D_1, D_2, D_3 be the principal directions and corresponding principal values of a rate of deformation tensor $\mathbf{D}$. Further, let $d\mathbf{x}^{(1)} = ds_1\mathbf{e}_1$, $d\mathbf{x}^{(2)} = ds_2\mathbf{e}_2$ and $d\mathbf{x}^{(3)} = ds_3\mathbf{e}_3$ be three material elements. Consider the material derivative $(D/Dt)\left\{d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} \times d\mathbf{x}^{(3)}\right\}$ and show that $\frac{1}{dV}\frac{D(dV)}{Dt} = D_1 + D_2 + D_3$, where $dV = ds_1 ds_2 ds_3$.

Ans. Since the principal directions are (or can always be chosen to be) mutually perpendicular, therefore, $d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} \times d\mathbf{x}^{(3)} = ds_1 ds_2 ds_3 = dV$ **.**

$$\rightarrow \frac{D(dV)}{Dt} = \frac{D(ds_1 ds_2 ds_3)}{Dt} = ds_2 ds_3 \frac{D(ds_1)}{Dt} + ds_1 ds_3 \frac{D(ds_2)}{Dt} + ds_1 ds_2 \frac{D(ds_3)}{Dt},$$

$$\rightarrow \frac{1}{dV}\frac{D(dV)}{Dt} = \frac{1}{ds_1}\frac{D(ds_1)}{Dt} + \frac{1}{ds_2}\frac{D(ds_2)}{Dt} + \frac{1}{ds_3}\frac{D(ds_3)}{Dt} = D_{11} + D_{22} + D_{33}.$$

3.48 Consider a material element $d\mathbf{x} = ds\mathbf{n}$ (a) Show that $(D/Dt)\mathbf{n} = \mathbf{Dn} + \mathbf{Wn} - (\mathbf{n} \cdot \mathbf{Dn})\mathbf{n}$, where $\mathbf{D}$ is rate of deformation tensor and $\mathbf{W}$ is the spin tensor. (b) Show that if $\mathbf{n}$ is an eigenvector of $\mathbf{D}$, then,

$$\frac{D\mathbf{n}}{Dt} = \mathbf{Wn} = \boldsymbol{\omega} \times \mathbf{n}$$

Ans. (a) $\frac{D}{Dt}(ds\mathbf{n}) = ds\frac{D\mathbf{n}}{Dt} + \mathbf{n}\frac{Dds}{Dt} = ds\left(\frac{D\mathbf{n}}{Dt} + \mathbf{n}\frac{1}{ds}\frac{Dds}{Dt}\right) = ds\left(\frac{D\mathbf{n}}{Dt} + (\mathbf{n} \cdot \mathbf{Dn})\mathbf{n}\right)$. [see Eq.(3.13.12)]. We also have, $\frac{D}{Dt}(ds\mathbf{n}) = \frac{D}{Dt}(d\mathbf{x}) = (\nabla\mathbf{v})d\mathbf{x} = ds(\nabla\mathbf{v})\mathbf{n}$, therefore,

$$(\nabla\mathbf{v})\mathbf{n} = \left(\frac{D\mathbf{n}}{Dt} + \mathbf{n}(\mathbf{n} \cdot \mathbf{Dn})\right) \rightarrow \frac{D\mathbf{n}}{Dt} = (\nabla\mathbf{v})\mathbf{n} - \mathbf{n}(\mathbf{n} \cdot \mathbf{Dn}) = (\mathbf{D} + \mathbf{W})\mathbf{n} - \mathbf{n}(\mathbf{n} \cdot \mathbf{Dn}).$$

(b) If $\mathbf{n}$ is an eigenvector of $\mathbf{D}$, then $\mathbf{Dn} = \lambda\mathbf{n}$, therefore,

$\frac{D\mathbf{n}}{Dt} = (\mathbf{D} + \mathbf{W})\mathbf{n} - \mathbf{n}(\mathbf{n} \cdot \mathbf{Dn}) = \lambda\mathbf{n} + \mathbf{Wn} - \mathbf{n}\lambda = \mathbf{Wn}$. That is, $\frac{D\mathbf{n}}{Dt} = \mathbf{Wn}$.

Since $\mathbf{W}$ is antisymmetric $\rightarrow$ $\mathbf{Wn} = \boldsymbol{\omega} \times \mathbf{n}$, where $\boldsymbol{\omega}$ is the dual vector for $\mathbf{W}$. Thus

$\frac{D\mathbf{n}}{Dt} = \mathbf{Wn} = \boldsymbol{\omega} \times \mathbf{n}$.

That is, the principal axes of $\mathbf{D}$ rotates with an angular velocity given by the dual vector of the spin tensor.

3.49 Given the following velocity field: $v_1 = k(x_2 - 2)^2 x_3$, $v_2 = -x_1 x_2$, $v_3 = kx_1 x_3$ for an incompressible fluid, determine the value of k, such that the equation of mass conservation is satisfied.

Ans. $\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}=0 \to 0-x_1+kx_1=0 \to k=1$

3.50 Given the velocity field in cylindrical coordinates: $v_r = f(r,\theta),\ \ v_\theta = v_z = 0$. For an incompressible material, from the conservation of mass principle, obtain the most general form of the function $f(r,\theta)$.

Ans. The equation of continuity for an incompressible material is [see Eq.(3.15.11)]:

$\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}=0 \to \frac{\partial f}{\partial r}+\frac{f}{r}=0 \to \frac{1}{r}\frac{\partial}{\partial r}(fr)=0, \to fr=g(\theta)$.

Therefore, $f=g(\theta)/r$, where $g(\theta)$ is an arbitrary function of θ.

3.51 An incompressible fluid undergoes a two-dimensional motion with $v_r = k\cos\theta/\sqrt{r}$. From the consideration of the principle of conservation of mass, find v_θ, subject to the condition that $v_\theta = 0$ at $\theta = 0$.

Ans.

$v_r=\frac{k\cos\theta}{\sqrt{r}} \to \frac{\partial v_r}{\partial r}=(k\cos\theta)\left(-\frac{1}{2}\right)\frac{1}{r^{3/2}}$, $\frac{v_r}{r}=\frac{(k\cos\theta)}{r^{3/2}} \to \frac{\partial v_r}{\partial r}+\frac{v_r}{r}=\left(\frac{1}{2}\right)\frac{(k\cos\theta)}{r^{3/2}}$.

The equation of continuity for an incompressible fluid is [see Eq.(3.15.11)]: $\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}=0$. Thus,

$\frac{\partial v_\theta}{\partial \theta}=-\left(\frac{k}{2}\right)\frac{\cos\theta}{\sqrt{r}} \to v_\theta=-\left(\frac{k}{2}\right)\frac{\sin\theta}{\sqrt{r}}+f(r)$. Since $v_\theta=0$ at $\theta=0$, Therefore,

$f(r)=0$. Thus, $v_\theta=-\left(\frac{k}{2}\right)\frac{\sin\theta}{\sqrt{r}}$.

3.52 Are the following two velocity fields isochoric (i.e., no change of volume)?

(i) $\mathbf{v}=\frac{x_1\mathbf{e}_1+x_2\mathbf{e}_2}{r^2},\ \ r^2=x_1^2+x_2^2$ and (ii) $\mathbf{v}=\frac{-x_2\mathbf{e}_1+x_1\mathbf{e}_2}{r^2},\ \ r^2=x_1^2+x_2^2$

Ans. (i) With $v_1=x_1/r^2,\ v_2=x_2/r^2,\ r^2=x_1^2+x_2^2$,

$\frac{\partial v_1}{\partial x_1}=\frac{1}{r^2}-\frac{2x_1}{r^3}\frac{\partial r}{\partial x_1}=\frac{1}{r^2}-\frac{2x_1^2}{r^4}$. $\left(r^2=x_1^2+x_2^2 \to 2r\frac{\partial r}{\partial x_1}=2x_1,\quad 2r\frac{\partial r}{\partial x_2}=2x_2\right)$.

$\frac{\partial v_2}{\partial x_2}=\frac{1}{r^2}-\frac{2x_2}{r^3}\frac{\partial r}{\partial x_2}=\frac{1}{r^2}-\frac{2x_2^2}{r^4},\quad \frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}=\frac{2}{r^2}-\frac{2x_1^2}{r^4}-\frac{2x_2^2}{r^4}=\frac{2}{r^2}-\frac{2}{r^2}=0.$

(ii) $v_1=-x_2/r^2,\ v_2=x_1/r^2,\ r^2=x_1^2+x_2^2$

$$\frac{\partial v_1}{\partial x_1}=\frac{2x_2}{r^3}\frac{\partial r}{\partial x_1}=\frac{2x_2x_1}{r^4}\quad\left(r^2=x_1^2+x_2^2\to 2r\frac{\partial r}{\partial x_1}=2x_1\right)$$

$$\frac{\partial v_2}{\partial x_2}=-\frac{2x_1}{r^3}\frac{\partial r}{\partial x_2}=-\frac{2x_1x_2}{r^4},\quad \frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}=\frac{2x_2x_1}{r^4}-\frac{2x_1x_2}{r^4}=0.$$

3.53 Given that an incompressible and inhomogeneous fluid has a density field given by $\rho=kx_2$. From the consideration of the principle of conservation of mass, find the permissible form of velocity field for a two dimensional flow $(v_3=0)$.

Ans. Since the fluid is incompressible, therefore,

$$\frac{D\rho}{Dt}=0\to\frac{\partial\rho}{\partial t}+v_1\frac{\partial\rho}{\partial x_1}+v_2\frac{\partial\rho}{\partial x_2}=0\to 0+v_1(0)+v_2k=0\to v_2=0.$$

The conservation of mass equation of an incompressible fluid in two dimensional flow is

$$\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}=0\to\frac{\partial v_1}{\partial x_1}=0\to v_1=f(x_2),\ v_2=0.$$

3.54 Consider the velocity field: $\mathbf{v}=\frac{\alpha x_1}{1+kt}\mathbf{e}_1$. From the consideration of the principle of conservation of mass, (a) Find the density if it depends only on time t, i.e., $\rho=\rho(t)$, with $\rho(0)=\rho_o$. (b) Find the density if it depends only on x_1, i.e., $\rho=\hat{\rho}(x_1)$, with $\hat{\rho}(x_o)=\rho^*$.

Ans.(a) Equation of conservation of mass is

$$\frac{\partial\rho}{\partial t}+v_1\frac{\partial\rho}{\partial x_1}+v_2\frac{\partial\rho}{\partial x_2}+v_3\frac{\partial\rho}{\partial x_3}+\rho\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)=0.$$ With $v_1=\frac{\alpha x_1}{1+kt},\ v_2=v_3=0$,

$$\to\frac{d\rho}{dt}+\rho\frac{\alpha}{1+kt}=0\to\int_{\rho_o}^{\rho}\frac{d\rho}{\rho}=-\alpha\int_0^t\frac{dt}{1+kt}\to\ln\frac{\rho}{\rho_o}=-\frac{\alpha}{k}\ln(1+kt)\to\frac{\rho}{\rho_o}=(1+kt)^{-\alpha/k}.$$

(b) with $\rho=\rho(x_1)$ and $v_1=\frac{\alpha x_1}{1+kt},\ v_2=v_3=0$

$$\frac{\partial\rho}{\partial t}+v_1\frac{\partial\rho}{\partial x_1}+v_2\frac{\partial\rho}{\partial x_2}+v_3\frac{\partial\rho}{\partial x_3}+\rho\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)=0\to\frac{\alpha x_1}{1+kt}\frac{d\rho}{dx_1}+\rho\frac{\alpha}{1+kt}=0$$

$$\to x_1\frac{d\rho}{dx_1}+\rho=0,\to\int_{\rho^*}^{\rho}\frac{d\rho}{\rho}=-\int_{x_o}^{x_1}\frac{dx_1}{x_1}\to\ln\frac{\rho}{\rho^*}=-\ln\frac{x_1}{x_o}\to\rho=\rho^*\frac{x_o}{x_1}$$

where ρ_o is the density at $x_1=x_o$.

3.55 Given the velocity field: $\mathbf{v}=\alpha\left(x_1t\mathbf{e}_1+x_2t\mathbf{e}_2\right)$. From the consideration of the principle of conservation of mass, determine how the fluid density varies with time, if in a spatial description, it is a function of time only.

Ans. Equation of conservation of mass is

$$\frac{\partial \rho}{\partial t}+v_1\frac{\partial \rho}{\partial x_1}+v_2\frac{\partial \rho}{\partial x_2}+v_3\frac{\partial \rho}{\partial x_3}+\rho\left(\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}+\frac{\partial v_3}{\partial x_3}\right)=0$$

. With $v_1=\alpha x_1 t,\ \ v_2=\alpha x_2 t,\ v_3=0$,

$$\frac{d\rho}{dt}+\rho(\alpha t+\alpha t)=0\rightarrow\int_{\rho_o}^{\rho}\frac{d\rho}{\rho}=-2\alpha\int_0^t tdt\rightarrow\ln\frac{\rho}{\rho_o}=-\alpha t^2\rightarrow\rho=\rho_o e^{-\alpha t^2}.$$

3.56 Show that $\frac{\partial W_{im}}{\partial X_k}=\frac{\partial E_{ik}}{\partial X_m}-\frac{\partial E_{km}}{\partial X_i}$, where $E_{im}=\frac{1}{2}\left(\frac{\partial u_i}{\partial X_m}+\frac{\partial u_m}{\partial X_i}\right)$ is the strain tensor and $W_{im}=\frac{1}{2}\left(\frac{\partial u_i}{\partial X_m}-\frac{\partial u_m}{\partial X_i}\right)$ is the rotation tensor.

Ans.

$$\frac{\partial W_{im}}{\partial X_k}=\frac{\partial}{\partial X_k}\frac{1}{2}\left(\frac{\partial u_i}{\partial X_m}-\frac{\partial u_m}{\partial X_i}\right)=\frac{1}{2}\left(\frac{\partial^2 u_i}{\partial X_m\partial X_k}-\frac{\partial^2 u_m}{\partial X_i\partial X_k}\right)=$$

$$\frac{1}{2}\left(\frac{\partial^2 u_i}{\partial X_m\partial X_k}+\frac{\partial^2 u_k}{\partial X_m\partial X_i}-\frac{\partial^2 u_k}{\partial X_m\partial X_i}-\frac{\partial^2 u_m}{\partial X_i\partial X_k}\right)=$$

$$\frac{1}{2}\left(\frac{\partial}{\partial X_m}\left(\frac{\partial u_i}{\partial X_k}+\frac{\partial u_k}{\partial X_i}\right)-\frac{\partial}{\partial X_i}\left(\frac{\partial u_k}{\partial X_m}+\frac{\partial u_m}{\partial X_k}\right)\right)=\frac{\partial E_{ik}}{\partial x_m}-\frac{\partial E_{km}}{\partial x_i}$$

3.57 Check whether or not the following distribution of the state of strain satisfies the compatibility conditions:

$$[\mathbf{E}]=k\begin{bmatrix} X_1+X_2, & X_1 & X_2 \\ X_1 & X_2+X_3 & X_3 \\ X_2 & X_3 & X_1+X_3 \end{bmatrix},\ \ k=10^{-4}$$

Ans. Yes. We note that the given E_{ij} are linear in X_1, X_2 and X_3 and the terms in the compatibility conditions all involve second derivatives with respect to X_i, therefore these conditions are obviously satisfied by the given strain components.

3.58 Check whether or not the following distribution of the state of strain satisfies the compatibility conditions:

$$[\mathbf{E}]=k\begin{bmatrix} X_1^2 & X_2^2+X_3^2 & X_1X_3 \\ X_2^2+X_3^2 & 0 & X_1 \\ X_1X_3 & X_1 & X_2^2 \end{bmatrix},\ \ k=10^{-4}$$

Ans.

$$\frac{\partial^2 E_{11}}{\partial X_2^2}+\frac{\partial^2 E_{22}}{\partial X_1^2}=2\frac{\partial^2 E_{12}}{\partial X_1 X_2}\rightarrow 0+0=0,\text{OK}$$

$$\frac{\partial^2 E_{22}}{\partial X_3^2}+\frac{\partial^2 E_{33}}{\partial X_2^2}=2\frac{\partial^2 E_{23}}{\partial X_2 X_3}\rightarrow 0+2k\neq 0,\text{ not satisfied}$$

The given strain components are not compatible.

3.59 Does the displacement field: $u_1=\sin X_1,\ u_2=X_1^3X_2,\ u_3=\cos X_3$ correspond to a compatible strain field?

Ans. Yes. The displacement field obviously exists. In fact, the displacement field *is given*. There is no need to check the compatibility conditions. Whenever a displacement field is given, there is never any problem of compatibility of strain components.

3.60 Given the strain field: $E_{12}=E_{21}=kX_1X_2,\ k=10^{-4}$ and all other $E_{ij}=0$.

(a) Check the equations of compatibility for this strain field and (b) by attempting to integrate the strain field, show that there does not exist a continuous displacement field for this strain field.

Ans. (a) $\frac{\partial^2 E_{11}}{\partial X_2^2}+\frac{\partial^2 E_{22}}{\partial X_1^2}=2\frac{\partial^2 E_{12}}{\partial X_1 X_2}\rightarrow 0+0\neq 2k$. This compatibility condition is not satisfied.

(b) $E_{11}=0\rightarrow\frac{\partial u_1}{\partial X_1}=0\rightarrow u_1=u_1\left(X_2,X_3\right)$. Also, $E_{22}=0\rightarrow\frac{\partial u_2}{\partial X_2}=0\rightarrow u_2=u_2\left(X_1,X_3\right)$.

Now, $2E_{12}=\frac{\partial u_1}{\partial X_2}+\frac{\partial u_2}{\partial X_1}\rightarrow 2kX_1X_2=\frac{\partial u_1\left(X_2,X_3\right)}{\partial X_2}+\frac{\partial u_2\left(X_1,X_3\right)}{\partial X_1}=f\left(X_2,X_3\right)+g\left(X_1,X_3\right)$,

That is,

$2kX_1X_2=f\left(X_2,X_3\right)+g\left(X_1,X_3\right)$. Clearly, there is no way this equation can be satisfied, because the right side can not have terms of the form of X_1X_2.

3.61 Given the following strain components:

$$E_{11}=\frac{1}{\alpha}f\left(X_2,X_3\right),\ E_{22}=E_{33}=-\frac{\nu}{\alpha}f\left(X_2,X_3\right),\ E_{12}=E_{13}=E_{23}=0.$$

Show that for the strains to be compatible, $f\left(X_2,X_3\right)$ must be linear in X_2 and X_3.

Ans

$$\frac{\partial^2 E_{11}}{\partial X_2^2}+\frac{\partial^2 E_{22}}{\partial X_1^2}=2\frac{\partial^2 E_{12}}{\partial X_1 X_2}\rightarrow\frac{1}{\alpha}\frac{\partial^2 f\left(X_2,X_3\right)}{\partial X_2^2}=0,\quad\frac{\partial^2 E_{11}}{\partial X_3^2}+\frac{\partial^2 E_{33}}{\partial X_1^2}=2\frac{\partial^2 E_{13}}{\partial X_1 X_3}\rightarrow\frac{1}{\alpha}\frac{\partial^2 f\left(X_2,X_3\right)}{\partial X_3^2}=0$$

,

$$\frac{\partial^2 E_{11}}{\partial X_2\partial X_3}=\frac{\partial}{\partial X_1}\left(-\frac{\partial E_{23}}{\partial X_1}+\frac{\partial E_{31}}{\partial X_2}+\frac{\partial E_{12}}{\partial X_3}\right)\rightarrow\frac{1}{\alpha}\frac{\partial^2 f\left(X_2,X_3\right)}{\partial X_2\partial X_3}=0,\text{ Thus,}$$

$$\frac{\partial^2 f(X_2,X_3)}{\partial X_2^2}=0,\quad \frac{\partial^2 f(X_2,X_3)}{\partial X_3^2}=0,\quad \frac{\partial^2 f(X_2,X_3)}{\partial X_2 \partial X_3}=0\,.$$ $\rightarrow f(X_2,X_3)$ is a linear function of X_2 and X_3. We note also

$$\frac{\partial^2 E_{22}}{\partial X_3^2}+\frac{\partial^2 E_{33}}{\partial X_2^2}=-\frac{\nu}{\alpha}\left(\frac{\partial^2 f}{\partial X_3^2}+\frac{\partial^2 f}{\partial X_2^2}\right)=0=2\frac{\partial^2 E_{23}}{\partial X_1 X_3},$$

$$\frac{\partial^2 E_{22}}{\partial X_3 \partial X_1}=-\frac{\nu}{\alpha}\frac{\partial^2 f}{\partial X_3 \partial X_1}=0=\frac{\partial}{\partial X_2}\left(-\frac{\partial E_{31}}{\partial X_2}+\frac{\partial E_{12}}{\partial X_3}+\frac{\partial E_{23}}{\partial X_1}\right),$$

$$\frac{\partial^2 E_{33}}{\partial X_1 \partial X_2}=-\frac{\nu}{\alpha}\frac{\partial^2 f}{\partial X_1 \partial X_2}=0=\frac{\partial}{\partial X_3}\left(-\frac{\partial E_{12}}{\partial X_3}+\frac{\partial E_{23}}{\partial X_1}+\frac{\partial E_{31}}{\partial X_2}\right).$$

Thus, if $f(X_2,X_3)$ is a linear function of X_2 and X_3, then all compatibility equations are satisfied.

3.62 In cylindrical coordinates (r,θ,z), consider a differential volume bounded by the three pairs of faces: $r=r$ and $r=r+dr$; $\theta=\theta$ and $\theta=\theta+d\theta$; $z=z$ and $z=z+dz$. The rate at which mass is flowing into the volume across the face $r=r$ is given by $\rho v_r(rd\theta)(dz)$ and similar expressions for the other faces. By demanding that the net rate of inflow of mass must be equal to the rate of increase of mass inside the differential volume, obtain the equation of conservation of mass in cylindrical coordinates. Check your answer with Eq. (3.15.7).

Ans. Mass flux across the face $r=r$ into the differential volume dV is $(\rho v_r)(rd\theta)dz$. That across the face $r=r+dr$ out of the volume is $(\rho v_r)_{r=r+dr}(r+dr)d\theta dz$. Thus ,
the net mass flux into dV through the pair of faces $r=r$ and $r=r+dr$ is

$$(\rho v_r)_{r=r}(rd\theta)dz-(\rho v_r)_{r=r+dr}(r+dr)d\theta dz=\left[(\rho v_r)_{r=r}-(\rho v_r)_{r=r+dr}\right]rd\theta dz$$
$$-(\rho v_r)_{r=r+dr}\,drd\theta dz\,.$$

Now, $\left[(\rho v_r)_{r=r}-(\rho v_r)_{r=r+dr}\right]rd\theta dz=-\left[\dfrac{\partial(\rho v_r)}{\partial r}\right]dr(rd\theta dz)$ and

$-(\rho v_r)_{r=r+dr}\,drd\theta dz=-\left[(\rho v_r)+d(\rho v_r)\right]drd\theta dz=-(\rho v_r)drd\theta dz$, where we have dropped the higher order term involving $\left[d(\rho v_r)\right]drd\theta dz$ which approaches zero in the limit compared to the terms involving only three differentials. Thus, the net mass flux into dV through the pair of faces $r=r$ and $r=r+dr$ is

$$\left\{-r\left(\frac{\partial \rho v_r}{\partial r}\right)-(\rho v_r)\right\}drd\theta dz\,.$$ Similarly,

the net mass flux into dV through the pair of faces $\theta=\theta$ and $\theta=\theta+d\theta$ is

$$-\left(\frac{\partial \rho v_\theta}{\partial \theta}\right)d\theta(drdz),$$

and the net mass flux into dV through the pair of faces $z=z$ and $z=z+dz$ is

$-\left(\frac{\partial \rho v_z}{\partial z}\right)dz\left[dr\left(rd\theta\right)\right]$

Thus, the total influx of mass through these three pairs of faces is:

$-\left\{\left(\frac{\partial \rho v_r}{\partial r}\right)+\frac{\left(\rho v_r\right)}{r}+\frac{1}{r}\left(\frac{\partial \rho v_\theta}{\partial \theta}\right)+\left(\frac{\partial \rho v_z}{\partial z}\right)\right\}dr\left(rd\theta\right)dz$

On the other hand, the rate of increase of mass inside dV is $\frac{\partial}{\partial t}\left(\rho rd\theta drdz\right)=\frac{\partial \rho}{\partial t}rd\theta drdz$.

Therefore, the conservation of mass principle gives,

$-\left\{\left(\frac{\partial \rho v_r}{\partial r}\right)+\frac{\left(\rho v_r\right)}{r}+\frac{1}{r}\left(\frac{\partial \rho v_\theta}{\partial \theta}\right)+\left(\frac{\partial \rho v_z}{\partial z}\right)\right\}dr\left(rd\theta\right)dz=\frac{\partial \rho}{\partial t}rd\theta drdz$, That is:

$\frac{\partial \rho}{\partial t}+\frac{\partial \rho v_r}{\partial r}+\frac{\rho v_r}{r}+\frac{1}{r}\left(\frac{\partial \rho v_\theta}{\partial \theta}\right)+\left(\frac{\partial \rho v_z}{\partial z}\right)=0$, Or,

$\left\{\frac{\partial \rho}{\partial t}+v_r\frac{\partial \rho}{\partial r}+\frac{v_\theta}{r}\left(\frac{\partial \rho}{\partial \theta}\right)+v_z\frac{\partial \rho}{\partial z}\right\}+\rho\left(\frac{\partial v_r}{\partial r}+\frac{v_r}{r}+\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial v_z}{\partial z}\right)=0$. This is the same as Eq.(3.15.7).

3.63 Given the following deformation in rectangular Cartesian coordinates:

$$x_1=3X_3,\;\; x_2=-X_1,\;\; x_3=-2X_2$$

Determine (a) the deformation gradient $\mathbf{F}$, (b) the right Cauchy-Green tensor $\mathbf{C}$ and the right stretch tensor $\mathbf{U}$, (c) the left Cauchy-Green tensor $\mathbf{B}$, (d) the rotation tensor $\mathbf{R}$, (e) the Lagrangean strain tensor $\mathbf{E}^*$ (f) the Euler strain tensor $\mathbf{e}^*$, (g) ratio of deformed volume to initial volume, (h) the deformed area (magnitude and its normal) for the area whose normal was in the direction of $\mathbf{e}_2$ and whose magnitude was unity for the undeformed area.

Ans. (a) $\left[\mathbf{F}\right]=\begin{bmatrix}0&0&3\\-1&0&0\\0&-2&0\end{bmatrix}$, (b) $\left[\mathbf{C}\right]=\left[\mathbf{F}\right]^{\mathrm{T}}\left[\mathbf{F}\right]=\begin{bmatrix}0&-1&0\\0&0&-2\\3&0&0\end{bmatrix}\begin{bmatrix}0&0&3\\-1&0&0\\0&-2&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&4&0\\0&0&9\end{bmatrix}$,

$\left[\mathbf{U}\right]=\left[\mathbf{C}\right]^{1/2}=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$. (The only positive definite root).

(c) $\left[\mathbf{B}\right]=\left[\mathbf{F}\right]\left[\mathbf{F}\right]^{\mathrm{T}}=\begin{bmatrix}0&0&3\\-1&0&0\\0&-2&0\end{bmatrix}\begin{bmatrix}0&-1&0\\0&0&-2\\3&0&0\end{bmatrix}=\begin{bmatrix}9&0&0\\0&1&0\\0&0&4\end{bmatrix}$.

(d) $\left[\mathbf{R}\right]=\left[\mathbf{F}\right]\left[\mathbf{U}\right]^{-1}=\begin{bmatrix}0&0&3\\-1&0&0\\0&-2&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1/2&0\\0&0&1/3\end{bmatrix}=\begin{bmatrix}0&0&1\\-1&0&0\\0&-1&0\end{bmatrix}$.

(e) $\left[\mathbf{E}^*\right]=\frac{1}{2}[\mathbf{C}-\mathbf{I}]=\frac{1}{2}\begin{bmatrix}0&0&0\\0&3&0\\0&0&8\end{bmatrix}=\begin{bmatrix}0&0&0\\0&3/2&0\\0&0&4\end{bmatrix}$, (f) $\left[\mathbf{e}^*\right]=\frac{1}{2}\left[\mathbf{I}-\mathbf{B}^{-1}\right]=\begin{bmatrix}4/9&0&0\\0&0&0\\0&0&3/8\end{bmatrix}$.

(g) $\frac{\Delta V}{\Delta V_o}=\sqrt{\det\mathbf{B}}=\sqrt{(9)(1)(4)}=6$,

(h) $d\mathbf{A}=dA_o\left(\det\mathbf{F}\right)\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\mathbf{n}_o$, $dA_o=1$, $\det\mathbf{F}=6$, $[\mathbf{F}]^{-1}=\frac{1}{6}\begin{bmatrix}0&-6&0\\0&0&-3\\2&0&0\end{bmatrix}$, $\mathbf{n}_o=\mathbf{e}_2$,

$$[d\mathbf{A}]=dA_o\left[\left(\det\mathbf{F}\right)\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\mathbf{n}_o\right]=(1)(6)\frac{1}{6}\begin{bmatrix}0&0&2\\-6&0&0\\0&-3&0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}0\\0\\-3\end{bmatrix}\rightarrow d\mathbf{A}=-3\mathbf{e}_3.$$

3.64 Do the previous problem for the following deformation:
$x_1=2X_2,\ x_2=3X_3,\ x_3=X_1$.

Ans. (a) $[\mathbf{F}]=\begin{bmatrix}0&2&0\\0&0&3\\1&0&0\end{bmatrix}$. (b) $[\mathbf{C}]=[\mathbf{F}]^{\mathrm{T}}[\mathbf{F}]=\begin{bmatrix}0&0&1\\2&0&0\\0&3&0\end{bmatrix}\begin{bmatrix}0&2&0\\0&0&3\\1&0&0\end{bmatrix}=\begin{bmatrix}1&0&0\\0&4&0\\0&0&9\end{bmatrix}$.

$[\mathbf{U}]=[\mathbf{C}]^{1/2}=\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}$. (The only positive definite root).

(c) $[\mathbf{B}]=[\mathbf{F}][\mathbf{F}]^{\mathrm{T}}=\begin{bmatrix}0&2&0\\0&0&3\\1&0&0\end{bmatrix}\begin{bmatrix}0&0&1\\2&0&0\\0&3&0\end{bmatrix}=\begin{bmatrix}4&0&0\\0&9&0\\0&0&1\end{bmatrix}$.

(d) $[\mathbf{R}]=[\mathbf{F}][\mathbf{U}]^{-1}=\begin{bmatrix}0&2&0\\0&0&3\\1&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\0&1/2&0\\0&0&1/3\end{bmatrix}=\begin{bmatrix}0&1&0\\0&0&1\\1&0&0\end{bmatrix}$.

(e) $\left[\mathbf{E}^*\right]=\frac{1}{2}[\mathbf{C}-\mathbf{I}]=\frac{1}{2}\begin{bmatrix}0&0&0\\0&3&0\\0&0&8\end{bmatrix}=\begin{bmatrix}0&0&0\\0&3/2&0\\0&0&4\end{bmatrix}$, (f) $\left[\mathbf{e}^*\right]=\frac{1}{2}\left[\mathbf{I}-\mathbf{B}^{-1}\right]=\begin{bmatrix}3/8&0&0\\0&4/9&0\\0&0&0\end{bmatrix}$.

(g) $\frac{\Delta V}{\Delta V_o}=\sqrt{\det\mathbf{B}}=\sqrt{(4)(9)(1)}=6$.

(h) $d\mathbf{A}=dA_o\left(\det\mathbf{F}\right)\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\mathbf{n}_o$, $dA_o=1$, $\det\mathbf{F}=6$, $[\mathbf{F}]^{-1}=\frac{1}{6}\begin{bmatrix}0&0&6\\3&0&0\\0&2&0\end{bmatrix}$, $\mathbf{n}_o=\mathbf{e}_2$

$$[d\mathbf{A}] = dA_o\left[(\det\mathbf{F})\left(\mathbf{F}^{-1}\right)^T \mathbf{n}_o\right] = (1)(6)\frac{1}{6}\begin{bmatrix} 0 & 3 & 0 \\ 0 & 0 & 2 \\ 6 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \rightarrow d\mathbf{A} = 3\mathbf{e}_1$$

3.65 Do Prob. 3.63 for the following deformation:
$x_1 = X_1,\ x_2 = 3X_3,\ x_3 = -2X_2$

Ans. (a) $[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & -2 & 0 \end{bmatrix}$.

(b) $[\mathbf{C}] = [\mathbf{F}]^T[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$, The only positive definite root is $[\mathbf{U}] = [\mathbf{C}]^{1/2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

(c) $[\mathbf{B}] = [\mathbf{F}][\mathbf{F}]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

(d) $[\mathbf{R}] = [\mathbf{F}][\mathbf{U}]^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$,

(e) $\left[\mathbf{E}^*\right] = \frac{1}{2}[\mathbf{C} - \mathbf{I}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, (f) $\left[\mathbf{e}^*\right] = \frac{1}{2}\left[\mathbf{I} - \mathbf{B}^{-1}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 4/9 & 0 \\ 0 & 0 & 3/8 \end{bmatrix}$.

(g) $\frac{\Delta V}{\Delta V_o} = \sqrt{\det\mathbf{B}} = \sqrt{(1)(9)(4)} = 6$.

(h) $d\mathbf{A} = dA_o(\det\mathbf{F})\left(\mathbf{F}^{-1}\right)^T\mathbf{n}_o,\ dA_o = 1,\ \det\mathbf{F} = 6,\ [\mathbf{F}]^{-1} = \frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix}, \mathbf{n}_o = \mathbf{e}_2$

$$[d\mathbf{A}] = dA_o\left[(\det\mathbf{F})\left(\mathbf{F}^{-1}\right)^T \mathbf{n}_o\right] = (1)(6)\frac{1}{6}\begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & -3 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \rightarrow d\mathbf{A} = -3\mathbf{e}_3$$

3.66 Do Prob. 3.63 for the following deformation:

$x_1 = 2X_2,\ x_2 = -X_1,\ x_3 = 3X_3$

Ans. (a) $[\mathbf{F}] = \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. (b) $[\mathbf{C}] = [\mathbf{F}]^{\mathrm{T}}[\mathbf{F}] = \begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{bmatrix}$,

$[\mathbf{U}] = [\mathbf{C}]^{1/2} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. (The only positive definite root).

(c) $[\mathbf{B}] = [\mathbf{F}][\mathbf{F}]^{\mathrm{T}} = \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 0 & -1 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix}$.

(d) $[\mathbf{R}] = [\mathbf{F}][\mathbf{U}]^{-1} = \begin{bmatrix} 0 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(e) $\left[\mathbf{E}^*\right] = \frac{1}{2}[\mathbf{C} - \mathbf{I}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3/2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$, (f) $\left[\mathbf{e}^*\right] = \frac{1}{2}\left[\mathbf{I} - \mathbf{B}^{-1}\right] = \begin{bmatrix} 3/8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4/9 \end{bmatrix}$

(g) $\dfrac{\Delta V}{\Delta V_{\mathrm{o}}} = \sqrt{\det \mathbf{B}} = \sqrt{(4)(1)(9)} = 6$.

(h) $d\mathbf{A} = dA_{\mathrm{o}}\left(\det \mathbf{F}\right)\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\mathbf{n}_{\mathrm{o}}$, $dA_{\mathrm{o}} = 1$, $\det \mathbf{F} = 6$, $[\mathbf{F}]^{-1} = \frac{1}{6}\begin{bmatrix} 0 & -6 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, $\mathbf{n}_{\mathrm{o}} = \mathbf{e}_2$

$$[d\mathbf{A}] = dA_{\mathrm{o}}(\det \mathbf{F})[\mathbf{F}^{-1}]^{\mathrm{T}}[\mathbf{n}_{\mathrm{o}}] = (1)(6)\frac{1}{6}\begin{bmatrix} 0 & 3 & 0 \\ -6 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \rightarrow d\mathbf{A} = 3\mathbf{e}_1$$

3.67 Given $x_1 = X_1 + 3X_2,\ x_2 = X_2,\ x_3 = X_3$.

Obtain (a) the deformation gradient $\mathbf{F}$ and the right Cauchy-Green tensor $\mathbf{C}$, (b) The eigenvalues and eigenvector of $\mathbf{C}$, (c) the matrix of the stretch tensor $\mathbf{U}$ and $\mathbf{U}^{-1}$ with respect to the $\mathbf{e}_i$-basis and (d) the rotation tensor $\mathbf{R}$ with respect to the $\mathbf{e}_i$-basis.

Ans. (a) $[\mathbf{F}] = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $[\mathbf{C}] = [\mathbf{F}]^{\mathrm{T}}[\mathbf{F}] = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 10 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) the characteristic equation is

$$\begin{vmatrix} 1-\lambda & 3 & 0 \\ 3 & 10-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \to (1-\lambda)\left(\lambda^2 - 11\lambda + 1\right) = 0,$$

$$\lambda_{1,2} = \frac{11 \pm \sqrt{121-4}}{2} \to \lambda_1 = 10.908326, \quad \lambda_2 = 0.0916735, \quad \lambda_3 = 1$$

For $\lambda_1 = 10.908326$,

$$(1-\lambda_1)\alpha_1 + 3\alpha_2 = 0 \to \alpha_2 = -(1-\lambda_1)\alpha_1 / 3 = 3.302775\alpha_1,$$

$$\mathbf{n}_1 = \frac{1}{3.450843}(\mathbf{e}_1 + 3.302775\mathbf{e}_2) = 0.289785\mathbf{e}_1 + 0.957093\mathbf{e}_2.$$

For $\lambda_2 = 0.0916735$,

$$(1-\lambda_2)\alpha_1 + 3\alpha_2 = 0 \to \alpha_2 = -(1-\lambda_2)\alpha_1 / 3 = -0.3027755\alpha_1,$$

$$\mathbf{n}_2 = \frac{1}{1.044832}(\mathbf{e}_1 - 0.3027755\mathbf{e}_2) = 0.957093\mathbf{e}_1 - 0.289784\mathbf{e}_2.$$

For $\lambda_3 = 1, \mathbf{n}_3 = \mathbf{e}_3$,

(c) The matrices with respect to the principal axes are as follows

$$[\mathbf{C}]_{\mathbf{n}_i} = \begin{bmatrix} 10.9083 & 0 & 0 \\ 0 & 0.0916735 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to [\mathbf{U}]_{\mathbf{n}_i} = \begin{bmatrix} 3.30277 & 0 & 0 \\ 0 & 0.302774 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\left[\mathbf{U}^{-1}\right]_{\mathbf{n}_i} = \begin{bmatrix} 0.302774 & 0 & 0 \\ 0 & 3.302772 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrices with respect to the $\mathbf{e}_i$-basis are given by the formula $[\mathbf{U}]_{\{\mathbf{e}_i\}} = [\mathbf{Q}]^{\mathrm{T}} [\mathbf{U}]_{\{\mathbf{n}_i\}} [\mathbf{Q}]$:

$$[\mathbf{U}]_{\mathbf{e}_i} = \begin{bmatrix} 0.289785 & 0.957093 & 0 \\ 0.957093 & -0.289785 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3.30277 & 0 & 0 \\ 0 & 0.302774 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.289785 & 0.957093 & 0 \\ 0.957093 & -0.289785 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.554704 & 0.832057 & 0 \\ 0.832057 & 3.05087 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\left[\mathbf{U}^{-1}\right]_{\mathbf{e}_i} = \begin{bmatrix} 0.289785 & 0.957093 & 0 \\ 0.957093 & -0.289785 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.302774 & 0 & 0 \\ 0 & 3.302772 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0.289785 & 0.957093 & 0 \\ 0.957093 & -0.289785 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3.050852 & -0.832052 & 0 \\ -0.832052 & 0.554701 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

(d) $$[\mathbf{R}]_{\mathbf{e}_i} = [\mathbf{F}][\mathbf{U}^{-1}] = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3.050852 & -0.832052 & 0 \\ -0.832052 & 0.554701 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0.55470 & 0.83205 & 0 \\ -0.83205 & 0.55470 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

3.68 Verify that with respect to rectangular Cartesian base vectors, the right stretch tensor **U** and the rotation tensor **R** for the simple shear deformation , $x_1 = X_1 + kX_2,\ x_2 = X_2,\ x_3 = X_3$,

are given by: With $f = (1+k^2/4)^{-1/2}$,

$$[\mathbf{U}] = \begin{bmatrix} f & kf/2 & 0 \\ kf/2 & (1+k^2/2)f & 0 \\ 0 & 0 & 1 \end{bmatrix},\ [\mathbf{R}] = \begin{bmatrix} f & kf/2 & 0 \\ -kf/2 & f & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Ans. $$[\mathbf{RU}] = \begin{bmatrix} f & kf/2 & 0 \\ -kf/2 & f & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} f & kf/2 & 0 \\ kf/2 & (1+k^2/2)f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} f^2 + (kf/2)(kf/2) & f(kf/2) + (kf/2)(1+k^2/2)f & 0 \\ (-kf/2)f + f(kf/2) & (-kf/2)(kf/2) + f(1+k^2/2)f & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} f^2(1+k^2/4) & kf^2(1+k^2/4) & 0 \\ 0 & f^2(1+k^2/4) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \text{the given } [\mathbf{F}]$$

Since the decomposition of **F** is unique, therefore, the given **R** and **U** are the rotation and the stretch tensor respectively.

3.69 Let $d\mathbf{X}^{(1)} = dS_1\mathbf{N}^{(1)},\ d\mathbf{X}^{(2)} = dS_2\mathbf{N}^{(2)}$ be two material elements at a point *P*. Show that if θ denotes the angle between their respective deformed elements $d\mathbf{x}^{(1)} = ds_1\mathbf{m}$ and $d\mathbf{x}^{(2)} = ds_2\mathbf{n}$,

then, $\cos\theta = \dfrac{C_{\alpha\beta}N_\alpha^{(1)}N_\beta^{(2)}}{\lambda_1\lambda_2}$, where $\mathbf{N}^{(1)} = N_\alpha^{(1)}\mathbf{e}_\alpha,\ \mathbf{N}^{(2)} = N_\alpha^{(2)}\mathbf{e}_\alpha,\ \lambda_1 = \dfrac{ds_1}{dS_1}$ and $\lambda_2 = \dfrac{ds_2}{dS_2}$.

Ans. $d\mathbf{x}^{(1)} \cdot d\mathbf{x}^{(2)} = \mathbf{F}d\mathbf{X}^{(1)} \cdot \mathbf{F}d\mathbf{X}^{(2)} = d\mathbf{X}^{(1)} \cdot \mathbf{F}^{\mathrm{T}}\mathbf{F}d\mathbf{X}^{(2)} = d\mathbf{X}^{(1)} \cdot \mathbf{C}d\mathbf{X}^{(2)}$,

$\to ds_1 ds_2 \cos\theta = dS_1 dS_2 \mathbf{N}^{(1)} \cdot \mathbf{C}\mathbf{N}^{(2)} = dS_1 dS_2 (N_\alpha^{(1)}\mathbf{e}_\alpha) \cdot \mathbf{C}(N_\beta^{(2)}\mathbf{e}_\beta)$,

$$\to \cos\theta = \frac{dS_1 dS_2}{ds_1 ds_2} N_\alpha^{(1)} N_\beta^{(2)} \mathbf{e}_\alpha \cdot \mathbf{C}\mathbf{e}_\beta = \frac{C_{\alpha\beta}N_\alpha^{(1)}N_\beta^{(2)}}{\lambda_1\lambda_2}.$$

3.70 Given the following right Cauchy-Green deformation tensor at a point

$$[\mathbf{C}]=\begin{bmatrix}9 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 0.36\end{bmatrix}$$

(a) Find the stretch for the material elements which were in the direction of $\mathbf{e}_1,\mathbf{e}_2$, and $\mathbf{e}_3$. (b) Find the stretch for the material element which was in the direction of $\mathbf{e}_1+\mathbf{e}_2$. (c) Find $\cos\theta$, where θ is the angle between $d\mathbf{x}^{(1)}$ and $d\mathbf{x}^{(2)}$ where $d\mathbf{X}^{(1)}=dS_1\mathbf{e}_1$ and $d\mathbf{X}^{(2)}=dS_2\mathbf{e}_1$ deform to $d\mathbf{x}^{(1)}=ds_1\mathbf{m}$ and $d\mathbf{x}^{(2)}=ds_2\mathbf{n}$.

Ans. (a) For the elements which were in $\mathbf{e}_1,\mathbf{e}_2$, and $\mathbf{e}_3$ direction, the stretches are $\sqrt{C_{11}},\sqrt{C_{22}},\sqrt{C_{33}}$, that is, 3, 2 and 0.6 respectively.

(b) Let $\mathbf{e}_1' =\frac{1}{\sqrt{2}}(\mathbf{e}_1+\mathbf{e}_2)\rightarrow C_{11}' =\frac{1}{2}\begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}9 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 0.36\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}9\\4\\0\end{bmatrix}=\frac{13}{2}.$

That is, the stretch for $d\mathbf{X}=dS\mathbf{e}_1'$ is $(ds/dS)=\sqrt{C_{11}'}=\sqrt{13/2}$.

(c) $C_{12}=0\rightarrow\cos\theta=0\rightarrow\theta=90^{\circ}$. There is no change in angle. (note, $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ are principal axes for $\mathbf{C}$.

3.71 Given the following large shear deformation:

$$x_1=X_1+X_2,\ \ x_2=X_2,\ \ x_3=X_3.$$

(a) Find the stretch tensor $\mathbf{U}$ (Hint: use the formula given in problem 3.68) and verify that $\mathbf{U}^2=\mathbf{C}$, the right Cauchy-Green deformation tensor. (b) What is the stretch for the element which was in the direction $\mathbf{e}_2$?

(c) Find the stretch for an element which was in the direction of $\mathbf{e}_1+\mathbf{e}_2$.

(d) What is the angle between the deformed elements of $dS_1\mathbf{e}_1$ and $dS_2\mathbf{e}_2$?.

Ans. (a) For $x_1=X_1+kX_2,\ x_2=X_2,\ x_3=X_3$, from Prob. 3.68, we have

$$[\mathbf{U}]=\begin{bmatrix}f & kf/2 & 0\\ kf/2 & (1+k^2/2)f & 0\\ 0 & 0 & 1\end{bmatrix}\text{ where } f=\left(1+\frac{k^2}{4}\right)^{-\frac{1}{2}}.\text{ Thus, with } k=1,\ \ f=2/\sqrt{5}$$

$$[\mathbf{U}]=\begin{bmatrix}f & f/2 & 0\\ f/2 & (3/2)f & 0\\ 0 & 0 & 1\end{bmatrix}=f\begin{bmatrix}1 & 1/2 & 0\\ 1/2 & 3/2 & 0\\ 0 & 0 & 1/f\end{bmatrix}=\frac{2}{\sqrt{5}}\begin{bmatrix}1 & 1/2 & 0\\ 1/2 & 3/2 & 0\\ 0 & 0 & \sqrt{5}/2\end{bmatrix}.$$

$$[\mathbf{U}][\mathbf{U}]=\frac{4}{5}\begin{bmatrix}1 & 1/2 & 0\\ 1/2 & (3/2) & 0\\ 0 & 0 & \sqrt{5}/2\end{bmatrix}\begin{bmatrix}1 & 1/2 & 0\\ 1/2 & (3/2) & 0\\ 0 & 0 & \sqrt{5}/2\end{bmatrix}=\begin{bmatrix}1 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}=[\mathbf{C}].$$

(b) The stretch for the element which was in the direction $\mathbf{e}_2$ is $\sqrt{C_{22}}=\sqrt{2}$.

(c) Let $\mathbf{e}_1'=(\mathbf{e}_1+\mathbf{e}_2)/\sqrt{2}$,

$$C_{11}'=\frac{1}{2}\begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}=\frac{1}{2}\begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}2\\ 3\\ 0\end{bmatrix}=\frac{5}{2}\to\frac{ds}{dS}=\sqrt{5/2}.$$

(d) $\left(\dfrac{ds_1}{dS_1}\right)\left(\dfrac{ds_2}{dS_2}\right)\cos\theta=C_{12}\to(1)\left(\sqrt{2}\right)\cos\theta=1\to\cos\theta=\dfrac{1}{\sqrt{2}}\to\theta=45^{\circ}$.

3.72 Given the following large shear deformation:

$$x_1=X_1+2X_2,\ x_2=X_2,\ x_3=X_3$$

(a) Find the stretch tensor $\mathbf{U}$ (Hint: use the formula given in problem 3.68) and verify that $\mathbf{U}^2=\mathbf{C}$, the right Cauchy-Green deformation tensor.

(b) What is the stretch for the element which was in the direction $\mathbf{e}_2$.

(c) Find the stretch for an element which was in the direction of $\mathbf{e}_1+\mathbf{e}_2$.

(d) What is the angle between the deformed elements of $dS_1\mathbf{e}_1$ and $dS_2\mathbf{e}_2$.

Ans. For $x_1=X_1+kX_2,\ x_2=X_2,\ x_3=X_3$, from Prob. 3.68, we have

$$[\mathbf{U}]=\begin{bmatrix}f & kf/2 & 0\\ kf/2 & \left(1+k^2/2\right)f & 0\\ 0 & 0 & 1\end{bmatrix}\text{ where } f=\left(1+\frac{k^2}{4}\right)^{-\frac{1}{2}}.$$ Thus, with $k=2$, $f=1/\sqrt{2}$

$$[\mathbf{U}]=\begin{bmatrix}f & kf/2 & 0\\ kf/2 & \left(1+k^2/2\right)f & 0\\ 0 & 0 & 1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0\\ 1 & 3 & 0\\ 0 & 0 & \sqrt{2}\end{bmatrix}.$$

$$[\mathbf{R}]=\begin{bmatrix}f & kf/2 & 0\\ -kf/2 & f & 0\\ 0 & 0 & 1\end{bmatrix}=\frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1 & 0\\ -1 & 1 & 0\\ 0 & 0 & \sqrt{2}\end{bmatrix}.$$

$$[\mathbf{U}][\mathbf{U}]=\left(\frac{1}{\sqrt{2}}\right)^2\begin{bmatrix}1 & 1 & 0\\ 1 & 3 & 0\\ 0 & 0 & \sqrt{2}\end{bmatrix}\begin{bmatrix}1 & 1 & 0\\ 1 & 3 & 0\\ 0 & 0 & \sqrt{2}\end{bmatrix}=\begin{bmatrix}1 & 2 & 0\\ 2 & 5 & 0\\ 0 & 0 & 1\end{bmatrix}=[\mathbf{C}].$$

(b) The stretch for the element which was in the direction $\mathbf{e}_2$ is $\sqrt{C_{22}}=\sqrt{5}$.

(c) Let $\mathbf{e}_1'=(\mathbf{e}_1+\mathbf{e}_2)/\sqrt{2}$,

$$C'_{11} = \frac{1}{2}\begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}1 & 2 & 0\\ 2 & 5 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1\\1\\0\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & 1 & 0\end{bmatrix}\begin{bmatrix}3\\7\\0\end{bmatrix} = 5 \rightarrow \frac{ds}{dS} = \sqrt{5} = 2.236.$$

(d) $\left(\frac{ds_1}{dS_1}\right)\left(\frac{ds_2}{dS_2}\right)\cos\theta = C_{12} \rightarrow (1)\left(\sqrt{5}\right)\cos\theta = 2 \rightarrow \cos\theta = \frac{2}{\sqrt{5}}$.

3.73 Show that for any tensor $\mathbf{A}(X_1, X_2, X_3)$, $\frac{\partial}{\partial X_m}\det\mathbf{A} = (\det\mathbf{A})\left(\mathbf{A}^{-1}\right)_{nj}\frac{\partial A_{jn}}{\partial X_m}$

Ans

$$|\mathbf{A}| = \begin{vmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{vmatrix} \rightarrow \frac{\partial|\mathbf{A}|}{\partial X_m} = \begin{vmatrix}\frac{\partial A_{11}}{\partial X_m} & \frac{\partial A_{12}}{\partial X_m} & \frac{\partial A_{13}}{\partial X_m}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{vmatrix} + \begin{vmatrix}A_{11} & A_{12} & A_{13}\\ \frac{\partial A_{21}}{\partial X_m} & \frac{\partial A_{22}}{\partial X_m} & \frac{\partial A_{23}}{\partial X_m}\\ A_{31} & A_{32} & A_{33}\end{vmatrix} + \begin{vmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ \frac{\partial A_{31}}{\partial X_m} & \frac{\partial A_{32}}{\partial X_m} & \frac{\partial A_{33}}{\partial X_m}\end{vmatrix}$$

.

Let A^c_{ij} denote the cofactor of A_{ij}, i.e., $A^c_{11} = \begin{vmatrix}A_{22} & A_{23}\\ A_{32} & A_{33}\end{vmatrix}, A^c_{12} = -\begin{vmatrix}A_{21} & A_{23}\\ A_{31} & A_{33}\end{vmatrix}$ etc.

Then, $\frac{\partial|\mathbf{A}|}{\partial X_m} = \frac{\partial A_{11}}{\partial X_m}A^c_{11} + \frac{\partial A_{12}}{\partial X_m}A^c_{12} + \frac{\partial A_{13}}{\partial X_m}A^c_{13} + \frac{\partial A_{21}}{\partial X_m}A^c_{21} + \frac{\partial A_{22}}{\partial X_m}A^c_{22} + \ldots$

That is, $\frac{\partial|\mathbf{A}|}{\partial X_m} = \frac{\partial A_{ij}}{\partial X_m}A^c_{ij}$. On the other hand, $\left(\mathbf{A}^{-1}\right)_{ij} = \frac{A^c_{ji}}{\det\mathbf{A}} \rightarrow A^c_{ji} = \det\mathbf{A}\left(\mathbf{A}^{-1}\right)_{ij}$.

Thus, $\frac{\partial|\mathbf{A}|}{\partial X_m} = \det\mathbf{A}\left(\mathbf{A}^{-1}\right)_{ji}\frac{\partial A_{ij}}{\partial X_m} = \det\mathbf{A}\left(\mathbf{A}^{-1}\right)_{jn}\frac{\partial A_{nj}}{\partial X_m}$.

3.74 Show that if $\mathbf{TU} = \mathbf{0}$, where the eigenvalues of $\mathbf{U}$ are all positive (nonzero), then $\mathbf{T} = \mathbf{0}$.

Ans. Using the eigenvectors of $\mathbf{U}$ as basis, we have,

$$[\mathbf{TU}] = [\mathbf{T}][\mathbf{U}] = \begin{bmatrix}T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{bmatrix}\begin{bmatrix}\lambda_1 & 0 & 0\\ 0 & \lambda_2 & 0\\ 0 & 0 & \lambda_3\end{bmatrix} = \begin{bmatrix}\lambda_1 T_{11} & \lambda_2 T_{12} & \lambda_3 T_{13}\\ \lambda_1 T_{21} & \lambda_2 T_{22} & \lambda_3 T_{23}\\ \lambda_1 T_{31} & \lambda_2 T_{32} & \lambda_3 T_{33}\end{bmatrix}$$

Thus, $\mathbf{TU} = \mathbf{0}$ gives, all $T_{ij} = 0$, that is, $\mathbf{T} = \mathbf{0}$.

3.75 Derive Eq. (3.29.21), that is, $B_{\theta\theta} = \left(\frac{r\partial\theta}{\partial r_o}\right)^2 + \left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)^2 + \left(\frac{r\partial\theta}{\partial z_o}\right)^2$

Ans. $B_{\theta\theta} = \mathbf{e}_\theta \cdot \mathbf{B}\mathbf{e}_\theta = \mathbf{e}_\theta \cdot \mathbf{F}\mathbf{F}^{\mathrm{T}}\mathbf{e}_\theta$. From Eq. (3.29.15). we have,

$$\mathbf{F}^{\mathrm{T}}\mathbf{e}_\theta = \frac{r\partial\theta}{\partial r_o}\mathbf{e}_r^o + \frac{r\partial\theta}{r_o\partial\theta_o}\mathbf{e}_\theta^o + \frac{r\partial\theta}{\partial z_o}\mathbf{e}_z^o, \text{ thus,}$$

$$B_{\theta\theta} = \mathbf{e}_\theta \cdot \mathbf{F}\left(\frac{r\partial\theta}{\partial r_o}\mathbf{e}_r^o + \frac{r\partial\theta}{r_o\partial\theta_o}\mathbf{e}_\theta^o + \frac{r\partial\theta}{\partial z_o}\mathbf{e}_z^o\right) = \frac{r\partial\theta}{\partial r_o}\mathbf{e}_\theta \cdot \mathbf{F}\mathbf{e}_r^o + \frac{r\partial\theta}{r_o\partial\theta_o}\mathbf{e}_\theta \cdot \mathbf{F}\mathbf{e}_\theta^o + \frac{r\partial\theta}{\partial z_o}\mathbf{e}_\theta \cdot \mathbf{F}\mathbf{e}_z^o.$$

Since, $\mathbf{e}_\theta \cdot \mathbf{F}\mathbf{e}_r^o = \frac{r\partial\theta}{\partial r_o}$, $\mathbf{e}_\theta \cdot \mathbf{F}\mathbf{e}_\theta^o = \frac{r\partial\theta}{r_o\partial\theta_o}$, $\mathbf{e}_\theta \cdot \mathbf{F}\mathbf{e}_z^o = \frac{r\partial\theta}{\partial z_o}$, [See Eq. (3.29.10)], therefore,

$$B_{\theta\theta} = \left(\frac{r\partial\theta}{\partial r_o}\right)^2 + \left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)^2 + \left(\frac{r\partial\theta}{\partial z_o}\right)^2.$$

3.76 Derive Eq. (3.29.23), i.e., $B_{rz} = \left(\frac{\partial r}{\partial r_o}\right)\left(\frac{\partial z}{\partial r_o}\right) + \left(\frac{\partial r}{r_o\partial\theta_o}\right)\left(\frac{\partial z}{r_o\partial\theta_o}\right) + \left(\frac{\partial r}{\partial z_o}\right)\left(\frac{\partial z}{\partial z_o}\right)$

Ans. $B_{rz} = \mathbf{e}_r \cdot \mathbf{B}\mathbf{e}_z = \mathbf{e}_r \cdot \mathbf{F}\mathbf{F}^{\mathrm{T}}\mathbf{e}_z$, from Eq. (3.29.16), we have,

$$\mathbf{F}^{\mathrm{T}}\mathbf{e}_z = \frac{\partial z}{\partial r_o}\mathbf{e}_r^o + \frac{\partial z}{r_o\partial\theta_o}\mathbf{e}_\theta^o + \frac{\partial z}{\partial z_o}\mathbf{e}_z^o, \text{ thus,}$$

$$B_{rz} = \mathbf{e}_r \cdot \mathbf{F}\left(\frac{\partial z}{\partial r_o}\mathbf{e}_r^o + \frac{\partial z}{r_o\partial\theta_o}\mathbf{e}_\theta^o + \frac{\partial z}{\partial z_o}\mathbf{e}_z^o,\right) = \frac{\partial z}{\partial r_o}\mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_r^o + \frac{\partial z}{r_o\partial\theta_o}\mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_\theta^o + \frac{\partial z}{\partial z_o}\mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_z^o.$$

From Eq. (3.29.9), $\mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_r^o = \frac{\partial r}{\partial r_o}$, $\mathbf{e}_r \cdot \mathbf{F}\mathbf{e}_\theta^o = \frac{\partial r}{r_o\partial\theta_o}$, $\mathbf{e}_\theta \cdot \mathbf{F}\mathbf{e}_z^o = \frac{\partial r}{\partial z_o}$, thus,

$$B_{rz} = \left(\frac{\partial r}{\partial r_o}\right)\left(\frac{\partial z}{\partial r_o}\right) + \left(\frac{\partial r}{r_o\partial\theta_o}\right)\left(\frac{\partial z}{r_o\partial\theta_o}\right) + \left(\frac{\partial r}{\partial z_o}\right)\left(\frac{\partial z}{\partial z_o}\right).$$

3.77 From $r_o = r_o(r,\theta,z,t)$, $\theta_o = \theta_o(r,\theta,z,t)$, $z_o = z_o(r,\theta,z,t)$,derive the components of $\mathbf{B}^{-1}$ with respect to the basis at $\mathbf{x}$.

Ans. From $d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x}$, where $d\mathbf{x} = dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + dz\mathbf{e}_z$ and $d\mathbf{X} = dr_o\mathbf{e}_r^o + r_o d\theta_o\mathbf{e}_\theta^o + dz_o\mathbf{e}_z^o$, we have, $dr_o\mathbf{e}_r^o + r_o d\theta_o\mathbf{e}_\theta^o + dz_o\mathbf{e}_z^o = \mathbf{F}^{-1}\left(dr\mathbf{e}_r + rd\theta\mathbf{e}_\theta + dz\mathbf{e}_z\right)$

$$\to dr_o = dr\left(\mathbf{e}_r^o \cdot \mathbf{F}^{-1}\mathbf{e}_r\right) + rd\theta\left(\mathbf{e}_r^o \cdot \mathbf{F}^{-1}\mathbf{e}_\theta\right) + dz\left(\mathbf{e}_r^o \cdot \mathbf{F}^{-1}\mathbf{e}_z\right)$$

$$\to \frac{\partial r_o}{\partial r}dr + \frac{\partial r_o}{\partial\theta}d\theta + \frac{\partial r_o}{\partial z}dz = dr\left(\mathbf{e}_r^o \cdot \mathbf{F}^{-1}\mathbf{e}_r\right) + rd\theta\left(\mathbf{e}_r^o \cdot \mathbf{F}^{-1}\mathbf{e}_\theta\right) + dz\left(\mathbf{e}_r^o \cdot \mathbf{F}^{-1}\mathbf{e}_z\right)$$

$$\to \mathbf{e}_r^o \cdot \mathbf{F}^{-1}\mathbf{e}_r = \frac{\partial r_o}{\partial r},\ \mathbf{e}_r^o \cdot \mathbf{F}^{-1}\mathbf{e}_\theta = \frac{\partial r_o}{r\partial\theta},\ \mathbf{e}_r^o \cdot \mathbf{F}^{-1}\mathbf{e}_z = \frac{\partial r_o}{\partial z}.$$

Similarly,

$$\mathbf{e}_\theta^o\cdot\mathbf{F}^{-1}\mathbf{e}_r=\frac{r_o\partial\theta_o}{\partial r},\ \ \mathbf{e}_\theta^o\cdot\mathbf{F}^{-1}\mathbf{e}_\theta=\frac{r_o\partial\theta_o}{r\partial\theta},\ \ \mathbf{e}_\theta^o\cdot\mathbf{F}^{-1}\mathbf{e}_z=\frac{r_o\partial\theta_o}{\partial z}.$$

$$\mathbf{e}_z^o\cdot\mathbf{F}^{-1}\mathbf{e}_r=\frac{\partial z_o}{\partial r},\ \ \mathbf{e}_z^o\cdot\mathbf{F}^{-1}\mathbf{e}_\theta=\frac{\partial z_o}{r\partial\theta},\ \ \mathbf{e}_z^o\cdot\mathbf{F}^{-1}\mathbf{e}_z=\frac{\partial z_o}{\partial z}.$$

Thus,

$$\mathbf{F}^{-1}\mathbf{e}_r=\frac{\partial r_o}{\partial r}\mathbf{e}_r^o+\frac{r_o\partial\theta_o}{\partial r}\mathbf{e}_\theta^o+\frac{\partial z_o}{\partial r}\mathbf{e}_z^o,\ \ \mathbf{F}^{-1}\mathbf{e}_\theta=\frac{\partial r_o}{r\partial\theta}\mathbf{e}_r^o+\frac{r_o\partial\theta_o}{r\partial\theta}\mathbf{e}_\theta^o+\frac{\partial z_o}{r\partial\theta}\mathbf{e}_z^o$$

$$\mathbf{F}^{-1}\mathbf{e}_z=\frac{\partial r_o}{\partial z}\mathbf{e}_r^o+\frac{r_o\partial\theta_o}{\partial z}\mathbf{e}_\theta^o+\frac{\partial z_o}{\partial z}\mathbf{e}_z^o.$$

Also, we have,

$$\mathbf{e}_r\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_r^o=\mathbf{e}_r^o\cdot\mathbf{F}^{-1}\mathbf{e}_r=\frac{\partial r_o}{\partial r},\quad \mathbf{e}_\theta\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_r^o=\mathbf{e}_r^o\cdot\mathbf{F}^{-1}\mathbf{e}_\theta=\frac{\partial r_o}{r\partial\theta},$$

$$\mathbf{e}_z\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_r^o=\mathbf{e}_r^o\cdot\left(\mathbf{F}^{-1}\right)\mathbf{e}_z=\frac{\partial r_o}{\partial z}.$$

Thus,

$$\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_r^o=\frac{\partial r_o}{\partial r}\mathbf{e}_r+\frac{\partial r_o}{r\partial\theta}\mathbf{e}_\theta+\frac{\partial r_o}{\partial z}\mathbf{e}_z,\ \ \left(\mathbf{F}^{-1}\right)^T\mathbf{e}_\theta^o=\frac{r_o\partial\theta_o}{\partial r}\mathbf{e}_r+\frac{r_o\partial\theta_o}{r\partial\theta}\mathbf{e}_\theta+\frac{r_o\partial\theta_o}{\partial z}\mathbf{e}_z$$

$$\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_z^o=\frac{\partial z_o}{\partial r}\mathbf{e}_r+\frac{\partial z_o}{r\partial\theta}\mathbf{e}_\theta+\frac{\partial z_o}{\partial z}\mathbf{e}_z.$$

The components of $\mathbf{B}^{-1}$ with respect to the basis at $\mathbf{x}$ are:

$$B_{rr}^{-1}=\mathbf{e}_r\cdot\mathbf{B}^{-1}\mathbf{e}_r=\mathbf{e}_r\cdot\left(\mathbf{F}\mathbf{F}^T\right)^{-1}\mathbf{e}_r=\mathbf{e}_r\cdot\left(\mathbf{F}^{-1}\right)^T\left(\mathbf{F}^{-1}\mathbf{e}_r\right)$$

$$=\left(\frac{\partial r_o}{\partial r}\mathbf{e}_r\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_r^o+\frac{r_o\partial\theta_o}{\partial r}\mathbf{e}_r\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_\theta^o+\frac{\partial z_o}{\partial r}\mathbf{e}_r\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_z^o\right)=\left(\frac{\partial r_o}{\partial r}\right)^2+\left(\frac{r_o\partial\theta_o}{\partial r}\right)^2+\left(\frac{\partial z_o}{\partial r}\right)^2.$$

$$B_{\theta\theta}^{-1}=\mathbf{e}_\theta\cdot\mathbf{B}^{-1}\mathbf{e}_\theta=\mathbf{e}_\theta\cdot\left(\mathbf{F}\mathbf{F}^T\right)^{-1}\mathbf{e}_\theta=\mathbf{e}_\theta\cdot\left(\mathbf{F}^{-1}\right)^T\left(\mathbf{F}^{-1}\mathbf{e}_\theta\right)$$

$$=\frac{\partial r_o}{r\partial\theta}\mathbf{e}_\theta\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_r^o+\frac{r_o\partial\theta_o}{r\partial\theta}\mathbf{e}_\theta\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_\theta^o+\frac{\partial z_o}{r\partial\theta}\mathbf{e}_\theta\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_z^o=\left(\frac{\partial r_o}{r\partial\theta}\right)^2+\left(\frac{r_o\partial\theta_o}{r\partial\theta}\right)^2+\left(\frac{\partial z_o}{r\partial\theta}\right)^2.$$

$$B_{zz}^{-1}=\left(\frac{\partial r_o}{\partial z}\right)^2+\left(\frac{r_o\partial\theta_o}{\partial z}\right)^2+\left(\frac{\partial z_o}{\partial z}\right)^2.$$

$$B_{r\theta}^{-1}=\mathbf{e}_r\cdot\mathbf{B}^{-1}\mathbf{e}_\theta=\mathbf{e}_r\cdot\left(\mathbf{F}\mathbf{F}^T\right)^{-1}\mathbf{e}_\theta=\mathbf{e}_r\cdot\left(\mathbf{F}^{-1}\right)^T\left(\mathbf{F}^{-1}\mathbf{e}_\theta\right)$$

$$=\frac{\partial r_o}{r\partial\theta}\mathbf{e}_r\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_r^o+\frac{r_o\partial\theta_o}{r\partial\theta}\mathbf{e}_r\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_\theta^o+\frac{\partial z_o}{r\partial\theta}\mathbf{e}_r\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_z^o=\left(\frac{\partial r_o}{r\partial\theta}\right)\left(\frac{\partial r_o}{\partial r}\right)+\left(\frac{r_o\partial\theta_o}{r\partial\theta}\right)\left(\frac{r_o\partial\theta_o}{\partial r}\right)+\left(\frac{\partial z_o}{r\partial\theta}\right)\left(\frac{\partial z_o}{\partial z}\right)$$

$$B_{rz}^{-1}=\mathbf{e}_r\cdot\mathbf{B}^{-1}\mathbf{e}_z=\mathbf{e}_r\cdot\left(\mathbf{F}\mathbf{F}^T\right)^{-1}\mathbf{e}_z=\mathbf{e}_r\cdot\left(\mathbf{F}^{-1}\right)^T\left(\mathbf{F}^{-1}\mathbf{e}_z\right)$$

$$=\frac{\partial r_o}{\partial z}\mathbf{e}_r\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_r^o+\frac{r_o\partial\theta_o}{\partial z}\mathbf{e}_r\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_\theta^o+\frac{\partial z_o}{\partial z}\mathbf{e}_r\cdot\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_z^o=\left(\frac{\partial r_o}{\partial r}\right)\left(\frac{\partial r_o}{\partial z}\right)+\left(\frac{r_o\partial\theta_o}{\partial r}\right)\left(\frac{r_o\partial\theta_o}{\partial z}\right)+\left(\frac{\partial z_o}{\partial r}\right)\left(\frac{\partial z_o}{\partial z}\right)$$

$$B_{\theta z}^{-1}=\left(\frac{\partial r_o}{r\partial\theta}\right)\left(\frac{\partial r_o}{\partial z}\right)+\left(\frac{r_o\partial\theta_o}{r\partial\theta}\right)\left(\frac{r_o\partial\theta_o}{\partial z}\right)+\left(\frac{\partial z_o}{r\partial\theta}\right)\left(\frac{\partial z_o}{\partial z}\right).$$

3.78 Derive Eq. (3.29.47), that is, $C_{\theta_o\theta_o} = \left(\frac{\partial r}{r_o\partial\theta_o}\right)^2 + \left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)^2 + \left(\frac{\partial z}{r_o\partial\theta_o}\right)^2$

Ans. $C_{\theta_o\theta_o} = \mathbf{e}_\theta^o \cdot \mathbf{C}\mathbf{e}_\theta^o = \mathbf{e}_\theta^o \cdot \mathbf{F}^T\mathbf{F}\mathbf{e}_\theta^o$. Now, $\mathbf{F}\mathbf{e}_\theta^o = \frac{\partial r}{r_o\partial\theta_o}\mathbf{e}_r + \frac{r\partial\theta}{r_o\partial\theta_o}\mathbf{e}_\theta + \frac{\partial z}{r_o\partial\theta_o}\mathbf{e}_z$[Eq.3.29.3], therefore,

$$C_{\theta_o\theta_o} = \mathbf{e}_\theta^o \cdot \mathbf{F}^T\left(\frac{\partial r}{r_o\partial\theta_o}\mathbf{e}_r + \frac{r\partial\theta}{r_o\partial\theta_o}\mathbf{e}_\theta + \frac{\partial z}{r_o\partial\theta_o}\mathbf{e}_z\right) = \frac{\partial r}{r_o\partial\theta_o}\mathbf{e}_\theta^o \cdot \mathbf{F}^T\mathbf{e}_r + \frac{r\partial\theta}{r_o\partial\theta_o}\mathbf{e}_\theta^o \cdot \mathbf{F}^T\mathbf{e}_\theta$$

$+\frac{\partial z}{r_o\partial\theta_o}\mathbf{e}_\theta^o \cdot \mathbf{F}^T\mathbf{e}_z$. Now, from Eqs. (3.29.14) (3.29.15) and (3.29.16),

$\mathbf{e}_\theta^o \cdot \mathbf{F}^T\mathbf{e}_r = \frac{\partial r}{r_o\partial\theta_o}, \quad \mathbf{e}_\theta^o \cdot \mathbf{F}^T\mathbf{e}_\theta = \frac{r\partial\theta}{r_o\partial\theta_o}, \quad \mathbf{e}_\theta^o \cdot \mathbf{F}^T\mathbf{e}_z = \frac{\partial z}{r_o\partial\theta_o}$, thus.

$$C_{\theta_o\theta_o} = \left(\frac{\partial r}{r_o\partial\theta_o}\right)^2 + \left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)^2 + \left(\frac{\partial z}{r_o\partial\theta_o}\right)^2$$

3.79 Derive Eq. (3.29.49), $C_{r_o\theta_o} = \left(\frac{\partial r}{r_o\partial\theta_o}\right)\left(\frac{\partial r}{\partial r_o}\right) + \left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)\left(\frac{r\partial\theta}{\partial r_o}\right) + \left(\frac{\partial z}{r_o\partial\theta_o}\right)\left(\frac{\partial z}{\partial r_o}\right)$

Ans. $C_{r_o\theta_o} = \mathbf{e}_r^o \cdot \mathbf{C}\mathbf{e}_\theta^o = \mathbf{e}_r^o \cdot \mathbf{F}^T\mathbf{F}\mathbf{e}_\theta^o$. Now, $\mathbf{F}\mathbf{e}_\theta^o = \frac{\partial r}{r_o\partial\theta_o}\mathbf{e}_r + \frac{r\partial\theta}{r_o\partial\theta_o}\mathbf{e}_\theta + \frac{\partial z}{r_o\partial\theta_o}\mathbf{e}_z$[Eq.(3.29.3)],

$$C_{r_o\theta_o} = \mathbf{e}_r^o \cdot \mathbf{F}^T\left(\frac{\partial r}{r_o\partial\theta_o}\mathbf{e}_r + \frac{r\partial\theta}{r_o\partial\theta_o}\mathbf{e}_\theta + \frac{\partial z}{r_o\partial\theta_o}\mathbf{e}_z\right) = \frac{\partial r}{r_o\partial\theta_o}\mathbf{e}_r^o \cdot \mathbf{F}^T\mathbf{e}_r + \frac{r\partial\theta}{r_o\partial\theta_o}\mathbf{e}_r^o \cdot \mathbf{F}^T\mathbf{e}_\theta$$

$+\frac{\partial z}{r_o\partial\theta_o}\mathbf{e}_r^o \cdot \mathbf{F}^T\mathbf{e}_z$. From Eqs. (3.29.14), (3.29.15) and (3.29.16)

$\mathbf{e}_r^o \cdot \mathbf{F}^T\mathbf{e}_r = \frac{\partial r}{\partial r_o}, \mathbf{e}_r^o \cdot \mathbf{F}^T\mathbf{e}_\theta = \frac{r\partial\theta}{\partial r_o}, \mathbf{e}_r^o \cdot \mathbf{F}^T\mathbf{e}_z = \frac{\partial z}{\partial r_o}$,

Thus, $C_{r_o\theta_o} = \left(\frac{\partial r}{r_o\partial\theta_o}\right)\left(\frac{\partial r}{\partial r_o}\right) + \left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)\left(\frac{r\partial\theta}{\partial r_o}\right) + \left(\frac{\partial z}{r_o\partial\theta_o}\right)\left(\frac{\partial z}{\partial r_o}\right)$.

3.80 Derive the components of $\mathbf{C}^{-1}$ with respect to the bases at $\mathbf{X}$.

Ans.

$$C^{-1}_{r_o r_o} = \mathbf{e}_r^o \cdot \left(\mathbf{F}^T\mathbf{F}\right)^{-1}\mathbf{e}_r^o = \mathbf{e}_r^o \cdot \mathbf{F}^{-1}\left(\mathbf{F}^{-1}\right)^T\mathbf{e}_r^o = \frac{\partial r_o}{\partial r}\mathbf{e}_r^o \cdot \mathbf{F}^{-1}\mathbf{e}_r + \frac{\partial r_o}{r\partial\theta}\mathbf{e}_r^o \cdot \mathbf{F}^{-1}\mathbf{e}_\theta + \frac{\partial r_o}{\partial z}\mathbf{e}_r^o \cdot \mathbf{F}^{-1}\mathbf{e}_z$$

$=\frac{\partial r_o}{\partial r}\frac{\partial r_o}{\partial r}+\frac{\partial r_o}{r\partial\theta}\frac{\partial r_o}{r\partial\theta}+\frac{\partial r_o}{\partial z}\frac{\partial r_o}{\partial z}$. [See Eqs.(3.29.30), (3.29.31) and (3.29.32)].

$$C^{-1}_{r_o\theta_o}=\mathbf{e}^o_r\cdot\left(\mathbf{F}^T\mathbf{F}\right)^{-1}\mathbf{e}^o_\theta=\mathbf{e}^o_r\cdot\mathbf{F}^{-1}\left(\mathbf{F}^{-1}\right)^T\mathbf{e}^o_\theta==\frac{r_o\partial\theta_o}{\partial r}\mathbf{e}^o_r\cdot\mathbf{F}^{-1}\mathbf{e}_r+\frac{r_o\partial\theta_o}{r\partial\theta}\mathbf{e}^o_r\cdot\mathbf{F}^{-1}\mathbf{e}_\theta+\frac{r_o\partial\theta_o}{\partial z}\mathbf{e}^o_r\cdot\mathbf{F}^{-1}\mathbf{e}_z$$

$$=\left(\frac{r_o\partial\theta_o}{\partial r}\right)\left(\frac{\partial r_o}{\partial r}\right)+\left(\frac{r_o\partial\theta_o}{r\partial\theta}\right)\left(\frac{\partial r_o}{r\partial\theta}\right)+\left(\frac{r_o\partial\theta_o}{\partial z}\right)\left(\frac{\partial r_o}{\partial z}\right).$$

The other components can be similarly derived.

3.81 Derive components of $\mathbf{B}$ with respect to the basis $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$ at $\mathbf{x}$ for the pathline equations given by $r=r(X,Y,Z,t),\quad \theta=\theta(X,Y,Z,t),\quad z=z(X,Y,Z,t)$.

Ans. From $d\mathbf{x}=dr\mathbf{e}_r+rd\theta\mathbf{e}_\theta+dz\mathbf{e}_z$ and $d\mathbf{X}=dX\mathbf{e}_X+dY\mathbf{e}_Y+dZ\mathbf{e}_Z$ and $r=r(X,Y,Z,t),\quad \theta=\theta(X,Y,Z,t),\quad z=z(X,Y,Z,t)$, we have,

$$d\mathbf{x}=\mathbf{F}d\mathbf{X}\rightarrow d\mathbf{x}=dr\mathbf{e}_r+rd\theta\mathbf{e}_\theta+dz\mathbf{e}_z=dX\mathbf{F}\mathbf{e}_X+dY\mathbf{F}\mathbf{e}_Y+dZ\mathbf{F}\mathbf{e}_Z$$

$$\rightarrow\left(\frac{\partial r}{\partial X}dX+\frac{\partial r}{\partial Y}dY+\frac{\partial r}{\partial Z}dZ\right)\mathbf{e}_r+\left(\frac{r\partial\theta}{\partial X}dX+\frac{r\partial\theta}{\partial Y}dY+\frac{r\partial\theta}{\partial Z}dZ\right)\mathbf{e}_\theta$$

$$+\left(\frac{\partial z}{\partial X}dX+\frac{\partial z}{\partial Y}dY+\frac{\partial z}{\partial Z}dZ\right)\mathbf{e}_z=dX\mathbf{F}\mathbf{e}_X+dY\mathbf{F}\mathbf{e}_Y+dZ\mathbf{F}\mathbf{e}_Z$$

$$\rightarrow\mathbf{F}\mathbf{e}_X=\frac{\partial r}{\partial X}\mathbf{e}_r+\frac{r\partial\theta}{\partial X}\mathbf{e}_\theta+\frac{\partial z}{\partial X}\mathbf{e}_z,\quad \mathbf{F}\mathbf{e}_Y=\frac{\partial r}{\partial Y}\mathbf{e}_r+\frac{r\partial\theta}{\partial Y}\mathbf{e}_\theta+\frac{\partial z}{\partial Y}\mathbf{e}_z,$$

$$\mathbf{F}\mathbf{e}_Z=\frac{\partial r}{\partial Z}\mathbf{e}_r+\frac{r\partial\theta}{\partial Z}\mathbf{e}_\theta+\frac{\partial z}{\partial Z}\mathbf{e}_z\text{, and}$$

$$\mathbf{e}_X\cdot\mathbf{F}^T\mathbf{e}_r=\mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_X=\frac{\partial r}{\partial X},\quad \mathbf{e}_Y\cdot\mathbf{F}^T\mathbf{e}_r=\mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_Y=\frac{\partial r}{\partial Y},etc.$$

$$\mathbf{F}^T\mathbf{e}_r=\frac{\partial r}{\partial X}\mathbf{e}_X+\frac{\partial r}{\partial Y}\mathbf{e}_Y+\frac{\partial r}{\partial Z}\mathbf{e}_Z,\ \mathbf{F}^T\mathbf{e}_\theta=\frac{r\partial\theta}{\partial X}\mathbf{e}_X+\frac{r\partial\theta}{\partial Y}\mathbf{e}_Y+\frac{r\partial\theta}{\partial Z}\mathbf{e}_Z,$$

$$\mathbf{F}^T\mathbf{e}_z=\frac{\partial z}{\partial X}\mathbf{e}_X+\frac{\partial z}{\partial Y}\mathbf{e}_Y+\frac{\partial z}{\partial Z}\mathbf{e}_Z.$$

The components of $\mathbf{B}$ are:

$$B_{rr}=\mathbf{e}_r\cdot\mathbf{F}\mathbf{F}^T\mathbf{e}_r=\frac{\partial r}{\partial X}\mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_X+\frac{\partial r}{\partial Y}\mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_Y+\frac{\partial r}{\partial Z}\mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_Z=\left(\frac{\partial r}{\partial X}\right)^2+\left(\frac{\partial r}{\partial Y}\right)^2+\left(\frac{\partial r}{\partial Z}\right)^2.$$

$$B_{r\theta}=\mathbf{e}_r\cdot\mathbf{F}\mathbf{F}^T\mathbf{e}_\theta=\frac{r\partial\theta}{\partial X}\mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_X+\frac{r\partial\theta}{\partial Y}\mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_Y+\frac{r\partial\theta}{\partial Z}\mathbf{e}_r\cdot\mathbf{F}\mathbf{e}_Z$$

$$=\left(\frac{\partial r}{\partial X}\right)\left(\frac{r\partial\theta}{\partial X}\right)+\left(\frac{\partial r}{\partial Y}\right)\left(\frac{r\partial\theta}{\partial Y}\right)+\left(\frac{\partial r}{\partial Z}\right)\left(\frac{r\partial\theta}{\partial Z}\right).$$

3.82 Derive the components of $\mathbf{B}^{-1}$ with respect to the basis $\{\mathbf{e}_r,\mathbf{e}_\theta,\mathbf{e}_z\}$ at $\mathbf{x}$ for the pathline equations given by $X=X(r,\theta,z,t),\quad Y=Y(r,\theta,z,t),\quad Z=Z(r,\theta,z,t)$.

Ans. From $d\mathbf{x}=dr\mathbf{e}_r+rd\theta\mathbf{e}_\theta+dz\mathbf{e}_z$ and $d\mathbf{X}=dX\mathbf{e}_X+dY\mathbf{e}_Y+dZ\mathbf{e}_Z$ and

$X = X(r,\theta,z,t), \quad Y = Y(r,\theta,z,t), \quad Z = Z(r,\theta,z,t)$, we have,

$$d\mathbf{X} = \mathbf{F}^{-1}d\mathbf{x} \to dX\mathbf{e}_{\text{X}} + dY\mathbf{e}_{\text{Y}} + dZ\mathbf{e}_{\text{Z}} = dr\mathbf{F}^{-1}\mathbf{e}_{\text{r}} + rd\theta\mathbf{F}^{-1}\mathbf{e}_{\theta} + dz\mathbf{F}^{-1}\mathbf{e}_{\text{z}}$$

$$\to\left(\frac{\partial X}{\partial r}dr + \frac{\partial X}{\partial\theta}d\theta + \frac{\partial X}{\partial z}dz\right)\mathbf{e}_{\text{X}} + \left(\frac{\partial Y}{\partial r}dr + \frac{\partial Y}{\partial\theta}d\theta + \frac{\partial Y}{\partial z}dz\right)\mathbf{e}_{\text{Y}}$$

$$+\left(\frac{\partial Z}{\partial r}dr + \frac{\partial Z}{\partial\theta}d\theta + \frac{\partial Z}{\partial z}dz\right)\mathbf{e}_{\text{Z}} = dr\mathbf{F}^{-1}\mathbf{e}_{\text{r}} + rd\theta\mathbf{F}^{-1}\mathbf{e}_{\theta} + dz\mathbf{F}^{-1}\mathbf{e}_{\text{z}}.$$

Thus,

$$\mathbf{F}^{-1}\mathbf{e}_{\text{r}} = \frac{\partial X}{\partial r}\mathbf{e}_{\text{X}} + \frac{\partial Y}{\partial r}\mathbf{e}_{\text{Y}} + \frac{\partial Z}{\partial r}\mathbf{e}_{\text{Z}}, \quad \mathbf{F}^{-1}\mathbf{e}_{\theta} = \frac{\partial X}{r\partial\theta}\mathbf{e}_{\text{X}} + \frac{\partial Y}{r\partial\theta}\mathbf{e}_{\text{Y}} + \frac{\partial Z}{r\partial\theta}\mathbf{e}_{\text{Z}}$$

$$\mathbf{F}^{-1}\mathbf{e}_{\text{z}} = \frac{\partial X}{\partial z}\mathbf{e}_{\text{X}} + \frac{\partial Y}{\partial z}\mathbf{e}_{\text{Y}} + \frac{\partial Z}{\partial z}\mathbf{e}_{\text{Z}}.$$

and

$$\mathbf{e}_{\text{r}}\cdot\left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{e}_{\text{X}} = \mathbf{e}_{\text{X}}\cdot\mathbf{F}^{-1}\mathbf{e}_{\text{r}} = \frac{\partial X}{\partial r}, \quad \mathbf{e}_{\text{r}}\cdot\left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{e}_{\text{Y}} = \mathbf{e}_{\text{Y}}\cdot\mathbf{F}^{-1}\mathbf{e}_{\text{r}} = \frac{\partial Y}{\partial r},$$

$$\mathbf{e}_{\text{r}}\cdot\left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{e}_{\text{Z}} = \mathbf{e}_{\text{Z}}\cdot\mathbf{F}^{-1}\mathbf{e}_{\text{r}} = \frac{\partial Z}{\partial r}, \quad \text{etc. that is,}$$

$$\left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{e}_{\text{X}} = \frac{\partial X}{\partial r}\mathbf{e}_{\text{r}} + \frac{\partial X}{r\partial\theta}\mathbf{e}_{\theta} + \frac{\partial X}{\partial z}\mathbf{e}_{\text{z}}, \quad \left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{e}_{\text{Y}} = \frac{\partial Y}{\partial r}\mathbf{e}_{\text{r}} + \frac{\partial Y}{r\partial\theta}\mathbf{e}_{\theta} + \frac{\partial Y}{\partial z}\mathbf{e}_{\text{z}}$$

$$\left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{e}_{\text{Z}} = \frac{\partial Z}{\partial r}\mathbf{e}_{\text{r}} + \frac{\partial Z}{r\partial\theta}\mathbf{e}_{\theta} + \frac{\partial Z}{\partial z}\mathbf{e}_{\text{z}}.$$

Thus,

$$B_{rr}^{-1} = \mathbf{e}_{\text{r}}\cdot\left(\mathbf{F}\mathbf{F}^{\text{T}}\right)^{-1}\mathbf{e}_{\text{r}} = \mathbf{e}_{\text{r}}\cdot\left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{F}^{-1}\mathbf{e}_{\text{r}} = \frac{\partial X}{\partial r}\mathbf{e}_{\text{r}}\cdot\left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{e}_{\text{X}} + \frac{\partial Y}{\partial r}\mathbf{e}_{\text{r}}\cdot\left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{e}_{\text{Y}}$$

$$+\frac{\partial Z}{\partial r}\mathbf{e}_{\text{r}}\cdot\left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{e}_{\text{Z}} = \left(\frac{\partial X}{\partial r}\right)^2 + \left(\frac{\partial Y}{\partial r}\right)^2 + \left(\frac{\partial Z}{\partial r}\right)^2.$$

$$B_{r\theta}^{-1} = \mathbf{e}_{\text{r}}\cdot\left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{F}^{-1}\mathbf{e}_{\theta} = \frac{\partial X}{r\partial\theta}\mathbf{e}_{\text{r}}\cdot\left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{e}_{\text{X}} + \frac{\partial Y}{r\partial\theta}\mathbf{e}_{\text{r}}\cdot\left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{e}_{\text{Y}}$$

$$+\frac{\partial Z}{r\partial\theta}\mathbf{e}_{\text{r}}\cdot\left(\mathbf{F}^{-1}\right)^{\text{T}}\mathbf{e}_{\text{Z}} = \left(\frac{\partial X}{\partial r}\right)\left(\frac{\partial X}{r\partial\theta}\right) + \left(\frac{\partial Y}{\partial r}\right)\left(\frac{\partial Y}{r\partial\theta}\right) + \left(\frac{\partial Z}{\partial r}\right)\left(\frac{\partial Z}{r\partial\theta}\right).,$$

etc.

3.83 Verify that (a) the components of $\mathbf{B}$ with respect to $\{\mathbf{e}_{\text{r}}, \mathbf{e}_{\theta}, \mathbf{e}_{\text{z}}\}$ can be obtained from $\left[\mathbf{F}\mathbf{F}^{\text{T}}\right]$ and (b) the component of $\mathbf{C}$, with respect to $\{\mathbf{e}_{\text{r}}^{\text{o}}, \mathbf{e}_{\theta}^{\text{o}}, \mathbf{e}_{\text{z}}^{\text{o}}\}$ can be obtained from $\left[\mathbf{F}^{\text{T}}\mathbf{F}\right]$, where $[\mathbf{F}]$ is the matrix of the two points deformation gradient tensor given in Eq. (3.29.12).

Ans.

(a) Eq. (3.29.12) $\to$

$$\left[\mathbf{F}\mathbf{F}^{\mathrm{T}}\right]=\begin{bmatrix}\frac{\partial r}{\partial r_o} & \frac{\partial r}{r_o\partial\theta_o} & \frac{\partial r}{\partial z_o}\\ \frac{r\partial\theta}{\partial r_o} & \frac{r\partial\theta}{r_o\partial\theta_o} & \frac{r\partial\theta}{\partial z_o}\\ \frac{\partial z}{\partial r_o} & \frac{\partial z}{r_o\partial\theta_o} & \frac{\partial z}{\partial z_o}\end{bmatrix}\begin{bmatrix}\frac{\partial r}{\partial r_o} & \frac{r\partial\theta}{\partial r_o} & \frac{\partial z}{\partial r_o}\\ \frac{\partial r}{r_o\partial\theta_o} & \frac{r\partial\theta}{r_o\partial\theta_o} & \frac{\partial z}{r_o\partial\theta_o}\\ \frac{\partial r}{\partial z_o} & \frac{r\partial\theta}{\partial z_o} & \frac{\partial z}{\partial z_o}\end{bmatrix}\rightarrow$$

$$B_{rr}=\left(\frac{\partial r}{\partial r_o}\right)^2+\left(\frac{\partial r}{r_o\partial\theta_o}\right)^2+\left(\frac{\partial r}{\partial z_o}\right)^2,\quad B_{r\theta}=\frac{\partial r}{\partial r_o}\frac{r\partial\theta}{\partial r_o}+\frac{\partial r}{r_o\partial\theta_o}\frac{r\partial\theta}{r_o\partial\theta_o}+\frac{\partial r}{\partial z_o}\frac{r\partial\theta}{\partial z_o}\ \text{etc.}$$

(b)

$$\left[\mathbf{F}^{\mathrm{T}}\mathbf{F}\right]=\begin{bmatrix}\frac{\partial r}{\partial r_o} & \frac{r\partial\theta}{\partial r_o} & \frac{\partial z}{\partial r_o}\\ \frac{\partial r}{r_o\partial\theta_o} & \frac{r\partial\theta}{r_o\partial\theta_o} & \frac{\partial z}{r_o\partial\theta_o}\\ \frac{\partial r}{\partial z_o} & \frac{r\partial\theta}{\partial z_o} & \frac{\partial z}{\partial z_o}\end{bmatrix}\begin{bmatrix}\frac{\partial r}{\partial r_o} & \frac{\partial r}{r_o\partial\theta_o} & \frac{\partial r}{\partial z_o}\\ \frac{r\partial\theta}{\partial r_o} & \frac{r\partial\theta}{r_o\partial\theta_o} & \frac{r\partial\theta}{\partial z_o}\\ \frac{\partial z}{\partial r_o} & \frac{\partial z}{r_o\partial\theta_o} & \frac{\partial z}{\partial z_o}\end{bmatrix}\rightarrow$$

$$C_{r_or_o}=\left(\frac{\partial r}{\partial r_o}\right)^2+\left(\frac{r\partial\theta}{\partial r_o}\right)^2+\left(\frac{\partial z}{\partial r_o}\right)^2,\quad C_{r_o\theta_o}=\left(\frac{\partial r}{\partial r_o}\right)\left(\frac{\partial r}{r_o\partial\theta_o}\right)+\left(\frac{r\partial\theta}{\partial r_o}\right)\left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)+\left(\frac{\partial z}{\partial r_o}\right)\left(\frac{\partial z}{r_o\partial\theta_o}\right).$$

3.84 Given $r=r_o,\ \ \theta=\theta_o+kz_o,\ \ z=z_o$. (a) Obtain the components of the Left Cauchy-Green tensor $\mathbf{B}$, with respect to the basis at the current configuration (r,θ,z). (b) Obtain the components of the right Cauchy-Green tensor $\mathbf{C}$ with respect to the basis at the reference configuration.

Ans. (a), Using Eqs (3.29.19) to (3.29.24). we obtain

$$B_{rr}=\left(\frac{\partial r}{\partial r_o}\right)^2+\left(\frac{\partial r}{r_o\partial\theta_o}\right)^2+\left(\frac{\partial r}{\partial z_o}\right)^2=1,$$

$$B_{\theta\theta}=\left(\frac{r\partial\theta}{\partial r_o}\right)^2+\left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)^2+\left(\frac{r\partial\theta}{\partial z_o}\right)^2=\left(\frac{r}{r_o}\right)^2+(kr)^2=1+(kr)^2,$$

$$B_{zz}=\left(\frac{\partial z}{\partial r_o}\right)^2+\left(\frac{\partial z}{r_o\partial\theta_o}\right)^2+\left(\frac{\partial z}{\partial z_o}\right)^2=1,$$

$$B_{r\theta}=\left(\frac{r\partial\theta}{\partial r_o}\right)\left(\frac{\partial r}{\partial r_o}\right)+\left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)\left(\frac{\partial r}{r_o\partial\theta_o}\right)+\left(\frac{r\partial\theta}{\partial z_o}\right)\left(\frac{\partial r}{\partial z_o}\right)=0,$$

$$B_{rz}=\left(\frac{\partial r}{\partial r_o}\right)\left(\frac{\partial z}{\partial r_o}\right)+\left(\frac{\partial r}{r_o\partial\theta_o}\right)\left(\frac{\partial z}{r_o\partial\theta_o}\right)+\left(\frac{\partial r}{\partial z_o}\right)\left(\frac{\partial z}{\partial z_o}\right)=0,$$

$$B_{z\theta}=\left(\frac{\partial z}{\partial r_o}\right)\left(\frac{r\partial\theta}{\partial r_o}\right)+\left(\frac{\partial z}{r_o\partial\theta_o}\right)\left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)+\left(\frac{\partial z}{\partial z_o}\right)\left(\frac{r\partial\theta}{\partial z_o}\right)=rk.$$

.

Thus, $\left[\mathbf{B}\right]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1+\left(rk\right)^2 & rk\\ 0 & rk & 1\end{bmatrix}$.

(b) Using Eqs.(3. 29.43) to (3. 29.51), we have,

$$C_{r_o r_o}=\left(\frac{\partial r}{\partial r_o}\right)^2+\left(\frac{r\partial\theta}{\partial r_o}\right)^2+\left(\frac{\partial z}{\partial r_o}\right)^2=1,\ C_{\theta_o\theta_o}=\left(\frac{\partial r}{r_o\partial\theta_o}\right)^2+\left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)^2+\left(\frac{\partial z}{r_o\partial\theta_o}\right)^2=1,$$

$$C_{z_o z_o}=\left(\frac{\partial r}{\partial z_o}\right)^2+\left(\frac{r\partial\theta}{\partial z_o}\right)^2+\left(\frac{\partial z}{\partial z_o}\right)^2=1+\left(rk\right)^2,$$

$$C_{r_o\theta_o}=\left(\frac{\partial r}{r_o\partial\theta_o}\right)\left(\frac{\partial r}{\partial r_o}\right)+\left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)\left(\frac{r\partial\theta}{\partial r_o}\right)+\left(\frac{\partial z}{r_o\partial\theta_o}\right)\left(\frac{\partial z}{\partial r_o}\right)=0,$$

$$C_{r_o z_o}=\left(\frac{\partial r}{\partial r_o}\right)\left(\frac{\partial r}{\partial z_o}\right)+\left(\frac{r\partial\theta}{\partial r_o}\right)\left(\frac{r\partial\theta}{\partial z_o}\right)+\left(\frac{\partial z}{\partial r_o}\right)\left(\frac{\partial z}{\partial z_o}\right)=0,$$

$$C_{z_o\theta_o}=\left(\frac{\partial r}{\partial z_o}\right)\left(\frac{\partial r}{r_o\partial\theta_o}\right)+\left(\frac{r\partial\theta}{\partial z_o}\right)\left(\frac{r\partial\theta}{r_o\partial\theta_o}\right)+\left(\frac{\partial z}{\partial z_o}\right)\left(\frac{\partial z}{r_o\partial\theta_o}\right)=rk.$$

Thus, $\left[\mathbf{C}\right]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & rk\\ 0 & rk & 1+\left(rk\right)^2\end{bmatrix}$.

3.85 Given $r=\left(2aX+b\right)^{1/2}$, $\theta=Y/a$, $z=Z$, where $\left(r,\theta,z\right)$ are cylindrical coordinates for the cuurent configuration and $\left(X,Y,Z\right)$ are rectangular coordinates for the reference configuration. (a) Obtain the components of $\left[\mathbf{B}\right]$ with respect to the basis at the current configuration and (b) calculate the change of volume.

Ans. (a) Using Eqs.(3.29.59) to (3.29.64), we have,

$$B_{rr}=\left(\frac{\partial r}{\partial X}\right)^2+\left(\frac{\partial r}{\partial Y}\right)^2+\left(\frac{\partial r}{\partial Z}\right)^2=\left(\frac{a}{r}\right)^2,\ B_{\theta\theta}=\left(\frac{r\partial\theta}{\partial X}\right)^2+\left(\frac{r\partial\theta}{\partial Y}\right)^2+\left(\frac{r\partial\theta}{\partial Z}\right)^2=\left(\frac{r}{a}\right)^2$$

$$B_{zz}=\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial Y}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2=1,$$

$$B_{r\theta}=\left(\frac{\partial r}{\partial X}\right)\left(\frac{r\partial\theta}{\partial X}\right)+\left(\frac{\partial r}{\partial Y}\right)\left(\frac{r\partial\theta}{\partial Y}\right)+\left(\frac{\partial r}{\partial Z}\right)\left(\frac{r\partial\theta}{\partial Z}\right)=0,$$

$$B_{rz}=\left(\frac{\partial r}{\partial X}\right)\left(\frac{\partial z}{\partial X}\right)+\left(\frac{\partial r}{\partial Y}\right)\left(\frac{\partial z}{\partial Y}\right)+\left(\frac{\partial r}{\partial Z}\right)\left(\frac{\partial z}{\partial Z}\right)=0,$$

$$B_{\theta z}=\left(\frac{r\partial\theta}{\partial X}\right)\left(\frac{\partial z}{\partial X}\right)+\left(\frac{r\partial\theta}{\partial Y}\right)\left(\frac{\partial z}{\partial Y}\right)+\left(\frac{r\partial\theta}{\partial Z}\right)\left(\frac{\partial z}{\partial Z}\right)=0.$$

Thus, $[\mathbf{B}] = \begin{bmatrix} (a/r)^2 & 0 & 0 \\ 0 & (r/a)^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $\det \mathbf{B} = 1 = 1$, thus, there is no change of volume.

3.86 Given $r = r(X),\quad \theta = g(Y),\quad z = h(Z)$, where (r,θ,z) and (X,Y,Z) are cylindrical and rectangular Cartesian coordinate with respect to the current and the reference configuration respectively. Obtain the components of the right Cauchy-Green Tensor $\mathbf{C}$ with respect to the basis at the reference configuration.

Ans. Using Eqs.(3.29.68) etc. we have,

$$C_{XX} = \left(\frac{\partial r}{\partial X}\right)^2 + \left(\frac{r\partial\theta}{\partial X}\right)^2 + \left(\frac{\partial z}{\partial X}\right)^2 = \left(r'(X)\right)^2,\ C_{YY} = \left(rg'(Y)\right)^2, C_{ZZ} = \left(h'(Z)\right)^2$$

$$C_{XY} = \left(\frac{\partial r}{\partial X}\right)\left(\frac{\partial r}{\partial Y}\right) + \left(\frac{r\partial\theta}{\partial X}\right)\left(\frac{r\partial\theta}{\partial Y}\right) + \left(\frac{\partial z}{\partial X}\right)\left(\frac{\partial z}{\partial Y}\right) = 0,\ C_{YZ} = 0,\ C_{XZ} = 0.$$

$$[\mathbf{C}] = \begin{bmatrix} (r'(X))^2 & 0 & 0 \\ 0 & (g'(Y))^2 & 0 \\ 0 & 0 & (h'(Z))^2 \end{bmatrix}, \text{ where } r'(X) \equiv dr/dX, etc.,$$

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4.1 The state of stress at a certain point in a body is given by: $[\mathbf{T}] = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 0 \end{bmatrix}_{\mathbf{e}_i}$ $MPa.$.

On each of the coordinate planes (with normal in $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ directions), (a) what is the normal stress and (b) what is the total shearing stress

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Ans. (a) The normal stress on the $\mathbf{e}_1$ plane (i.e., the plane whose normal is in the direction $\mathbf{e}_1$) is $1\ MPa.$, on the $\mathbf{e}_2$ plane is $4\ MPa.$, and on the $\mathbf{e}_3$ plane is $0\ MPa.$

(b) The total shearing stress on the $\mathbf{e}_1$ plane is $\sqrt{2^2+3^2}=\sqrt{13}$ =3.61 $MPa.$ On the $\mathbf{e}_2$ plane is $\sqrt{2^2+5^2}=\sqrt{29}$ =5.39 $MPa.$, and on the $\mathbf{e}_3$ plane is $\sqrt{3^2+5^2}=\sqrt{34}=5.83\ MPa$.

4.2 The state of stress at a certain point in a body is given by: $[\mathbf{T}] = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 0 \\ 3 & 0 & -1 \end{bmatrix}_{\mathbf{e}_i}$ $MPa.$

(a) Find the stress vector at a point on the plane whose normal is in the direction of $2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3$. (b) Determine the magnitude of the normal and shearing stresses on this plane.

Ans. (a) The stress vector on the plane is $\mathbf{t} = \mathbf{Tn}$, where $\mathbf{n} = (2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3)/3$. Thus

$$[\mathbf{t}] = \frac{1}{3}\begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 0 \\ 3 & 0 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}, \rightarrow \mathbf{t} = (5\mathbf{e}_1 + 6\mathbf{e}_2 + 5\mathbf{e}_3)/3\ MPa.$$

(b) Normal stress $T_n = \mathbf{n}\cdot\mathbf{t} = (1/9)(2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3)\cdot(5\mathbf{e}_1 + 6\mathbf{e}_2 + 5\mathbf{e}_3) = 3\ MPa.$

Magnitude of shearing stress $T_s = \sqrt{|\mathbf{t}|^2 - T_n^2} = \sqrt{86/9 - 9} = 0.745\ MPa.$ Or,

$\mathbf{T}_s = \mathbf{t} - T_n\mathbf{n} = (5\mathbf{e}_1 + 6\mathbf{e}_2 + 5\mathbf{e}_3)/3 - 3(1/3)(2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3) = (-\mathbf{e}_1 + 2\mathbf{e}_3)/3 \rightarrow T_s = \sqrt{5}/3 = 0.745$

4.3 Do the previous problem for a plane passing through the point and parallel to the plane $x_1 - 2x_2 + 3x_3 = 4$.

Ans. (a) The normal to the plane is $\mathbf{n} = (\mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3)/\sqrt{14}$. $[\mathbf{t}] = [\mathbf{T}][\mathbf{n}] \rightarrow$

$$\rightarrow [\mathbf{t}] = \frac{1}{\sqrt{14}}\begin{bmatrix} 2 & -1 & 3 \\ -1 & 4 & 0 \\ 3 & 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} = \frac{1}{\sqrt{14}}\begin{bmatrix} 13 \\ -9 \\ 0 \end{bmatrix}, \rightarrow \mathbf{t} = (1/\sqrt{14})(13\mathbf{e}_1 - 9\mathbf{e}_2) = 3.47\mathbf{e}_1 - 2.41\mathbf{e}_2\ MPa.$$

(b) Normal stress $T_n = \mathbf{n}\cdot\mathbf{t} = \frac{1}{14}\begin{bmatrix} 1 & -2 & 3 \end{bmatrix}\begin{bmatrix} 13 \\ -9 \\ 0 \end{bmatrix} = 31/14 = 2.21\ MPa.$

Magnitude of shearing stress $T_s = \sqrt{|\mathbf{t}|^2 - T_n^2} = \sqrt{250/14 - (2.21)^2} = 3.60\ MPa.$ Or,
$\mathbf{T}_s = \mathbf{t} - T_n\mathbf{n} = (3.47\mathbf{e}_1 - 2.41\mathbf{e}_2) - (2.21/\sqrt{14})(\mathbf{e}_1 - 2\mathbf{e}_2 + 3\mathbf{e}_3) = 2.88\mathbf{e}_1 - 1.23\mathbf{e}_2 - 1.77\mathbf{e}_3$,
$\rightarrow T_s = 3.60\ MPa.$

4.4 The stress distribution in a certain body is given by

$$[\mathbf{T}] = \begin{bmatrix} 0 & 100x_1 & -100x_2 \\ 100x_1 & 0 & 0 \\ -100x_2 & 0 & 0 \end{bmatrix} MPa.$$

Find the stress vector acting on a plane which passes through the point $\left(1/2,\ \sqrt{3}/2,\ 3\right)$ and is tangent to the circular cylindrical surface $x_1^2 + x_2^2 = 1$ at that point.

Ans. Let $f = x_1^2 + x_2^2$, then the unit normal to the circle $f = 1$ at a point (x_1, x_2) is given by
$\mathbf{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x_1\mathbf{e}_1 + 2x_2\mathbf{e}_2}{\sqrt{4x_1^2 + 4x_2^2}} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2$. At the point $\left(1/2,\ \sqrt{3}/2,\ 3\right)$, $\mathbf{n} = \frac{1}{2}\left(\mathbf{e}_1 + \sqrt{3}\mathbf{e}_2\right)$.

and $[\mathbf{T}] = \begin{bmatrix} 0 & 50 & -50\sqrt{3} \\ 50 & 0 & 0 \\ -50\sqrt{3} & 0 & 0 \end{bmatrix}$, thus,

$$[\mathbf{t}] = \begin{bmatrix} 0 & 50 & -50\sqrt{3} \\ 50 & 0 & 0 \\ -50\sqrt{3} & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/2 \\ \sqrt{3}/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 25\sqrt{3} \\ 25 \\ -25\sqrt{3} \end{bmatrix} \rightarrow \mathbf{t} = 25\sqrt{3}\mathbf{e}_1 + 25\mathbf{e}_2 - 25\sqrt{3}\mathbf{e}_3\ MPa.$$

4.5 Given $T_{11} = 1\ MPa.$, $T_{22} = -1\ MPa.$, and all other $T_{ij} = 0$ at a point in a continuum. (a) Show that the only plane on which the stress vector is zero is the plane with normal in the $\mathbf{e}_3$ direction. (b) Give three planes on which there is no normal stress acting.

Ans. (a) $[\mathbf{t}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} n_1 \\ -n_2 \\ 0 \end{bmatrix} \rightarrow \mathbf{t} = n_1\mathbf{e}_1 - n_2\mathbf{e}_2$. $\mathbf{t} = \mathbf{0} \rightarrow n_1 = n_2 = 0 \rightarrow \mathbf{t} = \mathbf{e}_3$.

(b) $T_n = \mathbf{n} \cdot \mathbf{t} = n_1^2 - n_2^2$. Thus, the plane with $n_1^2 - n_2^2 = 0$ has no normal stress. These include $\mathbf{n} = \mathbf{e}_3$, $\mathbf{n} = (\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2}$, $\mathbf{n} = (\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$ etc.

4.6 For the following state of stress $[\mathbf{T}] = \begin{bmatrix} 10 & 50 & -50 \\ 50 & 0 & 0 \\ -50 & 0 & 0 \end{bmatrix} MPa.$, find T'_{11} and T'_{13} where $\mathbf{e}'_1$ is in the direction of $\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3$ and $\mathbf{e}'_2$ is in the direction of $\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3$.

Ans. $\mathbf{e}'_1 = (\mathbf{e}_1 + 2\mathbf{e}_2 + 3\mathbf{e}_3)/\sqrt{14}$, $\quad \mathbf{e}'_2 = (\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3)/\sqrt{3}$, thus

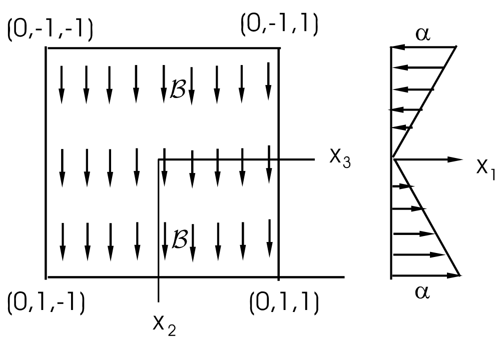
$$T'_{11} = \frac{1}{14}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}10 & 50 & -50\\ 50 & 0 & 0\\ -50 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\2\\3\end{bmatrix} = \frac{1}{14}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}-40\\ 50\\ -50\end{bmatrix} = -90/14 = -6.43\ MPa.$$

$\mathbf{e}'_3 = \mathbf{e}'_1 \times \mathbf{e}'_2 = (-5\mathbf{e}_1 + 4\mathbf{e}_2 - \mathbf{e}_3)/\sqrt{42}$, therefore,

$$T'_{13} = \frac{1}{\sqrt{588}}\begin{bmatrix}1 & 2 & 3\end{bmatrix}\begin{bmatrix}10 & 50 & -50\\ 50 & 0 & 0\\ -50 & 0 & 0\end{bmatrix}\begin{bmatrix}-5\\4\\-1\end{bmatrix} = 450/\sqrt{588} = 18.6\ MPa.$$

4.7 Consider the following stress distribution $[\mathbf{T}] = \begin{bmatrix}\alpha x_2 & \beta & 0\\ \beta & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ where α and β are constants. (a) Determine and sketch the distribution of the stress vector acting on the square in the $x_1 = 0$ plane with vertices located at $(0,1,1),(0,-1,1),(0,1,-1)$ and $(0,-1,-1)$. (b) Find the total resultant force and moment about the origin of the stress vectors acting on the square of part (a).

Ans. (a) The normal to the plane $x_1 = 0$ is $\mathbf{e}_1$, thus, $\mathbf{t}_{\mathbf{e}_1} = \alpha x_2 \mathbf{e}_1 + \beta \mathbf{e}_2$. On the plane, there is a constant shearing stress β in the $\mathbf{e}_2$ direction and a linear distribution of normal stress αx_2, (see figure).



(b)

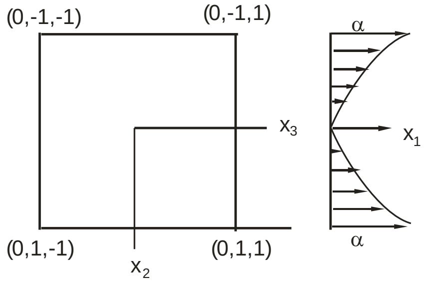
$$\mathbf{F}_R = \int \mathbf{t}dA = \int_{-1}^{1}\int_{-1}^{1}(\alpha x_2 \mathbf{e}_1 + \beta \mathbf{e}_2)dx_2 dx_3 = 0\mathbf{e}_1 + 4\beta \mathbf{e}_2.$$

$$\mathbf{M}_o = \int \mathbf{x}\times\mathbf{t}dA = \int_{-1}^{1}\int_{-1}^{1}(x_2\mathbf{e}_2 + x_3\mathbf{e}_3)\times(\alpha x_2\mathbf{e}_1 + \beta\mathbf{e}_2)dx_2 dx_3$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left(-\beta x_3\mathbf{e}_1 + \alpha x_2 x_3 \mathbf{e}_2 - \alpha x_2^2 \mathbf{e}_3\right)dx_2 dx_3 = 0\mathbf{e}_1 + 0\mathbf{e}_2 - \alpha(2)\left[\frac{x_2^3}{3}\right]_{-1}^{1}\mathbf{e}_3 = -\frac{4\alpha}{3}\mathbf{e}_3$$

4.8 Do the previous problem if the stress distribution is given by $T_{11} = \alpha x_2^2$ and all other $T_{ij} = 0$.

Ans. (a) The normal to the plane $x_1 = 0$ is $\mathbf{e}_1$, thus, $\mathbf{t}_{\mathbf{e}_1} = \alpha x_2^2 \mathbf{e}_1$. On the plane, there is a parabolic distribution of normal stress αx_2^2, (see figure).



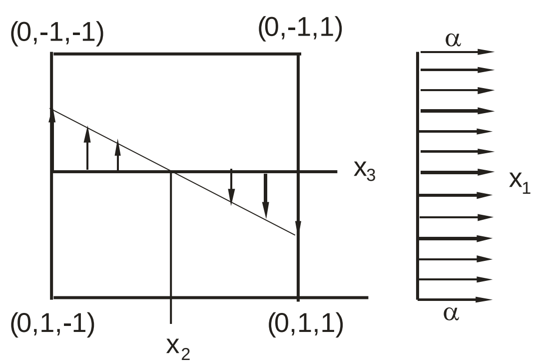
(b) $\mathbf{F}_R = \int \mathbf{t} dA = \int_{-1}^{1}\int_{-1}^{1} \left(\alpha x_2^2 \mathbf{e}_1\right) dx_2 dx_3 = \frac{4\alpha}{3}\mathbf{e}_1$.

$$\mathbf{M}_o = \int \mathbf{x} \times \mathbf{t} dA = \int_{-1}^{1}\int_{-1}^{1} \left(x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3\right) \times \left(\alpha x_2^2 \mathbf{e}_1\right) dx_2 dx_3$$

$$= \int_{-1}^{1}\int_{-1}^{1} \left(\alpha x_2^2 x_3 \mathbf{e}_2 - \alpha x_2^3 \mathbf{e}_3\right) dx_2 dx_3 = 0\mathbf{e}_2 + 0\mathbf{e}_3$$

4.9 Do problem 4.7 for the stress distribution: $T_{11} = \alpha,\ T_{12} = T_{21} = \alpha x_3$ and all other $T_{ij} = 0$.

Ans. (a) The normal to the plane $x_1 = 0$ is $\mathbf{e}_1$, thus, $\mathbf{t}_{\mathbf{e}_1} = \alpha \mathbf{e}_1 + \alpha x_3 \mathbf{e}_2$. On the plane, there is a constant normal stress of α and a linear distribution of shearing stress $\alpha x_3 \mathbf{e}_2$, (see figure).



(b) $\mathbf{F}_R = \int \mathbf{t} dA = \int_{-1}^{1}\int_{-1}^{1} \left(\alpha \mathbf{e}_1 + \alpha x_3 \mathbf{e}_2\right) dx_2 dx_3 = 4\alpha \mathbf{e}_1$.

$$\mathbf{M}_o = \int \mathbf{x}\times\mathbf{t}dA = \int_{-1}^{1}\int_{-1}^{1}\left(x_2\mathbf{e}_2 + x_3\mathbf{e}_3\right)\times\left(\alpha\mathbf{e}_1 + \alpha x_3\mathbf{e}_2\right)dx_2dx_3$$

$$= \int_{-1}^{1}\int_{-1}^{1}\left(-\alpha x_3^2\mathbf{e}_1 + \alpha x_3\mathbf{e}_2 - \alpha x_2\mathbf{e}_3\right)dx_2dx_3 = -\frac{4\alpha}{3}\mathbf{e}_1$$

4.10 Consider the following stress distribution for a circular cylindrical bar:

$$[\mathbf{T}] = \begin{bmatrix} 0 & -\alpha x_3 & \alpha x_2 \\ -\alpha x_3 & 0 & 0 \\ \alpha x_2 & 0 & 0 \end{bmatrix}$$

(a) What is the distribution of the stress vector on the surfaces defined by (i) the lateral surface $x_2^2 + x_3^2 = 4$, (ii) the end face $x_1 = 0$, and (iii) the end face $x_1 = \ell$? (b) Find the total resultant force and moment on the end face $x_1 = \ell$.

Ans. (a) The outward unit normal vector to the lateral surface $x_2^2 + x_3^2 = 4$ is given by $\mathbf{n}_1 = \dfrac{x_2\mathbf{e}_2 + x_3\mathbf{e}_3}{2}$. The outward unit normal vector to $x_1 = 0$ is $\mathbf{n}_2 = -\mathbf{e}_1$ and that to $x_1 = \ell$ is $\mathbf{n}_3 = \mathbf{e}_1$. Thus,

$$\left[\mathbf{t}_{\mathbf{n}_1}\right] = \frac{1}{2}\begin{bmatrix} 0 & -\alpha x_3 & \alpha x_2 \\ -\alpha x_3 & 0 & 0 \\ \alpha x_2 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \mathbf{t}_{\mathbf{n}_1} = 0 .$$

$$\mathbf{t}_{\mathbf{n}_2} = -\mathbf{T}\mathbf{e}_1 = -\left(-\alpha x_3\mathbf{e}_2 + \alpha x_2\mathbf{e}_3\right) = \alpha x_3\mathbf{e}_2 - \alpha x_2\mathbf{e}_3$$

$$\mathbf{t}_{\mathbf{n}_3} = \mathbf{T}\mathbf{e}_1 = \left(-\alpha x_3\mathbf{e}_2 + \alpha x_2\mathbf{e}_3\right) = -\alpha x_3\mathbf{e}_2 + \alpha x_2\mathbf{e}_3$$

(b) On the end face $x_1 = \ell$, $\mathbf{t}_{\mathbf{n}_3} = \mathbf{T}\mathbf{e}_1 = -\alpha x_3\mathbf{e}_2 + \alpha x_2\mathbf{e}_3$

$$\mathbf{F}_R = \int \mathbf{t}_{\mathbf{n}_3}dA = \int\left(-\alpha x_3\mathbf{e}_2 + \alpha x_2\mathbf{e}_3\right)dA = \left(-\alpha\mathbf{e}_2\right)\int x_3dA + \left(\alpha\mathbf{e}_3\right)\int x_2dA = 0 .$$

[note: the axes are axes of symmetry, the integrals are clearly zero].

$$\mathbf{M}_o = \int \mathbf{x}\times\mathbf{t}dA = \int\left(x_2\mathbf{e}_2 + x_3\mathbf{e}_3\right)\times\left(-\alpha x_3\mathbf{e}_2 + \alpha x_2\mathbf{e}_3\right)dA$$

$$= \int\left(\alpha x_2^2 + \alpha x_3^2\right)dA\mathbf{e}_1 = \alpha\int_0^2 r^2 2\pi r dr\mathbf{e}_1 = 2\pi\alpha\int_0^2 r^3 dr\mathbf{e}_1 = 8\pi\alpha\mathbf{e}_1$$

4.11 An elliptical bar with lateral surface defined by $x_2^2 + 2x_3^2 = 1$ has the following stress distribution: $[\mathbf{T}] = \begin{bmatrix} 0 & -2x_3 & x_2 \\ -2x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix}$. (a) Show that the stress vector at any point (x_1, x_2, x_3) on the lateral surface is zero. (b) Find the resultant force, and resultant moment, about the origin O, of the stress vector on the left end face $x_1 = 0$.

Note: $\int x_2^2 dA = \frac{\pi}{4\sqrt{2}}$ and $\int x_3^2 dA = \frac{\pi}{8\sqrt{2}}$.

Ans. (a) The outward unit normal vector to the lateral surface $x_2^2 + 2x_3^2 = 1$ is given by:

$$\mathbf{n}_1 = \frac{x_2\mathbf{e}_2 + 2x_3\mathbf{e}_3}{\sqrt{x_2^2 + 4x_3^2}}, \text{ thus, } [\mathbf{t}] = [\mathbf{T}][\mathbf{n}_1] = \begin{bmatrix} 0 & -2x_3 & x_2 \\ -2x_3 & 0 & 0 \\ x_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ x_2 \\ 2x_3 \end{bmatrix} \frac{1}{\sqrt{x_2^2 + 4x_3^2}} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

(b) On the left end face $x_1 = 0$, $\mathbf{n} = -\mathbf{e}_1$, the stress vector is $\mathbf{t} = 2x_3\mathbf{e}_2 - x_2\mathbf{e}_3$,

$$\mathbf{F}_R = \int \mathbf{t} dA = \int (2x_3\mathbf{e}_2 - x_2\mathbf{e}_3) dA = (2\mathbf{e}_2) \int x_3 dA - (\mathbf{e}_3) \int x_2 dA = 0.$$

[note: the axes are axes of symmetry, the integrals are clearly zero]]

$$\mathbf{M}_o = \int \mathbf{x} \times \mathbf{t} dA = \int (x_2\mathbf{e}_2 + x_3\mathbf{e}_3) \times (2x_3\mathbf{e}_2 - x_2\mathbf{e}_3) dA$$

$$= -\int (x_2^2 + 2x_3^2) \mathbf{e}_1 dA = -\left\{\frac{\pi}{4\sqrt{2}} + \frac{2\pi}{8\sqrt{2}}\right\} \mathbf{e}_1 = -\frac{\pi}{2\sqrt{2}} \mathbf{e}_1.$$

4.12 For any stress state $\mathbf{T}$, we define the deviatoric stress $\mathbf{S}$ to be $\mathbf{S} = \mathbf{T} - (T_{kk}/3)\mathbf{I}$, where T_{kk} is the first invariant of the stress tensor $\mathbf{T}$. (a) Show that the first invariant of the deviatoric stress vanishes. (b) Given the stress tensor $[\mathbf{T}] = 100\begin{bmatrix} 6 & 5 & -2 \\ 5 & 3 & 4 \\ -2 & 4 & 9 \end{bmatrix} kPa.$, evaluate $\mathbf{S}$. (c) Show that the principal directions of the stress tensor coincide with those of the deviatoric stress tensor.

Ans. (a) From $\mathbf{S} = \mathbf{T} - (T_{kk}/3)\mathbf{I}$, we have, $\text{tr}\mathbf{S} = \text{tr}\mathbf{T} - (T_{kk}/3)\text{tr}\mathbf{I} = T_{kk} - (T_{kk}/3)(3) = 0$.

(b) $$[\mathbf{S}] = 100\begin{bmatrix} 6 & 5 & -2 \\ 5 & 3 & 4 \\ -2 & 4 & 9 \end{bmatrix} - (1800/3)[\mathbf{I}] = \begin{bmatrix} 0 & 500 & -200 \\ 500 & -300 & 400 \\ -200 & 400 & 300 \end{bmatrix} kPa.$$

(c) Let $\mathbf{n}$ be an eigenvector of $\mathbf{T}$, then $\mathbf{Tn} = \lambda\mathbf{n}$. Now $\mathbf{Sn} = \mathbf{Tn} - (T_{kk}/3)\mathbf{In} = \lambda\mathbf{n} - (T_{kk}/3)\mathbf{n}$, that is $\mathbf{Sn} = \lambda'\mathbf{n}$ where $\lambda' = \lambda - (T_{kk}/3)$. Thus, $\mathbf{n}$ is also an eigenvector of $\mathbf{S}$ with eigenvalue $\lambda - (T_{kk}/3)$.

4.13 An octahedral stress plane is one whose normal makes equal angles with each of the principal axes of stress. (a) How many independent octahedral planes are there at each point? (b) Show that the normal stress on an octahedral plane is given by one-third the first stress invariant. (c) Show that the shearing stress on the octahedral plane is given by

$T_s = \frac{1}{3}\left[(T_1 - T_2)^2 + (T_2 - T_3)^2 + (T_3 - T_1)^2\right]^{1/2}$, where T_1, T_2, T_3 are principal values of the stress tensor.

Ans. (a) There are four independent octahedral planes. They are given by the following unit normal vectors:

$$\mathbf{n}_1 = \frac{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3}{\sqrt{3}}, \mathbf{n}_2 = \frac{\mathbf{e}_1 + \mathbf{e}_2 - \mathbf{e}_3}{\sqrt{3}}, \mathbf{n}_3 = \frac{\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3}{\sqrt{3}}, \mathbf{n}_4 = \frac{\mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3}{\sqrt{3}}$$

We note that $-\frac{\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3}{\sqrt{3}}$ gives the same plane as $\mathbf{n}_1$, etc.

(b) Using the principal directions as the orthonormal basis, the matrix of $\mathbf{T}$ is diagonal, i.e., $[\mathbf{T}] = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix}$. The normal to an octahedral plane is $\frac{\mathbf{e}_1 \pm \mathbf{e}_2 \pm \mathbf{e}_3}{\sqrt{3}}$, thus,

$T_n = \mathbf{n} \cdot \mathbf{Tn} = \frac{1}{3}[1 \quad \pm 1 \quad \pm 1] \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix} \begin{bmatrix} 1 \\ \pm 1 \\ \pm 1 \end{bmatrix}$ where in this equation, the row matrix and column matrix of $\mathbf{n}$ have the same elements, that is if the row matrix is $[1 \quad -1 \quad 1]$ then the column matrix is $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Thus, $T_n = \frac{1}{3}[1 \quad \pm 1 \quad \pm 1] \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix} \begin{bmatrix} 1 \\ \pm 1 \\ \pm 1 \end{bmatrix} = \frac{1}{3}(T_1 + T_2 + T_3)$.

(c) $T_s^2 = |\mathbf{t_n}|^2 - T_n^2 = \frac{1}{3}\left(T_1^2 + T_2^2 + T_3^3\right) - \frac{1}{9}\left(T_1^2 + T_2^2 + T_3^3 + 2T_1T_2 + 2T_1T_3 + 2T_2T_3\right)$

$= \frac{2}{9}\left(T_1^2 + T_2^2 + T_3^3 - T_1T_2 - T_1T_3 - T_2T_3\right) = \frac{1}{9}\left[(T_1 - T_2)^2 + (T_2 - T_3)^2 + (T_3 - T_1)^2\right]$

That is, $T_s = \frac{1}{3}\left[(T_1 - T_2)^2 + (T_2 - T_3)^2 + (T_3 - T_1)^2\right]^{1/2}$

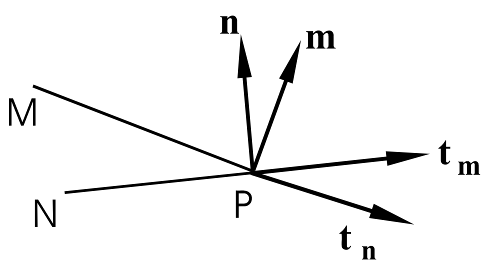
4.14 (a) Let $\mathbf{m}$ and $\mathbf{n}$ be two unit vectors that define two planes M and N that pass through a point P. For an arbitrary state of stress defined at the point P, show that the component of the stress vector $\mathbf{t_m}$ in the $\mathbf{n}$-direction is equal to the component of the stress vector $\mathbf{t_n}$ in the $\mathbf{m}$ direction. (b) If $\mathbf{m} = \mathbf{e}_1$ and $\mathbf{n} = \mathbf{e}_2$, what does the results of (a) reduce to?

Ans. (a) The component of the stress vector $\mathbf{t_m}$ in the $\mathbf{n}$-direction is $\mathbf{n} \cdot \mathbf{t_m} = \mathbf{n} \cdot \mathbf{Tm}$ and the component of the stress vector $\mathbf{t_n}$ in the $\mathbf{m}$ direction is $\mathbf{m} \cdot \mathbf{t_n} = \mathbf{m} \cdot \mathbf{Tn} = \mathbf{n} \cdot \mathbf{T}^{\mathrm{T}}\mathbf{m}$. Since $\mathbf{T}$ is symmetric, therefore, $\mathbf{n} \cdot \mathbf{T}^{\mathrm{T}}\mathbf{m} = \mathbf{n} \cdot \mathbf{Tm}$, therefore, $\mathbf{n} \cdot \mathbf{t_m} = \mathbf{m} \cdot \mathbf{t_n}$.

(b) If $\mathbf{m} = \mathbf{e}_1$ and $\mathbf{n} = \mathbf{e}_2$, then $\mathbf{e}_1 \cdot \mathbf{t}_{\mathbf{e}_2} = \mathbf{e}_2 \cdot \mathbf{t}_{\mathbf{e}_1} \rightarrow T_{12} = T_{21}$.

4.15 Let $\mathbf{m}$ be a unit vector that defines a plane M passing through a point P. Show that the stress vector on any plane that contains the stress traction $\mathbf{t_m}$, lies in the M plane.

Ans. Referring to the figure below, where $\mathbf{m}$ is perpendicular to the plane M, and $\mathbf{t_m}$ is the stress vector for the plane. Let N be any plane which contains the vector $\mathbf{t_m}$ and let $\mathbf{n}$ be the unit vector perpendicular to the plane N. Then $\mathbf{t_n} = \mathbf{Tn}$. We wish to show that $\mathbf{t_n}$ is perpendicular to $\mathbf{m}$.



Now, $\mathbf{t_n}\cdot\mathbf{m} = \mathbf{Tn}\cdot\mathbf{m} = \mathbf{n}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{m} = \mathbf{n}\cdot\mathbf{Tm} = \mathbf{n}\cdot\mathbf{t_m} = 0,$ because $\mathbf{t_m}$ is on the N plane.
Thus, $\mathbf{t_n}\cdot\mathbf{m} = 0,$ so that $\mathbf{t_n}$ lies on the M plane.

4.16 Let $\mathbf{t_m}$ and $\mathbf{t_n}$ be stress vectors on planes defined by the unit vector $\mathbf{m}$ and $\mathbf{n}$ respectively and pass through the point P. Show that if $\mathbf{k}$ is a unit vector that determines a plane that contains $\mathbf{t_m}$ and $\mathbf{t_n}$, then $\mathbf{t_k}$ is perpendicular to $\mathbf{m}$ and $\mathbf{n}$.

Ans. Since $\mathbf{k}$ is a unit vector that determines a plane that contains $\mathbf{t_m}$ **and** $\mathbf{t_n}$, therefore,

$\mathbf{k} = \dfrac{\mathbf{t_m}\times\mathbf{t_n}}{|\mathbf{t_m}\times\mathbf{t_n}|}$. Since $\mathbf{t_k} = \mathbf{Tk}$, $\mathbf{t_m} = \mathbf{Tm}$, and $\mathbf{t_n} = \mathbf{Tn}$, therefore,

$$\mathbf{m}\cdot\mathbf{t_k} = \mathbf{m}\cdot\mathbf{Tk} = \mathbf{k}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{m} = \mathbf{k}\cdot\mathbf{Tm} = \mathbf{k}\cdot\mathbf{t_m} = \frac{\mathbf{t_m}\cdot\mathbf{t_m}\times\mathbf{t_n}}{|\mathbf{t_m}\times\mathbf{t_n}|} = 0$$
, similarly,

$$\mathbf{n}\cdot\mathbf{t_k} = \mathbf{n}\cdot\mathbf{Tk} = \mathbf{k}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{n} = \mathbf{k}\cdot\mathbf{Tn} = \mathbf{k}\cdot\mathbf{t_n} = \frac{\mathbf{t_n}\cdot\mathbf{t_m}\times\mathbf{t_n}}{|\mathbf{t_m}\times\mathbf{t_n}|} = 0.$$

4.17 Given the function $f(x,y) = 4 - x^2 - y^2$, find the maximum value of f subjected to the constraint that $x + y = 2$.

Ans. Let $g(x,y) = 4 - x^2 - y^2 + \lambda(x + y - 2)$, then we have the following three equations to solve for x, y and λ:

$$\frac{\partial g}{\partial x} = -2x + \lambda = 0,\quad \frac{\partial g}{\partial y} = -2y + \lambda = 0 \text{ and } x + y = 2.$$

Thus, $-2x + \lambda = 0 \to \lambda = 2x,\quad -2y + \lambda = 0 \to \lambda = 2y$, therefore, $x = y \to$
$x + y = 2 \to 2x = 2 \to x = 1 = y$. That is, $f_{\max}$ occurs at $x = y = 1$. That is,
$f_{\max} = 4 - (1)^2 - (1)^2 = 2$

4.18 True or false:

(i) Symmetry of stress tensor is not valid if the body has an angular acceleration.
(ii) On the plane of maximum normal stress, the shearing stress is always zero.

Ans. (i) False. (ii) True.

4.19 True or false:

(i) On the plane of maximum shearing stress, the normal stress is always zero.

(ii) A plane with its normal in the direction of $\mathbf{e}_1 + 2\mathbf{e}_2 - 2\mathbf{e}_3$ has a stress vector $\mathbf{t} = 50\mathbf{e}_1 + 100\mathbf{e}_2 - 100\mathbf{e}_3$ MPa. It is a principal plane.

Ans. (i) Not true in general. Maybe true in some special cases.

(ii) True. We note that $\mathbf{t} = 50\mathbf{e}_1 + 100\mathbf{e}_2 - 100\mathbf{e}_3 = 50\left(\mathbf{e}_1 + 2\mathbf{e}_2 - 2\mathbf{e}_3\right)$. Therefore, $\mathbf{t}$ is normal to the plane, so that there is no shearing stress on the plane. That is, it is a principal plane.

4.20 Why can the following two matrices not represent the same stress tensor?

$$\begin{bmatrix} 100 & 200 & 40 \\ 200 & 0 & 0 \\ 40 & 0 & -50 \end{bmatrix} MPa., \begin{bmatrix} 40 & 100 & 60 \\ 100 & 100 & 0 \\ 60 & 0 & 20 \end{bmatrix} MPa..$$

Ans. The first scalar invariant for the first matrix is 50 MPa. The first scalar invariant for the second matrix is 160 MPa. They are not the same, therefore, they can not represent the same stress tensor.

4.21 Given $\left[\mathbf{T}\right] = \begin{bmatrix} 0 & 100 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa.$ (a) Find the magnitude of shearing stress on the plane whose normal is in the direction of $\mathbf{e}_1 + \mathbf{e}_2$. (b) Find the maximum and minimum normal stresses and the planes on which they act. (c) Find the maximum shearing stress and the plane on which it acts.

Ans. (a) Let $\mathbf{n} = \dfrac{1}{\sqrt{2}}\left(\mathbf{e}_1 + \mathbf{e}_2\right)$. Then $\mathbf{t_n} = \dfrac{1}{\sqrt{2}}\begin{bmatrix} 0 & 100 & 0 \\ 100 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \dfrac{100}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ i.e.,

$\mathbf{t_n} = 100\mathbf{n} \rightarrow$ shearing stress $T_s = 0$.

(b) The characteristic equation is $\begin{vmatrix} 0-\lambda & 100 & 0 \\ 100 & 0-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = 0 \rightarrow -\lambda\left(\lambda^2 - 100^2\right) = 0$

$\rightarrow \lambda_1 = 100MPa.,\ \lambda_2 = -100MPa.,\ \lambda_3 = 0$. The maximum normal stress is 100 $MPa.$ and the minimum normal stress is $-100MPa$.

For $\lambda_1 = 100MPa.,\ -100\alpha_1 + 100\alpha_2 = 0,$ so that $\alpha_1 = \alpha_2$, $\mathbf{n}_1 = (\mathbf{e}_1 + \mathbf{e}_2)/\sqrt{2}$.

For $\lambda_2 = -100MPa., 100\alpha_1 + 100\alpha_2 = 0 \rightarrow \alpha_1 = -\alpha_2$, $\mathbf{n}_2 = (\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$.

(c) $\left(T_s\right)_{\max} = \dfrac{\left(T_n\right)_{\max} - \left(T_n\right)_{\min}}{2} = \dfrac{100 - (-100)}{2} = 100$ MPa. The maximum shearing stress acts on the planes $\mathbf{n} = (\mathbf{n}_1 \pm \mathbf{n}_2)/\sqrt{2}$, i.e., on the planes $\mathbf{e}_1$ and $\mathbf{e}_2$.

4.22 Show the equation for the normal stress on the plane of maximum shearing stress is $T_n = \dfrac{(T_n)_{\max} + (T_n)_{\min}}{2}$.

Ans. Let $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ be the principal axes of the stress tensor with principal values $T_1 > T_2 > T_3$, then $[\mathbf{T}] = \begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix}$. On the plane $\mathbf{n} = (\mathbf{n}_1 \pm \mathbf{n}_3)/\sqrt{2}$, the shearing stress is a maximum. On this plane, the normal stress is:

$$T_n = \mathbf{n} \cdot \mathbf{T}\mathbf{n} \to T_n = \frac{1}{2}[1 \quad 0 \quad \pm 1]\begin{bmatrix} T_1 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & T_3 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \pm 1 \end{bmatrix} = \frac{T_1 + T_3}{2} = \frac{(T_n)_{\max} + (T_n)_{\min}}{2}$$

4.23 The stress components at a point are given by: $T_{11} = 100MPa., T_{22} = 300MPa.,$ $T_{33} = 400MPa$. $T_{12} = T_{13} = T_{23} = 0$. (a) Find the maximum shearing stress and the planes on which they act. (b) Find the normal stress on these planes. (c) Are there any plane/planes on which the normal stress is $500MPa.$?

Ans. (a) The maximum normal stress is clearly $T_{33} = 400MPa$**.,** acting on the $\mathbf{e}_3$ plane and the minimum normal stress is clearly $T_{11} = 100MPa.$**,** acting on the $\mathbf{e}_1$ plane. Thus, the maximum shearing stress is $(T_s)_{\max} = \dfrac{400-100}{2} = 150\ MPa.$, acting on the plane $\mathbf{n} = \dfrac{1}{\sqrt{2}}(\mathbf{e}_1 \pm \mathbf{e}_3)$.

(b) $$T_n = \frac{1}{2}[1 \quad 0 \quad \pm 1]\begin{bmatrix} 100 & 0 & 0 \\ 0 & 300 & 0 \\ 0 & 0 & 400 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ \pm 1 \end{bmatrix} = \frac{1}{2}[1 \quad 0 \quad \pm 1]\begin{bmatrix} 100 \\ 0 \\ \pm 400 \end{bmatrix} = 250\ MPa.$$

Note: We can also use the result of Prob. 4.22 to obtain $T_n = \dfrac{T_{\max} + T_{\min}}{2} = \dfrac{400+100}{2} = 250\ MPa.$

(c) No, because $T_{\max} = 400MPa$

4.24 The principal values of a stress tensor $\mathbf{T}$ are $T_1 = 10MPa.$, $T_2 = -10MPa.$ and $T_3 = 30MPa.$ If the matrix of the stress is given by: $[\mathbf{T}] = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & T_{33} \end{bmatrix} \times 10\ MPa.$, find the values of T_{11} and T_{33}.

Ans.

$I_1 = 10 - 10 + 30 = 10(T_{11} + 1 + T_{33}) \to 2 = (T_{11} + T_{33}) \to T_{11} = 2 - T_{33}$ (i)

$I_3 = (10)(-10)(30) = (10)^3(T_{11}T_{33} - 4T_{11}) \to -3 = (T_{11}T_{33} - 4T_{11})$ (ii)

(i) and (ii) $\to -3 = (2 - T_{33})(T_{33} - 4) \to T_{33}^2 - 6T_{33} + 5 = 0.$ Thus,

$T_{33} = [6 \pm \sqrt{36-20}]/2 = 3 \pm 2$. Thus, T_{33} is either 5 or 1.
To determine which is the correct value, we check

$I_2 = (-10)(30) + (10)(-10) + (10)(30) = 10^2\left[(T_{33}-4) + T_{11}T_{33} + T_{11}\right]$

$\rightarrow T_{33} + T_{11}T_{33} + T_{11} - 3 = 0$. (iii)

Try $T_{33} = 1$ first, from (i), $T_{11} = 1$, so that (iii) is clearly satisfied. Next try $T_{33} = 5$, eq (i) gives $T_{11} = 2 - 5 = -3$, then left side of (iii) becomes $5 + (-3)(5) - 3 - 3 = -16 \neq 0$.
Thus, $T_{33} = 1$ and $T_{11} = 1$.

4.25 If the state of stress at a point is: $[\mathbf{T}] = \begin{bmatrix} 300 & 0 & 0 \\ 0 & -200 & 0 \\ 0 & 0 & 400 \end{bmatrix} kPa.$, find (a) the magnitude of the shearing stress on the plane whose normal is in the direction of $(2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3)$ and (b) the maximum shearing stress.

Ans. (a) Let $\mathbf{n} = \frac{1}{3}(2\mathbf{e}_1 + 2\mathbf{e}_2 + \mathbf{e}_3)$, then

$$[\mathbf{t_n}] = \frac{1}{3}\begin{bmatrix} 300 & 0 & 0 \\ 0 & -200 & 0 \\ 0 & 0 & 400 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 600 \\ -400 \\ 400 \end{bmatrix} \rightarrow \mathbf{t_n} = \frac{100}{3}(6\mathbf{e}_1 - 4\mathbf{e}_2 + 4\mathbf{e}_3)$$

$$T_n = \mathbf{n}\cdot\mathbf{t_n} = \frac{100}{9}(12 - 8 + 4) = \frac{800}{9} = 88.89 kPa.$$

$$T_s^2 = |\mathbf{t_n}|^2 - T_n^2 = \frac{10^4 \times 68}{9} - \frac{10^4 \times 64}{81} = 10^4 \times 6.76 \rightarrow T_s = 260\ kPa.$$

(b) $(T_s)_{\max} = \dfrac{400 - (-200)}{2} = 300\ kPa.$

4.26 Given $[\mathbf{T}] = \begin{bmatrix} 1 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} MPa.$ (a) Find the stress vector on the plane whose normal is in the direction of $\mathbf{e}_1 + \mathbf{e}_2$. (b) Find the normal stress on the same plane. (c) Find the magnitude of the shearing stress on the same plane. (d) Find the maximum shearing stress and the planes on which this maximum shearing stress acts.

Ans. (a) Let $\mathbf{n} = \frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$, then $[\mathbf{t_n}] = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 4 & 0 \\ 4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} \rightarrow \mathbf{t_n} = \frac{5}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2)$.

(b) $T_n = \mathbf{n}\cdot\mathbf{t_n} = \frac{1}{2}(5+5) = 5\ MPa.$ (c) $T_s^2 = |\mathbf{t_n}|^2 - T_n^2 = 25 - 25 = 0$

(d) The characteristic equation is $(1-\lambda)\left[(1-\lambda)^2-4^2\right]=0 \to \lambda_1=5,\ \lambda_2=-3,\ \lambda_3=1$. Thus,
$(T_n)_{\max}=5\ MPa.$ and $(T_n)_{\min}=-3\ MPa.$
For $(T_n)_{\max}=5\ MPa.$ $(1-5)\alpha_1+4\alpha_2=0 \to \alpha_1=\alpha_2 \to \mathbf{n_1}=\left(1/\sqrt{2}\right)(\mathbf{e}_1+\mathbf{e}_2)$.
For $(T_n)_{\min}=-3\ MPa.$ $(1+3)\alpha_1+4\alpha_2=0 \to \alpha_1=-\alpha_2 \to \mathbf{n_2}=\left(1/\sqrt{2}\right)(\mathbf{e}_1-\mathbf{e}_2)$.
Thus, $(T_s)_{\max}=\dfrac{5-(-3)}{2}=4\ MPa.$, acting on the plane whose normal is
$\mathbf{n}=\left(1/\sqrt{2}\right)(\mathbf{n}_1\pm\mathbf{n}_2)\to \mathbf{n}=\mathbf{e}_1$ and $\mathbf{n}=\mathbf{e}_2$.

4.27 The stress state in which the only non-vanishing stress components are a single pair of shearing stresses is called simple shear. Take $T_{12}=T_{21}=\tau$ and all other $T_{ij}=0$. (a) Find the principal values and principal directions of this stress state. (b) Find the maximum shearing stress and planes on which it acts.

Ans. (a) With $[\mathbf{T}]=\begin{bmatrix}0&\tau&0\\ \tau&0&0\\ 0&0&0\end{bmatrix}$, the characteristic equation is
$\lambda\left(\lambda^2-\tau^2\right)=0\to\lambda_1=\tau,\ \lambda_2=-\tau,\ \lambda_3=0.$
For $\lambda_1=\tau,\ (0-\tau)\alpha_1+\tau\alpha_2=0\to\alpha_1=\alpha_2\to\mathbf{n_1}=\left(1/\sqrt{2}\right)(\mathbf{e}_1+\mathbf{e}_2)$.
For $\lambda_2=-\tau\ \to(0+\tau)\alpha_1+\tau\alpha_2=0\to\alpha_1=-\alpha_2\to\mathbf{n_2}=\left(1/\sqrt{2}\right)(\mathbf{e}_1-\mathbf{e}_2)$.
For $\lambda_3=0\to\mathbf{n}_3=\mathbf{e}_3$.
(b) $(T_s)_{\max}=\dfrac{\tau-(-\tau)}{2}=\tau$, acting on the plane whose normal is
$\mathbf{n}=\left(1/\sqrt{2}\right)(\mathbf{n}_1\pm\mathbf{n}_2)\to \mathbf{n}=\mathbf{e}_1$ and $\mathbf{n}=\mathbf{e}_2$.

4.28 The stress state in which only the three normal stress components do not vanish is called a tri-axial state of stress. Take $T_{11}=\sigma_1,\ T_{22}=\sigma_2,\ T_{33}=\sigma_3$ with $\sigma_1>\sigma_2>\sigma_3$ and all other $T_{ij}=0$. Find the maximum shearing stress and the plane on which it acts.

Ans. $(T_s)_{\max}=\dfrac{\sigma_1-\sigma_3}{2},\ \mathbf{n}=\dfrac{1}{\sqrt{2}}(\mathbf{e}_1\pm\mathbf{e}_3)$.

4.29 Show that the symmetry of the stress tensor is not valid if there are body moments per unit volume, as in the case of a polarized anisotropic dielectric solid.

Ans. Let $\mathbf{M}^* = M_1^*\mathbf{e}_1 + M_2^*\mathbf{e}_2 + M_3^*\mathbf{e}_3$ be the body moments per unit volume. Then referring to the figure shown below, the total moments of all the surface forces and the body force and body moment about the axis which passes through the center point A and parallel to the x_3 axis is :

$$\sum(M_c)_3 = T_{21}(\Delta x_2 \Delta x_3)(\Delta x_1/2) + (T_{21} + \Delta T_{21})(\Delta x_2 \Delta x_3)(\Delta x_1/2) -$$

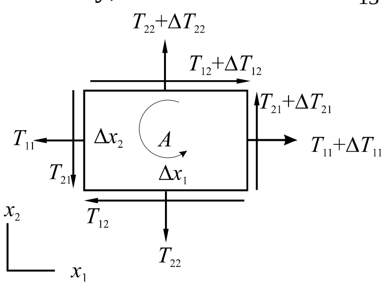
$$T_{12}(\Delta x_1 \Delta x_3)(\Delta x_2/2) - (T_{12} + \Delta T_{12})(\Delta x_1 \Delta x_3)(\Delta x_2/2) + M_3^*(\Delta x_1 \Delta x_2 \Delta x_3)$$

$$= (1/12)(\text{density})(\Delta x_1 \Delta x_2 \Delta x_3)\left[(\Delta x_1)^2 + (\Delta x_2)^2\right]\alpha_3$$

where α_3 is the x_3 components of the angular acceleration of the element. We now let $\Delta x_1 \to 0, \Delta x_2 \to 0, \Delta x_3 \to 0$ and drop all terms of small quantities of higher order than $(\Delta x_1 \Delta x_2 \Delta x_3)$, we obtain,

$$T_{21}(\Delta x_1 \Delta x_2 \Delta x_3) - T_{12}(\Delta x_1 \Delta x_2 \Delta x_3) + M_3^*(\Delta x_1 \Delta x_2 \Delta x_3) = 0 \to T_{12} - T_{21} = M_3^*,$$

Similarly, one can show that $T_{13} - T_{31} = M_2^*$ and $T_{23} - T_{32} = M_1^*$.



4.30 Given the following stress distribution: $[\mathbf{T}] = \begin{bmatrix} x_1 + x_2 & T_{12}(x_1, x_2) & 0 \\ T_{12}(x_1, x_2) & x_1 - 2x_2 & 0 \\ 0 & 0 & x_2 \end{bmatrix}$, find T_{12} so that the stress distribution is in equilibrium with zero body force and so that the stress vector on the plane $x_1 = 1$ is given by $\mathbf{t} = (1 + x_2)\mathbf{e}_1 + (5 - x_2)\mathbf{e}_2$.

Ans. The equations of equilibrium are $\dfrac{\partial T_{ij}}{\partial x_j} + \rho B_i = 0$. Now with $B_i = 0$, we have,

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} = 1 + \frac{\partial T_{12}}{\partial x_2} = 0 \to T_{12} = -x_2 + f(x_1).$$

$$\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} = \frac{\partial T_{12}}{\partial x_1} - 2 = 0 \to T_{12} = 2x_1 + g(x_2), \text{ thus, } \quad T_{12} = 2x_1 - x_2 + C.$$

$$\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} = 0 \to 0 = 0.$$

To determine C, we have, the stress vector on the plane $x_1 = 1$ is

$\mathbf{t} = \mathbf{T}\mathbf{e}_1 = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3 = \left[(x_1 + x_2)\mathbf{e}_1 + (2x_1 - x_2 + C)\mathbf{e}_2\right]_{x_1=1}$. Thus,

$$(1 + x_2)\mathbf{e}_1 + (2 - x_2 + C)\mathbf{e}_2 = (1 + x_2)\mathbf{e}_1 + (5 - x_2)\mathbf{e}_2 \to C = 3 \to T_{12} = 2x_1 - x_2 + 3.$$

4.31 Consider the following stress tensor: $[\mathbf{T}] = \alpha \begin{bmatrix} x_2 & -x_3 & 0 \\ -x_3 & 0 & -x_2 \\ 0 & -x_2 & T_{33} \end{bmatrix}$. Find an expression for T_{33} such that the stress tensor satisfies the equations of equilibrium in the presence of body force vector $\mathbf{B} = -g\mathbf{e}_3$, where g is a constant.

Ans. The equations of equilibrium are $\frac{\partial T_{ij}}{\partial x_j} + \rho B_i = 0$. With $B_1 = B_2 = 0,\ B_3 = -g$, we have,

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} + \rho B_1 = 0+0+0+0 = 0,\ \frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} + \rho B_2 = 0+0+0+0 = 0,$$

$$\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} + \rho B_3 = \alpha\left(-1 + \frac{\partial T_{33}}{\partial x_3}\right) - \rho g = 0 \rightarrow \frac{\partial T_{33}}{\partial x_3} = \frac{\rho g}{\alpha} + 1$$

$$\rightarrow T_{33} = \left(\frac{\rho g}{\alpha} + 1\right)x_3 + f(x_1, x_2)$$

4.32 In the absence of body forces, the equilibrium stress distribution for a certain body is $T_{11} = Ax_2,\ T_{12} = T_{21} = x_1,\ T_{22} = Bx_1 + Cx_2,\ T_{33} = (T_{11} + T_{22})/2,$ all other $T_{ij} = 0$. Also, the boundary plane $x_1 - x_2 = 0$ for the body is free of stress.

(a) Find the value of C and (b) determine the value of A and B.

Ans. (a) The equations of equilibrium are $\frac{\partial T_{ij}}{\partial x_j} + \rho B_i = 0$. With $B_i = 0,$, we have,

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} + \rho B_1 = 0+0+0+0 = 0,$$

$$\frac{\partial T_{21}}{\partial x_1} + \frac{\partial T_{22}}{\partial x_2} + \frac{\partial T_{23}}{\partial x_3} + \rho B_2 = 1 + C + 0 + 0 = 0 \rightarrow C = -1,$$

$$\frac{\partial T_{31}}{\partial x_1} + \frac{\partial T_{32}}{\partial x_2} + \frac{\partial T_{33}}{\partial x_3} + \rho B_3 = 0+0+0+0 = 0.$$

(b) The unit normal to the boundary plane $x_1 - x_2 = 0$ is $\mathbf{n} = (\mathbf{e}_1 - \mathbf{e}_2)/\sqrt{2}$. Thus, on this plane (note $x_1 = x_2$), we have,

$$[\mathbf{t}] = \frac{1}{\sqrt{2}}\begin{bmatrix} Ax_1 & x_1 & 0 \\ x_1 & Bx_1 - x_1 & 0 \\ 0 & 0 & (T_{11} + T_{22})/2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} Ax_1 - x_1 \\ x_1 - Bx_1 + x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ thus,}$$

$Ax_1 - x_1 = 0 \rightarrow A = 1$ and $x_1 - Bx_1 + x_1 = 0 \rightarrow B = 2$.

4.33 In the absence of body forces, do the following stress components satisfy the equations of equilibrium:

$$T_{11}=\alpha\left[x_2^2+\nu\left(x_1^2-x_2^2\right)\right],\ T_{22}=\alpha\left[x_1^2+\nu\left(x_2^2-x_1^2\right)\right],\ T_{33}=\alpha\nu\left(x_1^2+x_2^2\right),$$
$$T_{12}=T_{21}=-2\alpha\nu x_1x_2,\ T_{13}=T_{31}=0,\quad T_{23}=T_{32}=0.$$

Ans. The equations of equilibrium are $\frac{\partial T_{ij}}{\partial x_j}+\rho B_i=0$. With $B_i=0,$, we have,

$$\frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}+\frac{\partial T_{13}}{\partial x_3}+\rho B_1=2\alpha\nu x_1-2\alpha\nu x_1+0+0=0,$$
$$\frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3}+\rho B_2=-2\alpha\nu x_2+2\alpha\nu x_2+0+0=0$$
$$\frac{\partial T_{31}}{\partial x_1}+\frac{\partial T_{32}}{\partial x_2}+\frac{\partial T_{33}}{\partial x_3}+\rho B_3=0+0+0+0=0.$$ Yes, the equations of equilibrium are all satisfied.

4.34 Repeat the previous problem for the stress distribution

$$[\mathbf{T}]=\alpha\begin{bmatrix} x_1+x_2 & 2x_1-x_2 & 0\\ 2x_1-x_2 & x_1-3x_2 & 0\\ 0 & 0 & x_1\end{bmatrix}$$

Ans. The equations of equilibrium are $\frac{\partial T_{ij}}{\partial x_j}+\rho B_i=0$. With $B_i=0,$, we have,

$$\frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}+\frac{\partial T_{13}}{\partial x_3}+\rho B_1=0\to\alpha\left(1-1+0\right)=0\to 0=0,$$
$$\frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3}+\rho B_2=\alpha(2-3+0)\neq 0$$

No, the second equation of equilibrium is not satisfied.

4.35 Suppose that the stress distribution has the form (called a plane stress state)

$$[\mathbf{T}]=\begin{bmatrix} T_{11}\left(x_1,x_2\right) & T_{12}\left(x_1,x_2\right) & 0\\ T_{12}\left(x_1,x_2\right) & T_{22}\left(x_1,x_2\right) & 0\\ 0 & 0 & 0\end{bmatrix}$$

(a) If the state of stress is in equilibrium, can the body forces be dependent on x_3 ? (b) If we introduce a function $\varphi\left(x_1,x_2\right)$ such that $T_{11}=\frac{\partial^2\varphi}{\partial x_2^2}$, $T_{22}=\frac{\partial^2\varphi}{\partial x_1^2}$ and $T_{12}=-\frac{\partial^2\varphi}{\partial x_1\partial x_2}$, What should be the function $\varphi\left(x_1,x_2\right)$ for the equilibrium equations to be satisfied in the absence of body forces?

Ans. (a) $\frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}+\frac{\partial T_{13}}{\partial x_3}+\rho B_1=\frac{\partial T_{11}\left(x_1,x_2\right)}{\partial x_1}+\frac{\partial T_{12}\left(x_1,x_2\right)}{\partial x_2}+\rho B_1=0$.

$$\frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3}+\rho B_2=\frac{\partial T_{21}(x_1,x_2)}{\partial x_1}+\frac{\partial T_{22}(x_1,x_2)}{\partial x_2}+\rho B_2=0.$$

Thus, B_1 and B_2 must be independent of x_3.

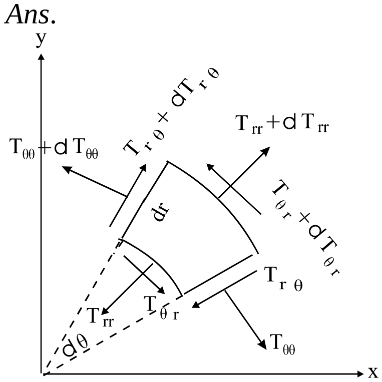
(b) $$\frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}+\frac{\partial T_{13}}{\partial x_3}+\rho B_1\to\frac{\partial}{\partial x_1}\left(\frac{\partial^2\varphi}{\partial x_2^2}\right)-\frac{\partial}{\partial x_2}\left(\frac{\partial^2\varphi}{\partial x_1\partial x_2}\right)+0+0=\frac{\partial}{\partial x_1}\left(\frac{\partial^2\varphi}{\partial x_2^2}\right)-\frac{\partial}{\partial x_1}\left(\frac{\partial^2\varphi}{\partial x_2^2}\right)=0.$$

$$\frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3}+\rho B_2=-\frac{\partial}{\partial x_1}\left(\frac{\partial^2\varphi}{\partial x_1\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial^2\varphi}{\partial x_1^2}\right)+0+0=-\frac{\partial}{\partial x_2}\left(\frac{\partial^2\varphi}{\partial x_1^2}\right)+\frac{\partial}{\partial x_2}\left(\frac{\partial^2\varphi}{\partial x_1^2}\right)=0.$$

Thus, the equations of equilibrium are satisfied for any function $\varphi(x_1,x_2)$ which is continuous up to the third derivatives.

4.36 In cylindrical coordinates (r,θ,z), consider a differential volume of material bounded by the three pairs of faces : $r=r$ and $r=r+dr$; $\theta=\theta$ and $\theta=\theta+d\theta$; $z=z$ and $z=z+dz$. Derive the r and θ equations of motion in cylindrical coordinates and compare the equations with those given in Section 4.8.

Ans.



From the free body diagram above, we have,

$$\sum F_r=-T_{rr}(rd\theta)dz+(T_{rr}+dT_{rr})((r+dr)d\theta)dz-T_{r\theta}drdz\cos(d\theta/2)$$
$$+(T_{r\theta}+dT_{r\theta})drdz\cos(d\theta/2)-T_{\theta\theta}drdz\sin(d\theta/2)-(T_{\theta\theta}+dT_{\theta\theta})drdz\sin(d\theta/2)$$
$$-T_{rz}(rd\theta)dr+(T_{rz}+dT_{rz})(rd\theta)dr+\rho B_r(rd\theta)drdz=\left[\rho(rd\theta)drdz\right]a_r.$$

Now, $\cos\left(\frac{d\theta}{2}\right)=1+\frac{1}{2!}\left(\frac{d\theta}{2}\right)^2+...$ and $\sin\left(\frac{d\theta}{2}\right)=\left(\frac{d\theta}{2}\right)-\frac{1}{3!}\left(\frac{d\theta}{2}\right)^3+...$ and keeping only terms involving products of three differentials (i.e., terms involving product of 4 or more differentials drop out in the limit when these differentials approach zero), we have,

$$T_{rr}drd\theta dz+(dT_{rr})rd\theta dz+dT_{r\theta}drdz-2T_{\theta\theta}drdz(d\theta/2)$$
$$+(dT_{rz})(rd\theta)dr+\rho B_r(rd\theta)drdz=\left[\rho(rd\theta)drdz\right]a_r$$

Dividing the equation by $rd\theta drdz$, we get,

$$\frac{T_{rr}}{r}+\frac{\partial T_{rr}}{\partial r}+\frac{\partial T_{r\theta}}{rd\theta}-\frac{T_{\theta\theta}}{r}+\frac{\partial T_{rz}}{\partial z}+\rho B_r=\rho a_r.$$ This is Eq. (4.8.1)

Next,

$$\sum F_\theta = -T_{\theta r}\left(rd\theta\right)dz + \left(T_{\theta r} + dT_{\theta r}\right)\left(\left(r+dr\right)d\theta\right)dz + T_{r\theta}drdz\sin\left(d\theta/2\right)$$
$$+\left(T_{r\theta} + dT_{r\theta}\right)drdz\sin\left(d\theta/2\right) - T_{\theta\theta}drdz\cos\left(d\theta/2\right) + \left(T_{\theta\theta} + dT_{\theta\theta}\right)drdz\cos\left(d\theta/2\right)$$
$$-T_{\theta z}\left(rd\theta\right)dr + \left(T_{\theta z} + dT_{\theta z}\right)\left(rd\theta\right)dr + \rho B_\theta\left(rd\theta\right)drdz = \left[\rho\left(rd\theta\right)drdz\right]a_\theta .$$

Again, $\cos\left(\frac{d\theta}{2}\right) = 1 + \frac{1}{2!}\left(\frac{d\theta}{2}\right)^2 + ...$and $\sin\left(\frac{d\theta}{2}\right) = \left(\frac{d\theta}{2}\right) - \frac{1}{3!}\left(\frac{d\theta}{2}\right)^3 + ...$ and keeping only terms involving products of three differentials, we have,

$$\left(T_{\theta r}\right)drd\theta dz + \left(dT_{\theta r}\right)rd\theta dz + 2T_{r\theta}drdz\left(d\theta/2\right) + \left(dT_{\theta\theta}\right)drdz + \left(dT_{\theta z}\right)rd\theta dr$$
$$+\rho B_\theta\left(rd\theta\right)drdz = \left[\rho\left(rd\theta\right)drdz\right]a_\theta$$

Dividing the equation by $rd\theta drdz$, we get,

$\frac{T_{\theta r}}{r} + \frac{\partial T_{\theta r}}{\partial r} + \frac{T_{r\theta}}{r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta} + \frac{\partial T_{\theta z}}{\partial z} + \rho B_\theta = \rho a_\theta$, this is Eq. (4.8.2).

4.37 Verify that the following stress field satisfies the *z*-equation of equilibrium in the absence of body forces:

$$T_{rr} = A\left(\frac{z}{R^3} - \frac{3r^2z}{R^5}\right),\ T_{\theta\theta} = \frac{Az}{R^3},\ T_{zz} = -A\left(\frac{z}{R^3} + \frac{3z^3}{R^5}\right),\ T_{rz} = -A\left(\frac{r}{R^3} + \frac{3rz^2}{R^5}\right),\ T_{r\theta} = T_{z\theta} = 0$$
$$R^2 = r^2 + z^2$$

Ans. The z equation of equilibrium in cylindrical coordinate is:

$\frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial\theta} + \frac{T_{zr}}{r} + \frac{\partial T_{zz}}{\partial z} + \rho B_z = 0$. Now $\frac{\partial R}{\partial r} = \frac{r}{R}$, $\frac{\partial R}{\partial z} = \frac{z}{R}$ so that

$$\frac{\partial T_{zr}}{\partial r} = -A\left(\frac{\partial}{\partial r}\frac{r}{R^3} + \frac{\partial}{\partial r}\frac{3rz^2}{R^5}\right) = -A\left(\frac{1}{R^3} - \frac{3r}{R^4}\frac{\partial R}{\partial r} + \frac{3z^2}{R^5} - \frac{15rz^2}{R^6}\frac{\partial R}{\partial r}\right)$$
$$= -A\left(\frac{1}{R^3} - \frac{3r^2}{R^5} + \frac{3z^2}{R^5} - \frac{15r^2z^2}{R^7}\right) = -A\left(\frac{1}{R^3} - \frac{3r^2}{R^5} + \frac{3z^2}{R^5} - \frac{15r^2z^2}{R^7}\right),\quad \frac{\partial T_{z\theta}}{\partial\theta} = 0$$
$$\frac{T_{rz}}{r} = -A\left(\frac{1}{R^3} + \frac{3z^2}{R^5}\right),$$
$$\frac{\partial T_{zz}}{\partial z} = -A\left(\frac{1}{R^3} - \frac{3z}{R^4}\frac{\partial R}{\partial z} + \frac{9z^2}{R^5} - \frac{15z^3}{R^6}\frac{\partial R}{\partial z}\right) = -A\left(\frac{1}{R^3} - \frac{3z^2}{R^5} + \frac{9z^2}{R^5} - \frac{15z^4}{R^7}\right)$$

Thus, $\frac{\partial T_{zr}}{\partial r} + \frac{T_{rz}}{r} + \frac{\partial T_{zz}}{\partial z} = -A\left(\frac{3}{R^3} - \frac{3r^2}{R^5} - \frac{3z^2}{R^5} + \frac{3z^2}{R^5} + \frac{3z^2}{R^5} + \frac{9z^2}{R^5} - \frac{15r^2z^2}{R^7} - \frac{15z^4}{R^7}\right)$

$$= -A\left(\frac{3}{R^3} - \frac{3r^2}{R^5} - \frac{3z^2}{R^5} + \frac{3z^2}{R^5} + \frac{3z^2}{R^5} + \frac{9z^2}{R^5} - \frac{15r^2z^2}{R^7} - \frac{15z^4}{R^7}\right)$$
$$= -A\left(\frac{3}{R^3} - \frac{3\left(r^2+z^2\right)}{R^5} + \frac{15z^2}{R^5} - \frac{15z^2\left(r^2+z^2\right)}{R^7}\right) = -A\left(\frac{3}{R^3} - \frac{3R^2}{R^5} + \frac{15z^2}{R^5} - \frac{15z^2R^2}{R^7}\right) = 0 .$$

4.38 Given the following stress field in cylindrical coordinates:

$$T_{rr}=-\frac{3Pzr^2}{2\pi R^5},\ T_{zz}=-\frac{3Pz^3}{2\pi R^5},\ T_{rz}=-\frac{3Pz^2r}{2\pi R^5},\ T_{\theta\theta}=T_{r\theta}=T_{z\theta}=0,\ \ R^2=r^2+z^2$$

Verify that the state of stress satisfies the equations of equilibrium in the absence of body forces.

Ans.

$$\frac{\partial T_{rr}}{\partial r}+\frac{\partial T_{r\theta}}{rd\theta}+\frac{T_{rr}-T_{\theta\theta}}{r}+\frac{\partial T_{rz}}{\partial z}=\frac{\partial}{\partial r}\left(-\frac{3Pzr^2}{2\pi R^5}\right)+\frac{1}{r}\left(-\frac{3Pzr^2}{2\pi R^5}\right)+\frac{\partial}{\partial z}\left(-\frac{3Pz^2r}{2\pi R^5}\right)$$

$$=-\frac{3Pz}{2\pi R^5}\frac{\partial}{\partial r}\left(r^2\right)+\left(-\frac{3Pzr^2}{2\pi}\right)\frac{\partial}{\partial r}\left(\frac{1}{R^5}\right)+\frac{1}{r}\left(-\frac{3Pzr^2}{2\pi R^5}\right)+\left(-\frac{3Pr}{2\pi R^5}\right)\frac{\partial}{\partial z}z^2+\left(-\frac{3Pz^2r}{2\pi}\right)\frac{\partial}{\partial z}\left(\frac{1}{R^5}\right)$$

$$=-\frac{3Pzr}{\pi R^5}+\left(\frac{15Pzr^2}{2\pi R^6}\right)\frac{\partial R}{\partial r}-\left(\frac{3Pzr}{2\pi R^5}\right)+\left(-\frac{3P\,zr}{\pi R^5}\right)+\left(\frac{15Pz^2r}{2\pi R^6}\right)\frac{\partial R}{\partial z}$$

$$=-\frac{15Pzr}{2\pi R^5}+\left(\frac{15Pzr^3}{2\pi R^7}\right)+\left(\frac{15Pz^3r}{2\pi R^7}\right)=-\frac{15Pzr}{2\pi R^5}+\left(\frac{15Pzr\left(r^2+z^2\right)}{2\pi R^7}\right)=0\,.$$

$$\frac{\partial T_{\theta r}}{\partial r}+\frac{2T_{r\theta}}{r}+\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial\theta}+\frac{\partial T_{\theta z}}{\partial z}=0+0+0+0=0$$

$$\frac{\partial T_{zr}}{\partial r}+\frac{1}{r}\frac{\partial T_{z\theta}}{\partial\theta}+\frac{T_{zr}}{r}+\frac{\partial T_{zz}}{\partial z}+\rho B_z=\frac{\partial}{\partial r}\left(-\frac{3Pz^2r}{2\pi R^5}\right)+\frac{1}{r}\left(-\frac{3Pz^2r}{2\pi R^5}\right)+\frac{\partial}{\partial z}\left(-\frac{3Pz^3}{2\pi R^5}\right)$$

$$=\left(-\frac{3Pz^2}{2\pi R^5}\right)+\left(-\frac{3Pz^2r}{2\pi}\right)\frac{\partial}{\partial r}\left(\frac{1}{R^5}\right)-\frac{3Pz^2}{2\pi R^5}-\frac{9Pz^2}{2\pi R^5}-\left(\frac{3Pz^3}{2\pi}\right)\frac{\partial}{\partial z}\left(\frac{1}{R^5}\right)$$

$$=\left(-\frac{3Pz^2}{2\pi R^5}\right)+\left(\frac{15Pz^2r}{2\pi R^6}\right)\frac{\partial R}{\partial r}-\frac{3Pz^2}{2\pi R^5}-\frac{9Pz^2}{2\pi R^5}+\left(\frac{15Pz^3}{2\pi R^6}\right)\frac{\partial R}{\partial z}$$

$$=\left(-\frac{3Pz^2}{2\pi R^5}\right)+\left(\frac{15Pz^2r^2}{2\pi R^7}\right)-\frac{3Pz^2}{2\pi R^5}-\frac{9Pz^2}{2\pi R^5}+\left(\frac{15Pz^4}{2\pi R^7}\right)=\left(-\frac{15Pz^2}{2\pi R^5}\right)+15Pz^2\left(\frac{r^2+z^2}{2\pi R^7}\right)=0$$

4.39 For the stress field given in Example 4.9.1, determine the constants A and B if the inner cylindrical wall is subjected to a uniform pressure p_i and the outer cylindrical wall is subjected to a uniform pressure p_o.

Ans. The given stress field is:

$$T_{rr}=A+\frac{B}{r^2},\ \ T_{\theta\theta}=A-\frac{B}{r^2},\ \ T_{zz}=\text{constant and } T_{r\theta}=T_{rz}=T_{\theta z}=0\,.$$

On the outer wall, $r=r_o$, and $T_{rr}=-p_o$, and on the inner wall, $r=r_i$, and $T_{rr}=-p_i$, therefore, we have, $-p_o=A+\dfrac{B}{r_o^2}$ (i) and $-p_i=A+\dfrac{B}{r_i^2}$ (ii).

$$\rightarrow p_o-p_i=\frac{B}{r_i^2}-\frac{B}{r_o^2}\rightarrow\ \ B=\frac{\left(p_o-p_i\right)r_i^2r_o^2}{\left(r_o^2-r_i^2\right)},\ \ A=\frac{\left(p_ir_i^2-p_or_o^2\right)}{\left(r_o^2-r_i^2\right)}\,.$$

4.40 Verify that Eq. (4.8.4) to (4.8.6) are satisfied by the stress field given in Example 4.9.2 in the absence of body forces.

Ans. The given stress field is: $T_{rr}=A-\frac{2B}{r^3},\ T_{\theta\theta}=T_{\phi\phi}=A+\frac{B}{r^3},\quad T_{r\theta}=T_{r\phi}=T_{\theta\phi}=0$.

$$\frac{1}{r^2}\frac{\partial\left(r^2T_{rr}\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(T_{r\theta}\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi}-\frac{T_{\theta\theta}+T_{\phi\phi}}{r}$$

$$=\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2A-\frac{2B}{r}\right)+0+0-\ \frac{2}{r}\left(A+\frac{B}{r^3}\right)=\frac{2}{r}\left(A+\frac{B}{r^3}\right)+0+0-\frac{2}{r}\left(A+\frac{B}{r^3}\right)=0.$$

$$\frac{1}{r^3}\frac{\partial\left(r^3T_{\theta r}\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(T_{\theta\theta}\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\theta\phi}}{\partial\phi}+\frac{T_{r\theta}-T_{\theta r}-T_{\phi\phi}\cot\theta}{r}$$

$$=0+0+0+\frac{T_{\theta\theta}\cot\theta}{r}-\frac{T_{\phi\phi}\cot\theta}{r}=0.$$

$$\frac{1}{r^3}\frac{\partial\left(r^3T_{\phi r}\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(T_{\phi\theta}\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial\phi}+\frac{T_{r\phi}-T_{\phi r}+T_{\theta\phi}\cot\theta}{r}=0+0+0+\ 0=0\,.$$

4.41 In Example 4.9.2, if the spherical shell is subjected to an inner pressure p_i and an outer pressure p_{o}, determine the constant A and B.

Ans. From the example, we have, $T_{rr}=A-\frac{2B}{r^3}$, thus, $-p_{\text{o}}=A-\frac{2B}{r_{\text{o}}^3}$ and $-p_i=A-\frac{2B}{r_{\text{i}}^3}$

$$\rightarrow A=-\frac{p_{\text{o}}r_{\text{o}}^3-p_ir_{\text{i}}^3}{\left(r_{\text{o}}^3-r_{\text{i}}^3\right)}\text{ and }B=-\frac{\left(p_{\text{o}}-p_i\right)r_{\text{o}}^3r_{\text{i}}^3}{2\left(r_{\text{o}}^3-r_{\text{i}}^3\right)}.$$

4.42 The equilibrium configuration of a body is described by:

$x_1=16X_1,\ x_2=-\frac{1}{4}X_2,\ x_3=-\frac{1}{4}X_3$. If the Cauchy stress tensor is given by:

$T_{11}=1000\ MPa.$, and all other $T_{ij}=0$, (a) calculate the first Piola –Kirchhoff stress tensor and the corresponding pseudo stress vector for the plane whose undeformed plane is $\mathbf{e}_1$ plane and (b) calculate the second Piola-Kirchhoff tensor and the corresponding pseudo stress vector for the same plane.

Ans. From $x_1=16X_1,\ x_2=-\frac{1}{4}X_2,\ x_3=-\frac{1}{4}X_3$, we obtain the deformation gradient $\mathbf{F}$ and its inverse as:

$$[\mathbf{F}]=\begin{bmatrix}16 & 0 & 0\\ 0 & -1/4 & 0\\ 0 & 0 & -1/4\end{bmatrix},\quad [\mathbf{F}^{-1}]=\begin{bmatrix}1/16 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -4\end{bmatrix}\text{ and }\det\mathbf{F}=1.$$

(a) The first Piola-Kirchhoff stress tensor is, from $\mathbf{T}_o=(\det\mathbf{F})\mathbf{T}(\mathbf{F}^{-1})^{\mathrm{T}}$:

$$[\mathbf{T}_o]=(\det\mathbf{F})[\mathbf{T}]\left[(\mathbf{F}^{-1})^{\mathrm{T}}\right]=(1)\begin{bmatrix}1000 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1/16 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -4\end{bmatrix}=\begin{bmatrix}1000/16 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}MPa.$$

For a unit area in the deformed state in the $\mathbf{e}_1$ direction, its undeformed area is

$$(dA_o\mathbf{n}_o)=\frac{1}{\det\mathbf{F}}\mathbf{F}^{\mathrm{T}}\mathbf{n}=(1)\begin{bmatrix}16 & 0 & 0\\ 0 & -1/4 & 0\\ 0 & 0 & -1/4\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}16\\0\\0\end{bmatrix}\rightarrow dA_o\mathbf{n}_o=16\mathbf{e}_1.$$

That is, its undeformed plane is also $\mathbf{e}_1$ plane. The corresponding pseudo stress vector is given by

$\mathbf{t_o}=\mathbf{T_o n_o}$, where $\mathbf{n}_o=\mathbf{e}_1$. Thus $\mathbf{t_o}=\mathbf{T_o n_o}\rightarrow[\mathbf{t_o}]=\begin{bmatrix}1000/16 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}\rightarrow\mathbf{t_o}=(1000/16)\mathbf{e}_1$

We note that the pseudo stress vector is in the same direction as the Cauchy stress vector and the intensity of the pseudo stress vector is 1/16 of the Cauchy stress vector simply because the undeformed area is 16 times the deformed area and both areas have the same normal direction.

(b)The second Piola-Kirchhoff stress tensor is, from $\tilde{\mathbf{T}}=(\det\mathbf{F})\mathbf{F}^{-1}\mathbf{T}(\mathbf{F}^{-1})^{\mathrm{T}}$

$$[\tilde{\mathbf{T}}]=(\det\mathbf{F})[\mathbf{F}^{-1}][\mathbf{T}]\left[(\mathbf{F}^{-1})^{\mathrm{T}}\right]=(1)\begin{bmatrix}1/16 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -4\end{bmatrix}\begin{bmatrix}1000 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1/16 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -4\end{bmatrix}$$

$$\begin{bmatrix}1/16 & 0 & 0\\ 0 & -4 & 0\\ 0 & 0 & -4\end{bmatrix}\begin{bmatrix}1000/16 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}1000/256 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}MPa.$$

The corresponding pseudo stress vector is given $\tilde{\mathbf{t}}=\tilde{\mathbf{T}}\mathbf{n_o}$, where $\mathbf{n}_o=\mathbf{e}_1$. Thus,

$\tilde{\mathbf{t}}=\tilde{\mathbf{T}}\mathbf{n_o}\rightarrow\tilde{\mathbf{t}}=(1000/256)\mathbf{e}_1$.

4.43 Can the following equations represent a physically acceptable deformation of a body? Give reason.

$$x_1=-\frac{1}{2}X_1,\quad x_2=\frac{1}{2}X_3,\quad x_3=-4X_2.$$

Ans. $[\mathbf{F}]=\begin{bmatrix}-1/2 & 0 & 0\\ 0 & 0 & 1/2\\ 0 & -4 & 0\end{bmatrix}\rightarrow\det\mathbf{F}=-1$. The given equations are not acceptable as a physically acceptable deformation because it gives a negative ratio of deformed volume to the undeformed volume.

4.44 The deformation of a body is described by:
$x_1 = 4X_1,\ x_2 = -(1/4)X_2,\ x_3 = -(1/4)X_3$. (a) For a unit cube with sides along the coordinate axes what is its deformed volume? What is the deformed area of the $\mathbf{e}_1$ face of the cube? (b) If the Cauchy stress tensor is given by: $T_{11} = 100\ MPa.,$ and all other $T_{ij} = 0$, calculate the first Piola –Kirchhoff stress tensor and the corresponding pseudo stress vector for the plane whose undeformed plane is $\mathbf{e}_1$ plane. (c) Calculate the second Piola-Kirchhoff tensor and the corresponding pseudo stress vector for the plane whose undeformed plane is $\mathbf{e}_1$ plane. Also, calculate the pseudo differential force for the same plane.

Ans. From $x_1 = 4X_1,\ x_2 = -(1/4)X_2,\ x_3 = -(1/4)X_3$, we have (a)

$$[\mathbf{F}] = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1/4 & 0 \\ 0 & 0 & -1/4 \end{bmatrix} \rightarrow \det\mathbf{F} = 1/4\text{, thus } dV = (\det\mathbf{F})dV_o \rightarrow dV = (1/4)(1) = 1/4\,.$$

$$d\mathbf{A} = dA_o(\det\mathbf{F})\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\mathbf{n}_o = (1)(1/4)\begin{bmatrix} 1/4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = (1/4)\begin{bmatrix} 1/4 \\ 0 \\ 0 \end{bmatrix} \rightarrow d\mathbf{A} = (1/16)\mathbf{e}_1$$

That is, the deformed volume is 1/4 of its original volume and the $\mathbf{e}_1$ face of unit area deformed into an area 1/16 of it original area and remain in the same direction. These results are quite obvious from the geometry of the deformation.

(b) The first PK stress tensor is:

$$[\mathbf{T}_o] = (\det\mathbf{F})[\mathbf{T}]\left[\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\right] = \left(\frac{1}{4}\right)\begin{bmatrix} 100 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1/4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix} = \begin{bmatrix} 100/16 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa.$$

The corresponding pseudo stress vector for $\mathbf{e}_1$-plane in the deformed state, whose undeformed plane is also $\mathbf{e}_1$-plane, is given by $\mathbf{t}_o = \mathbf{T}_o\mathbf{n}_o$, where $\mathbf{n}_o = \mathbf{e}_1$, that is $\mathbf{t}_\mathbf{o} = (100/16)\mathbf{e}_1 MPa.$ The Cauchy stress vector on the $\mathbf{e}_1$ face in the deformed state is $\mathbf{t} = 100\mathbf{e}_1 MPa.$ Clearly the Cauchy stress vector has a larger magnitude because the area in the deformed state is 1/16 of the undeformed area.

(c) The second PK stress tensor is:

$$[\tilde{\mathbf{T}}] = [\mathbf{F}^{-1}][\mathbf{T}_o] = \begin{bmatrix} 1/4 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & -4 \end{bmatrix}\begin{bmatrix} 100/16 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 100/64 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa.$$

The corresponding pseudo stress vector for the $\mathbf{e}_1$-plane in the deformed state, whose undeformed plane is also $\mathbf{e}_1$-plane, is given by $\tilde{\mathbf{t}} = \tilde{\mathbf{T}}\mathbf{n}_\mathbf{o}$, where $\mathbf{n}_o = \mathbf{e}_1$. Thus, $\tilde{\mathbf{t}} = (100/64)\mathbf{e}_1 MPa.$ The pseudo force $d\tilde{\mathbf{f}}$ is related to the force $d\mathbf{f}\,(=\mathbf{t}_{\circ}dA_{\circ}=100/16\mathbf{e}_1$ for $dA_{\circ}=1)$ by the formula

$$d\tilde{\mathbf{f}}=\mathbf{F}^{-1}d\mathbf{f}\text{ , Thus, }\left[d\tilde{\mathbf{f}}\right]=\left[\mathbf{F}^{-1}\right]\left[d\mathbf{f}\right]=\begin{bmatrix}1/4&0&0\\0&-4&0\\0&0&-4\end{bmatrix}\begin{bmatrix}100/16\\0\\0\end{bmatrix}\rightarrow d\tilde{\mathbf{f}}=\left(\frac{100}{64}\right)\mathbf{e}_1$$

4.45 The deformation of a body is described by:
$x_1=X_1+kX_2,\ x_2=X_2,\ x_3=X_3$. (a) For a unit cube with sides along the coordinate axes what is its deformed volume? What is the deformed area of the $\mathbf{e}_1$ face of the cube? (b) If the Cauchy stress tensor is given by: $T_{12}=T_{21}=100\ MPa.$,and all other $T_{ij}=0$, calculate the first Piola – Kirchhoff stress tensor and the corresponding pseudo stress vector for the plane whose undeformed plane is $\mathbf{e}_1$ plane and compare it with the Cauchy stress vector in the deformed state. (c) Calculate the second Piola-Kirchhoff tensor and the corresponding pseudo stress vector for the plane whose undeformed plane is $\mathbf{e}_1$ plane. Also, calculate the pseudo differential force for the same plane.

Ans. From $x_1=X_1+kX_2,\ x_2=X_2,\ x_3=X_3$, we have (a)

$$[\mathbf{F}]=\begin{bmatrix}1&k&0\\0&1&0\\0&0&1\end{bmatrix}\rightarrow\det\mathbf{F}=1\text{, thus }dV=(\det\mathbf{F})dV_{\text{o}}\rightarrow dV=dV_{\text{o}}=1.$$

$$d\mathbf{A}=dA_{\text{o}}(\det\mathbf{F})\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\mathbf{n}_{\text{o}}=(1)(1)\begin{bmatrix}1&0&0\\-k&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=\begin{bmatrix}1\\-k\\0\end{bmatrix}\rightarrow d\mathbf{A}=\mathbf{e}_1-k\mathbf{e}_2$$

That is, the deformed volume is the same as its original volume and the $\mathbf{e}_1$ face of unit area deformed into an area $\sqrt{1+k^2}$ of it original area and whose normal is in the direction of $\mathbf{e}_1-k\mathbf{e}_2$. These results are quite obvious from the geometry of the deformation.

(b) The first PK stress tensor is:

$$[\mathbf{T}_{\text{o}}]=(\det\mathbf{F})[\mathbf{T}]\left[\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\right]=\begin{bmatrix}0&100&0\\100&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1&0&0\\-k&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}-100k&100&0\\100&0&0\\0&0&0\end{bmatrix}MPa.$$

The corresponding pseudo stress vector for the $(\mathbf{e}_1-k\mathbf{e}_2)$ plane, whose undeformed plane is the $\mathbf{e}_1$ plane, is given by $\mathbf{t}_{\text{o}}=\mathbf{T}_{\text{o}}\mathbf{n}_{\text{o}}$, where $\mathbf{n}_{\text{o}}=\mathbf{e}_1$. Thus, $\mathbf{t_o}=100(-k\mathbf{e}_1+\mathbf{e}_2)\ MPa.$ The Cauchy stress vector on the $(\mathbf{e}_1-k\mathbf{e}_2)$ face in the deformed configuration is

$$[\mathbf{t}]=[\mathbf{T}][\mathbf{n}]=\frac{1}{\sqrt{1+k^2}}\begin{bmatrix}0&100&0\\100&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}1\\-k\\0\end{bmatrix}\rightarrow\mathbf{t}=\frac{100}{\sqrt{1+k^2}}(-k\mathbf{e}_1+\mathbf{e}_2)$$

The Cauchy stress vector has a smaller magnitude because the deformed area is $\sqrt{1+k^2}$ times the undeformed area.

(c) The second PK stress tensor is:

$$\left[\tilde{\mathbf{T}}\right]=\left[\mathbf{F}^{-1}\right]\left[\mathbf{T}_o\right]=\begin{bmatrix}1 & -k & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}-100k & 100 & 0\\100 & 0 & 0\\0 & 0 & 0\end{bmatrix}=\begin{bmatrix}-200k & 100 & 0\\100 & 0 & 0\\0 & 0 & 0\end{bmatrix}MPa.$$

The corresponding pseudo stress vector for the $\mathbf{e}_1 - k\mathbf{e}_2$ plane, whose undeformed plane is the $\mathbf{e}_1$ plane, is given by $\tilde{\mathbf{t}} = \tilde{\mathbf{T}}\mathbf{n_o}$, where $\mathbf{n}_o = \mathbf{e}_1$. Thus, $\tilde{\mathbf{t}} = 100\left(-2k\mathbf{e}_1 + \mathbf{e}_2\right)MPa.$ The pseudo force $d\tilde{\mathbf{f}}$ is related to the force $d\mathbf{f}\left(=\mathbf{t}_o dA_o = 100\left(-k\mathbf{e}_1 + \mathbf{e}_2\right) \text{ for } dA_o = 1\right)$ by the formula $d\tilde{\mathbf{f}} = \mathbf{F}^{-1}d\mathbf{f}$,

Thus, $\left[d\tilde{\mathbf{f}}\right]=\left[\mathbf{F}^{-1}\right]\left[d\mathbf{f}\right]=100\begin{bmatrix}1 & -k & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\begin{bmatrix}-k\\1\\0\end{bmatrix}\rightarrow d\tilde{\mathbf{f}} = 100\left(-2k\mathbf{e}_1 + \mathbf{e}_2\right)$.

4.46 The deformation of a body is described by:
$x_1 = 2X_1,\ x_2 = 2X_2,\ x_3 = 2X_3$. (a) For a unit cube with sides along the coordinate axes, what is its deformed volume? What is the deformed area of the $\mathbf{e}_1$ face of the cube? (b) If the Cauchy stress tensor is given by: $\begin{bmatrix}100 & 0 & 0\\0 & 100 & 0\\0 & 0 & 100\end{bmatrix}$ *Mpa.*, calculate the first Piola –Kirchhoff stress tensor and the corresponding pseudo stress vector for the plane whose undeformed plane is the $\mathbf{e}_1$ plane and compare it with the Cauchy stress vector on its deformed plane, (c) calculate the second Piola-Kirchhoff tensor and the corresponding pseudo stress vector for the plane whose undeformed plane is the $\mathbf{e}_1$ plane. Also, calculate the pseudo differential force for the same plane.

Ans. From, $x_1 = 2X_1,\ x_2 = 2X_2,\ x_3 = 2X_3$ we have (a)

$$\left[\mathbf{F}\right]=\begin{bmatrix}2 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{bmatrix}\rightarrow \det\mathbf{F} = 8\text{, thus } dV = \left(\det\mathbf{F}\right)dV_o \rightarrow dV = 8dV_o = 8.$$

$$d\mathbf{A} = dA_o\left(\det\mathbf{F}\right)\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\mathbf{n}_o = (1)(8)\begin{bmatrix}1/2 & 0 & 0\\0 & 1/2 & 0\\0 & 0 & 1/2\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=(8)\begin{bmatrix}1/2\\0\\0\end{bmatrix}\rightarrow d\mathbf{A} = 4\mathbf{e}_1$$

(b) The first PK stress tensor is:

$$\left[\mathbf{T}_o\right]=\left(\det\mathbf{F}\right)\left[\mathbf{T}\right]\left[\left(\mathbf{F}^{-1}\right)^{\mathrm{T}}\right]=(8)\begin{bmatrix}100 & 0 & 0\\0 & 100 & 0\\0 & 0 & 100\end{bmatrix}\begin{bmatrix}1/2 & 0 & 0\\0 & 1/2 & 0\\0 & 0 & 1/2\end{bmatrix}=\begin{bmatrix}400 & 0 & 0\\0 & 400 & 0\\0 & 0 & 400\end{bmatrix}MPa.$$

The corresponding pseudo stress vector for the $\mathbf{e}_1$ plane in the deformed state, whose undeformed plane is also $\mathbf{e}_1$ plane, is $\mathbf{t_o} = 400\mathbf{e}_1$ *MPa.* The Cauchy stress vector on the $\mathbf{e}_1$ plane is $\mathbf{t} = 100\mathbf{e}_1$ *MPa.* The Cauchy stress vector has a smaller magnitude because the area is four times larger.

(c) The second PK stress tensor is:

$$\left[\tilde{\mathbf{T}}\right]=\left[\mathbf{F}^{-1}\right]\left[\mathbf{T}_o\right]=\begin{bmatrix}1/2 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1/2\end{bmatrix}\begin{bmatrix}400 & 0 & 0\\ 0 & 400 & 0\\ 0 & 0 & 400\end{bmatrix}=\begin{bmatrix}200 & 0 & 0\\ 0 & 200 & 0\\ 0 & 0 & 200\end{bmatrix}MPa.$$

The corresponding pseudo stress vector for the $\mathbf{e}_1$ plane in the deformed state, whose undeformed plane is also $\mathbf{e}_1$ plane, is $\tilde{\mathbf{t}}=200\mathbf{e}_1 MPa.$ The pseudo force $d\tilde{\mathbf{f}}$ is related to the force $d\mathbf{f}\left(=\mathbf{t}_\circ dA_\circ=400\mathbf{e}_1 \text{ for } dA_\circ=1\right)$ by the formula $d\tilde{\mathbf{f}}=\mathbf{F}^{-1}d\mathbf{f}$. Thus,

$$\left[d\tilde{\mathbf{f}}\right]=\left[\mathbf{F}^{-1}\right]\left[d\mathbf{f}\right]=400\begin{bmatrix}1/2 & 0 & 0\\ 0 & 1/2 & 0\\ 0 & 0 & 1/2\end{bmatrix}\begin{bmatrix}1\\ 0\\ 0\end{bmatrix}\rightarrow d\tilde{\mathbf{f}}=200\mathbf{e}_1.$$

CHAPTER 5, PART A

5.1 Show that the null vector is the only isotropic vector. (Hint: Assume that $\mathbf{a}$ is an isotropic vector, and use a simple change of basis to equate the primed and unprimed components).

Ans. For an isotropic $\mathbf{a}$, by definition, $[\mathbf{a}]_{\mathbf{e}_i}=[\mathbf{a}]_{\mathbf{e}'_i}$, where $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ are any two orthonormal bases. That is $[\mathbf{a}]_{\mathbf{e}'_i}=[\mathbf{Q}]^{\mathrm{T}}_{\mathbf{e}_i}[\mathbf{a}]_{\mathbf{e}_i}\rightarrow[\mathbf{a}]_{\mathbf{e}_i}=[\mathbf{Q}]^{\mathrm{T}}_{\mathbf{e}_i}[\mathbf{a}]_{\mathbf{e}_i}$ for all $[\mathbf{Q}]_{\mathbf{e}_i}$.

Method I. Choose $[\mathbf{Q}]=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}$, then $\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&-1&0\\0&0&-1\end{bmatrix}\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}$ gives

$a_1=-a_1=0, a_2=-a_2=0, a_3=-a_3=0$. In other words, the only isotropic vector is the null vector.

Method II. The matrix equation $[\mathbf{a}]_{\mathbf{e}_i}=[\mathbf{Q}]^{\mathrm{T}}_{\mathbf{e}_i}[\mathbf{a}]_{\mathbf{e}_i}$, with the same basis for each matrix, is equivalent to the equation $\mathbf{a}=\mathbf{Q}^{\mathrm{T}}\mathbf{a}$. That is, $\mathbf{a}$ is an eigenvector for $\mathbf{Q}^{\mathrm{T}}$ for any orthogonal tensor $\mathbf{Q}$. But clearly, there is no non-zero vector which is an eigenvector for all orthogonal tensors.

5.2 Show that the most general isotropic second-order tensor is of the form of $\alpha\mathbf{I}$, where α is a scalar and $\mathbf{I}$ is the identity tensor.

Ans. For an isotropic $\mathbf{T}$, by definition, $[\mathbf{T}]_{\mathbf{e}_i}=[\mathbf{T}]_{\mathbf{e}'_i}$, where $\{\mathbf{e}_i\}$ and $\{\mathbf{e}'_i\}$ are any two orthonormal bases. Choose $[\mathbf{Q}]=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}$, then $[\mathbf{T}]_{\mathbf{e}_i}=[\mathbf{Q}]^{\mathrm{T}}_{\mathbf{e}_i}[\mathbf{T}]_{\mathbf{e}_i}[\mathbf{Q}]_{\mathbf{e}_i}$ gives

$$\begin{bmatrix}T_{11}&T_{12}&T_{13}\\T_{21}&T_{22}&T_{23}\\T_{31}&T_{32}&T_{33}\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}T_{11}&T_{12}&T_{13}\\T_{21}&T_{22}&T_{23}\\T_{31}&T_{32}&T_{33}\end{bmatrix}\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}-T_{11}&T_{12}&T_{13}\\-T_{21}&T_{22}&T_{23}\\-T_{31}&T_{32}&T_{33}\end{bmatrix}$$

$$=\begin{bmatrix}T_{11}&-T_{12}&-T_{13}\\-T_{21}&T_{22}&T_{23}\\-T_{31}&T_{32}&T_{33}\end{bmatrix}\rightarrow T_{12}=-T_{12}=0,\quad T_{21}=-T_{21}=0,\quad T_{13}=-T_{13}=0\quad T_{31}=-T_{31}=0.$$

Next, the choice of $[\mathbf{Q}]=\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$ gives,

$$\begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & T_{23} \\ 0 & T_{32} & T_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & T_{23} \\ 0 & T_{32} & T_{33} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & -T_{22} & T_{23} \\ 0 & -T_{32} & T_{33} \end{bmatrix}$$

$$= \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & -T_{23} \\ 0 & -T_{32} & T_{33} \end{bmatrix} \rightarrow T_{23} = -T_{23} = 0, \quad T_{32} = -T_{32} = 0.$$

Next, the choice of $[\mathbf{Q}] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ gives,

$$\begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} T_{22} & 0 & 0 \\ 0 & T_{11} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} \rightarrow T_{11} = T_{22}$$

Finally, the choice of $[\mathbf{Q}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ gives $\begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{11} & 0 \\ 0 & 0 & T_{33} \end{bmatrix} = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & T_{33} & 0 \\ 0 & 0 & T_{11} \end{bmatrix} \rightarrow T_{11} = T_{33}.$

Thus, $T_{11} = T_{22} = T_{33} = \alpha$ and $T_{12} = T_{21} = T_{13} = T_{31} = T_{23} = T_{32} = 0 \rightarrow [\mathbf{T}] = \alpha[\mathbf{I}]$

5.3 For an isotropic linearly elastic body, (a) verify the $\mu = \mu(\lambda, E_Y)$ as given in Table 5.1. (b) Obtain the value of μ as $E_Y / \lambda \rightarrow 0$

Ans. From Table 5.1, $\lambda = \dfrac{\mu(E_Y - 2\mu)}{3\mu - E_Y} \rightarrow 2\mu^2 + (3\lambda - E_Y)\mu - E_Y\lambda = 0$

$$\rightarrow \mu = \frac{-(3\lambda - E_Y) + \sqrt{(3\lambda - E_Y)^2 + 8E_Y\lambda}}{4}.$$

(b) $\mu = \dfrac{-(3\lambda - E_Y) + (3\lambda - E_Y)\sqrt{1 + 8(E_Y/\lambda)/(3 - E_Y/\lambda)^2}}{4}$

As $E_Y / \lambda \rightarrow 0$, $\sqrt{1 + 8(E_Y/\lambda)/(3 - E_Y/\lambda)^2} \rightarrow 1 + 4(E_Y/\lambda)/9$, where we have used the binomial theorem. Thus,

$$\mu \rightarrow \{-(3\lambda - E_Y) + (3\lambda - E_Y)[1 + 4(E_Y/\lambda)/9]\}/4 = (3\lambda - E_Y)(E_Y/\lambda)/9 = (3 - E_Y/\lambda)(E_Y)/9$$

Thus, as $E_Y / \lambda \rightarrow 0$, $\mu \rightarrow E_Y / 3$.

5.4 From $\lambda = \dfrac{\nu E_Y}{(1+\nu)(1-2\nu)}$, $\lambda = \dfrac{2\mu\nu}{(1-2\nu)}$ and $k = \dfrac{\lambda(1+\nu)}{3\nu}$ obtain $\mu = \mu(E_Y, \nu)$ and $k = k(\mu, \nu)$

$$Ans.\ \lambda = \frac{\nu E_Y}{(1+\nu)(1-2\nu)} = \frac{2\mu\nu}{(1-2\nu)} \rightarrow \mu = \frac{E_Y}{2(1+\nu)}.$$

$$\lambda = \frac{2\mu\nu}{(1-2\nu)} = \frac{3\nu k}{1+\nu} \rightarrow k = \frac{2\mu(1+\nu)}{3(1-2\nu)}.$$

5.5 Show that for an incompressible material ($\nu \rightarrow 1/2$) that

(a) $\mu = E_Y/3,\ \lambda \rightarrow \infty,\ k \rightarrow \infty,$ but $k - \lambda = 2\mu/3$

(b) $\mathbf{T} = 2\mu\mathbf{E} + (T_{kk}/3)\mathbf{I}$ where T_{kk} is constitutively indeterminate.

Ans. (a) From Table 5.1, we have

$$\mu = \frac{E_Y}{2(1+\nu)} = \frac{E_Y}{2(1+1/2)} = \frac{E_Y}{3}, \quad \lambda = \frac{\nu E_Y}{(1+\nu)(1-2\nu)} \rightarrow \infty, \quad \lambda + \frac{2}{3}\mu = k \rightarrow\ k - \lambda = \frac{2}{3}\mu.$$

(b) In general, $\mathbf{T} = \lambda e\mathbf{I} + 2\mu\mathbf{E}$. Now, from Eq.(5.4.2), we have

$e = \dfrac{T_{kk}}{(2\mu + 3\lambda)}$. As $\nu \rightarrow 1/2$, $\lambda \rightarrow \infty$, and $\lambda e \rightarrow \dfrac{T_{kk}}{3}$ so that $\mathbf{T} = \dfrac{T_{kk}}{3}\mathbf{I} + 2\mu\mathbf{E}$.

We note that because of incompressibility, T_{kk} will be constitutively indeterminate. It becomes determinate when the boundary condition(s) is (are) taken into account.

5.6 Given $A_{ijkl} = \delta_{ij}\delta_{kl}$ and $B_{ijkl} = \delta_{ik}\delta_{jl}$. (a) Obtain A_{11jk} and B_{11jk}. (b) Identity those A_{11jk} that are different from B_{11jk}.

Ans. (a) $A_{11kl} = \delta_{11}\delta_{kl} = \delta_{kl}, \quad B_{11kl} = \delta_{1k}\delta_{1l}$.

(b)

$$A_{1111} = A_{1122} = A_{1133} = 1, \ \text{ all other } A_{11kl} = 0, \quad B_{1111} = 1, \quad \text{all other } B_{11kl} = 0.$$

$$A_{1122} \neq B_{1122},\ A_{1133} \neq B_{1133}.$$

5.7 Show that for an anisotropic linear elastic material, the principal directions of stress and strain are in general not coincident.

Ans. We have, $T_{ij} = C_{ijkl}E_{kl}$. Let $\mathbf{e}_i$ be the principal basis for $\mathbf{E}$, then $[\mathbf{E}]_{\mathbf{e}_i}$ is diagonal. Thus, $T_{12} = C_{12kl}E_{kl} = C_{1211}E_{11} + C_{1222}E_{22} + C_{1233}E_{33}$. This equation shows that in general, $T_{12} \neq 0$. Similarly, in general $T_{13} \neq 0$ and $T_{23} \neq 0$. Thus the matrix of $\mathbf{T}$ is not diagonal with respect to the principal basis of $\mathbf{E}$.

5.8 If the Lamé Constants for a material are:

$$\lambda = 119.2 GPa\ (17.3\times 10^6\ psi), \mu = 79.2 GPa\ (11.5\times 10^6\ psi)$$

Find Young's modulus, Poisson's ratio and the bulk modulus.

Ans. From Table 5.1, we have,

$$E_Y = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu} = \frac{79.2\left[3(119.2)+2(79.2)\right]}{119.2+79.2} = 206\ GPa\ \left(30\times 10^6\ psi\right)$$

$$\nu = \frac{\lambda}{2(\lambda+\mu)} = \frac{119.2}{2(119.2+79.2)} = 0.3,$$

$$k = \lambda + 2\mu/3 = 119.2 + 2(79.2)/3 = 172\ GPa\ \ (25\times 10^6\ psi).$$

5.9 Given Young's modulus $E_Y = 103\ GPa$ and Poisson's ratio $\nu = 0.34$. Find the Lamé constants λ and μ. Also find the bulk modulus.

Ans.

$$\lambda = \frac{\nu E_Y}{(1+\nu)(1-2\nu)} = \frac{0.34(103)}{(1.34)(0.32)} = 81.7\ GPa\ \left(11.8\times 10^6\ psi\right)$$

$$\mu = \frac{E_Y}{2(1+\nu)} = \frac{103}{2\times 1.34} = 38.4\ GPa\ \left(5.56\times 10^6\ psi\right)$$

$$k = \lambda + 2\mu/3 = 81.7 + 2(38.4)/3 = 107.3\ GPa\ \left(15.6\times 10^6\ psi\right)$$

5.10 Given Young's modulus $E_Y = 193\ GPa$., shear modulus $\mu = 76\ GPa$. Find Poisson's ratio ν, Lamé constant λ and the bulk modulus k.

Ans. $\nu = \frac{E_Y}{2\mu} - 1 = \frac{193}{2(76)} - 1 = 0.27,\ \lambda = \frac{2\mu\nu}{1-2\nu} = \frac{2(76)(0.27)}{1-0.54} = 89.1\ GPa\ \left(12.9\times 10^6 psi\right)$

$$k = \lambda + 2\mu/3 = 89.1 + 2(76)/3 = 140\ GPa\ \left(20.3\times 10^6 psi\right)$$

5.11 The components of strain at a point of structural steel are:

$$E_{11} = 36\times 10^{-6},\ E_{22} = 40\times 10^{-6},\ E_{33} = 25\times 10^{-6}$$

$$E_{12} = 12\times 10^{-6},\ E_{23} = 0,\ E_{13} = 30\times 10^{-6}$$

Find the stress components. $\lambda = 119.2\ GPa\ \left(17.3\times 10^6 psi\right),\ \mu = 79.2\ GPa\ \left(11.5\times 10^6 psi\right)$

Ans. From $\mathbf{T} = \lambda e\mathbf{I} + 2\mu\mathbf{E}$, we have, with $e = (36+40+25)\times 10^{-6} = 101\times 10^{-6}$

$$[\mathbf{T}] = (119.2)(101)\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\times 10^{-6} + 2(79.2)\begin{bmatrix}36 & 12 & 30\\12 & 40 & 0\\30 & 0 & 25\end{bmatrix}\times 10^{-6} = \begin{bmatrix}17.7 & 1.9 & 4.75\\1.9 & 18.4 & 0\\4.75 & 0 & 16.0\end{bmatrix} MPa.$$

5.12 Do the previous problem if the strain components are:

$$E_{11} = 100\times 10^{-6},\ E_{22} = -200\times 10^{-6},\ E_{33} = 100\times 10^{-6}$$

$$E_{12} = -100\times 10^{-6},\ E_{23} = 0,\ E_{13} = 0$$

Ans. From $\mathbf{T} = \lambda e\mathbf{I} + 2\mu\mathbf{E}$, we have, with $e = (100 - 200 + 100)\times 10^{-6} = 0$

$$[\mathbf{T}] = 2(79.2)\begin{bmatrix}100 & -100 & 0\\-100 & -200 & 0\\0 & 0 & 100\end{bmatrix}\times 10^{-6} = \begin{bmatrix}15.8 & -15.8 & 0\\-15.8 & -31.7 & 0\\0 & 0 & 15.8\end{bmatrix} MPa.$$

5.13 An isotropic elastic body $(E_Y = 207GPa,\quad \mu = 79.2GPa)$ has a uniform state of stress

$$\text{given by: } [\mathbf{T}] = \begin{bmatrix}100 & 40 & 60\\40 & -200 & 0\\60 & 0 & 200\end{bmatrix} MPa.$$

(a) What are the strain components?
(b) What is the total change of volume for a five centimeter cube of the material?

Ans. (a) We would like to use the equation $E_{ij} = \frac{1}{E_Y}\left[(1+\nu)T_{ij} - \nu(T_{kk})\delta_{ij}\right]$, therefore, we first obtain $\nu = \frac{E_Y}{2\mu} - 1 = \frac{207}{2(79.2)} - 1 = 0.306$, then obtain $T_{kk} = T_{11} + T_{22} + T_{33} = 100MPa$. Thus,

$$[\mathbf{E}] = \frac{1}{E_Y}\left\{1.31\begin{bmatrix}100 & 40 & 60\\40 & -200 & 0\\60 & 0 & 200\end{bmatrix} - (0.306)(100)\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}\right\}$$

$$= \frac{1}{207\times 10^3}\begin{bmatrix}100 & 52.4 & 78.6\\52.4 & -292 & 0\\78.6 & 0 & 231\end{bmatrix} = \begin{bmatrix}0.483 & 0.253 & 0.380\\0.253 & -1.41 & 0\\0.380 & 0 & 1.12\end{bmatrix}\times 10^{-3}$$

(b) Dilatation $e = E_{kk} = (0.483 - 1.41 + 1.12)\times 10^{-3} = 0.193\times 10^{-3}$.

Total change of volume $= \Delta V = (V)(e) = (5^3)(0.193\times 10^{-3}) = 24.1\times 10^{-3}\ cm^3$.

5.14 An isotropic elastic sphere $(E_Y = 207GPa, \quad \mu = 79.2GPa)$ of 5 cm radius is under the uniform stress field

$$[\mathbf{T}] = \begin{bmatrix} 6 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

Find the change of volume for the sphere.

Ans. $\nu = \dfrac{E_Y}{2\mu} - 1 = \dfrac{207}{2(79.2)} - 1 = 0.306$, $E_{ij} = \dfrac{1}{E_Y}\left[(1+\nu)T_{ij} - \nu(T_{kk})\delta_{ij}\right]$ gives

$$[\mathbf{E}] = \frac{1}{E_Y}\left\{1.31\begin{bmatrix} 6 & 2 & 0 \\ 2 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} - (0.306)(3)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right\} = \begin{bmatrix} 3.35 & 1.26 & 0 \\ 1.26 & -2.34 & 0 \\ 0 & 0 & -0.443 \end{bmatrix} \times 10^{-5}$$

Thus, $e = 0.567 \times 10^{-5} \rightarrow \Delta V = (0.567)\left(\dfrac{4\pi 5^3}{3}\right) \times 10^{-5} = 2.96 \times 10^{-3}$

5.15 Given a motion

$x_1 = X_1 + k(X_1 + X_2), \quad x_2 = X_2 + k(X_1 - X_2)$, show that for a function $f(a,b) = ab$

(a) $f(x_1, x_2) = f(X_1, X_2) + \mathrm{O}(k)$, $\dfrac{\partial f(x_1, x_2)}{\partial x_1} = \dfrac{\partial f(X_1, X_2)}{\partial X_1} + \mathrm{O}(k)$,

where $\mathrm{O}(k) \rightarrow 0$ as $k \rightarrow 0$

Ans. (a) $f(x_1, x_2) = x_1 x_2 = \left[X_1 + k(X_1 + X_2)\right]\left[X_2 + k(X_1 - X_2)\right]$

$= X_1 X_2 + k\left\{X_1(X_1 - X_2) + X_2(X_1 + X_2)\right\} + k^2(X_1 + X_2)(X_1 - X_2)$.

That is, $f(x_1, x_2) = X_1 X_2 + \mathrm{O}(k)$, where $\mathrm{O}(k) \rightarrow 0$ as $k \rightarrow 0$, i.e.

$f(x_1, x_2) = f(X_1, X_2) + \mathrm{O}(k) \rightarrow f(x_1, x_2) \approx f(X_1, X_2)$ as $k \rightarrow 0$.

(b) $f(x_1, x_2) = x_1 x_2 \rightarrow \dfrac{\partial f}{\partial x_1} = x_2 = X_2 + k(X_1 - X_2) = X_2 + \mathrm{O}(k)$, and

$f(X_1, X_2) = X_1 X_2 \rightarrow \dfrac{\partial f}{\partial X_1} = X_2$. Thus, $\dfrac{\partial f}{\partial x_1} \approx \dfrac{\partial f}{\partial X_1}$ as $k \rightarrow 0$.

5.16 Do the previous problem for $f(a,b) = a^2 + b^2$

Ans. (a) $f(x_1, x_2) = x_1^2 + x_2^2 = \left[X_1 + k(X_1 + X_2)\right]^2 + \left[X_2 + k(X_1 - X_2)\right]^2$

$= X_1^2 + X_2^2 + 2k\left\{X_1(X_1 + X_2) + X_2(X_1 - X_2)\right\} + k^2\left\{(X_1 + X_2)^2 + (X_1 - X_2)^2\right\}$.

That is, $f(x_1, x_2) = X_1^2 + X_2^2 + \mathrm{O}(k)$, where $\mathrm{O}(k) \to 0$ as $k \to 0$, i.e.

$$f(x_1, x_2) = f(X_1, X_2) + \mathrm{O}(k) \to f(x_1, x_2) \approx f(X_1 X_2) \text{ as } k \to 0.$$

(b) $f(x_1, x_2) = x_1^2 + x_2^2 \to \dfrac{\partial f}{\partial x_1} = 2x_1 = 2X_1 + 2k(X_1 + X_2) = 2X_1 + \mathrm{O}(k)$ and

$f(X_1, X_2) = X_1^2 + X_2^2 \to \dfrac{\partial f}{\partial X_1} = 2X_1$. Thus, $\dfrac{\partial f}{\partial x_1} \approx \dfrac{\partial f}{\partial X_1}$ as $k \to 0$.

5.17 Given the following displacement field in an isotropic linearly elastic solid:

$$u_1 = kX_3X_2,\ u_2 = kX_3X_1,\ u_3 = k\left(X_1^2 - X_2^2\right),\ k = 10^{-4}$$

(a) Find the stress components and (b) in the absence of body forces, is the state of stress a possible equilibrium stress field?

Ans.

(a) $$[\nabla \mathbf{u}] = \begin{bmatrix} 0 & kX_3 & kX_2 \\ kX_3 & 0 & kX_1 \\ 2kX_1 & -2kX_2 & 0 \end{bmatrix} \to [\mathbf{E}] = \frac{k}{2}\begin{bmatrix} 0 & 2X_3 & (2X_1 + X_2) \\ 2X_3 & 0 & (X_1 - 2X_2) \\ (2X_1 + X_2) & (X_1 - 2X_2) & 0 \end{bmatrix}$$

Thus, $$E_{kk} = 0 \to [\mathbf{T}] = 2\mu[\mathbf{E}] = \mu k \begin{bmatrix} 0 & 2X_3 & (2X_1 + X_2) \\ 2X_3 & 0 & (X_1 - 2X_2) \\ (2X_1 + X_2) & (X_1 - 2X_2) & 0 \end{bmatrix}$$

Since the displacement components are small (of the order of k), therefore, $x_i \approx X_i$, so that

$$T_{11} = T_{22} = T_{33} = 0, \quad T_{12} = T_{21} = 2\mu k x_3, \quad T_{13} = T_{31} = \mu k(2x_1 + x_2), \quad T_{23} = T_{32} = \mu k(x_1 - 2x_2).$$

(b) Substituting the above stress components into the equations of equilibrium, we have,

$\dfrac{\partial T_{11}}{\partial x_1} + \dfrac{\partial T_{12}}{\partial x_2} + \dfrac{\partial T_{13}}{\partial x_3} = 0 \to 0 + 0 + 0 = 0$, $\dfrac{\partial T_{21}}{\partial x_1} + \dfrac{\partial T_{22}}{\partial x_2} + \dfrac{\partial T_{23}}{\partial x_3} = 0 \to 0 + 0 + 0 = 0$ and

$\dfrac{\partial T_{31}}{\partial x_1} + \dfrac{\partial T_{32}}{\partial x_2} + \dfrac{\partial T_{33}}{\partial x_3} = 0 \to 2\mu k - 2\mu k + 0 = 0$. Thus, all equations of equilibrium are satisfied.

Since the stress field is obtained from a given displacement field, therefore, the state of stress is a possible equilibrium stress field.

5.18 Given the following displacement field in an isotropic linearly elastic solid:

$$u_1 = kX_2X_3,\ u_2 = kX_1X_3,\ u_3 = kX_1X_2,\ k = 10^{-4}$$

(a) Find the stress components and (b) in the absence of body forces, is the state of stress a possible equilibrium stress field?

Ans.

(a) $\left[\nabla\mathbf{u}\right]=\begin{bmatrix}0 & kX_3 & kX_2\\ kX_3 & 0 & kX_1\\ kX_2 & kX_1 & 0\end{bmatrix}=\left[\mathbf{E}\right]$, Thus, $E_{kk}=0\rightarrow\left[\mathbf{T}\right]=2\mu\left[\mathbf{E}\right]=2\mu k\begin{bmatrix}0 & X_3 & X_2\\ X_3 & 0 & X_1\\ X_2 & X_1 & 0\end{bmatrix}$

Since the displacement components are small (of the order of k), therefore, $x_i\approx X_i$, so that

$$T_{11}=T_{22}=T_{33}=0,\quad T_{12}=T_{21}=2\mu kx_3,\quad T_{13}=T_{31}=2\mu kx_2,\quad T_{23}=T_{32}=2\mu kx_1.$$

(b) Substituting the above stress components into the equations of equilibrium, we have,

$\frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}+\frac{\partial T_{13}}{\partial x_3}=0\rightarrow 0+0+0=0$, $\frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3}=0\rightarrow 0+0+0=0$ and

$\frac{\partial T_{31}}{\partial x_1}+\frac{\partial T_{32}}{\partial x_2}+\frac{\partial T_{33}}{\partial x_3}=0\rightarrow 0+0+0=0$. Thus, all equations of equilibrium are satisfied. Since the stress field is obtained from a given displacement field, therefore, the state of stress is a possible equilibrium stress field.

5.19 Given the following displacement field in an isotropic linearly elastic solid:

$$u_1=kX_2X_3,\ u_2=kX_1X_3,\ u_3=k\left(X_1X_2+X_3^2\right),\ k=10^{-4}$$

(a) Find the stress components and (b) in the absence of body forces, is the state of stress a possible equilibrium stress field?

Ans.

(a) $\left[\nabla\mathbf{u}\right]=\begin{bmatrix}0 & kX_3 & kX_2\\ kX_3 & 0 & kX_1\\ kX_2 & kX_1 & 2kX_3\end{bmatrix}=\left[\mathbf{E}\right]$, Thus, $E_{kk}=2kX_3\rightarrow\left[\mathbf{T}\right]=\lambda\left(2kX_3\right)\mathbf{I}+2\mu\left[\mathbf{E}\right]$

Since the displacement components are small (of the order of k), therefore, $x_i\approx X_i$, so that

$$\left[\mathbf{T}\right]=2k\begin{bmatrix}\lambda x_3 & \mu x_3 & \mu x_2\\ \mu x_3 & \lambda x_3 & \mu x_1\\ \mu x_2 & \mu x_1 & \left(\lambda+2\mu\right)x_3\end{bmatrix}.$$

(b) Substituting the above stress components into the equations of equilibrium, we have,

$\frac{\partial T_{11}}{\partial x_1}+\frac{\partial T_{12}}{\partial x_2}+\frac{\partial T_{13}}{\partial x_3}=0\rightarrow 0+0+0=0$, $\frac{\partial T_{21}}{\partial x_1}+\frac{\partial T_{22}}{\partial x_2}+\frac{\partial T_{23}}{\partial x_3}=0\rightarrow 0+0+0=0$ and

$\frac{\partial T_{31}}{\partial x_1}+\frac{\partial T_{32}}{\partial x_2}+\frac{\partial T_{33}}{\partial x_3}=0\rightarrow 0+0+\left(\lambda+2\mu\right)\neq 0$. Thus, the stress field is ***not*** an equilibrium stress field in the absence of body forces. The given state of stress is not a possible equilibrium stress field.

5.20 Show that for any function $f(s)$, the displacement $u_1 = f(s)$ where $s = x_1 \pm c_L t$ satisfies the wave equation $\dfrac{\partial^2 u_1}{\partial t^2} = c_L^2 \dfrac{\partial^2 u_1}{\partial x_1^2}$

Ans.

$$\frac{\partial u_1}{\partial t} = \frac{df}{ds}\frac{\partial s}{\partial t} = \frac{df}{ds}(\pm c_L) \rightarrow \frac{\partial^2 u_1}{\partial t^2} = (\pm c_L)\frac{d^2 f}{ds^2}\frac{\partial s}{\partial t} = c_L^2 \frac{d^2 f}{ds^2},$$

$$\frac{\partial u_1}{\partial x_1} = \frac{df}{ds}\frac{\partial s}{\partial x_1} = \frac{df}{ds} \rightarrow \frac{\partial^2 u_1}{\partial x_1^2} = \frac{d^2 f}{ds^2}\frac{\partial s}{\partial x_1} = \frac{d^2 f}{ds^2}.$$

Thus, $\dfrac{\partial^2 u_1}{\partial t^2} = c_L^2 \dfrac{d^2 f}{ds^2} = c_L^2 \dfrac{\partial^2 u_1}{\partial x_1^2}$

5.21 Calculate the ratio of the phase velocities c_L / c_T for Poisson 's ratio equal $1/3$, 0.49 and 0.499.

Ans. From Table 5.1, we have $\lambda = \dfrac{2\mu\nu}{1-2\nu} \rightarrow \lambda + 2\mu = \dfrac{2\mu(1-\nu)}{1-2\nu} \rightarrow \dfrac{\mu}{\lambda+2\mu} = \dfrac{1-2\nu}{2(1-\nu)}$,

Thus, $\dfrac{c_L}{c_T} = \sqrt{\dfrac{\lambda+2\mu}{\mu}} = \sqrt{\dfrac{2(1-\nu)}{1-2\nu}}$. Thus, for

$\nu = 1/3,\ c_L / c_T = \sqrt{2(2/3)/(1-2/3)} = \sqrt{(4/3)/(1/3)} = 2.$

$\nu = 0.49,\ c_L / c_T = \sqrt{2(0.51)/(1-0.98)} = \sqrt{1.02/0.02} = 7.14.$

$\nu = 0.499,\ c_L / c_T = \sqrt{2(0.501)/(1-0.998)} = \sqrt{1.002/0.002} = 22.4.$

5.22 Assume a displacement which depends only x_2 and t, i.e., $u_i = u_i(x_2, t), i = 1,2,3$. Obtain the differential equations which $u_i(x_2, t)$ must satisfy in order to be a possible motion in the absence of body forces.

Ans. From the Navier equations, we have,

$e = \partial u_2 / \partial x_2 \rightarrow \partial e / \partial x_1 = 0,\ \partial e / \partial x_2 = \partial^2 u_2 / \partial x_2^2,\ \partial e / \partial x_3 = 0.$ Thus,

$\rho_o (\partial u_1^2 / \partial t^2) = \mu(\partial^2 u_1 / \partial x_2^2) \rightarrow (\partial u_1^2 / \partial t^2) = c_T^2 (\partial^2 u_1 / \partial x_2^2),$

$\rho_o (\partial u_2^2 / \partial t^2) = (\lambda + \mu)(\partial^2 u_2 / \partial x_2^2) + \mu(\partial^2 u_2 / \partial x_2^2),$

$\rightarrow \rho_o (\partial u_2^2 / \partial t^2) = (\lambda + 2\mu)(\partial^2 u_2 / \partial x_2^2) \rightarrow (\partial u_2^2 / \partial t^2) = c_L^2 \left(\partial^2 u_2 / \partial x_2^2\right)$

$\rho_o (\partial u_3^2 / \partial t^2) = \mu(\partial^2 u_3 / \partial x_2^2) \rightarrow (\partial u_3^2 / \partial t^2) = c_T^2 (\partial^2 u_3 / \partial x_2^2).$

5.23 Consider a linear elastic medium. Assume the following form for the displacement field:

$$u_1 = \varepsilon\left[\sin\beta\left(x_3 - ct\right) + \alpha\sin\beta\left(x_3 + ct\right)\right], \quad u_2 = u_3 = 0$$

(a) What is the nature of this elastic wave (longitudinal, transverse, direction of propagation?)
(b) Find the strains, stresses and determine under what condition(s), the equations of motion are satisfied in the absence of body forces.
(c)Suppose that there is boundary at $x_3 = 0$ that is traction free. Under what condition(s) will the above motion satisfy this boundary condition for all time.
(d) Suppose that there is boundary at $x_3 = \ell$ that is also traction free. What further conditions will be imposed on the above motion to satisfy this boundary condition for all time.

Ans. (a) Transverse wave, propagating in the $\mathbf{e}_3$ direction.

(b)The only nonzero strain components are:

$$E_{13} = E_{31} = \left(1/2\right)\left(\partial u_1 / \partial x_3\right) = \left(\varepsilon\beta / 2\right)\left[\cos\beta\left(x_3 - ct\right) + \alpha\cos\beta\left(x_3 + ct\right)\right].$$

The only nonzero stress components are:

$$T_{13} = T_{31} = \mu\left(\partial u_1 / \partial x_3\right) = \left(\varepsilon\mu\beta\right)\left[\cos\beta\left(x_3 - ct\right) + \alpha\cos\beta\left(x_3 + ct\right)\right],$$

x_1 equation of motion is: $\rho_0\left(\partial^2 u_1 / \partial t^2\right) = \partial T_{13} / \partial x_3 \rightarrow -\rho_0\beta^2 c^2 u_1 = -\beta^2\mu u_1 \rightarrow c^2 = \mu / \rho_0$.

The other two equations are 0=0.

(c) The boundary condition on $x_3 = 0$ is:

$$\mathbf{T}\left(-\mathbf{e}_3\right) = 0 \rightarrow T_{13}\left(0,t\right) = 0 \rightarrow \left[\cos\beta ct + \alpha\cos\beta ct\right] = 0 \rightarrow \alpha = -1.$$

(d) The boundary condition on $x_3 = \ell$ is,

$$\mathbf{T}\left(\mathbf{e}_3\right) = 0 \rightarrow T_{13}\left(\ell,t\right) = 0 \rightarrow \left[\cos\beta\left(\ell - ct\right) - \cos\beta\left(\ell + ct\right)\right] = 0, \text{[note } \alpha = -1\text{]}.$$

$$\rightarrow 2\sin\beta\ell\sin\beta ct = 0 \rightarrow \sin\beta\ell = 0 \rightarrow \beta = n\pi / \ell, n = 1,2,3...$$

5.24 Do the previous problem (Prob. 5.23) if the boundary $x_3 = 0$ is fixed (no motion) and $x_3 = \ell$ is still traction free.

Ans. (a) and (b) are the same as in the previous problem.

(c) The boundary condition on $x_3 = 0$ is: $u_1\left(0,t\right) = 0 \rightarrow u_1 = \varepsilon\left[-\sin\beta ct + \alpha\sin\beta ct\right] = 0 \rightarrow \alpha = 1$.

(d) The boundary condition on $x_3 = \ell$ is:

$$\mathbf{T}\left(\mathbf{e}_3\right) = 0 \rightarrow T_{13}\left(\ell,t\right) = 0 \rightarrow \left[\cos\beta\left(\ell - ct\right) + \cos\beta\left(\ell + ct\right)\right] = 0$$

$$\rightarrow 2\cos\beta\ell\cos\beta ct = 0 \rightarrow \cos\beta\ell = 0 \rightarrow \beta = n\pi / \left(2\ell\right), n = 1,3,5...$$

5.25 Do Problem 5.23 if the boundary $x_3 = 0$ and $x_3 = \ell$ are both rigidly fixed (no motion)

Ans. (a) and (b) are the same as in the previous problem 5.23.

(c) The boundary condition on $x_3=0$ is: $u_1(0,t)=0 \to u_1=\varepsilon[-\sin\beta ct+\alpha\sin\beta ct]=0 \to \alpha=1$

(d) The boundary condition on $x_3=\ell$ is:

$$u_1(\ell,t)=0 \to u_1=\varepsilon\left[\sin\beta(\ell-ct)+\sin\beta(\ell+ct)\right]=0$$
$$\to \sin\beta\ell\cos\beta ct \to 0 \to \beta=n\pi/\ell, n=1,2,3...$$

5.26 Do Problem 5.23, if the assumed displacement field is of the form:

$$u_3=\varepsilon\left[\sin\beta(x_3-ct)+\alpha\sin\beta(x_3+ct)\right],\quad u_1=u_2=0$$

Ans. (a) Longitudinal, propagating in the $\mathbf{e}_3$ direction.

(b)The only nonzero strain components are:

$$E_{33}=(\partial u_3/\partial x_3)=\varepsilon\beta\left[\cos\beta(x_3-ct)+\alpha\cos\beta(x_3+ct)\right].$$

The nonzero stress components are:

$$T_{11}=T_{22}=\lambda(\partial u_3/\partial x_3),\ T_{33}=\lambda(\partial u_3/\partial x_3)+2\mu(\partial u_3/\partial x_3)=(\lambda+2\mu)(\partial u_3/\partial x_3),$$

where $(\partial u_3/\partial x_3)=(\varepsilon\beta)\left[\cos\beta(x_3-ct)+\alpha\cos\beta(x_3+ct)\right]$.

x_3 equation of motion is:

$$\rho_0\left(\partial^2 u_3/\partial t^2\right)=\partial T_{33}/\partial x_3 \to -\rho_0\beta^2c^2u_3=-\beta^2(\lambda+2\mu)u_3 \to c^2=(\lambda+2\mu)/\rho_0.$$

The other two equations are 0=0.

(c) The boundary condition on $x_3=0$ is:

$$\mathbf{T}(-\mathbf{e}_3)=0 \to T_{33}(0,t)=0 \to (\lambda+2\mu)(\varepsilon\beta)\left[\cos\beta(ct)+\alpha\cos\beta(ct)\right]=0 \to \alpha=-1.$$

(d) The boundary condition on $x_3=\ell$ is:

$$\mathbf{T}(\mathbf{e}_3)=0 \to T_{33}(\ell,t)=0 \to \left[\cos\beta(\ell-ct)-\cos\beta(\ell+ct)\right]=0.\text{[Note } \alpha=-1\text{]},$$
$$\to 2\sin\beta\ell\sin\beta ct=0 \to \sin\beta\ell=0 \to \beta=n\pi/\ell, n=1,2,3...$$

5.27 Do the previous problem, Problem 5.26, if the boundary $x_3=0$ is fixed (no motion) and $x_3=\ell$ is traction free ($\mathbf{t}=\mathbf{0}$).

Ans. (a) and (b) are the same as the previous problem, problem 5.26.

(c) The boundary condition on $x_3=0$ is: $u_3(0,t)=0 \to u_3=\varepsilon[-\sin\beta ct+\alpha\sin\beta ct]=0 \to \alpha=1$.

(d) The boundary condition on $x_3=\ell$ is, with $\alpha=1$,

$$\mathbf{T}(\mathbf{e}_3)=0 \to T_{33}(\ell,t)=0 \to \left[\cos\beta(\ell-ct)+\cos\beta(\ell+ct)\right]=0$$
$$\to 2\cos\beta\ell\cos\beta ct=0 \to \cos\beta\ell=0 \to \beta=n\pi/(2\ell), n=1,3,5...$$

5.28 Do Problem 5.26, if the boundary $x_3=0$ and boundary $x_3=\ell$ are both rigidly fixed.

Ans. (a) (b) and (c) are the same as in Prob. 5.27, with $\alpha = 1$.

(d) The boundary condition on $x_3 = \ell$ is

$$u_3(\ell, t) = 0 \rightarrow u_3 = \varepsilon\left[\sin\beta(\ell - ct) + \sin\beta(\ell + ct)\right] = 0$$

$$\rightarrow \sin\beta\ell\cos\beta ct = 0 \rightarrow \sin\beta\ell = 0 \rightarrow \beta = n\pi/\ell, n = 1, 2, 3...$$

5.29 Consider the displacement field: $u_i = u_i(x_1, x_2, x_3, t)$. In the absence of body forces, (a) obtain the governing equation for u_i for the case where the motion is equivoluminal and (b) obtain the governing equation for the dilatation e for the case where the motion is irrotational $\left(\partial u_i/\partial x_j = \partial u_j/\partial x_i\right)$.

Ans. From the Navier equations of motion, Eq. (5.6.4)

$$\rho_o \frac{\partial^2 u_i}{\partial t^2} = \rho_o B_i + (\lambda + \mu)\frac{\partial e}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$, we have

(a) with $e = 0$ and $B_i = 0$, $\rho_o \frac{\partial^2 u_i}{\partial t^2} = \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$.

(b) For irrotational motion $\frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i} \rightarrow \frac{\partial^2 u_i}{\partial x_j \partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial u_j}{\partial x_i} = \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} = \frac{\partial e}{\partial x_i}$. Thus,

$$\rho_o \frac{\partial^2 u_i}{\partial t^2} = (\lambda + \mu)\frac{\partial e}{\partial x_i} + \mu\frac{\partial e}{\partial x_i} = (\lambda + 2\mu)\frac{\partial e}{\partial x_i} \rightarrow \rho_o \frac{\partial^2}{\partial t^2}\frac{\partial u_i}{\partial x_i} = (\lambda + 2\mu)\frac{\partial^2 e}{\partial x_i \partial x_i}$$

$$\rightarrow \frac{\partial^2 e}{\partial t^2} = \frac{(\lambda + 2\mu)}{\rho_o}\frac{\partial^2 e}{\partial x_i \partial x_i}.$$

5.30 (a) Write a displacement field for an infinite train of longitudinal waves propagating in the direction of $3\mathbf{e}_1 + 4\mathbf{e}_2$. (b) Write a displacement field for an infinite train of transverse waves propagating in the direction of $3\mathbf{e}_1 + 4\mathbf{e}_2$ and polarized in the x_1x_2 plane.

Ans. Let $\mathbf{e}_n = (1/5)(3\mathbf{e}_1 + 4\mathbf{e}_2)$, then $\mathbf{x}\cdot\mathbf{e}_n = (1/5)(3x_1 + 4x_2)$. Also, $\mathbf{e}_t = \pm(1/5)(4\mathbf{e}_1 - 3\mathbf{e}_2)$

(a) Equation 5.10.8 of Example 5.10.3 gives $\mathbf{u} = \varepsilon\sin\left(\frac{2\pi}{\ell}\mathbf{x}\cdot\mathbf{e}_n - c_L t - \eta\right)\mathbf{e}_n$. Thus,

$$u_1 = \frac{3\varepsilon}{5}\sin\left[\frac{2\pi}{\ell}\left(\frac{3x_1}{5} + \frac{4x_2}{5} - c_L t - \eta\right)\right],\ u_2 = \frac{4\varepsilon}{5}\sin\left[\frac{2\pi}{\ell}\left(\frac{3x_1}{5} + \frac{4x_2}{5} - c_L t - \eta\right)\right],\ u_3 = 0$$

(b) Equation 5.10.10 of Example 5.10.3 gives $\mathbf{u} = \varepsilon\sin\left(\frac{2\pi}{\ell}\mathbf{x}\cdot\mathbf{e}_n - c_T t - \eta\right)\mathbf{e}_t$. Thus,

$$u_1 = \pm\frac{4\varepsilon}{5}\sin\left[\frac{2\pi}{\ell}\left(\frac{3x_1}{5} + \frac{4x_2}{5} - c_T t - \eta\right)\right],\ u_2 = \mp\frac{3\varepsilon}{5}\sin\left[\frac{2\pi}{\ell}\left(\frac{3x_1}{5} + \frac{4x_2}{5} - c_T t - \eta\right)\right],\ u_3 = 0$$

5.31 Solve for ε_2 and ε_3 in terms of ε_1 from the following two algebra equations:

$$\varepsilon_2(\cos 2\alpha_1)+\varepsilon_3 n(\sin 2\alpha_3)=\varepsilon_1\cos 2\alpha_1 \text{ (i) } \varepsilon_2\sin 2\alpha_1-\varepsilon_3\frac{1}{n}(\cos 2\alpha_1)=-\varepsilon_1\sin 2\alpha_1, \text{ (ii)}$$

Ans.

$$\Delta=\begin{vmatrix}\cos 2\alpha_1 & n(\sin 2\alpha_3)\\ \sin 2\alpha_1 & -\frac{1}{n}(\cos 2\alpha_1)\end{vmatrix}=-\frac{1}{n}\left[(\cos 2\alpha_1)^2+n^2(\sin 2\alpha_3)\sin 2\alpha_1\right]$$

Thus,

$$\begin{bmatrix}\varepsilon_2\\ \varepsilon_3\end{bmatrix}=\frac{1}{\Delta}\begin{bmatrix}-\frac{1}{n}(\cos 2\alpha_1) & -n(\sin 2\alpha_3)\\ -\sin 2\alpha_1 & \cos 2\alpha_1\end{bmatrix}\begin{bmatrix}\varepsilon_1\cos 2\alpha_1\\ -\varepsilon_1\sin 2\alpha_1\end{bmatrix}$$

$$=-\frac{\varepsilon_1}{n\Delta}\begin{bmatrix}(\cos 2\alpha_1)^2-n^2(\sin 2\alpha_3)(\sin 2\alpha_1)\\ n\sin 4\alpha_1\end{bmatrix}$$

That is,

$$\varepsilon_2=\varepsilon_1\frac{(\cos 2\alpha_1)^2-n^2(\sin 2\alpha_3)(\sin 2\alpha_1)}{(\cos 2\alpha_1)^2+n^2(\sin 2\alpha_3)\sin 2\alpha_1},\quad \varepsilon_3=\varepsilon_1\frac{n\sin 4\alpha_1}{(\cos 2\alpha_1)^2+n^2(\sin 2\alpha_3)\sin 2\alpha_1}$$

5.32 A transverse elastic wave of amplitude ε_1 incidents on a traction free plane boundary. If the Poisson's ratio $\nu=1/3$, determine the amplitudes and angles of reflection of the reflected waves for the following two incident angles (a) $\alpha_1=0$ and (b) $\alpha_1=15^\circ$.

Ans. From Eq. (5.11.14), we have, for $\nu=1/3$

$$n=c_T/c_L=\sqrt{(1-2\nu)/2(1-\nu)}=\sqrt{(1-2/3)/2(1-1/3)}=\sqrt{(1/3)/2(2/3)}=1/2.$$

Thus, $(1/2)\sin\alpha_3=\sin\alpha_1$. Using this equation, and Equations

$$\varepsilon_2=\varepsilon_1\frac{(\cos 2\alpha_1)^2-n^2(\sin 2\alpha_3)(\sin 2\alpha_1)}{(\cos 2\alpha_1)^2+n^2(\sin 2\alpha_3)\sin 2\alpha_1},\quad \varepsilon_3=\varepsilon_1\frac{n\sin 4\alpha_1}{(\cos 2\alpha_1)^2+n^2(\sin 2\alpha_3)\sin 2\alpha_1}$$

we have,

(a) $\alpha_1=0\rightarrow\alpha_2=0$ and $(1/2)\sin\alpha_3=\sin\alpha_1=0\rightarrow\alpha_3=0.$ Also, the above equations give $\varepsilon_2=\varepsilon_1$, and $\varepsilon_3=0$,. That is, there is no reflected longitudinal wave. There is only a reflected transverse wave of the same amplitude which completely cancels out the incident transverse wave.

(b) $\alpha_1=15^\circ\rightarrow\alpha_2=15^\circ$ and $\sin\alpha_3=2\sin 15^\circ=0.5176\rightarrow\alpha_3=31.17^\circ$,

$$\varepsilon_2 = \varepsilon_1 \frac{(\cos 30^o)^2 - (1/4)(\sin 62.34^o)\left(\sin 30^o\right)}{\left(\cos 30^o\right)^2 + (1/4)(\sin 62.34^o)\left(\sin 30^o\right)} = 0.742\varepsilon_1$$

$$\varepsilon_3 = \varepsilon_1 \frac{\sin 60^0 / 2}{\left(\cos 30^o\right)^2 + (1/4)(\sin 62.34^o)\left(\sin 30^o\right)} = 0.503\varepsilon_1$$

That, the reflected transverse wave has an amplitude $\varepsilon_2 = 0.742\varepsilon_1$, with a reflected angle of $\alpha_2 = \alpha_1 = 15^o$. The reflected longitudinal wave has an amplitude $\varepsilon_3 = 0.503\varepsilon_1$,with a reflected angle of $\alpha_3 = 31.17^o$.

5.33 Referring to Figure 5.11.1 (Section 5.11), consider a transverse elastic wave incident on a traction-free plane surface $(x_2 = 0)$ with an angle of incident α_1 with the x_2 axis and polarized normal to $x_1 x_2$, the plane of incidence. Show that the boundary condition at $x_2 = 0$ can be satisfied with only a reflected transverse wave that is similarly polarized. What is the relation of the amplitudes, wavelengths, and direction of propagation of the incident and reflected wave?

Ans. Let the plane of incidence be $x_1 x_2$ plane with the angle of incidence of the transverse wave be α_1. That is, $\mathbf{e}_{n_1} = \sin\alpha_1 \mathbf{e}_1 - \cos\alpha_1 \mathbf{e}_2$. The waves are polarized normal to the plane of incidence, therefore, $u_1 = u_2 = 0$, and $u_3 = \varepsilon_1 \sin\varphi_1 + \varepsilon_2 \sin\varphi_2$, with

$$\varphi_1 = \frac{2\pi}{\ell_1}(x_1 \sin\alpha_1 - x_2\cos\alpha_1 - c_T t - \eta_1), \quad \varphi_2 = \frac{2\pi}{\ell_2}(x_1\sin\alpha_2 + x_2\cos\alpha_2 - c_T t - \eta_2)$$

The nonzero stress components are:

$$T_{13} = T_{31} = \mu \partial u_3 / \partial x_1 = 2\pi\mu\left[(\varepsilon_1/\ell_1)\cos\varphi_1 \sin\alpha_1 + (\varepsilon_2/\ell_2)\cos\varphi_2\sin\alpha_2\right],$$

$$T_{23} = T_{32} = \mu\partial u_3/\partial x_2 = 2\pi\mu\left[-(\varepsilon_1/\ell_1)\cos\varphi_1\cos\alpha_1 + (\varepsilon_2/\ell_2)\cos\varphi_2\cos\alpha_2\right].$$

The x_3 equation of motion $\rho_o\left(\partial^2 u_3/\partial t^2\right) = \partial T_{31}/\partial x_1 + \partial T_{32}/\partial x_2$ gives:

$$\rho_o(2\pi c_T)^2\left[\left(\varepsilon_1/\ell_1^2\right)\sin\varphi_1 + \left(\varepsilon_2/\ell_2^2\right)\sin\varphi_2\right] = (2\pi)^2\mu\left[\left(\varepsilon_1/\ell_1^2\right)\sin\varphi_1 + \left(\varepsilon_2/\ell_2^2\right)\sin\varphi_2\right]$$

$$\to \rho_o c_T^2 = \mu \to c_T^2 = \mu/\rho_o.$$

The traction free boundary at $x_2 = 0$ requires that $T_{12} = T_{22} = T_{32} = 0$ on the surface, thus,

$$\left[-(\varepsilon_1/\ell_1)\cos\varphi_1\cos\alpha_1 + (\varepsilon_2/\ell_2)\cos\varphi_2\cos\alpha_2\right]_{x_2=0} = 0, \text{ where}$$

$$\varphi_1 = \frac{2\pi}{\ell_1}(x_1\sin\alpha_1 - c_T t - \eta_1), \quad \varphi_2 = \frac{2\pi}{\ell_2}(x_1\sin\alpha_2 - c_T t - \eta_2).$$

Thus, the boundary condition is satisfied if $\alpha_1 = \alpha_2$, $\varepsilon_1 = \varepsilon_2$, $\ell_1 = \ell_2$, $\eta_1 = \eta_2$.

5.34 Do the problem of Section 5.11.(Reflection of Plane Elastic Waves, Figure 5.11-1) for the case where the boundary $x_2 = 0$ is fixed.

Ans. As in Section 5.11, we assume

$$u_1 = (\cos\alpha_1)\varepsilon_1 \sin\varphi_1 + (\cos\alpha_2)\varepsilon_2 \sin\varphi_2 + (\sin\alpha_3)\varepsilon_3 \sin\varphi_3$$

$$u_2 = (\sin\alpha_1)\varepsilon_1 \sin\varphi_1 - (\sin\alpha_2)\varepsilon_2 \sin\varphi_2 + (\cos\alpha_3)\varepsilon_3 \sin\varphi_3, \quad u_3 = 0$$

where

$$\varphi_1 = \frac{2\pi}{\ell_1}(x_1 \sin\alpha_1 - x_2 \cos\alpha_1 - c_T t - \eta_1), \ \varphi_2 = \frac{2\pi}{\ell_2}(x_1 \sin\alpha_2 + x_2 \cos\alpha_2 - c_T t - \eta_2)$$

$$\varphi_3 = \frac{2\pi}{\ell_3}(x_1 \sin\alpha_3 + x_2 \cos\alpha_3 - c_L t - \eta_3)$$

The equations of motion are satisfied with $(c_L)^2 = (\lambda + 2\mu)/\rho_0, \ (c_T)^2 = \mu/\rho_0$

Now, at $x_2 = 0$,

$$\left[(\cos\alpha_1)\varepsilon_1 \sin\varphi_1 + (\cos\alpha_2)\varepsilon_2 \sin\varphi_2 + (\sin\alpha_3)\varepsilon_3 \sin\varphi_3\right]_{x_2=0} = 0$$

$$\left[(\sin\alpha_1)\varepsilon_1 \sin\varphi_1 - (\sin\alpha_2)\varepsilon_2 \sin\varphi_2 + (\cos\alpha_3)\varepsilon_3 \sin\varphi_3\right]_{x_2=0} = 0$$

Thus, at $x_2 = 0$, $\sin\varphi_1 = \sin\varphi_2 = \sin\varphi_3$, so that

$$\varphi_1 = \frac{2\pi}{\ell_1}(x_1 \sin\alpha_1 - c_T t - \eta_1) = \frac{2\pi}{\ell_2}(x_1 \sin\alpha_2 - c_T t - \eta_2') = \frac{2\pi}{\ell_3}(x_1 \sin\alpha_3 - c_L t - \eta_3'),$$

$$\eta_2' = \eta_2 - (\pm p\ell_2), \eta_3' = \eta_3 - (\pm q\ell_2)$$

Thus, as in Section 5.11, we have, with $n = c_T / c_L$,

$$\ell_2 = \ell_1, \quad n\ell_3 = \ell_1, \ \alpha_1 = \alpha_2, \quad n\sin\alpha_3 = \sin\alpha_1, \ \eta_2' = \eta_1, \quad n\eta_3' = \eta_1.$$

However, the relations between the amplitudes are different. In fact, from

$$(\cos\alpha_1)\varepsilon_2 + (\sin\alpha_3)\varepsilon_3 = -(\cos\alpha_1)\varepsilon_1,$$

$$(\sin\alpha_1)\varepsilon_2 - (\cos\alpha_3)\varepsilon_3 = (\sin\alpha_1)\varepsilon_1.$$

we can obtain,

$$\varepsilon_2 = \frac{(\sin\alpha_3)(\sin\alpha_1) - (\cos\alpha_3)(\cos\alpha_1)}{(\sin\alpha_1)(\sin\alpha_3) + (\cos\alpha_1)(\cos\alpha_3)}, \quad \varepsilon_3 = \frac{-\sin 2\alpha_1}{(\sin\alpha_1)(\sin\alpha_3) + (\cos\alpha_1)(\cos\alpha_3)}.$$

5.35 A longitudinal elastic wave is incident on a fixed boundary $x_2 = 0$ with an incident angle of α_1 with the x_2 axis (similar to Fig. 5.11.1 of Section 5.11). (a) Show that in general, there are two reflected waves, one longitudinal and the other transverse (also polarized in the incident plane x_1x_2). (b)Find the amplitude ratio of reflected to incident elastic waves.

Ans. (a) Let

$$u_1 = (\sin\alpha_1)\varepsilon_1 \sin\varphi_1 + (\sin\alpha_2)\varepsilon_2 \sin\varphi_2 + (\cos\alpha_3)\varepsilon_3 \sin\varphi_3$$

$$u_2 = (-\cos\alpha_1)\varepsilon_1 \sin\varphi_1 + (\cos\alpha_2)\varepsilon_2 \sin\varphi_2 - (\sin\alpha_3)\varepsilon_3 \sin\varphi_3, \quad u_3 = 0, \text{where}$$

$$\varphi_1 = \frac{2\pi}{\ell_1}(x_1 \sin\alpha_1 - x_2 \cos\alpha_1 - c_L t - \eta_1),\ \varphi_2 = \frac{2\pi}{\ell_2}(x_1 \sin\alpha_2 + x_2 \cos\alpha_2 - c_L t - \eta_2)$$

$$\varphi_3 = \frac{2\pi}{\ell_3}(x_1 \sin\alpha_3 + x_2 \cos\alpha_3 - c_T t - \eta_3)$$

The equations of motion are satisfied with $(c_L)^2 = (\lambda + 2\mu)/\rho_0$, $(c_T)^2 = \mu/\rho_0$.

Now, at $x_2 = 0$,

$$\left[(\sin\alpha_1)\varepsilon_1 \sin\varphi_1 + (\sin\alpha_2)\varepsilon_2 \sin\varphi_2 + (\cos\alpha_3)\varepsilon_3 \sin\varphi_3\right]_{x_2=0} = 0$$

$$\left[(-\cos\alpha_1)\varepsilon_1 \sin\varphi_1 + (\cos\alpha_2)\varepsilon_2 \sin\varphi_2 - (\sin\alpha_3)\varepsilon_3 \sin\varphi_3\right]_{x_2=0} = 0$$

Thus, at $x_2 = 0$, $\sin\varphi_1 = \sin\varphi_2 = \sin\varphi_3$, so that

$$\varphi_1 = (2\pi/\ell_1)(x_1 \sin\alpha_1 - c_L t - \eta_1) = (2\pi/\ell_2)(x_1 \sin\alpha_2 - c_L t - \eta_2') = (2\pi/\ell_3)(x_1 \sin\alpha_3 - c_T t - \eta_3'),$$

$$\eta_2' = \eta_2 - (\pm p\ell_2), \eta_3' = \eta_3 - (\pm q\ell_3).$$

Thus, we have,

$$\varphi_1 = (2\pi/\ell_1)(x_1 \sin\alpha_1 - c_L t - \eta_1) = (2\pi/\ell_2)(x_1 \sin\alpha_2 - c_L t - \eta_2') = (2\pi/\ell_3)(x_1 \sin\alpha_3 - c_T t - \eta_3'),$$

$$\eta_2' = \eta_2 - (\pm p\ell_2), \eta_3' = \eta_3 - (\pm q\ell_2).$$

Thus,

$$\frac{\sin\alpha_1}{\ell_1} = \frac{\sin\alpha_2}{\ell_2} = \frac{\sin\alpha_3}{\ell_3},\quad \frac{c_L}{\ell_1} = \frac{c_L}{\ell_2} = \frac{c_T}{\ell_3},\quad \frac{\eta_1}{\ell_1} = \frac{\eta_2'}{\ell_2} = \frac{\eta_3'}{\ell_3}$$

So that $\alpha_1 = \alpha_2$, $\ell_2 = \ell_1$, $\ell_3 = n\ell_1$, $\sin\alpha_3 = n\sin\alpha_1$, $\eta_2' = \eta_1$, $\eta_3' = n\eta_1$

where $n = c_T/c_L$. We note that unlike the problem in Sect. 5.11, here $\ell_3 = n\ell_1$, $\sin\alpha_3 = n\sin\alpha_1$(instead of $n\ell_3 = \ell_1$, $n\sin\alpha_3 = \sin\alpha_1$). With $\sin\varphi_1 = \sin\varphi_2 = \sin\varphi_3$, we have

$$(\sin\alpha_1)\varepsilon_2 + (\cos\alpha_3)\varepsilon_3 = -(\sin\alpha_1)\varepsilon_1,\quad (\cos\alpha_1)\varepsilon_2 - (\sin\alpha_3)\varepsilon_3 = (\cos\alpha_1)\varepsilon_1$$

Thus,

$$\varepsilon_3/\varepsilon_1 = -\sin 2\alpha_1/\cos(\alpha_1 - \alpha_3),\quad \varepsilon_2/\varepsilon_1 = \cos(\alpha_1 + \alpha_3)/\cos(\alpha_1 - \alpha_3)$$

5.36 Do the previous problem (Prob. 5.35) for the case where $x_2 = 0$ is a traction free boundary

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Ans. Let

$$u_1 = (\sin\alpha_1)\varepsilon_1 \sin\varphi_1 + (\sin\alpha_2)\varepsilon_2 \sin\varphi_2 + (\cos\alpha_3)\varepsilon_3 \sin\varphi_3$$

$$u_2 = (-\cos\alpha_1)\varepsilon_1 \sin\varphi_1 + (\cos\alpha_2)\varepsilon_2 \sin\varphi_2 - (\sin\alpha_3)\varepsilon_3 \sin\varphi_3,\quad u_3 = 0, \text{where}$$

$$\varphi_1 = \frac{2\pi}{\ell_1}(x_1 \sin\alpha_1 - x_2 \cos\alpha_1 - c_L t - \eta_1),\ \varphi_2 = \frac{2\pi}{\ell_2}(x_1 \sin\alpha_2 + x_2 \cos\alpha_2 - c_L t - \eta_2)$$

$$\varphi_3 = \frac{2\pi}{\ell_3}(x_1 \sin\alpha_3 + x_2 \cos\alpha_3 - c_T t - \eta_3)$$

The equations of motion are satisfied with $(c_L)^2 = (\lambda + 2\mu)/\rho_0$, $(c_T)^2 = \mu/\rho_0$.

At $x_2 = 0$, $T_{21} = T_{22} = T_{23} = 0$. Thus,

$$\partial u_1/\partial x_2 + \partial u_2/\partial x_1 = 0 \text{ and } (\lambda+2\mu)(\partial u_2/\partial x_2) + \lambda(\partial u_1/\partial x_1) = 0.$$

That is,

$$(\varepsilon_1/\ell_1)\cos\varphi_1(-\sin 2\alpha_1) + (\varepsilon_2/\ell_2)\cos\varphi_2(\sin 2\alpha_2) + (\varepsilon_3/\ell_3)\cos\varphi_3(\cos 2\alpha_3) = 0 \qquad \text{(i)}$$

and

$$(\varepsilon_1/\ell_1)\left[\lambda + 2\mu(\cos\alpha_1)^2\right]\cos\varphi_1 + (\varepsilon_2/\ell_2)\left[\lambda + 2\mu(\cos\alpha_2)^2\right]\cos\varphi_2 + (\varepsilon_3/\ell_3)(2\mu)\left[(-\sin\alpha_3\cos\alpha_3)\right]\cos\varphi_3 = 0 \qquad \text{(ii)}$$

In order for the above two equations to be satisfied for all x_1 and t, we must have, at $x_2 = 0$ $\cos\varphi_1 = \cos\varphi_2 = \cos\varphi_3$, which gives

$$\frac{\sin\alpha_1}{\ell_1} = \frac{\sin\alpha_2}{\ell_2} = \frac{\sin\alpha_3}{\ell_3},\ \frac{c_L}{\ell_1} = \frac{c_L}{\ell_2} = \frac{c_T}{\ell_3},\ \frac{\eta_1}{\ell_1} = \frac{\eta_2 \mp p\ell_2}{\ell_2} = \frac{\eta_3 \mp q\ell_3}{\ell_3}$$

Thus, $\alpha_1 = \alpha_2,\ \ell_2 = \ell_1,\ \ell_3 = n\ell_1,\ \sin\alpha_3 = n\sin\alpha_1,\ \eta_2 \mp p\ell_2 = \eta_1,\ \eta_3 \mp q\ell_3 = n\eta_1$, where $n = c_T/c_L$.

(i) and (ii) now gives

$$(\varepsilon_1/\ell_1)(-\sin 2\alpha_1) + (\varepsilon_2/\ell_2)(\sin 2\alpha_1) + (\varepsilon_3/\ell_3)(\cos 2\alpha_3) = 0 \qquad \text{(iii)}$$

$$(\varepsilon_1/\ell_1)(\lambda + 2\mu\cos^2\alpha_1) + (\varepsilon_2/\ell_1)(\lambda + 2\mu\cos^2\alpha_1) - (\varepsilon_3/\ell_3)(2\mu)\sin\alpha_3\cos\alpha_3 = 0$$

$$\lambda + 2\mu\cos^2\alpha_1 = \lambda + 2\mu - 2\mu\sin^2\alpha_1 = \mu\left(\frac{\lambda+2\mu}{\mu} - 2\sin^2\alpha_1\right) = \qquad \text{(iv)}$$

$$(\varepsilon_2)\frac{\mu}{n}\left(1 - 2n^2\sin^2\alpha_1\right) - (\varepsilon_3/)(2\mu)\sin\alpha_3\cos\alpha_3 = -\varepsilon_1\frac{\mu}{n}\left(1 - 2n^2\sin^2\alpha_1\right)$$

Since

$$\lambda + 2\mu\cos^2\alpha_1 = \lambda + 2\mu - 2\mu\sin^2\alpha_1 = \mu\left(\frac{\lambda+2\mu}{\mu} - 2\sin^2\alpha_1\right) = \frac{\mu}{n^2}\left(1 - 2n^2\sin^2\alpha_1\right),$$

and $\ell_3 = n\ell_1$, therefore, (iii) and (iv) become

$$(n\sin 2\alpha_1)\varepsilon_2 + (\cos 2\alpha_3)\varepsilon_3 = \varepsilon_1 n\sin 2\alpha_1 \qquad \text{(v)}$$

$$\left(1 - 2n^2\sin^2\alpha_1\right)\varepsilon_2 - 2n\sin\alpha_3\cos\alpha_3\varepsilon_3 = -\left(1 - 2n^2\sin^2\alpha_1\right)\varepsilon_1 \qquad \text{(vi)}$$

(v) and (vi) give

$$\frac{\varepsilon_3}{\varepsilon_1} = \frac{2n\sin 2\alpha_1(1 - 2n^2\sin^2\alpha_1)}{n^2\sin 2\alpha_1\sin 2\alpha_3 + (1 - 2n^2\sin^2\alpha_1)\cos 2\alpha_3},$$

$$\frac{\varepsilon_2}{\varepsilon_1} = \frac{n^2\sin 2\alpha_1\sin 2\alpha_3 - (1 - 2n^2\sin^2\alpha_1)\cos 2\alpha_3}{n^2\sin 2\alpha_1\sin 2\alpha_3 + (1 - 2n^2\sin^2\alpha_1)\cos 2\alpha_3}$$

5.37 Verify that the thickness stretch vibration given by Eq.(5.12.3), i.e.,

$$u_1 = (A\cos kx_1 + B\sin kx_1)(C\cos c_L kt + D\sin c_L kt)$$

does satisfy the longitudinal wave equation $\partial^2 u_1/\partial t^2 = (c_L)^2\left(\partial^2 u_1/\partial x_1^2\right)$

Ans.

$$u_1 = (A\cos kx_1 + B\sin kx_1)(C\cos c_L kt + D\sin c_L kt),$$

$$\partial^2 u_1/\partial x_1^2 = -k^2(A\cos kx_1 + B\cos kx_1)(C\cos c_L kt + D\sin c_L kt)$$

$$\partial^2 u_1/\partial t^2 = -(kc_L)^2(A\cos kx_1 + B\cos kx_1)(C\cos c_L kt + D\sin c_L kt)$$

that is, $\partial^2 u_1/\partial x_1^2 = -k^2 u_1$ and $\partial^2 u_1/\partial t^2 = -(c_L k)^2 u_1$. Thus, $c_L^2 \partial^2 u_1/\partial x_1^2 = \partial^2 u_1/\partial t^2$

5.38 (a) Find the thickness-stretch vibration of a plate, where the left face ($x_1 = 0$) is subjected to a forced displacement $\mathbf{u} = (\alpha\cos\omega t)\mathbf{e}_1$ and the right face $x_1 = \ell$ is free. (b)Determine the values of ω that give resonance.

Ans. Let (a) $u_1 = (A\cos kx_1 + B\sin kx_1)(C\cos c_L kt + D\sin c_L kt)$. Using the boundary condition $\mathbf{u}(0,t) = (\alpha\cos\omega t)\mathbf{e}_1$, we have , $\alpha\cos\omega t = u_1(0,t) = AC\cos c_L kt + AD\sin c_L kt$

Thus, $AC = \alpha, \quad k = \omega/c_L, \quad D = 0 \rightarrow u_1 = (\alpha\cos kx_1 + BC\sin kx_1)\cos\omega t$.

At $x_1 = \ell$, $T_{11} = T_{12} = T_{13} = 0$. Now, $T_{11} = (\lambda + 2\mu)(\partial u_1/\partial x_1)$, thus $(\partial u_1/\partial x_1)_{x_1=\ell} = 0$, i.e.,

$$k(-\alpha\sin k\ell + BC\cos k\ell)\cos\omega t = 0 \rightarrow BC = \alpha\tan k\ell,$$

$$\rightarrow u_1 = \alpha[\cos(\omega x_1/c_L) + \tan(\omega\ell/c_L)\sin(\omega x_1/c_L)]\cos\omega t.$$

(b) Resonance occurs at : $\omega\ell/c_L = n\pi/2, \; n = 1,3,5...$

5.39 (a) Find the thickness stretch vibration if the $x_1 = 0$ face is being forced by a traction $\mathbf{t} = (\beta\cos\omega t)\mathbf{e}_1$ and the right hand face $x_1 = \ell$ is fixed. (b) Find the resonance frequencies.

Ans. (a) $u_1 = (A\cos kx_1 + B\sin kx_1)(C\cos c_L kt + D\sin c_L kt)$.

At

$$x_1 = 0, \mathbf{n} = -\mathbf{e}_1, \mathbf{t} = -\mathbf{T}\mathbf{e}_1 = -(T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 + T_{31}\mathbf{e}_3) = \beta\cos\omega t\mathbf{e}_1 \rightarrow T_{11} = -\beta\cos\omega t, \quad T_{21} = T_{31} = 0$$

Since $T_{11} = (\lambda + 2\mu)(\partial u_1/\partial x_1)$, therefore, the boundary condition at $x_1 = 0$ is:

$$(\lambda + 2\mu)(\partial u_1/\partial x_1)_{x_1=0} = -\beta\cos\omega t, \rightarrow (\lambda + 2\mu)k(B)(C\cos c_L kt + D\sin c_L kt) = -\beta\cos\omega t,$$

$$\rightarrow D = 0, \quad c_L k = \omega, \quad BC = -\frac{\beta}{k(\lambda + 2\mu)}, \quad \rightarrow u_1 = (AC\cos kx_1 - \frac{\beta}{k(\lambda + 2\mu)}\sin kx_1)\cos\omega t$$

At $x_1 = \ell$,

$$u_1 = (AC\cos k\ell - \frac{\beta}{k(\lambda + 2\mu)}\sin k\ell)\cos\omega t = 0 \rightarrow AC\cos k\ell - \frac{\beta}{k(\lambda + 2\mu)}\sin k\ell = 0 \rightarrow$$

$$AC = \frac{\beta}{k(\lambda + 2\mu)}\tan k\ell \rightarrow u_1 = \left[\frac{\beta c_L}{\omega(\lambda + 2\mu)}\tan\frac{\omega\ell}{c_L}\cos\frac{\omega x_1}{c_L} - \frac{\beta c_L}{\omega(\lambda + 2\mu)}\sin\frac{\omega x_1}{c_L}\right]\cos\omega t$$

(b) Resonance occurs at

$$\omega = n\pi c_L / (2\ell), n = 1,3,5...$$

5.40 (a) Find the thickness-shear vibration if the left hand face $x_1 = 0$ has a forced displacement $\mathbf{u} = (\alpha\cos\omega t)\mathbf{e}_3$ and the right-hand face $x_1 = \ell$ is fixed. (b) Find the resonance frequencies.

Ans. (a) Let $u_3 = (A\cos kx_1 + B\sin kx_1)(C\cos c_T kt + D\sin c_T kt), \quad u_1 = u_2 = 0$

In the absence of body forces, the x_3 Navier equation of motion (5.6.7) gives:

$$\rho_o \frac{\partial^2 u_3}{\partial t^2} = (\lambda + \mu)\frac{\partial e}{\partial x_3} + \mu\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_{2_1}^2} + \frac{\partial^2}{\partial x_3^2}\right)u_3 \to \rho_o \frac{\partial^2 u_3}{\partial t^2} = \mu \frac{\partial^2 u_3}{\partial x_1^2}$$

leads to $-\rho_o (c_T k)^2 u_3 = -\mu k^2 u_3 \to (c_T)^2 = \mu / \rho_o$.

The boundary condition at $x_1 = 0$,

$$u_3(0,t) = (\alpha\cos\omega t) \to u_3 = (A)(C\cos c_T kt + D\sin c_T kt) = \alpha\cos\omega t \to D = 0, \quad AC = \alpha, \quad k = \omega / c_T .$$

$$\to u_3 = (\alpha\cos kx_1 + BC\sin kx_1)\cos\omega t .$$

The boundary condition at $x_1 = \ell$,

$$u_3(\ell,t) = 0 \to u_3 = (\alpha\cos k\ell + BC\sin k\ell)\cos\omega t = 0 \to BC = -\alpha\cot k\ell .$$

$$\to u_3 = \alpha\left[\cos(\omega x_1 / c_T) - \cot(\omega\ell / c_T)\sin(\omega x_1 / c_T)\right]\cos\omega t .$$

(b) Resonance occurs at $\omega = n\pi c_T / \ell, n = 1,2,3...$

5.41 (a) Find the thickness-shear vibration if the left hand face $x_1 = 0$ has a forced displacement $\mathbf{u} = \alpha(\cos\omega t\mathbf{e}_2 + \sin\omega t\mathbf{e}_3)$ and the right-hand face $x_1 = \ell$ is fixed. (b) Find the resonance frequencies.

Ans.

(a) If the left hand face $x_1 = 0$ has a forced displacement $\mathbf{u} = \alpha\cos\omega t\mathbf{e}_2$ and the right-hand face $x_1 = \ell$ is fixed, it is clear from the result of the previous problem,

$$u_2 = \alpha\left[\cos(\omega x_1 / c_T) - \cot(\omega\ell / c_T)\sin(\omega x_1 / c_T)\right]\cos\omega t, \quad u_1 = u_3 = 0 .$$

If the left hand face $x_1 = 0$ has a forced displacement $\mathbf{u} = \alpha\sin\omega t\mathbf{e}_3$ and the right-hand face $x_1 = \ell$ is fixed, the displacement field is clearly given by

$$u_3 = \alpha\left[\cos(\omega x_1 / c_T) - \cot(\omega\ell / c_T)\sin(\omega x_1 / c_T)\right]\sin\omega t, \quad u_1 = u_2 = 0$$

Thus, the solution to the present problem can be obtained by superposition to be

$$u_1 = 0,$$

$$u_2 = \alpha\left[\cos(\omega x_1 / c_T) - \cot(\omega\ell / c_T)\sin(\omega x_1 / c_T)\right]\cos\omega t ,$$

$$u_3 = \alpha\left[\cos(\omega x_1 / c_T) - \cot(\omega\ell / c_T)\sin(\omega x_1 / c_T)\right]\sin\omega t .$$

(b) Resonance occurs at $\omega = n\pi c_T / \ell, n = 1,2,3...$

5.42 A cast iron bar, 200 cm long and 4 cm in diameter, is pulled by equal and opposite axial force P at its ends. (a) Find the maximum normal and shearing stresses if P=90,000N. (b) Find the total elongation and lateral contraction. $\left(E_Y = 103GPa., \nu = 0.3\right)$

Ans. $A = \pi(4\times10^{-2})^2 / 4 = 12.6\times10^{-4} m^2$.

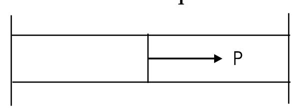
$(a)\left(T_n\right)_{\max} = P / A = 90,000 / \left(12.6\times10^{-4}\right) = 71.4\times10^6 N, \quad \left(T_s\right)_{\max} = P / \left(2A\right) = 35.7\times10^6 N$.

$(b)\ \delta_\ell = (P / A)(\ell / E_Y) = (71.4\times10^6)\times 2 / (103\times10^9) = 1.39\times10^{-3} m,$

$\delta_d = -\nu(P / A)(d / E_Y) = (0.3)(71.4\times10^6)\times(4\times10^{-2}) / (103\times10^9) = -0.832\times10^{-5} m.$

5.43 A composite bar, formed by welding two slender bars of equal length and equal cross-sectional area, is loaded by an axial load P as shown in Figure below. If Young's moduli of the two portions are $E_Y^{(1)}$ and $E_Y^{(2)}$, find how the applied force is distributed between the two halves.

Ans. Taking the whole bar as a free body, let P_1 be the compressive reactional force from the right wall to the bar and P_2 be the compressive reactional force from the left wall to the bar, then the equation of static equilibrium requires



$P = P_1 - P_2$. (i)

There is no net elongation of the composite bar, therefore,

$$\frac{P_1\ell}{AE_Y^{(1)}} + \frac{P_2\ell}{AE_Y^{(2)}} = 0 \quad \text{(ii)}$$

Combining Eq. (i) and (ii), we obtain

$$P_1 = \frac{P}{1 + (E_Y^{(2)} / E_Y^{(1)})}, \quad P_2 = \frac{-P}{1 + (E_Y^{(1)} / E_Y^{(2)})}. \quad \text{(iii)}$$

5.44 A bar of cross-sectional area A is stretched by a tensile force P at each end. (a) Determine the normal and shearing stresses on a plane with a normal vector which makes an angle α with the axis of the bar. (b) For what value of α are the normal and shearing stresses equal? (c) If the load carrying capacity of the bar is based on the shearing stress on the plane defined by $\alpha = \alpha_o$ to be less than τ_o what is the maximum allowable load P?

$$Ans.\ [\mathbf{T}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \mathbf{n} = \cos\alpha\mathbf{e}_1 + \sin\alpha\mathbf{e}_2,$$

(a) For the plane with a normal given by $\mathbf{n} = \cos\alpha\mathbf{e}_1 + \sin\alpha\mathbf{e}_2$, we have,

$$[\mathbf{t}] = [\mathbf{T}][\mathbf{n}] = \begin{bmatrix} \sigma & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{bmatrix} = \begin{bmatrix} \sigma\cos\alpha \\ 0 \\ 0 \end{bmatrix} \rightarrow \mathbf{t} = \sigma\cos\alpha\mathbf{e}_1 \rightarrow T_n = \mathbf{t}\cdot\mathbf{n} = \sigma\cos^2\alpha,$$

$$T_s^2 = |\mathbf{t}|^2 - T_n^2 = \sigma^2\cos^2\alpha - \sigma^2\cos^4\alpha = \sigma^2\cos^2\alpha\left(1-\cos^2\alpha\right) = \sigma^2\cos^2\alpha\sin^2\alpha$$

$$\rightarrow T_s = \sigma\sin 2\alpha / 2.$$

(b) $\dfrac{\sigma\sin 2\alpha}{2} = \sigma\cos^2\alpha \rightarrow \cos\alpha\sin\alpha = \cos^2\alpha \rightarrow \cos\alpha\left(\sin\alpha - \cos\alpha\right) = 0$.

Thus, (i) $\cos\alpha = 0 \rightarrow \alpha = \pi/2 \rightarrow T_s = T_n = 0$, and (ii) $\sin\alpha = \cos\alpha \rightarrow \alpha = \pi/4 \rightarrow T_s = T_n = \sigma/2$.

(c) $\dfrac{\sigma\sin 2\alpha_o}{2} \le \tau_o \rightarrow \sigma \le \dfrac{2\tau_o}{\sin 2\alpha_o}$. Max allowable $P \le \dfrac{2\tau_o}{\sin 2\alpha_o} A$

5.45 A cylindrical bar, whose lateral surface is constrained so that there can be no lateral expansion, is then loaded with an axial compressive stress $T_{11} = -\sigma$. (a) Find T_{22} and T_{33} in terms of σ and the Poisson's ratio ν, (b) show that the effective Young's modulus $\left(E_Y\right)_{eff} \equiv T_{11}/E_{11}$ is given by $\left(E_Y\right)_{eff} = (1-\nu)/(1-\nu-2\nu^2)$. [note misprint in text].

Ans. (a) $E_{22} = 0 \rightarrow T_{22} - \nu\left(T_{33} + T_{11}\right) = 0$, $E_{33} = 0 \rightarrow T_{33} - \nu\left(T_{11} + T_{22}\right) = 0$. Thus, $T_{22} - \nu T_{33} = -\nu\sigma$ and $T_{33} - \nu T_{22} = -\nu\sigma$. From these two equations, we have,

$$T_{22} = T_{33} = -\nu\sigma/(1-\nu).$$

(b) $$E_{11} = \frac{1}{E_Y}\left[T_{11} - \nu\left(T_{22} + T_{33}\right)\right] = \frac{1}{E_Y}\left[-\sigma + 2\nu\left(\frac{\nu\sigma}{1-\nu}\right)\right] = \frac{-\sigma}{E_Y}\left[1 - \left(\frac{2\nu^2}{1-\nu}\right)\right] = \frac{-\sigma}{E_Y}\left(\frac{1-\nu-2\nu^2}{1-\nu}\right)$$

Thus, $$\left(E_Y\right)_{eff} \equiv \frac{T_{11}}{E_{11}} \rightarrow \left(E_Y\right)_{eff} = \frac{-\sigma}{E_{11}} = \frac{E_Y\left(1-\nu\right)}{1-\nu-2\nu^2} = \frac{E_Y\left(1-\nu\right)}{\left(1-2\nu\right)\left(1+\nu\right)}.$$

5.46 Let the state of stress in a tension specimen be given by $T_{11} = \sigma$ and all other T_{ij} =0. (a) Find the components of the deviatoric stress defined by $\mathbf{T}^{o} = \mathbf{T} - \left(1/3\right)T_{kk}\mathbf{I}$. (b) Find the principal scalar invariants of $\mathbf{T}^{o}$

Ans. (a) $T_{kk} = T_{11} + T_{22} + T_{33} = \sigma \rightarrow T_{11}^{o} = T_{11} - T_{kk}/3 = \sigma - \sigma/3 = 2\sigma/3$.

$$T_{22}^{o} = T_{22} - T_{kk}/3 = -\sigma/3 = T_{33}^{o}, \quad T_{12}^{o} = T_{13}^{o} = T_{23}^{o} = 0.$$

(b)

$$I_1 = T_{11}^{o} + T_{22}^{o} + T_{33}^{o} = 2\sigma/3 - \sigma/3 - \sigma/3 = 0.$$

$$I_2 = T_{11}^{o}T_{22}^{o} + T_{22}^{o}T_{33}^{o} + T_{11}^{o}T_{33}^{o} = (2\sigma/3)(-\sigma/3) + (-\sigma/3)(-\sigma/3) + (2\sigma/3)(-\sigma/3) = -\sigma^2/3.$$

$$I_3 = T_{11}^{o}T_{22}^{o}T_{33}^{o} = (2\sigma/3)(-\sigma/3)(-\sigma/3) = 2\sigma^3/27.$$

5.47 A circular cylindrical bar of length ℓ hangs vertically under gravity force from the ceiling. Let x_1 axis coincides with the axis of the bar and points downward and let the point $(x_1, x_2, x_3) = (0,0,0)$ be fixed at the ceiling. (a) Verify that the following stress field satisfies the equations of equilibrium in the presence of the gravity force: $T_{11} = \rho g(\ell - x_1)$, all other $T_{ij} = 0$ and (b) verify that the boundary conditions of zero surface traction on the lateral face and the lower end face are satisfied and (c) obtained the resultant force of the surface traction at the upper face.

Ans. (a) The body force per unit volume is given by $\rho\mathbf{B} = \rho g\mathbf{e}_1$. Thus, with $T_{11} = \rho g(\ell - x_1)$, we have,

$$\frac{\partial T_{11}}{\partial x_1} + \frac{\partial T_{12}}{\partial x_2} + \frac{\partial T_{13}}{\partial x_3} + \rho g = -\rho g + 0 + 0 + \rho g = 0$$ and the other two equations are trivially satisfied.

(b) On the bottom end face $x_1 = \ell$, $\mathbf{n} = \mathbf{e}_1$, $\mathbf{t} = \mathbf{T}\mathbf{e}_1 = T_{11}\big|_{x_1=\ell}\mathbf{e}_1 = \rho g(\ell - \ell)\mathbf{e}_1 = \mathbf{0}$.

On the lateral face, $\mathbf{n} = n_2\mathbf{e}_2 + n_3\mathbf{e}_3$, $[\mathbf{t}] = \begin{bmatrix} T_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow \mathbf{t} = \mathbf{0}$.

(c) On the top face at $x_1 = 0$, $\mathbf{n} = -\mathbf{e}_1$, $\mathbf{t} = -\mathbf{T}\mathbf{e}_1 = -T_{11}\big|_{x_1=0}\mathbf{e}_1 = -\rho g(\ell - 0)\mathbf{e}_1 = -\rho g\ell\mathbf{e}_1$

Let the area of the face be A, then the resultant force is $\mathbf{t}A = -\rho g\ell A\mathbf{e}_1 = -W\mathbf{e}_1$ where $W = \rho g A\ell$ is the weight of the bar and the minus sign indicates that the resultant force at the ceiling is upward which balances the weight of the bar.

5.48 A circular steel shaft is subjected to twisting couples of $2700\ Nm$. The allowable tensile stress is $0.124\ GPa$. If the allowable shearing stress is 0.6 times the allowable tensile stress, what is the minimum allowable diameter?

Ans. $(T_n)_{\max} = (T_s)_{\max} = \dfrac{M_t a}{I_p} = M_t a/(\pi a^4/2) = \dfrac{2M_t}{\pi a^3}$. Thus

$$\frac{2(2700)}{\pi a^3} \le (0.6)\left(0.124\times10^9\right) \to a^3 \ge \frac{2(2700)}{\pi(0.6)\left(0.124\times10^9\right)} = 23.1\times10^{-6}\,m^3$$

$$\to a \ge 2.85\times10^{-2}\,m = 2.85\ cm \to d \ge 5.7\ cm.$$

5.49 In Figure 5P.2, a twisting torque M_t is applied to the rigid disc A. Find the twisting moments transmitted to the circular shafts on either side of the disc.

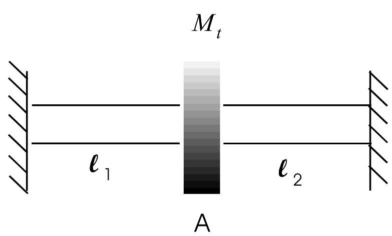


Figure 5P.2

Ans. Let M_1 and M_2 be the twisting moments transmitted to the left and the right shaft respectively. Then equilibrium of the disc demands that

$$M_1 + M_2 = M_t \qquad \text{(i)}$$

In addition, the disc is rigid, therefore, the angle of twist of the left shaft at the disc relative to the left wall must equal the angle of twist of the right shaft at the disc relative to the right wall, i.e.,

$$\frac{M_1\ell_1}{\mu I_p} = \frac{M_2\ell_2}{\mu I_p} \to M_1\ell_1 = M_2\ell_2 \qquad \text{(ii)}$$

Thus,

$$M_1 = \left(\frac{\ell_2}{\ell_1+\ell_2}\right)M_t, \quad M_2 = \left(\frac{\ell_1}{\ell_1+\ell_2}\right)M_t \qquad \text{(iii)}$$

for $\ell_1 = \ell_2$, $M_1 = M_2 = M_t/2$.

5.50 What needs to be changed in the solution for torsion of a solid circular bar obtained in Section 5.14 for it to be valid for torsion of a hollow circular bar with inner radius a and outer radius b?

Ans. The hollow circular bar differs from the solid circular bar in that there is an inner lateral surface which is also traction free. However, the normal to the inner lateral surface differs from that to the outer surface only by a sign so that the zero surface traction in the inner surface is also satisfied since that for the outer surface is satisfied. However, in calculating the resultant force and resultant moment due to the surface traction on the end faces, the integrals are now to be integrated over the circular ring area between by $r = a$ and $r = b$ rather than the whole solid circular area of radius b. Thus, the only change that needs to be made is that the polar area second moment I_p is now given by $I_p = \frac{\pi}{2}\left(b^4 - a^4\right)$.

5.51 A circular bar of radius r_o is under the action of an axial tensile load P and a twisting couple of M_t. (a) Determine the stress throughout the bar. (b) Find the maximum normal and shearing stress

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Ans. Superpose the solutions for tension and for torsion, we have, with $\sigma = \frac{P}{A},\ \beta \equiv \frac{M_t}{I_p}$

(a) $T_{11} = \sigma,\ T_{12} = T_{21} = -\beta x_3,\ T_{13} = T_{31} = \beta x_2$, all other $T_{ij} = 0$.

(b) The characteristic equation is

$$\begin{vmatrix} \sigma-\lambda & -\beta x_3 & \beta x_2 \\ -\beta x_3 & -\lambda & 0 \\ \beta x_2 & 0 & -\lambda \end{vmatrix} = 0 \rightarrow \sigma\lambda^2 - \lambda^3 + \lambda\left(\beta x_2\right)^2 + \lambda\left(\beta x_3\right)^2 = 0 \rightarrow \lambda\left(\lambda^2 - \sigma\lambda - \beta^2 r^2\right) = 0, \text{where } r^2 = x_2^2 + x_3^2$$

Thus, $\lambda_{1,2} = \frac{\sigma \pm \sqrt{\sigma^2 + 4\beta^2 r^2}}{2},\ \lambda_3 = 0$. Thus

$$\left(T_n\right)_{\max} = \frac{\sigma + \sqrt{\sigma^2 + 4\beta^2 r^2}}{2},\ T_s = \frac{1}{2}\sqrt{\sigma^2 + 4\beta^2 r^2}.$$

5.52 Compare the twisting torque which can be transmitted by a shaft with an elliptical cross-section having a major diameter equal to twice the minor diameter with a shaft of circular cross-section having a diameter equal to the major diameter of the elliptical shaft. Both shafts are of the same material. Also compare the unit twist (i.e., twist angle per unit length) under the same twisting moment. Assume that the maximum twisting moment which can be transmitted is controlled by the maximum shearing stress.

Ans. (a) For an elliptical shaft with major diameter $2b$ and minor diameter $2a$ (i.e., $b > a$),

$$\left(T_s\right)_{\max} = \frac{2\left(M_t\right)_{ell}}{\pi a^2 b}.$$

For a circular shaft with radius b, $\left(T_s\right)_{\max} = \frac{2\left(M_t\right)_{cir}}{\pi b^3}$, thus

$$\left(T_s\right)_{\max} = \frac{2\left(M_t\right)_{ell}}{\pi a^2 b} = \frac{2\left(M_t\right)_{cir}}{\pi b^3}, \qquad \rightarrow \frac{\left(M_t\right)_{ell}}{\left(M_t\right)_{cir}} = \left(\frac{a}{b}\right)^2 = \left(\frac{a}{2a}\right)^2 = \frac{1}{4}.$$

(b) $\alpha'_{ell} = \left(\frac{a^2 + b^2}{\mu\pi a^3 b^3}\right)M_t, \qquad \alpha'_{cir} = \left(\frac{2}{\mu\pi b^4}\right)M_t$, thus, $\frac{\alpha'_{ell}}{\alpha'_{cir}} = \frac{b\left(a^2 + b^2\right)}{2a^3} = \frac{(2a)\left(5a^2\right)}{2a^3} = 5$.

5.53 Repeat the previous problem except that the circular shaft has a diameter equal to the minor diameter of the elliptical shaft.

Ans. (a) For an elliptical shaft with major diameter $2b$ and minor diameter $2a$ (i.e., $b > a$),

$$\left(T_s\right)_{\max} = \frac{2\left(M_t\right)_{ell}}{\pi a^2 b}.$$

For a circular shaft with radius a, $\left(T_s\right)_{\max} = \dfrac{2\left(M_t\right)_{cir}}{\pi a^3}$, thus,

$$\left(T_s\right)_{\max} = \frac{2\left(M_t\right)_{ell}}{\pi a^2 b} = \frac{2\left(M_t\right)_{cir}}{\pi a^3}, \qquad \to \frac{\left(M_t\right)_{ell}}{\left(M_t\right)_{cir}} = \left(\frac{b}{a}\right) = \left(\frac{2a}{a}\right) = 2.$$

(b) $\alpha'_{ell} = \left(\dfrac{a^2+b^2}{\mu\pi a^3 b^3}\right)M_t, \quad \alpha'_{cir} = \left(\dfrac{2}{\mu\pi a^4}\right)M_t \to \quad \dfrac{\alpha'_{ell}}{\alpha'_{cir}} = \left(\dfrac{5a^2}{8a^6}\right)\left(\dfrac{a^4}{2}\right) = \dfrac{5}{16}.$

5.54 Consider torsion of a cylindrical bar with an equilateral triangular cross-section as shown in Fig. P.5.3. (a) Show that a warping function $\varphi = C\left(3x_2^2 x_3 - x_3^3\right)$ generate an equilibrium stress field. (b) Determine the constant C, so as to satisfy the traction free boundary condition on the lateral surface $x_2 = a$. With C so obtained, verify that the other two lateral surfaces are also traction free. (c) Evaluate the shear stress at the corners and along the line $x_3 = 0$.(d) Along the line $x_3 = 0$ where does the greatest shear stress occur?

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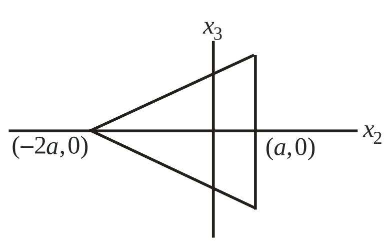
Ans. (a) $\dfrac{\partial^2\varphi}{\partial x_2^2} = 6Cx_3,\ \dfrac{\partial^2\varphi}{\partial x_3^2} = -6Cx_3,$ thus $\dfrac{\partial^2\varphi}{\partial x_2^2} + \dfrac{\partial^2\varphi}{\partial x_3^2} = 0$, so that equations of equilibrium are satisfied.



(b) For the lateral surface $x_2 = a$, $\mathbf{n}=\mathbf{e}_2, \mathbf{t} = \mathbf{T}\mathbf{e}_2 = T_{12}\mathbf{e}_1 = \left[-\mu x_3\alpha' + \mu\left(\partial\varphi/\partial x_2\right)\right]_{x_2=a}\mathbf{e}_1 = 0$

$$\to x_3\alpha' = \left[6Cx_2x_3\right]_{x_2=a} \to x_3\alpha' = 6Cax_3 \to C = \alpha'/6a.$$

On the lateral surface $x_3 = \left(1/\sqrt{3}\right)\left(x_2 + 2a\right) \to \sqrt{3}x_3 - x_2 = 2a \to \mathbf{n} = \left(1/2\right)\left(-\mathbf{e}_2 + \sqrt{3}\mathbf{e}_3\right)$

$\mathbf{t} = \mathbf{T}\mathbf{n} = \left(1/2\right)\left(-\mathbf{T}\mathbf{e}_2 + \sqrt{3}\mathbf{T}\mathbf{e}_3\right) = \left(1/2\right)\left(-T_{12} + \sqrt{3}T_{13}\right)\mathbf{e}_1$. Now, for $\varphi = \left(\alpha'/6a\right)\left(3x_2^2x_3 - x_3^3\right)$

$$T_{12} = -\mu x_3\alpha' + \mu(\partial\varphi/\partial x_2) = -\mu x_3\alpha' + \mu(\alpha'/a)(x_2x_3),$$
$$T_{13} = \mu x_2\alpha' + \mu(\partial\varphi/\partial x_3) = \mu x_2\alpha' + \mu(\alpha'/2a)(x_2^2 - x_3^2).,$$

Therefore, $-T_{12} + \sqrt{3}T_{13} = \frac{\mu\alpha'}{2a}\left[2ax_3 - 2x_2x_3 + 2\sqrt{3}ax_2 + \sqrt{3}\left(x_2^2 - x_3^2\right)\right].$ With $x_2 = \sqrt{3}x_3 - 2a$,

$$-T_{12} + \sqrt{3}T_{13} = \frac{\mu\alpha'}{2a}\left[2ax_3 - 2\left(\sqrt{3}x_3 - 2a\right)x_3 + 2\sqrt{3}a\left(\sqrt{3}x_3 - 2a\right) + \sqrt{3}\left(\left(\sqrt{3}x_3 - 2a\right)^2 - x_3^2\right)\right]$$
$$= \frac{\mu\alpha'}{2a}\left[2ax_3 + \left(-2\sqrt{3}x_3^2 + 4ax_3\right) + \left(6ax_3 - 4\sqrt{3}a^2\right) + \sqrt{3}\left(3x_3^2 - 4\sqrt{3}ax_3 + 4a^2 - x_3^2\right)\right]$$
$$= \frac{\mu\alpha'}{2a}\left[\left(2ax_3 + 4ax_3 + 6ax_3 - 12ax_3 - 4\sqrt{3}a^2 + 4\sqrt{3}a^2\right) + \left(-2\sqrt{3}x_3^2 + 3\sqrt{3}x_3^2 - \sqrt{3}x_3^2\right)\right] = 0.$$

That is, on $x_3 = \left(1/\sqrt{3}\right)\left(x_2 + 2a\right)$, $\mathbf{t} = \mathbf{0}$. Clearly, for the lateral surface

$$x_3 = -\left(1/\sqrt{3}\right)\left(x_2 + 2a\right),\ \mathbf{t} = \mathbf{0}.$$

(c) at the corner $(-2a, 0)$, $T_{12} = -\mu x_3\alpha' + \mu\left(\alpha'/a\right)\left(x_2x_3\right) = 0$ and

$$T_{13} = \mu x_2\alpha' + \mu\left(\alpha'/2a\right)\left(x_2^2 - x_3^2\right) = \left(-2a\right)\mu\alpha' + \mu\left(\alpha'/2a\right)\left(4a^2\right) = 0.$$

At the corners $\left(a, \pm\sqrt{3}a\right)$, $T_{12} = \mu\alpha'\left(x_3/a\right)\left(x_2 - a\right) = \mu\alpha'\left(\pm\sqrt{3}\right)\left(a - a\right) = 0$ and

$$T_{13} = \mu\alpha'\left[x_2 + \left(x_2^2 - x_3^2\right)/2a\right] = \mu\alpha'\left[a + \left(a^2 - 3a^2\right)/2a\right] = 0.$$

That is, the shear stress at all three corners are zero. Along $x_3 = 0$,

$$T_{12} = -\mu x_3\alpha' + \mu\left(\alpha'/a\right)\left(x_2x_3\right) = 0,$$
$$T_{13} = \mu\alpha'\left[x_2 + \left(x_2^2 - x_3^2\right)/2a\right] = \left(\mu\alpha'/2a\right)\left(2ax_2 + x_2^2\right).$$

(d) $dT_{13}/dx_2 = \left(\mu\alpha'/2a\right)\left(2a + 2x_2\right) = 0 \rightarrow x_2 = -a \rightarrow \left|\left(T_{13}\right)_{x_2=-a}\right| = \mu\alpha' a/2.$

But at $\left(x_2, x_3\right) = \left(a, 0\right)$, $T_{13} = \mu\alpha'\left[x_2 + \left(x_2^2 - x_3^2\right)/2a\right] = \mu\alpha'\left[a + \left(a^2\right)/2a\right] = \left(3a/2\right)\mu\alpha'$.

Thus, along $x_3 = 0$, the greatest shear stress occurs at $\left(x_2, x_3\right) = \left(a, 0\right)$ with $T_s = \left(3a/2\right)\mu\alpha'$.

5.55 Show from the compatibility equations that the Prandtl's stress function $\psi\left(x_2, x_3\right)$ for torsion problem must satisfy the equation $\frac{\partial^2\psi}{\partial x_3^3} + \frac{\partial^2\psi}{\partial x_2^2} = \text{constant}$

Ans. With $T_{12} = \frac{\partial\psi}{\partial x_3}$, $T_{13} = -\frac{\partial\psi}{\partial x_2}$, and all other $T_{ij} = 0$, we have, the nonzero strain components are: $E_{12} = \frac{1}{2\mu}\frac{\partial\psi}{\partial x_3}$, $E_{13} = -\frac{1}{2\mu}\frac{\partial\psi}{\partial x_2}$, all other $E_{ij} = 0$. All equations of compatibility are identically satisfied except the following two:

$$\frac{\partial^2 E_{22}}{\partial x_3 \partial x_1} = \frac{\partial}{\partial x_2}\left(-\frac{\partial E_{31}}{\partial x_2} + \frac{\partial E_{12}}{\partial x_3} + \frac{\partial E_{23}}{\partial x_1}\right), \quad \frac{\partial^2 E_{33}}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_3}\left(-\frac{\partial E_{12}}{\partial x_3} + \frac{\partial E_{23}}{\partial x_1} + \frac{\partial E_{31}}{\partial x_2}\right)$$

which leads to

$$\frac{\partial}{\partial x_2}\left(\frac{\partial^2 \psi}{\partial x_2^2} + \frac{\partial^3 \psi}{\partial x_3^3}\right) = 0\,, \quad \frac{\partial}{\partial x_3}\left(\frac{\partial^2 \psi}{\partial x_3^3} + \frac{\partial^2 \psi}{\partial x_2^2}\right) = 0 \rightarrow \frac{\partial^2 \psi}{\partial x_3^3} + \frac{\partial^2 \psi}{\partial x_2^2} = \text{constant}$$

5.56 Given that the Prandtl' stress function for a rectangular bar in torsion is given by

$$\psi = \left(\frac{32\mu\alpha' a^2}{\pi^3}\right)\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}(-1)^{(n-1)/2}\left\{1 - \frac{\cosh(n\pi x_3/2a)}{\cosh(n\pi b/2a)}\right\}\cos\frac{n\pi x_2}{2a}$$

The cross section is defined by $-a \le x_2 \le a$ and $-b \le x_3 \le b$. Assume $b > a$, (a) Find the maximum shearing stress. (b) Find the maximum normal stress and the plane it acts.

Ans. We know that when a rectangular membrane, fixed on its side, is subjected to a uniform pressure on one side of the membrane, the deformed surface has a maximum slope at the mid point of the longer side. Thus, based on the membrane analogy discussed in Example 5.17.3, on any plane $x_1 = \text{constant}$, the maximum shearing stress occurs on the mid point of the longer side. That is at the point $x_2 = a$ and $x_3 = 0$. From the given function $\psi(x_2, x_3)$, we obtain the stress components as

$$T_{12} = \frac{\partial \psi}{\partial x_3} = -\left(\frac{32\mu\alpha' a^2}{\pi^3}\right)\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}(-1)^{(n-1)/2}\left(\frac{n\pi}{2a}\right)\left\{\frac{\sinh(n\pi x_3/2a)}{\cosh(n\pi b/2a)}\right\}\cos\frac{n\pi x_2}{2a}.$$

$$T_{13} = -\frac{\partial \psi}{\partial x_2} = \left(\frac{32\mu\alpha' a^2}{\pi^3}\right)\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}(-1)^{(n-1)/2}\frac{n\pi}{2a}\left\{1 - \frac{\cosh(n\pi x_3/2a)}{\cosh(n\pi b/2a)}\right\}\sin\frac{n\pi x_2}{2a}.$$

$$At\ x_2 = a, x_3 = 0, \ \left(\text{note } \sin(n\pi/2) = (-1)^{(n+3)/2}, n = 1,3,5..\right)$$

$$T_{13} = \left(\frac{16\mu\alpha' a}{\pi^2}\right)\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\left\{1 - \frac{1}{\cosh(n\pi b/2a)}\right\}, \ T_{12} = 0$$

That is

$$(T_s)_{\max} = \left(\frac{16\mu\alpha' a}{\pi^2}\right)\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\left\{1 - \frac{1}{\cosh(n\pi b/2a)}\right\}.$$

Or, since $\sum_{n=1,3,5}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8}$, therefore $(T_s)_{\max} = 2\mu\alpha' a - \left(\frac{16\mu\alpha' a}{\pi^2}\right)\sum_{n=1,3,5}^{\infty}\frac{1}{n^2}\left\{\frac{1}{\cosh(n\pi b/2a)}\right\}$

Since at this point, the only nonzero stress components are T_{13} and $T_{31}(=T_{13})$, therefore, the characteristic equation is $-\lambda^3 + \lambda T_{13}^2 = 0$ so that the maximum normal stress is $(T_n)_{\max} = T_{13} = (T_s)_{\max}$, which acts on plane whose normal is in the direction $(1/\sqrt{2})(\mathbf{e}_1 \pm \mathbf{e}_3)$.

5.57 Obtain the relationship between the twisting moment M_t and the twist angle per unit length α' for a rectangular bar under torsion. Note: $1+\frac{1}{3^4}+\frac{1}{5^4}+...=\frac{\pi^4}{96}$.

Ans. We have, [see Eq. (5.18.10)],

$$\psi=\left(\frac{32\mu\alpha' a^2}{\pi^3}\right)\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}(-1)^{(n-1)/2}\left\{1-\frac{\cosh(n\pi x_3/2a)}{\cosh(n\pi b/2a)}\right\}\cos\frac{n\pi x_2}{2a}.$$

Thus, if let $A=\left(\frac{32\mu\alpha' a^2}{\pi^3}\right)$ and $F(x_3)=\frac{\cosh(n\pi x_3/2a)}{\cosh(n\pi b/2a)}$, then, we have,

$M_t=2\int\psi dA=$

$$\left(2A\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}(-1)^{(n-1)/2}(2b)\int_{-a}^{a}\cos\frac{n\pi x_2}{2a}dx_2\right)-\left(2A\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}(-1)^{(n-1)/2}\int_{-a}^{a}\cos\frac{n\pi x_2}{2a}\left[\int_{-b}^{b}F(x_3)dx_3\right]dx_2\right)$$

$\equiv M-N.$

Now, $\int_{-a}^{a}\cos\frac{n\pi x_2}{2a}dx_2=\left(\frac{2a}{n\pi}\right)2\sin\frac{n\pi}{2}=\left(\frac{4a}{n\pi}\right)(-1)^{(n+3)/2}$, $n=1,3,5$, therefore,

$$M=2A\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}(-1)^{(n-1)/2}(2b)\int_{-a}^{a}\cos\frac{n\pi x_2}{2a}dx_2=A\left(\frac{8a}{\pi}\right)(2b)\sum_{n=1,3,5}^{\infty}\frac{1}{n^4}=\left(\frac{32\mu\alpha'}{\pi^4}\right)(2a)^3(2b)\sum_{n=1,3,5}^{\infty}\frac{1}{n^4}$$

.

Next, $\int_{-b}^{b}F(x_3)dx_3=\int_{-b}^{b}\frac{\cosh(n\pi x_3/2a)}{\cosh(n\pi b/2a)}dx_3=\left(\frac{2a}{n\pi}\right)\frac{2\sinh(n\pi b/2a)}{\cosh(n\pi b/2a)}=\left(\frac{4a}{n\pi}\right)\tanh\frac{n\pi b}{2a}$, so that

$$N=2A\sum_{n=1,3,5}^{\infty}\frac{1}{n^3}(-1)^{(n-1)/2}\int_{-a}^{a}\cos\frac{n\pi x_2}{2a}\left[\int_{-b}^{b}F(x_3)dx_3\right]dx_2=\left(\frac{64\mu\alpha'(2a)^4}{\pi^5}\right)\sum_{n=1,3,5}^{\infty}\frac{1}{n^5}\tanh\frac{n\pi b}{2a}$$

Thus,

$$M_t\equiv M-N=\left(\frac{32\mu\alpha'}{\pi^4}\right)(2a)^3(2b)\sum_{n=1,3,5}^{\infty}\frac{1}{n^4}-\left(\frac{64\mu\alpha'(2a)^4}{\pi^5}\right)\sum_{n=1,3,5}^{\infty}\frac{1}{n^5}\tanh\frac{n\pi b}{2a}$$

$$=\left(\frac{32\mu\alpha'}{\pi^4}\right)(2a)^3(2b)\left(\frac{\pi^4}{96}\right)-\left(\frac{64\mu\alpha'(2a)^4}{\pi^5}\right)\sum_{n=1,3,5}^{\infty}\frac{1}{n^5}\tanh\frac{n\pi b}{2a}.$$

Or,

$$M_t=\left(\frac{\mu\alpha'}{3}\right)(2a)^3(2b)\left[1-\frac{192}{\pi^5}\left(\frac{a}{b}\right)\sum_{n=1,3,5}^{\infty}\frac{1}{n^5}\tanh\frac{n\pi b}{2a}\right]$$

5.58 In pure bending of a bar, let $\mathbf{M}_L = M_2\mathbf{e}_2 + M_3\mathbf{e}_3 = -\mathbf{M}_R$, where $\mathbf{e}_2$ and $\mathbf{e}_3$ are not along the principal axes, show that the flexural stress T_{11} is given by

$$T_{11} = -\frac{M_2 I_{23} + M_3 I_{22}}{\left(I_{33}I_{22} - I_{23}^2\right)} x_2 + \frac{M_2 I_{33} + M_3 I_{23}}{\left(I_{33}I_{22} - I_{23}^2\right)} x_3$$

Ans. Refer to Section 5.19, we had [see Eq.(5.19.4)(5.19.6) and (5.19.7)]

$T_{11} = \beta x_2 + \gamma x_3$, where $M_2 = \beta I_{23} + \gamma I_{22}$, $M_3 = -\beta I_{33} - \gamma I_{23}$

Solving the above two equations for β and γ, in terms of M_2 and M_3, we obtain

$$\beta = -\frac{M_2 I_{23} + M_3 I_{22}}{\left(I_{33}I_{22} - I_{23}^2\right)} \text{ and } \gamma = \frac{M_2 I_{33} + M_3 I_{23}}{\left(I_{33}I_{22} - I_{23}^2\right)}.$$

Thus, $T_{11} = -\dfrac{M_2 I_{23} + M_3 I_{22}}{\left(I_{33}I_{22} - I_{23}^2\right)} x_2 + \dfrac{M_2 I_{33} + M_3 I_{23}}{\left(I_{33}I_{22} - I_{23}^2\right)} x_3$.

5.59 From the strain components for pure bending

$$E_{11} = \frac{M_2 x_3}{I_{22}E_Y}, \quad E_{22} = E_{33} = -\frac{\nu M_2 x_3}{I_{22}E_Y}, \quad E_{12} = E_{13} = E_{23} = 0$$

Obtain the displacement field

Ans. Integration of $\partial u_1/\partial x_1 = Ax_3$, $\partial u_2/\partial x_2 = -\nu Ax_3$, $\partial u_3/\partial x_3 = -\nu Ax_3$, where $A \equiv \dfrac{M_2}{I_{22}E_Y}$ gives

$$u_1 = Ax_3x_1 + f_1(x_2,x_3), \quad u_2 = -\nu Ax_3x_2 + f_2(x_1,x_3), \quad u_3 = -\nu Ax_3^2/2 + f_3(x_1,x_2) \tag{i}$$

where $f_1(x_2,x_3)$, $f_2(x_1,x_3)$ and $f_3(x_1,x_2)$ are integration functions. Substituting (i) into $\partial u_1/\partial x_2 + \partial u_2/\partial x_1 = 0$, $\partial u_1/\partial x_3 + \partial u_3/\partial x_1 = 0$ and $\partial u_2/\partial x_3 + \partial u_3/\partial x_2 = 0$, we obtain

$$\begin{aligned}
\partial f_1(x_2,x_3)/\partial x_2 &= -\partial f_2(x_1,x_3)/\partial x_1 = g_1(x_3)\\
\partial f_1(x_2,x_3)/\partial x_3 &= -\partial f_3(x_1,x_2)/\partial x_1 = g_2(x_2)\\
\partial f_2(x_1,x_3)/\partial x_3 &= -\partial f_3(x_1,x_2)/\partial x_2 = g_3(x_1)
\end{aligned} \tag{ii}$$

where $g(x_1), g(x_2), g(x_3)$ are integration functions. Integrations of (ii) give,

$$f_1 = g_1(x_3)x_2 + g_4(x_3) \text{ and } f_1 = g_2(x_2)x_3 + g_6(x_2) \tag{iii}$$

$$-f_2 = g_1(x_3)x_1 + g_5(x_3), \text{ and } f_2 = g_3(x_1)x_3 + g_8(x_1) \tag{iv}$$

$$-f_3 = g_2(x_2)x_1 + g_7(x_2) \text{ and } -f_3 = g_3(x_1)x_2 + g_9(x_1) \tag{v}$$

From (iii), $g_1(x_3) = a_1x_3 + b_1$, $g_2(x_2) = a_1x_2 + b_2$, $g_4(x_3) = b_2x_3 + c_2$, $g_6(x_2) = b_1x_2 + c_2$ (vi)

From (iv) and (vi), $g_3(x_1) = -a_1x_1 + b_3$, $g_8(x_1) = -b_1x_1 + c_3$, $-g_5(x_3) = b_3x_3 + c_3$ (vii)

From (v) (vi),(vii), $a_1 = 0$, $g_9(x_1) = b_2x_1 + c_4$, $g_7(x_2) = b_3x_2 + c_4$ (viii)

Thus,

$$f_1 = b_1x_2 + b_2x_3 + c_2, \quad f_2 = b_3x_3 - b_1x_1 + c_3, \quad f_3 = -b_2x_1 - b_3x_2 - c_4 \tag{ix}$$

So that,

$$u_1 = \frac{M_2}{I_{22}E_Y} x_3 x_1 + b_1 x_2 + b_2 x_3 + c_2, \quad u_2 = -\frac{\nu M_2}{I_{22}E_Y} x_3 x_2 + b_3 x_3 - b_1 x_1 + c_3,$$
$$u_3 = -\frac{\nu M_2}{2I_{22}E_Y} x_3^2 - b_2 x_1 - b_3 x_2 - c_4. \qquad \text{(x)}$$

5.60 In pure bending of a bar, let $\mathbf{M}_L = M_2\mathbf{e}_2 + M_3\mathbf{e}_3 = -\mathbf{M}_R$, where $\mathbf{e}_2$and $\mathbf{e}_3$ are along the principal axes, show that the neutral axis, (that is, the axis on the cross section where the flexural stress T_{11} is zero) is, in general, not parallel to the couple vectors.

Ans. From Eq.(5.19.10), we have, $T_{11} = \frac{M_2 x_3}{I_{22}} - \frac{M_3 x_2}{I_{33}}$, thus the neutral axis is given by:

$\frac{M_2 x_3}{I_{22}} - \frac{M_3 x_2}{I_{33}} = 0$. That is, the neutral axis is given by $\frac{x_3}{x_2} = \left(\frac{I_{22}}{I_{33}}\right)\frac{M_3}{M_2}$. Thus, only when $I_{22} = I_{33}$ is the neutral axis parallel to the couple vector $\mathbf{M}_L = M_2\mathbf{e}_2 + M_3\mathbf{e}_3 = -\mathbf{M}_R$.

5.61 For plane strain problem, derive the bi-harmonic equation for the Airy stress function

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Ans. We have [Eq.(5.20.7)}

$$E_{11} = \frac{1}{E_Y}\left[\left(1-\nu^2\right)\frac{\partial^2\varphi}{\partial x_2^2} - \nu\left(1+\nu\right)\frac{\partial^2\varphi}{\partial x_1^2}\right], E_{22} = \frac{1}{E_Y}\left[\left(1-\nu^2\right)\frac{\partial^2\varphi}{\partial x_1^2} - \nu\left(1+\nu\right)\frac{\partial^2\varphi}{\partial x_2^2}\right],$$

$$E_{12} = -\frac{1}{E_Y}(1+\nu)\frac{\partial^2\varphi}{\partial x_1 \partial x_2}, \quad E_{13} = E_{23} = E_{33} = 0.$$

$$E_Y \frac{\partial^2 E_{11}}{\partial x_2^2} = \left[\left(1-\nu^2\right)\frac{\partial^4\varphi}{\partial x_2^4} - \nu\left(1+\nu\right)\frac{\partial^4\varphi}{\partial x_2^2 \partial x_1^2}\right], \; E_Y \frac{\partial^2 E_{22}}{\partial x_1^2} = \left[\left(1-\nu^2\right)\frac{\partial^4\varphi}{\partial x_1^4} - \nu\left(1+\nu\right)\frac{\partial^4\varphi}{\partial x_1^2 \partial x_2^2}\right],$$

$2E_Y \frac{\partial^2 E_{12}}{\partial x_2^2 \partial x_1^2} = -2(1+\nu)\frac{\partial^2\varphi}{\partial x_1^2 \partial x_2^2}$. Thus, the compatibility equation

$$\left(\frac{\partial^2 E_{11}}{\partial x_2^2} + \frac{\partial^2 E_{22}}{\partial x_2^2} - 2\frac{\partial^2 E_{12}}{\partial x_2^2 \partial x_1^2}\right) = 0 \rightarrow$$

$$\rightarrow \left[\left(1-\nu^2\right)\left\{\frac{\partial^4\varphi}{\partial x_2^4} + \frac{\partial^4\varphi}{\partial x_1^4}\right\} + \left\{2\left(1+\nu\right) - 2\nu\left(1+\nu\right)\right\}\frac{\partial^4\varphi}{\partial x_2^2 \partial x_1^2}\right] = 0,$$

$$\rightarrow \left(1-\nu^2\right)\left(\frac{\partial^4\varphi}{\partial x_2^4} + \frac{\partial^4\varphi}{\partial x_1^4} + 2\frac{\partial^4\varphi}{\partial x_2^2 \partial x_1^2}\right) = 0 \rightarrow \left(\frac{\partial^4\varphi}{\partial x_2^4} + \frac{\partial^4\varphi}{\partial x_1^4} + 2\frac{\partial^4\varphi}{\partial x_2^2 \partial x_1^2}\right) = 0$$

5.62 For plane stress problem, derive the bi-harmonic equation for the Airy stress function

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Ans.

From $E_{11}=\frac{1}{E_Y}\left(\frac{\partial^2\varphi}{\partial x_2^2}-\nu\frac{\partial^2\varphi}{\partial x_1^2}\right),\quad E_{22}=\frac{1}{E_Y}\left(\frac{\partial^2\varphi}{\partial x_1^2}-\nu\frac{\partial^2\varphi}{\partial x_2^2}\right),\ E_{12}=-\frac{(1+\nu)}{E_Y}\frac{\partial^2\varphi}{\partial x_1\partial x_2}$, [Eq. (5.22.3)]

we get

$$E_Y\left(\frac{\partial^2 E_{11}}{\partial x_2^2}+\frac{\partial^2 E_{22}}{\partial x_1^2}\right)=\frac{\partial^4\varphi}{\partial x_1^4}+\frac{\partial^4\varphi}{\partial x_2^4}-2\nu\frac{\partial^4\varphi}{\partial x_1^2\partial x_2^2},\quad 2E_Y\frac{\partial^2 E_{12}}{\partial x_1\partial x_2}=-2\frac{\partial^2\varphi}{\partial x_1^2\partial x_2^2}-2\nu\frac{\partial^2\varphi}{\partial x_1^2\partial x_2^2}$$

The compatibility equation $\frac{\partial^2 E_{11}}{\partial x_2^2}+\frac{\partial^2 E_{22}}{\partial x_1^2}=2\frac{\partial^2 E_{12}}{\partial x_1\partial x_2}$ then gives

$$\rightarrow\frac{\partial^4\varphi}{\partial x_1^4}+\frac{\partial^4\varphi}{\partial x_2^4}-2\nu\frac{\partial^4\varphi}{\partial x_1^2\partial x_2^2}=-2\frac{\partial^2\varphi}{\partial x_1^2\partial x_2^2}-2\nu\frac{\partial^2\varphi}{\partial x_1^2\partial x_2^2}\rightarrow \text{Thus, }\frac{\partial^4\varphi}{\partial x_1^4}+\frac{\partial^4\varphi}{\partial x_2^4}+2\frac{\partial^2\varphi}{\partial x_1^2\partial x_2^2}=0$$

5.63 Consider the Airy stress function $\varphi=\alpha_1x_1^2+\alpha_2x_1x_2+\alpha_3x_2^2$. (a) Verify that it satisfies the bi-harmonic equation. (b) Determine the in-plane stresses T_{11}, T_{12} *and* T_{22}. (c) Determine and sketch the tractions on the four rectangular boundaries $x_1=0, x_1=b, x_2=0, x_2=c$.(d) As a plane strain solution, determine T_{13}, T_{23}, T_{33} and all the strain components . (e) As a plane stress solution, determine T_{13}, T_{23}, T_{33}, and all the strain components .

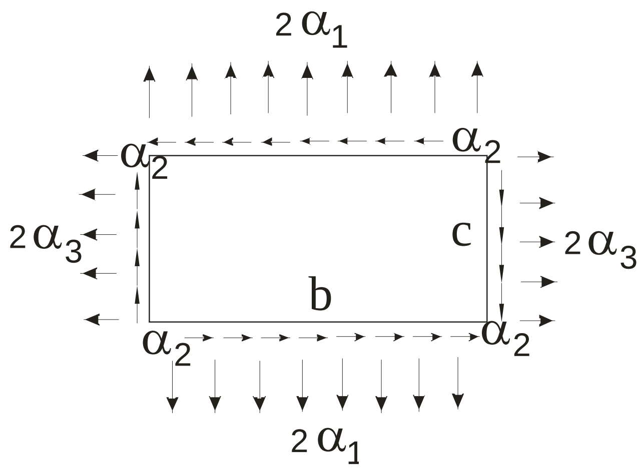
Ans. (a) $\partial^4\varphi/\partial x_1^4=0,\ \ \partial^4\varphi/\partial x_2^4=0,\ \ \partial^4\varphi/\partial x_1^2\partial x_2^2=0$, thus $\varphi=\alpha_1x_1^2+\alpha_2x_1x_2+\alpha_3x_2^2$ satisfies the bi-harmonic equation.

(b) $T_{11}=\partial^2\varphi/\partial x_2^2=2\alpha_3,\ \ T_{12}=-\partial^2\varphi/\partial x_1\partial x_2=-\alpha_2,\ \ T_{22}=\partial^2\varphi/\partial x_1^2=2\alpha_1$

(c)

On $x_1=0,\ \mathbf{t}=-\mathbf{T}\mathbf{e}_1=-\left(T_{11}\mathbf{e}_1+T_{21}\mathbf{e}_2\right)=-2\alpha_3\mathbf{e}_1+\alpha_2\mathbf{e}_2$, on $x_1=b,\ \mathbf{t}=\mathbf{T}\mathbf{e}_1=T_{11}\mathbf{e}_1+T_{21}\mathbf{e}_2=2\alpha_3\mathbf{e}_1-\alpha_2\mathbf{e}_2$,

on $x_2=0,\ \mathbf{t}=-\mathbf{T}\mathbf{e}_2=-\left(T_{12}\mathbf{e}_1+T_{22}\mathbf{e}_2\right)=\alpha_2\mathbf{e}_1-2\alpha_1\mathbf{e}_2$, on $x_1=c,\ \mathbf{t}=\mathbf{T}\mathbf{e}_2=T_{12}\mathbf{e}_1+T_{22}\mathbf{e}_2=-\alpha_2\mathbf{e}_1+2\alpha_1\mathbf{e}_2$.



(d) As a plane strain solution,

$$T_{33} = \nu\left(T_{11} + T_{22}\right) = 2\nu\left(\alpha_3 + \alpha_1\right),\ T_{13} = T_{23} = 0, \quad E_{13} = E_{23} = E_{33} = 0,$$

$$E_{11} = \left(1/E_Y\right)\left[(1-\nu^2)T_{11} - \nu\left(1+\nu\right)T_{22}\right] = 2\left(1/E_Y\right)\left[(1-\nu^2)\alpha_3 - \nu\left(1+\nu\right)\alpha_1\right],$$

$$E_{22} = \left(1/E_Y\right)\left[(1-\nu^2)T_{22} - \nu\left(1+\nu\right)T_{11}\right] = 2\left(1/E_Y\right)\left[(1-\nu^2)\alpha_1 - \nu\left(1+\nu\right)\alpha_3\right],$$

$$E_{12} = \left(1/E_Y\right)\left(1+\nu\right)T_{12} = -\left(1/E_Y\right)\left(1+\nu\right)\alpha_2,$$

(e) As a plane stress solution,

$$T_{33} = T_{13} = T_{23} = 0,\ \ E_{13} = E_{23} = 0,\ \ E_{11} = \left(1/E_Y\right)\left(T_{11} - \nu T_{22}\right) = 2\left(1/E_Y\right)\left(\alpha_3 - \nu\alpha_1\right),$$

$$E_{22} = \left(1/E_Y\right)\left(T_{22} - \nu T_{11}\right) = 2\left(1/E_Y\right)\left(\alpha_1 - \nu\alpha_3\right), \quad E_{12} = \left(1/E_Y\right)\left(1+\nu\right)T_{12} = -\alpha_2\left(1+\nu\right)/E_Y = -\alpha_2/2\mu.$$

$$E_{33} = \left(1/E_Y\right)\left[-\nu\left(T_{11} + T_{22}\right)\right] = -2\left(\nu/E_Y\right)\left(\alpha_3 + \alpha_1\right).$$

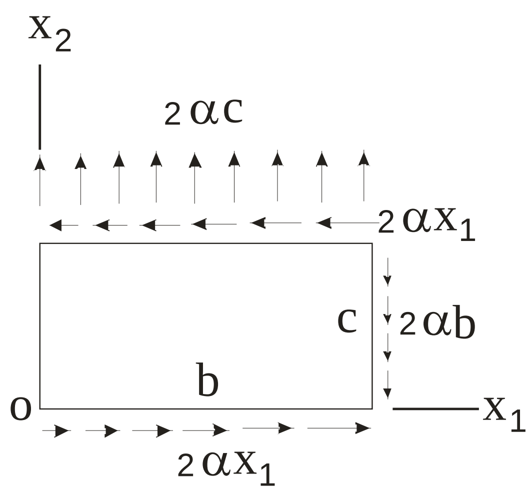
Note, for this problem, since $T_{11} + T_{22}$ is a linear function of x_1 and x_2, in fact, a constant, therefore, all the compatibility equations are satisfied so that E_{33} is meaningful and u_3 does exist.

5.64 Consider the Airy stress function $\varphi = \alpha x_1^2 x_2$. (a) Verify that it satisfies the bi-harmonic equation. (b) Determine the in-plane stresses T_{11}, T_{12} *and* T_{22}. (c) Determine and sketch the tractions on the four rectangular boundaries $x_1 = 0, x_1 = b, x_2 = 0, x_2 = c$. (d) As a plane strain solution, determine T_{13}, T_{23}, T_{33} and all the strain components. (e) As a plane stress solution, determine T_{13}, T_{23}, T_{33}. and all the strain components .

Ans. (a) $\partial^4\varphi/\partial x_1^4 = 0,\ \ \partial^4\varphi/\partial x_2^4 = 0,\ \ \partial^4\varphi/\partial x_1^2\partial x_2^2 = 0$, thus $\varphi = \alpha x_1^2 x_2$ satisfies the bi-harmonic equation.

(b) $T_{11} = \partial^2\varphi/\partial x_2^2 = 0,\ \ T_{12} = -\partial^2\varphi/\partial x_1\partial x_2 = -2\alpha x_1,\ \ T_{22} = \partial^2\varphi/\partial x_1^2 = 2\alpha x_2$

(c) On $x_1 = 0$, $\mathbf{t} = -\mathbf{T}\mathbf{e}_1 = -\left(T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2\right) = 2\alpha x_1\mathbf{e}_2 = 0$, On $x_1 = b$, $\mathbf{t} = \mathbf{T}\mathbf{e}_1 = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 = -2\alpha b\mathbf{e}_2$.
On $x_2 = 0$, $\mathbf{t} = -\mathbf{T}\mathbf{e}_2 = -\left(T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2\right) = 2\alpha x_1\mathbf{e}_1 - 2\alpha x_2\mathbf{e}_2 = 2\alpha x_1\mathbf{e}_1$.
On $x_2 = c$, $\mathbf{t} = \mathbf{T}\mathbf{e}_2 = T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 = -2\alpha x_1\mathbf{e}_1 + 2\alpha x_2\mathbf{e}_2 = -2\alpha x_1\mathbf{e}_1 + 2\alpha c\mathbf{e}_2$.



(d) As a plane strain solution,

$$T_{33} = \nu\left(T_{11} + T_{22}\right) = 2\nu\alpha x_2,\ T_{13} = T_{23} = 0.\quad E_{13} = E_{23} = E_{33} = 0,$$

$$E_{11} = \left(1/E_Y\right)\left[(1-\nu^2)T_{11} - \nu\left(1+\nu\right)T_{22}\right] = -\left[2\alpha\nu\left(1+\nu\right)/E_Y\right]x_2,$$

$$E_{22} = \left(1/E_Y\right)\left[(1-\nu^2)T_{22} - \nu\left(1+\nu\right)T_{11}\right] = \left[2\alpha(1-\nu^2)/E_Y\right]x_2,$$

$$E_{12} = \left(1/E_Y\right)\left(1+\nu\right)T_{12} = -\left[2\alpha\left(1+\nu\right)/E_Y\right]x_1.$$

(d) As a plane stress solution,

$$T_{33} = T_{13} = T_{23} = 0,\ \ E_{13} = E_{23} = 0,\ \ E_{33} = \left(1/E_Y\right)\left[-\nu\left(T_{11} + T_{22}\right)\right] = -2\alpha\left(\nu/E_Y\right)x_2.$$

$$E_{11} = \left(1/E_Y\right)\left(T_{11} - \nu T_{22}\right) = -\left(2\alpha\nu/E_Y\right)x_2,\ \ E_{22} = \left(1/E_Y\right)\left(T_{22} - \nu T_{11}\right) = \left(2\alpha/E_Y\right)x_2.$$

$$E_{12} = \left(1/E_Y\right)\left(1+\nu\right)T_{12} = -\left[2\alpha\left(1+\nu\right)/E_Y\right]x_1 = -\alpha x_1/\mu.$$

Note, for this problem, since $T_{11} + T_{22}$ is a linear function of x_1 and x_2, therefore, all the compatibility equations are satisfied so that E_{33} is meaningful and u_3 does exist.

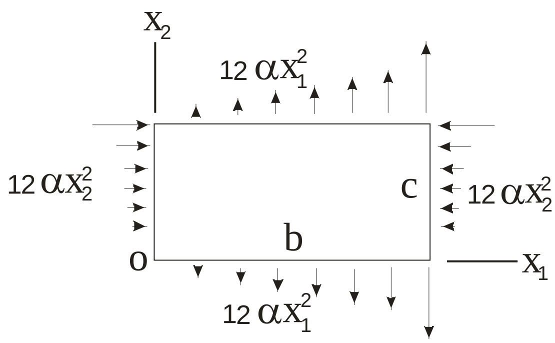
5.65 Consider the Airy stress function $\varphi = \alpha\left(x_1^4 - x_2^4\right)$. (a) Verify that it satisfies the bi-harmonic equation. (b) Determine the in-plane stresses T_{11}, T_{12} *and* T_{22}. (c) Determine and sketch the tractions on the four rectangular boundaries $x_1 = 0, x_1 = b, x_2 = 0, x_2 = c$. (d) As a plane strain solution, determine T_{13}, T_{23}, T_{33} and all the strain components . (e) As a plane stress solution, determine T_{13}, T_{23}, T_{33} and all the strain components .

Ans. (a) $\partial^4\varphi/\partial x_1^4 = 24\alpha,\ \ \partial^4\varphi/\partial x_2^4 = -24\alpha,\ \ \partial^4\varphi/\partial x_1^2\partial x_2^2 = 0$, thus $\varphi = \alpha\left(x_1^4 - x_2^4\right)$ satisfies the bi-harmonic equation.

(b) $T_{11} = \partial^2\varphi/\partial x_2^2 = -12\alpha x_2^2,\ \ T_{22} = \partial^2\varphi/\partial x_1^2 = 12\alpha x_1^2,\ \ \ T_{12} = -\partial^2\varphi/\partial x_1\partial x_2 = 0$.

(c) On $x_1 = 0,\ \mathbf{t} = -\mathbf{T}\mathbf{e}_1 = -\left(T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2\right) = 12\alpha x_2^2\mathbf{e}_1,\quad$ On $x_1 = b,\ \mathbf{t} = \mathbf{T}\mathbf{e}_1 = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 = -12\alpha x_2^2\mathbf{e}_1.$

On $x_2 = 0,\ \mathbf{t} = -\mathbf{T}\mathbf{e}_2 = -\left(T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2\right) = -12\alpha x_1^2\mathbf{e}_2,\quad$ On $x_2 = c,\ \mathbf{t} = \mathbf{T}\mathbf{e}_2 = T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 = 12\alpha x_1^2\mathbf{e}_2.$



(d) As a plane strain solution,

$T_{33} = \nu\left(T_{11} + T_{22}\right) = 12\alpha\nu\left(x_1^2 - x_2^2\right),\ T_{13} = T_{23} = 0,\quad E_{13} = E_{23} = E_{33} = 0,\quad E_{12} = \left(1/E_Y\right)\left(1+\nu\right)T_{12} = 0.$

$E_{11} = \left(1/E_Y\right)\left[(1-\nu^2)T_{11} - \nu\left(1+\nu\right)T_{22}\right] = -12\alpha\left(1/E_Y\right)\left[x_2^2(1-\nu^2) + \nu\left(1+\nu\right)x_1^2\right].$

$E_{22} = \left(1/E_Y\right)\left[(1-\nu^2)T_{22} - \nu\left(1+\nu\right)T_{11}\right] = 12\alpha\left(1/E_Y\right)\left[(1-\nu^2)x_1^2 + x_2^2\nu\left(1+\nu\right)\right].$

(d) As a plane stress solution,

$T_{33} = T_{13} = T_{23} = 0,\ \ E_{13} = E_{23} = 0,\quad E_{11} = \left(1/E_Y\right)\left(T_{11} - \nu T_{22}\right) = -12\alpha\left(1/E_Y\right)\left(x_2^2 + \nu x_1^2\right),$

$E_{22} = \left(1/E_Y\right)\left(T_{22} - \nu T_{11}\right) = 12\alpha\left(1/E_Y\right)\left(x_1^2 + \nu x_2^2\right),\quad E_{12} = \left(1/E_Y\right)\left(1+\nu\right)T_{12} = 0.$

.$E_{33} = \left(1/E_Y\right)\left[-\nu\left(T_{11} + T_{22}\right)\right] = 12\alpha\nu\left(1/E_Y\right)\left(x_2^2 - x_1^2\right).$

Since $T_{11} + T_{22}$ is not a linear function of x_1 and x_2, E_{33} is meaningless, because u_3 does not exist.

5.66 Consider the Airy's stress function $\varphi = \alpha x_1 x_2^2 + x_1 x_2^3$. (a) Verify that it satisfies the bi-harmonic equation. (b) Determine the in-plane stresses T_{11}, T_{12} *and* T_{22}. (c) Determine the condition necessary for the traction at $x_2 = c$ to vanish and (d) determine the tractions on the remaining boundaries $x_1 = 0, x_1 = b$ and $x_2 = 0$.

Ans. (a) $\partial^4\varphi/\partial x_1^4 = 0,\ \ \partial^4\varphi/\partial x_2^4 = 0,\ \ \partial^4\varphi/\partial x_1^2\partial x_2^2 = 0$, thus $\varphi = \alpha x_1 x_2^2 + x_1 x_2^3$ satisfies the bi-harmonic equation.

(b) $T_{11} = \partial^2\varphi/\partial x_2^2 = 2\alpha x_1 + 6x_1x_2,\ \ T_{22} = \partial^2\varphi/\partial x_1^2 = 0,\quad T_{12} = -\partial^2\varphi/\partial x_1\partial x_2 = -2\alpha x_2 - 3x_2^2$.

(c)On $x_2 = c$, $\mathbf{t} = \mathbf{T}\mathbf{e}_2 = T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2 = (-2\alpha c - 3c^2)\mathbf{e}_1, \rightarrow -2\alpha c - 3c^2 = 0 \rightarrow 2\alpha c = -3c^2 \rightarrow \alpha = -3c/2$
.

(d) On $x_1 = 0$, $\mathbf{t} = -\mathbf{T}\mathbf{e}_1 = -(T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2) = \left(2\alpha x_2 + 3x_2^2\right)\mathbf{e}_2 = 3x_2\left(x_2 - c\right)\mathbf{e}_2$.

On $x_1 = b$, $\mathbf{t} = \mathbf{T}\mathbf{e}_1 = T_{11}\mathbf{e}_1 + T_{21}\mathbf{e}_2 = 3b\left(2x_2 - c\right)\mathbf{e}_1 - 3x_2\left(x_2 - c\right)\mathbf{e}_2$.

On $x_2 = 0$, $\mathbf{t} = -\mathbf{T}\mathbf{e}_2 = -\left(T_{12}\mathbf{e}_1 + T_{22}\mathbf{e}_2\right) = 0$.

5.67 Obtain the in-plane displacement components for the plane stress solution for the cantilever beam from the following strain strain-displacement relations.

$$E_{11} = \frac{\partial u_1}{\partial x_1} = \frac{Px_1x_2}{E_Y I},\quad E_{22} = \frac{\partial u_2}{\partial x_2} = -\frac{\nu Px_1x_2}{E_Y I},\quad E_{12} = \left(\frac{P}{4\mu I}\right)\left(\frac{h^2}{4} - x_2^2\right).$$

Ans.

$$\frac{\partial u_1}{\partial x_1} = \frac{Px_1x_2}{E_Y I} \rightarrow u_1 = \frac{Px_1^2x_2}{2E_Y I} + f_1(x_2),\quad \frac{\partial u_2}{\partial x_2} = -\frac{\nu Px_1x_2}{E_Y I} \rightarrow u_2 = -\frac{\nu Px_1x_2^2}{2E_Y I} + f_2(x_1),$$

$$\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} = 2\left(\frac{P}{4\mu I}\right)\left(\frac{h^2}{4} - x_2^2\right) \rightarrow \frac{Px_1^2}{2E_Y I} + \frac{df_1}{dx_2} - \frac{\nu Px_2^2}{E_Y I} + \frac{df_2}{dx_1} = \left(\frac{P}{2\mu I}\right)\left(\frac{h^2}{4} - x_2^2\right),$$

$$\rightarrow \frac{Px_1^2}{2E_Y I} + \frac{df_2}{dx_1} = -\frac{df_1}{dx_2} + \frac{\nu Px_2^2}{E_Y I} - x_2^2\left(\frac{P}{2\mu I}\right) + \left(\frac{P}{2\mu I}\right)\left(\frac{h^2}{4}\right).$$

In the above equation, the left side is a function of x_1 only, right side is a function of x_2 only, thus both sides must equal to the same constant, say c_1. That is,

$$\frac{Px_1^2}{2E_Y I} + \frac{df_2}{dx_1} = c_1 \rightarrow \frac{df_2}{dx_1} = c_1 - \frac{Px_1^2}{2E_Y I} \rightarrow f_2 = c_1x_1 - \frac{Px_1^3}{6E_Y I} + c_2.$$

$$\frac{df_1}{dx_2} = \frac{\nu Px_2^2}{E_Y I} - x_2^2\left(\frac{P}{2\mu I}\right) + \left(\frac{P}{2\mu I}\right)\left(\frac{h^2}{4}\right) - c_1 \rightarrow f_1 = \frac{\nu Px_2^3}{6E_Y I} - \frac{x_2^3}{3}\left(\frac{P}{2\mu I}\right) + \left(\frac{P}{2\mu I}\right)\left(\frac{h^2}{4}\right)x_2 - c_1x_2 + c_3$$

Thus,

$$u_1 = \frac{Px_1^2x_2}{2E_Y I} + \frac{\nu Px_2^3}{6E_Y I} - \left(\frac{Px_2^3}{6\mu I}\right) + \left(\frac{P}{2\mu I}\right)\left(\frac{h}{2}\right)^2 x_2 - c_1x_2 + c_3,$$

$$u_2 = -\frac{\nu Px_1x_2^2}{2E_Y I} - \frac{Px_1^3}{6E_Y I} + c_1x_1 + c_2.$$

5.68 (a) Let the Airy stress function be of the form $\varphi = f(x_2)\cos\frac{m\pi x_1}{\ell}$. Show that the most general form of

$f(x_2)$ is $f(x_2) = C_1 \cosh \lambda_m x_2 + C_2 \sinh \lambda_m x_2 + C_3 x_2 \cosh \lambda_m x_2 + C_4 x_2 \sinh \lambda_m x_2$. (b) Is the answer the same if $\varphi = f(x_2)\sin \frac{m\pi x_1}{\ell}$?

Ans. (a) The function $\varphi(x_1, x_2)$ must satisfy the bi-harmonic equation. Now,

$$\frac{\partial^2 \varphi}{\partial x_1^2} = -\left(\frac{m\pi}{\ell}\right)^2 f(x_2)\cos\frac{m\pi x_1}{\ell}, \quad \frac{\partial^4 \varphi}{\partial x_1^4} = \left(\frac{m\pi}{\ell}\right)^4 f(x_2)\cos\frac{m\pi x_1}{\ell},$$

$$\frac{\partial^4 \varphi}{\partial x_1^2 \partial x_2^2} = -\left(\frac{m\pi}{\ell}\right)^2 \frac{d^2 f}{dx_2^2}\cos\frac{m\pi x_1}{\ell}, \quad \frac{\partial^4 \varphi}{\partial x_2^4} = \frac{d^4 f}{dx_2^4}\cos\frac{m\pi x_1}{\ell},$$

Thus,

$$\nabla^4 \varphi = \frac{\partial^4 \varphi}{\partial x_1^4} + 2\frac{\partial^4 \varphi}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 \varphi}{\partial x_2^4} = \cos\frac{m\pi x_1}{\ell}\left[\left(\frac{m\pi}{\ell}\right)^4 f(x_2) - 2\left(\frac{m\pi}{\ell}\right)^2 \frac{d^2 f}{dx_2^2} + \frac{d^4 f}{dx_2^4}\right] = 0 .$$

$$\text{Therefore, } \frac{d^4 f}{dx_2^4} - 2\lambda_m^2 \frac{d^2 f}{dx_2^2} + \lambda_m^4 f = 0, \quad \text{where } \lambda_m \equiv \frac{m\pi}{\ell} .$$

The characteristic equation for the above ODE is $D^4 - 2\lambda_m^2 D^2 + \lambda_m^4 = 0$. The roots of this equation consists of two sets of double roots. They are: $D = \lambda_m, \lambda_m, -\lambda_m, -\lambda_m$. Thus,

$$f(x_2) = C_1 \cosh \lambda_m x_2 + C_2 \sinh \lambda_m x_2 + C_3 x_2 \cosh \lambda_m x_2 + C_4 x_2 \sinh \lambda_m x_2 .$$

(b) Yes, the same

5.69 Consider a rectangular bar defined by $-\ell \le x_1 \le \ell, \quad -c \le x_2 \le c, \quad -b \le x_3 \le b$, where b/ℓ is very small. At the boundaries $x_2 = \pm c$, the bar is acted on by equal and opposite cosine normal stress $A_m \cos \lambda_m x_1$, where $\lambda_m = m\pi/\ell$ (per unit length in x_3 direction). (a) Obtain the in-plane stresses inside the bar. (b) Find the surface tractions at $x_1 = \pm\ell$. Under what conditions can these surface tractions be removed without affecting T_{22} and T_{12} (except near $x_1 = \pm\ell$)? How would T_{11} be affected by the removal. Hint: Assume $\varphi = f(x_2)\cos \lambda_m x_1$, where $\lambda_m = m\pi/\ell$ and use the results of the previous problem

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Ans. (a) Boundary conditions are

$$(T_{12})_{x_2 = \pm c} = 0, \qquad (T_{22})_{x_2 = \pm c} = A_m \cos \lambda_m x_1$$

Let $\varphi = f(x_2)\cos \lambda_m x_1$, where $\lambda_m = m\pi/\ell$. Then (see previous problem),

$$f(x_2) = C_1 \cosh \lambda_m x_2 + C_2 \sinh \lambda_m x_2 + C_3 x_2 \cosh \lambda_m x_2 + C_4 x_2 \sinh \lambda_m x_2 .$$

The in-plane stresses are:

$$T_{22}=\frac{\partial^2\varphi}{\partial x_1^2}=-(\lambda_m)^2 f(x_2)\cos\lambda_m x_1,\ T_{11}=\frac{\partial^2\varphi}{\partial x_2^2}=\left(d^2 f/dx_2^2\right)\cos\lambda_m x_1,$$

$T_{12}=-\frac{\partial}{\partial x_1}\frac{\partial\varphi}{\partial x_2}=\lambda_m(df/dx_2)\sin\lambda_m x_1$. Now, applying the boundary condition:

$$(T_{22})_{x_2=\pm c}=A_m\cos\lambda_m x_1\rightarrow-(\lambda_m)^2 f(\pm c)\cos\lambda_m x_1=A_m\cos\lambda_m x_1\rightarrow f(\pm c)=-A_m/(\lambda_m)^2.$$

From $f(\pm c)=-A_m/(\lambda_m)^2$, $f(+c)=f(-c)$, so that $C_2=C_3=0$ and

$$f(\pm c)=C_1\cosh\lambda_m c+C_4 c\sinh\lambda_m c=-A_m/\lambda_m^2 \qquad \text{(i)}$$

Applying the other boundary condition:

$$(T_{12})_{x_2=\pm c}=0\rightarrow(df/dx_2)_{x_2=\pm c}=0\rightarrow$$

$$C_1\lambda_m\sinh\lambda_m c+C_4(\sinh\lambda_m c+\lambda_m c\cosh\lambda_m c)=0 \qquad \text{(ii)}$$

(i) and (ii) give

$$C_1=-\frac{2A_m}{\lambda_m^2}\left[\frac{(\lambda_m c)\cosh\lambda_m c+\sinh\lambda_m c}{\sinh 2\lambda_m c+2\lambda_m c}\right],\quad C_4=\frac{2A_m}{\lambda_m^2}\left[\frac{\lambda_m\sinh\lambda_m c}{\sinh 2\lambda_m c+2\lambda_m c}\right].$$

With

$f(x_2)=C_1\cosh\lambda_m x_2+C_4 x_2\sinh\lambda_m x_2$, we have ,

$$T_{22}=-(\lambda_m)^2 f(x_2)\cos\lambda_m x_1=-(\lambda_m)^2\{C_1\cosh\lambda_m x_2+C_4\lambda_m x_2\sinh\lambda_m x_2\}\cos\lambda_m x_1$$

$$=2A_m\left[\frac{\{(\lambda_m c)\cosh\lambda_m c+\sinh\lambda_m c\}\cosh\lambda_m x_2-\{\lambda_m x_2\sinh\lambda_m x_2\sinh\lambda_m c\}\cos\lambda_m x_1}{\sinh 2\lambda_m c+2\lambda_m c}\right].$$

$$T_{12}=\lambda_m(df/dx_2)\sin\lambda_m x_1=\left[C_1\lambda_m^2\sinh\lambda_m x_2+C_4\lambda_m(\sinh\lambda_m x_2+\lambda_m x_2\cosh\lambda_m x_2)\right]\sin\lambda_m x_1$$

$$=2A_m\left[\frac{\{-(\lambda_m c)\cosh\lambda_m c\}\sinh\lambda_m x_2+\sinh\lambda_m c(\lambda_m x_2\cosh\lambda_m x_2)}{\sinh 2\lambda_m c+2\lambda_m c}\right]\sin\lambda_m x_1.$$

$$T_{11}=\frac{\partial^2\varphi}{\partial x_2^2}=\left(d^2 f/dx_2^2\right)\cos\lambda_m x_1=$$

$$2A_m\left[\frac{-(\lambda_m c)\cosh\lambda_m c\cosh\lambda_m x_2+\sinh\lambda_m c(\lambda_m x_2\sinh\lambda_m x_2+\cosh\lambda_m x_2)}{\sinh 2\lambda_m c+2\lambda_m c}\right]\cos\lambda_m x_1.$$

(b) Surface tractions at $x_1=\pm\ell$ are:

$$T_{12}(\pm\ell,x_2)=[]\sin m\pi=0\ .$$

$$T_{11}(\pm\ell,x_2)=2A_m\left[\frac{-(\lambda_m c)\cosh\lambda_m c\cosh\lambda_m x_2+\sinh\lambda_m c(\lambda_m x_2\sinh\lambda_m x_2+\cosh\lambda_m x_2)}{\sinh 2\lambda_m c+2\lambda_m c}\right]\cos m\pi$$

At $x_1=\pm\ell$, T_{11} is an even function of x_2, which gives rise to equal and opposite resultant force of magnitude F_R at the two ends. Removal of these resultants will have little effects on T_{12} and T_{22} , if ℓ / c is very large. However, T_{11} will need to be modified by subtracting the normal stress (F_R /Area) caused by the resultant forces.

5.70 Verify that the equations of equilibrium in polar coordinates are satisfied by

$$T_{rr} = \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2}, \quad T_{\theta\theta} = \frac{\partial^2 \varphi}{\partial r^2}, \quad T_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right).$$

Ans.

$$\frac{1}{r}\frac{\partial (rT_{rr})}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial \varphi}{\partial r} + \frac{1}{r}\frac{\partial^2 \varphi}{\partial \theta^2}\right) = \left(\frac{1}{r}\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^3 \varphi}{\partial r \partial \theta^2} - \frac{1}{r^3}\frac{\partial^2 \varphi}{\partial \theta^2}\right), \quad -\frac{T_{\theta\theta}}{r} = -\frac{1}{r}\frac{\partial^2 \varphi}{\partial r^2}$$

$$\frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} = -\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right)\right] = -\frac{1}{r}\frac{\partial}{\partial \theta}\left[\frac{1}{r}\left(\frac{\partial^2 \varphi}{\partial r \partial \theta}\right) - \frac{1}{r^2}\frac{\partial \varphi}{\partial \theta}\right] = -\left[\frac{1}{r^2}\left(\frac{\partial^3 \varphi}{\partial r \partial \theta^2}\right) - \frac{1}{r^3}\frac{\partial^2 \varphi}{\partial \theta^2}\right]$$

Thus, [See Eq.(4.8.1),

$$\frac{1}{r}\frac{\partial (rT_{rr})}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} - \frac{T_{\theta\theta}}{r}$$

$$= \left(\frac{1}{r}\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^3 \varphi}{\partial r \partial \theta^2} - \frac{1}{r^3}\frac{\partial^2 \varphi}{\partial \theta^2}\right) - \left[\frac{1}{r^2}\left(\frac{\partial^3 \varphi}{\partial r \partial \theta^2}\right) - \frac{1}{r^3}\frac{\partial^2 \varphi}{\partial \theta^2}\right] - \frac{1}{r}\frac{\partial^2 \varphi}{\partial r^2} = 0$$

Similarly,

$$\frac{1}{r^2}\frac{\partial (r^2 T_{\theta r})}{\partial r} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \varphi}{\partial \theta}\right)\right] = -\frac{1}{r^2}\frac{\partial}{\partial r}\left[r\left(\frac{\partial^2 \varphi}{\partial r \partial \theta}\right) - \frac{\partial \varphi}{\partial \theta}\right] = -\frac{1}{r}\left(\frac{\partial^3 \varphi}{\partial r^2 \partial \theta}\right)$$

$$\frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} = \frac{1}{r}\frac{\partial}{\partial \theta}\frac{\partial^2 \varphi}{\partial r^2} = \frac{1}{r}\frac{\partial^3 \varphi}{\partial \theta \partial r^2},$$

Thus, [See Eq.(4.8.2)] $\frac{1}{r^2}\frac{\partial (r^2 T_{\theta r})}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} = 0.$

5.71 From the transformation law : $\begin{bmatrix} T_{rr} & T_{r\theta} \\ T_{\theta r} & T_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

and

$T_{11} = \frac{\partial^2 \varphi}{\partial x_2^2}$, $T_{22} = \frac{\partial^2 \varphi}{\partial x_1^2}$ and $T_{12} = -\frac{\partial^2 \varphi}{\partial x_1 \partial x_2}$, obtain $T_{rr} = \frac{1}{r}\frac{\partial \varphi}{\partial r} + \left(\frac{1}{r^2}\right)\frac{\partial^2 \varphi}{\partial \theta^2}$

Ans.

$$\begin{bmatrix} T_{rr} & T_{r\theta} \\ T_{\theta r} & T_{\theta\theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} T_{11}\cos^2\theta + 2T_{12}\sin\theta\cos\theta + T_{22}\sin^2\theta & (T_{22}-T_{11})\sin\theta\cos\theta + T_{12}\left(\cos^2\theta - \sin^2\theta\right) \\ (T_{22}-T_{11})\cos\theta\sin\theta + T_{12}\left(\cos^2\theta - \sin^2\theta\right) & \left(T_{11}\sin^2\theta + T_{22}\cos^2\theta - 2T_{12}\sin\theta\cos\theta\right) \end{bmatrix},$$

$$r^2 = x_1^2 + x_2^2 \rightarrow \partial r/\partial x_1 = x_1/r = \cos\theta, \;\; \partial r/\partial x_2 = x_2/r = \sin\theta,$$

$$\theta = \tan^{-1}\frac{x_2}{x_1} \rightarrow \partial\theta/\partial x_1 = \frac{-x_2}{x_1^2 + x_2^2} = -\frac{\sin\theta}{r}, \;\; \partial\theta/\partial x_2 = \frac{x_1}{x_1^2 + x_2^2} = \frac{\cos\theta}{r},$$

$$\frac{\partial\varphi}{\partial x_2} = \frac{\partial\varphi}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial\varphi}{\partial\theta}\frac{\partial\theta}{\partial x_2} = \frac{\partial\varphi}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\cos\theta,$$

$$T_{11} = \frac{\partial^2\varphi}{\partial x_2^2} = \frac{\partial}{\partial r}\left(\frac{\partial\varphi}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\cos\theta\right)\sin\theta + \frac{\partial}{\partial\theta}\left(\frac{\partial\varphi}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\cos\theta\right)\frac{\cos\theta}{r}$$

$$= \frac{\partial^2\varphi}{\partial r^2}\sin^2\theta + \cos\theta\sin\theta\left(\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} - \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta}\right) + \left(\frac{\partial^2\varphi}{\partial r\partial\theta}\sin\theta + \frac{\partial\varphi}{\partial r}\cos\theta + \frac{1}{r}\frac{\partial^2\varphi}{\partial\theta^2}\cos\theta - \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\sin\theta\right)\frac{\cos\theta}{r}$$

$$= \frac{\partial^2\varphi}{\partial r^2}\sin^2\theta + \frac{\cos^2\theta}{r}\frac{\partial\varphi}{\partial r} + \frac{\cos^2\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} - \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial\varphi}{\partial\theta},$$

$$\frac{\partial\varphi}{\partial x_1} = \frac{\partial\varphi}{\partial r}\frac{\partial r}{\partial x_1} + \frac{\partial\varphi}{\partial\theta}\frac{\partial\theta}{\partial x_1} = \frac{\partial\varphi}{\partial r}\cos\theta - \frac{\partial\varphi}{\partial\theta}\frac{\sin\theta}{r},$$

$$T_{22} = \frac{\partial^2\varphi}{\partial x_1^2} = \frac{\partial}{\partial r}\left(\frac{\partial\varphi}{\partial r}\cos\theta - \frac{\partial\varphi}{\partial\theta}\frac{\sin\theta}{r}\right)\cos\theta - \frac{\partial}{\partial\theta}\left(\frac{\partial\varphi}{\partial r}\cos\theta - \frac{\partial\varphi}{\partial\theta}\frac{\sin\theta}{r}\right)\frac{\sin\theta}{r}$$

$$= \left(\frac{\partial^2\varphi}{\partial r^2}\cos^2\theta + \frac{\sin^2\theta}{r}\frac{\partial\varphi}{\partial r} + \frac{\sin^2\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} + \frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\varphi}{\partial\theta}\right).$$

$$-T_{12} = \frac{\partial}{\partial x_2}\frac{\partial\varphi}{\partial x_1} = \frac{\partial}{\partial r}\left(\frac{\partial\varphi}{\partial r}\cos\theta - \frac{\partial\varphi}{\partial\theta}\frac{\sin\theta}{r}\right)\sin\theta + \frac{\partial}{\partial\theta}\left(\frac{\partial\varphi}{\partial r}\cos\theta - \frac{\partial\varphi}{\partial\theta}\frac{\sin\theta}{r}\right)\frac{\cos\theta}{r}$$

$$= \frac{\partial^2\varphi}{\partial r^2}\cos\theta\sin\theta - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\sin^2\theta}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{\cos^2\theta}{r^2}\frac{\partial\varphi}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial^2\varphi}{\partial\theta\partial r} - \frac{\sin^2\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta} - \frac{\sin\theta\cos\theta}{r}\frac{\partial\varphi}{\partial r}.$$

Thus,

$$T_{rr}=T_{11}\cos^2\theta+2T_{12}\sin\theta\cos\theta+T_{22}\sin^2\theta=$$

$$\left(\frac{\partial^2\varphi}{\partial r^2}\sin^2\theta+\frac{\cos^2\theta}{r}\frac{\partial\varphi}{\partial r}+\frac{\cos^2\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{2\cos\theta\sin\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}-\frac{2\cos\theta\sin\theta}{r^2}\frac{\partial\varphi}{\partial\theta}\right)\cos^2\theta$$

$$-2\begin{pmatrix}\frac{\partial^2\varphi}{\partial r^2}\cos\theta\sin\theta-\frac{\sin\theta\cos\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}+\frac{\sin^2\theta}{r^2}\frac{\partial\varphi}{\partial\theta}-\frac{\cos^2\theta}{r^2}\frac{\partial\varphi}{\partial\theta}+\frac{\cos^2\theta}{r}\frac{\partial^2\varphi}{\partial\theta\partial r}\\-\frac{\sin^2\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}-\frac{\sin\theta\cos\theta}{r}\frac{\partial\varphi}{\partial r}\end{pmatrix}\sin\theta\cos\theta$$

$$+\sin^2\theta\left(\frac{\partial^2\varphi}{\partial r^2}\cos^2\theta+\frac{\sin^2\theta}{r}\frac{\partial\varphi}{\partial r}+\frac{\sin^2\theta}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial\varphi}{\partial\theta}\right).$$

$$T_{rr}=\sin^2\theta\cos^2\theta\left(\frac{\partial^2\varphi}{\partial r^2}-2\frac{\partial^2\varphi}{\partial r^2}+\frac{\partial^2\varphi}{\partial r^2}\right)+\left(\cos^4\theta+\sin^4\theta+2\sin^2\theta\cos^2\theta\right)\frac{1}{r}\frac{\partial\varphi}{\partial r}$$

$$+\left(\cos^4\theta+\sin^4\theta+2\sin^2\theta\cos^2\theta\right)\frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2}-\frac{2}{r^2}\frac{\partial\varphi}{\partial\theta}\left(\sin^3\theta\cos\theta-\sin\theta\cos^3\theta+\cos^3\theta\sin\theta-\sin^3\theta\cos\theta\right)$$

$$\left(\sin\theta\cos^3\theta-\cos^3\theta\sin\theta+\sin^3\theta\cos\theta-\sin^3\theta\cos\theta\right)\frac{2}{r}\frac{\partial^2\varphi}{\partial r\partial\theta}.$$

That is,

$$T_{rr}=\frac{1}{r}\frac{\partial\varphi}{\partial r}+\left(\frac{1}{r^2}\right)\frac{\partial^2\varphi}{\partial\theta^2}.$$

5.72 Obtain the displacement field for the plane strain solution of the axis-symmetric stress distribution from that for the plane stress solution obtained in Section 5.28.

Ans. From Section 5. 29, we have, for plane stress solution, [See Eq.(5.29.15) and .(5.29.16) and note $E_Y=2\mu(1+\nu)$]

$$u_r=\frac{1}{2\mu(1+\nu)}\left[-\frac{A}{r}(1+\nu)+2B(1-\nu)r\ln r-(1+\nu)Br+2C(1-\nu)r\right]+H\sin\theta+G\cos\theta,$$

$$u_\theta=\frac{2Br\theta}{\mu(1+\nu)}+H\cos\theta-G\sin\theta+Fr.$$

To obtain the corresponding displacement field for the plane strain solution, we replace the Poisson ratio ν with $\nu/(1-\nu)$ in the above equation [see Section. 5.26]. That is,

$$(1+\nu)\to\left(1+\frac{\nu}{1-\nu}\right)=\frac{1}{1-\nu},\quad(1-\nu)\to\left(1-\frac{\nu}{1-\nu}\right)=\frac{1-2\nu}{1-\nu}.$$

Thus, for plane strain:

$$u_r = \frac{(1-\nu)}{2\mu}\left[-\frac{A}{r}\frac{1}{(1-\nu)} + 2B\frac{(1-2\nu)}{(1-\nu)} r\ln r - B\frac{1}{(1-\nu)} r + 2C\frac{(1-2\nu)}{(1-\nu)} r\right] + H\sin\theta + G\cos\theta$$
$$= \frac{1}{2\mu}\left[-\frac{A}{r} + 2B(1-2\nu) r\ln r - Br + 2C(1-2\nu) r\right] + H\sin\theta + G\cos\theta$$
$$= \frac{(1+\nu)}{E_Y}\left[-\frac{A}{r} + 2B(1-2\nu) r\ln r - Br + 2C(1-2\nu) r\right] + H\sin\theta + G\cos\theta.$$

and

$$u_\theta = \frac{2Br\theta(1-\nu)}{\mu} + H\cos\theta - G\sin\theta + Fr = \frac{4Br\theta(1-\nu)(1+\nu)}{E_Y} + H\cos\theta - G\sin\theta + Fr\ .$$

5.73 Let the Airy stress function be $\varphi = f(r)\sin n\theta$, find the differential equation for $f(r)$. Is this the same ODE for $f(r)$ if $\varphi = f(r)\cos n\theta$?

Ans.

$$\varphi = f(r)\sin n\theta \to \frac{\partial\varphi}{\partial r} = f'\sin n\theta \to \frac{\partial^2\varphi}{\partial r^2} = f''\sin n\theta;\quad \frac{\partial\varphi}{\partial\theta} = nf\cos n\theta \to \frac{\partial^2\varphi}{\partial\theta^2} = -n^2 f\sin n\theta\ .$$

Thus,

$$\frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial r^2} = \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial r^2}\right) f(r)\sin n\theta = \left(\frac{f'}{r} - \frac{1}{r^2}n^2 f + f''\right)\sin n\theta \equiv g(r)\sin n\theta$$

,

where $g(r) \equiv \left(\frac{f'}{r} - \frac{1}{r^2}n^2 f + f''\right) = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) f$.

Now,

$$\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} + \frac{\partial^2\varphi}{\partial r^2}\right) = \left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial r^2}\right) g(r)\sin n\theta$$
$$= \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) g(r)\sin n\theta = \left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) f(r)\sin n\theta = 0.$$

Therefore,

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right)\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r^2}\right) f(r) = 0\ .$$

The same equation will be obtained if $\varphi = f(r)\cos n\theta$

5.74 Obtain the four independent solutions for the following equation

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{n^2}{r}\right)\left(\frac{d^2 f}{dr^2} + \frac{1}{r}\frac{df}{dr} - \frac{n^2}{r} f\right) = 0$$

Ans. Let $f = r^m$.

$$\left(\frac{d^2 f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{n^2}{r^2}f\right)=\left[m(m-1)+m-n^2\right]r^{m-2}=\left(m^2-n^2\right)r^{m-2}$$

$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\frac{d}{dr}-\frac{n^2}{r^2}\right)\left(\frac{d^2 f}{dr^2}+\frac{1}{r}\frac{df}{dr}-\frac{n^2}{r^2}f\right)=\left(m^2-n^2\right)\left[(m-3)(m-2)+(m-2)-n^2\right]r^{m-4}=0$$

$$\rightarrow\left(m^2-n^2\right)\left[(m-2)^2-n^2\right]=0.$$

Thus, $m_1=+n,\ m_2=-n,\ m_3=2+n,\ m_4=2-n$.

For $n\neq 0$ and $n\neq 1$, the four independent solutions for f are: r^{+n}, r^{-n}, r^{+n+2} and r^{-n+2}.

For $n=0$, $m_1=m_2=0,\ m_3=m_4=2$. Two independent solutions for f are given by C and r^2. Additional solutions are given by

$$\left(\frac{d}{dn}r^n\right)_{n\to 0}=\left(r^n \ln r\right)_{n=0}=\ln r\text{, and }\left(\frac{d}{dn}r^{n+2}\right)_{n\to 0}=\left[r^{n+2}\ln r\right]_{n\to 0}=r^2\ln r.$$

The four independent solutions are: $C, r^2, \ln r$ and $r^2\ln r$.

For $n=1$, $m_1=m_4=1$, in addition to r, r^{-1}, r^3, we have, $\left(\frac{d}{dn}r^n\right)_{n\to 1}=\left(r^n\ln r\right)_{n=1}=r\ln r$

Thus, the four independent solutions are: r, r^{-1}, r^3 and $r\ln r$.

5.75 Evaluate $\left[\frac{d}{dn}\left(r^n\cos n\theta\right)\right]_{n=0}$, $\left[\frac{d}{dn}\left(r^n\sin n\theta\right)\right]_{n=0}$, $\left[\frac{d}{dn}\left(r^{-n+2}\cos n\theta\right)\right]_{n=1}$ *and* $\left[\frac{d}{dn}\left(r^n\cos n\theta\right)\right]_{n=1}$

Ans.

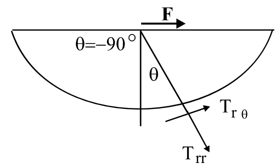
$$\left[\frac{d}{dn}\left(r^n\cos n\theta\right)\right]_{n=0}=\left[\left(r^n\ln r\right)\cos n\theta-r^n\theta\sin n\theta\right]_{n=0}=\ln r.$$

$$\left[\frac{d}{dn}\left(r^n\sin n\theta\right)\right]_{n=0}=\left[\left(r^n\ln r\right)\sin n\theta+r^n\theta\cos n\theta\right]_{n=0}=\theta.$$

$$\left[\frac{d}{dn}\left(r^{-n+2}\cos n\theta\right)\right]_{n=1}=\left[-r^{-n+2}\ln r\cos n\theta-r^{-n+2}\theta\sin n\theta\right]_{n=1}=-r\ln r\cos\theta-r\theta\sin\theta$$

$$\left[\frac{d}{dn}\left(r^n\cos n\theta\right)\right]_{n=1}=\left[r^n\ln r\cos n\theta-r^n\theta\sin n\theta\right]_{n=1}=r\ln r\cos\theta-r\theta\sin\theta$$

5.76 In the Flamont Problem (Sect. 5.37), if the concentrated line load F, acting at the origin on the surface of a 2D half-space (defined by $-\pi/2 \le \theta \le \pi/2$), is tangent to the surface and in the direction of $\theta = 90^o$, show that: $T_{rr} = -\left(\frac{2F}{\pi}\right)\frac{\sin\theta}{r}, \quad T_{\theta\theta} = T_{r\theta} = 0$



Ans. The boundary conditions are: $T_{\theta\theta} = T_{r\theta} = 0$ at $\theta = \pm\pi/2, r \ne 0$. (i),

$$\int_{-\pi/2}^{\pi/2} \left(T_{rr}\cos\theta - T_{r\theta}\sin\theta\right) r d\theta = 0 \quad \text{(ii)}, \qquad \int_{-\pi/2}^{\pi/2} \left(T_{rr}\sin\theta + T_{r\theta}\cos\theta\right) r d\theta = -F \ . \quad \text{(iii)}$$

From the stress field obtained in Sect. 5.37,

$$T_{rr} = r^{-1}\left(2B_5\cos\theta - 2\overline{B}_5\sin\theta\right), \quad T_{\theta\theta} = 0, \quad T_{r\theta} = 0, \quad \text{(iv)}$$

we obtain, from Eqs.(ii) and (iv) :

$$\int_{-\pi/2}^{\pi/2} \left(2B_5\cos^2\theta - \overline{B}_5\sin 2\theta\right) d\theta = 0 \rightarrow 2B_5 \int_{-\pi/2}^{\pi/2} \cos^2\theta d\theta = 0 \rightarrow B_5 = 0 \ .$$

From Eqs.(iii) and (iv)

$$\int_{-\pi/2}^{\pi/2} \left(B_5\sin 2\theta - 2\overline{B}_5\sin^2\theta\right) d\theta = -F \rightarrow -2\overline{B}_5 \int_{-\pi/2}^{\pi/2} \sin^2\theta d\theta = -F \rightarrow 2\overline{B}_5\frac{\pi}{2} = F \rightarrow 2\overline{B}_5 = \frac{2F}{\pi}$$

Thus

$$T_{rr} = -2\overline{B}_5\left(\frac{\sin\theta}{r}\right) = -\frac{2F}{\pi}\left(\frac{\sin\theta}{r}\right), \quad T_{\theta\theta} = 0, \quad T_{r\theta} = 0. \quad \text{(v)}$$

5.77 Verify that the displacement field for the Flamont Problem under a normal force P is given by

$$u_r = -\frac{P}{\pi E_Y}\left\{(1-\nu)\theta\sin\theta + 2\ln r\cos\theta\right\}, \quad u_\theta = \frac{P}{\pi E_Y}\left\{(1+\nu)\sin\theta + 2\ln r\sin\theta - (1-\nu)\theta\cos\theta\right\},$$

The 2D half space is defined by $-\pi/2 \le \theta \le \pi/2$.

Ans. From the given displacement field, we have,

$$E_{rr} = \partial u_r / \partial r = -\frac{2P}{E_Y\pi}\left(\frac{\cos\theta}{r}\right), \quad i.e., E_{rr} = \frac{T_{rr}}{E_Y}.$$

$$\frac{\partial u_\theta}{\partial\theta} + u_r = \frac{P}{\pi E_Y}\left\{(1+\nu)\cos\theta + 2\ln r\cos\theta - (1-\nu)\cos\theta + (1-\nu)\theta\sin\theta\right\}$$

$$-\frac{P}{\pi E_Y}\left\{(1-\nu)\theta\sin\theta + 2\ln r\cos\theta\right\} = \frac{P}{\pi E_Y}\left\{(1+\nu)\cos\theta - (1-\nu)\cos\theta\right\} = \frac{2\nu P\cos\theta}{\pi E_Y}.$$

That is, $E_{\theta\theta}=\frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta}+u_r\right)=\frac{2\nu P}{\pi E_Y}\frac{\cos\theta}{r},\quad i.e., E_{\theta\theta}=-\frac{\nu T_{rr}}{E_Y}.$

Next,

$$2E_{r\theta}=\frac{1}{r}\left(\frac{\partial u_r}{\partial\theta}-u_\theta\right)+\frac{\partial u_\theta}{\partial r}=-\frac{P\{(1-\nu)\sin\theta+(1+\nu)\sin\theta\}}{\pi E_Y r}+\frac{2P}{\pi E_Y}\frac{\sin\theta}{r}$$

$$=-\frac{2P}{\pi E_Y}\frac{\sin\theta}{r}+\frac{2P}{\pi E_Y}\frac{\sin\theta}{r}=0,\quad i.e., 2\mu E_{r\theta}=0=T_{r\theta}.$$

5.78 Show that Eq. (5.38.6), i.e., $\mathbf{u}=\mathbf{\Psi}-\frac{1}{4(1-\nu)}\nabla\left(\mathbf{x}\cdot\mathbf{\Psi}+\Phi\right)$

can also be written as:

$$2\mu\mathbf{u}=-4(1-\nu)\boldsymbol{\psi}+\nabla\left(\mathbf{x}\cdot\boldsymbol{\psi}+\phi\right) \text{ where } \mathbf{\Psi}=-\boldsymbol{\psi}\frac{2(1-\nu)}{\mu},\quad \Phi=-\phi\frac{2(1-\nu)}{\mu}$$

Ans. With $\mathbf{\Psi}=-\boldsymbol{\psi}\frac{2(1-\nu)}{\mu},\quad \Phi=-\phi\frac{2(1-\nu)}{\mu}$, we have, $\mathbf{x}\cdot\mathbf{\Psi}=-\frac{2(1-\nu)}{\mu}\mathbf{x}\cdot\boldsymbol{\psi}\rightarrow$.

$$\rightarrow\mathbf{u}=\mathbf{\Psi}-\frac{1}{4(1-\nu)}\nabla\left(\mathbf{x}\cdot\mathbf{\Psi}+\Phi\right)=-\frac{2(1-\nu)}{\mu}\boldsymbol{\psi}+\frac{1}{2\mu}\nabla\left(\mathbf{x}\cdot\boldsymbol{\psi}+\phi\right).$$

That is,

$$2\mu\mathbf{u}=-4(1-\nu)\boldsymbol{\psi}+\nabla\left(\mathbf{x}\cdot\boldsymbol{\psi}+\phi\right).$$

5.79 Show that with

$u_i=\Psi_i-\frac{1}{4(1-\nu)}\frac{\partial}{\partial x_i}\left(x_n\Psi_n+\Phi\right)$, the Navier Equations become :

$$-\frac{\mu}{2(1-2\nu)}\left(x_n\frac{\partial\nabla^2\Psi_n}{\partial x_i}-(1-4\nu)\nabla^2\Psi_i+\frac{\partial\nabla^2\Phi}{\partial x_i}\right)+B_i=0$$

Ans. $u_i=\Psi_i-\frac{1}{4(1-\nu)}\frac{\partial}{\partial x_i}\left(x_n\Psi_n+\Phi\right)=\Psi_i-\frac{1}{4(1-\nu)}\left(x_n\frac{\partial\Psi_n}{\partial x_i}+\Psi_i+\frac{\partial\Phi}{\partial x_i}\right)$

$$\rightarrow e=\frac{\partial u_i}{\partial x_i}=\frac{\partial \Psi_m}{\partial x_m}-\frac{1}{4(1-\nu)}\left(x_n\frac{\partial^2\Psi_n}{\partial x_m\partial x_m}+2\frac{\partial \Psi_m}{\partial x_m}+\frac{\partial^2\Phi}{\partial x_m\partial x_m}\right)$$

$$\frac{\partial e}{\partial x_i}=\frac{\partial}{\partial x_i}\frac{\partial \Psi_m}{\partial x_m}-\frac{1}{4(1-\nu)}\left(\frac{\partial}{\partial x_i}\left\{x_n\frac{\partial^2\Psi_n}{\partial x_m\partial x_m}\right\}+2\frac{\partial}{\partial x_i}\frac{\partial \Psi_m}{\partial x_m}+\frac{\partial}{\partial x_i}\frac{\partial^2\Phi}{\partial x_m\partial x_m}\right)$$

$$=\frac{\partial}{\partial x_i}\frac{\partial \Psi_m}{\partial x_m}\frac{(1-2\nu)}{2(1-\nu)}-\frac{1}{4(1-\nu)}\left(x_n\frac{\partial}{\partial x_i}\frac{\partial^2\Psi_n}{\partial x_m\partial x_m}+\nabla^2\Psi_i+\frac{\partial}{\partial x_i}\nabla^2\Phi\right)$$

$$\frac{\mu}{1-2\nu}\frac{\partial e}{\partial x_i}=\frac{\mu}{2(1-\nu)}\frac{\partial}{\partial x_i}\frac{\partial \Psi_m}{\partial x_m}-\frac{\mu}{4(1-2\nu)(1-\nu)}\left(x_n\frac{\partial}{\partial x_i}\frac{\partial^2\Psi_n}{\partial x_m\partial x_m}+\nabla^2\Psi_i+\frac{\partial}{\partial x_i}\nabla^2\Phi\right)$$

Also,

$$\mu\frac{\partial^2 u_i}{\partial x_j\partial x_j}=\mu\nabla^2\Psi_i-\frac{\mu}{4(1-\nu)}\left(x_n\frac{\partial}{\partial x_i}\nabla^2\Psi_n+2\frac{\partial}{\partial x_j}\frac{\partial \Psi_j}{\partial x_i}+\nabla^2\Psi_i+\frac{\partial}{\partial x_i}\nabla^2\Phi\right)$$

$$=-\frac{\mu}{2(1-\nu)}\frac{\partial^2\Psi_j}{\partial x_j\partial x_i}-\frac{\mu}{4(1-\nu)}\left(x_n\frac{\partial}{\partial x_i}\nabla^2\Psi_n+\nabla^2\Psi_i\left[1-4(1-\nu)\right]+\frac{\partial}{\partial x_i}\nabla^2\Phi\right)$$

Thus,

$$\mu\frac{\partial^2 u_i}{\partial x_j\partial x_j}+\frac{\mu}{1-2\nu}\frac{\partial e}{\partial x_i}=-\frac{\mu}{4(1-\nu)(1-2\nu)}\left((2-2\nu)x_n\frac{\partial}{\partial x_i}\nabla^2\Psi_n-2(1-4\nu)(1-\nu)\nabla^2\Psi_i+(2-2\nu)\frac{\partial}{\partial x_i}\nabla^2\Phi\right)$$

$$=-\frac{\mu}{2(1-2\nu)}\left(x_n\frac{\partial}{\partial x_i}\nabla^2\Psi_n-(1-4\nu)\nabla^2\Psi_i+\frac{\partial}{\partial x_i}\nabla^2\Phi\right)$$

i.e.,

$$\mu\frac{\partial^2 u_i}{\partial x_j\partial x_j}+\frac{\mu}{1-2\nu}\frac{\partial e}{\partial x_i}=-\frac{\mu}{2(1-2\nu)}\left(x_n\frac{\partial}{\partial x_i}\nabla^2\Psi_n-(1-4\nu)\nabla^2\Psi_i+\frac{\partial}{\partial x_i}\nabla^2\Phi\right),$$

so that the Navier Equations become:

$$-\frac{\mu}{2(1-2\nu)}\left(x_n\frac{\partial\nabla^2\Psi_n}{\partial x_i}-(1-4\nu)\nabla^2\Psi_i+\frac{\partial\nabla^2\Phi}{\partial x_i}\right)+B_i=0\,.$$

5.80 Consider the potential function given in Eq. (5.38.32) [See Example 5.38.5], i.e.,

$$\boldsymbol{\psi}=\psi(R)\mathbf{e}_R,\quad \phi=\phi(R),$$

where

$$\nabla^2\phi=\frac{d^2\phi}{dR^2}+\frac{2}{R}\frac{d\phi}{dR}=0\ \text{ and }\left(\frac{d^2\psi}{dR^2}+\frac{2}{R}\frac{d\psi}{dR}-\frac{2\psi}{R^2}\right)=0\,.$$

Show that these functions generate the following displacements, dilatation and stresses as given in Eq.(5.38.5) to (5.38.38):

(a) Displacements: $2\mu u_R = \left(R\frac{d\psi}{dR} + (-3+4\nu)\psi + \frac{d\phi}{dR} \right),\;\; u_\theta = u_\beta = 0$

(b) Dilation: $e = -\frac{(1-2\nu)}{\mu}\left\{ \frac{d\psi}{dR} + \frac{2\psi}{R} \right\}$

(c) Stresses: $T_{RR} = (2\nu-4)\frac{d\psi}{dR} + (2-4\nu)\frac{\psi}{R} + \frac{d^2\phi}{dR^2},\qquad T_{\beta\beta} = T_{\theta\theta} = -\left\{ (2\nu-1)\frac{d\psi}{dR} + \frac{3\psi}{R} - \frac{1}{R}\frac{d\phi}{dR} \right\}$

Ans. With $\mathbf{x} = R\mathbf{e}_R$, we have, $\mathbf{x}\cdot\boldsymbol{\psi} = R\psi$, thus $2\mu\mathbf{u} = -4(1-\nu)\boldsymbol{\psi} + \nabla(\mathbf{x}\cdot\boldsymbol{\psi} + \phi) \rightarrow$

$$2\mu\mathbf{u} = -4(1-\nu)\psi\mathbf{e}_R + \frac{d}{dR}(R\psi+\phi)\mathbf{e}_R = (-3+4\nu)\psi\mathbf{e}_R + \left(R\frac{d\psi}{dR} + \frac{d\phi}{dR} \right)\mathbf{e}_R\,,$$

$$\text{i.e., } 2\mu u_R = (-3+4\nu)\psi + R\frac{d\psi}{dR} + \frac{d\phi}{dR}\,,$$

(b) The non zero strain components are:

$$2\mu E_{RR} = 2\mu\frac{\partial u_R}{\partial R} = (-3+4\nu)\frac{d\psi}{dR} + \left(R\frac{d^2\psi}{dR^2} + \frac{d\psi}{dR} + \frac{d^2\phi}{dR^2} \right) = (-2+4\nu)\frac{d\psi}{dR} + R\frac{d^2\psi}{dR^2} + \frac{d^2\phi}{dR^2}\,.$$

$$\text{But } \frac{d^2\psi}{dR^2} + \frac{2}{R}\frac{d\psi}{dR} - \frac{2\psi}{R^2} = 0 \rightarrow \frac{d^2\psi}{dR^2} = \frac{2\psi}{R^2} - \frac{2}{R}\frac{d\psi}{dR}$$

$$\rightarrow 2\mu E_{RR} = (-2+4\nu)\frac{d\psi}{dR} + R\frac{d^2\psi}{dR^2} + \frac{d^2\phi}{dR^2} = -4(1-\nu)\frac{\partial\psi}{\partial R} + \frac{2\psi}{R} + \frac{d^2\phi}{dR^2}\,.$$

$$2\mu E_{\beta\beta} = 2\mu E_{\theta\theta} = \frac{2\mu u_R}{R} = (-3+4\nu)\frac{\psi}{R} + \frac{d\psi}{dR} + \frac{1}{R}\frac{d\phi}{dR}\,.$$

Therefore,

$$2\mu e = \rightarrow 2\mu e = -4(1-\nu)\frac{d\psi}{dR} + \frac{2\psi}{R} + \frac{d^2\phi}{dR^2} + \left\{ 2(-3+4\nu)\frac{\psi}{R} + 2\frac{d\psi}{dR} + 2\frac{1}{R}\frac{d\phi}{dR} \right\}$$

$$= (-2+4\nu)\frac{d\psi}{dR} + (-2+4\nu)\frac{2\psi}{R} + \left\{ \frac{d^2\phi}{dR^2} + \frac{2}{R}\frac{d\phi}{dR} \right\} = -2(1-2\nu)\left(\frac{d\psi}{dR} + \frac{2\psi}{R} \right).$$

$$\rightarrow e = -\frac{(1-2\nu)}{\mu}\left(\frac{d\psi}{dR} + \frac{2\psi}{R} \right).$$

(c) the stresses are:

$$T_{RR} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{RR} = -2\nu\left(\frac{d\psi}{dR} + \frac{2\psi}{R} \right) - 4(1-\nu)\frac{d\psi}{dR} + \frac{2\psi}{R} + \frac{d^2\phi}{dR^2}$$

$$= (2\nu-4)\frac{d\psi}{dR} + (2-4\nu)\frac{\psi}{R} + \frac{d^2\phi}{dR^2}.$$

$$T_{\beta\beta} = T_{\theta\theta} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{\theta\theta} = -2\nu\left(\frac{d\psi}{dR} + \frac{2\psi}{R} \right) + (-3+4\nu)\frac{\psi}{R} + \frac{d\psi}{dR} + \frac{1}{R}\frac{d\phi}{dR}$$

$$= (1-2\nu)\frac{d\psi}{dR} - \frac{3\psi}{R} + \frac{1}{R}\frac{d\phi}{dR}.$$

5.81 Consider the following potential functions for axis-symmetric problems:

$\boldsymbol{\psi} = \mathbf{0}$, $\phi = \phi(r,z) = \hat{\phi}(R,\beta)$, $\nabla^2\phi = \nabla^2\hat{\phi} = 0$,

where (r,θ,z) and (R,θ,β) are cylindrical and spherical coordinates respectively with z as the axis of symmetry, θ the longitudinal angle and β the angle between z axis and $\mathbf{e}_R$ (the azimuthal angle). Shows that these functions generate the following displacements, dilatation and stresses:

Cylindrical coordinates

(a) Displacements: $2\mu u_r = \dfrac{\partial\phi}{\partial r}$, $u_\theta = 0$, $2\mu u_z = \dfrac{\partial\phi}{\partial z}$

(b) Dilation: $e = 0$

$$\text{(c) } T_{rr} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{rr} = \frac{\partial^2\phi}{\partial r^2},\quad T_{\theta\theta} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{\theta\theta} = \frac{1}{r}\frac{\partial\phi}{\partial r}$$

$$T_{zz} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{zz} = \frac{\partial^2\phi}{\partial z^2},\quad E_{r\theta} = 0, E_{\theta z} = 0,\quad T_{rz} = 2\mu E_{rz} = \frac{\partial^2\phi}{\partial r\partial z}$$

Spherical coordinates:

(d) Displacements: $2\mu u_R = \dfrac{\partial\hat{\phi}}{\partial R}$, $u_\theta = 0$, $2\mu u_\beta = \dfrac{1}{R}\dfrac{\partial\hat{\phi}}{\partial\beta}$

(e) Dilation: $e = 0$

(f) Stresses:

$$T_{RR} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{RR} = \frac{\partial^2\hat{\phi}}{\partial R^2},\quad T_{\beta\beta} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{\beta\beta} = \frac{1}{R^2}\frac{\partial^2\hat{\phi}}{\partial\beta^2} + \frac{1}{R}\frac{\partial\hat{\phi}}{\partial R}$$

$$T_{\theta\theta} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{\theta\theta} = \frac{1}{R}\frac{\partial\hat{\phi}}{\partial R} + \frac{\cot\beta}{R^2}\frac{\partial\hat{\phi}}{\partial\beta},\quad T_{R\theta} = 0,\quad T_{\theta\beta} = 0$$

$$T_{R\beta} = 2\mu E_{R\beta} = \frac{1}{R}\frac{\partial^2\phi}{\partial\beta\partial R} - \frac{1}{R^2}\frac{\partial\phi}{\partial\beta}$$

Ans. With $\mathbf{x} = r\mathbf{e}_r + z\mathbf{e}_z = R\mathbf{e}_R$, $\boldsymbol{\psi} = \mathbf{0}$, $\phi = \phi(r,z) = \hat{\phi}(R,\beta)$ we have,[see Eqs.(2.34.4) and (2.35.15)]

$$2\mu\mathbf{u} = -4(1-\nu)\boldsymbol{\psi} + \nabla(\mathbf{x}\cdot\boldsymbol{\psi} + \phi) \rightarrow 2\mu\mathbf{u} = \frac{\partial\phi}{\partial r}\mathbf{e}_r + \frac{\partial\phi}{\partial z}\mathbf{e}_z = \frac{\partial\hat{\phi}}{\partial R}\mathbf{e}_R + \frac{1}{R}\frac{\partial\hat{\phi}}{\partial\beta}\mathbf{e}_\beta$$

That is, in cylindrical coordinates

$$2\mu u_r = \frac{\partial\phi}{\partial r},\quad u_\theta = 0,\quad 2\mu u_z = \frac{\partial\phi}{\partial z}$$

and in spherical coordinates;

$$2\mu u_R = \frac{\partial\hat{\phi}}{\partial R},\quad u_\theta = 0,\quad 2\mu u_\beta = \frac{1}{R}\frac{\partial\hat{\phi}}{\partial\beta}$$

(b) The non zero strain components are:

In cylindrical coordinates: [See Eqs.(3.7.20)]

$$2\mu E_{rr} = 2\mu \frac{\partial u_r}{\partial r} = \frac{\partial^2 \phi}{\partial r^2},\quad 2\mu E_{\theta\theta} = \frac{2\mu u_r}{r} = \frac{1}{r}\frac{\partial \phi}{\partial r},\quad 2\mu E_{zz} = 2\mu \frac{\partial u_z}{\partial z} = \frac{\partial^2 \phi}{\partial z^2}$$

$$E_{r\theta} = 0, E_{\theta z} = 0,\quad 2\mu E_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right) = \frac{\partial^2 \phi}{\partial r \partial z}$$

$$2\mu e = 2\mu\left(E_{rr} + E_{\theta\theta} + E_{zz}\right) = \frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial z^2} = \nabla^2 \phi = 0$$

In spherical coordinates: [see Eqs.(3.7.21)]

$$2\mu E_{RR} = 2\mu \frac{\partial u_R}{\partial R} = \frac{\partial^2 \hat{\phi}}{\partial R^2},\quad 2\mu E_{\beta\beta} = \frac{2\mu}{R}\frac{\partial u_\beta}{\partial \beta} + \frac{2\mu u_R}{R} = \frac{1}{R^2}\frac{\partial^2 \hat{\phi}}{\partial \beta^2} + \frac{1}{R}\frac{\partial \hat{\phi}}{\partial R}$$

$$2\mu E_{\theta\theta} = \frac{2\mu u_R}{R} + \frac{2\mu u_\beta \cot\beta}{R} = \frac{1}{R}\frac{\partial \hat{\phi}}{\partial R} + \frac{\cot\beta}{R^2}\frac{\partial \hat{\phi}}{\partial \beta},\quad E_{R\theta} = 0,\quad E_{\theta\beta} = 0$$

$$2\mu E_{R\beta} = \frac{2\mu}{2}\left(\frac{1}{R}\frac{\partial u_R}{\partial \beta} - \frac{u_\beta}{R} + \frac{\partial u_\beta}{\partial R}\right) = \frac{1}{2}\left(\frac{1}{R}\frac{\partial^2 \hat{\phi}}{\partial \beta \partial R} - \frac{1}{R^2}\frac{\partial \hat{\phi}}{\partial \beta} + \frac{1}{R}\frac{\partial^2 \hat{\phi}}{\partial R \partial \beta} - \frac{1}{R^2}\frac{\partial \hat{\phi}}{\partial \beta}\right)$$

$$= \frac{1}{2}\left(\frac{2}{R}\frac{\partial^2 \hat{\phi}}{\partial \beta \partial R} - \frac{2}{R^2}\frac{\partial \hat{\phi}}{\partial \beta}\right) = \left(\frac{1}{R}\frac{\partial^2 \hat{\phi}}{\partial \beta \partial R} - \frac{1}{R^2}\frac{\partial \hat{\phi}}{\partial \beta}\right)$$

$$2\mu e = 2\mu\left(E_{RR} + E_{\beta\beta} + E_{\theta\theta}\right) = \frac{\partial^2 \hat{\phi}}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 \hat{\phi}}{\partial \beta^2} + \frac{1}{R}\frac{\partial \hat{\phi}}{\partial R} + \frac{1}{R}\frac{\partial \hat{\phi}}{\partial R} + \frac{\cot\beta}{R^2}\frac{\partial \hat{\phi}}{\partial \beta}$$

$$\frac{\partial^2 \hat{\phi}}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2 \hat{\phi}}{\partial \beta^2} + \frac{2}{R}\frac{\partial \hat{\phi}}{\partial R} + \frac{\cot\beta}{R^2}\frac{\partial \hat{\phi}}{\partial \beta} = \nabla^2 \phi = 0 \quad \text{[see Eq.2.35.37)]}$$

(c) the stresses are:

In cylindrical coordinates:

$$T_{rr} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{rr} = \frac{\partial^2 \phi}{\partial r^2},\quad T_{\theta\theta} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{\theta\theta} = \frac{1}{r}\frac{\partial \phi}{\partial r}$$

$$T_{zz} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{zz} = \frac{\partial^2 \phi}{\partial z^2},\quad E_{r\theta} = 0, E_{\theta z} = 0,\quad T_{rz} = 2\mu E_{rz} = \frac{\partial^2 \phi}{\partial r \partial z}$$

In spherical coordinates:

$$T_{RR} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{RR} = \frac{\partial^2 \hat{\phi}}{\partial R^2},\quad T_{\beta\beta} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{\beta\beta} = \frac{1}{R^2}\frac{\partial^2 \hat{\phi}}{\partial \beta^2} + \frac{1}{R}\frac{\partial \hat{\phi}}{\partial R}$$

$$T_{\theta\theta} = \frac{2\mu\nu}{1-2\nu}e + 2\mu E_{\theta\theta} = \frac{1}{R}\frac{\partial \hat{\phi}}{\partial R} + \frac{\cot\beta}{R^2}\frac{\partial \hat{\phi}}{\partial \beta},\quad T_{R\theta} = 0,\quad T_{\theta\beta} = 0$$

$$T_{R\beta} = 2\mu E_{R\beta} = \frac{1}{R}\frac{\partial^2 \phi}{\partial \beta \partial R} - \frac{1}{R^2}\frac{\partial \phi}{\partial \beta}$$

These are the formulas given in Example 5.38.6.

5.82 For the potential functions given in Eq.(5.38.46), [see Example 5.38.7)], i.e., :

$$\boldsymbol{\psi} = \psi(R,\beta)\mathbf{e}_z, \quad \phi = 0, \text{ where } \nabla^2\psi = 0,$$

shows that these functions generate the following displacements u_i, dilatation e and the stresses T_{ij} (in spherical coordinates) as given in Eq.(5.38.47) to (5.38.50)::

(a) Displacements:

$$2\mu u_R = -\left\{(3-4\nu)\psi - \mathrm{R}\frac{\partial\psi}{\partial R}\right\}\cos\beta, \quad 2\mu u_\beta = \left\{(3-4\nu)\psi\sin\beta + \cos\beta\frac{\partial\psi}{\partial\beta}\right\}, \quad u_\theta = 0.$$

(b) Dilation: $2\mu e = -(2-4\nu)\left(\cos\beta\frac{\partial\psi}{\partial R} - \frac{\sin\beta}{R}\frac{\partial\psi}{\partial\beta}\right).$

(c) Stresses

$$T_{RR} = -\left(2(1-\nu)\cos\beta\frac{\partial\psi}{\partial R} - R\cos\beta\frac{\partial^2\psi}{\partial R^2} - \frac{2\nu\sin\beta}{R}\frac{\partial\psi}{\partial\beta}\right),$$

$$T_{\beta\beta} = -\left((2\nu-1)\cos\beta\frac{\partial\psi}{\partial R} - (2-2\nu)\frac{\sin\beta}{R}\frac{\partial\psi}{\partial\beta} - \frac{\cos\beta}{R}\frac{\partial^2\psi}{\partial\beta^2}\right),$$

$$T_{\theta\theta} = -\left((2\nu-1)\frac{\partial\psi}{\partial R}\cos\beta - \left((2\nu-1)\sin\beta + \frac{1}{\sin\beta}\right)\frac{\partial\psi}{R\partial\beta}\right),$$

$$T_{R\beta} = -\left[2(1-\nu)\frac{1}{R}\cos\beta\frac{\partial\psi}{\partial\beta} - \cos\beta\frac{\partial^2\psi}{\partial\beta\partial R} - \sin\beta(1-2\nu)\frac{\partial\psi}{\partial R}\right]. \; T_{R\theta} = T_{\theta\beta} = 0$$

Ans (a) with $\mathbf{x} = \mathrm{R}\,\mathbf{e}_\mathrm{R}$, $\mathbf{e}_\mathrm{R}\cdot\mathbf{e}_z = \cos\beta$ and

$$\mathbf{x}\cdot\boldsymbol{\psi} = \mathrm{R}\,\mathbf{e}_\mathrm{R}\cdot\psi\mathbf{e}_z = \psi\,\mathrm{R}\cos\beta, \quad \mathbf{e}_z = \cos\beta\mathbf{e}_\mathrm{R} - \sin\beta\mathbf{e}_\beta,$$

$$2\mu\mathbf{u} = -4(1-\nu)\boldsymbol{\psi} + \nabla(\mathbf{x}\cdot\boldsymbol{\psi} + \phi) = -4(1-\nu)\psi\mathbf{e}_z + \nabla(\mathrm{R}\,\psi\cos\beta) \to$$

$$2\mu\mathbf{u} = -4(1-\nu)\psi(\cos\beta\mathbf{e}_\mathrm{R} - \sin\beta\mathbf{e}_\beta) + \frac{\partial}{\partial R}(\psi\,\mathrm{R}\cos\beta)\mathbf{e}_\mathrm{R} + \frac{1}{R}\frac{\partial}{\partial\beta}(\psi\,\mathrm{R}\cos\beta)\mathbf{e}_\beta$$

$$= -4(1-\nu)\psi(\cos\beta\mathbf{e}_\mathrm{R} - \sin\beta\mathbf{e}_\beta) + \left[\psi\cos\beta\mathbf{e}_\mathrm{R} + \mathrm{R}\cos\beta\frac{\partial\psi}{\partial R}\mathbf{e}_\mathrm{R} + \cos\beta\frac{\partial\psi}{\partial\beta}\mathbf{e}_\beta - \psi\sin\beta\mathbf{e}_\beta\right]$$

$$= -\cos\beta\left[(3-4\nu)\psi - \mathrm{R}\frac{\partial\psi}{\partial R}\right]\mathbf{e}_\mathrm{R} + \left[(3-4\nu)\psi\sin\beta + \cos\beta\frac{\partial\psi}{\partial\beta}\right]\mathbf{e}_\beta.$$

(b) The strain components are:

$$2\mu E_{RR} = 2\mu\frac{\partial u_R}{\partial R} = -\left\{(3-4\nu)\frac{\partial\psi}{\partial R} - \mathrm{R}\frac{\partial^2\psi}{\partial R^2} - \frac{\partial\psi}{\partial R}\right\}\cos\beta = -\left\{(2-4\nu)\frac{\partial\psi}{\partial R} - \mathrm{R}\frac{\partial^2\psi}{\partial R^2}\right\}\cos\beta$$

$$2\mu E_{\beta\beta} = 2\mu\left(\frac{1}{R}\frac{\partial u_\beta}{\partial\beta} + \frac{u_R}{R}\right) = \frac{1}{R}\frac{\partial}{\partial\beta}\left\{(3-4\nu)\psi\sin\beta + \cos\beta\frac{\partial\psi}{\partial\beta}\right\} - \frac{1}{R}\left\{(3-4\nu)\psi - \mathrm{R}\frac{\partial\psi}{\partial R}\right\}\cos\beta$$

$$= \left\{(2-4\nu)\frac{\sin\beta}{R}\frac{\partial\psi}{\partial\beta} + (3-4\nu)\frac{\psi}{R}\cos\beta + \frac{\cos\beta}{R}\frac{\partial^2\psi}{\partial\beta^2}\right\} - \frac{1}{R}\left\{(3-4\nu)\psi - \mathrm{R}\frac{\partial\psi}{\partial R}\right\}\cos\beta$$

$$= (2-4\nu)\frac{\sin\beta}{R}\frac{\partial\psi}{\partial\beta} + \frac{\cos\beta}{R}\frac{\partial^2\psi}{\partial\beta^2} + \cos\beta\frac{\partial\psi}{\partial R}.$$

$$2\mu E_{\theta\theta} = 2\mu\left(\frac{u_R}{R} + \frac{u_\beta\cot\beta}{R}\right) = -\frac{1}{R}\left\{(3-4\nu)\psi - \mathrm{R}\frac{\partial\psi}{\partial R}\right\}\cos\beta + \frac{\cot\beta}{R}\left\{(3-4\nu)\psi\sin\beta + \cos\beta\frac{\partial\psi}{\partial\beta}\right\}$$

$$= \left(\frac{\partial\psi}{\partial R} + \frac{\cot\beta}{R}\frac{\partial\psi}{\partial\beta}\right)\cos\beta.$$

$$2\mu\left(2E_{R\beta}\right) = 2\mu\left(\frac{1}{R}\frac{\partial u_R}{\partial\beta} - \frac{u_\beta}{R} + \frac{\partial u_\beta}{\partial R}\right) = -\left\{(3-4\nu)\frac{\cos\beta}{R}\frac{\partial\psi}{\partial\beta} - \cos\beta\frac{\partial^2\psi}{\partial\beta\partial R}\right\} + \left\{(3-4\nu)\frac{\psi}{R} - \frac{\partial\psi}{\partial R}\right\}\sin\beta$$

$$-\left\{(3-4\nu)\psi\frac{\sin\beta}{R} + \frac{\cos\beta}{R}\frac{\partial\psi}{\partial\beta}\right\} + \left\{(3-4\nu)\sin\beta\frac{\partial\psi}{\partial R} + \cos\beta\frac{\partial^2\psi}{\partial R\partial\beta}\right\}$$

$$= -4(1-\nu)\frac{\cos\beta}{R}\frac{\partial\psi}{\partial\beta} + 2\cos\beta\frac{\partial^2\psi}{\partial R\partial\beta} + 2(1-2\nu)\sin\beta\frac{\partial\psi}{\partial R}.$$

i.e., $2\mu E_{R\beta} = -2(1-\nu)\frac{\cos\beta}{R}\frac{\partial\psi}{\partial\beta} + \cos\beta\frac{\partial^2\psi}{\partial R\partial\beta} + (1-2\nu)\sin\beta\frac{\partial\psi}{\partial R}$.

$$2\mu\left(E_{RR} + E_{\theta\theta} + E_{\beta\beta}\right) = -\left\{(2-4\nu)\frac{\partial\psi}{\partial R} - \mathrm{R}\frac{\partial^2\psi}{\partial R^2}\right\}\cos\beta + \left(\frac{\partial\psi}{\partial R} + \frac{\cot\beta}{R}\frac{\partial\psi}{\partial\beta}\right)\cos\beta$$

$$+(2-4\nu)\frac{\sin\beta}{R}\frac{\partial\psi}{\partial\beta} + \frac{\cos\beta}{R}\frac{\partial^2\psi}{\partial\beta^2} + \cos\beta\frac{\partial\psi}{\partial R}$$

$$= \left\{4\nu\frac{\partial\psi}{\partial R} + \mathrm{R}\frac{\partial^2\psi}{\partial R^2}\right\}\cos\beta + \frac{\cos\beta}{R}\frac{\partial^2\psi}{\partial\beta^2} + \left(\cos^2\beta + (2-4\nu)\sin^2\beta\right)\frac{1}{R\sin\beta}\frac{\partial\psi}{\partial\beta}$$

$$= \mathrm{R}\cos\beta\left(\frac{\partial^2\psi}{\partial R^2} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\beta^2} + \frac{\cot\beta}{R^2}\frac{\partial\psi}{\partial\beta}\right) + 4\nu\cos\beta\frac{\partial\psi}{\partial R} + (2-4\nu)\frac{\sin\beta}{R}\frac{\partial\psi}{\partial\beta}$$

$$= \mathrm{R}\cos\beta\left(-\frac{2}{R}\frac{\partial\psi}{\partial R}\right) + 4\nu\cos\beta\frac{\partial\psi}{\partial R} + (2-4\nu)\frac{\sin\beta}{R}\frac{\partial\psi}{\partial\beta} = -\left(2-4\nu\right)\left(\cos\beta\frac{\partial\psi}{\partial R} - \frac{\sin\beta}{R}\frac{\partial\psi}{\partial\beta}\right).$$

where we have used, the relation: $\frac{\partial^2\psi}{\partial R^2} + \frac{1}{R^2}\left(\frac{\partial^2\psi}{\partial\beta^2}\right) + \frac{2}{R}\frac{\partial\psi}{\partial R} + \frac{\cot\beta}{R^2}\frac{\partial\psi}{\partial\beta} = 0$.

Thus, $2\mu e = -\left(2-4\nu\right)\left(\cos\beta\frac{\partial\psi}{\partial R} - \frac{\sin\beta}{R}\frac{\partial\psi}{\partial\beta}\right)$.

(c) The stresses are:

with $\frac{2\mu\nu}{1-2\nu}e = -2\nu\left(\cos\beta\frac{\partial\psi}{\partial R} - \frac{\sin\beta}{R}\frac{\partial\psi}{\partial\beta}\right)$, we have,

$$T_{RR}=\frac{2\mu\nu}{1-2\nu}e+2\mu E_{RR}=-2\nu\left(\cos\beta\frac{\partial\psi}{\partial R}-\frac{\sin\beta}{R}\frac{\partial\psi}{\partial\beta}\right)-\left\{(2-4\nu)\frac{\partial\psi}{\partial R}-\mathrm{R}\frac{\partial^2\psi}{\partial R^2}\right\}\cos\beta$$

$$=(-2+2\nu)\cos\beta\frac{\partial\psi}{\partial R}+\frac{2\nu\sin\beta}{R}\frac{\partial\psi}{\partial\beta}+\mathrm{R}\cos\beta\frac{\partial^2\psi}{\partial R^2}.$$

Similarly,

$$T_{\beta\beta}=\frac{2\mu\nu}{1-2\nu}e+2\mu E_{\beta\beta}=-(2\nu-1)\cos\beta\frac{\partial\psi}{\partial R}+(2-2\nu)\frac{\sin\beta}{R}\frac{\partial\psi}{\partial\beta}+\frac{\cos\beta}{R}\frac{\partial^2\psi}{\partial\beta^2}.$$

$$T_{\theta\theta}=\frac{2\mu\nu}{1-2\nu}e+2\mu E_{\theta\theta}=-(2\nu-1)\frac{\partial\psi}{\partial R}\cos\beta+\left((2\nu-1)\sin\beta+\frac{1}{\sin\beta}\right)\frac{\partial\psi}{R\partial\beta}.$$

$$T_{R\beta}=2\mu E_{R\beta}=-2(1-\nu)\frac{\cos\beta}{R}\frac{\partial\psi}{\partial\beta}+\cos\beta\frac{\partial^2\psi}{\partial\beta\partial R}+\sin\beta(1-2\nu)\frac{\partial\psi}{\partial R}. \qquad T_{R\theta}=T_{\theta\beta}=0$$

5.83 Show that $(1/R)$ is a harmonic function (i.e., it satisfies the Laplace Equation $\nabla^2(1/R)=0$), where R is the radial distance from the origin.

Ans (a) $R^2=x_1^2+x_2^2+x_3^2$, therefore, $\frac{\partial R}{\partial x_1}=\frac{x_1}{R}$, $\frac{\partial R}{\partial x_2}=\frac{x_2}{R}$, $\frac{\partial R}{\partial x_3}=\frac{x_3}{R}$ so that

$$\frac{\partial}{\partial x_1}\left(\frac{1}{R}\right)=-\frac{1}{R^2}\frac{\partial R}{\partial x_1}=-\frac{x_1}{R^3}\text{ and }\frac{\partial^2}{\partial x_1^2}\left(\frac{1}{R}\right)=-\frac{1}{R^3}+\frac{3x_1}{R^4}\left(\frac{x_1}{R}\right)=-\frac{1}{R^3}+\frac{3x_1^2}{R^5}.$$

Similarly, $\frac{\partial^2}{\partial x_2^2}\left(\frac{1}{R}\right)=-\frac{1}{R^3}+\frac{3x_2^2}{R^5}$, $\frac{\partial^2}{\partial x_3^2}\left(\frac{1}{R}\right)=-\frac{1}{R^3}+\frac{3x_3^2}{R^5}$

Thus,

$$\frac{\partial^2}{\partial x_j\partial x_j}\left(\frac{1}{R}\right)=\left(\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)\left(\frac{1}{R}\right)=\left(-\frac{1}{R^3}+\frac{3x_1^2}{R^5}\right)+\left(-\frac{1}{R^3}+\frac{3x_2^2}{R^5}\right)+\left(-\frac{1}{R^3}+\frac{3x_3^2}{R^5}\right)$$

$$=-\frac{3}{R^3}+\frac{3(x_1^2+x_2^2+x_3^2)}{R^5}=-\frac{3}{R^3}+\frac{3}{R^3}=0.$$

5.84 In Kelvin's problem, we used the potential function $\boldsymbol{\Psi}=\psi\mathbf{e}_z$ where in cylindrical coordinates: $\psi=\frac{A}{R}$, $R^2=r^2+z^2$ .Using the results in Example 5.38.6, obtain the stresses.

Ans. $\psi=\frac{1}{R}$, $\frac{\partial\psi}{\partial z}=-\frac{z}{R^3}$, $\frac{\partial^2\psi}{\partial z^2}=\frac{3z^2}{R^5}-\frac{1}{R^3}$,

$$\frac{\partial\psi}{\partial r}=-\frac{r}{R^3}\to\frac{\partial^2\psi}{\partial r^2}=\frac{3r^2}{R^5}-\frac{1}{R^3},\quad\frac{\partial^2\psi}{\partial z\partial r}=\frac{\partial}{\partial z}\left(-\frac{r}{R^3}\right)=\frac{3rz}{R^5},$$

$$T_{rr} = -2\nu\frac{\partial\psi}{\partial z} + z\frac{\partial^2\psi}{\partial r^2} = -2\nu\left(-\frac{z}{R^3}\right) + \left(\frac{3r^2z}{R^5} - \frac{z}{R^3}\right) = -(1-2\nu)\frac{z}{R^3} + \frac{3r^2z}{R^5}.$$

$$T_{\theta\theta} = -2\nu\frac{\partial\psi}{\partial z} + \frac{z}{r}\frac{\partial\psi}{\partial r} = -2\nu\left(-\frac{z}{R^3}\right) + \frac{z}{r}\left(-\frac{r}{R^3}\right) = -(1-2\nu)\left(\frac{z}{R^3}\right).$$

$$T_{rz} = -(1-2\nu)\frac{\partial\psi}{\partial r} + z\frac{\partial^2\psi}{\partial r\partial z} = -(1-2\nu)\left(-\frac{r}{R^3}\right) + z\left(\frac{3rz}{R^5}\right) = (1-2\nu)\left(\frac{r}{R^3}\right) + \left(\frac{3rz^2}{R^5}\right).$$

$$T_{zz} = z\frac{\partial^2\psi}{\partial z^2} - 2(1-\nu)\frac{\partial\psi}{\partial z} = z\left(\frac{3z^2}{R^5} - \frac{1}{R^3}\right) + 2(1-\nu)\frac{z}{R^3} = \frac{3z^3}{R^5} + (1-2\nu)\frac{z}{R^3}.$$

5.85 Show that for $\varphi = C\ln(R+z)$, $R^2 = r^2 + z^2$,

$$\frac{\partial^2\varphi}{\partial r^2} = C\left\{\frac{z}{R^3} - \frac{1}{R(R+z)}\right\}.$$

Ans. $\frac{\partial\varphi}{\partial r} = \frac{C}{(R+z)}\frac{r}{R}$, $\frac{1}{r}\frac{\partial\varphi}{\partial r} = \frac{C}{R(R+z)}$.

$$\frac{\partial^2\varphi}{\partial r^2} = C\left\{\frac{r}{R}\frac{\partial}{\partial r}\frac{1}{(R+z)} + \frac{1}{(R+z)}\frac{\partial}{\partial r}\frac{r}{R}\right\} = \frac{C}{R(R+z)}\left\{-\frac{r^2}{R(R+z)} - \frac{r^2}{R^2} + 1\right\}$$

$$= \frac{C}{R(R+z)}\left\{-\frac{r^2}{R(R+z)} - \frac{r^2}{R^2} + \frac{R(R+z)}{R(R+z)}\right\} = \frac{C}{R(R+z)}\left\{\frac{R^2 - r^2 + zR}{R(R+z)} - \frac{r^2}{R^2}\right\}$$

$$= \frac{C}{R(R+z)}\left\{\frac{z}{R} - \frac{R^2 - z^2}{R^2}\right\} = \frac{C}{R(R+z)}\left\{\frac{Rz + z^2 - R^2}{R^2}\right\} = C\left\{\frac{z}{R^3} - \frac{1}{R(R+z)}\right\}.$$

5.86 Given the following potential functions:

$\boldsymbol{\psi} = (\partial\varphi/\partial z)\mathbf{e}_z$, $\phi = (1-\nu)\varphi$, where $\varphi = C\ln(R+z)$, $R^2 = r^2 + z^2$.

From the results of Example 5.38.4, and Eqs (i), (ii) (iii) of Section 5.40, obtain

$T_{rr} = C\left\{3r^2z/R^5 - (1-2\nu)/[R(R+z)]\right\}$, $T_{\theta\theta} = C(1-2\nu)\left\{-z/R^3 + 1/[R(R+z)]\right\}$.

$T_{zz} = C(3z^3/R^5)$, $T_{rz} = C(3rz^2)/R^5$.

Ans.

$$T_{rr} = z\frac{\partial^3\varphi}{\partial r^2\partial z} + 2\nu\left(\frac{1}{r}\frac{\partial\varphi}{\partial r}\right) + \frac{\partial^2\varphi}{\partial r^2}$$

$$= C\left\{z\left(\frac{3r^2}{R^5} - \frac{1}{R^3}\right) + \frac{2\nu}{R(R+z)} + \left(\frac{z}{R^3} - \frac{1}{R(R+z)}\right)\right\} = C\left\{\left(\frac{3r^2z}{R^5}\right) - \frac{1-2\nu}{R(R+z)}\right\},$$

$$T_{\theta\theta}=\left\{-2\nu\frac{\partial^2\varphi}{\partial z^2}+\frac{z}{r}\frac{\partial^2\varphi}{\partial r\partial z}+(1-2\nu)\frac{1}{r}\frac{\partial\varphi}{\partial r}\right\}$$

$$=-2\nu\left(-\frac{Cz}{R^3}\right)+\frac{z}{r}\left(-\frac{Cr}{R^3}\right)+(1-2\nu)\frac{C}{R(R+z}=-\frac{Cz}{R^3}\left(1-2\nu\right)+(1-2\nu)\frac{C}{R(R+z},$$

$$=C\left(1-2\nu\right)\left\{-\frac{z}{R^3}+\frac{1}{R(R+z)}\right\}$$

$$T_{zz}=z\frac{\partial^3\varphi}{\partial z^3}-\frac{\partial^2\varphi}{\partial z^2}=C\left\{-\left(\frac{z}{R^3}-\frac{3z^3}{R^5}\right)+\frac{z}{R^3}\right\}=C\frac{3z^3}{R^5},\quad T_{rz}=\left(z\frac{\partial^3\varphi}{\partial r\partial z^2}\right)=C\frac{3rz^2}{R^5},$$

$$T_{r\theta}=T_{\theta z}=0.$$

5.87 The stresses in Boussinesq problem in cylindrical coordinate are given by:

$$T_{rr}=-\frac{F_z}{2\pi}\left\{\frac{3r^2z}{R^5}-\frac{\left(1-2\nu\right)}{R(R+z)}\right\},\quad T_{\theta\theta}=-\frac{F_z\left(1-2\nu\right)}{2\pi}\left\{-\frac{z}{R^3}+\frac{1}{R(R+z)}\right\},$$

$$T_{zz}=-\frac{F_z}{2\pi}\frac{3z^3}{R^5},\quad T_{rz}=-\frac{F_z}{2\pi}\frac{3rz^2}{R^5},\ T_{r\theta}=T_{\theta z}=0.$$

Obtain the stresses in rectangular Cartesian coordinates.

Ans.

$$\begin{bmatrix}T_{xx}&T_{xy}&T_{xz}\\T_{yx}&T_{yy}&T_{yz}\\T_{zx}&T_{yz}&T_{zz}\end{bmatrix}=\begin{bmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}\begin{bmatrix}T_{rr}&0&T_{rz}\\0&T_{\theta\theta}&0\\T_{zr}&0&T_{zz}\end{bmatrix}\begin{bmatrix}\cos\theta&\sin\theta&0\\-\sin\theta&\cos\theta&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}T_{rr}\cos^2\theta+T_{\theta\theta}\sin^2\theta&\left(T_{rr}-T_{\theta\theta}\right)\sin\theta\cos\theta&T_{rz}\cos\theta\\\left(T_{rr}-T_{\theta\theta}\right)\sin\theta\cos\theta&T_{rr}\sin^2\theta+T_{\theta\theta}\cos^2\theta&T_{rz}\sin\theta\\T_{rz}\cos\theta&T_{rz}\sin\theta&1\end{bmatrix}.\quad \text{Thus,}$$

$$T_{xx}=T_{rr}\cos^2\theta+T_{\theta\theta}\sin^2\theta$$

$$=-\frac{F_z}{2\pi}\left\{\frac{3r^2z}{R^5}-\frac{\left(1-2\nu\right)}{R(R+z)}\right\}\cos^2\theta-\frac{F_z\left(1-2\nu\right)}{2\pi}\left\{-\frac{z}{R^3}+\frac{1}{R(R+z)}\right\}\sin^2\theta$$

$$=-\frac{F_z}{2\pi}\left[\frac{3x^2z}{R^5}-\frac{\left(1-2\nu\right)}{R(R+z)}\cos^2\theta-\frac{\left(1-2\nu\right)z}{R^3}\sin^2\theta+\frac{\left(1-2\nu\right)}{R(R+z)}\sin^2\theta\right]$$

$$=-\frac{F_z}{2\pi}\left[\frac{3x^2z}{R^5}-\frac{\left(1-2\nu\right)z}{R^3}+\frac{\left(1-2\nu\right)}{R(R+z)}\left\{\frac{z}{R^2}(R+z)\cos^2\theta-2\cos^2\theta+1\right\}\right]$$

$$=-\frac{F_z}{2\pi}\left[\frac{3x^2z}{R^5}-\frac{\left(1-2\nu\right)z}{R^3}+\frac{\left(1-2\nu\right)}{R(R+z)}+\frac{\left(1-2\nu\right)}{R(R+z)}\left\{\frac{z}{R^2}(R+z)\cos^2\theta-2\cos^2\theta\right\}\right]$$

$$=-\frac{F_z}{2\pi}\left[\frac{3x^2z}{R^5}-\frac{(1-2\nu)z}{R^3}+\frac{(1-2\nu)}{R(R+z)}+\frac{(1-2\nu)}{R(R+z)}\left\{\frac{R(z-R)+z^2-R^2}{R^2}\right\}\cos^2\theta\right]$$

$$=-\frac{F_z}{2\pi}\left[\frac{3x^2z}{R^5}-\frac{(1-2\nu)z}{R^3}+\frac{(1-2\nu)}{R(R+z)}+\frac{(1-2\nu)}{R(R+z)}\left\{\frac{R(z-R)+z^2-R^2}{R^2}\right\}\cos^2\theta\right]$$

Now,

$$r\cos\theta=x\rightarrow\cos\theta=x/r, r^2+z^2=R^2,\quad \cos^2\theta=\frac{x^2}{R^2-z^2}=\frac{x^2}{(R+z)(R-z)}$$

Therefore,

$$T_{xx}=-\frac{F_z}{2\pi}\left[\frac{3x^2z}{R^5}-\frac{(1-2\nu)z}{R^3}+\frac{(1-2\nu)}{R(R+z)}+\frac{(1-2\nu)}{R(R+z)}\left\{\frac{R(z-R)+z^2-R^2}{R^2}\right\}\frac{x^2}{(R+z)(R-z)}\right]$$

$$=-\frac{F_z}{2\pi}\left[\frac{3x^2z}{R^5}-\frac{(1-2\nu)z}{R^3}+\frac{(1-2\nu)}{R(R+z)}-\frac{(1-2\nu)}{R(R+z)}\left\{\frac{1}{R}\frac{x^2}{(R+z)}+\frac{x^2}{R^2}\right\}\right].$$

5.88 Obtain the variation of T_{zz} along the z axis for the case where the normal load on the surface of an elastic half-space is uniform with intensity q_o, and the loaded area is a circle of radius r_o with its center at the origin.

Ans. Using Eq.(5.41.3), we have,

$$T_{zz}=-\frac{3z^3}{2\pi}\int\frac{q_o 2\pi r'dr'}{R'^5}=-3z^3q_o\int_{r'=o}^{r_o}\frac{r'dr'}{R'^5},\quad\text{where}$$

$$R'^2=r'^2+z^2\rightarrow R'dR'=r'dr'\rightarrow T_{zz}=-3z^3q_o\int_z^{R_o}\frac{dR'}{R'^4}$$

$$\rightarrow T_{zz}=-3z^3q_o\left[-\frac{1}{3R'^3}\right]_{R'=z}^{R_o}=\frac{q_oz^3}{R_o^3}-q_o=\frac{q_oz^3}{(r_o^2+z^2)^{3/2}}-q_o\,.$$

5.89 For the potential function $\psi=D_1R^{-2}\cos\beta\mathbf{e}_z$, where (R,β,θ) are the spherical coordinates with β as the azimuthal angle. Find T_{RR} and $T_{\beta R}$.

Ans. $\psi=D_1R^{-2}\cos\beta\rightarrow\frac{\partial\psi}{\partial R}=-2D_1R^{-3}\cos\beta\rightarrow\frac{\partial^2\psi}{\partial R^2}=6D_1R^{-4}\cos\beta,\quad\frac{\partial\psi}{\partial\beta}=-D_1R^{-2}\sin\beta\,.$

From Example 5.38.7,

$$T_{RR} = -2(1-\nu)\cos\beta\frac{\partial\psi}{\partial R} + R\cos\beta\frac{\partial^2\psi}{\partial R^2} + \frac{2\nu\sin\beta}{R}\frac{\partial\psi}{\partial\beta}$$
$$= D_1\left[-2(1-\nu)\cos\beta\left(-2R^{-3}\cos\beta\right) + R\cos\beta\left(6R^{-4}\cos\beta\right) + \frac{2\nu\sin\beta}{R}\left(-R^{-2}\sin\beta\right)\right]$$
$$= D_1\left[2\left(5-\nu\right)\cos^2\beta - 2\nu\right]R^{-3}.$$

$$T_{R\beta} = -\frac{2(1-\nu)}{R}\cos\beta\frac{\partial\psi}{\partial\beta} + \cos\beta\frac{\partial^2\psi}{\partial\beta\partial R} + \sin\beta(1-2\nu)\frac{\partial\psi}{\partial R}$$
$$= D_1\left[-\frac{2(1-\nu)}{R}\cos\beta\left(-R^{-2}\sin\beta\right) + \cos\beta\left(2R^{-3}\sin\beta\right) + \sin\beta(1-2\nu)\left(-2R^{-3}\cos\beta\right)\right]$$
$$= D_1\left[2(1-\nu)R^{-3}\cos\beta\sin\beta + 2R^{-3}\sin\beta\cos\beta - 2R^{-3}\cos\beta\sin\beta(1-2\nu)\right]$$
$$= D_1R^{-3}\cos\beta\sin\beta\left[2(1-\nu) + 2 - 2(1-2\nu)\right] = 2D_1R^{-3}(1+\nu)\cos\beta\sin\beta.$$

5.90 For the potential function, $\tilde{\phi} = \tilde{\phi}\left(R,\beta\right) = C_1[R^{-3}\left(3\cos^2\beta - 1\right)/2] + C_2R^{-1}$, where (R,β,θ) are the spherical coordinates with β as the azimuthal angle, obtain T_{RR} and $T_{\beta R}$.

Ans. With

$$\tilde{\phi}\left(R,\beta\right) = \left(C_1/2\right)R^{-3}\left(3\cos^2\beta - 1\right) + C_2R^{-1} \rightarrow \frac{\partial\tilde{\phi}}{\partial R} = \left(C_1/2\right)\left(-3R^{-4}\right)\left(3\cos^2\beta - 1\right) + \left(C_2\right)\left(-R^{-2}\right)$$
$$\rightarrow \frac{\partial^2\tilde{\phi}}{\partial R^2} = 6C_1R^{-5}\left(3\cos^2\beta - 1\right) + 2C_2R^{-3}$$
$$and\ \frac{\partial\tilde{\phi}}{\partial\beta} = -3C_1R^{-3}\sin\beta\cos\beta \rightarrow \frac{\partial}{\partial R}\frac{\partial\tilde{\phi}}{\partial\beta} = 9C_1R^{-4}\sin\beta\cos\beta.$$

From Example 5.38.6, we have

$$T_{RR} = \frac{\partial^2\hat{\phi}}{\partial R^2} = 6C_1R^{-5}\left(3\cos^2\beta - 1\right) + 2C_2R^{-3}.$$
$$T_{R\beta} = \frac{1}{R}\left(\frac{\partial^2\hat{\phi}}{\partial\beta\partial R} - \frac{1}{R}\frac{\partial\hat{\phi}}{\partial\beta}\right) = \frac{1}{R}\left[9C_1R^{-4}\sin\beta\cos\beta - \frac{1}{R}\left(-3C_1R^{-3}\sin\beta\cos\beta\right)\right] = 12C_1R^{-5}\sin\beta\cos\beta$$

.

$$T_{R\theta} = T_{\theta\beta} = 0.$$

CHARTER 5, PART B

5.91 Demonstrate that if only E_2 and E_3 are nonzero, then Eq.(5.46.4) becomes

$$2U = \begin{bmatrix} E_2 & E_3 \end{bmatrix} \begin{bmatrix} C_{22} & C_{23} \\ C_{32} & C_{33} \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix}.$$

Ans. Eq. .(5.46.4) gives

$$2U = \begin{bmatrix} 0 & E_2 & E_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_{12}E_2 + C_{13}E_3 \\ C_{22}E_2 + C_{23}E_3 \\ C_{23}E_2 + C_{33}E_3 \\ C_{24}E_2 + C_{34}E_3 \\ C_{25}E_2 + C_{35}E_3 \\ C_{26}E_2 + C_{36}E_3 \end{bmatrix} = \left[E_2 \left(C_{22}E_2 + C_{23}E_3 \right) + E_3 \left(C_{23}E_2 + C_{33}E_3 \right) \right].$$

This is the same as

$$\begin{bmatrix} E_2 & E_3 \end{bmatrix} \begin{bmatrix} C_{22} & C_{23} \\ C_{23} & C_{33} \end{bmatrix} \begin{bmatrix} E_2 \\ E_3 \end{bmatrix} = \left[E_2 \left(C_{22}E_2 + C_{23}E_3 \right) + E_3 \left(C_{23}E_2 + C_{33}E_3 \right) \right].$$

5.92 Demonstrate that if only E_1 and E_3 are nonzero, then Eq.(5.46.4) becomes

$$2U = \begin{bmatrix} E_1 & E_3 \end{bmatrix} \begin{bmatrix} C_{11} & C_{13} \\ C_{31} & C_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_3 \end{bmatrix}$$

Ans.

$$2U = \begin{bmatrix} E_1 & 0 & E_3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} E_1 \\ 0 \\ E_3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \left[E_1 \left(C_{11}E_1 + C_{13}E_3 \right) + E_3 \left(C_{13}E_1 + C_{33}E_3 \right) \right]$$

This is the same as

$$\begin{bmatrix} E_1 & E_3 \end{bmatrix} \begin{bmatrix} C_{11} & C_{13} \\ C_{13} & C_{33} \end{bmatrix} \begin{bmatrix} E_1 \\ E_3 \end{bmatrix} = \frac{1}{2} \left[E_1 \left(E_1 C_{11} + C_{13}E_3 \right) + E_3 \left(C_{13}E_1 + C_{33}E_3 \right) \right] = 2U$$

5.93 Write stress strain laws for a monoclinic elastic solid in contracted notation, whose plane of symmetry is the x_1x_2 plane.

Ans. All $C_{ijkl} = 0$ where the indices $ijkl$ contain an odd number of 3. Therefore,

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix}.$$

5.94 Write stress strain laws for a monoclinic elastic solid in contracted notation, whose plane of symmetry is the x_1x_3 plane.

Ans. All $C_{ijkl}=0$ where the indices $ijkl$ contain an odd number of 2. Therefore,

$$[C]=\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ C_{12} & C_{22} & C_{23} & 0 & C_{25} & 0 \\ C_{13} & C_{23} & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & C_{46} \\ C_{15} & C_{25} & C_{35} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & C_{46} & 0 & C_{66} \end{bmatrix}$$

5.95 For transversely isotropic solid with $\mathbf{e}_3$ as the axis of transversely isotropy, show from the transformation law $C'_{ijkl}=Q_{mi}Q_{nj}Q_{rk}Q_{sl}C_{mnrs}$ that $C'_{1113}=0$ (See Sect.5.50)

Ans. Since $Q_{33}=1,\;\; Q_{13}=Q_{23}=Q_{31}=Q_{32}=0$, therefore,

$$C'_{1113}=Q_{m1}Q_{n1}Q_{r1}Q_{s3}C_{mnrs}=Q_{m1}Q_{n1}Q_{r1}Q_{33}C_{mnr3}=Q_{m1}Q_{n1}Q_{r1}C_{mnr3}=Q_{11}Q_{n1}Q_{r1}C_{1nr3}+Q_{21}Q_{n1}Q_{r1}C_{2nr3}$$
$$=Q_{11}Q_{11}Q_{r1}C_{11r3}+Q_{11}Q_{21}Q_{r1}C_{12r3}+Q_{21}Q_{11}Q_{r1}C_{21r3}+Q_{21}Q_{21}Q_{r1}C_{22r3}.$$

Now, all C_{ijkl} with odd number of either 1 or 2 are zero because $\mathbf{e}_1$ plane and $\mathbf{e}_2$ -plane are planes of material symmetry. Thus, $Q_{r1}C_{11r3}=Q_{r1}C_{12r3}=Q_{r1}C_{21r3}=Q_{r1}C_{22r3}=0$. Thus, $C'_{1113}=0$

5.96 Show that for a transversely isotropic elastic material with $\mathbf{e}_3$ as the axis of transverse isotropy, $C_{1133}=C_{2233}$ (see Sect.5.50) .

Ans. $\mathbf{e}'_1=\cos\beta\mathbf{e}_1+\sin\beta\mathbf{e}_2,\quad \mathbf{e}'_2=-\sin\beta\mathbf{e}_1+\cos\beta\mathbf{e}_2,\quad \mathbf{e}'_3=\mathbf{e}_3$

$Q_{11}=\cos\beta,\; Q_{12}=-\sin\beta,\;\; Q_{21}=\sin\beta,\;\; Q_{22}=\cos\beta,\;\; Q_{33}=1,\;\; Q_{31}=Q_{13}=Q_{23}=Q_{32}=0$. Thus,

$$C'_{1233}=Q_{m1}Q_{n2}Q_{r3}Q_{s3}C_{mnrs}=Q_{m1}Q_{n2}Q_{33}Q_{33}C_{mn33}=Q_{m1}Q_{n2}C_{mn33}=Q_{11}Q_{n2}C_{1n33}+Q_{21}Q_{n2}C_{2n33}$$
$$=Q_{11}Q_{12}C_{1133}+Q_{11}Q_{21}C_{1233}+Q_{21}Q_{12}C_{2133}+Q_{21}Q_{22}C_{2233}.$$

Now $C_{1233}=C_{2133}=0$ because $\mathbf{e}_1$ plane (as well as $\mathbf{e}_2$ -plane) is a plane of material symmetry. Thus,

$$C'_{1233}=Q_{11}Q_{12}C_{1133}+Q_{21}Q_{22}C_{2233}=-\cos\beta\sin\beta C_{1133}+\sin\beta\cos\beta C_{2233}$$
$$=\left(-C_{1133}+C_{2233}\right)\cos\beta\sin\beta.$$

Again, $C'_{1233}=0$, because $\mathbf{e}'_1$ is also a plane of symmetry. Thus $C_{1133}=C_{2233}$.

5.97 Show that for a transversely isotropic elastic material with $\mathbf{e}_3$ as the axis of transverse isotropy(see Sect.5.50)

$$(\sin\beta)^2C_{1111}+\left[(\cos\beta)^2-(\sin\beta)^2\right]C_{1122}+2\left[(\cos\beta)^2-(\sin\beta)^2\right]C_{1212}-(\cos\beta)^2C_{2222}=0.$$

Ans. Since $Q_{13}=Q_{31}=Q_{23}=Q_{32}=0$ and $C_{ijkl}=0$ when the indices $ijkl$ contain an odd number of either 1 or 2, therefore,

$$C'_{1222}=Q_{m1}Q_{n2}Q_{r2}Q_{s2}C_{mnrs}=Q_{11}Q_{n2}Q_{r2}Q_{s2}C_{1nrs}+Q_{21}Q_{n2}Q_{r2}Q_{s2}C_{2nrs}$$
$$=Q_{11}Q_{12}Q_{r2}Q_{s2}C_{11rs}+Q_{11}Q_{22}Q_{r2}Q_{s2}C_{12rs}+Q_{21}Q_{12}Q_{r2}Q_{s2}C_{21rs}+Q_{21}Q_{22}Q_{r2}Q_{s2}C_{22rs}$$
$$=Q_{11}Q_{12}Q_{12}Q_{12}C_{1111}+Q_{11}Q_{12}Q_{22}Q_{22}C_{1122}+Q_{11}Q_{22}Q_{12}Q_{22}C_{1212}+Q_{11}Q_{22}Q_{22}Q_{12}C_{1221}$$
$$+Q_{21}Q_{12}Q_{12}Q_{22}C_{2112}+Q_{21}Q_{12}Q_{22}Q_{12}C_{2121}+Q_{21}Q_{22}Q_{12}Q_{12}C_{2211}+Q_{21}Q_{22}Q_{22}Q_{22}C_{2222}.$$

Thus,

$$C'_{1222}=-\cos\beta(\sin\beta)^3C_{1111}-\sin\beta(\cos\beta)^3C_{1122}-(\cos\beta)^3\sin\beta C_{1212}-(\cos\beta)^3\sin\beta C_{1221}$$
$$+(\sin\beta)^3\cos\beta C_{2112}+(\sin\beta)^3\cos\beta C_{2121}+(\sin\beta)^3\cos\beta C_{2211}+(\cos\beta)^3\sin\beta C_{2222}$$
$$=-\cos\beta\sin\beta\left[\sin^2\beta C_{1111}+(\cos^2\beta-\sin^2\beta)C_{1122}+2(\cos^2\beta-\sin^2\beta)C_{1212}-\cos^2\beta C_{2222}\right].$$

where we have used $C_{ijkl}=C_{jikl}$, $C_{ijkl}=C_{jilk}$ and $C_{ijkl}=C_{klij}$.

Now, $C'_{1222}=0$ because $\mathbf{e}'_1$ is also a plane of symmetry, therefore,

$$\sin^2\beta C_{1111}+(\cos^2\beta-\sin^2\beta)C_{1122}+2(\cos^2\beta-\sin^2\beta)C_{1212}-\cos^2\beta C_{2222}=0.$$

5.98 In Section 5.50, we obtained the reduction in the elastic coefficients for a transversely isotropic elastic solid by demanding that each S_β plane is a plane of material symmetry. We can also obtain the same reduction by demanding the C'_{ijkl} be the same for all β. Use this procedure to obtain the result: $C_{1133}=C_{2233}$.

Ans. Since $Q_{31}=Q_{13}=Q_{32}=Q_{23}=0,\ Q_{33}=1$, therefore,

$$C'_{1133}=Q_{m1}Q_{n1}Q_{r3}Q_{s3}C_{mnrs}=Q_{m1}Q_{n1}Q_{33}Q_{33}C_{mn33}=Q_{m1}Q_{n1}C_{mn33}.$$

Now, $C_{ijkl}=0$ when the indices contain an odd number of either 1 or 2, therefore,

$$C'_{1133}=Q_{11}Q_{11}C_{1133}+Q_{21}Q_{21}C_{2233}=\cos^2\beta C_{1133}+\sin^2\beta C_{2233}.$$

Now, $C'_{1133}=C_{1133}$ for all β, therefore,

$$C_{1133}=\cos^2\beta C_{1133}+\sin^2\beta C_{2233}\rightarrow C_{1133}\sin^2\beta=\sin^2\beta C_{2233}.$$

Thus, $C_{1133} = C_{2233}$.

5.99 Invert the compliance matrix for a transversely isotropic elastic solid to obtain the relationship between C_{ij} and the engineering constants. That is, verify Eq. (5.53.2) and (5.53.3) by inverting the following matrix:

$$[A] = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_1 & -\nu_{31}/E_3 \\ -\nu_{21}/E_1 & 1/E_1 & -\nu_{31}/E_3 \\ -\nu_{13}/E_1 & -\nu_{13}/E_1 & 1/E_3 \end{bmatrix}$$

Ans. $[A] = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_1 & -\nu_{31}/E_3 \\ -\nu_{21}/E_1 & 1/E_1 & -\nu_{31}/E_3 \\ -\nu_{13}/E_1 & -\nu_{13}/E_1 & 1/E_3 \end{bmatrix}$

$$\Delta = \det[A] = \frac{1}{E_1^2 E_3}\left(1 - 2\nu_{21}\nu_{13}\nu_{31} - 2\nu_{13}\nu_{31} - \nu_{21}\nu_{21}\right) = \frac{1}{E_1^2 E_3}\left(1+\nu_{21}\right)\left(1-\nu_{21}-2\nu_{13}\nu_{31}\right).$$

Now, $\nu_{31}/E_3 = \nu_{13}/E_1 \to$

$$\Delta = \frac{1}{E_1^2 E_3}\left(1+\nu_{21}\right)\left(1-\nu_{21}-2\nu_{31}^2\left(E_1/E_3\right)\right) = \frac{1}{E_1^2 E_3}\left(1+\nu_{21}\right)D$$

where $D = \left(1-\nu_{21}-2\nu_{31}^2\left(E_1/E_3\right)\right)$.

$$C_{11} = \frac{1}{\Delta}\begin{vmatrix} 1/E_1 & -\nu_{31}/E_3 \\ -\nu_{13}/E_1 & 1/E_3 \end{vmatrix} = \frac{E_1^2 E_3}{D\left(1+\nu_{21}\right)}\frac{1}{E_1 E_3}\left(1-\nu_{31}\nu_{13}\right) = \frac{E_1}{\left(1+\nu_{21}\right)}\frac{\left[1-\nu_{31}^2(E_1/E_3)\right]}{D}$$

$$C_{22} = C_{11}$$

$$C_{33} = \frac{1}{\Delta}\begin{vmatrix} 1/E_1 & -\nu_{21}/E_1 \\ -\nu_{21}/E_1 & 1/E_3 \end{vmatrix} = \frac{1}{\Delta E_1 E_3}\left(1-\nu_{21}^2\left(E_3/E_1\right)\right) = \frac{E_1}{\left(1+\nu_{21}\right)}\frac{\left[1-\nu_{21}^2(E_3/E_1)\right]}{D}$$

$$C_{12} = -\frac{1}{\Delta}\begin{vmatrix} -\nu_{21}/E_1 & -\nu_{31}/E_3 \\ -\nu_{13}/E_1 & 1/E_3 \end{vmatrix} = \frac{1}{\Delta E_1 E_3}\left(\nu_{21}+\nu_{31}\nu_{13}\right) = \frac{E_1\left(\nu_{21}+\nu_{31}^2 E_1/E_3\right)}{\left(1+\nu_{21}\right)D}$$

$$C_{13} = \frac{1}{\Delta}\begin{vmatrix} -\nu_{21}/E_1 & -\nu_{31}/E_3 \\ 1/E_1 & -\nu_{31}/E_3 \end{vmatrix} = \frac{1}{\Delta E_1 E_3}\left(\nu_{21}\nu_{31}+\nu_{31}\right) = \frac{\nu_{31}E_1}{D}$$

$$C_{23} = -\frac{1}{\Delta}\begin{vmatrix} 1/E_1 & -\nu_{31}/E_3 \\ -\nu_{21}/E_1 & -\nu_{31}/E_3 \end{vmatrix} = \frac{1}{\Delta E_1 E_3}\left(\nu_{31}+\nu_{21}\nu_{31}\right) = \frac{\nu_{31}E_1}{D}$$

5.100 Obtain Eq.(5.53.6) from Eq. (5.53.2) and (5.53.3).

Ans. From

$$C_{11}=\frac{E_1}{(1+\nu_{21})}\frac{\left[1-\nu_{31}^2(E_1/E_3)\right]}{D} \text{ and } C_{12}=\frac{E_1\left(\nu_{21}+\nu_{31}^2E_1/E_3\right)}{(1+\nu_{21})D}$$

$$C_{11}-C_{12}=\frac{E_1\left[1-\nu_{31}^2(E_1/E_3)\right]}{(1+\nu_{21})D}-\frac{E_1\left(\nu_{21}+\nu_{31}^2E_1/E_3\right)}{(1+\nu_{21})D}$$

$$=\frac{E_1}{(1+\nu_{21})D}\left\{1-\nu_{21}-2\nu_{31}^2(E_1/E_3)\right\}=\frac{E_1}{(1+\nu_{21})D}\{D\}=\frac{E_1}{(1+\nu_{21})}$$

Thus, [see Eq.5.53.5], $2G_{12}=\dfrac{E_1}{(1+\nu_{21})}$.

5.101 Invert the compliance matrix for an orthotropic elastic solid to obtain the relationship between C_{ij} and the engineering constants.

Ans. Let $[A]^{-1}=\begin{bmatrix}\frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3}\\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3}\\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3}\end{bmatrix}^{-1}=\begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{12} & C_{22} & C_{23}\\ C_{13} & C_{23} & C_{33}\end{bmatrix}$

$\det[A]\equiv\Delta=\dfrac{\left[1-\nu_{12}\nu_{23}\nu_{31}-\nu_{13}\nu_{21}\nu_{32}-\nu_{13}\nu_{31}-\nu_{23}\nu_{32}-\nu_{21}\nu_{12}\right]}{E_1E_2E_3}$. Since

$\nu_{12}\nu_{23}\nu_{31}=\left(\dfrac{E_1\nu_{21}}{E_2}\right)\left(\dfrac{E_2\nu_{32}}{E_3}\right)\left(\dfrac{E_3\nu_{13}}{E_1}\right)=\nu_{21}\nu_{32}\nu_{13}$, therefore,

$$\Delta=\frac{\left[1-2\nu_{13}\nu_{21}\nu_{32}-\nu_{13}\nu_{31}-\nu_{23}\nu_{32}-\nu_{21}\nu_{12}\right]}{E_1E_2E_3}.$$

Next

$C_{11}=\dfrac{1}{\Delta}\begin{vmatrix}\frac{1}{E_2} & -\frac{\nu_{32}}{E_3}\\ -\frac{\nu_{23}}{E_2} & \frac{1}{E_3}\end{vmatrix}=\dfrac{1}{\Delta}\left(\dfrac{1}{E_2}\dfrac{1}{E_3}-\dfrac{\nu_{32}}{E_3}\dfrac{\nu_{23}}{E_2}\right)=\dfrac{1}{\Delta E_2E_3}(1-\nu_{32}\nu_{23})$, etc.,

$C_{12}=-\dfrac{1}{\Delta}\begin{vmatrix}-\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3}\\ -\frac{\nu_{23}}{E_2} & \frac{1}{E_3}\end{vmatrix}=\dfrac{1}{\Delta E_2E_3}(\nu_{21}+\nu_{31}\nu_{23})$, $\quad C_{13}=\dfrac{1}{\Delta}\begin{vmatrix}-\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3}\\ \frac{1}{E_2} & -\frac{\nu_{32}}{E_3}\end{vmatrix}=\dfrac{1}{\Delta E_2E_3}(\nu_{31}+\nu_{21}\nu_{32})$

$$C_{23} = -\frac{1}{\Delta}\begin{vmatrix} \frac{1}{E_1} & -\frac{\nu_{31}}{E_3} \\ -\frac{\nu_{12}}{E_1} & -\frac{\nu_{32}}{E_3} \end{vmatrix} = \frac{1}{\Delta E_1 E_3}\left(\nu_{32} + \nu_{31}\nu_{12}\right)$$

5.102 Obtain the restriction given in Eq.(5.54.8) for engineering constants for an orthotropic elastic solid

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Ans.

$$\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} \end{bmatrix} \to \det\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} \end{bmatrix} = \frac{1}{E_1E_2}\left(1-\nu_{21}\nu_{12}\right) \to 1-\nu_{21}\nu_{12} > 0\,,$$

But, $\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2} \to 1-\nu_{21}\nu_{12} = 1 - \frac{\nu_{21}\nu_{21}E_1}{E_2} > 0 \to \frac{E_2}{E_1} > \nu_{21}^2$. Also,

$$1-\nu_{21}\nu_{12} = 1-\frac{\nu_{12}^2 E_2}{E_1} > 0 \to \frac{E_1}{E_2} > \nu_{12}^2$$

Next, $\det\begin{bmatrix} \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} \\ -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} \end{bmatrix} = \frac{1}{E_2E_3}\left(1-\nu_{32}\nu_{23}\right) \to 1-\nu_{32}\nu_{23} > 0$.

But, $\frac{\nu_{32}}{E_3} = \frac{\nu_{23}}{E_2} \to 1-\nu_{32}\nu_{23} = 1-\nu_{23}^2\frac{E_3}{E_2} = 1-\nu_{32}^2\frac{E_2}{E_3}$,

$$1-\nu_{32}\nu_{23} > 0 \to \frac{E_2}{E_3} > \nu_{23}^2 \text{ and } \frac{E_3}{E_2} > \nu_{32}^2 .$$

Also,

$$\det\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{31}}{E_3} \\ -\frac{\nu_{13}}{E_1} & \frac{1}{E_3} \end{bmatrix} = \frac{1}{E_1E_3}\left(1-\nu_{31}\nu_{13}\right) = \frac{1}{E_1E_3}\left(1-\nu_{13}^2\frac{E_3}{E_1}\right) = \frac{1}{E_1E_3}\left(1-\nu_{31}^2\frac{E_1}{E_3}\right)$$

$$\left(1-\nu_{31}\nu_{13}\right) > 0 \to \nu_{13}^2 < \frac{E_1}{E_3} \text{ and } \nu_{31}^2 < \frac{E_3}{E_1} .$$

5.103 Write down all the restrictions for the engineering constants for a monoclinic solid in determinant form (no need to expand the determinant).

$$Ans.\ \begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{23} \\ 2E_{31} \\ 2E_{12} \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & \eta_{41}/G_4 & 0 & 0 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & \eta_{42}/G_4 & 0 & 0 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & \eta_{43}/G_4 & 0 & 0 \\ \eta_{14}/E_1 & \eta_{24}/E_2 & \eta_{34}/E_3 & 1/G_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/G_5 & \mu_{65}/G_6 \\ 0 & 0 & 0 & 0 & \mu_{56}/G_5 & 1/G_6 \end{bmatrix} \begin{bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{23} \\ T_{31} \\ T_{12} \end{bmatrix}$$

(i)

$$E_1 > 0, E_2 > 0, E_3 > 0, G_4 > 0, G_5 > 0, G_6 > 0$$

$$(ii) \begin{vmatrix} 1/E_1 & -\nu_{21}/E_2 \\ -\nu_{12}/E_1 & -\nu_{23}/E_2 \end{vmatrix} > 0, \begin{vmatrix} 1/E_2 & -\nu_{32}/E_3 \\ -\nu_{23}/E_2 & 1/E_3 \end{vmatrix} > 0, \begin{vmatrix} 1/E_3 & \eta_{43}/G_4 \\ \eta_{34}/E_3 & 1/G_4 \end{vmatrix} > 0$$

$$\begin{vmatrix} 1/G_5 & \mu_{65}/G_6 \\ \mu_{56}/E_2 & 1/G_6 \end{vmatrix} > 0, \begin{vmatrix} 1/E_1 & -\nu_{31}/E_3 \\ -\nu_{13}/E_1 & 1/E_3 \end{vmatrix} > 0, \begin{vmatrix} 1/E_2 & \eta_{42}/G_4 \\ \eta_{24}/E_2 & 1/G_4 \end{vmatrix} > 0, \begin{vmatrix} 1/E_1 & \eta_{41}/G_4 \\ \eta_{14}/E_1 & 1/G_4 \end{vmatrix} > 0$$

(iii)

$$\begin{vmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 \end{vmatrix} > 0, \begin{vmatrix} 1/E_2 & -\nu_{32}/E_3 & \eta_{42}/G_4 \\ -\nu_{23}/E_2 & 1/E_3 & \eta_{43}/G_4 \\ \eta_{24}/E_2 & \eta_{34}/E_3 & 1/G_4 \end{vmatrix} > 0$$

$$\begin{vmatrix} 1/E_1 & -\nu_{31}/E_3 & \eta_{41}/G_4 \\ -\nu_{13}/E & 1/E_3 & \eta_{43}/G_4 \\ \eta_{14}/E_1 & \eta_{34}/E_3 & 1/G_4 \end{vmatrix} > 0, \begin{vmatrix} 1/E_1 & -\nu_{21}/E_2 & \eta_{41}/G_4 \\ -\nu_{12}/E_1 & 1/E_2 & \eta_{42}/G_4 \\ \eta_{14}/E_1 & \eta_{24}/E_2 & 1/G_4 \end{vmatrix} > 0$$

(iv)

$$\begin{vmatrix} 1/E_1 & -\nu_{21}/E_2 & -\nu_{31}/E_3 & \eta_{41}/G_4 \\ -\nu_{12}/E_1 & 1/E_2 & -\nu_{32}/E_3 & \eta_{42}/G_4 \\ -\nu_{13}/E_1 & -\nu_{23}/E_2 & 1/E_3 & \eta_{43}/G_4 \\ \eta_{14}/E_1 & \eta_{24}/E_2 & \eta_{34}/E_3 & 1/G_4 \end{vmatrix} > 0$$

CHAPRTR 5, PART C

5.104 Show that if a tensor is objective, then its inverse is also objective .

Ans. Let $\mathbf{T}$ be an objective tensor, then in a change of frame: $\mathbf{x}^* = \mathbf{c}(t) + \mathbf{Q}(t)(\mathbf{x} - \mathbf{x}_o)$

$\mathbf{T}^* = \mathbf{Q}(t)\mathbf{T}\mathbf{Q}(t)^{\mathrm{T}}$. Taking the inverse of this equation, we get, since $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$.

$\mathbf{T}^{-1*} = \left(\mathbf{Q}(t)\mathbf{T}\mathbf{Q}(t)^{\mathrm{T}}\right)^{-1} = \mathbf{Q}(t)\mathbf{T}^{-1}\mathbf{Q}(t)^{\mathrm{T}}$. Thus, $\mathbf{T}^{-1}$ is objective.

5.105 Show that the rate of deformation tensor $\mathbf{D}=[\nabla\mathbf{v}+(\nabla\mathbf{v})^{\mathrm{T}}]/2$ is objective. [See Example 5.56.2)].

Ans. From Eq.(5.56.13), we have $\nabla^*\mathbf{v}^*=\mathbf{Q}(t)(\nabla\mathbf{v})\mathbf{Q}^{\mathrm{T}}(t)+\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$. Thus,

$(\nabla^*\mathbf{v}^*)^{\mathrm{T}}=\mathbf{Q}(t)(\nabla\mathbf{v})^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}(t)+\mathbf{Q}(\dot{\mathbf{Q}})^{\mathrm{T}}$. Now,

$(d/dt)(\mathbf{Q}\mathbf{Q}^{\mathrm{T}})=\mathbf{0}\rightarrow\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}+\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}}\rightarrow\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}=-\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}}$. Thus,

$$\nabla^*\mathbf{v}^*+(\nabla^*\mathbf{v}^*)^{\mathrm{T}}=\mathbf{Q}(t)(\nabla\mathbf{v})\mathbf{Q}^{\mathrm{T}}(t)+\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}+\mathbf{Q}(t)(\nabla\mathbf{v})^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}(t)-\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}=\mathbf{Q}(t)\left[(\nabla\mathbf{v})+(\nabla\mathbf{v})^{\mathrm{T}}\right]\mathbf{Q}^{\mathrm{T}}(t)$$

$$\rightarrow\mathbf{D}^*=\mathbf{Q}(t)\mathbf{D}\mathbf{Q}^{\mathrm{T}}(t).$$

5.106 Show that in a change of frame, the spin tensor $\mathbf{W}=[\nabla\mathbf{v}-(\nabla\mathbf{v})^{\mathrm{T}}]/2$ transforms in accordance with the equation $\mathbf{W}^*=\mathbf{Q}(t)\mathbf{W}\mathbf{Q}^{\mathrm{T}}(t)+\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$. [See Example 5.56.2)].

Ans. From Eq.(5.56.13), we have $\nabla^*\mathbf{v}^*=\mathbf{Q}(t)(\nabla\mathbf{v})\mathbf{Q}^{\mathrm{T}}(t)+\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$. Thus

$(\nabla^*\mathbf{v}^*)^{\mathrm{T}}=\mathbf{Q}(t)(\nabla\mathbf{v})^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}(t)+\mathbf{Q}(\dot{\mathbf{Q}})^{\mathrm{T}}$. Now,

$(d/dt)(\mathbf{Q}\mathbf{Q}^{\mathrm{T}})=\mathbf{0}=\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}+\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}}\rightarrow\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}}=-\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$, $\rightarrow(\nabla^*\mathbf{v}^*)^{\mathrm{T}}=\mathbf{Q}(t)(\nabla\mathbf{v})^{\mathrm{T}}\mathbf{Q}^{\mathrm{T}}(t)-\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$.

Thus, $(\nabla^*\mathbf{v}^*)-(\nabla^*\mathbf{v}^*)^{\mathrm{T}}=\mathbf{Q}(t)[(\nabla\mathbf{v})-(\nabla\mathbf{v})^{\mathrm{T}}]\mathbf{Q}^{\mathrm{T}}(t)+2\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$

$$\rightarrow\mathbf{W}^*=\mathbf{Q}(t)\mathbf{W}\mathbf{Q}^{\mathrm{T}}(t)+\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}.$$

5.107 Show that in a change of frame, the material derivative of an objective tensor $\mathbf{T}$ transforms in accordance with the equation $\dot{\mathbf{T}}^*=\dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^{\mathrm{T}}(t)+\mathbf{Q}(t)\dot{\mathbf{T}}\mathbf{Q}^{\mathrm{T}}(t)+\mathbf{Q}(t)\mathbf{T}\dot{\mathbf{Q}}^{\mathrm{T}}$, where a super-dot indicates material derivative. Thus the material derivative of an objective tensor $\mathbf{T}$ is not objective.

Ans. Since $\mathbf{T}$ is objective, therefore, in a change of frame, $\mathbf{T}^*=\mathbf{Q}(t)\mathbf{T}\mathbf{Q}^{\mathrm{T}}(t)$. Taking the material derivative of this equation and noting that $t^*=t$, we have,

$\dot{\mathbf{T}}^*=\dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^{\mathrm{T}}+\mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^{\mathrm{T}}+\mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^{\mathrm{T}}$. Since $\dot{\mathbf{T}}^*\neq\mathbf{Q}(t)\dot{\mathbf{T}}\mathbf{Q}^{\mathrm{T}}(t)$, therefore, $\dfrac{D\mathbf{T}}{Dt}$ is non-objective.

5.108 The second Rivlin-Ericksen tensor is defined by:

$\mathbf{A}_2=\dot{\mathbf{A}}_1+\mathbf{A}_1(\nabla\mathbf{v})+(\nabla\mathbf{v})^{\mathrm{T}}\mathbf{A}_1$, where $\dot{\mathbf{A}}_1\equiv\dfrac{D}{Dt}\mathbf{A}_1$, where $\mathbf{A}_1=2\mathbf{D}=\nabla\mathbf{v}+(\nabla\mathbf{v})^{\mathrm{T}}$. Show that $\mathbf{A}_2$ is objective. [See Prob.**Error! Reference source not found.** and **Error! Reference source not found.**].

Ans. From Prob. 5.105, we had, $\mathbf{D}^* = \mathbf{Q}(t)\mathbf{D}\mathbf{Q}^{\mathrm{T}}(t) \to \mathbf{A}_1{}^* = \mathbf{Q}(t)\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}(t)$.

$$\to \dot{\mathbf{A}}_1{}^* = \dot{\mathbf{Q}}(t)\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}(t) + \mathbf{Q}(t)\dot{\mathbf{A}}_1\mathbf{Q}^{\mathrm{T}}(t) + \mathbf{Q}(t)\mathbf{A}_1\dot{\mathbf{Q}}^{\mathrm{T}}(t). \qquad \text{(i)}$$

We also have, from Eq.(5.56.13), $\nabla^*\mathbf{v}^* = \mathbf{Q}(t)(\nabla\mathbf{v})\mathbf{Q}^{\mathrm{T}}(t) + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$.

Thus $\mathbf{A}_1{}^*\nabla^*\mathbf{v}^* + (\nabla^*\mathbf{v}^*)^{\mathrm{T}}\mathbf{A}_1{}^*$

$$= \mathbf{Q}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}[\mathbf{Q}(\nabla\mathbf{v})\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}] + [\mathbf{Q}(\nabla\mathbf{v})\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}]^{\mathrm{T}}\mathbf{Q}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}$$

$$= [\mathbf{Q}\mathbf{A}_1(\nabla\mathbf{v})\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}(\dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}})] + [\mathbf{Q}(\nabla\mathbf{v})^{\mathrm{T}}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}}\mathbf{Q}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}].$$

Since $(D/Dt)\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = 0 \to \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}} = 0 \to \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} = -\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}}$, therefore,

$$\mathbf{A}_1{}^*\nabla^*\mathbf{v}^* + (\nabla^*\mathbf{v}^*)^{\mathrm{T}}\mathbf{A}_1{}^* =$$

$$[\mathbf{Q}\mathbf{A}_1(\nabla\mathbf{v})\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}(\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}})] + [\mathbf{Q}(\nabla\mathbf{v})^{\mathrm{T}}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}} - \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}]$$

i.e.,

$$\mathbf{A}_1{}^*\nabla^*\mathbf{v}^* + (\nabla^*\mathbf{v}^*)^{\mathrm{T}}\mathbf{A}_1{}^* = [\mathbf{Q}\mathbf{A}_1(\nabla\mathbf{v})\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}(\nabla\mathbf{v})^{\mathrm{T}}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}] - \mathbf{Q}\mathbf{A}_1\dot{\mathbf{Q}}^{\mathrm{T}} - \dot{\mathbf{Q}}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}} \qquad \text{(ii)}$$

(i) and (ii) give

$$\to \dot{\mathbf{A}}_1{}^* + \mathbf{A}_1{}^*\nabla^*\mathbf{v}^* + (\nabla^*\mathbf{v}^*)^{\mathrm{T}}\mathbf{A}_1{}^*$$

$$= \dot{\mathbf{Q}}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{A}}_1\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{A}_1\dot{\mathbf{Q}}^{\mathrm{T}} + [\mathbf{Q}\mathbf{A}_1(\nabla\mathbf{v})\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}(\nabla\mathbf{v})^{\mathrm{T}}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}] - \mathbf{Q}\mathbf{A}_1\dot{\mathbf{Q}}^{\mathrm{T}} - \dot{\mathbf{Q}}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}}$$

$$= \mathbf{Q}\dot{\mathbf{A}}_1\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{A}_1(\nabla\mathbf{v})\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}(\nabla\mathbf{v})^{\mathrm{T}}\mathbf{A}_1\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\left[\dot{\mathbf{A}}_1 + \mathbf{A}_1(\nabla\mathbf{v}) + (\nabla\mathbf{v})^{\mathrm{T}}\mathbf{A}_1\right]\mathbf{Q}^{\mathrm{T}}.$$

Thus, $\dot{\mathbf{A}}_1 + \mathbf{A}_1(\nabla\mathbf{v}) + (\nabla\mathbf{v})^{\mathrm{T}}\mathbf{A}_1$ is objective.

5.109 The Jaumann Derivative of a second order objective tensor $\mathbf{T}$ is : $\dot{\mathbf{T}} + \mathbf{TW} - \mathbf{WT}$, where $\mathbf{W}$ is the spin tensor. Show that the Jaumann derivative of $\mathbf{T}$ is objective. [See Prob. 5.106 and Prob. 5.107]

Ans. We have, since $\mathbf{T}$ is objective, therefore, in a change of frame, $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}$.

In Prob.5.106, we had $\mathbf{W}^* = \mathbf{Q}(t)\mathbf{W}\mathbf{Q}^{\mathrm{T}}(t) + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}$ and in Prob. 5.107, we had

$\dot{\mathbf{T}}^* = \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^{\mathrm{T}}(t) + \mathbf{Q}(t)\dot{\mathbf{T}}\mathbf{Q}^{\mathrm{T}}(t) + \mathbf{Q}(t)\mathbf{T}\dot{\mathbf{Q}}^{\mathrm{T}}$. Also,

$$(D/Dt)\mathbf{Q}\mathbf{Q}^{\mathrm{T}} = \mathbf{0} \to \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}} \to \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}} = -\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}}$$

Thus,

$$\mathbf{T}^*\mathbf{W}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}(-\mathbf{Q}\dot{\mathbf{Q}}^{\mathrm{T}}) = \mathbf{Q}\mathbf{T}\mathbf{W}\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^{\mathrm{T}},$$

$$\mathbf{W}^*\mathbf{T}^* = \mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}} = \mathbf{Q}\mathbf{W}\mathbf{T}\mathbf{Q}^{\mathrm{T}} + \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^{\mathrm{T}}.$$

$$\to \mathbf{T}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{T}^* = \mathbf{Q}(\mathbf{TW} - \mathbf{WT})\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^{\mathrm{T}} - \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^{\mathrm{T}}.$$

Thus,

$$\dot{\mathbf{T}}^* + \mathbf{T}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{T}^* = \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^{\mathrm{T}} + \mathbf{Q}(\mathbf{T}\mathbf{W} - \mathbf{W}\mathbf{T})\mathbf{Q}^{\mathrm{T}} - \mathbf{Q}\mathbf{T}\dot{\mathbf{Q}}^{\mathrm{T}} - \dot{\mathbf{Q}}\mathbf{T}\mathbf{Q}^{\mathrm{T}}$$
$$= \mathbf{Q}\dot{\mathbf{T}}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}(\mathbf{T}\mathbf{W} - \mathbf{W}\mathbf{T})\mathbf{Q}^{\mathrm{T}}.$$

That is, $\dot{\mathbf{T}}^* + \mathbf{T}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{T}^* = \mathbf{Q}\left(\dot{\mathbf{T}} + \mathbf{T}\mathbf{W} - \mathbf{W}\mathbf{T}\right)\mathbf{Q}^{\mathrm{T}}$.

Therefore, the Jaumann derivative of $\mathbf{T}$, that is, $\left(\dot{\mathbf{T}} + \mathbf{T}\mathbf{W} - \mathbf{W}\mathbf{T}\right)$ is objective.

5.110 The second Piola Kirchhoff stress tensor $\tilde{\mathbf{T}}$ is related to the first Piola-Kirchhoff stress tensor $\mathbf{T}_o$ by the formula $\tilde{\mathbf{T}} = \mathbf{F}^{-1}\mathbf{T}_o$, or to the Cauchy stress tensor $\mathbf{T}$ by $\tilde{\mathbf{T}} = (\det\mathbf{F})\mathbf{F}^{-1}\mathbf{T}(\mathbf{F}^{-1})^{\mathrm{T}}$ Show that, in a change of frame, $\tilde{\mathbf{T}}^* = \tilde{\mathbf{T}}$. [See Example 5.56.3 and Example 5.57.1]

Ans. In Example 5.56.3 and Example 5.57.1, we obtained that in a change of frame, $\mathbf{F}^* = \mathbf{Q}(t)\mathbf{F}$ and $\mathbf{T}_o^* = \mathbf{Q}\mathbf{T}_o$. Thus,

$$\tilde{\mathbf{T}}^* = \mathbf{F}^{*-1}\,\mathbf{T}_o^* = (\mathbf{Q}(t)\mathbf{F})^{-1}\mathbf{Q}\mathbf{T}_o = \mathbf{F}^{-1}\mathbf{Q}^{-1}\mathbf{Q}\mathbf{T}_o = \mathbf{F}^{-1}\mathbf{T}_o. \text{ That is, } \tilde{\mathbf{T}}^* = \tilde{\mathbf{T}}.$$

5.111 Starting from the constitutive assumption that $\mathbf{T} = \mathbf{H}(\mathbf{F})$ and $\mathbf{T}^* = \mathbf{H}(\mathbf{F}^*)$, where $\mathbf{T}$ is Cauchy stress and $\mathbf{F}$ is deformation gradient, show that in order that the assumption be independent of observers, $\mathbf{H}(\mathbf{F})$ must transform in accordance with the equation $\mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}} = \mathbf{H}(\mathbf{Q}\mathbf{F})$. (b) Choose $\mathbf{Q} = \mathbf{R}^{\mathrm{T}}$ to obtain $\mathbf{T} = \mathbf{R}\mathbf{H}(\mathbf{U})\mathbf{R}^{\mathrm{T}}$, where $\mathbf{R}$ is the rotation tensor associated with $\mathbf{F}$ and $\mathbf{U}$ is the right stretch tensor. (c) Show that $\tilde{\mathbf{T}} = \mathbf{h}(\mathbf{U})$, where $\mathbf{h} = (\det\mathbf{U})\mathbf{U}^{-1}\mathbf{H}(\mathbf{U})\mathbf{U}^{-1}$. Since $\mathbf{C} = \mathbf{U}^2$, therefore, we may write $\mathbf{T} = \mathbf{f}(\mathbf{C})$.

Ans. (a) In a change of frame, $\mathbf{T}^* = \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}$ and $\mathbf{F}^* = \mathbf{Q}\mathbf{F}$, therefore,

$$\mathbf{T}^* = \mathbf{H}(\mathbf{F}^*) \to \mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}} = \mathbf{H}(\mathbf{Q}\mathbf{F}).$$

(b) From $\mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}} = \mathbf{H}(\mathbf{Q}\mathbf{F})$, with $\mathbf{Q} = \mathbf{R}^{\mathrm{T}} \to \mathbf{R}^{\mathrm{T}}\mathbf{T}\mathbf{R} = \mathbf{H}(\mathbf{R}^{\mathrm{T}}\mathbf{F})$. But $\mathbf{R}^{\mathrm{T}}\mathbf{F} = \mathbf{R}^{\mathrm{T}}(\mathbf{R}\mathbf{U}) = \mathbf{U}$ where $\mathbf{U}$ is the right stretch tensor. Therefore, $\mathbf{R}^{\mathrm{T}}\mathbf{T}\mathbf{R} = \mathbf{H}(\mathbf{R}^{\mathrm{T}}\mathbf{F}) \to \mathbf{T} = \mathbf{R}\mathbf{H}(\mathbf{U})\mathbf{R}^{\mathrm{T}}$.

(c) $\mathbf{F} = \mathbf{R}\mathbf{U} \to \mathbf{R} = \mathbf{F}\mathbf{U}^{-1} \to \mathbf{R}^{\mathrm{T}} = \mathbf{U}^{-1}\mathbf{F}^{\mathrm{T}}$, thus,

$$\mathbf{T} = \mathbf{R}\mathbf{H}(\mathbf{U})\mathbf{R}^{\mathrm{T}} \to \mathbf{T} = \mathbf{F}\mathbf{U}^{-1}\mathbf{H}(\mathbf{U})\mathbf{U}^{-1}\mathbf{F}^{\mathrm{T}} \to \mathbf{F}^{-1}\mathbf{T}(\mathbf{F}^{\mathrm{T}})^{-1} = \mathbf{U}^{-1}\mathbf{H}(\mathbf{U})\mathbf{U}^{-1}.$$

Now since $J = |\det\mathbf{F}| = \det\mathbf{U}$, we can write $J\mathbf{F}^{-1}\mathbf{T}(\mathbf{F}^{\mathrm{T}})^{-1} = (\det\mathbf{U})\mathbf{U}^{-1}\mathbf{H}(\mathbf{U})\mathbf{U}^{-1}$.

The left side of the above equation is the second Piola-Kirchhoff stress tensor $\tilde{\mathbf{T}}$ and the right side is a function of the right stretch tensor $\mathbf{U}$. Thus, $\tilde{\mathbf{T}} = \mathbf{h}(\mathbf{U})$, or since $\mathbf{U}^2 = \mathbf{C}$, one can write $\tilde{\mathbf{T}} = \mathbf{h}(\mathbf{C})$.

5.112 From $r=\left(2\alpha X+\beta\right)^{1/2}$, $\theta = cY$, $z=Z$, $\alpha = 1/c$, obtain the right Cauchy-Green deformation tensor $\mathbf{B}$.

Ans., we have, with $r=\left(2\alpha X+\beta\right)^{1/2}$, $\theta = cY$, $z=Z$,

$$\frac{\partial r}{\partial X}=\frac{1}{2}\left(2\alpha X+\beta\right)^{-1/2}\left(2\alpha\right)=\alpha\left(2\alpha X+\beta\right)^{-1/2}=\frac{\alpha}{r},\quad \frac{\partial r}{\partial Y}=\frac{\partial r}{\partial Z}=0$$

$$\frac{\partial \theta}{\partial X}=0,\quad \frac{\partial \theta}{\partial Y}=c,\quad \frac{\partial \theta}{\partial Z}=0;\qquad \frac{\partial z}{\partial X}=0,\quad \frac{\partial z}{\partial Y}=0,\quad \frac{\partial z}{\partial Z}=1$$

Thus, Using Eq.3.29.59 to 3.29.64,

$$B_{rr}=\left(\frac{\partial r}{\partial X}\right)^2+\left(\frac{\partial r}{\partial Y}\right)^2+\left(\frac{\partial r}{\partial Z}\right)^2=\left(\frac{\alpha}{r}\right)^2,\quad B_{\theta\theta}=\left(\frac{r\partial \theta}{\partial X}\right)^2+\left(\frac{r\partial \theta}{\partial Y}\right)^2+\left(\frac{r\partial \theta}{\partial Z}\right)^2=\left(rc\right)^2$$

$$B_{zz}=\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial Y}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2=1,\quad B_{r\theta}=\left(\frac{\partial r}{\partial X}\right)\left(\frac{r\partial \theta}{\partial X}\right)+\left(\frac{\partial r}{\partial Y}\right)\left(\frac{r\partial \theta}{\partial Y}\right)+\left(\frac{\partial r}{\partial Z}\right)\left(\frac{r\partial \theta}{\partial Z}\right)=0$$

$$B_{rz}=\left(\frac{\partial r}{\partial X}\right)\left(\frac{\partial z}{\partial X}\right)+\left(\frac{\partial r}{\partial Y}\right)\left(\frac{\partial z}{\partial Y}\right)+\left(\frac{\partial r}{\partial Z}\right)\left(\frac{\partial z}{\partial Z}\right)=0,$$

$$B_{\theta z}=\left(\frac{r\partial \theta}{\partial X}\right)\left(\frac{\partial z}{\partial X}\right)+\left(\frac{r\partial \theta}{\partial Y}\right)\left(\frac{\partial z}{\partial Y}\right)+\left(\frac{r\partial \theta}{\partial Z}\right)\left(\frac{\partial z}{\partial Z}\right)=0$$

5.113 From $r=\lambda_1 R$, $\theta=\Theta+KZ$, $z=\lambda_3 Z$, $\lambda_1^2\lambda_3=1$, obtain the right Cauchy-Green deformation tensor $\mathbf{B}$.

Ans. With $r=\lambda_1 R$, $\theta=\Theta+KZ$, $z=\lambda_3 Z$, $\lambda_1^2\lambda_3=1$, we have,

$$r=\lambda_1 R,\quad \theta=\Theta+KZ,\quad z=\lambda_3 Z,\quad \lambda_1^2\lambda_3=1,$$

$$\frac{\partial r}{\partial R}=\lambda_1,\quad \frac{\partial r}{\partial \Theta}=0,\quad \frac{\partial r}{\partial Z}=0,\quad \frac{\partial \theta}{\partial R}=0,\quad \frac{\partial \theta}{\partial \Theta}=1,\quad \frac{\partial \theta}{\partial Z}=K,$$

$$\frac{\partial z}{\partial R}=0,\quad \frac{\partial z}{\partial \Theta}=0,\quad \frac{\partial z}{\partial Z}=\lambda_3.$$

Using Eq. (3.29.19) to Eq. (3.29.24) and noting that $r_o\equiv R$, $\theta_o\equiv\Theta$, $z_o\equiv Z$,

$$B_{rr}=\left(\frac{\partial r}{\partial R}\right)^2+\left(\frac{\partial r}{R\partial \Theta}\right)^2+\left(\frac{\partial r}{\partial Z}\right)^2=(\lambda_1)^2.$$

$$B_{r\theta}=\left(\frac{r\partial \theta}{\partial R}\right)\left(\frac{\partial r}{\partial R}\right)+\left(\frac{r\partial \theta}{R\partial \Theta}\right)\left(\frac{\partial r}{R\partial \Theta}\right)+\left(\frac{r\partial \theta}{\partial Z}\right)\left(\frac{\partial r}{\partial Z}\right)=0=B_{\theta r}.$$

$$B_{\theta\theta}=\left(\frac{r\partial \theta}{\partial R}\right)^2+\left(\frac{r\partial \theta}{R\partial \Theta}\right)^2+\left(\frac{r\partial \theta}{\partial Z}\right)^2=\left(\frac{r}{R}\right)^2+\left(rK\right)^2=\left(\lambda_1\right)^2+\left(rK\right)^2.$$

$$B_{zz}=\left(\frac{\partial z}{\partial R}\right)^2+\left(\frac{\partial z}{R\partial \Theta}\right)^2+\left(\frac{\partial z}{\partial Z}\right)^2=(\lambda_3)^2.$$

$$B_{rz} = \left(\frac{\partial r}{\partial R}\right)\left(\frac{\partial z}{\partial R}\right) + \left(\frac{\partial r}{R\partial \Theta}\right)\left(\frac{\partial z}{R\partial \Theta}\right) + \left(\frac{\partial r}{\partial Z}\right)\left(\frac{\partial z}{\partial Z}\right) = 0 = B_{zr}\ .$$

$$B_{z\theta} = \left(\frac{\partial z}{\partial R}\right)\left(\frac{r\partial \theta}{\partial R}\right) + \left(\frac{\partial z}{R\partial \Theta}\right)\left(\frac{r\partial \theta}{R\partial \Theta}\right) + \left(\frac{\partial z}{\partial Z}\right)\left(\frac{r\partial \theta}{\partial Z}\right) = \left(\frac{\partial z}{\partial Z}\right)\left(\frac{r\partial \theta}{\partial Z}\right) = \lambda_3 rK = B_{\theta z}\ .$$

CHAPTER 6

6.1 In Figure P 6-1, the gate AB is rectangular with width $b = 60\ cm$ and length L= 4 m. The gate is hinged at the upper edge A. Neglect the weight of the gate, find the reactional force at B. Take the specific weight of water to be 9800 N/m^3 and neglect frictions.

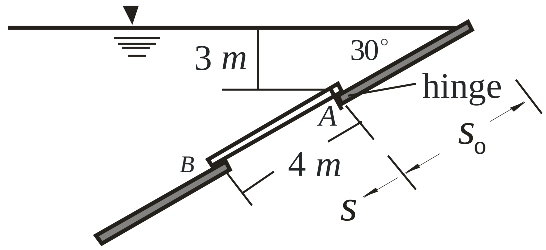

Figure P 6-1

Ans. Take the gate AB as a free body. With s_o measured from the water surface along the inclined plane to point A, s measured from point A along the length of the plate (AB) and $\alpha = 30^\circ$ as shown in the figure, we have,

$dF = pdA = \left[\rho g\left(s_o + s\right)\sin\alpha\right](bds)$, thus ,

$$\sum M_A = 0 \rightarrow R_B L = \int_A sdF = b\rho g\sin\alpha\int_0^L s\left(s_o + s\right)ds = b\rho g\sin\alpha\left\{s_o\frac{L^2}{2} + \frac{L^3}{3}\right\}.$$ Therefore,

$$R_B = bL\rho g\left\{\frac{s_o\sin\alpha}{2} + \frac{L\sin\alpha}{3}\right\} = (0.6)(4)(9800)\left\{\frac{3}{2} + \frac{4(0.5)}{3}\right\} = 5.1\times 10^4\ N.$$

6.2 The gate AB in Figure P 6-2 is 5 m long and 3 m wide. Neglect the weight of the gate, compute the water level h for which the gate will start to fall. Take the specific weight of water to be 9800 N/m^3.

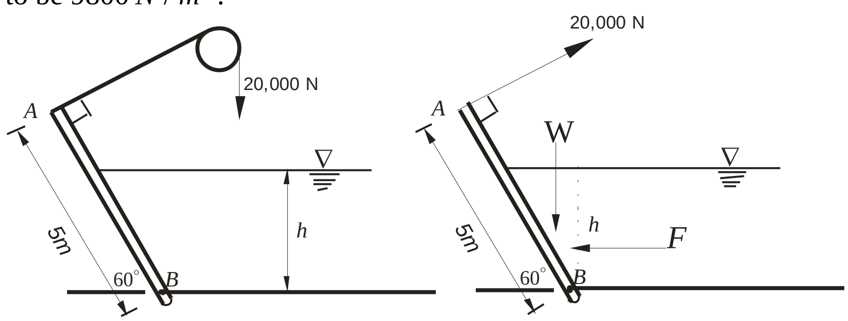

Figure P 6-2

Ans. Consider the gate plus the triangular region of water above the gate as the free body diagram. Then,

Horizontal force from water to gate: $F = \rho g(h/2)(bh) = \rho gbh^2/2$ acting at 1/3 from base.

Weight of water on gate: $W = \rho gb(1/2)[h(h\tan 30^o)] = \rho gbh^2/(2\sqrt{3})$.

$\sum M_B = 0 \to W(1/3)(h/\sqrt{3}) + Fh/3 = P(AB) \to$

$h^3 = \dfrac{9}{2}\dfrac{P(AB)}{\rho gb} = \dfrac{9}{2}\dfrac{(20000)(5)}{(9800)(3)} = 15.31 \to h = 2.48m.$

6.3 The liquids in the U-tube shown in Figure P 6-3 is in equilibrium. Find h_2 as a function of $\rho_1, \rho_2, \rho_3, h_1$ and h_3. The liquids are immiscible.

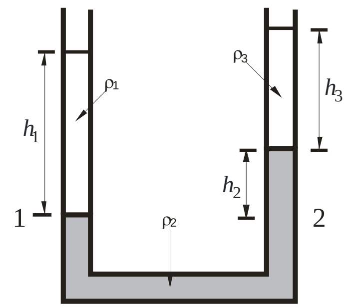

Figure P 6-3

Ans. $p_1 = \rho_1 gh_1,\ p_2 = \rho_3 gh_3 + \rho_2 gh_2$,

$p_1 = p_2 \to \rho_1 gh_1 = \rho_3 gh_3 + \rho_2 gh_2 \to h_2 = (\rho_1 h_1 - \rho_3 h_3)/\rho_2$.

6.4 In Figure P 6-4, ,the weight W_R is supported by the weight W_L, via the liquids in the container. The area under W_R is twice that under W_L. Find W_R in terms of W_L, ρ_1, ρ_2, A_L, and h $(\rho_2 < \rho_1$ and assume no mixing$)$.

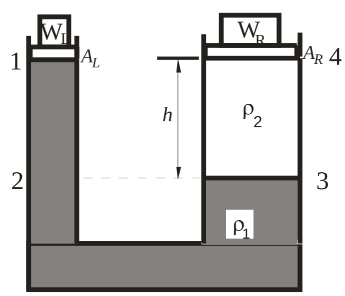

Figure P 6-4

Ans. $p_3 = p_2 = p_1 + \rho_1 gh$, $p_4 = p_3 - \rho_2 gh = p_1 + (\rho_1 - \rho_2)gh$

$W_R = p_4 A_R = p_1 A_R + (\rho_1 - \rho_2)ghA_R$, i.e.,

$W_R = 2p_1 A_L + 2(\rho_1 - \rho_2)ghA_L = 2W_L + 2(\rho_1 - \rho_2)ghA_L$

6.5 Referring to Figure P 6-5, the radius and length of the cylinder are r and L respectively, The specific weight of the liquid is γ .

(a) Find the buoyancy force on the cylinder and
(b) Find the resultant force on the cylindrical surface due to the water pressure. The centroid of a semi-circular area is $4r/3\pi$ from the diameter.

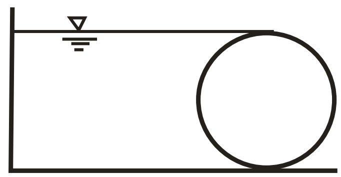

Figure P 6-5

Ans. **(a)** Buoyancy force is the net upward force due to the water pressure on the left half of the boundary of the cylinder which is submerged in the water. It is therefore equal to the weight of the water displaced by this left half. That is, Buoyancy force = $\gamma(\pi r^2/2)L$ **.**

(b) Horizontal water force: $F_x = \gamma(2r/2)(2rL) = \gamma(2r^2L)$. The line of action of F_x is $2r/3$ above the ground. The line of action of F_y (the buoyancy force) passes through the centroid of the semi-circular area, i.e., $4r/3\pi$ left of the diameter.

6.6 A glass of water moves vertically upward with a constant acceleration a . Find the pressure at a point whose depth from the surface of the water is h. Take the atmospheric pressure to be p_a .

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*Ans***.** Let z axis be appointing vertically upward, then

$$-\frac{dp}{dz} - \rho g = \rho a \rightarrow -\frac{dp}{dz} = \rho(g+a) \rightarrow p = -\rho(g+a)z + C .$$

At the instant of interest, let the origin be at the free surface, then $C = p_a$, the atmospheric pressure. Thus, $p - p_a = -\rho(g+a)z$. At a point which is at $z = -h$, $p - p_a = \rho(g+a)h$.

6.7 A glass of water moves with a constant acceleration $\boldsymbol{a}$ in the direction shown in Figure P 6-6. (a) Show that the free surface is a plane and find its angle of inclination and (b) find the pressure at the point A. Take the atmospheric pressure to be p_a .

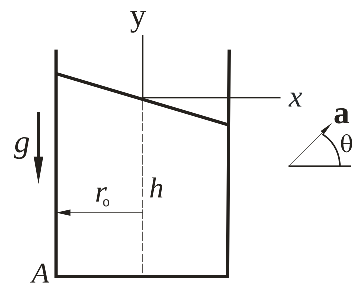

Figure P 6-6

Ans. (a) With respect to the coordinates shown, the governing equations are:

(i) $-\frac{\partial p}{\partial x}=\rho a\cos\theta$, (ii) $-\frac{\partial p}{\partial y}-\rho g=\rho a\sin\theta$, (iii) $-\frac{\partial p}{\partial z}=0$, thus

(iii) $\rightarrow p=p(x,y)$, (i) $\rightarrow p=-(\rho a\cos\theta)x+f(y)\rightarrow\frac{\partial p}{\partial y}=\frac{df}{dy}$.

(ii) $\rightarrow\frac{df}{dy}=-\rho(g+a\sin\theta)\rightarrow f=-\rho(g+a\sin\theta)y+C\rightarrow p=-(\rho a\cos\theta)x-\rho(g+a\sin\theta)y+C$

.

At the instant of interest, let the origin be at the center of the surface, then $C=p_a$ .and $p=-(\rho a\cos\theta)x-\rho(g+a\sin\theta)y+p_a$. On every point on the free surface, $p=p_a$, therefore, $-(\rho a\cos\theta)x-\rho(g+a\sin\theta)y=0$. Thus, the free surface is a plane. The angle of inclinations is given by $\tan\beta=\frac{dy}{dx}=-\frac{a\cos\theta}{g+a\sin\theta}$.

(b) At the point A, $x=-r_o, y=-h$. Thus, $p=(\rho a\cos\theta)r_o+\rho(g+a\sin\theta)h+p_a$

6.8 The slender U-tube shown in Figure P 6-7 moves horizontally to the right with an acceleration a. Determine the relation between a,ℓ and h.

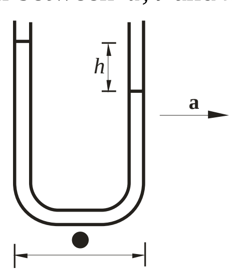

Figure P 6-7

Ans. The slope of the free surface is given by $-\frac{a}{g}$. Thus $-\frac{a}{g}=-\frac{h}{\ell}\rightarrow h=\frac{a\ell}{g}$.

6.9 A liquid in a container rotates with a constant angular velocity ω about a vertical axis. Show that the free surface is a paraboloid given by $z = r^2\omega^2/(2g)$ where the origin is on the axis of rotation and z is measured upward from the lowest point of the free surface.

Ans. Let z be pointing vertically upward with the origin at the lowest point of the free surface. We have,

$$\text{(i)} -\frac{\partial p}{\partial r} = \rho\left(-r\omega^2\right) \text{ and (ii)} -\frac{\partial p}{\partial z} - \rho g = 0 \quad \to p = -\rho g z + f(r) \to \frac{\partial p}{\partial r} = \frac{df}{dr}.$$

$$\text{(i)} \to \frac{\partial p}{\partial r} = \frac{df}{dr} = \rho r\omega^2 \to f = \frac{\rho r^2\omega^2}{2} + C, \ \to p = -\rho g z + \frac{\rho r^2\omega^2}{2} + C.$$

At $(r,z) = (0,0)$, $p = p_a$, therefore, $p = -\rho g z + \dfrac{\rho r^2\omega^2}{2} + p_a$. The free surface is characterized by $p = p_a$, therefore, the equation of the surface is: $z = \dfrac{r^2\omega^2}{2g}$.

6.10 The slender U-tube rotates with an angular velocity ω about the vertical axis shown in Figure P 6-8. Find the relation between $\delta h (\equiv h_1 - h_2), \omega, r_1$ and r_2.

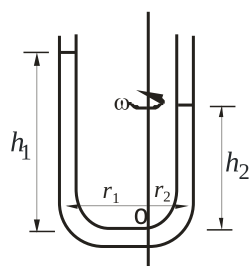

Figure P 6-8

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*Ans***.** The equation for the free surface is given by (see the previous problem) $z = r^2\omega^2/(2g)$, where the origin is on the axis of rotation and z is measured upward from the lowest point of the free surface. Thus, we have,

$$z_1 = \frac{r_1^2\omega^2}{2g} \text{ and } z_2 = \frac{r_2^2\omega^2}{2g} \to z_1 - z_2 = \left(r_1^2 - r_2^2\right)\frac{\omega^2}{2g}, \quad \text{but}, z_1 - z_2 = h_1 - h_2$$

$$\to h_1 - h_2 = (r_1^2 - r_2^2)\omega^2/(2g).$$

6.11 For minor altitude differences, the atmosphere can be assumed to have constant temperature. Find the pressure and density distribution for this case. The pressure p, density ρ and absolute temperature Θ are related by the ideal gas law $p = \rho R\Theta$.

Ans.: Let gravity be in the negative x_3 direction, then we have

$$\partial p/\partial x_1 = 0, \quad \partial p/\partial x_2 = 0, \quad \partial p/\partial x_3 = -\rho g \qquad \text{(i)}$$

Thus, p depends only on x_3. Let p_o denote the pressure at $x=0$, then, we have

$$\frac{dp}{dx_3}=-\rho g \rightarrow \frac{dp}{p}=-\frac{g}{R\Theta}dx_3 \rightarrow \ln p=-\frac{g}{R\Theta}x_3+\ln p_o \rightarrow p=p_o e^{-(g/R\Theta)x_3} \qquad \text{(ii)}$$

If ρ_o is the density at $x_3=0$, then $\rho=\rho_o e^{-(g/R\Theta)x_3}$.

6.12 In astrophysical applications, an atmosphere having the relation between the density ρ and the pressure p given by $p/p_o=(\rho/\rho_o)^n$, where p_o and ρ_o are some reference pressure and density, is known as a polytropic atmosphere. Find the distribution of pressure and density in a polytropic atmosphere.

Ans. Let z axis point upward, then $dp/dz=-\rho g$. From $p/p_o=(\rho/\rho_o)^n$, we have,

$\rho=Cp^{1/n}$, where $C=\rho_o p_o^{-1/n}$. Thus, $dp/dz=-Cp^{1/n}g \rightarrow p^{-1/n}dp=-Cgdz$

$\rightarrow \int_{p_o}^{p} p^{-1/n}dp=-\int_{z_o}^{z} Cgdz$. Thus,

(A) for $n \neq 1$,

$$[n/(n-1)]p^{(n-1)/n}\Big|_{p_o}^{p}=-Cgz\Big|_{z_o}^{z} \rightarrow p^{(n-1)/n}-p_o^{(n-1)/n}=-[(n-1)/n]Cg(z-z_o)\rightarrow$$

$$p^{(n-1)/n}=p_o^{(n-1)/n}-[(n-1)/n]\left(\rho_o p_o^{-1/n}\right)g(z-z_o)=p_o^{-1/n}\left[p_o-\{(n-1)/n\}\rho_o g(z-z_o)\right]$$

$$p=p_o^{-1/(n-1)}\left[p_o-\frac{n-1}{n}\rho_o g(z-z_o)\right]^{n/(n-1)}.$$

(B) for $n=1$,

$$\int_{p_o}^{p}\frac{dp}{p}=-\int_{z_o}^{z}Cgdz \rightarrow \ln(p/p_o)=-Cg(z-z_o)\rightarrow p=p_o\exp\left[-Cg(z-z_o)\right]$$

$$\rightarrow p=p_o\exp\left[-\rho_o p_o^{-1/n}g(z-z_o)\right]$$

6.13 Given the following velocity field for a Newtonian liquid with viscosity μ=0.982 *mPa.s* $\left(2.05\times10^{-5}\,\text{lb}\times\text{s/ft}^2\right)$:

$$v_1=-c(x_1+x_2),\ \ v_2=c(x_2-x_1),\ \ v_3=0,\ \ c=1\ s^{-1}$$

For a plane whose normal is in the $\mathbf{e}_1$ direction, (a) find the excess of the total normal compressive stress over the pressure *p*, and (b) find the magnitude of the shearing stress.

***Ans*.** (a) $T_{11}=-p+2\mu D_{11}\rightarrow(-T_{11})-p=-2\mu D_{11}$, where $D_{11}=\frac{\partial v_1}{\partial x_1}=-c=-1\ \text{s}^{-1}$,

Thus, $(-T_{11})-p=-2(0.982)(-1)=1.96$ *mPa.*

(b) $T_{12}=2\mu D_{12}=\mu\left(\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_1}\right)=-2c\mu=-1.96$ *mPa.* $T_{13}=2\mu D_{13}=\mu\left(\frac{\partial v_1}{\partial x_3}+\frac{\partial v_3}{\partial x_1}\right)=0$.

Thus, the magnitude of shearing stress = 1.96 *mPa*.

6.14 For a steady parallel flow of an incompressible linearly viscous fluid, if we take the flow direction to be $\mathbf{e}_3$, (a) show that the velocity field is of the form

$$v_1 = 0,\quad v_2 = 0 \quad \text{and} \quad v_3 = v(x_1, x_2)$$

(b) If $v(x_1, x_2) = kx_2$, find the normal and shear stresses on the plane whose normal is in the direction of $\mathbf{e}_2 + \mathbf{e}_3$ in terms of viscosity μ and pressure p.
(c) On what planes are the total normal stresses given by p.

Ans. (a) From the equation of continuity $\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0$, we get, $\frac{\partial v_3}{\partial x_3} = 0$, thus v_3 is independent of x_3 i.e., $v_3 = v(x_1, x_2)$.

(b) with $v_1 = 0,\ v_2 = 0$ and $v_3 = kx_2$, we have,

$$[\nabla \mathbf{v}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & k & 0 \end{bmatrix} \rightarrow [\mathbf{D}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k/2 \\ 0 & k/2 & 0 \end{bmatrix} \rightarrow [\mathbf{T}] = -p[\mathbf{I}] + 2\mu[\mathbf{D}] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & \mu k \\ 0 & \mu k & -p \end{bmatrix}$$

On the plane with $\mathbf{n} = (\mathbf{e}_2 + \mathbf{e}_3)/\sqrt{2}$,

$$[\mathbf{t}] = [\mathbf{T}][\mathbf{n}] = \frac{1}{\sqrt{2}} \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & \mu k \\ 0 & \mu k & -p \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ \mu k - p \\ \mu k - p \end{bmatrix} \rightarrow T_n = \mathbf{t} \cdot \mathbf{n} = \mu k - p.$$

$$T_s^2 = |\mathbf{t}|^2 - (T_n)^2 = \frac{1}{2}\left[(\mu k - p)^2 + (\mu k - p)^2\right] - (\mu k - p)^2 = 0.$$

(c)

$$[\mathbf{t}] = \begin{bmatrix} -p & 0 & 0 \\ 0 & -p & \mu k \\ 0 & \mu k & -p \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} -pn_1 \\ -pn_2 + \mu k n_3 \\ \mu k n_2 - pn_3 \end{bmatrix}, \qquad n_1^2 + n_2^2 + n_3^2 = 1$$

$\rightarrow T_n = \mathbf{t} \cdot \mathbf{n} = -pn_1^2 + \left(-pn_2^2 + \mu k n_3 n_2\right) + \left(\mu k n_2 n_3 - pn_3^2\right) = -p + 2\mu k n_3 n_2$. Thus,

$-p + 2\mu k n_3 n_2 = -p \rightarrow n_2 = 0$ and/or $n_3 = 0$

That is, on any plane $(n_1, 0, n_3)$ and $(n_1, n_2, 0)$, where $n_1^2 + n_2^2 + n_3^2 = 1$, the normal component of stress is $-p$, these include the three coordinate planes $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

6.15 Given the following velocity field for a Newtonian incompressible fluid with a viscosity $\mu = 0.96\ mPa.s$:

$$v_1 = k\left(x_1^2 - x_2^2\right),\ v_2 = -2kx_1x_2,\ v_3 = 0,\ k = 1\ s^{-1}m^{-1}.$$

At the point (1,2,1)*m* and on the plane whose normal is in the direction of $\mathbf{e}_1$,
(a) find the excess of the total normal compressive stress over the pressure *p* and
(b) find the magnitude of the shearing stress.

Ans. (a)

$$[\nabla \mathbf{v}] = \begin{bmatrix} 2kx_1 & -2kx_2 & 0 \\ -2kx_2 & -2kx_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\mathbf{D}], \quad [\mathbf{T}] = \begin{bmatrix} -p+4\mu kx_1 & -4\mu kx_2 & 0 \\ -4\mu kx_2 & -p-4\mu kx_1 & 0 \\ 0 & 0 & -p \end{bmatrix}.$$

At $(1,2,1)$ and for $k=1$, $[\mathbf{T}] = -\begin{bmatrix} -p+4\mu & -8\mu & 0 \\ -8\mu & -p-4\mu & 0 \\ 0 & 0 & -p \end{bmatrix}$.

On $\mathbf{e}_1$-plane $(-T_{11}) - p = -4\mu = -3.84\ mPa.$

(b) on the same plane, $|T_s| = 8\mu = 7.68\ mPa.$

6.16 Do Problem 6.15 except that the plane has a normal in the direction $3\mathbf{e}_1 + 4\mathbf{e}_2$.

Ans.

$$[\nabla \mathbf{v}] = \begin{bmatrix} 2kx_1 & -2kx_2 & 0 \\ -2kx_2 & -2kx_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\mathbf{D}] \to [\mathbf{T}] = \begin{bmatrix} -p+4\mu kx_1 & -4\mu kx_2 & 0 \\ -4\mu kx_2 & -p-4\mu kx_1 & 0 \\ 0 & 0 & -p \end{bmatrix}.$$

At $(1,2,1)$ and for $k=1$, $[\mathbf{T}] = -\begin{bmatrix} -p+4\mu & -8\mu & 0 \\ -8\mu & -p-4\mu & 0 \\ 0 & 0 & -p \end{bmatrix}$,

$$[\mathbf{t}] = [\mathbf{T}][\mathbf{n}] = \begin{bmatrix} -p+4\mu & -8\mu & 0 \\ -8\mu & -p-4\mu & 0 \\ 0 & 0 & -p \end{bmatrix} \begin{bmatrix} 3/5 \\ 4/5 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -3p-20\mu \\ -4p-40\mu \\ 0 \end{bmatrix}.$$

$$T_n = \mathbf{t}\cdot\mathbf{n} = \frac{1}{25}\left[(-3p-20\mu)3 + (-4p-40\mu)4\right] = -p - \frac{44\mu}{5}. \qquad (-T_n) - p = \frac{44\mu}{5}.$$

(b) $|\mathbf{t}|^2 = \frac{1}{25}\left[(-3p-20\mu)^2 + (-4p-40\mu)^2\right] = \frac{1}{25}\left(25p^2 + 440p\mu + 2000\mu^2\right)$

$$(T_n)^2 = p^2 + \frac{88\mu p}{5} + \frac{1936}{25}\mu^2. \qquad (T_s)^2 = |\mathbf{t}|^2 - (T_n)^2 = \frac{64}{25}\mu^2 \to T_s = \frac{8\mu}{5}.$$

6.17 Use the results of Sect. 2.34., chapter 2 and the constitutive equations for the Newtonian viscous fluid, verify the Navier Stokes Equation in the r-direction for cylindrical coordinates, i.e., Eq. (6.8.1).

Ans. For a Newtonian fluid, the stress tensor in cylindrical coordinates is given by:

$$[\mathbf{T}]=\begin{bmatrix} -p+2\mu\frac{\partial v_r}{\partial r} & \mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta}-\frac{\mathrm{v}_\theta}{r}+\frac{\partial v_\theta}{\partial r}\right) & \mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right) \\ T_{21} & -p+2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right) & \mu\left(\frac{\partial v_\theta}{\partial z}+\frac{1}{r}\frac{\partial v_z}{\partial \theta}\right) \\ T_{31} & T_{32} & -p+2\mu\frac{\partial v_z}{\partial z} \end{bmatrix}$$

The Equations of motion in terms of the stress components in the r-direction is [see Eq.(4.8.1)]:

$$\frac{\partial T_{rr}}{\partial r}+\frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta}+\frac{T_{rr}-T_{\theta\theta}}{r}+\frac{\partial T_{rz}}{\partial z}+\rho B_r=\rho a_r$$

We also have the equation of continuity [see Eq.(2.34.6) or Eq.(6.8.4)]:

$$\frac{\partial v_r}{\partial r}+\left(\frac{1}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r}{r}\right)+\frac{\partial v_z}{\partial z}=0$$

Now, $T_{rr}=-p+2\mu\frac{\partial v_r}{\partial r}\rightarrow\frac{\partial T_{rr}}{\partial r}=-\frac{\partial p}{\partial r}+2\mu\frac{\partial^2 v_r}{\partial r^2}$

$$T_{r\theta}=\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta}-\frac{\mathrm{v}_\theta}{r}+\frac{\partial v_\theta}{\partial r}\right)\rightarrow\frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta}=\mu\left(\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{1}{r^2}\frac{\partial \mathrm{v}_\theta}{\partial \theta}+\frac{1}{r}\frac{\partial^2 \mathrm{v}_\theta}{\partial r\partial \theta}\right)$$

$$\frac{T_{rr}-T_{\theta\theta}}{r}=2\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{v_r}{r^2}\right)$$

$$T_{rz}=\mu\left(\frac{\partial v_r}{\partial z}+\frac{\partial v_z}{\partial r}\right)\rightarrow\frac{\partial T_{rz}}{\partial z}=\mu\left(\frac{\partial^2 v_r}{\partial z^2}+\frac{\partial^2 v_z}{\partial r\partial z}\right)$$

Thus,

$$(\mathrm{div}\mathbf{T})_r=\frac{\partial T_{rr}}{\partial r}+\frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta}+\frac{T_{rr}-T_{\theta\theta}}{r}+\frac{\partial T_{rz}}{\partial z}=-\frac{\partial p}{\partial r}+2\mu\frac{\partial^2 v_r}{\partial r^2}$$

$$+\mu\left(\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{1}{r^2}\frac{\partial \mathrm{v}_\theta}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial \theta}\right)+2\mu\left(\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{v_r}{r^2}\right)+\mu\left(\frac{\partial^2 v_r}{\partial z^2}+\frac{\partial^2 v_z}{\partial r\partial z}\right)$$

$$=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\partial^2 v_r}{\partial z^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{v_r}{r^2}\right]$$

$$+\mu\left(\frac{\partial^2 v_r}{\partial r^2}-\frac{1}{r^2}\frac{\partial \mathrm{v}_\theta}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial \theta}-\frac{v_r}{r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}+\frac{\partial^2 v_z}{\partial r\partial z}\right)$$

But using the equation of continuity, we have,

$$\left(\frac{\partial^2 v_r}{\partial r^2}-\frac{1}{r^2}\frac{\partial \mathrm{v}_\theta}{\partial \theta}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial \theta}-\frac{v_r}{r^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}+\frac{\partial^2 v_z}{\partial r\partial z}\right)=\frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial r}+\frac{1}{r}\frac{\partial \mathrm{v}_\theta}{\partial \theta}+\frac{v_r}{r}+\frac{\partial v_z}{\partial z}\right)=0$$

Thus,

$(\mathrm{div}\mathbf{T})_r+\rho B_r=\rho a_r\rightarrow$

$$-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}+\frac{\partial^2 v_r}{\partial z^2}+\frac{1}{r}\frac{\partial v_r}{\partial r}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{v_r}{r^2}\right]+\rho B_r=\rho a_r$$

6.18 Use the results of Sect. 2.35, chapter 2 and the constitutive equations for the Newtonian viscous fluid, verify Navier-Stokes Equation in the r-direction in spherical coordinates, i.e., Eqs. (6.8.5).

Ans. For a Newtonian fluid, the stress tensor in spherical coordinates is given by:

$$[\mathbf{T}]=\begin{bmatrix} -p+2\mu\frac{\partial v_r}{\partial r} & \mu\left[\left(\frac{1}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r}\right)+\frac{\partial v_\theta}{\partial r}\right] & \mu\left[\left(\frac{1}{r\sin\theta}\frac{\partial v_r}{\partial\phi}-\frac{v_\phi}{r}\right)+\frac{\partial v_\phi}{\partial r}\right] \\ T_{\theta\phi} & -p+2\mu\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{v_r}{r}\right) & \mu\left[\left(\frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\phi}-\frac{v_\phi\cot\theta}{r}\right)+\frac{1}{r}\frac{\partial v_\phi}{\partial\theta}\right] \\ T_{\phi r} & T_{\phi\theta} & -p+2\mu\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{v_r}{r}+\frac{v_\theta\cot\theta}{r}\right) \end{bmatrix}$$

The equation of motion in the r-direction is given by [see Eq.(4.8.4)]

$$\frac{1}{r^2}\frac{\partial\left(r^2T_{rr}\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(T_{r\theta}\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi}-\frac{T_{\theta\theta}+T_{\phi\phi}}{r}+\rho B_r=\rho a_r\,.$$

We also have the equation of continuity [see Eq.(2.35.26) or Eq.(6.8.8)]

$$\frac{\partial v_r}{\partial r}+\left(\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{2v_r}{r}\right)+\left(\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{v_\theta\cot\theta}{r}\right)=0\,.$$

Now,

$$\frac{1}{r^2}\frac{\partial\left(r^2T_{rr}\right)}{\partial r}=-\frac{\partial p}{\partial r}-\frac{2p}{r}+2\mu\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{2}{r}\frac{\partial v_r}{\partial r}\right],$$

$$\frac{1}{r\sin\theta}\frac{\partial\left(T_{r\theta}\sin\theta\right)}{\partial\theta}$$

$$=\mu\left[\left(\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}\right)+\frac{1}{r}\frac{\partial v_\theta}{\partial r}\right]\cot\theta+\mu\left[\left(\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}\right)+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta}\right].$$

$$\frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi}=\mu\left[\left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}-\frac{1}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)+\frac{1}{r\sin\theta}\frac{\partial^2 v_\phi}{\partial r\partial\phi}\right]$$

$$-\frac{T_{\theta\theta}+T_{\phi\phi}}{r}=+\frac{2p}{r}-2\mu\left(\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{2v_r}{r^2}\right)-2\mu\left(\frac{1}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{v_\theta\cot\theta}{r^2}\right).$$

Thus.

$$\frac{1}{r^2}\frac{\partial\left(r^2T_{rr}\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(T_{r\theta}\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi}-\frac{T_{\theta\theta}+T_{\phi\phi}}{r}=$$

$$-\frac{\partial p}{\partial r}-\frac{2p}{r}+2\mu\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{2}{r}\frac{\partial v_r}{\partial r}\right]+\mu\left[\left(\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}-\frac{v_\theta}{r^2}\right)+\frac{1}{r}\frac{\partial v_\theta}{\partial r}\right]\cot\theta$$

$$+\mu\left[\left(\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}\right)+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta}\right]+\mu\left[\left(\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}-\frac{1}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}\right)+\frac{1}{r\sin\theta}\frac{\partial^2 v_\phi}{\partial r\partial\phi}\right]$$

$$+\frac{2p}{r}-2\mu\left(\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{2v_r}{r^2}\right)-2\mu\left(\frac{1}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{v_\theta\cot\theta}{r^2}\right)$$

$$=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{2}{r}\frac{\partial v_r}{\partial r}-\frac{2v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}\cot\theta+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}\right)$$

$$+\mu\left(-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{2v_\theta\cot\theta}{r^2}\right)$$

$$+\mu\left[\frac{\partial^2 v_r}{\partial r^2}+\frac{2}{r}\frac{\partial v_r}{\partial r}-\frac{2v_r}{r^2}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r}\frac{\partial v_\theta}{\partial r}\cot\theta-\frac{v_\theta}{r^2}\cot\theta-\frac{1}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{1}{r\sin\theta}\frac{\partial^2 v_\phi}{\partial r\partial\phi}\right]$$

Now, differentiate the equation of continuity with respect to r, we have,

$\frac{\partial}{\partial r}\left(\frac{\partial v_r}{\partial r}+\frac{2v_r}{r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}+\frac{v_\theta\cot\theta}{r}\right)=0$, that is,

$$\frac{\partial^2 v_r}{\partial r^2}+\frac{2}{r}\frac{\partial v_r}{\partial r}-\frac{2v_r}{r^2}+\frac{1}{r}\frac{\partial^2 v_\theta}{\partial r\partial\theta}-\frac{1}{r^2}\frac{\partial v_\theta}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial^2 v_\phi}{\partial r\partial\phi}-\frac{1}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}+$$

$$\frac{\cot\theta}{r}\frac{\partial v_\theta}{\partial r}-\frac{v_\theta\cot\theta}{r^2}=0$$

Thus, $\frac{1}{r^2}\frac{\partial\left(r^2T_{rr}\right)}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial\left(T_{r\theta}\sin\theta\right)}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial T_{r\phi}}{\partial\phi}-\frac{T_{\theta\theta}+T_{\phi\phi}}{r}$

$$=-\frac{\partial p}{\partial r}+\mu\left(\frac{\partial^2 v_r}{\partial r^2}+\frac{2}{r}\frac{\partial v_r}{\partial r}-\frac{2v_r}{r^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2}+\frac{1}{r^2}\frac{\partial v_r}{\partial\theta}\cot\theta+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}\right)$$

$$+\mu\left(-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}-\frac{2v_\theta\cot\theta}{r^2}\right)$$

$$=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2v_r\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}\right]$$

$$+\mu\left[-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right)\right]$$

Finally, the Navier-Stokes equation in r direction is:

$$=-\frac{\partial p}{\partial r}+\mu\left[\frac{\partial}{\partial r}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 v_r\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial v_r}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 v_r}{\partial\phi^2}\right]$$
$$+\mu\left[-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial\phi}-\frac{2}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(v_\theta\sin\theta\right)\right]+\rho B_r=\rho a_r$$

6.19 Show that for a steady flow, the streamline containing a point P coincides with the pathline for a particle which passes through the point P at some time t.

Ans. For a steady flow, the velocity at every point on a streamline does not change with time. Therefore, any particle, which is at a point P on the streamline at a given time t, will move along the streamline at all time. That is, its pathline coincides with the streamline containing the point P. We can also demonstrate this mathematically as follows:
For steady flow, the velocity field is independent of time, that is, $\mathbf{v}=\mathbf{v}(\mathbf{x})$. Let $\mathbf{x}=\mathbf{x}(t)$ be the pathline, then, the differential system for the pathline is:

$$\frac{d\mathbf{x}}{dt}=\mathbf{v}\{\mathbf{x}(t)\},\text{ subjected to the condition }\mathbf{x}(t_o)=\mathbf{x_o}\qquad(1)$$

Let $\mathbf{x}=\mathbf{x}(s)$ be the parametric equation for the streamline passing though $\mathbf{x_o}$, then the differential system for the streamline is:

$$\frac{d\mathbf{x}}{ds}=\mathbf{v}\{\mathbf{x}(s)\},\text{ subjected to the condition }\mathbf{x}(s_o)=\mathbf{x_o}\qquad(2)$$

The two differential systems are identical. They determine the same curve.

6.20 Given the two dimensional velocity field

$$v_1=\frac{kx_1x_2}{1+kx_2t},\quad v_2=0$$

(a) Find the streamline at time t, which passes through the spatial point (α_1,α_2) and,
(b) find the pathline equation $\mathbf{x}=\mathbf{x}(t)$ for a particle which is at (X_1,X_2) at time t_o

----

Ans. (a) Since the flow is in $\mathbf{e}_1$ direction only, therefore, both the streamline and the pathline are straight line in the $\mathbf{e}_1$ direction. The streamline equation which passes through the spatial point (α_1,α_2) is simply $x_2=\alpha_2$.
(b) The pathline for a particle which is at (X_1,X_2) at time t_o is simply $x_2=X_2$. To find the time history of the particle along the pathline, i.e., to find $\mathbf{x}=\mathbf{x}(t)$ with $\mathbf{X}=\mathbf{x}(t_o)$,we have,

$$\frac{dx_1}{dt}=\frac{kx_1X_2}{1+kX_2t}\rightarrow\int_{X_1}^{x_1}\frac{dx_1}{x_1}=\int_{t_o}^{t}\frac{kX_2}{1+kX_2t}dt\rightarrow\ln\frac{x_1}{X_1}=\ln\frac{1+kX_2t}{1+kX_2t_o},$$

$$\rightarrow x_1=\frac{1+kX_2t}{1+kX_2t_o}X_1\text{ and }x_2=X_2.$$

6.21 Given the two dimensional flow: $v_1=kx_2,\ v_2=0$

(a) Obtain the streamline passing through the point (α_1, α_2). (b) Obtain the pathline for the particle which is at (X_1, X_2) at $t = 0$, including the time history of the particle along the pathline

----

Ans. (a) The streamline is clearly $x_2 = \alpha_2$.

(b) The pathline for the particle which is at (X_1, X_2) at time 0 is simply $x_2 = X_2$. To find the time history of the particle along the pathline, i.e., to find $\mathbf{x} = \mathbf{x}(t)$ with $\mathbf{X} = \mathbf{x}(0)$, we have,

$$\frac{dx_2}{dt} = 0 \rightarrow x_2 = X_2, \rightarrow \frac{dx_1}{dt} = kX_2 \rightarrow x_1 = X_1 + kX_2 t, \quad 0 \le t \le \infty .$$

6.22 Do Prob. 6.21 for the following velocity field : $v_1 = \omega x_2, \ v_2 = -\omega x_1$.

Ans. (a) From $\frac{dx_1}{ds} = \omega x_2, \ \frac{dx_2}{ds} = -\omega x_1 \rightarrow \frac{dx_1}{dx_2} = -\frac{x_2}{x_1} \rightarrow x_1 dx_1 + x_2 dx_2 = 0$,

$\rightarrow x_1^2 + x_2^2 = C \rightarrow x_1^2 + x_2^2 = \alpha_1^2 + \alpha_2^2$. The streamline is a circle.

(b) Since the flow is steady, clearly, the pathline is also a circle given by $x_1^2 + x_2^2 = X_1^2 + X_2^2$. To find the time history of the particle along the pathline, i.e., to find $\mathbf{x} = \mathbf{x}(t)$ with $\mathbf{X} = \mathbf{x}(0)$, we have,

$$\frac{dx_1}{dt} = \omega x_2, \ \frac{dx_2}{dt} = -\omega x_1 \rightarrow \frac{d^2 x_1}{dt^2} = \omega \frac{dx_2}{dt} = -\omega^2 x_1 \rightarrow \frac{d^2 x_1}{dt^2} + \omega^2 x_1 = 0,$$

$$\rightarrow x_1 = A \sin \omega t + B \cos \omega t, \rightarrow x_2 = \frac{1}{\omega}\frac{dx_1}{dt} = A \cos \omega t - B \sin \omega t .$$

at $t = 0, \ x_1 = X_1, \ x_2 = X_2$, thus, $x_1 = X_2 \sin \omega t + X_1 \cos \omega t, \rightarrow x_2 = X_2 \cos \omega t - X_1 \sin \omega t$

6.23 Given the following velocity field in polar coordinates (r, θ): $v_r = \frac{Q}{2\pi r}, v_\theta = 0$.

(a) Obtain the streamline passing through the point (r_o, θ_o),

(b) Obtain the pathline for the particle which is at (R, Θ) at $t = 0$, including the time history of the particle along the pathline

----

Ans. Both the streamline and the path lines are radial lines with θ =constant.

(a) the streamline passing through the point $r = r_o, \theta = \theta_o$ is $\theta = \theta_o$.

(b) the pathline for the particle which is at (R, Θ) at $t = 0$ is $\theta = \Theta$. To find the time history of the particle, we have,

$$\frac{d\theta}{dt} = 0 \rightarrow \theta = \Theta, \quad \frac{dr}{dt} = \frac{Q}{2\pi r} \rightarrow \int_R^r r dr = \int_0^t \frac{Q}{2\pi} dt \rightarrow r^2 = R^2 + \frac{Q}{\pi} t .$$

6.24 Do Prob. 6.23 for the following velocity field in polar coordinates (r, θ): $v_r = 0, v_\theta = C / r$.

Ans. Both the streamline and the path lines are circles $r = \text{constant}$.
(a) the streamline passing through the point $r = r, \theta = \theta_o$ is $r = r_o$
(b) the pathline for the particle which is at (R, Θ) at $t = 0$ is $r = R$. To find the time history of the particle, we have, $\frac{dr}{dt} = 0 \rightarrow r = R \rightarrow R\frac{d\theta}{dt} = \frac{C}{R} \rightarrow \frac{d\theta}{dt} = \frac{C}{R^2} \rightarrow \theta = \frac{C}{R^2}t + \Theta$.

6.25 From the Navier-stokes equations, obtain Eq. (6.11.2) for the velocity distribution of the plane Couette flow.

Ans. With x_2 axis pointing vertically upward, we have
$v_1 = v(x_2), v_2 = 0, v_3 = 0$ and $a_1 = a_2 = a_3 = 0$, thus, with $p = p(x_2)$, the Navier-Stoke's equation in the x_1 direction become

$$0 = \mu\frac{d^2v}{dx_2^2} \rightarrow v = C_1x_2 + C_2. \quad \text{At } x_2 = 0, v = 0, \rightarrow C_2 = 0. \quad \text{At } x_2 = d, v = v_o \rightarrow C_1 = \frac{v_o}{d} \rightarrow v = \frac{v_o}{d}x_2.$$

6.26 For the plane Couette flow, if in addition to the movement of the upper plate, there is also an applied negative pressure gradient $\partial p / \partial x_1$, obtain the velocity distribution. Also obtain the volume flow rate per unit width.

Ans. With x_2 axis pointing vertically upward, we have
$v_1 = v(x_2), v_2 = 0, v_3 = 0$ and $a_1 = a_2 = a_3 = 0$, thus, the Navier-Stoke's Equations become,

$$0 = -\frac{\partial p}{\partial x_1} + \mu\frac{d^2v}{dx_2^2} \rightarrow \frac{\partial}{\partial x_1}\left(\frac{\partial p}{\partial x_1}\right) = 0$$

$$0 = -\frac{\partial p}{\partial x_2} \rightarrow \quad \frac{\partial}{\partial x_1}\left(\frac{\partial p}{\partial x_2}\right) = \frac{\partial}{\partial x_2}\left(\frac{\partial p}{\partial x_1}\right) = 0$$

$$0 = -\frac{\partial p}{\partial x_3} \rightarrow \quad \frac{\partial}{\partial x_1}\left(\frac{\partial p}{\partial x_3}\right) = \frac{\partial}{\partial x_3}\left(\frac{\partial p}{\partial x_1}\right) = 0$$

Thus $\frac{\partial p}{\partial x_1} = \text{constant} \equiv -\alpha$

$$\frac{d^2v}{dx_2^2} = -\frac{\alpha}{\mu} \rightarrow v = -\left(\frac{\alpha}{\mu}\right)\frac{x_2^2}{2} + C_1x_2 + C_2, \quad \text{At } x_2 = 0, v = 0 \rightarrow C_2 = 0. \ \text{At } x_2 = d, v = v_o \rightarrow$$

$$v_o = -\left(\frac{\alpha}{\mu}\right)\frac{d^2}{2} + C_1 d \rightarrow C_1 = \frac{v_o}{d} + \left(\frac{\alpha}{\mu}\right)\frac{d}{2} \rightarrow$$

$$v = -\left(\frac{\alpha}{\mu}\right)\frac{x_2^2}{2} + \left\{\frac{v_o}{d} + \left(\frac{\alpha}{\mu}\right)\frac{d}{2}\right\}x_2 = \left(\frac{\alpha}{2\mu}\right)\left(x_2 d - x_2^2\right) + \frac{v_o}{d}x_2 .$$

The volume flow rate per unit width is given by

$$Q=\int_0^d v(x_2)dx_2=\int_0^d\left[-\left(\frac{\alpha}{\mu}\right)\frac{x_2^2}{2}+\left\{\frac{v_o}{d}+\left(\frac{\alpha}{\mu}\right)\frac{d}{2}\right\}x_2\right]dx_2$$

$$=-\left(\frac{\alpha}{\mu}\right)\frac{d^3}{6}+\left\{\frac{v_o}{d}+\left(\frac{\alpha}{\mu}\right)\frac{d}{2}\right\}\frac{d^2}{2}=\left(\frac{\alpha}{\mu}\right)\left(\frac{d^3}{12}\right)+\left\{\frac{v_o}{d}\right\}\frac{d^2}{2}$$

6.27 Obtain the steady uni-directional flow of an incompressible viscous fluid layer of uniform depth d flowing down an inclined plane which makes an angle θ with the horizontal.

Ans. With x_2 axis normal to the flow and pointing away from the fluid and x_1 axis in the flow direction, we are looking for the velocity field in the following form: $v_1=v(x_2), v_2=0, v_3=0$, which clearly satisfies the continuity equation. Now the N-S equations give

$$0=-\frac{\partial p}{\partial x_1}+\rho g\sin\theta+\mu\frac{d^2v}{dx_2^2}\rightarrow\frac{\partial}{\partial x_1}\left(\frac{\partial p}{\partial x_1}\right)=0\,.$$

$$0=-\frac{\partial p}{\partial x_2}-\rho g\cos\theta\rightarrow\frac{\partial}{\partial x_1}\frac{\partial p}{\partial x_2}=\frac{\partial}{\partial x_2}\frac{\partial p}{\partial x_1}=0$$

$$0=-\frac{\partial p}{\partial x_3}=0\rightarrow\frac{\partial}{\partial x_1}\frac{\partial p}{\partial x_3}=\frac{\partial}{\partial x_3}\frac{\partial p}{\partial x_1}=0\,.$$ Thus

$$\frac{\partial p}{\partial x_1}=C\,.$$

The constant C can be determined from the pressure condition on the free surface ($x_2=d$), where pressure $p=p_a$, the atmospheric pressure which is independent of x_1, thus

$\partial p/\partial x_1=0\rightarrow C=0$ so that $\partial p/\partial x_1=0$

for the whole flow field. Thus,

$$0=\rho g\sin\theta+\mu\frac{d^2v}{dx_2^2}\rightarrow\mu\frac{d^2v}{dx_2^2}=-\rho g\sin\theta\rightarrow\mu\frac{dv}{dx_2}=-\rho g\sin\theta x_2+C_1$$

$$\rightarrow\mu v=-\rho g\sin\theta\frac{x_2^2}{2}+C_1x_2+C_2\,.$$ At $x_2=0, v=0$ (non slip condition) $\rightarrow C_2=0$.

At $x_2=d$, shear stress $T_{12}=0\rightarrow\mu dv/dx_2=0\rightarrow \mathrm{C}_1=\rho \mathrm{gd}\sin\theta$.

$$\rightarrow\mu v=\rho\mathrm{g}\sin\theta\left(\mathrm{d}-\frac{x_2}{2}\right)x_2\,.$$

6.28 A layer of water ($\rho g=62.4lb/ft^3$) flows down an inclined plane ($\theta=30^{\circ}$) with a uniform thickness of $0.1ft$. Assuming the flow to be laminar, what is the pressure at any point on the inclined plane. Take the atmospheric pressure to be zero.

Ans. With flow in the x_1 direction, the N-S equation in the x_2 direction (pointing away from the inclined plane) gives, [note: p is independent of x_1 and x_3, see Prob. 6.27]

$$-\partial p/\partial x_2-\rho g\cos\theta=0\rightarrow p=-(\rho g\cos\theta)x_2+C\,.$$

At $x_2 = d$, $p = p_a = 0 \to (\rho g \cos\theta) d = C, \to p = (\rho g \cos\theta)(d - x_2)$.

At $x_2 = 0 \to p = (\rho g \cos\theta)(d) = (62.4 \cos 30^{o})(0.1) = 5.40\ lb/ft^2$.

We can also obtain the same result by using the fact that the piezometric head $p/(\rho g) + z$ =constant for any points on the same plane perpendicular to the direction of the flow (see example 6.7.2), therefore $\dfrac{p_a}{\rho g} + z_a = \dfrac{p_b}{\rho g} + z_b \to p_b = \rho g (z_a - z_b) = \rho g d \cos\theta$.

6.29 Two layers of liquids with viscosities μ_1 and μ_2, densities ρ_1 and ρ_2 respectively, and with equal depths b, flow steadily between two fixed horizontal parallel plates. Find the velocity distribution for this steady uni-directional flow. Neglect body forces.

Ans. We are looking for velocity fields in the two layers in the following form corresponding to the uni-directional steady laminar flows:

For the top layer: $v_1^{(t)} = v^{(t)}(x_2)$, $v_2^{(t)} = v_3^{(t)} = 0$.

For the bottom layer: $v_1^{(b)} = v^{(b)}(x_2)$, $v_2^{(b)} = v_3^{(b)} = 0$.

From the N-S equations for the top layer, we have

$$0 = -\partial p^{(t)}/\partial x_1 + \mu_1 d^2 v^{(t)}/dx_2^2 \to (\partial/\partial x_1)(\partial p^{(t)}/\partial x_1) = 0$$

$$0 = -\partial p^{(t)}/\partial x_2 \to (\partial/\partial x_1)(\partial p^{(t)}/\partial x_2) = (\partial/\partial x_2)(\partial p^{(t)}/\partial x_1) = 0$$

$$0 = -\partial p^{(t)}/\partial x_3 \to (\partial/\partial x_1)(\partial p^{(t)}/\partial x_3) = (\partial/\partial x_3)(\partial p^{(t)}/\partial x_1) = 0$$

Thus, $\partial p^{(t)}/\partial x_1 = -\alpha_1$ (a constant). Now,

$$\mu_1 d^2 v^{(t)}/dx_2^2 = -\alpha_1 \to \mu_1 dv^{(t)}/dx_2 = -\alpha_1 x_2 + A_1 \to \mu_1 v^{(t)} = -(\alpha_1 x_2^2/2) + A_1 x_2 + B_1.$$

Similarly from the N-S equations for the bottom layer, we have,

$\partial p^{(b)}/\partial x_1 = -\alpha_2$ (a constant).

$$\mu_2 dv^{(b)}/dx_2 = -\alpha_2 x_2 + A_2 \to \mu_2 v^{(b)} = -(\alpha_2 x_2^2/2) + A_2 x_2 + B_2.$$

The constants A_1, B_1, A_2, B_2 will be determined from the boundary and the interface conditions:

At $x_2 = b$ (the top plate), $v^{(t)} = 0$, $0 = -(\alpha_1 b^2/2) + A_1 b + B_1$ (1)

At $x_2 = -b$ (the bottom plate), $v^{(b)} = 0$, $0 = -(\alpha_2 b^2/2) - A_2 b + B_2$ (2)

At $x_2 = 0$ (the interface), there is no slip between the two layers of flow, i.e., $v^{(t)} = v^{(b)} \to$
$B_1/\mu_1 = B_2/\mu_2$ (3)

Also, according to Newton's 3rd law, the action and reaction at the interface between the two fluid must be equal and opposite, that is, both the shear stress and the normal stress must by continuous at $x_2 = 0$. Since $T_{12}^{(t)}\Big|_{x_2=0} = \mu_1 \left(dv^{(t)}/dx_2\right)_{x_2=0} = A_1$, $T_{12}^{(b)}\Big|_{x_2=0} = \mu_2 \left(dv^{(t)}/dx_2\right)_{x_2=0} = A_2$

Therefore, $A_1 = A_2$, (4)

and $T_{22}^{(t)}\Big|_{x_2=0} = -p^{(t)}(x_1,0)$, $T_{22}^{(b)}\Big|_{x_2=0} = -p^{(b)}(x_1,0)$, $T_{22}^{(t)}\Big|_{x_2=0} = T_{22}^{(b)}\Big|_{x_2=0} \to$

$$p^{(t)}(x_1,0) = p^{(b)}(x_1,0) \to (\partial p^{(t)}/\partial x_1)(x_1,0) = (\partial p^{(b)}/\partial x_1)(x_1,0) \to \alpha_1 = \alpha_2 \equiv \partial p/\partial x_1.$$

Now, Eqs.(1)(2)(3)(4) determine the four constants A_1, B_1, A_2, B_2 as a function of $\alpha = -\partial p/\partial x_1$:

$$A_1 = A_2 = \alpha b\left\{\frac{\mu_2-\mu_1}{2(\mu_1+\mu_2)}\right\},\ B_1 = b^2\alpha\left(\frac{\mu_1}{\mu_1+\mu_2}\right), B_2 = b^2\alpha\left(\frac{\mu_2}{\mu_1+\mu_2}\right).\text{ Thus,}$$

$$\mu_1 v^{(t)} = -\alpha\left[\frac{x_2^2}{2} - \frac{b}{2}\left(\frac{\mu_2-\mu_1}{\mu_1+\mu_2}\right)x_2 - b^2\left(\frac{\mu_1}{\mu_1+\mu_2}\right)\right],$$

$$\mu_2 v^{(b)} = -\alpha\left[\frac{x_2^2}{2} - \frac{b}{2}\left(\frac{\mu_2-\mu_1}{\mu_1+\mu_2}\right)x_2 - b^2\left(\frac{\mu_2}{\mu_1+\mu_2}\right)\right].$$

6.30 For the Couette flow of Section 6.15, (a) obtain the shear stress at any point inside the fluid (b) obtain the shear stress on the outer and inner cylinder (c) obtain the torque which must be applied to the cylinders to maintain the flow.

Ans. (a) Eq.(6.15.4) and (6.15.7), give $v_\theta = Ar + B/r$, where $B = r_1^2 r_2^2(\Omega_1-\Omega_2)/(r_2^2-r_1^2)$.

Thus, $T_{r\theta} = T_{\theta r} = 2\mu D_{r\theta} = \mu r\dfrac{d}{dr}\left(\dfrac{v_\theta}{r}\right) = -\dfrac{2\mu B}{r^2} = -\dfrac{2\mu r_1^2 r_2^2(\Omega_1-\Omega_2)}{r_2^2-r_1^2}\dfrac{1}{r^2}$

(b) On the outer wall: $r = r_2$, the shear stress is $T_{r\theta} = T_{\theta r} = 2\mu r_1^2(\Omega_2-\Omega_1)/(r_2^2-r_1^2)$,

On the inner wall, $r = r_1$, the shear stress is $T_{r\theta} = T_{\theta r} = 2\mu r_2^2(\Omega_2-\Omega_1)/(r_2^2-r_1^2)$.

(c) On the outer wall, per unit height, the torque is given by

$$(\boldsymbol{M})_{r_2} = \left[(T_{r\theta})_{r=r_2}(2\pi r_2)(1)\right]r_2\mathbf{e}_\theta = \frac{2\mu r_1^2(\Omega_2-\Omega_1)}{r_2^2-r_1^2}\left(2\pi r_2^2\right)\mathbf{e}_\theta = \frac{4\pi\mu r_1^2 r_2^2(\Omega_2-\Omega_1)}{r_2^2-r_1^2}\mathbf{e}_\theta$$

The torque on the inner wall is equal and opposite to that on the outer wall.

6.31 Verify the equation $\beta^2 = \rho\omega/2\mu$ for the oscillating plane problem of Section 6.16.

Ans. With $v = \alpha e^{-\beta x_2}\cos(\omega t - \beta x_2 + \varepsilon)$, $\partial v/\partial t = -\omega\alpha e^{-\beta x_2}\sin(\omega t - \beta x_2 + \varepsilon)$,

$\partial v/\partial x_2 = -\beta\alpha e^{-\beta x_2}\cos(\omega t - \beta x_2 + \varepsilon) + \beta\alpha e^{-\beta x_2}\sin(\omega t - \beta x_2 + \varepsilon)$ and

$\partial^2 v/\partial x_2^2 = \beta^2\alpha e^{-\beta x_2}\cos(\omega t - \beta x_2 + \varepsilon) - \beta^2\alpha e^{-\beta x_2}\sin(\omega t - \beta x_2 + \varepsilon)$

$-\beta^2\alpha e^{-\beta x_2}\sin(\omega t - \beta x_2 + \varepsilon) - \beta^2\alpha e^{-\beta x_2}\cos(\omega t - \beta x_2 + \varepsilon)$

$= -2\beta^2\alpha e^{-\beta x_2}\sin(\omega t - \beta x_2 + \varepsilon)$

Thus $\rho\partial v/\partial t = \mu\partial^2 v/\partial x_2^2 \rightarrow$

$-\rho\omega\alpha e^{-\beta x_2}\sin(\omega t - \beta x_2 + \varepsilon) = -\mu 2\beta^2\alpha e^{-\beta x_2}\sin(\omega t - \beta x_2 + \varepsilon)$

$\rightarrow \rho\omega = \mu 2\beta^2 \rightarrow \beta^2 = \rho\omega/(2\mu)$.

6.32 Consider the flow of an incompressible viscous fluid through the annular space between two concentric horizontal cylinders. The radii are a and b. (a) Find the flow field if there is no variation of pressure in the axial direction and if the inner and the outer cylinders have axial velocities v_a and v_b respectively and (b) find the flow field if there is a pressure gradient in the axial direction and both cylinders are fixed. Take body forces to be zero.

Ans. (a) We look for the following form of velocity field in cylindrical coordinates:
$v_r = 0, v_\theta = 0, v_z = v(r)$ and $\partial p / \partial z = 0$. The N-S equations give, in the absence of body forces

$$0 = -\frac{\partial p}{\partial r}, \quad 0 = -\frac{1}{r}\frac{\partial p}{\partial \theta}, \quad 0 = \mu\left(\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr}\right).$$

The first two equations together with $\partial p / \partial z = 0$ give, $p = \text{constant}$.

$$0 = \mu\left(\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr}\right) \to \frac{1}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = 0 \to r\frac{dv}{dr} = C \to \frac{dv}{dr} = \frac{C}{r} \to v = C\ln r + D$$

At $r = a,\ v = v_a$ and at $r = b,\ v = v_b$, thus,
$v_a = C\ln a + D$ and $v_b = C\ln b + D \to v_a - v_b = C\ln(a/b)$.

$$\to C = \frac{v_a - v_b}{\ln(a/b)} \quad \text{and} \quad D = \frac{v_a \ln b - v_b \ln a}{\ln(b/a)}. \quad \text{So that,}$$

$$v = \frac{v_a - v_b}{\ln(a/b)}\ln r + \frac{v_a \ln b - v_b \ln a}{\ln(b/a)}.$$

(b) $0 = -\partial p/\partial r$ and $0 = -(1/r)\partial p / \partial \theta \to p = p(z)$,

$$0 = -\frac{dp}{dz} + \mu\left(\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr}\right) \to \frac{d^2 p}{dz^2} = 0 \to \frac{dp}{dz} = \text{constant} \equiv -\alpha,$$

$$\to \mu\left(\frac{d^2 v}{dr^2} + \frac{1}{r}\frac{dv}{dr}\right) = -\alpha \to \frac{\mu}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right) = -\alpha \to \frac{d}{dr}\left(r\frac{dv}{dr}\right) = -\frac{\alpha r}{\mu} \to$$

$$\to r\frac{dv}{dr} = -\frac{\alpha}{\mu}\frac{r^2}{2} + C \to \frac{dv}{dr} = -\frac{\alpha}{\mu}\frac{r}{2} + \frac{C}{r} \to v = -\frac{\alpha r^2}{4\mu} + C\ln r + D.$$

The boundary conditions are: $v(a) = v(b) = 0 \to$

$$0 = -\frac{\alpha a^2}{4\mu} + C\ln a + D \text{ and } 0 = -\frac{\alpha b^2}{4\mu} + C\ln b + D \to$$

$$C = \frac{\alpha}{4\mu}\frac{\left(a^2 - b^2\right)}{\ln(a/b)} = -\frac{1}{4\mu}\frac{dp}{dz}\frac{\left(a^2 - b^2\right)}{\ln(a/b)}, \quad D = -\frac{1}{4\mu}\frac{dp}{dz}\frac{\left(a^2 \ln b - b^2 \ln a\right)}{\ln(b/a)}$$

$$v = \frac{1}{4\mu}\frac{dp}{dz}\left[r^2 + \frac{\left(a^2 - b^2\right)}{\ln(b/a)}\ln r + \frac{\left(b^2 \ln a - a^2 \ln b\right)}{\ln(b/a)}\right].$$

6.33 Show that for the velocity field : $v_x = v(y,z),\ v_y = v_z = 0$,

the Navier-Stokes equations, with $\rho \mathbf{B} = \mathbf{0}$, reduces to $\dfrac{\partial^2 v}{\partial y^2} + \dfrac{\partial^2 v}{\partial z^2} = \dfrac{1}{\mu}\dfrac{dp}{dx} = \beta = \text{constant}$.

Ans. With $v_x = v(y,z),\ v_y = v_z = 0$, we have,

$a_x = \frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} = 0+0+0+0 = 0,$ and $a_y = a_z = 0$, thus, the N-S equations in the absence of body forces are:

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right), \quad 0 = -\frac{\partial p}{\partial y}, \quad 0 = -\frac{\partial p}{\partial z}.$$ Thus, $p = p(x)$, and

$$\mu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) = \frac{dp}{dx} \to \frac{\partial}{\partial x}\mu\left(\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}\right) = \frac{d^2 p}{dx^2} \to 0 = \frac{d^2 p}{dx^2} \to,$$

$$\frac{dp}{dx} = \text{constant} \to \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} = \frac{1}{\mu}\frac{dp}{dx} \equiv \beta.$$

6.34 Given the velocity field in the form of

$v_x = v = A\left(y^2/a^2 + z^2/b^2\right) + B$, $v_y = v_z = 0$. Find A and B for the steady laminar flow of a Newtonian fluid in a pipe having an elliptical cross section given by $y^2/a^2 + z^2/b^2 = 1$. Assume no body forces and use the governing equation obtained in the previous problem.

Ans. The governing equation is [see the previous problem] :

$\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} = \frac{1}{\mu}\frac{dp}{dx} \equiv \beta$. Now, $v_x = v = A\left(y^2/a^2 + z^2/b^2\right) + B \to$

$$\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} = A\left(\frac{2}{a^2} + \frac{2}{b^2}\right) = 2A\left(\frac{a^2+b^2}{a^2b^2}\right) \to 2A\left(\frac{a^2+b^2}{a^2b^2}\right) = \beta \to A = \beta\frac{a^2b^2}{2\left(a^2+b^2\right)}$$

On the boundary $y^2/a^2 + z^2/b^2 = 1$, no slip condition requires that $v_x = 0$, therefore, $A(1) + B = 0 \to B = -A$, thus,

$$v_x = A\left[\left(\frac{y^2}{a^2} + \frac{z^2}{b^2}\right) - 1\right] = \frac{a^2b^2\beta}{2\left(a^2+b^2\right)}\left[\left(\frac{y^2}{a^2} + \frac{z^2}{b^2}\right) - 1\right].$$

6.35 Given the velocity field in the form of

$$v_x = A\left(z + \frac{b}{2\sqrt{3}}\right)\left(z + \sqrt{3}y - \frac{b}{\sqrt{3}}\right)\left(z - \sqrt{3}y - \frac{b}{\sqrt{3}}\right) + B, \quad v_y = v_z = 0$$

Find A and B for the steady laminar flow of a Newtonian fluid in a pipe having an equilateral triangular cross-section defined by the planes:

$$z + \frac{b}{2\sqrt{3}} = 0, \quad z + \sqrt{3}y - \frac{b}{\sqrt{3}} = 0, \quad z - \sqrt{3}y - \frac{b}{\sqrt{3}} = 0.$$

Assume no body forces and use the governing equation obtained in Prob. 6.33.

Ans. The governing equation is $\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} = \frac{1}{\mu}\frac{dp}{dx} \equiv \beta$. [See problem 6.33]

With $v_x = A\left(z + b/(2\sqrt{3})\right)\left(z + \sqrt{3}y - b/\sqrt{3}\right)\left(z - \sqrt{3}y - b/\sqrt{3}\right) + B$,

$\partial v_x / \partial y = A\left(z + b/(2\sqrt{3})\right)\left\{\sqrt{3}\left(z - \sqrt{3}y - b/\sqrt{3}\right) - \sqrt{3}\left(z + \sqrt{3}y - b/\sqrt{3}\right)\right\}$

$= A\left(z + b/(2\sqrt{3})\right)\left(-6y\right) \rightarrow$

$\partial^2 v_x / \partial y^2 = -6A\left(z + b/(2\sqrt{3})\right)$.

Let $f(z) = \left(z + b/(2\sqrt{3})\right), \quad g(y,z) = \left(z + \sqrt{3}y - b/\sqrt{3}\right), \quad h(y,z) = \left(z - \sqrt{3}y - b/\sqrt{3}\right)$,

then $v_x = Af(z)g(y,z)h(y,z) + B \rightarrow \partial v_x / \partial z = Ag(y,z)h(y,z) + Af(z)\left\{h(y,z) + g(y,z)\right\}$

Now, $h(y,z) + g(y,z) = \left(z + \sqrt{3}y - b/\sqrt{3}\right) + \left(z - \sqrt{3}y - b/\sqrt{3}\right) = 2\left(z - b/\sqrt{3}\right)$

$f(z)\left\{h(y,z) + g(y,z)\right\} = 2\left(z - b/\sqrt{3}\right)\left(z + b/(2\sqrt{3})\right) = \left(2z^2 - zb/\sqrt{3} - b^2/3\right)$

$g(y,z)h(y,z) = \left(z - b/\sqrt{3}\right)^2 - 3y^2 = z^2 - 2bz/\sqrt{3} + b^2/3 - 3y^2$. Thus,

$$\frac{\partial v_x}{\partial z} = Ag(y,z)h(y,z) + Af(z)\left\{h(y,z) + g(y,z)\right\}$$

$$= A\left(z^2 - 2bz/\sqrt{3} + b^2/3 - 3y^2\right) + A\left(2z^2 - zb/\sqrt{3} - b^2/3\right) = A\left(3z^2 - 3bz/\sqrt{3} - 3y^2\right)$$

$\partial^2 v_x / \partial z^2 = A\left(6z - 3b/\sqrt{3}\right) = 6A\left(z - b/(2\sqrt{3})\right)$. Thus,

$$\frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} = -6A\left(z + b/(2\sqrt{3})\right) + 6A\left(z - b/(2\sqrt{3})\right) = -6A\left(b/\sqrt{3}\right) = \frac{1}{\mu}\frac{dp}{dx} \equiv \beta,$$

from which, $A = -\beta/(2\sqrt{3}b)$. The non-slip condition on the boundary requires $B = 0$.

6.36 For the steady-state, time dependent parallel flow of water (density $\rho = 10^3 Kg/m^3$, viscosity, $\mu = 10^{-3} Ns/m^2$) near an oscillating plate, calculate the wave length for $\omega = 2\ cps$.

Ans. $v = ae^{-\beta x_2}\left(\cos \omega t - \beta x_2 + \varepsilon\right)$, the wave length is given by $2\pi/\beta$, where

$\beta = \sqrt{\frac{\rho\omega}{2\mu}}$. Here we have, $\rho = 10^3 Kg/m^3$, $\omega = 4\pi\ rad/s$, $\mu = 10^{-3} Ns/m^2$, thus

$$\beta = \sqrt{\frac{\rho\omega}{2\mu}} = \sqrt{\frac{\left(10^3\right)\left(4\pi\right)}{2\left(10^{-3}\right)}} = 10^3\sqrt{2\pi}\ m^{-1} \rightarrow \text{wave length} = \frac{2\pi}{\beta} = \frac{\sqrt{2\pi}}{10^3} = 2.51\times 10^{-3} m$$

6.37 The space between two concentric spherical shells is filled with an incompressible Newtonian fluid. The inner shell (radius r_i) is fixed; the outer shell (radius r_o) rotates with an angular velocity Ω about a diameter. Find the velocity distribution. Assume the flow to be laminar without secondary flow.

Ans. We look for solution in the form of $v_r = 0,\ \ v_\theta = 0,\ \ v_\phi = f(r)\sin\theta$. This velocity field clearly satisfies the continuity equation [see Section 6.8, Eq.(6.8.8)]:

$$\frac{\partial v_r}{\partial r} + \frac{2v_r}{r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\theta \cot\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} = 0 .$$

The N-S equations in spherical coordinates give[see Section 6.8]:

$$-\frac{v_\phi^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad (1), \quad -\frac{v_\phi^2 \cot\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} \quad (2),$$

$$0 = -\frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi} + \frac{\mu}{\rho}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial v_\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}\left(v_\phi \sin\theta\right)\right)\right] \quad (3).$$

$$\text{Eq.(1)}\to \frac{\partial^2 p}{\partial\phi\partial r} = 0,\ \ \text{eq.(2)}\to \frac{\partial^2 p}{\partial\phi\partial\theta} = 0 \ \text{ and } \ \text{eq.(3)}\to \frac{\partial^2 p}{\partial\phi\partial\phi} = 0 .$$

Thus, $\partial p/\partial\phi = \text{constant}$. The constant must be zero, otherwise p will not be single-valued. Eq. (3) now becomes, with $\partial p/\partial\phi = 0$ and $v_\phi = f(r)\sin\theta$,

$$0 = \frac{\sin\theta}{r^2}\frac{d}{dr}\left(r^2\frac{df}{dr}\right) - \frac{2f(r)}{r^2}\sin\theta . \ \text{ That is, } \frac{d}{dr}\left(r^2\frac{df}{dr}\right) - 2f = 0 \to r^2\frac{d^2 f}{dr^2} + 2r\frac{df}{dr} - 2f = 0 .$$

The general solution of this equation is: $f = Ar + B/r^2$. Thus, $v_\phi = \left(Ar + B/r^2\right)\sin\theta$.

The inner shell (radius r_i) is fixed; therefore, at $r = r_i,\ \ v_\phi = 0 \to \quad 0 = \left(Ar_i + B/r_i^2\right) \quad (4)$

The outer shell (radius r_o) rotates with an angular velocity Ω, therefore, at

$$r = r_o,\ \ v_\phi = (r_o\sin\theta)\Omega \to (r_o\sin\theta)\Omega = \left(Ar_o + B/r_o^2\right)\sin\theta \to r_o\Omega = \left(Ar_o + B/r_o^2\right) \quad (5)$$

Equations (4) and (5) are two equations for the two unknowns A and B:

$$A = \frac{\Omega r_o^3}{\left(r_o^3 - r_i^3\right)},\ \ B = -\frac{r_o^3 r_i^3}{\left(r_o^3 - r_i^3\right)}\Omega,\ \text{ and } \ v_\phi = \left(Ar + \frac{B}{r^2}\right)\sin\theta .$$

6.38 Consider the following velocity field in cylindrical coordinates for an incompressible fluid:

$$v_r = v(r),\ \ v_\theta = 0,\ \ v_z = 0$$

(a) Show that $v_r = \dfrac{A}{r}$ where A is a constant so that the equation of conservation of mass is satisfied. (b) If the rate of mass flow through the circular cylindrical surface of radius r and unit length (in z direction) is Q_m, determine the constant A in terms of Q_m.

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*Ans. (*a) The equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0 . \text{ Thus, } \frac{1}{r}\frac{d}{dr}(rv_r) = 0 \to rv_r = A \to v_r = \frac{A}{r}.$$

(b) $v_r(2\pi r)(1) = Q_m \to v_r = \dfrac{Q_m}{2\pi r} \to \dfrac{A}{r} = \dfrac{Q_m}{2\pi r} \to A = \dfrac{Q_m}{2\pi}$.

6.39 Given the following velocity field in cylindrical coordinates for an incompressible fluid:

$$v_r = v(r,\theta),\ \ v_\theta = 0,\ \ v_z = 0$$

(a) Show that $v_r = f(\theta)/r$, where $f(\theta)$ is any function of θ. (b) In the absence of body forces, show that

$$\frac{d^2 f}{d\theta^2} + 4f + \frac{\rho f^2}{\mu} + k = 0, \quad p = 2\mu\frac{f}{r^2} + \frac{k\mu}{2r^2} + C, \text{ where } k \text{ and } C \text{ are constants.}$$

Ans. (a). The equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} + \frac{\partial v_z}{\partial z} = 0. \text{ Thus, } \frac{1}{r}\frac{\partial}{\partial r}(rv_r) = 0 \to rv_r = f(\theta) \to v_r = f(\theta)/r$$

(b).The N-S equation in the r direction gives

$$\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\left(\frac{\partial v_r}{\partial\theta} - v_\theta\right) + v_z\frac{\partial v_r}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + B_r$$

$$+\frac{\mu}{\rho}\left[\frac{\partial^2 v_r}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\theta^2} + \frac{\partial^2 v_r}{\partial z^2} + \frac{1}{r}\frac{\partial v_r}{\partial r} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta} - \frac{v_r}{r^2}\right] \to -\frac{f^2}{r^3} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \frac{\mu}{\rho}\left(\frac{1}{r^3}\frac{d^2 f}{d\theta^2}\right) \quad (1)$$

The N-S equation in the z direction gives $-\partial p/\partial z = 0 \to p$ is independent of z.

The N-S equation in the θ direction gives

$$= -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \frac{\mu}{\rho}\left(\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right) \to -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \frac{\mu}{\rho}\frac{2}{r^3}\left(\frac{df}{d\theta}\right) = 0 \to \frac{\partial p}{\partial\theta} = \frac{2\mu}{r^2}\left(\frac{df}{d\theta}\right)$$

$$\to p = \frac{2\mu f}{r^2} + g(r) \to \frac{\partial p}{\partial r} = -\frac{4\mu f}{r^3} + \frac{dg}{dr}. \text{ Thus, Eq.(1) gives:}$$

$$-\frac{\rho f^2}{\mu} = -\frac{r^3}{\mu}\left(\frac{dg}{dr}\right) + \frac{d^2 f}{d\theta^2} + 4f \to \left(\frac{d^2 f}{d\theta^2}\right) + 4f + \frac{\rho f^2}{\mu} = \frac{r^3}{\mu}\left(\frac{dg}{dr}\right).$$

The left side of the above equation is a function of θ, the right side is a function of r, therefore, they must be equal to a constant, say, $-k$, i.e.,

$$\left(\frac{d^2 f}{d\theta^2}\right) + 4f + \frac{\rho f^2}{\mu} = \frac{r^3}{\mu}\left(\frac{dg}{dr}\right) = -k \to g = \frac{\mu k}{2r^2} + C. \quad \text{Therefore,}$$

$$\left(\frac{d^2 f}{d\theta^2}\right) + 4f + \frac{\rho f^2}{\mu} + k = 0, \quad p = \frac{2\mu f}{r^2} + \frac{k\mu}{2r^2} + C.$$

6.40 Consider the steady two dimensional channel flow of an incompressible Newtonian fluid under the action of an applied negative pressure gradient $\partial p/\partial x_1$, as well as the movement of the top plate with velocity v_o in its own plane.[See Prob. 6.26]. Determine the temperature distribution for this flow due to viscous dissipation when both plates are maintained at the same fixed temperature θ_o. Assume constant physical properties.

Ans. From the result of Prob. 6.26, we have $v_1 = v_o x_2/d + (\alpha/2\mu)\left(x_2 d - x_2^2\right), \ v_2 = v_3 = 0.$

Let the temperature distribution be denoted by $\Theta = \Theta(x_2)$. From Eq. (6.18.3), we have,

$$\rho c\frac{D\Theta}{Dt} = \Phi_{inc} + \kappa\frac{\partial^2\Theta}{\partial x_j\partial x_j}, \text{ where } \Phi_{inc} = 2\mu\left(D_{11}^2 + D_{22}^2 + D_{33}^2 + 2D_{12}^2 + 2D_{13}^2 + 2D_{23}^2\right), \text{ represents}$$

the heat generated through viscous forces. For this problem, only D_{12} is nonzero, thus,

$$\Phi_{inc} = 4\mu D_{12}^2 = (4\mu)\left(\frac{1}{2}\right)^2\left[\frac{v_o}{d} + \left(\frac{\alpha}{2\mu}\right)(d - 2x_2)\right]^2 = \mu\left[\frac{v_o}{d} + \left(\frac{\alpha}{2\mu}\right)(d - 2x_2)\right]^2.$$

$$\frac{\partial^2\Theta}{\partial^2 x_2} = -\frac{\mu}{\kappa}\left[\frac{v_o}{d} + \left(\frac{\alpha}{2\mu}\right)(d - 2x_2)\right]^2 \to \frac{\partial\Theta}{\partial x_2} = \frac{\mu^2}{3\kappa\alpha}\left[\frac{v_o}{d} + \left(\frac{\alpha}{2\mu}\right)(d - 2x_2)\right]^3 + C \to$$

$$\Theta = -\frac{\mu^3}{12\kappa\alpha^2}\left[\frac{v_o}{d} + \left(\frac{\alpha}{2\mu}\right)(d - 2x_2)\right]^4 + Cx_2 + D.$$

At $x_2 = 0$, $\Theta = \theta_o \to \theta_o = -\frac{\mu^3}{12\kappa\alpha^2}\left(\frac{v_o}{d} + \frac{\alpha d}{2\mu}\right)^4 + D.$

At $x_2 = d$, $\Theta = \theta_o \to \theta_o = -\frac{\mu^3}{12\kappa\alpha^2}\left(\frac{v_o}{d} - \frac{\alpha d}{2\mu}\right)^4 + Cd + D.$

Thus,

$$D = \theta_o + \frac{\mu^3}{12\kappa\alpha^2}\left(\frac{v_o}{d} + \frac{\alpha d}{2\mu}\right)^4 \text{ and } 0 = \frac{\mu^3}{12\kappa\alpha^2}\left\{\left(\frac{v_o}{d} + \frac{\alpha d}{2\mu}\right)^4 - \left(\frac{v_o}{d} - \frac{\alpha d}{2\mu}\right)^4\right\} + Cd$$

$$\to C = -\frac{\mu^3}{12\kappa\alpha^2 d}\left\{\left(\frac{v_o}{d} + \frac{\alpha d}{2\mu}\right)^4 - \left(\frac{v_o}{d} - \frac{\alpha d}{2\mu}\right)^4\right\}.$$

6.41 Determine the temperature distribution in the plane Poiseuille flow where the bottom plate is kept at a constant temperature Θ_1 and the top plate Θ_2. Include the heat generated by viscous dissipation.

Ans. For the plane Poiseuille flow [see Eq.(6.12.9)],

$v_1 = (\alpha / 2\mu)(b^2 - x_2^2)$, $v_2 = v_3 = 0, \alpha \equiv -\partial p / \partial x_1 > 0$

Let the temperature distribution be denoted by $\Theta = \Theta(x_2)$. From Eq. (6.18.3), we have,

$\rho c \frac{D\Theta}{Dt} = \Phi_{inc} + \kappa \frac{\partial^2\Theta}{\partial x_j \partial x_j}$, where $\Phi_{inc} = 2\mu\left(D_{11}^2 + D_{22}^2 + D_{33}^2 + 2D_{12}^2 + 2D_{13}^2 + 2D_{23}^2\right)$, represents the heat generated through viscous forces. For this problem, only D_{12} is nonzero,

$D_{12} = \frac{1}{2}\frac{\partial v_1}{\partial x_2} = -\frac{1}{2}\frac{\alpha}{\mu}x_2 \to \Phi_{inc} = 4\mu D_{12}^2 = 4\mu\left(-\frac{1}{2}\frac{\alpha}{\mu}x_2\right)^2 = \frac{\alpha^2}{\mu}x_2^2$. Thus,

$$\rho c \frac{D\Theta}{Dt} = \Phi_{inc} + \kappa \frac{\partial^2\Theta}{\partial x_j \partial x_j} \to 0 = \frac{\alpha^2}{\mu}x_2^2 + \kappa\frac{d^2\Theta}{dx_2^2} \to \frac{d^2\Theta}{dx_2^2} = -\frac{1}{\kappa\mu}\left(\frac{\partial p}{\partial x_1}\right)^2 x_2^2$$

$$\to \frac{d\Theta}{dx_2} = -\frac{1}{\kappa\mu}\left(\frac{\partial p}{\partial x_1}\right)^2 \frac{x_2^3}{3} + C \to \Theta = -\frac{1}{\kappa\mu}\left(\frac{\partial p}{\partial x_1}\right)^2 \frac{x_2^4}{12} + Cx_2 + D.$$

At $x_2 = +b, \quad \Theta = \Theta_2 \rightarrow \Theta_2 = -\frac{1}{\kappa\mu}\left(\frac{\partial p}{\partial x_1}\right)^2 \frac{b^4}{12} + Cb + D$

At $x_2 = -b, \quad \Theta = \Theta_1 \rightarrow \Theta_1 = -\frac{1}{\kappa\mu}\left(\frac{\partial p}{\partial x_1}\right)^2 \frac{b^4}{12} - Cb + D$. Thus,

$$D = \frac{\Theta_1 + \Theta_2}{2} + \frac{1}{12\kappa\mu}\left(\frac{\partial p}{\partial x_1}\right)^2 b^4. \quad C = \frac{\Theta_2 - \Theta_1}{2b}.$$

6.42 Determine the temperature distribution in the steady laminar flow between two coaxial cylinders (Couette flow) if the temperatures at the inner and the outer cylinders are kept at the same fixed temperature θ_o.

Ans. For Couette flow, we have, $v_r = 0, \; v_\theta = Ar + \frac{B}{r}, \; v_z = 0$, where

$A = \frac{\Omega_2 r_2^2 - \Omega_1 r_1^2}{r_2^2 - r_1^2}, \quad B = \frac{r_1^2 r_2^2 (\Omega_1 - \Omega_2)}{r_2^2 - r_1^2}$. The only nonzero rate of deformation is

$$D_{r\theta} = \frac{1}{2}\left\{\frac{1}{r}\left(\frac{\partial v_r}{\partial \theta} - \mathrm{v}_\theta\right) + \frac{\partial v_\theta}{\partial r}\right\} = \frac{1}{2}\left\{-\frac{\mathrm{v}_\theta}{r} + \frac{\partial v_\theta}{\partial r}\right\} = -\frac{B}{r^2}.$$

$$\Phi_{inc} = 2\mu\left(2D_{r\theta}^2\right) = 4\mu\left(-\frac{B}{r^2}\right)^2 = \frac{4\mu B^2}{r^4}. \quad \rho c \frac{D\Theta}{Dt} = \Phi_{inc} + \kappa \frac{\partial^2 \Theta}{\partial x_j \partial x_j} \rightarrow$$

$$\rightarrow 0 = \frac{4\mu B^2}{r^4} + \kappa \frac{1}{r}\frac{d}{dr}\left(r\frac{d\Theta}{dr}\right) \rightarrow \frac{d}{dr}\left(r\frac{d\Theta}{dr}\right) = -\frac{4\mu B^2}{\kappa r^3} \rightarrow \frac{d\Theta}{dr} = \frac{2\mu B^2}{\kappa r^3} + \frac{C}{r}$$

$$\rightarrow \Theta = -\frac{\mu B^2}{\kappa r^2} + C \ln r + D.$$

At $r = r_i, \quad \Theta = \Theta_o \rightarrow \Theta_o = -\frac{\mu B^2}{\kappa r_i^2} + C \ln r_i + D.$

At $r = r_o, \quad \Theta = \Theta_o \rightarrow \Theta_o = -\frac{\mu B^2}{\kappa r_o^2} + C \ln r_o + D.$

Thus,

$$C = \left(\frac{r_o^2 - r_i^2}{r_i^2 r_o^2}\right)\left(\frac{\mu B^2}{\kappa}\right) / \left(\ln \frac{r_i}{r_o}\right), \quad D = \left[\Theta_o \ln \frac{r_o}{r_i} + \frac{\mu B^2}{\kappa r_i^2 r_o^2}\left\{r_o^2 \ln r_o - r_i^2 \ln r_i\right\}\right] / \ln \frac{r_o}{r_i}.$$

6.43 Show that the dissipation function for a compressible fluid can be written as that given in Eq. (6.17.10).

Ans. From

$\Phi = \lambda(D_{11} + D_{22} + D_{33})^2 + 2\mu\left(D_{11}^2 + D_{22}^2 + D_{33}^2 + 2D_{12}^2 + 2D_{13}^2 + 2D_{23}^2\right)$, we get,

$$\Phi=(\lambda+2\mu)\left(D_{11}^2+D_{22}^2+D_{33}^2\right)+2\lambda\left(D_{11}D_{22}+D_{11}D_{33}+D_{22}D_{33}\right)+4\mu\left(D_{12}^2+D_{13}^2+D_{23}^2\right)$$

We now verify that this is the same as

$$\bar{\Phi}=(\lambda+2\mu/3)(D_{11}+D_{22}+D_{33})^2+\frac{2}{3}\mu\left[(D_{11}-D_{22})^2+(D_{11}-D_{33})^2+(D_{22}-D_{33})^2\right]$$

$$+4\mu\left(D_{12}^2+D_{13}^2+D_{23}^2\right)$$

Expanding the above equation, we have,

$$\bar{\Phi}=(\lambda+2\mu/3)\left(D_{11}^2+D_{22}^2+D_{33}^2+2D_{11}D_{22}+2D_{11}D_{33}+2D_{22}D_{33}\right)$$

$$+(2\mu/3)\left[2\left(D_{11}^2+D_{22}^2+D_{33}^2\right)-\left(2D_{11}D_{22}+2D_{11}D_{33}+2D_{22}D_{33}\right)\right]+4\mu\left(D_{12}^2+D_{13}^2+D_{23}^2\right).$$

i.e.,

$$\bar{\Phi}=(\lambda+2\mu/3)\left(D_{11}^2+D_{22}^2+D_{33}^2\right)+\left(\lambda+\frac{2}{3}\mu\right)\left(2D_{11}D_{22}+2D_{11}D_{33}+2D_{22}D_{33}\right)$$

$$+(4\mu/3)\left(D_{11}^2+D_{22}^2+D_{33}^2\right)-(2\mu/3)\left(2D_{11}D_{22}+2D_{11}D_{33}+2D_{22}D_{33}\right)+4\mu\left(D_{12}^2+D_{13}^2+D_{23}^2\right)$$

$$=\left(D_{11}^2+D_{22}^2+D_{33}^2\right)(\lambda+2\mu/3+4\mu/3)+2\left(D_{11}D_{22}+D_{11}D_{33}+D_{22}D_{33}\right)(\lambda+2\mu/3-2\mu/3)$$

$$+4\mu\left(D_{12}^2+D_{13}^2+D_{23}^2\right).$$

That is

$$\bar{\Phi}=\left(D_{11}^2+D_{22}^2+D_{33}^2\right)(\lambda+2\mu)+2\lambda\left(D_{11}D_{22}+D_{11}D_{33}+D_{22}D_{33}\right)+4\mu\left(D_{12}^2+D_{13}^2+D_{23}^2\right)$$

Thus, $\bar{\Phi}=\Phi$

6.44 Given the velocity field of a linearly viscous fluid

$$v_1=kx_1,\quad v_2=-kx_2,\quad v_3=0$$

(a) Show that the velocity field is irrotational. (b) Find the stress tensor. (c) Find the acceleration field. (d) Show that the velocity field satisfies the Navier-Stokes equations by finding the pressure distribution directly from the equations. Neglect body forces. Take $p=p_o$ at the origin. (e) Use the Bernoulli equation to find the pressure distribution. (f) Find the rate of dissipation of mechanical energy into heat. (g) If $x_2=0$ is a fixed boundary, what condition is not satisfied by the velocity field.

Ans. (a) $[\nabla\mathbf{v}]=\begin{bmatrix}k&0&0\\0&-k&0\\0&0&0\end{bmatrix}\rightarrow[\mathbf{D}]=\begin{bmatrix}k&0&0\\0&-k&0\\0&0&0\end{bmatrix},[\mathbf{W}]=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$

therefore, the flow is irrotational.

(b) $T_{11}=-p+2\mu k,\quad T_{22}=-p-2\mu k,\quad T_{33}=-p,\quad T_{12}=T_{13}=T_{23}=0$.

(c) $\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}k&0&0\\0&-k&0\\0&0&0\end{bmatrix}\begin{bmatrix}kx_1\\-kx_2\\0\end{bmatrix}=\begin{bmatrix}k^2x_1\\k^2x_2\\0\end{bmatrix}$.

(d)

$$\rho\left(k^2x_1\right)=-\frac{\partial p}{\partial x_1}+\mu\left(\frac{\partial^2 v_1}{\partial x_1^2}+\frac{\partial^2 v_1}{\partial x_2^2}+\frac{\partial^2 v_1}{\partial x_3^2}\right)\rightarrow \rho k^2 x_1=-\frac{\partial p}{\partial x_1}$$

$$\rho\left(k^2x_2\right)=-\frac{\partial p}{\partial x_2}+\mu\left(\frac{\partial^2 v_2}{\partial x_1^2}+\frac{\partial^2 v_2}{\partial x_2^2}+\frac{\partial^2 v_2}{\partial x_3^2}\right)\rightarrow \rho k^2 x_2=-\frac{\partial p}{\partial x_2} \text{ and}$$

$0=-\dfrac{\partial p}{\partial x_3}\rightarrow p$ is independent of x_3.

Thus,

$$\rho k^2 x_1=-\frac{\partial p}{\partial x_1}\rightarrow p=-\frac{\rho k^2 x_1^2}{2}+f(x_2)\rightarrow \frac{\partial p}{\partial x_2}=\frac{df(x_2)}{dx_2}\rightarrow -\rho k^2 x_2=\frac{df(x_2)}{dx_2}$$

$$\rightarrow f=-\frac{\rho k^2 x_2^2}{2}+C\rightarrow p=-\frac{\rho k^2}{2}\left(x_1^2+x_2^2\right)+C. \quad \text{At } x_1=x_2=0, p=p_o\rightarrow C=p_o$$

That is, $p=-\left(\rho k^2/2\right)\left(x_1^2+x_2^2\right)+p_o\rightarrow p=-\left(\rho/2\right)\left(v_1^2+v_2^2\right)+p_o$.

(e) From the Bernoulli Equation, we have

$$\frac{p}{\rho}+\frac{v^2}{2}=\text{constant}\rightarrow\frac{p}{\rho}+\frac{v_1^2+v_2^2}{2}=\left[\frac{p}{\rho}+\frac{v_1^2+v_2^2}{2}\right]_{\text{at origin}}=\frac{p_o}{\rho}+0$$

$$\rightarrow p=p_o-\rho k^2\left(x_1^2+x_2^2\right)/2.$$

(f)

$$\Phi=\lambda\left(D_{11}+D_{22}+D_{33}\right)^2+2\mu\left(D_{11}^2+D_{22}^2+D_{33}^2+2D_{12}^2+2D_{13}^2+2D_{23}^2\right)=2\mu\left(k^2+k^2\right)=4\mu k^2$$

(h) if $x_2=0$ is a fixed boundary, then $\mathbf{v}$ must be zero there. But $\mathbf{v}=k\left(x_1\mathbf{e}_1-x_2\mathbf{e}_2\right)=kx_1\mathbf{e}_1\neq 0$ at $x_2=0$, therefore the non slip boundary condition at $x_2=0$ is not satisfied for a viscous fluid.

6.45 Do Problem 6.44 for the following velocity field: $v_1=k\left(x_1^2-x_2^2\right),\ \ v_2=-2kx_1x_2,\ \ v_3=0$.

Ans.

(a)

$$[\nabla\mathbf{v}]=\begin{bmatrix}2kx_1 & -2kx_2 & 0\\ -2kx_2 & -2kx_1 & 0\\ 0 & 0 & 0\end{bmatrix}\rightarrow[\mathbf{D}]=2k\begin{bmatrix}x_1 & -x_2 & 0\\ -x_2 & -x_1 & 0\\ 0 & 0 & 0\end{bmatrix},[\mathbf{W}]=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

therefore, the flow is irrotational

(b) $T_{11}=-p+4\mu kx_1,\ \ T_{22}=-p-4\mu kx_1,\ \ T_{33}=-p,\ \ T_{12}=T_{13}=T_{23}=0$

(c)

$$\begin{bmatrix}a_1\\a_2\\a_3\end{bmatrix}=\begin{bmatrix}2kx_1 & -2kx_2 & 0\\ -2kx_2 & -2kx_1 & 0\\ 0&0&0\end{bmatrix}\begin{bmatrix}k\left(x_1^2-x_2^2\right)\\ -2kx_1x_2\\ 0\end{bmatrix}=\begin{bmatrix}2k^2x_1\left(x_1^2+x_2^2\right)\\ 2k^2x_2\left(x_1^2+x_2^2\right)\\ 0\end{bmatrix}$$

(d)

$$\rho\left[2k^2x_1\left(x_1^2+x_2^2\right)\right]=-\frac{\partial p}{\partial x_1}+\mu\left(\frac{\partial^2 v_1}{\partial x_1^2}+\frac{\partial^2 v_1}{\partial x_2^2}+\frac{\partial^2 v_1}{\partial x_3^2}\right)\rightarrow\rho\left[2k^2x_1\left(x_1^2+x_2^2\right)\right]=-\frac{\partial p}{\partial x_1}$$

$$\rho\left[2k^2x_2\left(x_1^2+x_2^2\right)\right]=-\frac{\partial p}{\partial x_2}+\mu\left(\frac{\partial^2 v_2}{\partial x_1^2}+\frac{\partial^2 v_2}{\partial x_2^2}+\frac{\partial^2 v_2}{\partial x_3^2}\right)\rightarrow\rho\left[2k^2x_2\left(x_1^2+x_2^2\right)\right]=-\frac{\partial p}{\partial x_2}$$

$0=-\dfrac{\partial p}{\partial x_3}\rightarrow p$ is independent of x_3

Thus,

$$\rho\left[2k^2x_1\left(x_1^2+x_2^2\right)\right]=-\frac{\partial p}{\partial x_1}\rightarrow p=-\rho k^2\left(\frac{x_1^4}{2}+x_1^2x_2^2\right)+f(x_2)\rightarrow\frac{\partial p}{\partial x_2}=-2\rho k^2x_1^2x_2+\frac{df}{dx_2}$$

and

$$\rho\left[2k^2x_2\left(x_1^2+x_2^2\right)\right]=-\frac{\partial p}{\partial x_2}\rightarrow-\rho\left[2k^2x_2\left(x_1^2+x_2^2\right)\right]=-2\rho k^2x_1^2x_2+\frac{df}{dx_2}$$

$$\rightarrow-2\rho k^2x_2^3=\frac{df}{dx_2}\rightarrow f=-\frac{\rho k^2x_2^4}{2}+C.$$

Since $p=p_o$ at origin, therefore, $C=p_o$,

$$\rightarrow p=-\frac{\rho k^2}{2}\left(x_1^4+2x_1^2x_2^2+x_2^4\right)+p_o\rightarrow p=-\frac{\rho k^2}{2}\left(x_1^2+x_2^2\right)^2+p_o.$$

Or, since

$$v_1=k\left(x_1^2-x_2^2\right),\quad v_2=-2kx_1x_2,\quad v_1^2+v_2^2=k^2\left[\left(x_1^2-x_2^2\right)^2+4x_1^2x_2^2\right]=k^2\left(x_1^2+x_2^2\right)^2$$

$$\rightarrow p=-\rho\left(v_1^2+v_2^2\right)/2+p_o.$$

(e) From the Bernoulli Equation, we have

$$\frac{p}{\rho}+\frac{v^2}{2}=\text{constant}\rightarrow\frac{p}{\rho}+\frac{v_1^2+v_2^2}{2}=\left[\frac{p}{\rho}+\frac{v_1^2+v_2^2}{2}\right]_{\text{at origin}}=\frac{p_o}{\rho}+0$$

$$\rightarrow p=p_o-\rho k^2\left(x_1^2+x_2^2\right)^2/2.$$

(f)

$$\Phi=\lambda\left(D_{11}+D_{22}+D_{33}\right)^2+2\mu\left(D_{11}^2+D_{22}^2+2D_{12}^2\right)=0+2\mu\left(8k^2x_1^2+8k^2x_2^2\right)=16\mu k^2\left(x_1^2+x_2^2\right)\text{(h)}$$

if $x_2=0$ is a fixed boundary, then $\mathbf{v}$ must be zero there. But $\mathbf{v}=kx_1^2\mathbf{e}_1\neq0$ at $x_2=0$, therefore the non slip boundary condition at $x_2=0$ is not satisfied for a viscous fluid.

6.46 Obtain the vorticity vector for the plane Poiseuille flow.

Ans. With $v_1=v(x_2)=(\alpha/2\mu)\left(b^2-x_2^2\right)$, where $\alpha=-\partial p/\partial x_1$ and $v_2=v_3=0$, the spin tensor is

$$[\mathbf{W}]=[\nabla\mathbf{v}]^A=\frac{1}{2}\begin{bmatrix}0&\partial v_1/\partial x_2&0\\-\partial v_1/\partial x_2&0&0\\0&0&0\end{bmatrix}$$

The vorticity tensor is $2\mathbf{W}$ and the vorticity vector is twice the axial vector

$$\boldsymbol{\varsigma} = 2\boldsymbol{\omega} = 2\left(W_{32}\mathbf{e}_1 + W_{13}\mathbf{e}_2 + W_{21}\mathbf{e}_3\right) = 2\left(-\frac{1}{2}\frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_3 = -\frac{\partial v_1}{\partial x_2}\mathbf{e}_3 = \frac{\alpha x_2}{\mu}\mathbf{e}_3 = -\frac{\mathbf{1}}{\mu}\frac{\partial p}{\partial x_1}x_2\mathbf{e}_3 .$$

6.47 Obtain the vorticity vector for the Hagen-Poiseuille flow.

Ans. With $v_z = \frac{\alpha}{4\mu}\left(\frac{d^2}{4} - r^2\right),\ v_r = v_\theta = 0,\ \alpha = -\frac{\partial p}{\partial z}$, the spin tensor is

$$[\mathbf{W}] = [\nabla\mathbf{v}]^A = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{1}{r}\frac{\partial v_r}{\partial\theta} - \frac{v_\theta}{r} - \frac{\partial v_\theta}{\partial r}\right) & \frac{1}{2}\left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right) \\ W_{21} & 0 & \frac{1}{2}\left(\frac{\partial v_\theta}{\partial z} - \frac{1}{r}\frac{\partial v_z}{\partial\theta}\right) \\ W_{31} & W_{32} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{2}\left(\frac{\partial v_z}{\partial r}\right) \\ 0 & 0 & 0 \\ \frac{1}{2}\left(\frac{\partial v_z}{\partial r}\right) & 0 & 0 \end{bmatrix}$$

The vorticity tensor is $2\mathbf{W}$ and the vorticity vector is twice the axial vector

$$\boldsymbol{\varsigma} = 2\boldsymbol{\omega} = 2\left(W_{z\theta}\mathbf{e}_r + W_{rz}\mathbf{e}_\theta + W_{\theta r}\mathbf{e}_z\right) = 2W_{rz}\mathbf{e}_\theta = -\left(\frac{\partial v_z}{\partial r}\right)\mathbf{e}_\theta = \frac{\alpha r}{2\mu}\mathbf{e}_\theta = -\frac{1}{2\mu}\frac{\partial p}{\partial z}r\mathbf{e}_\theta .$$

6.48 For a two-dimensional flow of an incompressible fluid, we can express the velocity components in terms of a scalar function ψ (known as the Lagrange stream function) by the relations $v_x = \frac{\partial\psi}{\partial y},\ v_y = -\frac{\partial\psi}{\partial x}$. (a) Show that the equation of conservation of mass is automatically satisfied for any $\psi(x, y)$ which has continuous second partial derivatives.
(b) Show that for two-dimensional flow of an incompressible fluid, ψ =constants are streamlines.
(c) If the velocity field is irrotational, then $v_i = -\frac{\partial\varphi}{\partial x_i}$ where φ is known as the velocity potential. Show that the curves of constant velocity potential $\varphi = \text{constant}$ and the streamline ψ =constant are orthogonal to each other. (d) Obtain the only nonzero vorticity component in terms of ψ.

Ans. (a) With $v_x = \frac{\partial\psi}{\partial y}$ and $v_y = -\frac{\partial\psi}{\partial x}$, we have, $\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} = \frac{\partial}{\partial x}\frac{\partial\psi}{\partial y} - \frac{\partial}{\partial y}\frac{\partial\psi}{\partial x} = 0.$

(b) From $\psi(x, y) = C$, we have,

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy = 0 \to \left(\frac{dy}{dx}\right)_{\psi=\text{constant}} = -\frac{\partial\psi/\partial x}{\partial\psi/\partial y} \to \left(\frac{dy}{dx}\right)_{\psi=\text{constant}} = \frac{v_y}{v_x}$$

Thus, $\psi(x, y) = C$ are streamlines.

(c) From

$$\varphi(x, y) = C \to d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy = 0 \to \left(\frac{dy}{dx}\right)_{\varphi=\text{constant}} = -\frac{\partial\varphi/\partial x}{\partial\varphi/\partial y} \to \left(\frac{dy}{dx}\right)_{\varphi=\text{constant}} = -\frac{v_x}{v_y}$$

Thus, $\rightarrow\left(\frac{dy}{dx}\right)_{\varphi=\text{constant}}\left(\frac{dy}{dx}\right)_{\psi=\text{constant}}=-1$.

(d)

$$\boldsymbol{\varsigma}=2\boldsymbol{\omega}=2\left(W_{zy}\mathbf{e}_{\mathrm{x}}+W_{xz}\mathbf{e}_{\mathrm{y}}+W_{yx}\mathbf{e}_{\mathrm{z}}\right)=\left(\frac{\partial v_y}{\partial x}-\frac{\partial v_x}{\partial y}\right)\mathbf{e}_{\mathrm{z}}=-\left(\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial x^2}\right)\mathbf{e}_{\mathrm{z}}.$$

6.49 Show that $\psi=V_{\mathrm{o}}y\left(1-\frac{a^2}{x^2+y^2}\right)$ represents a two-dimensional irrotational flow of an inviscid fluid.

Ans. With $\psi=V_{\mathrm{o}}y\left(1-\frac{a^2}{x^2+y^2}\right)$, we have,

$$\frac{\partial\psi}{\partial x}=V_{\mathrm{o}}y\left(\frac{2xa^2}{\left(x^2+y^2\right)^2}\right)\rightarrow\frac{\partial^2\psi}{\partial x^2}=V_{\mathrm{o}}y\left(\frac{2a^2}{\left(x^2+y^2\right)^2}+\frac{(-2)(2x)^2a^2}{\left(x^2+y^2\right)^3}\right)$$

$$\frac{\partial^2\psi}{\partial x^2}=V_{\mathrm{o}}y\left(2a^2\right)\left(\frac{1}{\left(x^2+y^2\right)^2}-\frac{4x^2}{\left(x^2+y^2\right)^3}\right)=V_o\left(\frac{2a^2y_{\mathrm{o}}}{\left(x^2+y^2\right)^2}-\frac{8a^2yx^2}{\left(x^2+y^2\right)^3}\right)$$

$$\frac{\partial\psi}{\partial y}=V_{\mathrm{o}}\left(1-\frac{a^2}{x^2+y^2}\right)+V_{\mathrm{o}}\left(\frac{2y^2a^2}{\left(x^2+y^2\right)^2}\right)$$

$$\frac{\partial^2\psi}{\partial y^2}=V_{\mathrm{o}}\left(\frac{2ya^2}{\left(x^2+y^2\right)^2}\right)+V_{\mathrm{o}}\left(\frac{2(2y)a^2}{\left(x^2+y^2\right)^2}-\frac{2y^2a^2(2)(2y)}{\left(x^2+y^2\right)^3}\right)$$

$$\frac{\partial^2\psi}{\partial y^2}=V_{\mathrm{o}}\left(\frac{2ya^2}{\left(x^2+y^2\right)^2}\right)+V_{\mathrm{o}}\left(\frac{4ya^2}{\left(x^2+y^2\right)^2}-\frac{8y^3a^2}{\left(x^2+y^2\right)^3}\right).$$ Thus,

$$\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial x^2}=V_o\left(\frac{2a^2y_{\mathrm{o}}}{\left(x^2+y^2\right)^2}-\frac{8a^2yx^2}{\left(x^2+y^2\right)^3}\right)+V_{\mathrm{o}}\left(\frac{2ya^2}{\left(x^2+y^2\right)^2}\right)+V_{\mathrm{o}}\left(\frac{4ya^2}{\left(x^2+y^2\right)^2}-\frac{8y^3a^2}{\left(x^2+y^2\right)^3}\right)$$

$$V_o\left(\frac{8a^2y}{\left(x^2+y^2\right)^2}\right)-V_{\mathrm{o}}\left(\frac{8a^2yx^2}{\left(x^2+y^2\right)^3}+\frac{8y^3a^2}{\left(x^2+y^2\right)^3}\right)=V_o\left(\frac{8a^2y}{\left(x^2+y^2\right)^2}\right)-V_{\mathrm{o}}\left(\frac{8a^2y\left(x^2+y^2\right)}{\left(x^2+y^2\right)^3}\right)=0$$

Therefore, the given stream function ψ represents a two-dimensional irrotational flow of an inviscid fluid.

6.50 Referring to Figure P 6-9, compute the maximum possible flow of water. Take the atmospheric pressure to be 93.1 $kPa.$, the specific weight of water 9810 N/m^3, and the vapor pressure 17.2 $kPa.$ Assume the fluid to be inviscid. Find the length ℓ for this rate of discharge.

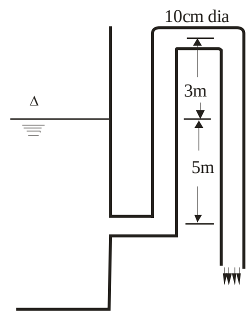

Figure P 6-9

Ans. The Bernoulli equation gives, with point 1 at the reservoir top and point 2 at the highest point inside the tube, we have,

$p_1/\rho + v_1^2/2 + g(0) = p_2/\rho + v_2^2/2 + g(3)$. Thus, assuming v_1 to be very small and negligible, we have, with $p_1 = 93{,}100\ Pa., p_2 = 17{,}200\ Pa., \rho = 1000\ kg/m^3$ and $g = 9.81 m/s^2$.

$$v_2^2/2 = (p_1 - p_2)/\rho - 3g = (93{,}100 - 17{,}200)/1000 - 3(9.81) = 46.47$$

$$\rightarrow v_2 = 9.64\ m/s. \quad Q_{\max} = v_2 A = 9.64\left(\pi(0.1)^2/4\right) = 0.0757\ m^3/s.$$

With point 3 at the exit, we have, $p_2/\rho + v_2^2/2 + g(0) = p_3/\rho + v_3^2/2 + g(-\ell)$

now, $v_2 = v_3,\ p_2 = p_v$ (the vapor pressure), $p_3 = p_a$ (atm.pressure)

$$\ell = (p_a - p_v)/(\rho g) = (93{,}100 - 17{,}200)/9810 = 7.74\ m.$$

6.51 Water flows upward through a vertical pipeline which tapers from cross sectional area A_1 to area A_2 in a distance of h. If the pressure at the beginning and end of the constriction are p_1 and p_2 respectively. Determine the flow rate Q in terms of ρ, A_1, A_2, p_1, p_2 and h. Assume the fluid to be inviscid.

Ans. Let the lower point be denoted as point 1, and the upper point denoted as point 2, we have

$$p_1/\rho + v_1^2/2 + g(0) = p_2/\rho + v_2^2/2 + g(h) \rightarrow (p_1 - p_2)/\rho - g(h) = \left(v_2^2 - v_1^2\right)/2$$

Let Q be the flow rate, then $Q = A_1 v_1 = A_2 v_2$ and

$$\frac{p_1 - p_2}{\rho} - g(h) = \frac{1}{2}\left[\left(\frac{Q}{A_2}\right)^2 - \left(\frac{Q}{A_1}\right)^2\right] \rightarrow Q^2 = \frac{2\left[(p_1 - p_2) - \rho g h\right] A_2^2 A_1^2}{\rho\left(A_1^2 - A_2^2\right)}$$

$$\rightarrow Q = A_1 A_2 \sqrt{\frac{2\left[(p_1 - p_2) - \rho g(h)\right]}{\rho\left(A_1^2 - A_2^2\right)}}$$

6.52 Verify that the equation of conservation of mass is automatically satisfied if the velocity components in cylindrical coordinates are given by

$$v_r = -\frac{1}{\rho r}\frac{\partial \psi}{\partial z},\ \ v_z = \frac{1}{\rho r}\frac{\partial \psi}{\partial r},\ \ v_\theta = 0$$

where the density ρ is a constant and ψ is any function of r and z having continuous second partial derivatives.

Ans. The equation of continuity is

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0.$$ With $v_r = -\frac{1}{\rho r}\frac{\partial \psi}{\partial z},\ \ v_z = \frac{1}{\rho r}\frac{\partial \psi}{\partial r},\ \ v_\theta = 0$, we have,

$$\frac{1}{r}\frac{\partial}{\partial r}(rv_r) = -\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{1}{\rho r}\frac{\partial \psi}{\partial z}\right) = -\frac{1}{\rho r}\left(\frac{\partial^2 \psi}{\partial r \partial z}\right),\quad \frac{\partial v_z}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1}{\rho r}\frac{\partial \psi}{\partial r}\right) = \frac{1}{\rho r}\left(\frac{\partial^2 \psi}{\partial z \partial r}\right)$$

Thus, the equation of continuity is automatically satisfied for any function $\psi(x, y)$.

6.53 From the constitutive equation for a compressible fluid $T_{ij} = -p\delta_{ij} - (2\mu/3)\Delta\delta_{ij} + 2\mu D_{ij} + k\Delta\delta_{ij}$, $\Delta = \partial v_j / \partial x_j$, derive the equation

$$\rho\frac{Dv_i}{Dt} = \rho B_i - \frac{\partial p}{\partial x_i} + \frac{\mu}{3}\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right) + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j} + k\frac{\partial}{\partial x_i}\left(\frac{\partial v_j}{\partial x_j}\right)$$

Ans.

$$\frac{\partial T_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_j}\delta_{ij} - \frac{2}{3}\mu\frac{\partial \Delta}{\partial x_j}\delta_{ij} + 2\mu\frac{\partial}{\partial x_j}\frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) + k\frac{\partial \Delta}{\partial x_j}\delta_{ij}$$

$$= -\frac{\partial p}{\partial x_i} - \frac{2}{3}\mu\frac{\partial \Delta}{\partial x_i} + \mu\left(\frac{\partial^2 v_i}{\partial x_j \partial x_j} + \frac{\partial \Delta}{\partial x_i}\right) + k\frac{\partial \Delta}{\partial x_i}$$

That is,

$$\frac{\partial T_{ij}}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{\mu}{3}\frac{\partial \Delta}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j} + k\frac{\partial \Delta}{\partial x_i}.$$ Thus, $\rho\frac{Dv_i}{Dt} = \rho B_i + \frac{\partial T_{ij}}{\partial x_j} \rightarrow$

$$\rho\frac{Dv_i}{Dt} = \rho B_i - \frac{\partial p}{\partial x_i} + \frac{\mu}{3}\frac{\partial \Delta}{\partial x_i} + \mu\frac{\partial^2 v_i}{\partial x_j \partial x_j} + k\frac{\partial \Delta}{\partial x_i}$$

6.54 Show that for a one-dimensional, steady, adiabatic flow of an ideal gas, the ratio of temperature Θ_1/Θ_2 at sections 1 and 2 is given by

$$\frac{\Theta_1}{\Theta_2} = \frac{1 + \frac{1}{2}(\gamma - 1)M_1^2}{1 + \frac{1}{2}(\gamma - 1)M_2^2}$$

where γ is the ratio of specific heat, M_1 and M_2 are local Mach number at section 1 and section 2 respectively.

Ans. $\frac{p}{\rho}\left(\frac{\gamma}{\gamma-1}\right)+\frac{v^2}{2}=\text{constant}$, we have $\frac{p_1}{\rho_1}\left(\frac{\gamma}{\gamma-1}\right)+\frac{v_1^2}{2}=\frac{p_2}{\rho_2}\left(\frac{\gamma}{\gamma-1}\right)+\frac{v_2^2}{2}$.

In terms of the Mach numbers $M_1=v_1/c_1$ and $M_2=v_2/c_2$, we have,

$$\frac{p_1}{\rho_1}\left(\frac{\gamma}{\gamma-1}\right)+\frac{c_1^2M_1^2}{2}=\frac{p_2}{\rho_2}\left(\frac{\gamma}{\gamma-1}\right)+\frac{c_2^2M_2^2}{2}.$$

For an ideal gas, $p=\rho R\Theta$, and $c^2=\frac{\gamma p}{\rho}=\gamma R\Theta$, therefore,

$$\frac{\rho_1 R\Theta_1}{\rho_1}\left(\frac{\gamma}{\gamma-1}\right)+\frac{\gamma R\Theta_1M_1^2}{2}=\frac{\rho_2 R\Theta_2}{\rho_2}\left(\frac{\gamma}{\gamma-1}\right)+\frac{\gamma R\Theta_2M_2^2}{2}\to$$

$$\left(\frac{\gamma}{\gamma-1}\right)\Theta_1-\left(\frac{\gamma}{\gamma-1}\right)\Theta_2=\frac{\gamma\Theta_2M_2^2}{2}-\frac{\gamma\Theta_1M_1^2}{2}\to\Theta_1+\frac{1}{2}(\gamma-1)\Theta_1M_1^2=\Theta_2+\frac{1}{2}(\gamma-1)\Theta_2M_2^2\to$$

$$\Theta_1\left[1+\frac{1}{2}(\gamma-1)M_1^2\right]=\Theta_2\left[1+\frac{1}{2}(\gamma-1)M_2^2\right]\to\frac{\Theta_1}{\Theta_2}=\frac{1+M_2^2(\gamma-1)/2}{1+M_1^2(\gamma-1)/2}.$$

6.55 Show that for a compressible fluid in isothermal flow with no external work,

$\frac{dM^2}{M^2}=2\frac{dv}{v}$, where M is the Mach number. (Assume perfect gas).

Ans. Since $M^2\equiv v^2/c^2$ and $c^2=\gamma R\Theta$ for ideal gas, therefore, $M^2\equiv v^2/(\gamma R\Theta)$

For isothermal flow, $\Theta=\text{constant}$, therefore,

$$M^2\equiv\frac{v^2}{\gamma R\Theta}\to dM^2=\frac{2vdv}{\gamma R\Theta}\to\frac{dM^2}{M^2}=\frac{2vdv}{\gamma R\Theta}\frac{\gamma R\Theta}{v^2}=\frac{2dv}{v}.$$

6.56 Show that for a perfect gas flowing through a duct of constant cross sectional area at constant temperature $\frac{dp}{p}=-\frac{1}{2}\frac{dM^2}{M^2}$. [Use the results of the last problem].

Ans. We have, from $\rho Av=\text{constant}$, $(d\rho)v+\rho(dv)=0\to d\rho/\rho=-dv/v$

Since $\Theta=\text{constant}$, therefore, $p=\rho R\Theta\to dp=R\Theta d\rho\to\frac{dp}{p}=\frac{R\Theta d\rho}{\rho R\Theta}=\frac{d\rho}{\rho}$

Thus, $dp/p=-dv/v$. From the results of last problem, we have, $dM^2/M^2=2dv/v$, therefore, $dp/p=-(1/2)(dM^2/M^2)$.

6.57 For the flow of a compressible inviscid fluid around a thin body in a uniform stream of speed V_m in the x_1 direction, we let the velocity potential be $\varphi=-V_o(x_1+\varphi_1)$, where φ_1 is assumed to be very small. Show that for steady flow the equation governing φ_1 is, with

$M_o = V_o / c_o$, $\left(1-M_o^2\right)\dfrac{\partial^2\varphi_1}{\partial x_1^2}+\dfrac{\partial^2\varphi_1}{\partial x_2^2}+\dfrac{\partial^2\varphi_1}{\partial x_3^2}=0$.

Ans. For steady flow, the equation of continuity is $v_i\dfrac{\partial\rho}{\partial x_i}+\rho\dfrac{\partial v_i}{\partial x_i}=0$, in terms of the potential function φ, we have, $\rightarrow -\dfrac{\partial\varphi}{\partial x_i}\dfrac{\partial\rho}{\partial x_i}-\rho\dfrac{\partial^2\varphi}{\partial x_i\partial x_i}=0$. (i).

The equation of motion is:

$$v_j\frac{\partial v_i}{\partial x_j}=-\frac{1}{\rho}\frac{\partial p}{\partial x_i}=-\frac{1}{\rho}\frac{\partial p}{\partial\rho}\frac{\partial\rho}{\partial x_i}=-\frac{1}{\rho}c^2\frac{\partial\rho}{\partial x_i},\quad\left[\text{note}: c^2=\frac{\partial p}{\partial\rho}=\text{local sound speed}\right]$$

which becomes,

$$\frac{\partial\varphi}{\partial x_j}\frac{\partial^2\varphi}{\partial x_j\partial x_i}=-\frac{1}{\rho}c^2\frac{\partial\rho}{\partial x_i}\quad\rightarrow\frac{\partial\rho}{\partial x_i}=-\frac{\rho}{c^2}\frac{\partial\varphi}{\partial x_j}\frac{\partial^2\varphi}{\partial x_j\partial x_i}\qquad\text{(ii)}$$

(ii) into (i) $\rightarrow\dfrac{\partial\varphi}{\partial x_i}\left(\dfrac{\partial\varphi}{\partial x_j}\dfrac{\partial^2\varphi}{\partial x_j\partial x_i}\right)-c^2\dfrac{\partial^2\varphi}{\partial x_i\partial x_i}=0$ (iii). Now, with $\varphi=-V_o\left(x_1+\varphi_1\right)$, we have,

$$\frac{\partial\varphi}{\partial x_i}=-V_o\left(\delta_{i1}+\frac{\partial\varphi_1}{\partial x_i}\right),\quad\rightarrow\frac{\partial\varphi}{\partial x_i}\left(\frac{\partial\varphi}{\partial x_j}\frac{\partial^2\varphi}{\partial x_j\partial x_i}\right)=-V_o\left(\delta_{i1}+\frac{\partial\varphi_1}{\partial x_i}\right)\left[V_o\left(\delta_{j1}+\frac{\partial\varphi_1}{\partial x_j}\right)V_o\frac{\partial^2\varphi_1}{\partial x_j\partial x_i}\right]$$

$$\approx -V_o^3\left(\delta_{i1}+\frac{\partial\varphi_1}{\partial x_i}\right)\left[\delta_{j1}\frac{\partial^2\varphi_1}{\partial x_j\partial x_i}\right]\approx -V_o^3\delta_{i1}\delta_{j1}\frac{\partial^2\varphi_1}{\partial x_j\partial x_i}=-V_o^3\frac{\partial^2\varphi_1}{\partial x_1\partial x_1}.$$

Thus, Eq.(iii) $\rightarrow -V_o^3\dfrac{\partial^2\varphi}{\partial x_1^2}+c^2V_o\left(\dfrac{\partial^2\varphi_1}{\partial x_1^2}+\dfrac{\partial^2\varphi_1}{\partial x_2^2}+\dfrac{\partial^2\varphi_1}{\partial x_3^2}\right)=0$,

$$\rightarrow\rightarrow\left(1-M_o^2\right)\frac{\partial^2\varphi_1}{\partial x_1^2}+\left(\frac{\partial^2\varphi_1}{\partial x_2^2}+\frac{\partial^2\varphi_1}{\partial x_3^2}\right)=0.$$

6.58 For a one dimensional steady flow of a compressible fluid through a convergent channel, obtain (a) the critical pressure and (b) the corresponding velocity. That is, verify equation (6.30.7) and Eq. (6.30.8)

Ans. (a) From Eq. (6.30.6),

$$\frac{dm}{dt}=A_2\left[\frac{2\gamma}{\gamma-1}p_1\rho_1\left\{\left(\frac{p_2}{p_1}\right)^{\frac{2}{\gamma}}-\left(\frac{p_2}{p_1}\right)^{\frac{\gamma+1}{\gamma}}\right\}\right]^{\frac{1}{2}},\qquad\text{(i)}$$

we have,

$$\frac{d\left(dm/dt\right)}{d\left(p_2/p_1\right)}=\frac{1}{2}\left(\frac{dm}{dt}\right)^{-\frac{1}{2}}\frac{2\gamma}{\gamma-1}p_1\rho_1\left\{\frac{2}{\gamma}\left(\frac{p_2}{p_1}\right)^{\frac{2-\gamma}{\gamma}}-\frac{\gamma+1}{\gamma}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}}\right\}. \qquad \text{(ii)}$$

Thus, $\dfrac{d\left(dm/dt\right)}{d\left(p_2/p_1\right)}=0$ gives,

$$\frac{2}{\gamma}\left(\frac{p_2}{p_1}\right)^{\frac{2-\gamma}{\gamma}}-\frac{\gamma+1}{\gamma}\left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}}=0,\ \text{or}\ \left(\frac{p_2}{p_1}\right)^{\frac{1}{\gamma}}\left[\frac{2}{\gamma}\left(\frac{p_2}{p_1}\right)^{\frac{1-\gamma}{\gamma}}-\frac{\gamma+1}{\gamma}\right]=0\,. \qquad \text{(iii)}$$

That is,

$$\frac{2}{\gamma}\left(\frac{p_2}{p_1}\right)^{\frac{1-\gamma}{\gamma}}=\frac{\gamma+1}{\gamma},\ \text{therefore}\ \left(\frac{p_2}{p_1}\right)_{critical}=\left(\frac{2}{\gamma+1}\right)^{\frac{\gamma}{\gamma-1}}. \qquad \text{(iv)}$$

(b) Substituting this critical pressure into Eq. (6.30.4) for the velocity, we get

$$v_2^2=\frac{2\gamma}{\gamma-1}\frac{p_1}{\rho_1}\left(1-\left(\frac{p_2}{p_1}\right)_{crit}^{\frac{\gamma-1}{\gamma}}\right)=\frac{2\gamma}{\gamma-1}\frac{p_1}{\rho_1}\left(1-\left(\frac{\gamma+1}{2}\right)^{-1}\right)=\frac{2\gamma}{\gamma-1}\frac{p_1}{\rho_1}\left(\frac{\gamma-1}{\gamma+1}\right)=\frac{p_1}{\rho_1}\left(\frac{2\gamma}{\gamma+1}\right).$$

From $\dfrac{p_1}{p_2}=\left(\dfrac{\rho_2}{\rho_1}\right)^{-\gamma}\rightarrow\left(\dfrac{p_1}{\rho_1}\right)\left(\dfrac{\rho_1}{\rho_2}\right)\left(\dfrac{\rho_2}{p_2}\right)=\left(\dfrac{\rho_2}{\rho_1}\right)^{-\gamma}\rightarrow\left(\dfrac{p_1}{\rho_1}\right)=\left(\dfrac{\rho_2}{\rho_1}\right)^{1-\gamma}\left(\dfrac{p_2}{\rho_2}\right)$. But

$\left(\dfrac{\rho_2}{\rho_1}\right)=\left(\dfrac{p_2}{p_1}\right)^{\frac{1}{\gamma}}\rightarrow\left(\dfrac{\rho_2}{\rho_1}\right)^{1-\gamma}=\left(\dfrac{p_2}{p_1}\right)^{\frac{1-\gamma}{\gamma}}\rightarrow\left(\dfrac{p_1}{\rho_1}\right)=\left(\dfrac{p_2}{p_1}\right)^{\frac{1-\gamma}{\gamma}}\left(\dfrac{p_2}{\rho_2}\right)$. Now, at

$\left(\dfrac{p_2}{p_1}\right)_{critical}=\left(\dfrac{\gamma+1}{2}\right)^{\frac{\gamma}{1-\gamma}}$, we have, $\left(\dfrac{p_1}{\rho_1}\right)=\dfrac{\gamma+1}{2}\left(\dfrac{p_2}{\rho_2}\right)$. Thus,

$v_2^2=\gamma\left(\dfrac{p_2}{\rho_2}\right)$=speed of sound at section (2) .

CHAPTER 7

7.1 Verify the divergence theorem $\int_S \mathbf{v}\cdot\mathbf{n}dS = \int_V \text{div}\,\mathbf{v}\, dV$ for the vector field $\mathbf{v} = 2x\mathbf{e}_1 + z\mathbf{e}_2$, by considering the region bounded by $x=0,\ x=2,\ .\ y=0,\ y=2,\ \ z=0,\ z=2$.

Ans. With $\mathbf{v} = 2x\mathbf{e}_1 + z\mathbf{e}_2$, we have

For the face $x=0$, $\mathbf{n}=-\mathbf{e}_1$, $\mathbf{v}\cdot\mathbf{n} = -2\text{x} = 0$, $\int \mathbf{v}\cdot\mathbf{n}dS = 0$.

For the face $x=2$, $\mathbf{n}=+\mathbf{e}_1$, $\mathbf{v}\cdot\mathbf{n} = +2\text{x} = 4$, $\int \mathbf{v}\cdot\mathbf{n}dS = 4A = 4(4) = 16$.

For the face $y=0$, $\mathbf{n}=-\mathbf{e}_2$, $\mathbf{v}\cdot\mathbf{n} = -\text{z}$, $\int \mathbf{v}\cdot\mathbf{n}dS = -\int_0^2 z(2dz) = -2\left[z^2/2\right]_0^2 = -4$.

For the face $y=2$, $\mathbf{n}=+\mathbf{e}_2$, $\mathbf{v}\cdot\mathbf{n} = \text{z}$, $\int \mathbf{v}\cdot\mathbf{n}dS = \int_0^2 z(2dz) = +4$.

For the face $z=0$, $\mathbf{n}=-\mathbf{e}_3$, $\mathbf{v}\cdot\mathbf{n} = 0$, $\int \mathbf{v}\cdot\mathbf{n}dS = 0$.

For the face $z=2$, $\mathbf{n}=+\mathbf{e}_3$, $\mathbf{v}\cdot\mathbf{n} = 0$, $\int \mathbf{v}\cdot\mathbf{n}dS = 0$.

Thus, $\int_S \mathbf{v}\cdot\mathbf{n}dS = 16 - 4 + 4 = 16$ and $\int(\text{div}\,\mathbf{v})dV = \int 2dV = 2(2\times2\times2) = 16.$

So, $\int_S \mathbf{v}\cdot\mathbf{n}dS = \int_V \text{div}\,\mathbf{v}dV$.

7.2 Verify the divergence theorem $\int_S \mathbf{v}\cdot\mathbf{n}dS = \int_V \text{div}\,\mathbf{v}\, dV$ for the vector field, which in cylindrical coordinates, is $\mathbf{v} = 2r\mathbf{e}_r + z\mathbf{e}_z$, by considering the region bounded by $r=2$, $z=0$ and $z=4$.

Ans. For the cylindrical surface $r=2$,

$\mathbf{n}=\mathbf{e}_r,\rightarrow\ \ \mathbf{v}\cdot\mathbf{n} = 2r = 2(2) = 4,\ \ \rightarrow \int \mathbf{v}\cdot\mathbf{n}dS = 4\int dS = 4S = 4\left[(2\pi 2)(4)\right] = 64\pi$.

For the end face $z=0$, $\mathbf{n}=-\mathbf{e}_z,\rightarrow\ \ \mathbf{v}\cdot\mathbf{n} = -z_{z=0} = 0,\ \ \rightarrow \int \mathbf{v}\cdot\mathbf{n}dS = 0$.

For the end face $z=4$, $\mathbf{n}=\mathbf{e}_z,\rightarrow\ \ \mathbf{v}\cdot\mathbf{n} = z_{z=4} = 4,\ \ \rightarrow \int \mathbf{v}\cdot\mathbf{n}dS = 4S = 4\pi 2^2 = 16\pi$.

Therefore, $\int_S \mathbf{v}\cdot\mathbf{n}dS = 64\pi + 16\pi = 80\pi$.

$$\text{div}\,\mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{v_r}{r} + \frac{\partial v_z}{\partial z} = \frac{\partial(2r)}{\partial r} + \frac{2r}{r} + \frac{\partial z}{\partial z} = 2+2+1 = 5.$$

$\int_V \text{div}\,\mathbf{v}dV = 5\left(\pi 2^2\right)(4) = 80\pi$, thus, $\int_S \mathbf{v}\cdot\mathbf{n}dS = \int_V \text{div}\mathbf{v}\, dV$.

7.3 Verify the divergence theorem $\int_S \mathbf{v}\cdot\mathbf{n}dS = \int_V \text{div}\mathbf{v}\, dV$ for the vector field, which in spherical coordinates is $\mathbf{v} = 2r\mathbf{e}_r$, by considering the region bounded by the spherical surface $r=2$.

Ans. For the spherical surface at $r = 2$,

$\mathbf{n} = \mathbf{e}_r, \rightarrow \mathbf{v}\cdot\mathbf{n} = 2r = 2(2) = 4, \rightarrow \int \mathbf{v}\cdot\mathbf{n}dS = \int 4dS = 4S = 4\left(4\pi 2^2\right) = 64\pi$

On the other hand [see Eq.(2.35.26)],

$\text{div}\,\mathbf{v} = \dfrac{1}{r^2}\dfrac{d\left(r^2 v_r\right)}{dr} = \dfrac{1}{r^2}\dfrac{d\left(2r^3\right)}{dr} = 6, \quad \int \text{div}\,\mathbf{v}\,dV = 6\left(\dfrac{4\pi 2^3}{3}\right) = 64\pi$.

7.4 Show that

$$\int_S \mathbf{x}\cdot\mathbf{n}\,dS = 3V.$$

where $\mathbf{x}$ is the position vector and V is the volume enclosed by the boundary S.

Ans. $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$, $\text{div}\,\mathbf{x} = \dfrac{\partial x_1}{\partial x_1} + \dfrac{\partial x_2}{\partial x_2} + \dfrac{\partial x_3}{\partial x_3} = 1+1+1 = 3$.

Thus $\int_S \mathbf{x}\cdot\mathbf{n}dS = \int_V \text{div}\,\mathbf{x}\,dV = 3V$.

7.5 (a) Consider the vector field $\mathbf{v} = \varphi\mathbf{a}$, where φ is a given scalar field and $\mathbf{a}$ is an arbitrary constant vector (independent of position). Using the divergence theorem, prove that

$$\int_V \nabla\varphi\,dV = \int_S \varphi\mathbf{n}\,dS.$$

(b) Show that for any closed surface S that

$$\int_S \mathbf{n}dS = 0.$$

Ans. (a) With $\mathbf{v} = \varphi\mathbf{a}$, $\mathbf{v}\cdot\mathbf{n} = \varphi\mathbf{a}\cdot\mathbf{n} \rightarrow \int \mathbf{v}\cdot\mathbf{n}dS = \mathbf{a}\cdot\int\varphi\mathbf{n}dS$,

$\text{div}\,\mathbf{v} = \text{div}\left(\varphi\mathbf{a}\right) = \dfrac{\partial\varphi a_i}{\partial x_i} = a_i\dfrac{\partial\varphi}{\partial x_i} = \mathbf{a}\cdot\nabla\varphi$.

Thus, $\int_S \mathbf{v}\cdot\mathbf{n}dS = \int_S \text{div}\,\mathbf{v}dV \rightarrow \mathbf{a}\cdot\int_S \varphi\mathbf{n}dS = \mathbf{a}\cdot\int_V \nabla\varphi\,dV$. Since $\mathbf{a}$ is arbitrary, therefore,

$\int_S \varphi\mathbf{n}dS = \int_V \nabla\varphi\,dV$.

(b) Take $\varphi = 1$ in the results of part (a), we have $\int_S \mathbf{n}dS = 0$.

7.6 A stress field $\mathbf{T}$ is in equilibrium with a body force $\rho\mathbf{B}$. Using the divergence theorem, show that for any volume V and boundary surface S, that

$$\int_S \mathbf{t}dS + \int_V \rho\mathbf{B}dV = 0.$$

where $\mathbf{t}$ is the stress vector. That is, the total resultant force is equipollent to zero.

Ans. The stress vector $\mathbf{t}$ is related to the stress tensor $\mathbf{T}$ by $\mathbf{t} = \mathbf{Tn}$, therefore,

$\int_S \mathbf{t}dS = \int_S \mathbf{Tn}dS = \int_V \text{div}\,\mathbf{T}dV$, thus, $\int_S \mathbf{t}dS + \int_V \rho\mathbf{B}dV = \int_V \left(\text{div}\mathbf{T} + \rho\mathbf{B}\right)dV$.

But in equilibrium, $(\text{div}\mathbf{T}+\rho\mathbf{B})=0$, therefore, $\int_S \mathbf{t}dS+\int_V \rho\mathbf{B}dV=0$.

7.7 Let $\mathbf{u^*}$ define an infinitesimal strain field $\mathbf{E^*}=\frac{1}{2}\left[\nabla\mathbf{u^*}+(\nabla\mathbf{u^*})^{\mathrm{T}}\right]$ and let $\mathbf{T^{**}}$ be the symmetric stress tensor in static equilibrium with a body force $\rho\mathbf{B^{**}}$ and a surface traction $\mathbf{t^{**}}$. Using the divergence theorem, verify the following identity (theory of virtual work).

$$\int_S \mathbf{t^{**}}\cdot\mathbf{u^*}\,dS+\int_V(\rho\mathbf{B^{**}})\cdot\mathbf{u^*}\,dV=\int_V T_{ij}^{**}E_{ij}^{*}dV.$$

Ans.

$$\int_S \mathbf{t^{**}}\cdot\mathbf{u^*}\,dS=\int_S(\mathbf{T^{**}}\mathbf{n})\cdot\mathbf{u^*}\,dS=\int_S \mathbf{n}\cdot(\mathbf{T^{**}})^{\mathrm{T}}\mathbf{u^*}\,dS=\int_V \text{div}\left[(\mathbf{T^{**}})^{\mathrm{T}}\mathbf{u^*}\,dV\right].$$

$$=\int_V \text{div}(\mathbf{T^{**}}\mathbf{u^*})dV$$

Now, $\text{div}(\mathbf{T^{**}}\mathbf{u^*})=\frac{\partial(T_{ij}^{**}u_j^*)}{\partial x_i}=\frac{\partial T_{ij}^{**}}{\partial x_i}u_j^*+T_{ij}^{**}\frac{\partial u_j^*}{\partial x_i}=\text{div}\mathbf{T^{**}}\cdot\mathbf{u^*}+T_{ij}^{**}\frac{\partial u_j^*}{\partial x_i}$, therefore,

$$\int_S \mathbf{t^{**}}\cdot\mathbf{u^*}\,dS+\int_V \rho\mathbf{B^{**}}\cdot\mathbf{u^*}dV=\int\left[(\text{div}\mathbf{T^{**}}+\rho\mathbf{B^{**}})\cdot\mathbf{u^*}+T_{ij}^{**}\partial u_j^*/\partial x_i\right]dV=\int T_{ij}^{**}\partial u_j^*/\partial x_i dV.$$

Now, since $T_{ij}^{**}=T_{ji}^{**}$, therefore,

$$T_{ij}^{**}E_{ij}^{*}=\frac{1}{2}T_{ij}^{**}\left(\frac{\partial u_i^*}{\partial x_j}+\frac{\partial u_j^*}{\partial x_i}\right)=\frac{1}{2}T_{ij}^{**}\frac{\partial u_i^*}{\partial x_j}+\frac{1}{2}T_{ij}^{**}\frac{\partial u_j^*}{\partial x_i}=\frac{1}{2}T_{ij}^{**}\frac{\partial u_i^*}{\partial x_j}+\frac{1}{2}T_{ji}^{**}\frac{\partial u_j^*}{\partial x_i}$$

$$=T_{ij}^{**}\frac{\partial u_i^*}{\partial x_j}=T_{ji}^{**}\frac{\partial u_i^*}{\partial x_j}=T_{ij}^{**}\frac{\partial u_j^*}{\partial x_i}.$$

Thus,

$$\int_S \mathbf{t^{**}}\cdot\mathbf{u^*}\,dS+\int_V(\rho\mathbf{B^*})\cdot\mathbf{u^*}\,dV=\int_V T_{ij}^{**}E_{ij}^{*}dV$$

7.8 Using the equations of motion and the divergence theorem, verify the following rate of work identity. Assume the stress tensor to be symmetric.

$$\int_S \mathbf{t}\cdot\mathbf{v}dS+\int_V \rho\mathbf{B}\cdot\mathbf{v}dV=\int_V \rho\frac{D}{Dt}\left(\frac{v^2}{2}\right)dV+\int_V T_{ij}D_{ij}dV$$

Ans. $\int_S \mathbf{t}\cdot\mathbf{v}dS=\int_S \mathbf{Tn}\cdot\mathbf{v}dS=\int_S \mathbf{n}\cdot\mathbf{T}^{\mathrm{T}}\mathbf{v}dS=\int_V \text{div}(\mathbf{T}^{\mathrm{T}}\mathbf{v})dV=\int_V \text{div}(\mathbf{Tv})dV$

Now, $\text{div}(\mathbf{Tv})=\frac{\partial T_{ij}v_j}{\partial x_i}=\frac{\partial T_{ij}}{\partial x_i}v_j+T_{ij}\frac{\partial v_j}{\partial x_i}=\text{div}\mathbf{T}\cdot\mathbf{v}+T_{ij}\frac{\partial v_j}{\partial x_i}$, therefore,

$$\int_S \mathbf{t}\cdot\mathbf{v}dS+\int_V \rho\mathbf{B}\cdot\mathbf{v}dV=\int_V[(\text{div}\mathbf{T}+\rho\mathbf{B})\cdot\mathbf{v}+T_{ij}\partial v_j/\partial x_i]dV=\int_V \rho(D\mathbf{v}/Dt)\cdot\mathbf{v}dV+T_{ij}\partial v_j/\partial x_i.$$

Now, $\left(\rho\frac{D\mathbf{v}}{Dt}\right)\cdot\mathbf{v}=\frac{1}{2}\rho\frac{D(\mathbf{v}\cdot\mathbf{v})}{Dt}=\rho\frac{D}{Dt}\left(\frac{v^2}{2}\right)$ and $T_{ij}D_{ij}=\frac{1}{2}T_{ij}\left(\frac{\partial v_i}{\partial x_j}+\frac{\partial v_j}{\partial x_i}\right)=T_{ij}\frac{\partial v_i}{\partial x_j}=T_{ij}\frac{\partial v_j}{\partial x_i}$.

Therefore, $\int_S \mathbf{t}\cdot\mathbf{v}dS + \int_V \rho\mathbf{B}\cdot\mathbf{v}dV = \int_V \rho\frac{D}{Dt}\left(\frac{v^2}{2}\right)dV + \int_V T_{ij}D_{ij}dV$.

7.9 Consider the velocity and density fields[1]

$$\mathbf{v} = \alpha x_1\mathbf{e}_1,\quad \rho = \rho_o e^{-\alpha(t-t_o)}$$

(a) Check the equation of mass conservation.
(b) Compute the mass and rate of increase of mass in the cylindrical control volume of cross-section A and bounded by $x_1 = 0$ and $x_1 = 3$.
(c) Compute the net mass inflow into the control volume of part (b). Does the net mass inflow equal the rate of mass increase inside the control volume?

Ans. (a). $\frac{D\rho}{Dt} + \rho\,\text{div}\,\mathbf{v} = -\alpha\rho_o e^{-\alpha(t-t_o)} + \rho_o e^{-\alpha(t-t_o)}(\alpha) = 0$. That is, the conservation of mass equation is satisfied.
(b) Inside the control volume,
$m = \int \rho dV = \int_0^3 \rho_o e^{-\alpha(t-t_o)} A dx_1 = 3\rho_o e^{-\alpha(t-t_o)} A$, and $dm/dt = -3\alpha\rho_o e^{-\alpha(t-t_o)} A$. That is, the mass inside the volume is decreasing at the rate of $3\alpha\rho_o e^{-\alpha(t-t_o)} A$.
(c). Rate of inflow from the face $x_1 = 0$ is zero because at $x_1 = 0$, $v_1 = 0$.
Rate of outflow from the face $x_1 = 3$ is given by $\rho v_1\big|_{x_1=3} A = 3\alpha\rho_o e^{-\alpha(t-t_o)} A$. There is no flow across the cylindrical surface because flow is only in the x_1 direction. Thus, the rate of outflow exactly equals the rate of decrease of mass inside the volume.

7.10 (a) Check that the motion

$$x_1 = X_1 e^{\alpha(t-t_o)},\quad x_2 = X_2,\quad x_3 = X_3$$

corresponds to the velocity field $\mathbf{v} = \alpha x_1\mathbf{e}_1$.

(b) For a density field $\rho = \rho_o e^{-\alpha(t-t_o)}$, verify that the mass contained in the material volume that was coincident with the control volume of Prob. 7.9 at time t_o, remains a constant at all times, as it should (conservation of mass).
(c) Compute the total linear momentum for the material volume of part (b).
(d) Compute the force acting on the material volume

Ans. (a) $v_1 = \frac{\partial x_1}{\partial t} = \alpha X_1 e^{\alpha(t-t_o)} = \alpha x_1,\quad v_2 = \frac{\partial x_2}{\partial t} = 0,\quad v_3 = \frac{\partial x_3}{\partial t} = 0$, i.e., $\mathbf{v} = \alpha x_1\mathbf{e}_1$.
(b) The particles which are at $x_1 = 0$ at time t_o have the material coordinates $X_1 = 0$. These particles remain at $x_1 = 0$ at all time. The particles which are at $x_1 = 3$ at time t_o have the material

[1] It should be remarked that, for a real fluid, to achieve the given velocity and density fields in this and some other problems may require body force distributions and/or a pressure density relationship that are not realistic.

coordinates $X_1 = 3$. These particles move in such a way that $x_1 = 3e^{\alpha(t-t_o)}$. Thus, to find the mass inside this material volume as a function of time, we have

$$M = \int_0^{3e^{\alpha(t-t_o)}} \rho_o e^{-\alpha(t-t_o)} A dx_1 = \rho_o e^{-\alpha(t-t_o)} A\left[3e^{\alpha(t-t_o)}\right] = 3\rho_o A.$$

(c) Linear momentum in the material volume

$$\mathbf{P} = \int_0^{3e^{\alpha(t-t_o)}} \rho_o e^{-\alpha(t-t_o)} v_1 A dx_1 \mathbf{e}_1 = \int_0^{3e^{\alpha(t-t_o)}} \rho_o e^{-\alpha(t-t_o)} \alpha x_1 A dx_1 \mathbf{e}_1$$

$$= \rho_o \alpha e^{-\alpha(t-t_o)} A \int_0^{3e^{\alpha(t-t_o)}} x_1 dx_1 \mathbf{e}_1 = \rho_o \alpha e^{-\alpha(t-t_o)} A\left[\frac{9e^{2\alpha(t-t_o)}}{2}\right]\mathbf{e}_1 = \frac{9}{2} A\rho_o \alpha e^{\alpha(t-t_o)} \mathbf{e}_1.$$

(d) Force acting on the material volume

$$\mathbf{F} = \frac{d\mathbf{P}}{dt} = \frac{9}{2} A\rho_o \alpha^2 e^{\alpha(t-t_o)} \mathbf{e}_1.$$

We see that both the linear momentum and the force increase exponentially with time. This is due to the given data of density and velocity fields, which describe the space occupied by the fixed material increases exponentially with time, $\left[0 \le x_1 \le 3e^{\alpha(t-t_o)}\right]$, while the density decreases exponentially $\left[\rho_o e^{-\alpha(t-t_o)}\right]$ to conserve the mass. We note also that at $t = t_o$, the materials occupy the space between $x_1 = 0$ and $x_1 = 3$, and $\mathbf{F} = \frac{9}{2} A\rho_o \alpha^2 \mathbf{e}_1$.

7.11 Do Problem 7.9 for the velocity field $\mathbf{v} = \alpha x_1 \mathbf{e}_1$ and the density field $\rho = k\rho_o / x_1$ and for the cylindrical control volume bounded by $x_1 = 1$ and $x_1 = 3$.

Ans. (a). $\frac{D\rho}{Dt} + \rho \operatorname{div} \mathbf{v} = v_1 \frac{\partial \rho}{\partial x_1} + \rho \frac{\partial v_1}{\partial x_1} = \alpha x_1 \left(-k\frac{\rho_o}{x_1^2}\right) + k\frac{\rho_o}{x_1}(\alpha) = 0$. That is, the conservation of mass equation is satisfied.

(b) Inside the control volume, $m = \int \rho dV = \int_1^3 k\frac{\rho_o}{x_1} A dx_1 = kA\rho_o \ln 3$, and $\frac{dm}{dt} = 0$.

(c). Rate of inflow from the face $x_1 = 1$ is $\left[\rho v_1 A\right]_{x_1} = \left[k\frac{\rho_o}{x_1}\alpha x_1 A\right]_{x_1=1} = k\alpha\rho_o A$.

Rate of outflow from the face $x_1 = 3$ is given by $\left[\rho v_1 A\right]_{x_1=3} = \left[k\frac{\rho_o}{x_1}\alpha x_1 A\right]_{x_1=3} = k\alpha\rho_o A$. There is no flow across the cylindrical surface because flow is only in the x_1 direction. The net mass inflow is 0, which is equal to the rate of increase of mass inside the control volume.

7.12 The center of mass $\mathbf{x}_{c.m}$ of a material volume is defined by the equation

$$m\mathbf{x}_{c.m} = \int_{V_m} \mathbf{x}\rho dV, \text{ where } m = \int_{V_m} \rho dV$$

Demonstrate that the linear momentum principle may be written in the form

$$\int_S \mathbf{t}dS + \int_V \rho\mathbf{B}dV = m\mathbf{a}_{c.m}$$

where $\mathbf{a}_{c.m}$ is the acceleration of the mass center.

Ans. We have from the principle of linear momentum: $\int_S \mathbf{t}dS + \int_V \rho\mathbf{B}dV = \frac{D}{Dt}\int_{V_m} \rho\mathbf{v}dV$

Now, since $\frac{D}{Dt}(\rho dV) = 0$, therefore, $\int \rho\mathbf{v}dV = \int \frac{D}{Dt}(\rho\mathbf{x}dV) = \frac{D}{Dt}\int(\rho\mathbf{x}dV) = \frac{D}{Dt}m\mathbf{x}_{c.m} = m\mathbf{v}_{c.m}$.

Therefore, $\int_S \mathbf{t}dS + \int_V \rho\mathbf{B}dV = \frac{D}{Dt}m\mathbf{v}_{c.m} = m\mathbf{a}_{c.m}$.

7.13 Consider the following velocity field and density field

$$\mathbf{v} = \frac{\alpha x_1}{1+\alpha t}\mathbf{e}_1, \quad \rho = \frac{\rho_o}{1+\alpha t}$$

(a) Compute the total linear momentum and rate of increase of linear momentum in a cylindrical control volume of cross-sectional area A and bounded by the planes $x_1 = 1$ and $x_1 = 3$.
(b) Compute the net rate of outflow of linear momentum from the control volume of (a)
(c) Compute the total force on the material in the control volume.
(d) Compute the total kinetic energy and rate of increase of kinetic energy for the control volume of (a).
(e) Compute the net rate of outflow of kinetic energy from the control volume.

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*Ans***.** (a) Linear momentum is

$$\mathbf{P} = \int \rho\mathbf{v}dV = \int_1^3 \frac{\rho_o}{1+\alpha t}\frac{\alpha x_1}{1+\alpha t}Adx_1\mathbf{e}_1 = \frac{\rho_o A\alpha}{(1+\alpha t)^2}\int_1^3 x_1 dx_1\mathbf{e}_1 = \left(\frac{9}{2}-\frac{1}{2}\right)\frac{\rho_o A\alpha}{(1+\alpha t)^2}\mathbf{e}_1 = \frac{4\rho_o A\alpha}{(1+\alpha t)^2}\mathbf{e}_1.$$

Rate of increase of linear momentum inside the control volume is

$$\frac{d\mathbf{P}}{dt} = -\frac{8\rho_o A\alpha^2}{(1+\alpha t)^3}\mathbf{e}_1.$$

(b) Net rate of outflow of linear momentum in $\mathbf{e}_1$ direction

$$=\left(\rho A v_1^2\right)_{x_1=3} - \left(\rho A v_1^2\right)_{x_1=1} = \frac{\rho_o A}{1+\alpha t}\left(\frac{9\alpha^2}{(1+\alpha t)^2} - \frac{\alpha^2}{(1+\alpha t)^2}\right) = \frac{8\alpha^2\rho_o A}{(1+\alpha t)^3}.$$

(c) Total force = Rate of inc. of **P** inside control volume + net outflux of **P**

$$= -\frac{8\rho_o A\alpha^2}{(1+\alpha t)^3}\mathbf{e}_1 + \frac{8\alpha^2\rho_o A}{(1+\alpha t)^3}\mathbf{e}_1 = 0.$$

(d) Total kinetic energy inside the control volume

$$K.E. = \int \frac{1}{2}\rho v^2 dV = \frac{1}{2}\int_1^3 \frac{\rho_o}{1+\alpha t}\left(\frac{\alpha x_1}{1+\alpha t}\right)^2 Adx_1 = \frac{1}{2}\frac{\rho_o A\alpha^2}{(1+\alpha t)^3}\int_1^3 x_1^2 dx_1 = \frac{13}{3}\frac{\rho_o A\alpha^2}{(1+\alpha t)^3}$$

$$\frac{d}{dt}K.E = -\frac{13\rho_o A\alpha^3}{(1+\alpha t)^4}.$$

(e) Net rate of outflow of kinetic energy from the control volume=

$$\left(\frac{1}{2}\rho A v_1^3\right)_{x_1=3} - \left(\frac{1}{2}\rho A v_1^3\right)_{x_1=1} = \frac{1}{2}\frac{\rho_o A}{1+\alpha t}\left(\frac{27\alpha^3}{(1+\alpha t)^3} - \frac{\alpha^3}{(1+\alpha t)^3}\right) = \frac{13\alpha^3\rho_o A}{(1+\alpha t)^4}.$$

7.14 Consider the velocity and density fields

$$\mathbf{v} = \alpha x_1 \mathbf{e}_1, \quad \rho = \rho_o e^{-\alpha(t-t_o)}$$

For an arbitrary time t, consider the material contained in the cylindrical control volume of cross-sectional area A , bounded by $x_1 = 0$ and $x_1 = 3$.

(a) Determine the linear momentum and rate of increase of linear momentum in this control volume.

(b) Determine the outflux of linear momentum.

(c) Determine the net resultant force that is acting on the material contained in the control volume.

Ans. (a) Linear momentum inside the control volume:

$$\mathbf{P} = \int \rho \mathbf{v} dV = \int_0^3 \rho_o e^{-\alpha(t-t_o)} v_1 A dx_1 \mathbf{e}_1 = \int_0^3 \rho_o e^{-\alpha(t-t_o)} \alpha x_1 A dx_1 \mathbf{e}_1$$

$$= \rho_o \alpha e^{-\alpha(t-t_o)} A \int_0^3 x_1 dx_1 \mathbf{e}_1 = \rho_o \alpha e^{-\alpha(t-t_o)} A\left[\frac{9}{2}\right]\mathbf{e}_1 = \frac{9}{2} A\rho_o \alpha e^{-\alpha(t-t_o)}\mathbf{e}_1.$$

Rate of increase of linear momentum inside the control volume

$$\frac{d\mathbf{P}}{dt} = -\frac{9}{2}\alpha^2 A\rho_o e^{-\alpha(t-t_o)}\mathbf{e}_1.$$

(b) Out flux of linear momentum from the control volume in the $\mathbf{e}_1$ direction:

$$\left(\rho A v_1^2\right)_{x_1=3} - \left(\rho A v_1^2\right)_{x_1=0} = \left(\rho A \alpha^2 x_1^2\right)_{x_1=3} - \left(\rho A \alpha^2 x_1^2\right)_{x_1=0} = 9A\rho_o \alpha^2 e^{-\alpha(t-t_o)}.$$

(c) The total force = rate of increase of linear momentum inside the control volume + net momentum outflux from the control volume. Thus,

$$\mathbf{F} = -\frac{9}{2}\alpha^2 A\rho_o e^{-\alpha(t-t_o)}\mathbf{e}_1 + 9A\rho_o\alpha^2 e^{-\alpha(t-t_o)}\mathbf{e}_1 = \frac{9}{2}A\rho_o\alpha^2 e^{-\alpha(t-t_o)}\mathbf{e}_1$$

We see that the force exerted on the material within the control volume decreases exponentially with time. This is due to the given data of density field and velocity field, which states that within the *fixed space* defined by $0 \le x_1 \le 3$, the density decreases exponentially with time while speed at each spatial point is independent of time. We also note that at $t = t_o$, $\mathbf{F} = \frac{9}{2}A\rho_o\alpha^2\mathbf{e}_1$, the same results was obtained in Problem 7.10.

7.15 Do Problem 7.14 for the same velocity field, $\mathbf{v} = \alpha x_1 \mathbf{e}_1$ but with $\rho = k\frac{\rho_o}{x_1}$ and the cylindrical control volume bounded by $x_1 = 1$ and $x_1 = 3$.

Ans. (a) Linear momentum inside the control volume:

$$\mathbf{P} = \int \rho \mathbf{v} dV = \int_1^3 (k\rho_o / x_1) v_1 A dx_1 \mathbf{e}_1 = \int_1^3 (k\rho_o / x_1)(\alpha x_1) A dx_1 \mathbf{e}_1 = \rho_o k\alpha A \int_1^3 dx_1 \mathbf{e}_1 = 2\rho_o k\alpha A \mathbf{e}_1$$

Rate of increase of linear momentum inside the control volume

$$\frac{d\mathbf{P}}{dt} = \mathbf{0}.$$

(b) Out flux of linear momentum from the control volume in the $\mathbf{e}_1$ direction:

$$\left(\rho A v_1^2\right)_{x_1=3} - \left(\rho A v_1^2\right)_{x_1=1} = \left(k\left(\rho_o / x_1\right) A\alpha^2 x_1^2\right)_{x_1=3} - \left(k\left(\rho_o / x_1\right) A\alpha^2 x_1^2\right)_{x_1=1} = 2k\rho_o A\alpha^2 .$$

(c) The total force = rate of increase of linear momentum inside the control volume + net momentum outflux from the control volume $\mathbf{F} = 0 + 2k\rho_o A\alpha^2 \mathbf{e}_1 = 2k\rho_o A\alpha^2 \mathbf{e}_1$

7.16 Consider the flow field $\mathbf{v} = k(x\mathbf{e}_1 - y\mathbf{e}_2)$ with ρ =constant. For a control volume defined by $x = 0, x = 2, y = 0, y = 2, z = 0, z = 2$, determine the net resultant force and moment about the origin (note misprint in text) that are acting on the material contained in this volume.

Ans. Since the flow is steady, the resultant force = net linear momentum outflux through the three pairs of faces:

(i) through $x = 0$ and $x = 2$,

$$\left(\int \rho v_1 \mathbf{v} dA\right)_{x=2} - \left(\int \rho v_1 \mathbf{v} dA\right)_{x=0} = k^2 \left[\iint \rho x (x\mathbf{e}_1 - y\mathbf{e}_2) dydz\right]_{x=2} - k^2 \left[\iint \rho x (x\mathbf{e}_1 - y\mathbf{e}_2) dydz\right]_{x=0}$$

$$= k^2 \rho \int_{y=0}^{2} \left[\int_{z=0}^{2} (4\mathbf{e}_1 - 2y\mathbf{e}_2) dz\right] dy = 2k^2 \rho \left[\int_{y=0}^{2} (4\mathbf{e}_1 - 2y\mathbf{e}_2) dy\right] \qquad \text{i}$$

$$= 2k^2 \rho (8\mathbf{e}_1 - 4\mathbf{e}_2) = k^2 \rho (16\mathbf{e}_1 - 8\mathbf{e}_2).$$

(ii) through $y = 0$ and $y = 2$,

$$\left(\int \rho v_2 \mathbf{v} dA\right)_{y=2} - \left(\int \rho v_2 \mathbf{v} dA\right)_{y=0} = k^2 \left[\iint \rho(-y)(x\mathbf{e}_1 - y\mathbf{e}_2) dxdz\right]_{y=2} - k^2 \left[\iint \rho(-y)(x\mathbf{e}_1 - y\mathbf{e}_2) dxdz\right]_{y=0}$$

$$= k^2 \rho \int_{x=0}^{2} \left[\int_{z=0}^{2} (-2x\mathbf{e}_1 + 4\mathbf{e}_2) dz\right] dx = 2k^2 \rho \int_{x=0}^{2} (-2x\mathbf{e}_1 + 4\mathbf{e}_2) dx$$

$$= 2k^2 \rho \left[-x^2 \mathbf{e}_1 + 4x\mathbf{e}_2\right]_{x=2} = 2k^2 \rho (-4\mathbf{e}_1 + 8\mathbf{e}_2) = k^2 \rho (-8\mathbf{e}_1 + 16\mathbf{e}_2).$$

(iii) through $z = 0$ and $z = 2$,

$$\left(\int \rho v_3 \mathbf{v} dA\right)_{z=2} - \left(\int \rho v_3 \mathbf{v} dA\right)_{z=0} = 0, \ (v_3 = 0) .$$

Thus, the total net force

$$\mathbf{F} = k^2 \rho (16\mathbf{e}_1 - 8\mathbf{e}_2) + k^2 \rho (-8\mathbf{e}_1 + 16\mathbf{e}_2) = k^2 \rho (8\mathbf{e}_1 + 8\mathbf{e}_2) .$$

The flow is steady, the resultant moment about a point = net moment of momentum outflux about the same point. Take the point to be the origin, then

(i) through $x=0$ and $x=2$,

$$\left(\int \rho v_1(\mathbf{r}\times\mathbf{v})dA\right)_{x=2}-\left(\int \rho v_1(\mathbf{r}\times\mathbf{v})dA\right)_{x=0}=k^2\left[\int\int \rho x(x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3)\times(x\mathbf{e}_1-y\mathbf{e}_2)dydz\right]_{x=2}$$

$$=k^2\rho\left[\int\int\left(xyz\mathbf{e}_1+x^2z\mathbf{e}_2-2x^2y\mathbf{e}_3\right)dydz\right]_{x=2}=k^2\rho\int_{y=0}^{2}\left[\int_{z=0}^{2}(2yz\mathbf{e}_1+4z\mathbf{e}_2-8y\mathbf{e}_3)dz\right]dy \qquad \text{(i)}$$

$$=k^2\rho\int_{y=0}^{2}(4y\mathbf{e}_1+8\mathbf{e}_2-16\mathbf{e}_3)dy=k^2\rho(8\mathbf{e}_1+16\mathbf{e}_2-32\mathbf{e}_3).$$

through $y=0$ and $y=2$,

$$\left(\int \rho v_2(\mathbf{r}\times\mathbf{v})dA\right)_{y=2}-\left(\int \rho v_2(\mathbf{r}\times\mathbf{v})dA\right)_{y=0}=k^2\left[\int\int \rho(-y)(x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3)\times(x\mathbf{e}_1-y\mathbf{e}_2)dxdz\right]_{y=2}$$

$$=k^2\rho\left[\int\int\left(-y^2z\mathbf{e}_1-xyz\mathbf{e}_2+2xy^2\mathbf{e}_3\right)dxdz\right]_{y=2}=k^2\rho\int_{x=0}^{2}\left[\int_{z=0}^{2}(-4z\mathbf{e}_1-2xz\mathbf{e}_2+8x\mathbf{e}_3)dz\right]dx$$

$$=k^2\rho\int_{y=0}^{2}(-8\mathbf{e}_1-4x\mathbf{e}_2+16x\mathbf{e}_3)dx=k^2\rho(-16\mathbf{e}_1-8\mathbf{e}_2+32\mathbf{e}_3).$$

(iii) through $z=0$ and $z=2$,

$$\left(\int \rho v_3(\mathbf{r}\times\mathbf{v})dA\right)_{z=2}-\left(\int \rho v_3(\mathbf{r}\times\mathbf{v})dA\right)_{z=0}=0$$

Thus, $\mathbf{M}_o=k^2\rho(8\mathbf{e}_1+16\mathbf{e}_2-32\mathbf{e}_3)+k^2\rho(-16\mathbf{e}_1-8\mathbf{e}_2+32\mathbf{e}_3)=k\rho(-8\mathbf{e}_1+8\mathbf{e}_2)$.

7.17 For Hagen-Poiseuille flow in a pipe, $\mathbf{v}=C\left(r_o^2-r^2\right)\mathbf{e}_1$. Calculate the momentum flux across a cross-section. For the same flow rate, if the velocity is assumed to be uniform, what is the momentum flux across a cross section? Compare the two results.

Ans. Momentum flux across a cross section

$$=\int \rho v_1^2 dA\mathbf{e}_1=\rho C^2\int_o^{r_o}\left(r_o^2-r^2\right)^2(2\pi r)dr\mathbf{e}_1=\left(\rho C^2\pi r_o^6/3\right)\mathbf{e}_1$$

Volume flow rate is $Q=\int v_1 dA=C\int_o^{r_o}\left(r_o^2-r^2\right)(2\pi r)dr=C\left(\pi r_o^4/2\right)$.

The uniform flow which has the same flow rate Q is given by : $\mathbf{v}=(Q/\pi r_o^2)\mathbf{e}_1=(Cr_o^2/2)\mathbf{e}_1$.
The momentum flux across a cross section for this uniform flow is given by
$\rho Qv_1\mathbf{e}_1=(\pi\rho C^2 r_o^6/4)\mathbf{e}_1$.

Thus, the momentum flux for the Hagen-Poiseuille flow is $\frac{4}{3}$ that of the uniform flow.

7.18 Consider a steady flow of an incompressible viscous fluid of density ρ, flowing up a vertical pipe of radius R. At the lower section of the pipe, the flow is uniform with a speed v_l and a pressure p_l .After flowing upward through a distance ℓ, the flow becomes fully developed with a parabolic velocity distribution at the upper section, where the pressure is p_u. Obtain an expression for the fluid pressure drop p_l-p_u between the two sections in terms of ρ, R and the frictional force F_f, exerted on the fluid column from the wall though viscosity.

Ans. Let the control volume encloses the fluid between the two sections. The linear momentum theorem states that: For steady flow, Force on the fluid= Momentum outflux – momentum influx.

The force on the fluid in the control volume is given by: $\left(p_l - p_u\right)A - \rho g\left(A\ell\right) - F_f$.

The momentum influx through the lower section = $\rho v_l^2\left(\pi R^2\right)$.

The momentum outflux through the upper section $\int \rho v_u^2\left(2\pi r dr\right)$, where $v_u = C(R^2 - r^2)$. The constant C can be obtain as follows

$Q = \int_0^R v_u 2\pi r dr = 2\pi C\int_0^R (R^2 - r^2) r dr = \pi C R^4 / 2 = v_l\left(\pi R^2\right) \rightarrow C = 2v_l / R^2$.

Thus, $v_u = \dfrac{2v_l}{R^2}(R^2 - r^2)$.

Momentum outflux $\int_o^R \rho v_u^2\left(2\pi r dr\right) = \rho 2\pi\left(2v_l / R^2\right)^2 \int_o^R (R^2 - r^2)^2\left(r dr\right)$

$= \left(8\rho\pi v_l^2 / R^4\right)\int_o^R (R^2 - r^2)^2\left(r dr\right) = \left(8\rho\pi v_l^2 / R^4\right)\left(R^6 / 6\right) = 4\rho\pi v_l^2 R^2 / 3$.

Thus, $\left(p_l - p_u\right)A - \rho g\left(A\ell\right) - F_f = 4\rho\pi v_l^2 R^2 / 3 - \rho v_l^2(\pi R^2) = \rho v_l^2(\pi R^2) / 3$

$\rightarrow \left(p_l - p_u\right)(\pi R^2) = \rho v_l^2(\pi R^2) / 3 + F_f + \rho g(\pi R^2)\left(\ell\right)$. That is,

$\left(p_l - p_u\right) = \rho v_l^2 / 3 + F_f / (\pi R^2) + \rho g\ell$.

7.19 A pile of chain on a table falls through a hole from the table under the action of gravity. Derive the differential equation governing the hanging length x . [Assume the pile is large compared with the hanging portion]

Ans. Using a control volume $\left(Vc\right)_2$ [see Fig. 7.6-1 in Section7.6] enclosing the hanging down portion x of the chain, we can obtain the same equation as that given in Eq. (iv) of Section 7.6, i.e., with μ denoting m / ℓ, mass per unit length:

$$\mu g x - T = \mu x d^2 x / dt^2 \qquad (1)$$

where T is the tension on the chain at the hole. Next, using a control volume enclosing the pile above the table, then, since the particles of the chain pile stay essentially at rest at any given instant (except those near the hole), we can assume that the rate of change of momentum inside the control volume is zero (quasi-static approximation). Further, we assume that the net force acting at the pile is the tension T at the hole (the reaction of the supporting table exactly balances the weight of the pile). Then, the momentum principle gives:

$$T = \mu\left(dx / dt\right)^2 \qquad (2).$$

Equations (1) and (2) give

$$g x = x\frac{d^2 x}{dt^2} + \left(\frac{dx}{dt}\right)^2 \qquad (3)$$

We note that this equation is a good approximation when the length of the pile is large compared with the hanging portion x . Eventually, when the pile reduces to essentially a flat straight segment on the table, Eq. (vi) of Section 7.6 becomes a better approximation.

7.20 A water jet of 5 $cm.$ diameter moves at 12 $m/\sec$, impinges on a curved vane which deflects it 60^o from its original direction. Neglect the weight, obtain the force exerted by the liquid on the vane. (see Fig. 7.6-2 of Example 7.6.2).

Ans. Referring to Fig. 7.6-2, we have, $v_o = 12\ m/s,\ \theta = 60^o$, volume flow rate

$$Q = v_o(\pi d^2/4) = 12\left(\pi(5\times10^{-2})^2/4\right) = 235.6\times10^{-4} m^3, \quad \rho Q v_o = (998)(235.6\times10^{-4})(12) = 282N$$

Thus, force on the jet =

$$-\rho Q v_o\left(1-\cos 60^o\right)\mathbf{e}_1 + \rho Q v_o \sin 60^o \mathbf{e}_2 = -282(0.5)\mathbf{e}_1 + 282(0.866)\mathbf{e}_2 = -141\mathbf{e}_1 + 244\mathbf{e}_2 N.$$

Force on the vane from the jet is $141\mathbf{e}_1 - 244\mathbf{e}_2 N$.

7.21 A horizontal pipeline of 10 $cm.$ diameter bends through 90^o, and while bending, changes its diameter to 5 $cm.$ The pressure in the 10 $cm.$ pipe is 140 $kPa.$ Estimate the resultant force on the bends when 0.005 $m^3/\sec$ .of water is flowing in the pipeline.

Ans. Let (v_u, p_u, A_u) and (v_d, p_d, A_d) denote upstream and downstream (speed, pressure and cross-sectional area) respectively and Q the volume discharge. We have, $Q = 0.005\ m^3/s$,

$$v_u = 0.005/\left(\pi(0.1)^2/4\right) = 0.6366\ m/s, \quad v_d = 0.005/\left(\pi(0.05)^2/4\right) = 2.546\ m/s$$

Upstream pressure $p_u = 140,000\ Pa$. Down stream pressure can be obtained from Bernoulli Equation: $\dfrac{p_u}{\rho} + \dfrac{v_u^2}{2} = \dfrac{p_d}{\rho} + \dfrac{v_d^2}{2}$. Thus,

$$p_d = p_u + \frac{\rho}{2}\left(v_u^2 - v_d^2\right) = 140,000 + \frac{998}{2}\left(0.6366^2 - 2.546^2\right) = 137,000.$$

Let $\mathbf{e}_1$ be the direction of the incoming flow and $\mathbf{e}_2$ be the direction after the 90^o bend, then, we have,

Momentum outflux = $\rho Q \mathbf{v}_d = (998)(.005)(2.546)\mathbf{e}_2 = 12.7\mathbf{e}_2$.

Momentum influx = $\rho Q v_u = (998)(.005)(0.6366) = 3.18\mathbf{e}_1$.

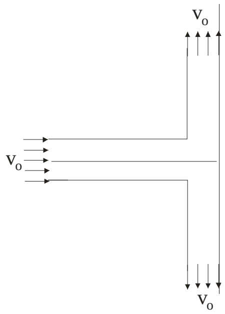
Momentum principle gives: $p_u A_u \mathbf{e}_1 - p_d A_d \mathbf{e}_2 + \mathbf{F_w} = \rho Q v_d \mathbf{e}_2 - \rho Q v_u \mathbf{e}_1$.

Force on water $\mathbf{F_w} = -(\rho Q v_u + p_u A_u)\mathbf{e}_1 + (\rho Q v_d + p_d A_d)\mathbf{e}_2$

$$= -\left(3.177 + 140,000(\pi(0.1)^2/4)\right)\mathbf{e}_1 + \left(12.7 + 137000\pi(0.05)^2/4\right)\mathbf{e}_2 = -1100\mathbf{e}_1 + 282\mathbf{e}_2 N.$$

Thus, the force from water to the bend is $-\mathbf{F_w} = 1100\mathbf{e}_1 - 282\mathbf{e}_2 N.$

7.22 Figure P7.1 shows a steady water jet of area A impinging onto the flat wall. Find the force exerted on the wall. Neglect weight and viscosity of water.



Ans. Let the control volume be coincident with the outline of the flow shown in the figure.

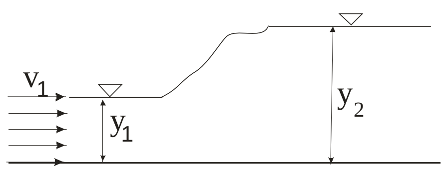
Force on the liquid $\mathbf{F}_L$ = momentum outflux-momentum influx = $\mathbf{0} - \rho Q v_o \mathbf{e}_1$

Force on the wall = $\rho Q v_o \mathbf{e}_1 = \rho A v_o^2 \ \mathbf{e}_1$.

7.23 Frequently in open channel flow, a high speed flow "jumps" to a low speed flow with an abrupt rise in the water surface. This is known as a **hydraulic jump**. Referring to Fig. p7.2, if the flow rate is Q per unit width, show that when the jump occurs, the relation between y_1 and y_2 .is given by

$$y_2 = -\frac{y_1}{2} + \frac{1}{2} y_1 \sqrt{1 + \frac{8v_1^2}{g y_1}}$$

Assume the flow before and after the jump is uniform and the pressure distribution is hydrostatic.



Ans. Use a control volume enclosing the water with an upstream section before the jump and a downstream section after the jump. According to the momentum principle, the force on the fluid per unit width is given by (neglect friction from the ground and air)

$F_x = \rho g y_1^2 / 2 - \rho g y_2^2 / 2 = \rho Q v_2 - \rho Q v_1$, thus , $(g/2)\left(y_1^2 - y_2^2\right) = Q\left(v_2 - v_1\right) = v_1 y_1 \left(v_2 - v_1\right)$.

Conservation of mass gives: $v_1 y_1 = v_2 y_2 \rightarrow v_2 - v_1 = v_1 y_1 / y_2 - v_1 = v_1\left(y_1 - y_2\right)/ y_2$. Therefore, we have, $(g y_2 / 2)\left(y_1^2 - y_2^2\right) = v_1^2 y_1 \left(y_1 - y_2\right)$.

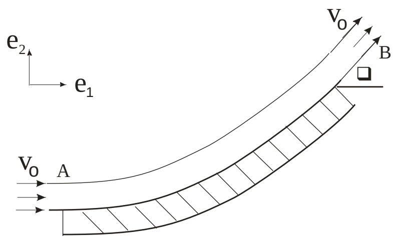
The above equation shows that $y_1 = y_2$ is a root for the equation. This solution corresponds to a flow without a jump. To look for the jump solution, we eliminate the factor $\left(y_1 - y_2\right)$ and obtain

$$g y_2 \left(y_1 + y_2\right)/2 = v_1^2 y_1 \rightarrow y_2^2 + y_1 y_2 - (2v_1^2 / g) y_1 = 0$$

$$y_2 = (1/2)\left[-y_1 + \sqrt{y_1^2 + (8v_1^2 y_1 / g)}\right] = -y_1/2 + \frac{1}{2} y_1 \sqrt{1 + \left(8v_1^2 / g y_1\right)}$$

7.24 If the curved vane of Example 7.6.2 moves with a velocity $v < v_o$ in the same direction as the oncoming jet, find the resultant force exerted on the vane by the jet.

Ans. Fig. 7.6-2 of Example 7.6.2 is reproduced below.



Let the control volume surrounding the jet moves with the vane, then the flow is steady with respect to the moving control volume.
Momentum outflux relative to the control volume =

$$\rho Q(v_o - v)\left(\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2\right) = \rho A(v_o - v)^2 \left(\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2\right)$$

Momentum influx relative to the control volume is

$$\rho Q(v_o - v)\mathbf{e}_1 = \rho A(v_o - v)^2 \mathbf{e}_1$$

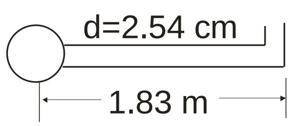
Thus, since the control volume moves with a constant speed, there is no extra term to be added to the momentum equation for the fixed control volume case. Thus, force acting on the jet is

$$\mathbf{F}_{\text{jet}} = \rho A(v_o - v)^2 \left(\cos\theta \mathbf{e}_1 + \sin\theta \mathbf{e}_2\right) - \rho A(v_o - v)^2 \mathbf{e}_1 = \rho A(v_o - v)^2 \left[\left(\cos\theta - 1\right)\mathbf{e}_1 + \sin\theta \mathbf{e}_2\right]$$

and the force on the vane is

$$\mathbf{F}_{\text{vane}} = \rho A(v_o - v)^2 \left[\left(1 - \cos\theta\right)\mathbf{e}_1 - \sin\theta \mathbf{e}_2\right].$$

7.25 For the half-arm sprinkler shown in Fig. P7.3, find the angular speed if $Q = 0.566 m^3 / \text{sec}$. Neglect friction.



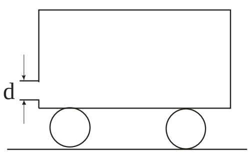
Ans. Let the control volume V_c rotate with the arm. Then, relative to the control volume, the outflux of moment of momentum about an axis passing through O and perpendicular to the plane of the paper is $\rho Q\left(Q / A\right) r_o \mathbf{e}_3$, where r_o is the length of the arm. There is no influx of moment of momentum about the same axis since the inflow is parallel to it. Since the control volume is rotating with an angular velocity ω about the same axis, we need to add terms to the left hand side of Eq. (7.9.8) , the moment of momentum principle. The terms that need to be added are given in Eq. (7.9.9). With $\boldsymbol{\omega} = \omega \mathbf{e}_3$ and $\mathbf{x} = x\mathbf{e}_1$, we have, $\boldsymbol{\omega} \times \mathbf{x} = x\omega \mathbf{e}_2$, $\boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{x}\right) = -x\omega^2 \mathbf{e}_1$ so that $\mathbf{x} \times \boldsymbol{\omega} \times \left(\boldsymbol{\omega} \times \mathbf{x}\right) = 0$. We also have, $\mathbf{a}_o = 0$ and $\dot{\omega} = 0$, therefore, the only non-zero term is

$$-2\int \mathbf{x} \times \left(\boldsymbol{\omega} \times \mathbf{v}\right) dm = -2\int x\mathbf{e}_1 \times \left(\omega \mathbf{e}_3 \times \left(Q / A\right)\mathbf{e}_1\right) \rho A dx = -2\rho\omega Q \int_0^{r_o} x dx \mathbf{e}_3 = -\rho\omega Q r_o^2 \mathbf{e}_3 .$$

Adding this term to Eq. (7.9.8), whose left hand side is zero (because frictional torque is neglected) and whose right hand side is the net moment of momentum outflux, we have,

$-\rho\omega Q r_o^2 \mathbf{e}_3 = \rho Q(Q/A) r_o \mathbf{e}_3 \rightarrow \omega = -Q/(A r_o)$. Now, $A = \pi(2.54\times10^{-2})^2/4 = 5.067\times10^{-4}\,m^2$, therefore, $\omega = -Q/(A r_o) = -0.566/[(5.067\times10^{-4})(1.83)] = -610.4\; rad/s$.
The minus sign means the rotation is clockwise looking from the top.

7.26 The tank car shown in fig. P7.4 contains water and compressed air which is regulated to force a water jet out of the nozzle at a constant rate of $Q\; m^3/\sec$. The diameter of the jet is $d\;\; cm.$, the initial total mass of the tank car is M_o. Neglecting frictional forces, find the velocity of the car as a function of time.



Ans. Let the control volume V_c encloses the whole tank car and moves with the car. Then relative to the control volume, the momentum outflux is

$$-\rho Q\left(4Q/\pi d^2\right)\mathbf{e}_1 = -4\rho Q^2/(\pi d^2)\mathbf{e}_1.$$

There is no momentum influx. Since the control volume moves with the car which has an acceleration $a_o\mathbf{e}_1$, therefore, the momentum principle in the $\mathbf{e}_1$ direction takes the form [see Eq.(7.8.20}: (with al frictional/resistance force neglected):
$-\left(M_o - \rho Q t\right)(dv/dt) = -4\rho Q^2/(\pi d^2)$. $\rightarrow dv/dt = [4\rho Q^2/(\pi d^2)]/\left(M_o - \rho Q t\right)$.
Integrating, we have, $v = -[4Q/(\pi d^2)]\ln\left(M_o - \rho Q t\right) + C$.
If the initial velocity is zero then we have

$$v = [4Q/(\pi d^2)]\left[-\ln\left(M_o - \rho Q t\right) + \ln M_o\right].$$

7.27 For the one dimensional problem discussed in Section 7.10,

(a) from the continuity equation $\rho_1 v_1 = \rho_2 v_2$ and the momentum equation $p_1 - p_2 = \rho_2 v_2^2 - \rho_1 v_1^2$, obtain

$$\frac{v_2}{v_1} = 1 - \frac{1}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right)$$

(b) From the energy equation $\frac{\gamma}{\gamma-1}\frac{p_1}{\rho_1} + \frac{1}{2}v_1^2 = \frac{\gamma}{\gamma-1}\frac{p_2}{\rho_2} + \frac{1}{2}v_2^2$, obtain

$$1 + \frac{\gamma-1}{2}\frac{v_1^2}{a_1^2} = \frac{p_2}{p_1}\left(\frac{v_2}{v_1}\right) + \frac{\gamma-1}{2}\frac{v_1^2}{a_1^2}\left(\frac{v_2^2}{v_1^2}\right)$$

(c) From the results of (a) and (b), obtain

$$\left(\frac{p_2}{p_1}\right)^2 - \frac{2}{\gamma+1}\left(1 + \gamma M_1^2\right)\left(\frac{p_2}{p_1}\right) - \frac{2}{\gamma+1}\left(\frac{\gamma-1}{2} - \gamma M_1^2\right) = 0.$$

Ans. (a)

Using $\rho_1 v_1 = \rho_2 v_2,\ p_1 - p_2 = \rho_2 v_2^2 - \rho_1 v_1^2 \rightarrow p_1 - p_2 = \rho_1 v_1 (v_2 - v_1) \rightarrow$

$$1 - \frac{p_2}{p_1} = \frac{\rho_1 v_1 (v_2 - v_1)}{p_1} = \left(\frac{\rho_1 v_1^2}{p_1}\right)\frac{(v_2 - v_1)}{v_1} = \left(\frac{\rho_1 v_1^2}{p_1}\right)\left(\frac{v_2}{v_1} - 1\right) \rightarrow \frac{1 - p_2/p_1}{\rho_1 v_1^2 / p_1} + 1 = \frac{v_2}{v_1}$$

$$\rightarrow \frac{1 - p_2/p_1}{\gamma v_1^2 (\rho_1/\gamma p_1)} + 1 = \frac{v_2}{v_1} \rightarrow \frac{1 - p_2/p_1}{\gamma v_1^2 / a_1^2} + 1 = \frac{v_2}{v_1}.$$ That is, $\frac{v_2}{v_1} = 1 - \frac{1}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right)$.

(b)

$$\frac{\gamma}{\gamma - 1}\frac{p_1}{\rho_1} + \frac{1}{2}v_1^2 = \frac{\gamma}{\gamma - 1}\frac{p_2}{\rho_2} + \frac{1}{2}v_2^2 \rightarrow 1 + \frac{\gamma - 1}{2\gamma}\frac{\rho_1}{p_1}v_1^2 = \frac{p_2}{\rho_2}\frac{\rho_1}{p_1} + \frac{\gamma - 1}{2\gamma}\frac{\rho_1}{p_1}v_2^2$$

$$\rightarrow 1 + \frac{\gamma - 1}{2}\frac{v_1^2}{a_1^2} = \frac{p_2}{p_1}\left(\frac{\rho_1}{\rho_2}\right) + \frac{\gamma - 1}{2}\frac{v_2^2}{a_1^2}, \rightarrow\rightarrow 1 + \frac{\gamma - 1}{2}\frac{v_1^2}{a_1^2} = \frac{p_2}{p_1}\left(\frac{v_2}{v_1}\right) + \frac{\gamma - 1}{2}\frac{v_1^2}{a_1^2}\left(\frac{v_2^2}{v_1^2}\right).$$

$\left[\text{note } \gamma p_1/\rho_1 = a_1^2 \text{ and } \rho_1 v_1 = \rho_2 v_2\right]$.

(c)

Using $\frac{v_2}{v_1} = 1 - \frac{1}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right)$, the equation $1 + \frac{\gamma - 1}{2}\frac{v_1^2}{a_1^2} = \frac{p_2}{p_1}\left(\frac{v_2}{v_1}\right) + \frac{\gamma - 1}{2}\frac{v_1^2}{a_1^2}\left(\frac{v_2^2}{v_1^2}\right) \rightarrow$

$$1 + \frac{\gamma - 1}{2}M_1^2 = \frac{p_2}{p_1}\left[1 - \frac{1}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right)\right] + \frac{\gamma - 1}{2}M_1^2\left[1 - \frac{1}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right)\right]^2$$

$$= \left[\frac{1}{\gamma M_1^2}\left(-\left(\frac{p_2}{p_1}\right)^2 + \frac{p_2}{p_1}\right) + \frac{p_2}{p_1}\right] + \frac{\gamma - 1}{2}M_1^2\left[1 - \frac{2}{\gamma M_1^2}\left(\frac{p_2}{p_1} - 1\right) + \frac{1}{\gamma^2 M_1^4}\left(\frac{p_2}{p_1} - 1\right)^2\right]$$

$$= \frac{1}{\gamma M_1^2}\left(\frac{p_2}{p_1}\right)^2\left[-1 + \frac{\gamma - 1}{2\gamma}\right] + \frac{p_2}{p_1}\frac{1}{\gamma M_1^2}\left[\frac{\gamma M_1^2 + 1}{\gamma}\right] + \frac{\gamma - 1}{2\gamma}\frac{1}{\gamma M_1^2}\left[\gamma^2 M_1^4 + 2\gamma M_1^2 + 1\right].$$

Thus,

$$1 + \frac{\gamma - 1}{2}M_1^2 = \frac{1}{\gamma M_1^2}\left(\frac{p_2}{p_1}\right)^2\left[\frac{-(\gamma + 1)}{2\gamma}\right] + \frac{p_2}{p_1}\frac{1}{\gamma M_1^2}\left[\frac{\gamma M_1^2 + 1}{\gamma}\right] + \frac{\gamma - 1}{2\gamma}\frac{1}{\gamma M_1^2}\left[\gamma^2 M_1^4 + 2\gamma M_1^2 + 1\right].$$

Rearranging,

$$\frac{\gamma - 1}{2}M_1^2 = \frac{1}{\gamma M_1^2}\left(\frac{p_2}{p_1}\right)^2\left[\frac{-(\gamma + 1)}{2\gamma}\right] + \frac{p_2}{p_1}\frac{1}{\gamma M_1^2}\left[\frac{\gamma M_1^2 + 1}{\gamma}\right]$$

$$+ \frac{\gamma - 1}{2}M_1^2 + \frac{\gamma - 1}{2\gamma}\frac{1}{\gamma M_1^2}\left[2\gamma M_1^2 + 1 - \frac{2\gamma^2 M_1^2}{(\gamma - 1)}\right]$$

$$\rightarrow 0 = \frac{1}{\gamma M_1^2}\left(\frac{p_2}{p_1}\right)^2\left[\frac{-(\gamma + 1)}{2\gamma}\right] + \frac{p_2}{p_1}\frac{1}{\gamma M_1^2}\left[\frac{\gamma M_1^2 + 1}{\gamma}\right] + \frac{\gamma - 1}{2\gamma}\frac{1}{\gamma M_1^2}\left[1 - \frac{2\gamma M_1^2}{(\gamma - 1)}\right]$$

That is, $\left(\frac{p_2}{p_1}\right)^2 - \frac{2(\gamma M_1^2 + 1)}{(\gamma + 1)}\frac{p_2}{p_1} - \frac{2}{(\gamma + 1)}\left(\frac{(\gamma - 1)}{2} - \gamma M_1^2\right) = 0$.

The above equation has two solutions:

$$(1)\, p_2 = p_1$$

$$(2)\, p_2 = \frac{1}{\gamma+1}\left[2\rho_1 v_1^2 - (\gamma-1)\,p_1\right], or \;\; p_2 = \frac{1}{\gamma+1}\left[2\gamma M_1^2 - (\gamma-1)\right] p_1$$

CHAPTER 8

8.1 Show that for an incompressible Newtonian fluid in Couette flow, the pressure at the outer cylinder $(r=R_o)$ is always larger than that at the inner cylinder. That is, obtain

$$\left[-T_{rr}\left(R_o\right)\right]-\left[-T_{rr}\left(R_i\right)\right]=\rho\int_{R_i}^{R_o} r\omega^2\left(r\right)dr$$

Ans. In Couette flow, $v_r=v_z=0$ and $v_\theta=Ar+\frac{B}{r}$, [see Eq. (6.15.4) and (6.15.7)]. Thus,

$T_{rr}=T_{\theta\theta}=T_{zz}=-p$, $T_{rz}=T_{\theta z}=0$, and $T_{r\theta}=\mu\left(\frac{dv_\theta}{dr}-\frac{v_\theta}{r}\right)=$ function of r only.

Thus, the r-equation of motion $\frac{\partial T_{rr}}{\partial r}+\frac{1}{r}\frac{\partial T_{r\theta}}{\partial\theta}+\frac{T_{rr}-T_{\theta\theta}}{r}+\frac{\partial T_{rz}}{\partial z}=-\rho r\omega^2$ becomes:

$\frac{\partial T_{rr}}{\partial r}=-\rho r\omega^2$. Now, $\int_{R_i}^{R_o}\frac{\partial T_{rr}}{\partial r}dr=-\rho\int_{R_i}^{R_o} r\omega^2\left(r\right)dr$. Thus,.

$\left[-T_{rr}\left(R_o\right)\right]-\left[-T_{rr}\left(R_i\right)\right]=\rho\int_{R_i}^{R_o} r\omega^2\left(r\right)dr$. The right hand side of this last equation is always positive.

8.2 Show that the constitutive equation

$$\boldsymbol{\tau}=\boldsymbol{\tau}_1+\boldsymbol{\tau}_2+\boldsymbol{\tau}_3,\ \text{with}\ \ \boldsymbol{\tau}_n+\lambda_n\partial\boldsymbol{\tau}_n/\partial t=2\mu_n\mathbf{D},\ \ n=1,2,3$$

is equivalent to

$$\boldsymbol{\tau}+a_1\partial\boldsymbol{\tau}/\partial t+a_2\partial^2\boldsymbol{\tau}/\partial t^2+a_3\partial^3\boldsymbol{\tau}/\partial t^3=b_o\mathbf{D}+b_1\partial\mathbf{D}/\partial t+b_2\partial^2\mathbf{D}/\partial t^2$$

where

$a_1=\left(\lambda_1+\lambda_2+\lambda_3\right), a_2=\left(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1\right), a_3=\lambda_1\lambda_2\lambda_3$

$b_o=2\left(\mu_1+\mu_2+\mu_3\right), b_1=2\left[\mu_1\left(\lambda_2+\lambda_3\right)+\mu_2\left(\lambda_1+\lambda_3\right)+\mu_3\left(\lambda_2+\lambda_1\right)\right]$

$b_2=2\left(\mu_1\lambda_2\lambda_3+\mu_2\lambda_1\lambda_3+\mu_3\lambda_1\lambda_2\right)$

Ans.

$$\left(\sum_{i=1}^3\lambda_i\right)\frac{\partial\boldsymbol{\tau}}{\partial t}=\left(\sum_{i=1}^3\lambda_i\right)\frac{\partial}{\partial t}\left(\sum_{j=1}^3\boldsymbol{\tau}_j\right)=\left(\sum_{i=1}^3\lambda_i\right)\left(\sum_{j=1}^3\frac{\partial\boldsymbol{\tau}_j}{\partial t}\right)=\sum_{i=1}^3\sum_{j=1}^3\lambda_i\frac{\partial\boldsymbol{\tau}_j}{\partial t}$$

$$=\sum_{i=1}^3\lambda_i\frac{\partial\boldsymbol{\tau}_i}{\partial t}+\sum_{i=1}^3\sum_{\substack{j=1\\ j\neq i}}^3\lambda_i\frac{\partial\boldsymbol{\tau}_j}{\partial t}=\sum_{i=1}^3\left(2\mu_i\mathbf{D}-\boldsymbol{\tau}_i\right)+\sum_{i=1}^3\sum_{\substack{j=1\\ j\neq i}}^3\lambda_i\frac{\partial\boldsymbol{\tau}_j}{\partial t}$$

$$=2\mathbf{D}\left(\sum_{i=1}^3\mu_i\right)-\sum_{i=1}^3\boldsymbol{\tau}_i+\sum_{i=1}^3\sum_{\substack{j=1\\ j\neq i}}^3\lambda_i\frac{\partial\boldsymbol{\tau}_j}{\partial t}=2\mathbf{D}\sum_{i=1}^3\mu_i-\boldsymbol{\tau}+\sum_{i=1}^3\sum_{\substack{j=1\\ j\neq i}}^3\lambda_i\frac{\partial\boldsymbol{\tau}_j}{\partial t}$$

That is,

$$\left(\sum_{i=1}^{3}\lambda_i\right)\frac{\partial\boldsymbol{\tau}}{\partial t}=2\mathbf{D}\left(\sum_{i=1}^{3}\mu_i\right)-\boldsymbol{\tau}+\left(\lambda_1\frac{\partial\boldsymbol{\tau}_2}{\partial t}+\lambda_1\frac{\partial\boldsymbol{\tau}_3}{\partial t}\right)+\left(\lambda_2\frac{\partial\boldsymbol{\tau}_1}{\partial t}+\lambda_2\frac{\partial\boldsymbol{\tau}_3}{\partial t}\right)+\left(\lambda_3\frac{\partial\boldsymbol{\tau}_1}{\partial t}+\lambda_3\frac{\partial\boldsymbol{\tau}_2}{\partial t}\right)\qquad\text{(i)}$$

Next, we have,

$$\begin{aligned}&(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\partial^2\boldsymbol{\tau}/\partial t^2=(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\partial^2\boldsymbol{\tau}_1/\partial t^2+(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\partial^2\boldsymbol{\tau}_2/\partial t^2\\&+(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\partial^2\boldsymbol{\tau}_3/\partial t^2=(\lambda_2\lambda_1+\lambda_3\lambda_1)\partial^2\boldsymbol{\tau}_1/\partial t^2+\lambda_2\lambda_3\partial^2\boldsymbol{\tau}_1/\partial t^2+(\lambda_1\lambda_2+\lambda_2\lambda_3)\partial^2\boldsymbol{\tau}_2/\partial t^2\\&+\lambda_3\lambda_1\partial^2\boldsymbol{\tau}_2/\partial t^2+(\lambda_2\lambda_3+\lambda_3\lambda_1)\partial^2\boldsymbol{\tau}_3/\partial t^2+\lambda_1\lambda_2\partial^2\boldsymbol{\tau}_3/\partial t^2.\end{aligned}$$

i.e.,

$$\begin{aligned}&(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\partial^2\boldsymbol{\tau}/\partial t^2=\left[2\mu_1(\lambda_2+\lambda_3)+2\mu_2(\lambda_1+\lambda_3)+2\mu_3(\lambda_2+\lambda_1)\right]\partial\mathbf{D}/\partial t\\&-(\lambda_2+\lambda_3)\partial\boldsymbol{\tau}_1/\partial t-(\lambda_1+\lambda_3)\partial\boldsymbol{\tau}_2/\partial t-(\lambda_2+\lambda_1)\partial\boldsymbol{\tau}_3/\partial t+\lambda_2\lambda_3\partial^2\boldsymbol{\tau}_1/\partial t^2+\lambda_3\lambda_1\partial^2\boldsymbol{\tau}_2/\partial t^2\qquad\text{(ii)}\\&+\lambda_1\lambda_2\partial^2\boldsymbol{\tau}_3/\partial t^2.\end{aligned}$$

Finally, we have,

$$\begin{aligned}&\lambda_1\lambda_2\lambda_3\frac{\partial^3\boldsymbol{\tau}}{\partial t^3}=\lambda_1\lambda_2\lambda_3\frac{\partial^3\boldsymbol{\tau}_1}{\partial t^3}+\lambda_1\lambda_2\lambda_3\frac{\partial^3\boldsymbol{\tau}_2}{\partial t^3}+\lambda_1\lambda_2\lambda_3\frac{\partial^3\boldsymbol{\tau}_3}{\partial t^3}\\&=\lambda_2\lambda_3\left(2\mu_1\frac{\partial^2\mathbf{D}}{\partial t^2}-\frac{\partial^2\boldsymbol{\tau}_1}{\partial t^2}\right)+\lambda_1\lambda_3\left(2\mu_2\frac{\partial^2\mathbf{D}}{\partial t^2}-\frac{\partial^2\boldsymbol{\tau}_2}{\partial t^2}\right)+\lambda_1\lambda_2\left(2\mu_3\frac{\partial^2\mathbf{D}}{\partial t^2}-\frac{\partial^2\boldsymbol{\tau}_3}{\partial t^2}\right)\\&=2\lambda_2\lambda_3\mu_1\frac{\partial^2\mathbf{D}}{\partial t^2}+2\lambda_1\lambda_3\mu_2\frac{\partial^2\mathbf{D}}{\partial t^2}+2\lambda_1\lambda_2\mu_3\frac{\partial^2\mathbf{D}}{\partial t^2}-\lambda_2\lambda_3\frac{\partial^2\boldsymbol{\tau}_1}{\partial t^2}-\lambda_1\lambda_3\frac{\partial^2\boldsymbol{\tau}_2}{\partial t^2}-\lambda_1\lambda_2\frac{\partial^2\boldsymbol{\tau}_3}{\partial t^2}\end{aligned}$$

that is, $\lambda_1\lambda_2\lambda_3\frac{\partial^3\boldsymbol{\tau}}{\partial t^3}=2\lambda_2\lambda_3\mu_1\frac{\partial^2\mathbf{D}}{\partial t^2}+2\lambda_1\lambda_3\mu_2\frac{\partial^2\mathbf{D}}{\partial t^2}+2\lambda_1\lambda_2\mu_3\frac{\partial^2\mathbf{D}}{\partial t^2}-\lambda_2\lambda_3\frac{\partial^2\boldsymbol{\tau}_1}{\partial t^2}-\lambda_1\lambda_3\frac{\partial^2\boldsymbol{\tau}_2}{\partial t^2}-\lambda_1\lambda_2\frac{\partial^2\boldsymbol{\tau}_3}{\partial t^2}$ (iii)

Thus, (i) + (ii)+ (iii) gives

$$\begin{aligned}&\boldsymbol{\tau}+(\lambda_1+\lambda_2+\lambda_3)\partial\boldsymbol{\tau}/\partial t+(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1)\partial^2\boldsymbol{\tau}/\partial t^2+\lambda_1\lambda_2\lambda_3\partial^3\boldsymbol{\tau}/\partial t^3\\&=2\mathbf{D}(\mu_1+\mu_2+\mu_3)+2\left[\mu_1(\lambda_2+\lambda_3)+\mu_2(\lambda_1+\lambda_3)+\mu_3(\lambda_2+\lambda_1)\right]\partial\mathbf{D}/\partial t\\&+2(\mu_1\lambda_2\lambda_3+\mu_2\lambda_1\lambda_3+\mu_3\lambda_1\lambda_2)\partial^2\mathbf{D}/\partial t^2.\end{aligned}$$

That is,

$$\boldsymbol{\tau}+a_1\frac{\partial\boldsymbol{\tau}}{\partial t}+a_2\frac{\partial^2\boldsymbol{\tau}}{\partial t^2}+a_3\frac{\partial^3\boldsymbol{\tau}}{\partial t^3}=b_o\mathbf{D}+b_1\frac{\partial\mathbf{D}}{\partial t}+b_2\frac{\partial^2\mathbf{D}}{\partial t^2}.$$

where

$$\begin{aligned}&a_1=(\lambda_1+\lambda_2+\lambda_3),a_2=(\lambda_1\lambda_2+\lambda_2\lambda_3+\lambda_3\lambda_1),a_3=\lambda_1\lambda_2\lambda_3\\&b_o=2(\mu_1+\mu_2+\mu_3),b_1=2\left[\mu_1(\lambda_2+\lambda_3)+\mu_2(\lambda_1+\lambda_3)+\mu_3(\lambda_2+\lambda_1)\right]\\&b_2=2(\mu_1\lambda_2\lambda_3+\mu_2\lambda_1\lambda_3+\mu_3\lambda_1\lambda_2)\end{aligned}$$

8.3 Obtain the force-displacement relationship for the Kelvin-Voigt solid, which consists of a dashpot (with damping coefficient η) and a spring (with spring constant G) connected in parallel. Also, obtain its relaxation function.

Ans. Since the spring and the dashpot are connected in parallel, therefore, the total force is given by: $S=S_{sp}+S_{dash}$ and the total displacement ε is given by $\varepsilon=\varepsilon_{sp}=\varepsilon_{dash}$. Now,

$S_{sp} = G\varepsilon$ and $S_{dash} = \eta \dfrac{d\varepsilon}{dt}$, therefore, $S = G\varepsilon + \eta \dfrac{d\varepsilon}{dt}$. To find the relaxation function, we let $\varepsilon = \varepsilon_o H(t)$, where $H(t)$ is the Heaviside function. Then $S = G\varepsilon_o H(t) + \eta\varepsilon_o \delta(t)$. Thus, the relaxation function is $S/\varepsilon_o = GH(t) + \eta\delta(t)$.

8.4 (a) Obtain the force-displacement relationship for a dashpot (damping coefficient η_o) and a Kelvin-Voigt solid (damping coefficient η and spring constant G, see the previous problem) connected in series. (b) Obtain its relaxation function.

Ans. (a) Let S_{kv} and S_d be the force transmitted by the Kelvin-Voight element and the dashpot respectively and let ε_{kv} and ε_d be the elongation of the Kelvin-Voight element and the dashpot respectively. Then we have, the total force is given by $S = S_d = S_{kv}$ (i) and the total displacement is given by $\varepsilon = \varepsilon_d + \varepsilon_{kv}$ (ii), where $S_d = \eta_o \dfrac{d\varepsilon_d}{dt} = S = S_{kv} = G\varepsilon_{kv} + \eta \dfrac{d\varepsilon_{kv}}{dt}$ (iii). From (ii) and (iii) we have, $\dfrac{d\varepsilon}{dt} = \dfrac{d\varepsilon_d}{dt} + \dfrac{d\varepsilon_{kv}}{dt} = \dfrac{S}{\eta_o} + \dfrac{1}{\eta}\left(S - G\varepsilon_{kv}\right) = \dfrac{S}{\eta_o} + \dfrac{S}{\eta} - \dfrac{G}{\eta}\left(\varepsilon - \varepsilon_d\right)$ (iv). Thus,

$\dfrac{d^2\varepsilon}{dt^2} = \left(\dfrac{\eta + \eta_o}{\eta\eta_o}\right)\dfrac{dS}{dt} - \dfrac{G}{\eta}\dfrac{d\varepsilon}{dt} + \dfrac{G}{\eta}\dfrac{d\varepsilon_d}{dt}$, or, $\dfrac{\eta\eta_o}{G}\dfrac{d^2\varepsilon}{dt^2} = \dfrac{\left(\eta + \eta_o\right)}{G}\dfrac{dS}{dt} - \eta_o \dfrac{d\varepsilon}{dt} + S$.

Thus, the force-displacement relationship is given by:

$$S + \frac{\left(\eta + \eta_o\right)}{G}\frac{dS}{dt} = \eta_o \frac{d\varepsilon}{dt} + \frac{\eta\eta_o}{G}\frac{d^2\varepsilon}{dt^2}. \qquad \text{(v)}$$

(b) Let $\varepsilon = \varepsilon_o H(t)$, where $H(t)$ is Heaviside function. Then Eq. (v) gives

$$\frac{dS}{dt} + \frac{G}{\left(\eta + \eta_o\right)}S = \frac{\varepsilon_o\eta_o G}{\left(\eta + \eta_o\right)}\delta(t) + \frac{\varepsilon_o\eta\eta_o}{\left(\eta + \eta_o\right)}\frac{d\delta}{dt}, \qquad \text{(vi)}$$

where $\delta(t)$ is the Dirac function. The integration factor for this ODE is $\exp\left[Gt/\left(\eta + \eta_o\right)\right]$.

Thus, $\dfrac{d}{dt}\left[Se^{\frac{Gt}{\eta+\eta_o}}\right] = \dfrac{\varepsilon_o\eta_o G}{\left(\eta + \eta_o\right)}e^{\frac{Gt}{\eta+\eta_o}}\delta(t) + \dfrac{\varepsilon_o\eta\eta_o}{\left(\eta + \eta_o\right)}e^{\frac{Gt}{\eta+\eta_o}}\dfrac{d\delta}{dt}$ and

$$Se^{\frac{Gt}{\eta+\eta_o}} = \frac{\varepsilon_o\eta_o G}{\left(\eta + \eta_o\right)}\int_{t=-\infty}^{t} e^{\frac{Gt}{\eta+\eta_o}}\delta(t)dt + \frac{\varepsilon_o\eta\eta_o}{\left(\eta + \eta_o\right)}\int_{-\infty}^{t} e^{\frac{Gt}{\eta+\eta_o}}\frac{d\delta}{dt}dt$$

$$= \frac{\varepsilon_o\eta_o G}{\left(\eta + \eta_o\right)} + \frac{\varepsilon_o\eta\eta_o}{\left(\eta + \eta_o\right)}\left\{\left[e^{Gt/(\eta+\eta_o)}\delta(t)\right]_{-\infty}^{t} - \frac{G}{\eta + \eta_o}\int_{-\infty}^{t}\delta e^{Gt/(\eta+\eta_o)}dt\right\}.$$ That is,

$$Se^{\frac{Gt}{\eta+\eta_o}} = \frac{\varepsilon_o\eta_o G}{\left(\eta + \eta_o\right)} + \frac{\varepsilon_o\eta\eta_o}{\left(\eta + \eta_o\right)}\left\{e^{\frac{Gt}{\eta+\eta_o}}\delta(t) - \frac{G}{\eta + \eta_o}\right\} = \frac{\varepsilon_o\eta_o^2 G}{\left(\eta + \eta_o\right)^2} + \left\{\frac{\varepsilon_o\eta\eta_o}{\left(\eta + \eta_o\right)}e^{\frac{Gt}{\eta+\eta_o}}\delta(t)\right\}.$$

Thus, the relaxation function is $\dfrac{S}{\varepsilon_o} = \dfrac{\eta_o^2 G}{\left(\eta + \eta_o\right)^2}e^{\frac{-Gt}{\eta+\eta_o}} + \dfrac{\eta\eta_o}{\left(\eta + \eta_o\right)}\delta(t)$.

8.5 A linear Maxwell fluid, defined by Eq. (8.1.2), is between two parallel plates which are one unit apart. Staring from rest, at time $t=0$, the top plate is given a displacement $u=v_o t$ while the bottom plate remains fixed. Neglect inertia effects, obtain the shear stress history.

Ans. The velocity field for the fluid in this motion is given by (inertia neglected) $v_1 = v_o H(t)x_2$, $v_2 = v_3 = 0$, where $H(t)$ is the Heaviside Function. The only non-zero rate of deformation component is $D_{12} = v_o H(t)/2$. Thus, from the constitutive equation for the linear Maxwell fluid, we obtain, $S_{12} + \lambda \frac{dS_{12}}{dt} = \mu v_o H(t)$. Thus, $\frac{d}{dt}\left[S_{12}e^{t/\lambda}\right] = \frac{\mu v_o}{\lambda} e^{t/\lambda} H(t)$. That is, $S_{12}e^{t/\lambda} = \frac{\mu v_o}{\lambda}\int_{-\infty}^{t} e^{t/\lambda} H(t)\,dt = \frac{\mu v_o}{\lambda}\int_0^t e^{t/\lambda}dt = \frac{\mu v_o}{\lambda}\left[\lambda e^{t/\lambda}\right]_0^t = \mu v_o\left(e^{t/\lambda}-1\right)$. Thus, the shear stress history is: $S_{12} = \mu v_o\left(1-e^{-t/\lambda}\right)$.

8.6 Obtain Eq. (8.3.1) i.e., $\mathbf{S} = 2\int_{-\infty}^{t}\phi(t-t')\mathbf{D}(t')dt'$, where $\phi(t) = (\mu/\lambda)e^{-t/\lambda}$, by solving the linear non-homogeneous ordinary differential equation $\mathbf{S} + \lambda \frac{d\mathbf{S}}{dt} = 2\mu\mathbf{D}$.

Ans. The integration factor for this ODE is $\exp[t/\lambda]$. That is the equation can be written as; $\frac{d}{dt}\left(\mathbf{S}e^{t/\lambda}\right) = \frac{2\mu}{\lambda}e^{t/\lambda}\mathbf{D}$. Thus, $\mathbf{S}e^{t/\lambda} = (2\mu/\lambda)\int_{-\infty}^{t} e^{t/\lambda}\mathbf{D}(t)\,dt \rightarrow$

$\rightarrow \mathbf{S} = e^{-t/\lambda}(2\mu/\lambda)\int_{-\infty}^{t} e^{t'/\lambda}\mathbf{D}(t')\,dt'$. That is,

$$\mathbf{S} = 2\int_{-\infty}^{t}\frac{\mu}{\lambda}e^{-(t-t')/\lambda}\mathbf{D}(t')\,dt' \equiv 2\int_{-\infty}^{t}\phi(t-t')\mathbf{D}(t')\,dt'.$$

8.7 Show that for the linear Maxwell fluid, defined by Eq. (8.1.2), $\int_{-\infty}^{t}\phi(t-t')J(t')\,dt' = t$, where $\phi(t)$ is the relaxation function and $J(t)$ is the creep compliance function.

Ans. Let $S_{12} = S_o H(t)$ be applied to the top plate of a channel of unit depth in which is the linear Maxwell fluid. [$H(t)$ is the unit step function, i.e., Heaviside function]. Neglecting inertia, the velocity field is $v(x_2) = v_o x_2$, where v_o is the velocity of the top plate. Then from the constitutive equation $\mathbf{S} + \lambda d\mathbf{S}/dt = 2\mu\mathbf{D}$, we obtain $S_o + \lambda S_o\delta(t) = 2\mu D_{12} = 2\mu(v_o/2) = \mu du_o/dt$, where $u_o(t)$ is the displacement of the top plate. From $\frac{du_o}{dt} = \frac{S_o}{\mu} + \frac{\lambda}{\mu}S_o\delta(t)$, we obtain $\int_o^t \frac{du_o}{dt}dt = \int_o^t \frac{S_o}{\mu}dt + \frac{\lambda}{\mu}\int_o^t S_o\delta(t) \rightarrow u_o = \frac{S_o}{\mu}t + \frac{\lambda}{\mu}S_o = \frac{S_o}{\mu}(t+\lambda)$. Thus, the creep compliance function is: $J(t) = u_o/S_o = (t+\lambda)/\mu$.

Since the relaxation function is $\phi(t) = (\mu/\lambda)e^{-t/\lambda}$, therefore,

$$\int_{-\infty}^{t}\phi(t-t')J(t')dt'=\int_{-\infty}^{t}(\mu/\lambda)e^{-(t-t')/\lambda}(t'+\lambda)/\mu dt'=(1/\lambda)\int_{-\infty}^{t}e^{-(t-t')/\lambda}(t'+\lambda)dt'$$
$$=\int_{-\infty}^{t}e^{-(t-t')/\lambda}dt'+(1/\lambda)\int_{-\infty}^{t}e^{-(t-t')/\lambda}(t')dt'.$$

Now $\int_{-\infty}^{t}e^{-(t-t')/\lambda}dt'=[\lambda e^{-(t-t')/\lambda}]_{t'=-\infty}^{t}=\lambda$ and

$$\int_{-\infty}^{t}e^{-(t-t')/\lambda}(t')dt'=\int_{-\infty}^{t}(\lambda de^{-(t-t')/\lambda})(t')dt'=\left[\lambda e^{-(t-t')/\lambda}t'\right]_{t'=-\infty}^{t}-\lambda\int_{-\infty}^{t}e^{-(t-t')/\lambda}dt'=\lambda t-\lambda^2.$$

Thus, $\int_{-\infty}^{t}\phi(t-t')J(t')dt'=\lambda+\frac{1}{\lambda}\left(\lambda t-\lambda^2\right)=\lambda+(t-\lambda)=t$.

8.8 Obtain the storage modulus and loss modulus for the linear Maxwell fluid with a continuous relaxation spectrum defined by Eq. (8.4.1), i.e., $\phi(t)=\int_{0}^{\infty}\frac{H(\lambda)}{\lambda}e^{-t/\lambda}d\lambda$.

Ans. Let the shear strain be: $\gamma_{12}=\gamma_o e^{i\omega t}$. For this strain history, the rate of deformation history is given by $2D_{12}=\frac{d\gamma_{12}}{dt}=i\omega\gamma_o e^{i\omega t}$. Thus, from the constitutive equation,

$\mathbf{S}=2\int_{-\infty}^{t}\phi(t-t')\mathbf{D}(t')dt'$, we have $S_{12}=2\int_{-\infty}^{t}\phi(t-t')D_{12}(t')dt'=i\omega\gamma_o\int_{-\infty}^{t}\phi(t-t')e^{i\omega t'}dt'$.

With $\phi(t)=\int_{0}^{\infty}\left[H(\lambda)/\lambda\right]e^{-t/\lambda}d\lambda$, we have,

$$S_{12}=i\omega\gamma_o\int_{-\infty}^{t}\int_{0}^{\infty}\frac{H(\lambda)}{\lambda}e^{-(t-t')/\lambda}e^{i\omega t'}d\lambda dt'=i\omega\gamma_o\int_{\lambda=0}^{\infty}\frac{H(\lambda)e^{-t/\lambda}}{\lambda}\int_{t'=-\infty}^{t}e^{t'/\lambda}e^{i\omega t'}dt'd\lambda$$

Now, $\int_{t'=-\infty}^{t}e^{t'/\lambda}e^{i\omega t'}dt'=\int_{t'=-\infty}^{t}e^{(1+i\lambda\omega)t'/\lambda}dt'=\frac{\lambda}{(1+i\lambda\omega)}e^{(1+i\lambda\omega)t/\lambda}$. Thus,

$\frac{S_{12}}{\gamma_o}=i\omega e^{i\omega t}\int_{\lambda=0}^{\infty}\frac{H(\lambda)}{(1+i\lambda\omega)}d\lambda\equiv G^*e^{i\omega t}$, where $G^*=i\omega\int_{\lambda=0}^{\infty}\frac{H(\lambda)}{(1+i\lambda\omega)}d\lambda$ is the complex modulus.

Now,

$$G^*=i\omega\int_{\lambda=0}^{\infty}\frac{H(\lambda)}{(1+i\lambda\omega)}d\lambda=i\omega\int_{\lambda=0}^{\infty}\frac{(1-i\lambda\omega)H(\lambda)}{(1+i\lambda\omega)(1-i\lambda\omega)}d\lambda$$
$$=\int_{\lambda=0}^{\infty}\frac{(i\omega+\lambda\omega^2)H(\lambda)}{(1+\lambda^2\omega^2)}d\lambda=\int_{\lambda=0}^{\infty}\frac{\lambda\omega^2H(\lambda)}{(1+\lambda^2\omega^2)}d\lambda+i\int_{\lambda=0}^{\infty}\frac{\omega H(\lambda)}{(1+\lambda^2\omega^2)}d\lambda$$

Thus, $G'=\int_{\lambda=0}^{\infty}\frac{\lambda^2\omega^2H(\lambda)}{\lambda(1+\lambda^2\omega^2)}d\lambda,\quad G''=\int_{\lambda=0}^{\infty}\frac{\lambda\omega H(\lambda)}{\lambda(1+\lambda^2\omega^2)}d\lambda$

8.9 Show that the viscosity μ of a linear Maxwell fluid, define by $\mathbf{S}=2\int_{-\infty}^{t}\phi(t-t')\mathbf{D}(t')dt'$, is related to the relaxation function $\phi(t)$ and the memory function $f(s)$ by the relation $\mu=\int_{0}^{\infty}\phi(s)ds=-\int_{0}^{\infty}sf(s)ds$.

Ans. $S_{12} = 2\int_{t'=-\infty}^{t} \phi(t-t') D_{12}(t') dt' = -2\int_{s=\infty}^{0} \phi(s) D_{12}(t-s) ds = 2\int_{s=0}^{\infty} \phi(s) D_{12}(t-s) ds$.
For simple shearing flow, $v_1 = kx_2$, $v_2 = v_3 = 0$, $2D_{12} = k$, so that
$S_{12} = k\int_{s=0}^{\infty} \phi(s) ds \rightarrow \mu = S_{12}/k = \int_{s=0}^{\infty} \phi(s) ds$. Now, the memory function $f(s)$ is related to the relaxation function $\phi(s)$ by the relation $d\phi(s)/ds = f(s)$. Thus,

$$\mu = \int_0^\infty \phi(s) ds = \left[s\phi(s) \right]_0^\infty - \int_0^\infty s \frac{d\phi(s)}{ds} ds = -\int_0^\infty sf(s) ds .$$

8.10 Show that the relaxation function for the Jeffrey model [Eq. (8.2.7)] with $a_2 = 0$ is given by [note: Reference to Eq.(8.2.7) is missing in the problem statement in the text]

$$\phi(t) = \frac{S_{12}}{\gamma_o} = \frac{b_o}{2a_1}\left[\left(1 - \frac{b_1}{b_o a_1}\right) e^{-t/a_1} + \frac{b_1}{b_o}\delta(t) \right], \quad \delta(t) = \text{Dirac Function} .$$

Ans. Let the shear strain γ_{12} be given by $\gamma_{12} = \gamma_\circ H(t)$. Then $2D_{12} = d\gamma_{12}/dt = \gamma_o \delta(t)$, where $\delta(t)$ is Dirac function. From the constitutive equation, we have,

$$S_{12} + a_1 \frac{\partial S_{12}}{\partial t} = b_o \frac{\gamma_o}{2}\delta + b_1 \frac{\gamma_o}{2}\frac{\partial \delta}{\partial t} \rightarrow \frac{\partial S_{12}}{\partial t} + \frac{1}{a_1} S_{12} = \frac{\gamma_o}{2}\left(\frac{b_o}{a_1}\delta + \frac{b_1}{a_1}\frac{\partial \delta}{\partial t} \right)$$

$$\rightarrow 2\frac{\partial}{\partial t}\left(S_{12} e^{t/a_1} \right) = e^{t/a_1}\gamma_o \left(\frac{b_o}{a_1}\delta + \frac{b_1}{a_1}\frac{\partial \delta}{\partial t} \right)$$

$$\rightarrow 2\frac{S_{12}}{\gamma_o} e^{t/a_1} = \frac{b_o}{a_1}\int_{-\infty}^{t} e^{t/a_1}\delta(t) dt + \frac{b_1}{a_1}\int_{-\infty}^{t} e^{t/a_1}\frac{d\delta(t)}{dt} dt$$

$$= \frac{b_o}{a_1} + \frac{b_1}{a_1}\left[e^{t/a_1}\delta(t) \right]_{-\infty}^{t} - \frac{b_1}{a_1}\int_{-\infty}^{t} \frac{1}{a_1}\delta(t) e^{t/a_1} dt = \frac{b_o}{a_1} + \frac{b_1}{a_1} e^{t/a_1}\delta(t) - \frac{b_1}{a_1^2}$$

Thus, the relaxation function is:

$$\phi(t) \equiv \frac{S_{12}}{\gamma_o} = \frac{b_o}{2a_1}\left(1 - \frac{b_1}{a_1 b_o}\right) e^{-t/a_1} + \frac{b_1}{2a_1}\delta(t) = \frac{b_o}{2a_1}\left[\left(1 - \frac{b_1}{a_1 b_o}\right) e^{-t/a_1} + \frac{b_1}{b_o}\delta(t) \right]$$

8.11 Given the following velocity field: $v_1 = 0, \quad v_2 = v(x_1), \quad v_3 = 0$. Obtain (a) the particle pathline equations using the current time as the reference time, (b) the relative right Cauchy-Green deformation tensor and (c) the Rivlin-Ericksen tensors using the equation $\mathbf{C}_t = \mathbf{I} + (\tau - t)\mathbf{A}_1 + (\tau - t)^2 \mathbf{A}_2 / 2 + ..$ (d) the Rivlin-Ericksen tensor $\mathbf{A}_2$ using the recursive equation, $[\mathbf{A}_2] = [D\mathbf{A}_1/Dt] + [\mathbf{A}_1][\nabla \mathbf{v}] + [\nabla \mathbf{v}]^{\mathrm{T}}[\mathbf{A}_1]$ etc.

Ans. (a) Let $\mathbf{x}' = x_i'\mathbf{e}_i$ be the position at time τ of the particle which is at $\mathbf{x} = x_i\mathbf{e}_i$ at time t. Then $x_i' = x_i'(x_1, x_2, x_3, \tau)$ gives the pathline equation. Thus,

$$\frac{dx_1'}{d\tau} = v_1 = 0 \;\text{(i)}, \quad \frac{dx_2'}{d\tau} = v_2 = v(x_1') \;\text{(ii)}, \quad \frac{dx_3'}{d\tau} = 0 \;\text{(iii)}$$

with the initial conditions: $x_i = x_i'(x_1, x_2, x_3, t)$. Eq (i) gives $x_1' = f(x_1, x_2, x_3) = x_1$, Eq. (iii) gives $x_3' = g(x_1, x_2, x_3) = x_3$. Eq. (ii) becomes, $\frac{dx_2'}{d\tau} = v(x_1) \rightarrow x_2' = v(x_1)\tau + h(x_1, x_2, x_3)$,
$\rightarrow x_2 = v(x_1)t + h(x_1, x_2, x_3) \rightarrow h(x_1, x_2, x_3) = x_2 - v(x_1)t \rightarrow\rightarrow x_2' = x_2 + v(x_1)(\tau - t)$.
Thus, $x_1' = x_1, \quad x_2' = x_2 + v(x_1)(\tau - t), \quad x_3' = x_3$,

(b) $[\mathbf{F}_t] = [\nabla \mathbf{x}_t'] = \begin{bmatrix} 1 & 0 & 0 \\ (dv/dx_1)(\tau - t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ k(\tau - t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; k \equiv dv/dx_1$

$$[\mathbf{C}_t] = [\mathbf{F}_t]^{\mathrm{T}} [\mathbf{F}_t] = \begin{bmatrix} 1 & k(\tau - t) & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ k(\tau - t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 + k^2(\tau - t)^2 & k(\tau - t) & 0 \\ k(\tau - t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} (\tau - t) + \begin{bmatrix} 2k^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \frac{(\tau - t)^2}{2}$$

(c) $[\mathbf{A}_1] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{A}_2] = \begin{bmatrix} 2k^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad k = \frac{dv}{dx_1}$

(d) $[\mathbf{A}_2] = \left[\frac{D\mathbf{A}_1}{Dt}\right] + [\mathbf{A}_1][\nabla \mathbf{v}] + [\nabla \mathbf{v}]^{\mathrm{T}} [\mathbf{A}_1]$, where $[\mathbf{A}_1] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Thus,

$$\left[\frac{D\mathbf{A}_1}{Dt}\right] = \left[\frac{\partial \mathbf{A}_1}{\partial t}\right] + [\nabla \mathbf{A}_1][\mathbf{v}] \rightarrow \left[\frac{D\mathbf{A}_1}{Dt}\right]_{ij} = \left[\frac{\partial \mathbf{A}_1}{\partial t}\right]_{ij} + \frac{\partial (\mathbf{A}_1)_{ij}}{\partial x_k} v_k = 0 + 0 = 0.$$

$$[\mathbf{A}_1][\nabla \mathbf{v}] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} k^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\nabla \mathbf{v}]^{\mathrm{T}} [\mathbf{A}_1] = \begin{bmatrix} k^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, $[\mathbf{A}_2] = [\mathbf{A}_1][\nabla \mathbf{v}] + [\nabla \mathbf{v}]^{\mathrm{T}} [\mathbf{A}_1] = \begin{bmatrix} 2k^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

8.12 Given the following velocity field: $v_1 = -kx_1, \quad v_2 = kx_2, \quad v_3 = 0$. Obtain (a) the particle pathline equations using the current time as the reference time, (b) the relative right Cauchy-Green deformation tensor and (c) the Rivlin-Ericksen tensors using the equation $\mathbf{C}_t = \mathbf{I} + (\tau - t)\mathbf{A}_1 + (\tau - t)^2 \mathbf{A}_2 / 2 + ..$ (d) the Rivlin-Ericksen tensor $\mathbf{A}_2$ and $\mathbf{A}_3$ using the recursive equation, $[\mathbf{A}_2] = [D\mathbf{A}_1 / Dt] + [\mathbf{A}_1][\nabla \mathbf{v}] + [\nabla \mathbf{v}]^{\mathrm{T}} [\mathbf{A}_1]$ etc.

Ans. (a) (a) Let $\mathbf{x}' = x_i'\mathbf{e}_i$ be the position at time τ of the particle which is at $\mathbf{x} = x_i\mathbf{e}_i$ at time t.
Then $x_i' = x_i'(x_1, x_2, x_3, \tau)$ gives the pathline equation. Thus,

$$\frac{dx_1'}{d\tau} = v_1 = -kx_1' \quad \text{(i)}, \quad \frac{dx_2'}{d\tau} = v_2 = kx_2' \quad \text{(ii)}, \quad \frac{dx_3'}{d\tau} = 0 \quad \text{(iii)}$$

with the initial conditions: $x_i = x_i'(x_1, x_2, x_3, t)$. Now,

$$\frac{dx_1'}{d\tau} = -kx_1' \to \ln x_1' = -k\tau + g(x_1, x_2, x_3) \to g(x_1, x_2, x_3) = \ln x_1 + kt$$

$$\to \ln x_1' = -k\tau + \ln x_1 + kt \to \ln x_1' - \ln x_1 = -k(\tau - t) \to x_1' = x_1 e^{-k(\tau-t)}.$$

Similarly, $\dfrac{dx_2'}{d\tau} = kx_2' \to x_2' = x_2 e^{k(\tau-t)}$ and $x_3' = f(x_1, x_2, x_3) = x_3$

Thus, $x_1' = x_1 e^{-k(\tau-t)}, \quad x_2' = x_2 e^{k(\tau-t)}, \quad x_3' = x_3,$

(b) $$[\mathbf{F}_t] = [\nabla \mathbf{x}_t'] = \begin{bmatrix} e^{-k(\tau-t)} & 0 & 0 \\ 0 & e^{k(\tau-t)} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [\mathbf{C}_t] = [\mathbf{F}_t]^{\mathrm{T}} [\mathbf{F}_t] = \begin{bmatrix} e^{-2k(\tau-t)} & 0 & 0 \\ 0 & e^{2k(\tau-t)} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since $e^{\mp 2k(\tau-t)} = 1 \mp 2k(\tau - t) + \dfrac{4k^2}{2}(\tau - t)^2 \mp \dfrac{8k^3}{3!}(\tau - t)^2 + \ldots$, therefore,

$$[\mathbf{C}_t] = [\mathbf{I}] + (\tau - t)\begin{bmatrix} -2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 4k^2 & 0 & 0 \\ 0 & 4k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{(\tau - t)^2}{2} + \begin{bmatrix} -8k^3 & 0 & 0 \\ 0 & 8k^3 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{(\tau - t)^3}{3!} + \ldots$$

(c) $$[\mathbf{A}_1] = \begin{bmatrix} -2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{A}_2] = \begin{bmatrix} 4k^2 & 0 & 0 \\ 0 & 4k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad [\mathbf{A}_3] = \begin{bmatrix} -8k^3 & 0 & 0 \\ 0 & 8k^3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(d) with $v_1 = -kx_1, \quad v_2 = kx_2, \quad v_3 = 0$, $$[\nabla \mathbf{v}] = \begin{bmatrix} -k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix} \to [\mathbf{A}_1] = \begin{bmatrix} -2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\frac{D\mathbf{A}_1}{Dt}\right] = \left[\frac{\partial \mathbf{A}_1}{\partial t}\right] + [\nabla \mathbf{A}_1][\mathbf{v}] \to \left[\frac{D\mathbf{A}_1}{Dt}\right]_{ij} = \left[\frac{\partial \mathbf{A}_1}{\partial t}\right]_{ij} + \frac{\partial (\mathbf{A}_1)_{ij}}{\partial x_k} v_k = 0 + 0 = 0$$

$$[\mathbf{A}_2] = [\mathbf{A}_1][\nabla \mathbf{v}] + [\nabla \mathbf{v}]^{\mathrm{T}} [\mathbf{A}_1]$$

$$= \begin{bmatrix} -2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4k^2 & 0 & 0 \\ 0 & 4k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Next,

$$[\mathbf{A}_2][\nabla\mathbf{v}]=\begin{bmatrix}4k^2 & 0 & 0\\ 0 & 4k^2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}-k & 0 & 0\\ 0 & k & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}-4k^3 & 0 & 0\\ 0 & 4k^3 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

Thus, $[\mathbf{A}_3]=[0]+[\mathbf{A}_2][\nabla\mathbf{v}]+[\nabla\mathbf{v}]^{\mathrm{T}}[\mathbf{A}_2]=\begin{bmatrix}-8k^3 & 0 & 0\\ 0 & 8k^3 & 0\\ 0 & 0 & 0\end{bmatrix}.$

8.13 Given the following velocity field: $v_1=kx_1,\quad v_2=kx_2,\quad v_3=-2kx_3$. Obtain (a) the particle pathline equations using the current time as the reference time, (b) the relative right Cauchy-Green deformation tensor and (c) the Rivlin-Ericksen tensors using the equation $\mathbf{C}_t=\mathbf{I}+(\tau-t)\mathbf{A}_1+(\tau-t)^2\mathbf{A}_2/2+..$ (d) the Rivlin-Ericksen tensor $\mathbf{A}_2$ and $\mathbf{A}_3$ using the recursive equation, $[\mathbf{A}_2]=[D\mathbf{A}_1/Dt]+[\mathbf{A}_1][\nabla\mathbf{v}]+[\nabla\mathbf{v}]^{\mathrm{T}}[\mathbf{A}_1]$ etc.

Ans. (a) Let $\mathbf{x}'=x_i'\mathbf{e}_i$ be the position at time τ of the particle which is at $\mathbf{x}=x_i\mathbf{e}_i$ at time t. Then $x_i'(x_1,x_2,x_3,\tau)$ gives the pathline equation. Thus,

$$\frac{dx_1'}{d\tau}=v_1=kx_1' \quad \text{(i)},\quad \frac{dx_2'}{d\tau}=v_2=kx_2' \quad \text{(ii)},\quad \frac{dx_3'}{d\tau}=-2kx_3' \quad \text{(iii)}$$

with the initial conditions: $x_i=x_i'(x_1,x_2,x_3,t)$. Now,

$$\frac{dx_1'}{d\tau}=kx_1'\rightarrow \ln x_1'=k\tau+g(x_1,x_2,x_3)\rightarrow g(x_1,x_2,x_3)=\ln x_1-kt$$

$$\rightarrow \ln x_1'=k\tau+\ln x_1-kt\rightarrow \ln x_1'-\ln x_1=k(\tau-t)\rightarrow x_1'=x_1e^{k(\tau-t)}.$$

Similarly, $\frac{dx_2'}{d\tau}=kx_2'\rightarrow x_2'=x_2e^{k(\tau-t)}$ and $x_3'=x_3e^{-2k(\tau-t)}$. Thus,

$$x_1'=x_1e^{k(\tau-t)},\quad x_2'=x_2e^{k(\tau-t)},\quad x_3'=x_3e^{-2k(\tau-t)}$$

(b)

$$[\mathbf{F}_t]=[\nabla\mathbf{x}_t']=\begin{bmatrix}e^{k(\tau-t)} & 0 & 0\\ 0 & e^{k(\tau-t)} & 0\\ 0 & 0 & e^{-2k(\tau-t)}\end{bmatrix},\quad [\mathbf{C}_t]=[\mathbf{F}_t]^{\mathrm{T}}[\mathbf{F}_t]=\begin{bmatrix}e^{2k(\tau-t)} & 0 & 0\\ 0 & e^{2k(\tau-t)} & 0\\ 0 & 0 & e^{-4k(\tau-t)}\end{bmatrix}$$

Since

$$e^{2k(\tau-t)}=1+2k(\tau-t)+\frac{4k^2}{2}(\tau-t)^2+\frac{8k^3}{3!}(\tau-t)^3+...,$$

$$e^{-4k(\tau-t)}=1-4k(\tau-t)+\frac{16k^2}{2}(\tau-t)^2-\frac{64k^3}{3!}(\tau-t)^3+...,$$

therefore, $[\mathbf{C}_t]=[\mathbf{I}]$

$$+(\tau-t)\begin{bmatrix}2k&0&0\\0&2k&0\\0&0&-4k\end{bmatrix}+\begin{bmatrix}4k^2&0&0\\0&4k^2&0\\0&0&16k^2\end{bmatrix}\frac{(\tau-t)^2}{2}+\begin{bmatrix}8k^3&0&0\\0&8k^3&0\\0&0&-64k^3\end{bmatrix}\frac{(\tau-t)^3}{3!}+...$$

(c) $[\mathbf{A}_1]=\begin{bmatrix}2k&0&0\\0&2k&0\\0&0&-4k\end{bmatrix}$, $[\mathbf{A}_2]=\begin{bmatrix}4k^2&0&0\\0&4k^2&0\\0&0&16k^2\end{bmatrix}$, $[\mathbf{A}_3]=\begin{bmatrix}8k^3&0&0\\0&8k^3&0\\0&0&-64k^3\end{bmatrix}$

(d) with $v_1=kx_1,\quad v_2=kx_2,\quad v_3=-2kx_3$, $[\nabla\mathbf{v}]=\begin{bmatrix}k&0&0\\0&k&0\\0&0&-2k\end{bmatrix}\rightarrow[\mathbf{A}_1]=\begin{bmatrix}2k&0&0\\0&2k&0\\0&0&-4k\end{bmatrix}$

$$\left[\frac{D\mathbf{A}_1}{Dt}\right]=\left[\frac{\partial\mathbf{A}_1}{\partial t}\right]+[\nabla\mathbf{A}_1][\mathbf{v}]\rightarrow\left[\frac{D\mathbf{A}_1}{Dt}\right]_{ij}=\left[\frac{\partial\mathbf{A}_1}{\partial t}\right]_{ij}+\frac{\partial(\mathbf{A}_1)_{ij}}{\partial x_k}v_k=0+0=0$$

$$[\mathbf{A}_2]=[\mathbf{A}_1][\nabla\mathbf{v}]+[\nabla\mathbf{v}]^{\mathrm{T}}[\mathbf{A}_1]$$

$$=\begin{bmatrix}2k&0&0\\0&2k&0\\0&0&-4k\end{bmatrix}\begin{bmatrix}k&0&0\\0&k&0\\0&0&-2k\end{bmatrix}+\begin{bmatrix}k&0&0\\0&k&0\\0&0&-2k\end{bmatrix}\begin{bmatrix}2k&0&0\\0&2k&0\\0&0&-4k\end{bmatrix}=\begin{bmatrix}4k^2&0&0\\0&4k^2&0\\0&0&16k^2\end{bmatrix}.$$

Next,

$$[\mathbf{A}_2][\nabla\mathbf{v}]=\begin{bmatrix}4k^2&0&0\\0&4k^2&0\\0&0&16k^2\end{bmatrix}\begin{bmatrix}k&0&0\\0&k&0\\0&0&-2k\end{bmatrix}=\begin{bmatrix}4k^3&0&0\\0&4k^3&0\\0&0&-32k^3\end{bmatrix}.$$

Thus, $[\mathbf{A}_3]=[0]+[\mathbf{A}_2][\nabla\mathbf{v}]+[\nabla\mathbf{v}]^{\mathrm{T}}[\mathbf{A}_2]=\begin{bmatrix}8k^3&0&0\\0&8k^3&0\\0&0&-64k^3\end{bmatrix}$ Etc.

8.14 Given the following velocity field: $v_1=kx_2,\quad v_2=kx_1,\quad v_3=0$. Obtain (a) the particle pathline equations using the current time as the reference time, (b) the relative right Cauchy-Green deformation tensor and (c) the Rivlin-Ericksen tensors using the equation $\mathbf{C}_t=\mathbf{I}+(\tau-t)\mathbf{A}_1+(\tau-t)^2\mathbf{A}_2/2+..$ (d) the Rivlin-Ericksen tensor $\mathbf{A}_2$ and $\mathbf{A}_3$ using the recursive equation, $[\mathbf{A}_2]=[D\mathbf{A}_1/Dt]+[\mathbf{A}_1][\nabla\mathbf{v}]+[\nabla\mathbf{v}]^{\mathrm{T}}[\mathbf{A}_1]$ etc.

Ans. (a) Let $\mathbf{x}'=x_i'\mathbf{e}_i$ be the position at time τ of the particle which is at $\mathbf{x}=x_i\mathbf{e}_i$ at time t. Then $x_i'=x_i'(x_1,x_2,x_3,\tau)$ gives the pathline equation. Thus,

$$\frac{dx_1'}{d\tau}=v_1=kx_2'\ \text{(i)},\quad \frac{dx_2'}{d\tau}=v_2=kx_1'\ \text{(ii)},\quad \frac{dx_3'}{d\tau}=0\ \text{(iii)}$$

with the initial conditions: $x_i = x_i'\left(x_1, x_2, x_3, t\right)$. Now,

$$\text{(i)} \rightarrow \frac{dx_1'}{d\tau} = kx_2' \rightarrow \frac{d^2x_1'}{d\tau^2} = k\frac{dx_2'}{d\tau} = k^2x_1' \rightarrow \frac{d^2x_1'}{d\tau^2} - k^2x_1' = 0 \rightarrow$$

$$x_1' = A\sinh k\tau + B\cosh k\tau \rightarrow x_1 = A\sinh kt + B\cosh kt \qquad \text{(iv)}$$

$$\text{(ii)} \rightarrow x_2' = \frac{1}{k}\frac{dx_1'}{d\tau} = A\cosh k\tau + B\sinh k\tau \rightarrow x_2 = A\cosh kt + B\sinh kt \quad \text{(v)}$$

(iv) and (v) gives $A = -x_1\sinh kt + x_2\cosh kt, \quad B = x_1\cosh kt - x_2\sinh kt$

$$x_1' = x_1\left(\cosh kt\cosh k\tau - \sinh kt\sinh k\tau\right) + x_2\left(\cosh kt\sinh k\tau - \sinh kt\cosh k\tau\right)$$

$$x_2' = x_1\left(\cosh kt\sinh k\tau - \sinh kt\cosh k\tau\right) + x_2\left(\cosh kt\cosh k\tau - \sinh kt\sinh k\tau\right)$$

That is,

$$x_1' = x_1\cosh k\left(\tau - t\right) + x_2\sinh k\left(\tau - t\right), \quad x_2' = x_1\sinh k\left(\tau - t\right) + x_2\cosh k\left(\tau - t\right), \quad x_3' = x_3$$

$$\left[\mathbf{F}_t\right] = \left[\nabla\mathbf{x}_t'\right] = \begin{bmatrix} \cosh k\left(\tau - t\right) & \sinh k\left(\tau - t\right) & 0 \\ \sinh k\left(\tau - t\right) & \cosh k\left(\tau - t\right) & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

(b)

$$\left[\mathbf{C}_t\right] = \left[\mathbf{F}_t\right]^{\mathrm{T}}\left[\mathbf{F}_t\right] = \begin{bmatrix} \cosh^2 x + \sinh^2 x & 2\cosh x\sinh x & 0 \\ 2\cosh x\sinh x & \sinh^2 x + \cosh^2 x & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad x \equiv k\left(\tau - t\right)$$

Since

$$\cosh x = 1 + \frac{x^2}{2} + \mathrm{O}(x^4), \quad \sinh x = x + \frac{x^3}{6} + \mathrm{O}(x^5)$$

$$\cosh^2 x = 1 + x^2 + \mathrm{O}(x^4), \quad \sinh^2 x = x^2 + \mathrm{O}(x^4), \quad \sinh x\cosh x = x + \frac{2x^3}{3} + \mathrm{O}(x^5)$$

$$\left[\mathbf{C}_t\right] = \begin{bmatrix} 1 + 2x^2 + .. & 2x + \frac{4}{3}x^3 + ... & 0 \\ 2x + \frac{4}{3}x^3 + ... & 1 + 2x^2 + ... & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left[\mathbf{I}\right] + \begin{bmatrix} 0 & 2k & 0 \\ 2k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\left(\tau - t\right)$$

$$+ \begin{bmatrix} 4k^2 & 0 & 0 \\ 0 & 4k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{\left(\tau - t\right)^2}{2} + \begin{bmatrix} 0 & 8k^3 & 0 \\ 8k^3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\frac{\left(\tau - t\right)^3}{6} + ...$$

Thus,

(c)

$$\left[\mathbf{A}_1\right] = \begin{bmatrix} 0 & 2k & 0 \\ 2k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \left[\mathbf{A}_2\right] = \begin{bmatrix} 4k^2 & 0 & 0 \\ 0 & 4k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \left[\mathbf{A}_3\right] = \begin{bmatrix} 0 & 8k^3 & 0 \\ 8k^3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \textit{etc.}$$

(d) with $v_1 = kx_2, \quad v_2 = kx_1, \quad v_3 = 0$, $[\nabla \mathbf{v}] = \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow [\mathbf{A}_1] = \begin{bmatrix} 0 & 2k & 0 \\ 2k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$$\left[\frac{D\mathbf{A}_1}{Dt}\right] = \left[\frac{\partial \mathbf{A}_1}{\partial t}\right] + [\nabla \mathbf{A}_1][\mathbf{v}] \rightarrow \left[\frac{D\mathbf{A}_1}{Dt}\right]_{ij} = \left[\frac{\partial \mathbf{A}_1}{\partial t}\right]_{ij} + \frac{\partial (\mathbf{A}_1)_{ij}}{\partial x_k} v_k = 0 + 0 = 0 .$$

$$[\mathbf{A}_2] = [\mathbf{A}_1][\nabla \mathbf{v}] + [\nabla \mathbf{v}]^{\mathrm{T}} [\mathbf{A}_1]$$

$$[\mathbf{A}_1][\nabla \mathbf{v}] = \begin{bmatrix} 0 & 2k & 0 \\ 2k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2k^2 & 0 & 0 \\ 0 & 2k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow [\mathbf{A}_2] = \begin{bmatrix} 4k^2 & 0 & 0 \\ 0 & 4k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Next,

$$[\mathbf{A}_2][\nabla \mathbf{v}] = \begin{bmatrix} 4k^2 & 0 & 0 \\ 0 & 4k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & k & 0 \\ k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4k^3 & 0 \\ 4k^3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, $[\mathbf{A}_3] = [0] + [\mathbf{A}_2][\nabla \mathbf{v}] + [\nabla \mathbf{v}]^{\mathrm{T}} [\mathbf{A}_2] = \begin{bmatrix} 0 & 8k^3 & 0 \\ 8k^3 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

8.15 Given the velocity field in cylindrical coordinates: $v_r = 0, \;\; v_\theta = 0, \;\; v_z = v(r)$, obtain the second Rivlin-Ericksen tensors $\mathbf{A}_N$, $N = 2, 3, \ldots$ using the recursive formula.

Ans. $[\nabla \mathbf{v}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ k & 0 & 0 \end{bmatrix}, \quad k = \frac{dv}{dr}, \quad [\mathbf{A}_1] = [\nabla \mathbf{v}] + [\nabla \mathbf{v}]^{\mathrm{T}} = \begin{bmatrix} 0 & 0 & k \\ 0 & 0 & 0 \\ k & 0 & 0 \end{bmatrix}$

Since $(\mathbf{A}_1)_{ij}$ =constant, independent of time and space, therefore

$$\left[\left(\frac{D\mathbf{A}_1}{Dt}\right)_{ij}\right] = \left[\left(\frac{\partial \mathbf{A}_1}{\partial t}\right)_{ij} + (\nabla \mathbf{A}_1)_{ijk} v_k\right] = [0].$$

$$[\mathbf{A}_2]=\left[\mathbf{A}_1(\nabla\mathbf{v})+(\nabla\mathbf{v})^{\mathrm{T}}\mathbf{A}_1\right]$$

$$=\begin{bmatrix}0&0&k\\0&0&0\\k&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\k&0&0\end{bmatrix}+\begin{bmatrix}0&0&k\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&k\\0&0&0\\k&0&0\end{bmatrix}=\begin{bmatrix}2k^2&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

$$[\mathbf{A}_3]=\left[\mathbf{A}_2(\nabla\mathbf{v})+(\nabla\mathbf{v})^{\mathrm{T}}\mathbf{A}_2\right]$$

$$=\begin{bmatrix}2k^2&0&0\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}0&0&0\\0&0&0\\k&0&0\end{bmatrix}+\begin{bmatrix}0&0&k\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}2k^2&0&0\\0&0&0\\0&0&0\end{bmatrix}=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}.$$

Thus, $\mathbf{A}_N=0, N=3,4...$

8.16 Using the equations given in Appendix 8.1 for cylindrical coordinates, verify that the $rr\theta$ component of the third order tensor $\nabla\mathbf{T}$ is given by:

$$(\nabla\mathbf{T})_{rr\theta}=\left[\frac{1}{r}\frac{\partial T_{rr}}{\partial\theta}-\frac{T_{\theta r}+T_{r\theta}}{r}\right]$$

Ans. From the equations

$$(\nabla T)_{ijm}h_m=\frac{\partial T_{ij}}{\partial x_m}+T_{qj}\Gamma_{qmi}+T_{iq}\Gamma_{qmj}\quad\text{no sum on } m\text{, sum on } q,$$

and $h_r=1,\ h_\theta=r,\ h_z=1;\ \Gamma_{r\theta\theta}=1, \Gamma_{\theta\theta r}=-1$, all other $\Gamma_{ijk}=0$

we have,

$$(\nabla T)_{rr\theta}h_\theta=\frac{\partial T_{rr}}{\partial\theta}+T_{qr}\Gamma_{q\theta r}+T_{rq}\Gamma_{q\theta r}=\frac{\partial T_{rr}}{\partial\theta}+T_{\theta r}\Gamma_{\theta\theta r}+T_{r\theta}\Gamma_{\theta\theta r}\text{, thus,}$$

$$(\nabla T)_{rr\theta}r=\frac{\partial T_{rr}}{\partial\theta}+T_{\theta r}(-1)+T_{r\theta}(-1)\rightarrow(\nabla T)_{rr\theta}=\frac{1}{r}\frac{\partial T_{rr}}{\partial\theta}-\frac{T_{\theta r}+T_{r\theta}}{r}.$$

8.17 Using the equations given in Appendix 8.1 for cylindrical coordinates, verify that the $r\theta\theta$ component of the third order tensor $\nabla\mathbf{T}$ is given by:

$$(\nabla\mathbf{T})_{r\theta\theta}=\frac{1}{r}\frac{\partial T_{r\theta}}{\partial\theta}+\frac{T_{rr}-T_{\theta\theta}}{r}$$

Ans. From the equations

$$(\nabla T)_{ijm}h_m=\frac{\partial T_{ij}}{\partial x_m}+T_{qj}\Gamma_{qmi}+T_{iq}\Gamma_{qmj}\quad\text{no sum on } m\text{, sum on } q$$

and $h_r=1,\ h_\theta=r,\ h_z=1;\ \Gamma_{r\theta\theta}=1, \Gamma_{\theta\theta r}=-1$, all other $\Gamma_{ijk}=0.$

we have,

$$(\nabla T)_{r\theta\theta} h_\theta = \frac{\partial T_{r\theta}}{\partial \theta} + T_{q\theta}\Gamma_{q\theta r} + T_{rq}\Gamma_{q\theta\theta} \to (\nabla T)_{r\theta\theta} h_\theta = \frac{\partial T_{r\theta}}{\partial \theta} + T_{\theta\theta}\Gamma_{\theta\theta r} + T_{rr}\Gamma_{r\theta\theta}$$

$$\to (\nabla T)_{r\theta\theta} r = \frac{\partial T_{r\theta}}{\partial \theta} + T_{\theta\theta}(-1) + T_{rr}(1) \to (\nabla T)_{r\theta\theta} = \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{T_{rr} - T_{\theta\theta}}{r}.$$

8.18 Using the equations given in Appendix 8.1 for spherical coordinates, verify that the $rr\phi$ component of the third order tensor $\nabla \mathbf{T}$ is given by:

$$(\nabla \mathbf{T})_{rr\phi} = \frac{1}{r\sin\theta}\frac{\partial T_{rr}}{\partial \phi} - \frac{(T_{\phi r} + T_{r\phi})}{r}$$

Ans. From the equations

$$(\nabla T)_{ijm} h_m = \frac{\partial T_{ij}}{\partial x_m} + T_{qj}\Gamma_{qmi} + T_{iq}\Gamma_{qmj} \quad \text{no sum on } m\text{, sum on } q$$

and $h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta;\ \Gamma_{r\theta\theta} = 1,\ \Gamma_{r\phi\phi} = \sin\theta,$

$\Gamma_{\phi\phi r} = -\sin\theta, \Gamma_{\phi\phi\theta} = -\cos\theta,\ \Gamma_{\theta\theta r} = -1, \Gamma_{\theta\phi\phi} = \cos\theta$ all other $\Gamma_{ijk} = 0$

we have,

$$(\nabla T)_{rr\phi} h_\phi = \frac{\partial T_{rr}}{\partial \phi} + T_{qr}\Gamma_{q\phi r} + T_{rq}\Gamma_{q\phi r} \to (\nabla T)_{rr\phi} h_\phi = \frac{\partial T_{rr}}{\partial \phi} + T_{\phi r}\Gamma_{\phi\phi r} + T_{r\phi}\Gamma_{\phi\phi r}$$

$$\to (\nabla T)_{rr\phi}(r\sin\theta) = \frac{\partial T_{rr}}{\partial \phi} + T_{\phi r}(-\sin\theta) + T_{r\phi}(-\sin\theta) \to$$

$$\to (\nabla T)_{rr\phi} = \frac{1}{r\sin\theta}\frac{\partial T_{rr}}{\partial \phi} - \frac{T_{\phi r} + T_{r\phi}}{r}.$$

8.19 Using the equations given in Appendix 8.1 for spherical coordinates, verify that the $\phi\phi\phi$ component of the third order tensor $\nabla \mathbf{T}$ is given by:

$$\frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{(T_{r\phi} + T_{\phi r})}{r} + \frac{(T_{\theta\phi} + T_{\phi\theta})\cot\theta}{r}$$

Ans. From the equations

$$(\nabla T)_{ijm} h_m = \frac{\partial T_{ij}}{\partial x_m} + T_{qj}\Gamma_{qmi} + T_{iq}\Gamma_{qmj} \quad \text{no sum on } m\text{, sum on } q$$

$h_r = 1,\ h_\theta = r,\ h_\phi = r\sin\theta;\ \Gamma_{r\theta\theta} = 1,\ \Gamma_{r\phi\phi} = \sin\theta,$

$\Gamma_{\phi\phi r} = -\sin\theta, \Gamma_{\phi\phi\theta} = -\cos\theta,\ \Gamma_{\theta\theta r} = -1, \Gamma_{\theta\phi\phi} = \cos\theta$ all other $\Gamma_{ijk} = 0$

we have,

$$\left(\nabla T\right)_{\phi\phi\phi} h_\phi = \frac{\partial T_{\phi\phi}}{\partial \phi} + T_{q\phi}\Gamma_{q\phi\phi} + T_{\phi q}\Gamma_{q\phi\phi}$$

$$\rightarrow \left(\nabla T\right)_{\phi\phi\phi} h_\phi = \frac{\partial T_{\phi\phi}}{\partial \phi} + T_{r\phi}\Gamma_{r\phi\phi} + T_{\theta\phi}\Gamma_{\theta\phi\phi} + T_{\phi r}\Gamma_{r\phi\phi} + T_{\phi\theta}\Gamma_{\theta\phi\phi}$$

$$\rightarrow \left(\nabla T\right)_{\phi\phi\phi} h_\phi = \frac{\partial T_{\phi\phi}}{\partial \phi} + \left(T_{r\phi} + T_{\phi r}\right)\Gamma_{r\phi\phi} + \left(T_{\theta\phi} + T_{\phi\theta}\right)\Gamma_{\theta\phi\phi}$$

$$= \frac{\partial T_{\phi\phi}}{\partial \phi} + \left(T_{r\phi} + T_{\phi r}\right)\left(\sin\theta\right) + \left(T_{\theta\phi} + T_{\phi\theta}\right)\cos\theta$$

$$\rightarrow \left(\nabla T\right)_{\phi\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial T_{\phi\phi}}{\partial \phi} + \frac{\left(T_{r\phi} + T_{\phi r}\right)}{r} + \frac{\left(T_{\theta\phi} + T_{\phi\theta}\right)\cot\theta}{r}$$

8.20 Given the velocity field in cylindrical coordinates: $v_r = 0,\ \ v_\theta = v\left(r\right),\ \ v_z = 0$, obtain (a) the first Rivlin-Ericksen tensor $\mathbf{A}_1$ (b) $\nabla\mathbf{A}_1$ (c) the second Rivlin-Ericksen tensors $\mathbf{A}_2$, using the recursive formula..

Ans.
$$\left[\nabla\mathbf{v}\right] = \begin{bmatrix} \dfrac{\partial v_r}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_r}{\partial \theta} - \mathrm{v}_\theta\right) & \dfrac{\partial v_r}{\partial z} \\ \dfrac{\partial v_\theta}{\partial r} & \dfrac{1}{r}\left(\dfrac{\partial v_\theta}{\partial \theta} + v_r\right) & \dfrac{\partial v_\theta}{\partial z} \\ \dfrac{\partial v_z}{\partial r} & \dfrac{1}{r}\dfrac{\partial v_z}{\partial \theta} & \dfrac{\partial v_z}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{v(r)}{r} & 0 \\ \dfrac{dv}{dr} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\left[\mathbf{A}_1\right] = \left[\nabla\mathbf{v}\right] + \left[\nabla\mathbf{v}\right]^{\mathrm{T}} = \begin{bmatrix} 0 & k\left(r\right) & 0 \\ k\left(r\right) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \ k = \left(\frac{dv}{dr} - \frac{v(r)}{r}\right)$$

$$\left[\left(\frac{D\mathbf{A}_1}{Dt}\right)_{ij}\right] = \left[\left(\frac{\partial\mathbf{A}_1}{\partial t}\right)_{ij} + \left(\nabla\mathbf{A}_1\right)_{ijk} v_k\right]$$

$$= \left[\left(\nabla\mathbf{A}_1\right)_{ijk} v_k\right] = \begin{bmatrix} \left(\nabla\mathbf{A}_1\right)_{rr\theta} v_\theta & \left(\nabla\mathbf{A}_1\right)_{r\theta\theta} v_\theta & \left(\nabla\mathbf{A}_1\right)_{rz\theta} v_\theta \\ \left(\nabla\mathbf{A}_1\right)_{\theta r\theta} v_\theta & \left(\nabla\mathbf{A}_1\right)_{\theta\theta\theta} v_\theta & \left(\nabla\mathbf{A}_1\right)_{\theta z\theta} v_\theta \\ \left(\nabla\mathbf{A}_1\right)_{zr\theta} v_\theta & \left(\nabla\mathbf{A}_1\right)_{z\theta\theta} v_\theta & \left(\nabla\mathbf{A}_1\right)_{zz\theta} v_\theta \end{bmatrix}$$

The components of the third order tensor $\left(\nabla\mathbf{A}_1\right)$ can be obtained from Appendix 8.1 as:

$$\left(\nabla\mathbf{A}_1\right)_{rr\theta} = \left[\frac{1}{r}\frac{\partial A_{rr}}{\partial \theta} - \frac{A_{\theta r} + A_{r\theta}}{r}\right] = -\frac{2k}{r},\ \left(\nabla\mathbf{A}_1\right)_{r\theta\theta} = \frac{1}{r}\frac{\partial A_{r\theta}}{\partial \theta} + \frac{A_{rr} - A_{\theta\theta}}{r} = 0$$

$$\left(\nabla\mathbf{A}_1\right)_{rz\theta} = \frac{1}{r}\frac{\partial A_{rz}}{\partial \theta} - \frac{A_{\theta z}}{r} = 0$$

$$\left(\nabla\mathbf{A}_1\right)_{\theta r\theta} = \frac{1}{r}\frac{\partial A_{\theta r}}{\partial \theta} + \frac{A_{rr} - A_{\theta\theta}}{r} = 0,\ \left(\nabla\mathbf{A}_1\right)_{\theta\theta\theta} = \frac{2k}{r},\ \left(\nabla\mathbf{A}_1\right)_{\theta z\theta} = \frac{1}{r}\frac{\partial A_{\theta z}}{\partial \theta} + \frac{A_{rz}}{r} = 0$$

$\left(\nabla \mathbf{A}_1\right)_{zr\theta} = \frac{1}{r}\frac{\partial A_{zr}}{\partial \theta} - \frac{A_{z\theta}}{r} = 0$, $\left(\nabla \mathbf{A}_1\right)_{z\theta\theta} = \frac{1}{r}\frac{\partial A_{z\theta}}{\partial \theta} + \frac{A_{zr}}{r} = 0$, $\left(\nabla \mathbf{A}_1\right)_{zz\theta} = \frac{1}{r}\frac{\partial A_{zz}}{\partial \theta} = 0$

Thus, $\left[\left(\frac{D\mathbf{A}_1}{Dt}\right)_{ij}\right] = v_\theta \begin{bmatrix} -2k/r & 0 & 0 \\ 0 & 2k/r & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -2kv/r & 0 & 0 \\ 0 & 2kv/r & 0 \\ 0 & 0 & 0 \end{bmatrix}$,

$$[\mathbf{A}_1][\nabla \mathbf{v}] = \begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & -v(r)/r & 0 \\ dv/dr & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} kdv/dr & 0 & 0 \\ 0 & -kv/r & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$[\nabla \mathbf{v}]^{\mathrm{T}}[\mathbf{A}_1] = \begin{bmatrix} 0 & dv/dr & 0 \\ -v(r)/r & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & k(r) & 0 \\ k(r) & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} kdv/dr & 0 & 0 \\ 0 & -kv/r & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mathbf{A}_2 = \left[\frac{D\mathbf{A}_1}{Dt}\right] + [\mathbf{A}_1][\nabla \mathbf{v}] + [\nabla \mathbf{v}]^{\mathrm{T}}[\mathbf{A}_1] = 2\begin{bmatrix} k(dv/dr - v/r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 2k^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

8.21 Derive Eq. (8.11.3), i.e., $\mathbf{A}_{N+1} = \frac{D\mathbf{A}_N}{Dt} + \mathbf{A}_N(\nabla \mathbf{v}) + (\nabla \mathbf{v})^{\mathrm{T}}\mathbf{A}_N$.

Ans. We had {see Eq. (8.11.7)},

$\frac{D^N}{Dt^N}\left(ds^2\right) = d\mathbf{x}\cdot\mathbf{A}_N d\mathbf{x} \rightarrow \frac{D^{N+1}}{Dt^{N+1}}\left(ds^2\right) = \frac{Dd\mathbf{x}}{Dt}\cdot\mathbf{A}_N d\mathbf{x} + d\mathbf{x}\cdot\frac{D\mathbf{A}_N}{Dt}d\mathbf{x} + d\mathbf{x}\cdot\mathbf{A}_N\frac{Dd\mathbf{x}}{Dt}$. That is,

$$\frac{D^{N+1}}{Dt^{N+1}}\left(ds^2\right) == (\nabla \mathbf{v})d\mathbf{x}\cdot\mathbf{A}_N d\mathbf{x} + d\mathbf{x}\cdot\frac{D\mathbf{A}_N}{Dt}d\mathbf{x} + d\mathbf{x}\cdot\mathbf{A}_N(\nabla \mathbf{v})d\mathbf{x}$$

$$= d\mathbf{x}\cdot(\nabla \mathbf{v})^{\mathrm{T}}\mathbf{A}_N d\mathbf{x} + d\mathbf{x}\cdot\mathbf{A}_N(\nabla \mathbf{v})d\mathbf{x} + d\mathbf{x}\cdot\frac{D\mathbf{A}_N}{Dt}d\mathbf{x}$$

$$= d\mathbf{x}\cdot\left[(\nabla \mathbf{v})^{\mathrm{T}}\mathbf{A}_N + \mathbf{A}_N(\nabla \mathbf{v}) + \frac{D\mathbf{A}_N}{Dt}\right]d\mathbf{x} = d\mathbf{x}\cdot\mathbf{A}_{N+1}d\mathbf{x}.$$

Thus, $\mathbf{A}_{N+1} = \frac{D\mathbf{A}_N}{Dt} + \mathbf{A}_N(\nabla \mathbf{v}) + (\nabla \mathbf{v})^{\mathrm{T}}\mathbf{A}_N$

8.22 Let $\mathbf{S} \equiv D\mathbf{T}/Dt + \mathbf{TW} - \mathbf{WT}$, where $\mathbf{T}$ is an objective tensor and $\mathbf{W}$ is the spin tensor, show that $\mathbf{S}$ is objective, i.e., $\mathbf{S}^* = \mathbf{Q}(t)\mathbf{S}\mathbf{Q}^{\mathrm{T}}(t)$.

Ans. Since $\mathbf{T}$ is objective, therefore $\mathbf{T}^* = \mathbf{Q}(t)\mathbf{T}\mathbf{Q}^{\mathrm{T}}(t)$ and from Eq. (8.13.13),

$\mathbf{W}^* = (d\mathbf{Q}/dt)\mathbf{Q}^{\mathrm{T}}(t) + \mathbf{Q}(t)\mathbf{W}\mathbf{Q}^{\mathrm{T}}(t)$, therefore,

$$\mathbf{S}^* \equiv \frac{D\mathbf{T}^*}{Dt} + \mathbf{T}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{T}^*$$
$$= \frac{d\mathbf{Q}}{dt}\mathbf{T}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\left(\frac{D\mathbf{T}}{Dt}\right)\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{T}\frac{d\mathbf{Q}^{\mathrm{T}}}{dt} + \left[\mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}\right]\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}} + \left[\mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}}\right]$$
$$-\frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}}\left[\mathbf{Q}\mathbf{T}\mathbf{Q}^{\mathrm{T}}\right] - \left[\mathbf{Q}\mathbf{W}\mathbf{Q}^{\mathrm{T}}\mathbf{Q}(t)\mathbf{T}\mathbf{Q}^{\mathrm{T}}(t)\right].$$

Now, $\mathbf{Q}(t)\mathbf{Q}^{\mathrm{T}}(t) = \mathbf{I} \rightarrow \frac{d\mathbf{Q}}{dt}\mathbf{Q}^{\mathrm{T}}(t) = -\mathbf{Q}(t)\frac{d\mathbf{Q}^{\mathrm{T}}}{dt}$, therefore, the above equation becomes

$$\mathbf{S}^* = \frac{d\mathbf{Q}}{dt}\mathbf{T}\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\left(\frac{D\mathbf{T}}{Dt}\right)\mathbf{Q}^{\mathrm{T}} + \mathbf{Q}\mathbf{T}\frac{d\mathbf{Q}^{\mathrm{T}}}{dt} - \left[\mathbf{Q}\mathbf{T}\frac{d\mathbf{Q}^{\mathrm{T}}}{dt}\right] + \left[\mathbf{Q}\mathbf{T}\mathbf{W}\mathbf{Q}^{\mathrm{T}}\right]$$
$$-\left[\frac{d\mathbf{Q}}{dt}\mathbf{T}\mathbf{Q}^{\mathrm{T}}\right] - \left[\mathbf{Q}\mathbf{W}\mathbf{T}\mathbf{Q}^{\mathrm{T}}(t)\right].$$

That is,

$$\mathbf{S}^* = \mathbf{Q}(t)\left(\frac{D\mathbf{T}}{Dt} + \mathbf{T}\mathbf{W} - \mathbf{W}\mathbf{T}\right)\mathbf{Q}^{\mathrm{T}}(t) = \mathbf{Q}(t)\mathbf{S}\mathbf{Q}^{\mathrm{T}}(t).$$

8.23 Obtain the viscosity function and the two normal stress function for the nonlinear viscoelastic fluid defined by $\mathbf{S} = \int_0^\infty f_2(s)\left[\mathbf{I} - \mathbf{C}_t^{-1}(t-s)\right]ds$

Ans. For $v_1 = kx_2,\ \ v_2 = v_3 = 0$, we have [see Section 8.9, Eq.(8.9.12)]

$$\left[\mathbf{C}_t(\tau)\right] = \begin{bmatrix} 1 & k(\tau-t) & 0 \\ k(\tau-t) & k^2(\tau-t)^2+1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\quad \left[\mathbf{C}_t^{-1}(\tau)\right] = \begin{bmatrix} 1+k^2(\tau-t)^2 & -k(\tau-t) & 0 \\ -k(\tau-t) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\left[\mathbf{C}_t^{-1}(t-s)\right] = \begin{bmatrix} 1+k^2s^2 & ks & 0 \\ ks & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.\ \text{Thus,}\ \left[\mathbf{I} - \mathbf{C}_t^{-1}(t-s)\right] = \begin{bmatrix} -k^2s^2 & -ks & 0 \\ -ks & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$S_{12} = -k\int_0^\infty sf_2(s)ds \rightarrow \mu \equiv \frac{S_{12}}{k} = -\int_0^\infty sf_2(s)ds,.$$

$$S_{11} = -k^2\int_0^\infty s^2 f_2(s)ds,\ \ S_{22} = 0,\ \ S_{33} = 0,$$

$$\sigma_1 = S_{11} - S_{22} = -k^2\int_0^\infty s^2 f_2(s)ds,\ \ \sigma_2 = S_{22} - S_{33} = 0.$$

8.24 Derive the following transformation laws [Eqs.(8.13.8) and Eq. (8.13.12)] under a change of frame.

$$\mathbf{V}_t^* = \mathbf{Q}(\tau)\mathbf{V}_t\mathbf{Q}^{\mathrm{T}}(\tau) \text{ and } \mathbf{R}_t^* = \mathbf{Q}(\tau)\mathbf{R}_t\mathbf{Q}^{\mathrm{T}}(t)$$

Ans. Since $\mathbf{F}_t = \mathbf{V}_t\mathbf{R}_t$ and $\mathbf{F}_t^* = \mathbf{V}_t^*\mathbf{R}_t^*$, therefore, from $\mathbf{F}_t^*(\tau) = \mathbf{Q}(\tau)\mathbf{F}_t(\tau)\mathbf{Q}^{\mathrm{T}}(t)$, we get $\mathbf{V}_t^*\mathbf{R}_t^* = \mathbf{Q}(\tau)\mathbf{V}_t\mathbf{R}_t\mathbf{Q}^{\mathrm{T}}(t) = \left[\mathbf{Q}(\tau)\mathbf{V}_t\mathbf{Q}^{\mathrm{T}}(\tau)\right]\left[\mathbf{Q}(\tau)\mathbf{R}_t\mathbf{Q}^{\mathrm{T}}(t)\right]$, where $\left[\mathbf{Q}(\tau)\mathbf{V}_t\mathbf{Q}^{\mathrm{T}}(\tau)\right]$ is a symmetry tensor and $\left[\mathbf{Q}(\tau)\mathbf{R}_t\mathbf{Q}^{\mathrm{T}}(t)\right]$ is an orthogonal tensor. Therefore, the uniqueness of the polar decomposition leads to

$$\mathbf{V}_t^* = \mathbf{Q}(\tau)\mathbf{V}_t\mathbf{Q}^{\mathrm{T}}(\tau) \text{ and } \mathbf{R}_t^* = \mathbf{Q}(\tau)\mathbf{R}_t\mathbf{Q}^{\mathrm{T}}(t).$$

8.25 From $\breve{\mathbf{T}} \equiv \left[\dfrac{D\mathbf{J}_{\mathrm{L}}(\tau)}{D\tau}\right]_{\tau=t}$ and $\left[\dfrac{D\mathbf{F}_t(\tau)}{D\tau}\right]_{\tau=t} = \nabla\mathbf{v}$, show that $\breve{\mathbf{T}} = \overset{\mathrm{o}}{\mathbf{T}} + \mathbf{TD} + \mathbf{DT}$. [note misprint in the problem in text]

Ans. From $\mathbf{J}_{\mathrm{L}}(\tau) = \mathbf{F}_t^{\mathrm{T}}(\tau)\mathbf{T}(\tau)\mathbf{F}_t(\tau)$, we have,

$$\frac{D\mathbf{J}_{\mathrm{L}}(\tau)}{D\tau} = \frac{D\mathbf{F}_t^{\mathrm{T}}(\tau)}{D\tau}\mathbf{T}(\tau)\mathbf{F}_t(\tau) + \mathbf{F}_t^{\mathrm{T}}(\tau)\frac{D\mathbf{T}(\tau)}{D\tau}\mathbf{F}_t(\tau) + \mathbf{F}_t^{\mathrm{T}}(\tau)\mathbf{T}(\tau)\frac{D\mathbf{F}_t(\tau)}{D\tau}$$

Thus, $\left[\dfrac{D\mathbf{J}_{\mathrm{L}}(\tau)}{D\tau}\right]_{\tau=t} = (\nabla\mathbf{v})^{\mathrm{T}}\mathbf{T}(t) + \dfrac{D\mathbf{T}}{Dt} + \mathbf{T}(t)\nabla\mathbf{v}$ [Note $\mathbf{F}_t(t) = \mathbf{F}_t^{\mathrm{T}}(t) = \mathbf{I}$]

Now, $\nabla\mathbf{v} = \mathbf{D} + \mathbf{W}$, therefore,

$$\left[\frac{D\mathbf{J}_{\mathrm{L}}(\tau)}{D\tau}\right]_{\tau=t} = \frac{D\mathbf{T}}{Dt} + (\mathbf{D}+\mathbf{W})^{\mathrm{T}}\mathbf{T} + \mathbf{T}(\mathbf{D}+\mathbf{W}) = \frac{D\mathbf{T}}{Dt} + \mathbf{DT} + \mathbf{TD} + \left(\mathbf{W}^{\mathrm{T}}\mathbf{T} + \mathbf{TW}\right)$$

$$= \frac{D\mathbf{T}}{Dt} + \mathbf{DT} + \mathbf{TD} + (\mathbf{TW} - \mathbf{WT}) = \overset{\mathrm{o}}{\mathbf{T}} + \mathbf{DT} + \mathbf{TD}$$

8.26 Consider $\mathbf{J}_{\mathrm{U}}(\tau) = \mathbf{F}_t^{-1}(\tau)\mathbf{T}(\tau)\mathbf{F}_t^{-1\mathrm{T}}(\tau)$. Show that (a) $\left[D\mathbf{J}_{\mathrm{U}}(\tau)/D\tau\right]_{\tau=t}$ is objective and (b) $\left[D\mathbf{J}_{\mathrm{U}}(\tau)/D\tau\right]_{\tau=t} = \dfrac{D\mathbf{T}}{D\tau} - \mathbf{T}(\nabla\mathbf{v})^{\mathrm{T}} - (\nabla\mathbf{v})\mathbf{T} = \overset{\mathrm{o}}{\mathbf{T}} - (\mathbf{TD} + \mathbf{DT})$.

Ans. (a) Given $\mathbf{J}_{\mathrm{U}}(\tau) = \mathbf{F}_t^{-1}(\tau)\mathbf{T}(\tau)\left(\mathbf{F}_t^{-1}\right)^{\mathrm{T}}(\tau)$, and $\mathbf{J}_{\mathrm{U}}^*(\tau) = \mathbf{F}^{*\,-1}_{t}(\tau)\mathbf{T}^*(\tau)\left(\mathbf{F}^{*\,-1}_{t}\right)^{\mathrm{T}}(\tau)$.

In a change of frame (see Section 8.13. Eq.(8.13.6), $\mathbf{F}_t^*(\tau) = \mathbf{Q}(\tau)\mathbf{F}_t(\tau)\mathbf{Q}^{\mathrm{T}}(t)$, so that

$\mathbf{F}^{*\,-1}_{t}(\tau) = \mathbf{Q}(t)\mathbf{F}_t^{-1}(\tau)\mathbf{Q}^{\mathrm{T}}(\tau)$ and $\left(\mathbf{F}^{*\,-1}_{t}\right)^{\mathrm{T}}(\tau) = \mathbf{Q}(\tau)\left(\mathbf{F}_t^{-1}\right)^{\mathrm{T}}(\tau)\mathbf{Q}^{\mathrm{T}}(t)$.

Also, $\mathbf{T}^*(t) = \mathbf{Q}(t)\mathbf{T}(t)\mathbf{Q}^{\mathrm{T}}(t)$. Thus

$$\mathbf{J}_{\mathrm{U}}^*(\tau) = \mathbf{Q}(t)\mathbf{F}_t^{-1}(\tau)\mathbf{Q}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau)\mathbf{T}(\tau)\mathbf{Q}^{\mathrm{T}}(\tau)\mathbf{Q}(\tau)\left(\mathbf{F}_t^{-1}\right)^{\mathrm{T}}(\tau)\mathbf{Q}^{\mathrm{T}}(t)$$

$$= \mathbf{Q}(t)\mathbf{F}_t^{-1}(\tau)\mathbf{T}(\tau)\left(\mathbf{F}_t^{-1}\right)^{\mathrm{T}}(\tau)\mathbf{Q}^{\mathrm{T}}(t).$$ That is,

$\mathbf{J}_{\mathrm{U}}^*(\tau) = \mathbf{Q}(t)\mathbf{J}_{\mathbf{U}}(\tau)\,\mathbf{Q}^{\mathrm{T}}(t)$ and $D^N\mathbf{J}_{\mathrm{U}}^*(\tau)/D\tau^N = \mathbf{Q}(t)\left(D^N\mathbf{J}_{\mathbf{U}}(\tau)/D\tau^N\right)\mathbf{Q}^{\mathrm{T}}(t)$. Thus,

$$\left[\frac{D^N\mathbf{J}^*_{\text{U}}(\tau)}{D\tau^N}\right]_{\tau=t} = \mathbf{Q}(t)\left[\left(\frac{D^N\mathbf{J}_{\text{U}}(\tau)}{D\tau^N}\right)\right]_{\tau=\mathbf{t}}\mathbf{Q}^{\text{T}}(t).$$

(b) $\left[D\mathbf{J}_{\text{U}}(\tau)/D\tau\right]_{\tau=t} = \left[D\mathbf{F}_t^{-1}/D\tau\right]_{\tau=\mathbf{t}}\mathbf{T}(t) + \left[D\mathbf{T}/D\tau\right]_{\tau=t} + \mathbf{T}(t)\left[D\left(\mathbf{F}_t^{-1}\right)^{\text{T}}/D\tau\right]_{\tau=t}$.

Now, $\mathbf{F}_t(\tau)\mathbf{F}_t^{-1}(\tau) = \mathbf{I} \rightarrow \left[D\mathbf{F}_t/D\tau\right]\mathbf{F}_t^{-1}(\tau) + \mathbf{F}_t(\tau)\left[D\mathbf{F}_t^{-1}/D\tau\right] = 0 \rightarrow$

$\left[D\mathbf{F}_t^{-1}(\tau)/D\tau\right]_{\tau=t} = -\mathbf{F}_t^{-1}(t)\left[D\mathbf{F}_t/D\tau\right]_{\tau=\mathbf{t}}\mathbf{F}_t^{-1}(t) = -\left[D\mathbf{F}_t/D\tau\right]_{\tau=t} = -\left[(\nabla\mathbf{v})\mathbf{F}_t(\tau)\right]_{\tau=t}$

$= -(\nabla\mathbf{v})$. Thus,

$$\left[D\mathbf{J}_{\text{U}}(\tau)/D\tau\right]_{\tau=t} = \frac{D\mathbf{T}}{Dt} - \mathbf{T}(\nabla\mathbf{v})^{\text{T}} - (\nabla\mathbf{v})\mathbf{T} = \frac{D\mathbf{T}}{Dt} - \mathbf{T}(\mathbf{D}-\mathbf{W}) - (\mathbf{D}+\mathbf{W})\mathbf{T}.$$

That is, the upper convected derivative of $\mathbf{T}$ can be written:

$$\hat{\mathbf{T}} \equiv \left[\frac{D\mathbf{J}_{\text{U}}(\tau)}{D\tau}\right]_{\tau=t} = \frac{D\mathbf{T}}{Dt} + \mathbf{TW} - \mathbf{WT} - (\mathbf{TD}+\mathbf{DT}) = \overset{\circ}{\mathbf{T}} - (\mathbf{TD}+\mathbf{DT}).$$

8.27 Given the velocity field of a plane Couette flow: $v_1 = 0,\ v_2 = kx_1$. (a) For a Newtonian fluid, find the stress field $[\mathbf{T}]$ and the co-rotational stress rate $\left[\overset{\circ}{\mathbf{T}}\right]$. (b) Consider a change of frame (change of observer) described by:

$$\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix},\quad [\mathbf{Q}] = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}$$

Find $\left[\mathbf{v}^*\right],\left[\nabla^*\mathbf{v}^*\right],\left[\mathbf{D}^*\right]$ and $\left[\mathbf{W}^*\right]$.

(c) Find the co-rotational stress rate for the starred frame

(d) Verify that the two stress rates are related by the objective tensorial relation.

Ans.

(a) $[\nabla\mathbf{v}] = \begin{bmatrix} 0 & 0 \\ k & 0 \end{bmatrix},\ [\mathbf{D}] = \begin{bmatrix} 0 & k/2 \\ k/2 & 0 \end{bmatrix},\ [\mathbf{W}] = \begin{bmatrix} 0 & -k/2 \\ k/2 & 0 \end{bmatrix}$. Thus, stress tensor is

$$[\mathbf{T}] = \begin{bmatrix} -p & 0 \\ 0 & -p \end{bmatrix} + 2\mu\begin{bmatrix} 0 & k/2 \\ k/2 & 0 \end{bmatrix} = \begin{bmatrix} -p & \mu k \\ \mu k & -p \end{bmatrix}.$$

$$[\mathbf{TW}] - [\mathbf{WT}] = \begin{bmatrix} -p & \mu k \\ \mu k & -p \end{bmatrix}\begin{bmatrix} 0 & -k/2 \\ k/2 & 0 \end{bmatrix} - \begin{bmatrix} 0 & -k/2 \\ k/2 & 0 \end{bmatrix}\begin{bmatrix} -p & \mu k \\ \mu k & -p \end{bmatrix} = \begin{bmatrix} \mu k^2 & 0 \\ 0 & -\mu k^2 \end{bmatrix}$$

Co-rotational stress rate is: $\left[\overset{\circ}{\mathbf{T}}\right] = \left[\frac{D\mathbf{T}}{Dt}\right] + \begin{bmatrix} \mu k^2 & 0 \\ 0 & -\mu k^2 \end{bmatrix} = \begin{bmatrix} \mu k^2 & 0 \\ 0 & -\mu k^2 \end{bmatrix}$.

(b) From Eq. (5.56.12) of Chapter 5, we have,

$\left[v^*(\mathbf{x^*})\right] = [\mathbf{Qv}] + \left[(d\mathbf{Q}/dt)\mathbf{x}\right] = [\mathbf{Qv}] + \left[(d\mathbf{Q}/dt)\mathbf{Q}^{\text{T}}\mathbf{x^*}\right]$. Thus,

$$\begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} = \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix}\begin{bmatrix} 0 \\ v_2 \end{bmatrix} + \omega\begin{bmatrix} -\sin\omega t & -\cos\omega t \\ \cos\omega t & -\sin\omega t \end{bmatrix}\begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix}\begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}.$$ Since,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} \to x_1 = \cos\omega t x_1^* + \sin\omega t x_2^* \to v_2 = k\left(\cos\omega t x_1^* + \sin\omega t x_2^*\right)$$

Therefore,

$$\begin{bmatrix} v_1^* \\ v_2^* \end{bmatrix} = \begin{bmatrix} -v_2 \sin\omega t \\ v_2 \cos\omega t \end{bmatrix} + \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = k \begin{bmatrix} -\left(\cos\omega t \sin\omega t x_1^* + \sin^2\omega t x_2^*\right) \\ \left(\cos^2\omega t x_1^* + \sin\omega t \cos\omega t x_2^*\right) \end{bmatrix} + \omega \begin{bmatrix} -x_2^* \\ x_1^* \end{bmatrix}$$

from which, we get,

$$\left[\left(\nabla * \mathbf{v}*\right)\right] = k \begin{bmatrix} -\left(\sin 2\omega t\right)/2 & -\sin^2\omega t \\ \cos^2\omega t & \left(\sin 2\omega t\right)/2 \end{bmatrix} + \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

$$[\mathbf{D}*] = k \begin{bmatrix} -\left(\sin 2\omega t\right)/2 & \left(\cos 2\omega t\right)/2 \\ \left(\cos 2\omega t\right)/2 & \left(\sin 2\omega t\right)/2 \end{bmatrix}, \quad [\mathbf{W}*] = k \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix} + \omega \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

(c) For the Newtonian fluid, the stress field in the starred-frame is:

$$[\mathbf{T}^*] = \begin{bmatrix} -p - \mu k\left(\sin 2\omega t\right) & \mu k \cos 2\omega t \\ \mu k\left(\cos 2\omega t\right) & -p + \mu k\left(\sin 2\omega t\right) \end{bmatrix},$$

where the indeterminate pressure p is time independent. Thus,

$$\left[\frac{D\mathbf{T}*}{Dt}\right] = 2\mu\omega k \begin{bmatrix} -\left(\cos 2\omega t\right) & -\left(\sin 2\omega t\right) \\ -\left(\sin 2\omega t\right) & \left(\cos 2\omega t\right) \end{bmatrix}, \text{ and}$$

$$\left[\mathbf{T}^*\mathbf{W}^*\right] = \frac{k}{2} \begin{bmatrix} \mu k \cos 2\omega t & p + \mu k \sin 2\omega t \\ -p + \mu k\left(\sin 2\omega t\right) & -\mu k \cos 2\omega t \end{bmatrix} + \omega \begin{bmatrix} \mu k \cos 2\omega t & p + \mu k \sin 2\omega t \\ -p + \mu k \sin 2\omega t & -\mu k \cos 2\omega t \end{bmatrix}$$

$$\left[\mathbf{W}^*\mathbf{T}^*\right] = \frac{k}{2} \begin{bmatrix} -\mu k \cos 2\omega t & p - \mu k \sin 2\omega t \\ -p - \mu k \sin 2\omega t & \mu k \cos 2\omega t \end{bmatrix} + \omega \begin{bmatrix} -\mu k \cos 2\omega t & p - \mu k \sin 2\omega t \\ -p - \mu k \sin 2\omega t & \mu k \cos 2\omega t \end{bmatrix}$$

Thus,

$$\left[\mathbf{T}^*\mathbf{W}^*\right] - \left[\mathbf{W}^*\mathbf{T}^*\right] = k \begin{bmatrix} \mu k \cos 2\omega t & \mu k \sin 2\omega t \\ \mu k \sin 2\omega t & -\mu k \cos 2\omega t \end{bmatrix} + 2\mu\omega k \begin{bmatrix} \cos 2\omega t & \sin 2\omega t \\ \sin 2\omega t & -\cos 2\omega t \end{bmatrix}.$$

Thus, $\overset{o}{\mathbf{T}}{}^* = \left[D\mathbf{T}^*/Dt\right] + \left[\mathbf{T}^*\mathbf{W}^* - \mathbf{W}^*\mathbf{T}^*\right] = \mu k^2 \begin{bmatrix} \cos 2\omega t & \sin 2\omega t \\ \sin 2\omega t & -\cos 2\omega t \end{bmatrix}$

(d) $[\mathbf{Q}]\left[\overset{o}{\mathbf{T}}\right][\mathbf{Q}]^T$

$$= \begin{bmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{bmatrix} \begin{bmatrix} \mu k^2 & 0 \\ 0 & -\mu k^2 \end{bmatrix} \begin{bmatrix} \cos\omega t & \sin\omega t \\ -\sin\omega t & \cos\omega t \end{bmatrix} = \mu k^2 \begin{bmatrix} \cos 2\omega t & \sin 2\omega t \\ \sin 2\omega t & -\cos 2\omega t \end{bmatrix}$$

Thus, we have

$$\left[\overset{o}{\mathbf{T}}{}^*\right] = [\mathbf{Q}]\left[\overset{o}{\mathbf{T}}\right][\mathbf{Q}]^T.$$

8.28 Given the velocity field: $v_1 = -kx_1,\ v_2 = kx_2,\ v_3 = 0$. Obtain (a) the stress field for a second-order fluid (b) the co-rotational derivative of the stress tensor

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Ans. (a) $[\nabla \mathbf{v}] = \begin{bmatrix} -k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix} = [\mathbf{D}], \quad [\mathbf{W}] = [0]$,

$$[\mathbf{A}_1] = [2\mathbf{D}] = \begin{bmatrix} -2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix} \to [\mathbf{A}_1]^2 = \begin{bmatrix} -2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4k^2 & 0 & 0 \\ 0 & 4k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$[\mathbf{A}_2] = \left[D\mathbf{A}_1 / Dt + \mathbf{A}_1(\nabla \mathbf{v}) + (\nabla \mathbf{v})^{\mathrm{T}} \mathbf{A}_1 \right] = \left[\mathbf{A}_1(\nabla \mathbf{v}) + (\nabla \mathbf{v})^{\mathrm{T}} \mathbf{A}_1 \right]$$

$$= \begin{bmatrix} -2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4k^2 & 0 & 0 \\ 0 & 4k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The second-order fluid is defined by Eq.(8.18.6):

$$\mathbf{T} = -p\mathbf{I} + \mu_1 \mathbf{A}_1 + \mu_2 \mathbf{A}_1^2 + \mu_3 \mathbf{A}_2 \to$$

$$[\mathbf{T}] = -p[\mathbf{I}] + \mu_1 \begin{bmatrix} -2k & 0 & 0 \\ 0 & 2k & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu_2 \begin{bmatrix} 4k^2 & 0 & 0 \\ 0 & 4k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \mu_3 \begin{bmatrix} 4k^2 & 0 & 0 \\ 0 & 4k^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$T_{11} = -p - 2\mu_1 k + 4(\mu_2 + \mu_3)k^2, \quad T_{22} = -p + 2\mu_1 k + 4(\mu_2 + \mu_3)k^2, \quad T_{33} = -p\,.$$

To obtain the pressure p, we first calculate the acceleration:

$$[\mathbf{a}] = [\partial \mathbf{v} / \partial t] + [\nabla \mathbf{v}][\mathbf{v}] = \begin{bmatrix} -k & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -kx_1 \\ kx_2 \\ 0 \end{bmatrix} = \begin{bmatrix} k^2 x_1 \\ k^2 x_2 \\ 0 \end{bmatrix}$$

Equations of motion $\dfrac{\partial Tij}{\partial x_j} = \rho a_i$ then give

$$-\frac{\partial p}{\partial x_1} = \rho k^2 x_1, \quad -\frac{\partial p}{\partial x_2} = \rho k^2 x_2, \quad -\frac{\partial p}{\partial x_2} = 0, \text{ thus,}$$

$$p = -\rho k^2 (x_1^2 + x_2^2)/2 + \mathrm{C}\,.$$

(b) The co-rotational derivative of $\mathbf{T}$: $\mathring{\mathbf{T}} = D\mathbf{T}/Dt + \mathbf{TW} - \mathbf{WT}$. Since $\mathbf{W} = \mathbf{0}$,

$$\left[\left(\mathring{\mathbf{T}} \right)_{ij} \right] = \left[\left(\frac{D\mathbf{T}}{Dt} \right)_{ij} \right] = \left[\frac{\partial T_{ij}}{\partial x_1} v_1 + \frac{\partial T_{ij}}{\partial x_2} v_2 \right] = \left(-\frac{\partial p}{\partial x_1} v_1 - \frac{\partial p}{\partial x_2} v_2 \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= k\left(\frac{\partial p}{\partial x_1} x_1 - \frac{\partial p}{\partial x_2} x_2 \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \rho k\left(-k^2 x_1^2 + k^2 x_2^2 \right) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \rho k\left(v_2^2 - v_1^2 \right)[\mathbf{I}]$$

8.29 Show that the Lower Convected derivative of $\mathbf{A}_1$ is $\mathbf{A}_2$, i.e., $\breve{\mathbf{A}}_1 = \mathbf{A}_2$.

Ans. From Eq.(8.19.22),

$$\breve{\mathbf{A}}_1 = \mathring{\mathbf{A}}_1 + \mathbf{A}_1\mathbf{D} + \mathbf{D}\mathbf{A}_1 = \left(D\mathbf{A}_1/Dt + \mathbf{A}_1\mathbf{W} - \mathbf{W}\mathbf{A}_1\right) + \mathbf{A}_1\mathbf{D} + \mathbf{D}\mathbf{A}_1$$

$$= D\mathbf{A}_1/Dt + \mathbf{A}_1\left(\mathbf{W}+\mathbf{D}\right) + \left(\mathbf{D}-\mathbf{W}\right)\mathbf{A}_1 = D\mathbf{A}_1/Dt + \mathbf{A}_1\left(\nabla\mathbf{v}\right) + \left(\nabla\mathbf{v}\right)^{\mathrm{T}}\mathbf{A}_1 = \mathbf{A}_2.$$

8.30 The Reiner-Rivlin fluid is defined by the constitutive equation:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} = \phi_1\left(I_2, I_3\right)\mathbf{D} + \phi_2\left(I_2, I_3\right)\mathbf{D}^2$$

where I_i are the scalar invariants of $\mathbf{D}$. Obtain the stress components for this fluid in a simple shearing flow.

Ans. In a simple shearing flow, $v_1 = kx_2, \; v_2 = v_3 = 0$,

$$[\mathbf{D}] = \begin{bmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \left[\mathbf{D}^2\right] = \begin{bmatrix} k^2/4 & 0 & 0 \\ 0 & k^2/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad I_2 = -\frac{k^2}{4}, \quad I_3 = 0$$

$$[\mathbf{T}] = -p[\mathbf{I}] + \phi_1\left(k^2/4, 0\right)\begin{bmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \phi_2\left(k^2/4, 0\right)\begin{bmatrix} k^2/4 & 0 & 0 \\ 0 & k^2/4 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

8.31 The exponential of a tensor $\mathbf{A}$ is defined as: $\exp[\mathbf{A}] = \mathbf{I} + \sum_{1}^{N} \frac{1}{n!}\mathbf{A}^n$. If $\mathbf{A}$ is an objective tensor, is $\exp[\mathbf{A}]$ also objective?

Ans. Yes. Because

$$\mathbf{A}^* = \mathbf{Q}(t)\mathbf{A}\mathbf{Q}^{\mathrm{T}}(t) \rightarrow \left(\mathbf{A}^*\right)^2 = \mathbf{Q}(t)\mathbf{A}\mathbf{Q}^{\mathrm{T}}(t)\mathbf{Q}(t)\mathbf{A}\mathbf{Q}^{\mathrm{T}}(t) = \mathbf{Q}(t)\mathbf{A}^2\mathbf{Q}^{\mathrm{T}}(t)$$

$$\rightarrow \left(\mathbf{A}^*\right)^N = \mathbf{Q}(t)\mathbf{A}^N\mathbf{Q}^{\mathrm{T}}(t)$$

That is, $\left(\mathbf{A}^*\right)^N$ is objective for all N. As a consequence, $\exp[\mathbf{A}]$ is objective.

8.32 Why is it that the following constitutive equation is not acceptable:
$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \; \mathbf{S} = \alpha\left(\nabla\mathbf{v}\right)$, where $\mathbf{v}$ is velocity and α is a constant

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Ans. Because $\nabla\mathbf{v}$ is not objective.

8.33 Let $d\boldsymbol{a}$ and $d\mathbf{A}$ denote the differential area vectors at time τ and time t respectively. For an incompressible fluid, show that

$$\left[D^N da^2 / D\tau^N\right]_{\tau=t} = d\mathbf{A}\cdot\left[D^N \mathbf{C}_t^{-1} / D\tau^N\right]_{\tau=t} d\mathbf{A} \equiv -d\mathbf{A}\cdot\mathbf{M}_N d\mathbf{A}$$

where da is the magnitude of $d\boldsymbol{a}$ and the tensors $\mathbf{M}_N$ are known as the White-Metzner tensors.

Ans. From Eq. (3.27.12), we have, [note here $d\mathbf{A}$ is the reference area and $d\boldsymbol{a}$ is the area at the running time τ], $d\boldsymbol{a} = (\det\mathbf{F})\left(\mathbf{F}^{-1}\right)^{\mathrm{T}} d\mathbf{A}$. For an incompressible fluid, $(\det\mathbf{F}) = 1$, so that

$$d\boldsymbol{a} = \left(\mathbf{F}^{-1}\right)^{\mathrm{T}} d\mathbf{A},\quad d\boldsymbol{a}\cdot d\boldsymbol{a} = \left(\mathbf{F}^{-1}\right)^{\mathrm{T}} d\mathbf{A}\cdot\left(\mathbf{F}^{-1}\right)^{\mathrm{T}} d\mathbf{A} = d\mathbf{A}\cdot\left(\mathbf{F}^{-1}\right)\left(\mathbf{F}^{-1}\right)^{\mathrm{T}} d\mathbf{A} = d\mathbf{A}\cdot\left(\mathbf{F}^{\mathrm{T}}\mathbf{F}\right)^{-1} d\mathbf{A}.$$

That is, $da^2 = d\mathbf{A}\cdot\mathbf{C}_t^{-1} d\mathbf{A}$. Thus,

$$\left[\frac{D^N da^2}{D\tau^N}\right]_{\tau=t} = d\mathbf{A}\cdot\left[\frac{D^N \mathbf{C}_t^{-1}}{D\tau^N}\right]_{\tau=t} d\mathbf{A} \equiv -d\mathbf{A}\cdot\mathbf{M}_N d\mathbf{A},\ \text{where } \mathbf{M}_N = -\left[\frac{D^N \mathbf{C}_t^{-1}}{D\tau^N}\right]_{\tau=t}.$$

8.34 (a) Verify that Oldroyd's lower convected derivatives of the identity tensor $\mathbf{I}$ are the Rivlin-Ericksen tensor $\mathbf{A}_N$. (b) Verify that Oldroyd upper derivatives of the identity tensor are the negative White-Metzner tensors [see Prob. 8.33 for the definition of White-Metzner tensor].

Ans. (a) The Nth lower convected derivative of $\mathbf{T}$ is given by

$\left[D^N \mathbf{J}_L / D\tau^N\right]_{\tau=t}$, where $\mathbf{J}_L(\tau) = \mathbf{F}_t^{\mathrm{T}}(\tau)\mathbf{T}(\tau)\mathbf{F}_t(\tau)$. For $\mathbf{T} = \mathbf{I}$,

$$\mathbf{J}_L(\tau) = \mathbf{F}_t^{\mathrm{T}}(\tau)\mathbf{F}_t(\tau) = \mathbf{C}_t(\tau).\ \text{Thus,}\ \left[\frac{D^N \mathbf{J}_L}{D\tau^N}\right]_{\tau=t} = \left[\frac{D^N \mathbf{C}_t(\tau)}{D\tau^N}\right]_{\tau=t} = \mathbf{A}_N$$

(b) The Nth upper convected derivative of $\mathbf{T}$ is given by

$\left[D^N \mathbf{J}_U / D\tau^N\right]_{\tau=t}$, where $\mathbf{J}_U(\tau) = \mathbf{F}_t^{-1}(\tau)\mathbf{T}(\tau)\left(\mathbf{F}_t^{-1}\right)^{\mathrm{T}}(\tau)$. For $\mathbf{T} = \mathbf{I}$,

$$\mathbf{J}_U(\tau) = \mathbf{F}_t^{-1}(\tau)\left(\mathbf{F}_t^{-1}\right)^{\mathrm{T}}(\tau) = \mathbf{C}_t^{-1}(\tau).\ \text{Thus,}\ \left[\frac{D^N \mathbf{J}_U}{D\tau^N}\right]_{\tau=t} = \left[\frac{D^N \mathbf{C}_t^{-1}(\tau)}{D\tau^N}\right]_{\tau=t} = -\mathbf{M}_N.$$

8.35 Obtain the equation $\breve{\mathbf{T}} = D\mathbf{T}/Dt + \mathbf{T}\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\mathbf{T}$, where $\breve{\mathbf{T}}$ is the lower convected derivative of T.

Ans. By definition, the lower convected derivative is $\left[D\mathbf{J}_L(\tau)/D\tau\right]_{\tau=t}$, where

$$\mathbf{J}_L(\tau) = \mathbf{F}_t^{\mathrm{T}}(\tau)\mathbf{T}(\tau)\mathbf{F}_t(\tau).\ \text{Thus,}\ \left[D\mathbf{J}_L(\tau)/D\tau\right]_{\tau=t} = \left[D\mathbf{F}_t^{\mathrm{T}}/D\tau\right]_{\tau=t}\mathbf{T}(t)\mathbf{F}_t(t)$$

$$+\mathbf{F}_t^{\mathrm{T}}(t)\left[D\mathbf{T}/D\tau\right]_{\tau=t}\mathbf{F}_t(t) + \mathbf{F}_t^{\mathrm{T}}(t)\mathbf{T}(t)\left[D\mathbf{F}^{\mathrm{T}}/D\tau\right]_{\tau=t}.$$

Now, $\left[D\mathbf{F}_t^{\mathrm{T}}(\tau)/D\tau\right]_{\tau=t} = D\mathbf{F}_t^{\mathrm{T}}/Dt = \left(D\mathbf{F}_t/Dt\right)^{\mathrm{T}} = (\nabla\mathbf{v})^{\mathrm{T}}$ [see Eq.(8.12.3)] and $\mathbf{F}_t(t) = \mathbf{I}$, therefore,

$$\breve{\mathbf{T}} = \left[\frac{D\mathbf{J}_L(\tau)}{D\tau}\right]_{\tau=t} = \frac{D\mathbf{T}}{Dt} + \mathbf{T}\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\,\mathbf{T}.$$

8.36 Consider the following constitutive equation:
$\mathbf{S} + \lambda\left(D_*\mathbf{S}/Dt\right) = 2\mu\mathbf{D}$, where $\left(D_*\mathbf{S}/Dt\right) \equiv \overset{o}{\mathbf{S}} + \alpha\left(\mathbf{DS}+\mathbf{SD}\right)$ and $\overset{o}{\mathbf{S}}$ is co-rotational derivative of $\mathbf{S}$. Obtain the shear stress function and the two normal stress functions for this fluid.

Ans. With $v_1 = kx_2,\quad v_2 = v_3 = 0$, the rate of deformation tensor and spin tensor are:

$$[\mathbf{D}] = \begin{bmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\quad [\mathbf{W}] = \begin{bmatrix} 0 & k/2 & 0 \\ -k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ Since the flow is steady, $\frac{\partial\mathbf{S}}{\partial t} = 0$.

The co-rotational derivative is, for symmetric $\mathbf{S}$: $\overset{o}{\mathbf{S}} = \mathbf{SW} - \mathbf{WS} = \mathbf{SW} + (\mathbf{SW})^{\mathrm{T}}$. Now,

$$[\mathbf{SW}] = \frac{k}{2}\begin{bmatrix} -S_{12} & S_{11} & 0 \\ -S_{22} & S_{21} & 0 \\ -S_{32} & S_{31} & 0 \end{bmatrix},\quad [\mathbf{SW}]^{\mathrm{T}} = \frac{k}{2}\begin{bmatrix} -S_{12} & -S_{22} & -S_{32} \\ S_{11} & S_{21} & S_{31} \\ 0 & 0 & 0 \end{bmatrix},$$

$$\left[\overset{o}{\mathbf{S}}\right] = \frac{k}{2}\begin{bmatrix} -2S_{12} & S_{11}-S_{22} & -S_{32} \\ S_{11}-S_{22} & 2S_{21} & S_{31} \\ -S_{32} & S_{31} & 0 \end{bmatrix},\quad [\mathbf{SD}]+[\mathbf{DS}] = \frac{k}{2}\begin{bmatrix} 2S_{12} & S_{11}+S_{22} & S_{32} \\ S_{11}+S_{22} & 2S_{21} & S_{31} \\ S_{32} & S_{31} & 0 \end{bmatrix}.$$

Thus, $D_*\mathbf{S}/Dt \equiv \overset{o}{\mathbf{S}} + \alpha\left(\mathbf{DS}+\mathbf{SD}\right)$ gives

$$\left[\frac{D_*\mathbf{S}}{Dt}\right] = \frac{k}{2}\begin{bmatrix} 2(\alpha-1)S_{12} & (1+\alpha)S_{11}+(\alpha-1)S_{22} & (\alpha-1)S_{32} \\ (1+\alpha)S_{11}+(\alpha-1)S_{22} & 2(1+\alpha)S_{21} & (1+\alpha)S_{31} \\ (\alpha-1)S_{32} & (1+\alpha)S_{31} & 0 \end{bmatrix}$$

Therefore, $\mathbf{S} + \lambda\frac{D_*\mathbf{S}}{Dt} = 2\mu\mathbf{D} \rightarrow$

$$S_{11} + \lambda k(\alpha-1)S_{12} = 0 \quad \text{(i)},\quad S_{12} + \left(\frac{\lambda k}{2}\right)\left[(1+\alpha)S_{11} + (\alpha-1)S_{22}\right] = \mu k \quad \text{(ii)},$$

$$S_{13} + \frac{\lambda k}{2}(\alpha-1)S_{23} = 0 \quad \text{(iii)},\quad S_{22} + \lambda k(1+\alpha)S_{12} = 0 \quad \text{(iv)},$$

$$S_{23} + \frac{\lambda k}{2}(1+\alpha)S_{13} = 0 \quad \text{(v)},\qquad S_{33}=0 \quad \text{(vi)}.$$

Now, (iii), (v) and (vi) give $S_{13} = S_{23} = S_{33} = 0$. Eq. (i) gives $S_{11} = \lambda k(1-\alpha)S_{12}$, Eq.(iv) gives $S_{22} = -\lambda k(1+\alpha)S_{12}$, thus, with
$A(k) \equiv 1 + \left(1-\alpha^2\right)(\lambda k)^2$, we have,

$$S_{12} = \mu k / A(k),\quad S_{11} = \lambda\mu k^2(1-\alpha)/A(k),\quad S_{22} = -\lambda\mu k^2(1+\alpha)/A(k)$$

The shear stress function is $S_{12} = \mu k / A(k)$. The normal stress functions are:

$$\sigma_1 \equiv S_{11} - S_{22} = 2\lambda\mu k^2 / A(k),\quad \sigma_2 = S_{22} - S_{33} = -\lambda\mu k^2(1+\alpha)/A(k).$$

8.37 Obtain the apparent viscosity and the normal stress functions for the Oldroyd 3-constant fluid [see (C) of Section 8.20].

Ans. For the simple shearing flow,

$$[\mathbf{D}]=\begin{bmatrix}0 & k/2 & 0\\ k/2 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad [\mathbf{W}]=\begin{bmatrix}0 & k/2 & 0\\ -k/2 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}.$$

$$\overset{o}{\mathbf{S}}=[\mathbf{0}]+[\mathbf{SW}]-[\mathbf{WS}]=(k/2)\begin{bmatrix}-2S_{12} & S_{11}-S_{22} & -S_{32}\\ S_{11}-S_{22} & 2S_{21} & S_{31}\\ -S_{32} & S_{31} & 0\end{bmatrix},$$

$$[\mathbf{SD}]+[\mathbf{DS}]=(k/2)\begin{bmatrix}2S_{12} & S_{11}+S_{22} & S_{23}\\ S_{11}+S_{22} & 2S_{12} & S_{13}\\ S_{23} & S_{13} & 0\end{bmatrix},$$

$$\hat{\mathbf{S}}=\overset{o}{\mathbf{S}}-(\mathbf{SD}+\mathbf{DS})=(k/2)\begin{bmatrix}-4S_{12} & -2S_{22} & -2S_{23}\\ -2S_{22} & 0 & 0\\ -2S_{23} & 0 & 0\end{bmatrix},$$

$$\overset{o}{\mathbf{D}}=[\mathbf{0}]+[\mathbf{DW}]-[\mathbf{WD}]=\begin{bmatrix}-k^2/2 & 0 & 0\\ 0 & k^2/2 & 0\\ 0 & 0 & 0\end{bmatrix},\quad \mathbf{DD}=\frac{k^2}{4}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\hat{\mathbf{D}}=\overset{o}{\mathbf{D}}-2\mathbf{D}^2=\begin{bmatrix}-k^2 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix},\quad 2\mu\left(\mathbf{D}+\lambda_2\hat{\mathbf{D}}\right)=\begin{bmatrix}-2\lambda_2\mu k^2 & \mu k & 0\\ \mu k & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$\mathbf{S}+\lambda_1\hat{\mathbf{S}}=\begin{bmatrix}S_{11}-2k\lambda_1S_{12} & S_{12}-k\lambda_1S_{22} & S_{13}-k\lambda_1S_{23}\\ S_{12}-k\lambda_1S_{22} & S_{22} & S_{23}\\ S_{13}-k\lambda_1S_{23} & S_{23} & S_{33}\end{bmatrix}$$

$$\mathbf{S}+\lambda_1\hat{\mathbf{S}}=2\mu\left(\mathbf{D}+\lambda_2\hat{\mathbf{D}}\right)\rightarrow$$

$$\begin{bmatrix}S_{11}-2k\lambda_1S_{12} & S_{12}-k\lambda_1S_{22} & S_{13}-k\lambda_1S_{23}\\ S_{12}-k\lambda_1S_{22} & S_{22} & S_{23}\\ S_{13}-k\lambda_1S_{23} & S_{23} & S_{33}\end{bmatrix}=\begin{bmatrix}-2\lambda_2\mu k^2 & \mu k & 0\\ \mu k & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

Thus,

$S_{22}=S_{23}=S_{33}=S_{13}=0,\ \ S_{12}=\mu k,\ \ S_{11}-2k\lambda_1S_{12}=-2\lambda_2\mu k^2$, so that, we have,

$S_{12}=\mu k,\ \ S_{11}=2\mu k^2(\lambda_1-\lambda_2)$, all other $S_{ij}=0$. The apparent viscosity is

$\eta(k)=S_{12}/k=\mu,\ \ \sigma_1=T_{11}-T_{22}=2\mu k^2(\lambda_1-\lambda_2),\ \ \sigma_1=T_{22}-T_{33}=0$.

8.38 Obtain the apparent viscosity and the normal stress functions for the Oldroyd 4-constant fluid [see (D) of Section 8.20]

Ans. For the simple shearing flow

$$[\mathbf{D}] = \begin{bmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ [\mathbf{W}] = \begin{bmatrix} 0 & k/2 & 0 \\ -k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \overset{\circ}{\mathbf{S}} = \left(\frac{k}{2}\right) \begin{bmatrix} -2S_{12} & S_{11}-S_{22} & -S_{23} \\ S_{11}-S_{22} & 2S_{12} & S_{13} \\ -S_{32} & S_{31} & 0 \end{bmatrix},$$

$$\hat{\mathbf{S}} = \overset{\circ}{\mathbf{S}} - (\mathbf{SD} + \mathbf{DS}) = (k/2) \begin{bmatrix} -4S_{12} & -2S_{22} & -2S_{23} \\ -2S_{22} & 0 & 0 \\ -2S_{23} & 0 & 0 \end{bmatrix},$$

$$\overset{\circ}{\mathbf{D}} = [\mathbf{0}] + [\mathbf{DW}] - [\mathbf{WD}] = \begin{bmatrix} -k^2/2 & 0 & 0 \\ 0 & k^2/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \ \mathbf{DD} = \frac{k^2}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\hat{\mathbf{D}} = \overset{\circ}{\mathbf{D}} - 2\mathbf{D}^2 = \frac{k^2}{2} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} - \frac{k^2}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -k^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\mu_o (tr\mathbf{S})[\mathbf{D}] = \mu_o (S_{11} + S_{22} + S_{33}) \begin{bmatrix} 0 & k/2 & 0 \\ k/2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\mathbf{S} + \lambda_1 \hat{\mathbf{S}} + \mu_o (tr\mathbf{S}) \mathbf{D} = 2\mu \left(\mathbf{D} + \lambda_2 \hat{\mathbf{D}} \right) \rightarrow$$

$$\begin{bmatrix} S_{11} - 2k\lambda_1 S_{12} & S_{12} - k\lambda_1 S_{22} + \mu_o k (S_{11} + S_{22} + S_{33})/2 & S_{13} - k\lambda_1 S_{23} \\ S_{12} - k\lambda_1 S_{22} + \mu_o k (S_{11} + S_{22} + S_{33})/2 & S_{22} & S_{23} \\ S_{13} - k\lambda_1 S_{23} & S_{23} & S_{33} \end{bmatrix}_T$$

$$= \begin{bmatrix} -2\lambda_2 \mu k^2 & \mu k & 0 \\ \mu k & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

hus, $S_{22} = S_{23} = S_{33} = S_{13} = 0, \ S_{11} - 2k\lambda_1 S_{12} = -2\lambda_2 \mu k^2, \ S_{12} + \mu_o k S_{11}/2 = \mu k$

From which, we get, with $B(k) \equiv (1 + \lambda_1 \mu_o k^2)$,

$$S_{11} = 2\mu k^2 (\lambda_1 - \lambda_2)/B(k), \ S_{12} = \mu k \left(1 + \lambda_2 \mu_o k^2\right)/B(k).$$

Thus, the apparent viscosity is: $\eta(k) = S_{12}/k = \mu(1 + \lambda_2 \mu_o k^2)/B(k)$.

Normal stress functions are: $\sigma_1 = T_{11} - T_{22} = 2\mu k^2 (\lambda_1 - \lambda_2)/B(k), \ \sigma_2 = T_{22} - T_{33} = 0$.

8.39 Given $[\mathbf{Q}]=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}_{\{\mathbf{n}_i\}}$ and $[\mathbf{N}]=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}_{\{\mathbf{n}_i\}}$ and

$\mathbf{A}_1=k\left(\mathbf{N}+\mathbf{N}^{\mathrm{T}}\right)$ and $\mathbf{A}_2=2k^2\mathbf{N}^{\mathrm{T}}\mathbf{N}$. (a) Verify that $\mathbf{QA}_1\mathbf{Q}^{\mathrm{T}}=-\mathbf{A}_1$ and $\mathbf{QA}_2\mathbf{Q}^{\mathrm{T}}=\mathbf{A}_2$. (b) From $\mathbf{T}=-p\mathbf{I}+\mathbf{f}(\mathbf{A}_1,\mathbf{A}_2)$ and $\mathbf{Qf}(\mathbf{A}_1,\mathbf{A}_2)\mathbf{Q}^{\mathrm{T}}=\mathbf{f}\left(\mathbf{QA}_1\mathbf{Q}^{\mathrm{T}},\mathbf{QA}_2\mathbf{Q}^{\mathrm{T}}\right)$, show that $\mathbf{QT}(k)\mathbf{Q}^{\mathrm{T}}=\mathbf{T}(-k)$ and (c) From the results of part (b), show that the viscometric functions have the properties: $\mathbf{S}(k)=-\mathbf{S}(-k),\ \ \sigma_1(k)=\sigma_1(-k),\ \ \sigma_2(k)=\sigma_2(-k)$.

Ans. (a) $[\mathbf{A}_1]=k\left[\left(\mathbf{N}+\mathbf{N}^{\mathrm{T}}\right)\right]=\begin{bmatrix}0 & k & 0\\ k & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$ and $\mathbf{A}_2=2k^2\begin{bmatrix}0 & 0 & 0\\ 1 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 2k^2 & 0\\ 0 & 0 & 0\end{bmatrix}$

$$\left[\mathbf{QA}_1\mathbf{Q}^{\mathrm{T}}\right]=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & k & 0\\ k & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=-\begin{bmatrix}0 & k & 0\\ k & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=-[\mathbf{A}_1]$$

$$\left[\mathbf{QA}_2\mathbf{Q}^{\mathrm{T}}\right]=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 2k^2 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0 & 0\\ 0 & 2k^2 & 0\\ 0 & 0 & 0\end{bmatrix}=[\mathbf{A}_2]$$

(b)

$\mathbf{QT}(k)\mathbf{Q}^{\mathrm{T}}=\mathbf{Q}\left[-p\mathbf{I}+\mathbf{f}(\mathbf{A}_1,\mathbf{A}_2)\right]\mathbf{Q}^{\mathrm{T}}=-p\mathbf{I}+\mathbf{Qf}(\mathbf{A}_1,\mathbf{A}_2)\mathbf{Q}^{\mathrm{T}}$

$=-p\mathbf{I}+\mathbf{f}\left(\mathbf{QA}_1\mathbf{Q}^{\mathrm{T}},\mathbf{QA}_2\mathbf{Q}^{\mathrm{T}}\right)=-p\mathbf{I}+\mathbf{f}\left(-\mathbf{A}_1(k),\mathbf{A}_2(k)\right).$

Now, $\mathbf{A}_1(-k)=-\mathbf{A}_1(k)$ and $\mathbf{A}_2(-k)=\mathbf{A}_2(k)$, thus,.

$\mathbf{QT}(k)\mathbf{Q}^{\mathrm{T}}=-p\mathbf{I}+\mathbf{f}\left(\mathbf{A}_1(-k),\mathbf{A}_2(-k)\right)$ and $\mathbf{QT}(-k)\mathbf{Q}^{\mathrm{T}}=-p\mathbf{I}+\mathbf{f}\left(\mathbf{A}_1(k),\mathbf{A}_2(k)\right)$.

That is, $\mathbf{QT}(-k)\mathbf{Q}^{\mathrm{T}}=\mathbf{T}(k)$ and $\mathbf{QT}(k)\mathbf{Q}^{\mathrm{T}}=\mathbf{T}(-k)$

(c) $[\mathbf{Q}][\mathbf{T}(k)]\left[\mathbf{Q}^{\mathrm{T}}\right]=\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{bmatrix}\begin{bmatrix}-1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}T_{11} & -T_{12} & -T_{13}\\ -T_{21} & T_{22} & T_{23}\\ -T_{31} & T_{32} & T_{33}\end{bmatrix}$

$\mathbf{QT}(-k)\mathbf{Q}^{\mathrm{T}}=\mathbf{T}(k)\rightarrow T_{11}(-k)=T_{11}(k),\ \ T_{22}(-k)=T_{22}(k),\ \ T_{33}(-k)=T_{33}(k)$

$-T_{12}(-k)_{12}=T_{12}(k),\ \ -T_{13}(-k)=T_{13}(k),\ \ -T_{23}(-k)=T_{23}(k)$. Thus,

$\sigma_1(k)=\sigma_1(-k),\ \ \sigma_2(k)=\sigma_2(-k),\ S(k)=-S(-k)$. [Note, in viscometric flow, $T_{13}=T_{23}=0$].

8.40 For the velocity field given in example 8.21.2, i.e., $v_r=0,\ v_\theta=0,\ v_z=v(r)$, (a) obtain the stress components in terms of the shear stress function $S(k)$ and the normal stress functions $\sigma_1(k)$ and $\sigma_2(k)$, where $k=dv/dr$, (b) obtain the following velocity distribution for the

Poiseuille flow under a pressure gradient of ($-f$): $v(r)=\int_r^R \gamma(fr/2)dr$, where γ is the inverse shear stress function, and (c)obtain the relation $\gamma(Rf/2)=\left[1/(\pi R^3 f^2)\right]\partial\left(f^3 Q\right)/\partial f$.

Ans. (a) In example 8.21.2, we see that the velocity field $v_r=0,\ \ v_\theta=0,\ \ v_z=v(r)$ describes a viscometric flow with the nonzero Rivlin-Ericksen tensors given by

$$[\mathbf{A}_1]=\begin{bmatrix}0 & k(r) & 0\\ k(r) & 0 & 0\\ 0 & 0 & 0\end{bmatrix}_{\mathbf{n}_i},\quad [\mathbf{A}_2]=\begin{bmatrix}0 & 0 & 0\\ 0 & 2k^2(r) & 0\\ 0 & 0 & 0\end{bmatrix}_{\mathbf{n}_i},$$

where $\mathbf{n}_1=\mathbf{e}_z,\ \mathbf{n}_2=\mathbf{e}_r,\ \mathbf{n}_3=\mathbf{e}_\theta$ and $k(r)=dv/dr$ (see Example 8.10.2, but, note the differences in the order of bases). Thus the stress components with respect to the basis $\{\mathbf{n}_i\}$ are given by (See section 8.22):

$S_{zr}=\tau(k),\ \ S_{zz}-S_{rr}=\sigma_1(k),\ \ S_{rr}-S_{\theta\theta}=\sigma_2(k),\ \ S_{z\theta}=S_{r\theta}=0$.

(b) With S_{ij} depending only on r , the equations of motion become:

$$\frac{\partial S_{rr}}{\partial r}+\frac{S_{rr}-S_{\theta\theta}}{r}-\frac{\partial p}{\partial r}=0\ \ (i),\quad \frac{\partial p}{\partial\theta}=0\ \text{(ii)},\quad \frac{1}{r}\frac{\partial}{\partial r}(rS_{rz})-\frac{\partial p}{\partial z}=0\ \ (iii)$$

Eq. (i) gives $\frac{\partial}{\partial r}\left(\frac{\partial p}{\partial z}\right)=0$, Eq. (ii) gives $\frac{\partial}{\partial\theta}\left(\frac{\partial p}{\partial z}\right)=0$ and Eq. (iii) gives $\frac{\partial}{\partial z}\left(\frac{\partial p}{\partial z}\right)=0$

Thus, $\partial p/\partial z=$ a constant $\equiv -f$. Eq. (iii) becomes

$\frac{1}{r}\frac{\partial}{\partial r}(rS_{rz})=-f\rightarrow\frac{\partial}{\partial r}(rS_{rz})=-fr\rightarrow S_{rz}=-\frac{fr}{2}+\frac{C}{r}$. Since S_{rz} must be finite at $r=0$, thus, $C=0$ and $S_{rz}=-fr/2$. Now, $S_{rz}=\tau(k)$ where $\tau(k)$ is the shear stress function and $k=dv/dr$.

Thus, $\tau(k)=-fr/2$. Inverting this equation, we have, $k=\overset{-1}{\tau}(-fr/2)\ \equiv\gamma(-fr/2)$. Since $\tau(k)$ is an odd function of k , therefore, γ is also an odd function of k , so that

$k=-\gamma(fr/2)\rightarrow dv/dr=-\gamma(fr/2)\rightarrow dv=-\gamma(fr/2)dr$.

Thus, $v(R)-v(r)=-\int_r^R\gamma(fr/2)dr$. Since $v(R)=0$, therefore, $v(r)=\int_r^R\gamma(fr/2)dr$.

(c) The volume discharge is given by $Q=\int_0^R v(r)2\pi r dr$. Therefore,

$$Q=\pi\int_0^R v(r)dr^2=\pi\left\{\left[v(r)r^2\right]_0^R-\int_0^R r^2\frac{dv}{dr}dr\right\}=-\pi\int_0^R r^2\frac{dv}{dr}dr=\pi\int_0^R r^2\gamma(fr/2)dr$$

Thus, $Q/\pi=\int_0^R r^2\gamma(fr/2)dr$. Let $fr/2\equiv s\rightarrow dr\equiv 2ds/f$ and $r^2=4s^2/f^2$, then ,

$$Q/\pi=\int_{r=0}^R r^2\gamma(fr/2)dr=\int_{s=0}^{Rf/2}\left(8s^2/f^3\right)\gamma(s)ds\rightarrow f^3Q/\pi=\int_{s=0}^{Rf/2}8s^2\gamma(s)ds.$$

Differentiating the last equation with respect to f , we obtain

$$\frac{1}{\pi}\frac{\partial\left(f^3Q\right)}{\partial f}=\frac{\partial}{\partial f}\left(\frac{Rf}{2}\right)\left\{8\left(\frac{Rf}{2}\right)^2\gamma\left(\frac{Rf}{2}\right)\right\}=\left(\frac{R}{2}\right)8\left(\frac{Rf}{2}\right)^2\gamma\left(\frac{Rf}{2}\right)=R^3f^2\gamma\left(\frac{Rf}{2}\right).$$

Thus, $\gamma\left(\frac{Rf}{2}\right)=\frac{1}{\pi R^3 f^2}\frac{\partial\left(f^3Q\right)}{\partial f}$.