Impact Mechanics

Waves: General Considerations

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Some Remarks on the Validity of Approximate Theories

Longitudinal Waves

- For longitudinal wave propagation in a circular bar the elementary formula are valid if the ratio of the diameter of the bar to the wavelength transmitted is small.
- An approximate equation to allow for transverse radial motion of the elements is due to Love and Rayleigh.
- If the axial strain in the bar is $\partial u/\partial x$, the lateral strain is $-\nu\ \partial u/\partial x$.
- If the time rate of change of the axial strain is $\partial^2 u/\partial x\partial t$, then the time rate of change of the lateral strain is $-\nu\ \partial^2 u/\partial x\partial t$.
- Thus, at radius r, the radial speed of a particle is $\nu\ r\ \partial^2 u/\partial x\partial t$.

Longitudinal Waves

- Hence the kinetic energy in an element of radial thickness δr and length δx is

$$\frac{1}{2}(2\pi r.\,\delta r.\,\delta x)\rho.\left(\nu\, r\,\frac{\partial^2 u}{\partial x \partial t}\right)^2$$

- The total kinetic energy due to radial motion for a cylinder of radius a, is,

$$\int_0^a \frac{1}{2}\cdot 2\pi r.\,dr.\,dx.\,\rho.\,r^2\left(\nu\frac{\partial^2 u}{\partial x \partial t}\right)^2 = \frac{\pi a^4}{4}\rho\left(\nu\frac{\partial^2 u}{\partial x \partial t}\right)^2 dx$$

- The energy contained in a length of shaft δx, is,

$$\frac{1}{2}\rho\pi a^2\delta x\left(\frac{\partial u}{\partial t}\right)^2 + \rho\frac{\pi a^4}{4}\delta x\left(\nu\frac{\partial^2 u}{\partial x \partial t}\right)^2 + \frac{1}{2}E\pi a^2\left(\frac{\partial u}{\partial x}\right)^2\delta x$$

Longitudinal Waves

- The equation of motion is derived using Hamilton's principle,

$$\rho\left(\frac{\partial^2 u}{\partial t^2} - \frac{\upsilon^2 a^2}{2}\cdot\frac{\partial^4 u}{\partial x^2 \partial t^2}\right) = E\cdot\frac{\partial^2 u}{\partial x^2}$$

- An approximate solution due to Rayleigh yields,

$$\frac{c_p}{c_0} = 1 - \upsilon^2\cdot\pi^2\cdot\left(\frac{a}{\lambda}\right)^2$$

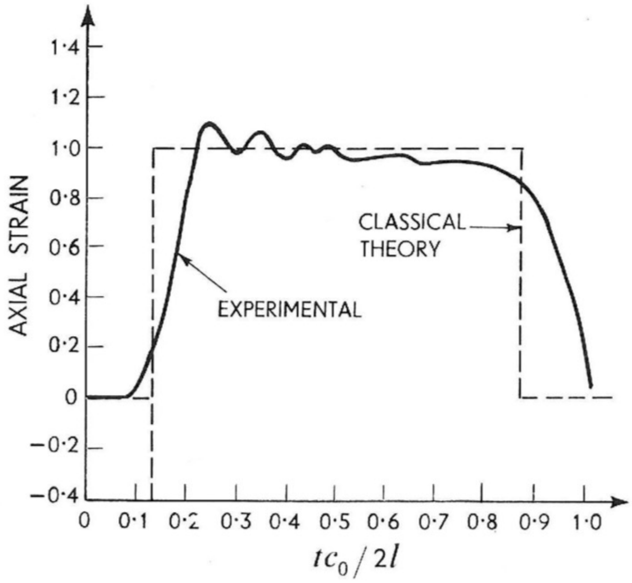
  - Where $c_0 = \sqrt{E/\rho}$ and λ is wavelength.

- According to this equation, the wave speed depends on their frequency.

- Therefore, a pulse containing a mixture of frequencies will be dispersed.

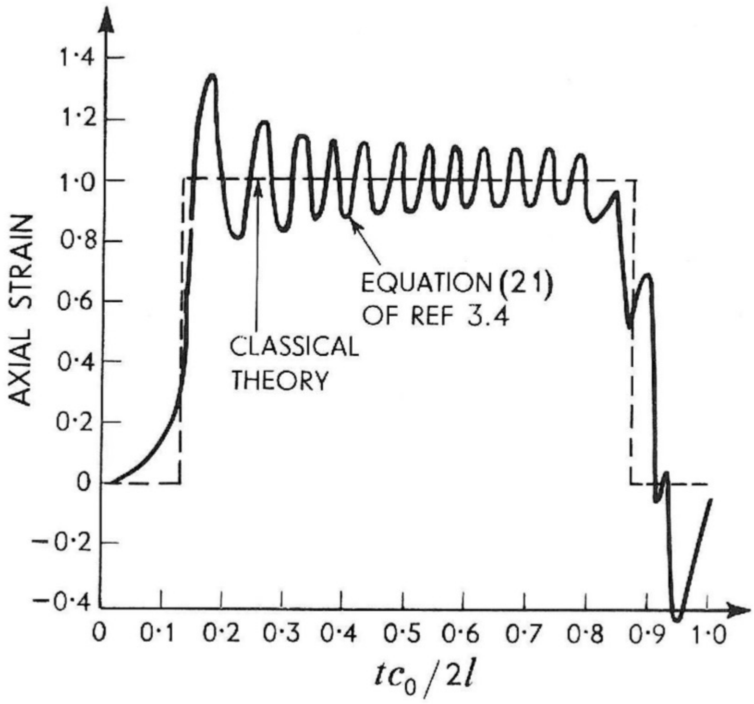
Longitudinal Waves

- This approximate solution shows that the elementary theory is reliable for waves whose length is several times greater than the bar radius: $0 \lesssim a/\lambda \lesssim 0.7$.
- Conway and Jakubowsky have given an analysis of coaxial impact of bars and compared the theoretical results with the experiment.



Longitudinal Waves

- This figure shows the theoretical result given by the authors.



Flexural Waves

- The elastic wave equation for a beam in flexure is shown to be,

$$\frac{\partial^2 w}{\partial t^2} = -\frac{EI}{\rho A} \cdot \frac{\partial^4 w}{\partial x^4}$$

- It is noted that no simple expression for the speed of propagation of flexural waves could be derived as for a longitudinal pulse.
- Considering a circular bar of radius a,

$$\frac{\partial^2 w}{\partial t^2} = -c_0^2 \cdot \frac{a^2}{4} \cdot \frac{\partial^4 w}{\partial x^4}$$

- Try as a solution,

$$w = A \sin \tfrac{2\pi}{\lambda} (x - c_p t)$$

Flexural Waves

- Where A denotes the wave amplitude, λ the wavelength and c_p the phase speed. It is found to be a solution if,

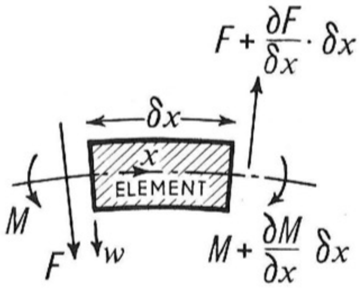
$$c_p = \frac{c_0}{(\lambda / \pi a)}$$

  - Clearly the wave speed is inversely proportional to the wavelength and infinitely short waves should travel at infinite speed; this is physically unacceptable, of course.

- This equation is only valid as long as the wavelength is much greater than any lateral dimension of the beam.
- When this is not so, it is necessary to include in the equation of motion a term allowing for the rotary motion about the neutral axis of elements of the bar.

Flexural Waves

- The equation of motion of an element when the rotary inertia is included:



$$\left(-F + \frac{\partial M}{\partial x} \cdot\right) dx = I_y \cdot dx \cdot \frac{\partial^2 \theta}{\partial t^2}$$

- Since,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right) \quad \text{and} \quad \frac{\partial^2 \theta}{\partial t^2} = \frac{\partial^3 w}{\partial t^2 \partial x}$$

- So, by differentiation,

$$-\frac{\partial F}{\partial x} = -\frac{\partial^2 M}{\partial x^2} + I_y \cdot \frac{\partial^4 w}{\partial t^2 \partial x^2}$$

- The equation of motion in lateral direction,

$$\frac{\partial F}{\partial x} = -\rho \cdot A \cdot \frac{\partial^2 w}{\partial t^2}$$

Flexural Waves

- From strength of materials, $M/EI = \partial^2 w/\partial x^2$, $I = Ak^2$. So,

$$\frac{\partial^2 M}{\partial x^2} = EAk^2 \cdot \frac{\partial^4 w}{\partial x^4}$$

- For a circular section beam, with $I_y = A\rho a^2/4$, and by substituting all these,

$$c_0^2 \cdot \frac{a^2}{4} \cdot \frac{\partial^4 w}{\partial x^4} - \frac{a^2}{4} \cdot \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\partial^2 w}{\partial t^2} = 0$$

- A solution to this is given by $w = A \sin 2\pi/\lambda\,(x - c_p t)$ with,

$$c_p = \frac{c_0}{1 + \left(\lambda^2/\pi^2 a^2\right)^{1/2}}$$

  - For very short wavelengths $a/\lambda \to \infty$, $c_P/c_0 \to 0$, not infinity; as does the previous solution.

Flexural Waves

- A feature still unaccounted for in the above analysis is the distortion of cross-sections (or elements) due to the presence of shear force.
- Inclusion of a term to accommodate this feature was first made by Timoshenko and results in 'exact' expressions for c_P/c_0 versus a/λ.
- There is very considerable discrepancy between (3.6), (3.7iv) and Timoshenko's results, when $a/\lambda > 0.1$.

Columns Under Impact

- An infinite, straight, uniform, elastic column subject to a compressive thrust P instantaneously applied and maintained constant has an equation of motion,

$$\rho A \cdot \frac{\partial^2 w}{\partial t^2} = -EI\frac{\partial^4 w}{\partial x^4} - \frac{\partial}{\partial x}\left(P\frac{\partial w}{\partial x}\right)$$

- The second term on the right hand side of the equation is the net transverse force due to P, $\partial w/\partial x$ being the local slope of the column center line.
- Thus,

$$EI\frac{\partial^4 w}{\partial x^4} + P\frac{\partial^2 w}{\partial x^2} + \rho A\frac{\partial^2 w}{\partial t^2} = 0$$

Columns Under Impact

- Small disturbances may be represented by

$$w = a \cdot \exp\left[2\pi i(\mu x - \nu t)\right]$$

  - where the wavelength $\lambda = 1/\mu$

- Substituting,

$$\nu^2 = \frac{4\pi^2 EI\mu^4 - P\mu^2}{\rho A}$$

- For sufficiently small values of λ, disturbances are propagated with speed ν/μ.
- A critical situation is reached when $\mu_c^2 = \mathrm{P}/(4\pi^2 EI)$ or $\lambda_c = 2\pi\sqrt{EI/P}$, which critical wavelength is not propagated.

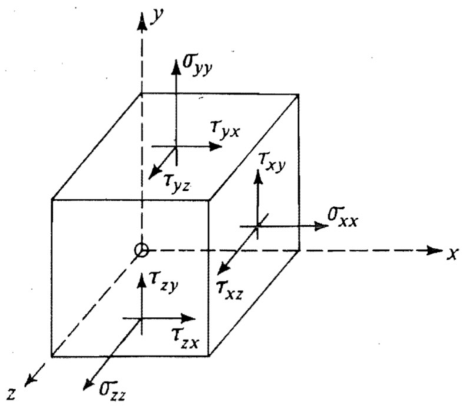
Columns Under Impact

- Perturbations where $\lambda > \lambda_c$ are not propagated and amplitude is increased without limit.
- For a solid circular shaft of diameter d, $\lambda_c = \pi d\sqrt{E/2\sigma}$.
  - This is the wave length of the most unstable wave so that at the onset of instability, waves of length λ_c are to be expected.
- The impulsive compression end-on of two identical elastic columns each moving with speed v, gives for the instability wave length $\lambda_c = \pi d\sqrt{c/2v}$.

Elastic Waves in an Isotropic Extended Medium

Body Waves

- Equations of motion:



$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + X\rho = \rho \frac{\partial^2 u}{\partial t^2}$$

$$Ee_{xx} = \sigma_{xx} - v(\sigma_{yy} + \sigma_{zz}) = (1+v)\sigma_{xx} - v(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$
$$Ee_{yy} = \sigma_{yy} - v(\sigma_{zz} + \sigma_{xx}) = (1+v)\sigma_{yy} - v(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$
$$Ee_{zz} = \sigma_{zz} - v(\sigma_{xx} + \sigma_{yy}) = (1+v)\sigma_{zz} - v(\sigma_{xx} + \sigma_{yy} + \sigma_{zz})$$

Body Waves

- Adding, $$Ee = (\sigma_{xx} + \sigma_{yy} + \sigma_{zz})(1 - 2v)$$
- Where
$$e = e_{xx} + e_{yy} + e_{zz} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$
- So,
$$\sigma_{xx} = \frac{Ee_{xx}}{1+v} + \frac{v}{1+v} \cdot \frac{E}{1-2v} \cdot e$$
$$\sigma_{xx} = 2G \cdot \frac{\partial u}{\partial x} + \lambda e$$
  - λ and μ are Lame' constants.
- Shear stresses: $$\tau_{xy} = G\gamma_{xy} = G\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$
$$\tau_{xz} = G\gamma_{xz} = G\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)$$

Body Waves

- Substituting,

$$\frac{\partial}{\partial x}\left(2G\frac{\partial u}{\partial x} + \lambda e\right) + G \cdot \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + G\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) + X\rho = \rho \cdot \frac{\partial^2 u}{\partial t^2}$$

$$2G \cdot \frac{\partial^2 u}{\partial x^2} + \lambda\frac{\partial e}{\partial x} + G\left(\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 v}{\partial y \partial x}\right) + G\left(\frac{\partial^2 w}{\partial z \partial x} + \frac{\partial^2 u}{\partial z^2}\right) + X\rho = \rho \cdot \frac{\partial^2 u}{\partial t^2}$$

$$G\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial z \partial x}\right) + \lambda\frac{\partial e}{\partial x} + G \cdot \nabla^2 u + X\rho = \rho\frac{\partial^2 u}{\partial t^2}$$

- Differentiating e with respect to x,

$$\frac{\partial e}{\partial x} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z}$$

Body Waves

- Hence,

$$(G+\lambda)\frac{\partial e}{\partial x} + G\cdot\nabla^2 u + X\rho = \rho\frac{\partial^2 u}{\partial t^2}$$

  - Similar expressions can be obtained for the y- and z-directions.

- For no body-forces and small strains:

$$(G+\lambda)\frac{\partial e}{\partial x} + G\cdot\nabla^2 u = \rho\frac{\partial^2 u}{\partial t^2}$$

Equivoluminal Waves

- Assume that this equation involves deformation in which there is no change of volume with x so that straining therefore involves only distortion and rotation.

- Thus with $\partial e/\partial x = 0$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{G}{\rho}\cdot\nabla^2 u$$

- For an arbitrary plane wave to be propagated with a speed c,

$$u = f_1(lx + my + nz - ct)$$

  - where l, m and n are the direction cosines of the normal to the plane. Then, $\frac{\partial u}{\partial t} = -cf_1'$; $\frac{\theta^2 u}{\partial t^2} = c^2 f_1''$

Equivoluminal Waves

$$\frac{\partial u}{\partial x} = l f_1' \quad \text{and} \quad \frac{\partial^2 u}{\partial x^2} = l^2 f_1''$$

$$\frac{\partial^2 u}{\partial y^2} = m^2 f_1'' \quad \text{and} \quad \frac{\partial^2 u}{\partial z^2} = n^2 \cdot f_1''$$

- Adding,

$$\nabla^2 u = f_1'' = \frac{\rho}{G} \cdot \frac{\partial^2 u}{\partial t^2} = \frac{\rho}{G} c^2 f_1''$$

- Thus $c_t = \sqrt{G/\rho}$.
- An equivoluminal body wave propagated with a speed of c_t.

Irrotational Waves

- Consider the formal modification to equation (3.15) when the straining propagated is irrotational, $\omega_x = \omega_y = \omega_z = 0$.

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0, \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = 0 \text{ and } \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

$$\begin{aligned}
\frac{\partial e}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) \\
&= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} \\
&= \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial x}\right) \\
&= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \nabla^2 u.
\end{aligned}$$

$$(\lambda + 2G) \cdot \nabla^2 u = \rho^2 \frac{\partial^2 u}{\partial t^2}$$

$$\frac{\partial^2 u}{\partial t^2} = \left(\frac{\lambda + 2G}{\rho}\right) \nabla^2 u$$

Irrotational Waves

- This is a wave equations with speed $c_d = \sqrt{(\lambda + 2G)/\rho}$.
- Thus only two types of body waves are propagated in an isotropic elastic solid and any general disturbance is evaluated by their superposition.
- In seismology, equivoluminal waves are referred to as S or shake waves, and irrotational waves as P or push waves.
- For a fluid medium $G = 0$ and thus $c_t = 0$, and $c_d = \sqrt{\lambda/\rho}$.
- Lord Kelvin in 1899 apparently first applied the terms irrotational and equivoluminal to describe these kinds of waves.
- Knowledge that two kinds of waves each moving with a different speed dates back to Poisson.

Surface Waves: Rayleigh Waves

- Whilst the waves discussed immediately above are body waves, there are well investigated surface waves; a simple example is a surface sea wave.
- It was shown by Rayleigh in 1887 that waves may propagate along the surface of an unbounded or semi-infinite solid at a speed, for $\nu = 1/4$ of $0.92c_t$, or for $\nu = 1/2$ of $0.96c_t$.
- The waves rapidly decrease, exponentially, in amplitude with depth below the surface and their speed is always slightly less than c_t.

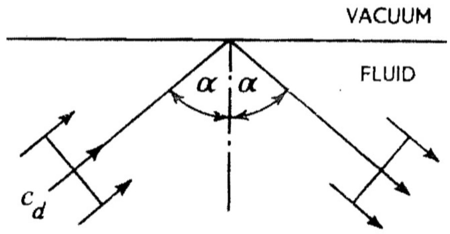
Surface Waves: Rayleigh Waves

- The amplitude of vibration increases and reaches a maximum at a depth of 0.076λ and thereafter decreases; λ is the wave length.
- At a depth of λ the amplitude is 0.19 of that at the surface.
- Rayleigh waves of high frequency are attenuated more rapidly with depth than low frequency waves—a kind of skin effect.
- The normal stress (perpendicular to the surface) is greatest at a depth 0.32λ. (for $\nu = 0.29$) whilst the normal stress parallel to the surface changes sign at a depth z of about 0.252λ.

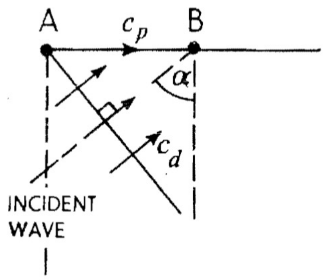
Reflection and, Refraction of Waves at an Interface

Fluid-Vacuum Interfaces

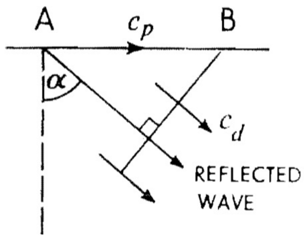
- The stress boundary or interfacial condition to be fulfilled in this circumstance is obviously that of zero normal stress at all times.



- The incident wave initiates the reflected wave and since it also has a speed of c_d, this implies that the angle of reflection must be the same as that of incidence.



- The amplitude or intensity of the incident wave is reversed in sign on reflection

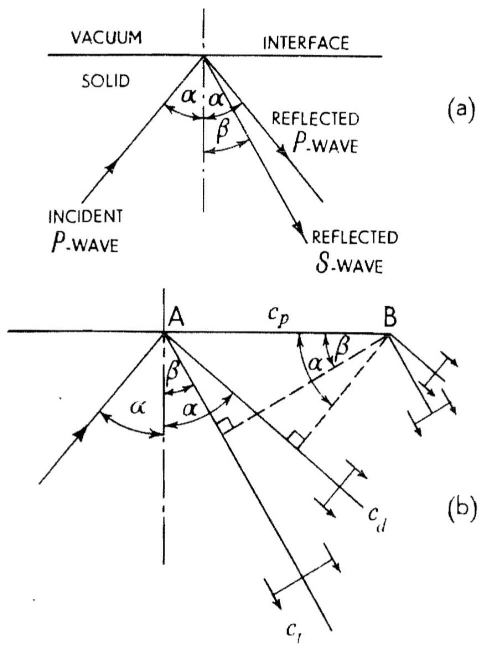


Fluid-Vacuum Interfaces

- Together the incident and reflected plane waves give the propagated wavefront and an 'interference' or 'standing' wave pattern is found for the direction normal to the plane of periodicity $\omega/c_d \sec\alpha$.
- Phase velocity: The phase speed at a boundary is the rate at which a point of constant phase travels along the interface.
  - an analogy is that of sea waves striking a straight wall at an angle.
- The rate at which the point of intersection of, say, a crest of a wave and the wall, moves along the wall is the boundary phase velocity, ${}_Bc_p = c_d/\sin\alpha$.

Solid-Vacuum Interfaces

- A plane P-wave incident on a plane boundary produces both reflected plane P-waves and plane S-waves.
- The angle of reflection of the reflected P-wave must equal that of the angle of incidence of the incident P-wave.
- And because both of the reflected waves, i.e. S and P, are initiated concurrently from the surface,



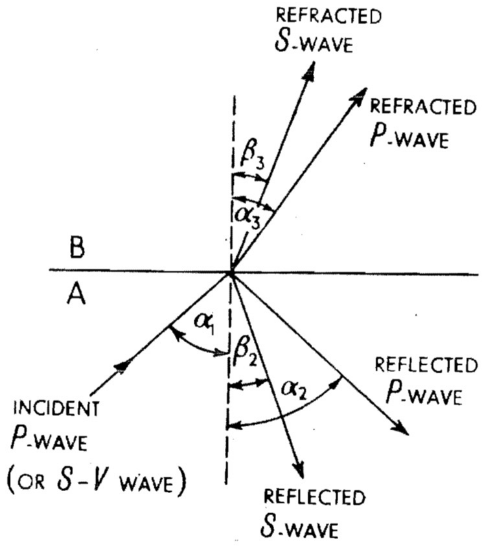
$$_B c_p = \frac{c_d}{\sin\alpha} = \frac{c_t}{\sin\beta}$$

Solid-Vacuum Interfaces

- In general the energy of the incident wave is distributed between the reflected S- and P-waves.
- Certainly if $\alpha = 0°$ there can obviously be no reflected S-wave.

Interfaces between two semi-infinite solids

- We suppose A and B to be 'welded' together at the interface so that on either side of the boundary the following four conditions must be satisfied:
  - (i) equality of normal displacements,
  - (ii) identical tangential displacements,
  - (iii) equality of (or continuity in) normal stresses, and
  - (iv) equality of (or continuity in) shear or tangential stresses.



Incident P-Waves

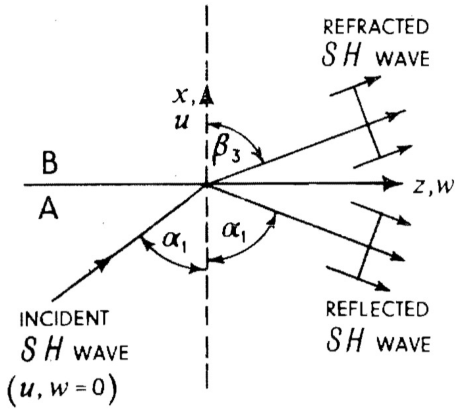
- If the angle of incidence is α_1, incident dilatational waves give rise to reflected and refracted waves in both A and B,

$$\frac{\sin\alpha_1}{c_{d_A}} = \frac{\sin\alpha_2}{c_{d_A}} = \frac{\sin\beta_2}{c_{t_A}} = \frac{\sin\alpha_3}{c_{d_B}} = \frac{\sin\beta_3}{c_{t_B}}$$

- Again the relations are clearly similar to those prevailing for geometrical optics.
- Expressions can be found for the amplitudes of the reflected and refracted waves in terms of the incident dilatation wave amplitude.

Incident S-Wave

- The behavior of plane S-waves at a boundary can be resolved in terms of S-waves which are horizontally/vertically polarized with respect to the plane boundary.
  - The former are referred to as SH-, and the latter as SV-waves.
- For an SH-wave, particle motion occurs parallel to Oy only; i.e. u=0 and w=0, and gives rise only to two S-waves, in particular,



$$\frac{\sin \alpha_1}{c_{t_A}} = \frac{\sin \beta_3}{c_{t_B}}$$

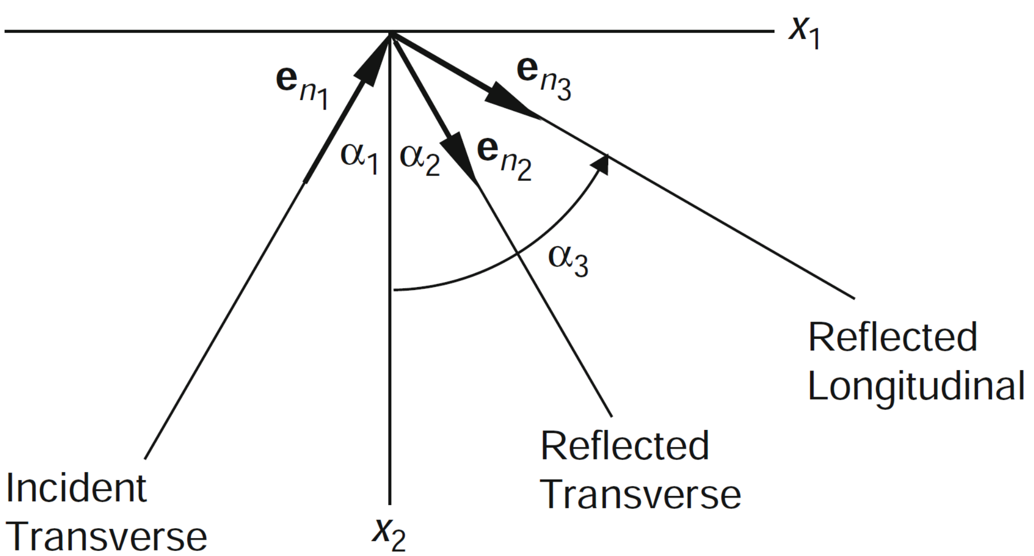
Incident S-Wave

- Since there is only motion parallel to Oy, there is thus no particle motion normal to the interface and hence no P-waves can result.
- Alternatively, in SV-waves, particle movement occurs parallel to the Oxz plane only, i.e. $v = 0$.
- The same relation between the various angles and speeds applies as before.
- Other cases such as behavior at a solid-liquid interface can easily be deduced as special cases of the solid-solid interface instance discussed above.

Wave Reflection: Exact Solution

Exact Solution

- We wish to study how an incident plane wave is reflected by the boundary.



Exact Solution

- Since $x_2=0$ is a free boundary, the surface traction on the plane is zero at all times.
- Thus, the boundary will generate reflection waves in such a way that when they are superposed on the incident wave, the stress vector on the boundary vanishes at all times.
- The reason for superposing not only a reflected transverse wave but also a longitudinal one is that if only one is superposed, the stress-free condition on the boundary in general cannot be met.

Exact Solution

- Let u_i denote the displacement components of the superposition of the three waves;

$$
\begin{aligned}
u_1 &= (\cos\alpha_1)\varepsilon_1 \sin\varphi_1 + (\cos\alpha_2)\varepsilon_2 \sin\varphi_2 + (\sin\alpha_3)\varepsilon_3 \sin\varphi_3,\\
u_2 &= (\sin\alpha_1)\varepsilon_1 \sin\varphi_1 - (\sin\alpha_2)\varepsilon_2 \sin\varphi_2 + (\cos\alpha_3)\varepsilon_3 \sin\varphi_3,\\
u_3 &= 0,
\end{aligned}
$$

$$
\begin{aligned}
\varphi_1 &= \frac{2\pi}{\ell_1}(x_1 \sin\alpha_1 - x_2 \cos\alpha_1 - c_T t - \eta_1),\\
\varphi_2 &= \frac{2\pi}{\ell_2}(x_1 \sin\alpha_2 + x_2 \cos\alpha_2 - c_T t - \eta_2),\\
\varphi_3 &= \frac{2\pi}{\ell_3}(x_1 \sin\alpha_3 + x_2 \cos\alpha_3 - c_L t - \eta_3).
\end{aligned}
$$

Exact Solution

- On the free boundary, $\boldsymbol{n} = -\boldsymbol{e}_2$, the condition $\boldsymbol{t} = \boldsymbol{0}$ yields,

$$T_{12} = T_{22} = T_{32} = 0.$$

- Using Hooke's law and noting that $u_3 = 0$ and u_2 does not depend on x_3, we easily see that the condition $T_{32} = 0$ is automatically satisfied. For the other two,

$$T_{12}/\mu = \partial u_1/\partial x_2 + \partial u_2/\partial x_1 = 0 \quad \text{on} \quad x_2 = 0,$$

$$T_{22} = (\lambda + 2\mu)(\partial u_2/\partial x_2) + \lambda \partial u_1/\partial x_1 = 0 \quad \text{on} \quad x_2 = 0.$$

$$\frac{T_{12}}{2\pi\mu} = \frac{\varepsilon_1}{\ell_1}\cos\varphi_1(\sin^2\alpha_1 - \cos^2\alpha_1) + \frac{\varepsilon_2}{\ell_2}\cos\varphi_2(\cos^2\alpha_2 - \sin^2\alpha_2) + \frac{\varepsilon_3}{\ell_3}\cos\varphi_3 \sin 2\alpha_3 = 0,$$

$$\frac{T_{22}}{2\pi} = -\frac{\varepsilon_1}{\ell_1}\mu \sin 2\alpha_1 \cos\varphi_1 - \frac{\varepsilon_2}{\ell_2}\mu \sin 2\alpha_2 \cos\varphi_2 + \frac{\varepsilon_3}{\ell_3}(\lambda + 2\mu\cos^2\alpha_3)\cos\varphi_3 = 0.$$

Exact Solution

- These equations are to be valid at $x_2 = 0$ for whatever values of x_1 and t; therefore, we must have,

$$\cos\varphi_1 = \cos\varphi_2 = \cos\varphi_3 \quad \text{at } x_2 = 0.$$

- Thus,

$$\varphi_1 = \varphi_2 \pm 2p\pi = \varphi_3 \pm 2q\pi, \quad p \text{ and } q \text{ are integers.}$$

$$\frac{2\pi}{\ell_1}(x_1 \sin\alpha_1 - c_T t - \eta_1) = \frac{2\pi}{\ell_2}(x_1 \sin\alpha_2 - c_T t - \eta_2') = \frac{2\pi}{\ell_3}(x_1 \sin\alpha_3 - c_L t - \eta_3'),$$

where $\eta_2' = \eta_2 - (\pm p\ell_2)$ and $\eta_3' = \eta_3 - (\pm q\ell_3)$.

- This equation can be satisfied for whatever values of x_1 and t only if,

$$\frac{\sin\alpha_1}{\ell_1} = \frac{\sin\alpha_2}{\ell_2} = \frac{\sin\alpha_3}{\ell_3}, \qquad \frac{c_T}{\ell_1} = \frac{c_T}{\ell_2} = \frac{c_L}{\ell_3}, \qquad \frac{\eta_1}{\ell_1} = \frac{\eta_2'}{\ell_2} = \frac{\eta_3'}{\ell_3}.$$

Exact Solution

- Thus, with $\dfrac{1}{n} \equiv \dfrac{c_L}{c_T} = \left(\dfrac{\lambda + 2\mu}{\mu}\right)^{1/2}$, we have

$$\ell_2 = \ell_1, \quad n\ell_3 = \ell_1,$$

$$\alpha_1 = \alpha_2, \quad n \sin \alpha_3 = \sin \alpha_1,$$

$$\eta_2' = \eta_1, \quad n\eta_3' = \eta_1.$$

- The reflected transverse wave has the same wavelength as that of the incident transverse wave and the angle of reflection is the same as the incident angle,
- The longitudinal wave has a different wave length and a different reflection angle depending on the so-called refraction index n.

Exact Solution

- It can be shown that $\dfrac{1}{n} \equiv \dfrac{c_L}{c_T} = \left(\dfrac{\lambda + 2\mu}{\mu}\right)^{1/2} = \left[\dfrac{2(1-\nu)}{1-2\nu}\right]^{1/2}$
- So the boundary conditions become

$$\varepsilon_1(-\cos 2\alpha_1) + \varepsilon_2(\cos 2\alpha_1) + \varepsilon_3 n(\sin 2\alpha_3) = 0,$$

$$\varepsilon_1 \sin 2\alpha_1 + \varepsilon_2 \sin 2\alpha_1 - \varepsilon_3 \frac{1}{n} \cos 2\alpha_1 = 0.$$

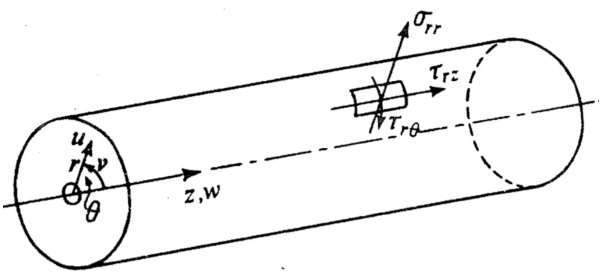
- These two equations uniquely determine the amplitudes of the reflected waves in terms of the incident amplitude

$$\varepsilon_2 = \frac{\cos^2 2\alpha_1 - n^2 \sin 2\alpha_1 \sin 2\alpha_3}{\cos^2 2\alpha_1 + n^2 \sin 2\alpha_1 \sin 2\alpha_3} \varepsilon_1,$$

$$\varepsilon_3 = \frac{n \sin 4\alpha_1}{\cos^2 2\alpha_1 + n^2 \sin 2\alpha_1 \sin 2\alpha_3} \varepsilon_1$$

Vibrations in Cylindrical Bars

Vibrations in Cylindrical Bars



- Equations of motion in cylindrical coordinates:

$$\rho \frac{\partial^2 u}{\partial t^2} = (\lambda + 2G)\frac{\partial e}{\partial r} - \frac{2G}{r}\frac{\partial \omega_z}{\partial \theta} + 2G \cdot \frac{\partial \omega_\theta}{\partial z}$$

$$\rho \frac{\partial^2 v}{\partial t^2} = (\lambda + 2G)\frac{1}{r} \cdot \frac{\partial e}{\partial \theta} - 2G\frac{\partial \omega_r}{\partial z} + 2G\frac{\partial \omega_z}{\partial r}$$

$$\rho \frac{\partial^2 w}{\partial t^2} = (\lambda + 2G) \cdot \frac{\partial e}{\partial z} - 2G\frac{\partial}{\partial r}(r\omega_\theta) + \frac{2G}{r}\frac{\partial \omega_r}{\partial \theta}$$

Vibrations in Cylindrical Bars

- The dilatation e in the cylindrical coordinates:

$$e = \frac{1}{r} \cdot \frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z}$$

- And ω_r, ω_θ and ω_z are rotation components:

$$\omega_r = \frac{1}{2}\left(\frac{1}{r} \cdot \frac{\partial w}{\partial \theta} - \frac{\partial v}{\partial z}\right)$$
$$\omega_\theta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial r}\right)$$
$$\omega_z = \frac{1}{2r}\left(\frac{\partial (rv)}{\partial r} - \frac{\partial u}{\partial \theta}\right)$$

Vibrations in Cylindrical Bars

- On the surface of the bar:

$$\sigma_{rr} = \tau_{r\theta} = \tau_{rz} = 0$$

- Using Hooke's law,

$$\sigma_{rr} = \lambda e + 2G \cdot \frac{\partial u}{\partial r}$$
$$\tau_{r\theta} = G\left[\frac{1}{r}\frac{\partial u}{\partial \theta} + r \cdot \frac{\partial}{\partial r}\left(\frac{v}{r}\right)\right]$$
$$\tau_{rz} = G\left[\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right]$$

Vibrations in Cylindrical Bars

- These equations can be satisfied by using displacement equations due to Pochhammer (1876) and Chree (1889) which represent an infinite train of waves:

$$u = U \cdot \exp\left[\mathrm{i} \cdot \frac{2\pi}{\lambda}(z + c_p t)\right]$$
$$v = V \cdot \exp\left[\mathrm{i} \cdot \frac{2\pi}{\lambda}(z + c_p t)\right]$$
$$w = W \cdot \exp\left[\mathrm{i} \cdot \frac{2\pi}{\lambda}(z + c_p t)\right]$$

  - λ and c_P denote wavelength and phase speed respectively, and U, V, W, are functions of r and θ only.

Longitudinal Waves

- It is assumed that particles of the cylinder move in an rOz plane only, i.e. $V = 0$, and also that functions U and W are independent of θ.
- Introducing these conditions into the equations of the previous section, a very complex equation for the frequency of wave transmissions is arrived at; and is written in terms of Bessel functions J_0 and J_1.
- By expanding the latter for the case when the bar radius to wavelength ratio is relatively small, Rayleigh arrived at,

$$f\lambda = \left(1 - \frac{1}{4}\nu^2 \frac{\pi^2}{\lambda^2} \cdot a^2\right)\sqrt{\frac{E}{\rho}} = c_p$$

$$c_p/c_L = 1 - \nu^2\pi^2(a/\lambda)^2$$

Longitudinal Waves

- In a bar which propagates very long waves, $a/\lambda \to 0$ and $c_p \to c_L = \sqrt{E/\rho}$.
- The second term shows that a band of waves propagated in accordance with the assumptions chosen is, in fact, dispersed.
- Short wavelength waves travel slower than long wavelength waves and hence, as time passes, the extent of a wave train is progressively increased.
- The results quoted only apply for a cylinder of infinite length.

Torsional Waves

- To transmit torsional waves, longitudinal and lateral displacements must be zero, i.e. $u = w = 0$, and motion about the cylinder axis must be symmetrical so that V must be independent of θ.
- The principal result which transpires is that $c_p = \sqrt{G/\rho}$ as arrived at by elementary theory.
- In this fundamental mode dispersion is absent.

Flexural Waves

- The displacement functions adapted for flexure, with the axis of the cylinder having a purely lateral motion are,

$$u = U \cdot \cos\theta \cdot \exp\left[\mathrm{i} \cdot \frac{2\pi}{\lambda}(z + ft)\right]$$

$$v = V \cdot \sin\theta \cdot \exp\left[\mathrm{i} \cdot \frac{2\pi}{\lambda}(z + ft)\right]$$

$$w = W \cdot \cos\theta \cdot \exp\left[\mathrm{i} \cdot \frac{2\pi}{\lambda}(z + ft)\right]$$

  - where U, V and W are functions of r only.

- Since any account of the treatment for this case is lengthy we shall not pursue discussion but simply refer to Love's treatment.

Propagation of a Short Pulse Through a Short Cylinder

Short Cylinder

- Kolsky has demonstrated that the propagation of a pulse through a short solid cylinder is complex.
- A small explosive charge was used to produce a pulse at the center of one end face of a short steel cylinder, the pulse duration being about 2 μsec.
- Both P- and S-waves were created by the explosive and spread out spherically from it.
- Waves arriving at a detector at the center of the lower face can travel by a number of paths of different length, so that the detected signal has many components corresponding to the different routes through the cylinder.

Short Cylinder

- The first signal to arrive is a P-wave following route 1 and then the same P-wave is reflected from the sides of the cylinder, i.e. via route 2.
- only the first few pulses of the explosive are clearly recognizable an separate.
- A propagated pulse obviously is difficult to interpret for there are then many paths producing similar time delays.