CHAPTER 7

Applications of Integration

> ‘It’s like this,’ he said. ‘When you go after honey with a balloon, the great thing is not to let the bees know you’re coming. Now if you have a green balloon, they might think you were only part of the tree and not notice you, and if you have a blue balloon, they might think you were only part of the sky and not notice you, and the question’ [said Winnie the Pooh] ‘is: Which is most likely?’

A. A. Milne 1882–1956
from *Winnie the Pooh*

> The entire world believes [in the Normal distribution], Mr. Lippmann told me one day, because the experimentalists believe that it is a theorem of mathematics, and mathematicians believe it is an experimental fact.

Henri Poincaré
Calcul des Probabilités, 1896, p.149

Introduction

Numerous quantities in mathematics, physics, economics, biology, and indeed any quantitative science can be conveniently represented by integrals. In addition to measuring plane areas, the problem that motivated the definition of the definite integral, we can use these integrals to express volumes of solids, lengths of curves, areas of surfaces, forces, work, energy, pressure, probabilities, dollar values of a stream of payments, and a variety of other quantities that are in one sense or another equivalent to areas under graphs.

In addition, as we saw previously, many of the basic principles that govern the behaviour of our world are expressed in terms of differential equations and initial-value problems. Indefinite integration is a key tool in the solution of such problems.

In this chapter we examine some of these applications. For the most part they are independent of one another, and for that reason some of the later sections in this chapter can be regarded as optional material. The material of Sections 7.1–7.3, however, should be regarded as core because these ideas will arise again in the study of multivariable calculus.

7.1 Volumes by Slicing—Solids of Revolution

In this section we show how volumes of certain three-dimensional regions (or *solids*) can be expressed as definite integrals and thereby determined. We will not attempt to give a definition of *volume* but will rely on our intuition and experience with solid objects to provide enough insight for us to specify the volumes of certain simple solids. For example, if the base of a rectangular box is a rectangle of length l and width w (and therefore area $A = lw$), and if the box has height h, then its volume is $V = Ah = lwh$. If l, w, and h are measured in *units* (e.g., centimetres), then the volume is expressed in *cubic units* (cubic centimetres, or cm^3).

What is a cylinder? The word "cylinder" has two different but related meanings in Mathematics. As used in this Section, it is a *solid object* lying between congruent bases in two parallel planes and inside a surface (the cylindrical wall) consisting of parallel line segments joining corresponding points on the boundaries of those bases. The second meaning for "cylinder" that we will encounter in Chapter 10 and later, extends the concept of the cylindrical wall of a solid cylinder. It is a *surface* consisting of a family of parallel straight lines in three dimensional space that intersect a plane perpendicular to those lines in a curve $\mathcal{C}$. In this case the cylinder is circular if $\mathcal{C}$ is a circle.

A rectangular box is a special case of a solid called a **cylinder**. (See Figure 7.1.) Such a solid has a flat base occupying a region R in a plane, and consists of all points on parallel straight line segments having one end in R and the other end in a (necessarily congruent) region in a second plane parallel to the plane of the base. Either of these regions can be called the **base** of the cylinder. The **cylindrical wall** is the surface consisting of the parallel line segments joining corresponding points on the boundaries of the two bases. A cylinder having a polygonal base (i.e., one bounded by straight lines) is usually called a **prism**. The height of any cylinder or prism is the perpendicular distance between the parallel planes containing the two bases. If this height is h units and the area of a base is A square units, then the volume of the cylinder or prism is $V = Ah$ cubic units.

We use the adjective **right** to describe a cylinder or prism if the parallel line segments that constitute it are perpendicular to the base planes; otherwise, the cylinder or prism is called **oblique**. For example, a right cylinder whose bases are circular disks of radius r units and whose height is h units is called a **right circular cylinder**; its volume is $V = \pi r^2 h$ cubic units. Obliqueness has no effect on the volume $V = Ah$ of a prism or cylinder since h is always measured in a direction perpendicular to the base.

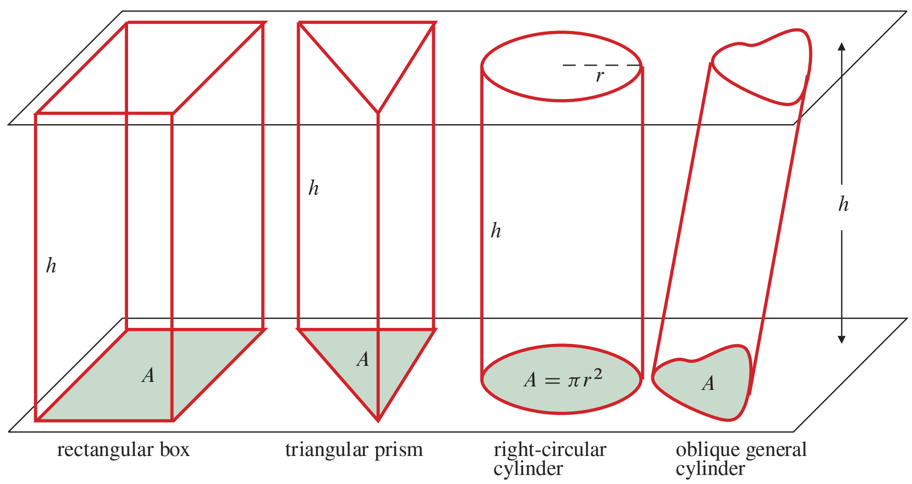


Figure 7.1 The volume of any prism or cylinder is the area A of its base times its height h (measured perpendicularly to the base): $V = Ah$

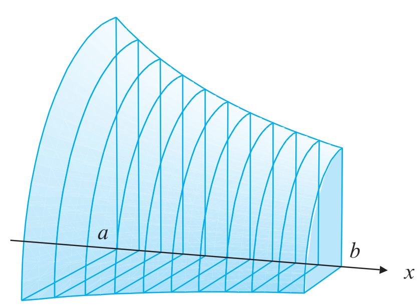


Figure 7.2 Slicing a solid perpendicularly to an axis

Volumes by Slicing

Knowing the volume of a cylinder enables us to determine the volumes of some more general solids. We can divide solids into thin "slices" by parallel planes. (Think of a loaf of sliced bread.) Each slice is approximately a cylinder of very small "height"; the height is the thickness of the slice. See Figure 7.2, where the height is measured horizontally in the x direction. If we know the cross-sectional area of each slice, we can determine its volume and sum these volumes to find the volume of the solid.

To be specific, suppose that the solid S lies between planes perpendicular to the x-axis at positions $x = a$ and $x = b$ and that the cross-sectional area of S in the plane perpendicular to the x-axis at x is a known function $A(x)$, for $a \le x \le b$. We assume that $A(x)$ is continuous on $[a, b]$. If $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$, then $P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$ is a partition of $[a, b]$ into n subintervals, and the planes perpendicular to the x-axis at $x_1, x_2, \ldots, x_{n-1}$ divide the solid into n slices of which the ith has thickness $\Delta x_i = x_i - x_{i-1}$. The volume ΔV_i of that slice lies between the maximum and minimum values of $A(x)\,\Delta x_i$ for values of x in $[x_{i-1}, x_i]$ (Figure 7.3), so, by the Intermediate-Value Theorem, for some c_i in $[x_{i-1}, x_i]$,

$$\Delta V_i = A(c_i)\,\Delta x_i.$$

The volume of the solid is therefore given by the Riemann sum

$$V = \sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} A(c_i)\,\Delta x_i.$$

Letting n approach infinity in such a way that $\max \Delta x_i$ approaches 0, we obtain the definite integral of $A(x)$ over $[a, b]$ as the limit of this Riemann sum. Therefore:

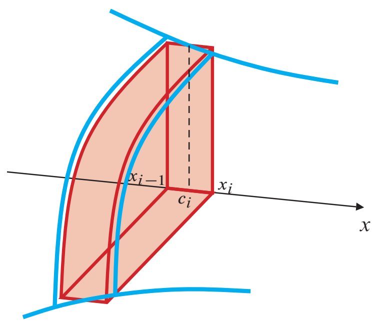


Figure 7.3 The volume of a slice

> The volume V of a solid between $x = a$ and $x = b$ having cross-sectional area $A(x)$ at position x is
>
> $$V = \int_a^b A(x)\,dx.$$

There is another way to obtain this formula and others of a similar nature. Consider a slice of the solid between the planes perpendicular to the x-axis at positions x and $x + \Delta x$. Since $A(x)$ is continuous, it doesn't change much in a short interval, so if Δx is small, then the slice has volume ΔV approximately equal to the volume of a cylinder of base area $A(x)$ and height Δx:

$$\Delta V \approx A(x)\,\Delta x.$$

The error in this approximation is small compared to the size of ΔV. This suggests, correctly, that the **volume element**, that is, the volume of an infinitely thin slice of thickness dx is $dV = A(x)\,dx$, and that the volume of the solid is the "sum" (i.e., the integral) of these volume elements between the two ends of the solid, $x = a$ and $x = b$ (see Figure 7.4):

$$V = \int_{x=a}^{x=b} dV, \qquad \text{where} \qquad dV = A(x)\,dx.$$

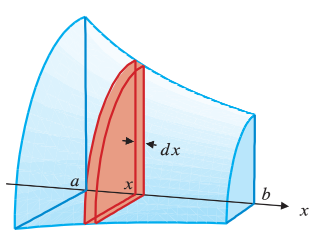


Figure 7.4 The volume element

We will use this *differential element* approach to model other applications that result in integrals rather than setting up explicit Riemann sums each time. Even though this argument does *not* constitute a proof of the formula, you are strongly encouraged to think of the formula this way; the volume is the integral of the volume elements.

Solids of Revolution

Many common solids have circular cross-sections in planes perpendicular to some axis. Such solids are called **solids of revolution** because they can be generated by rotating a plane region about an axis in that plane so that it sweeps out the solid. For example, a solid ball is generated by rotating a half-disk about the diameter of that half-disk (Figure 7.5(a)). Similarly, a solid right-circular cone is generated by rotating a right-angled triangle about one of its legs (Figure 7.5(b)).

If the region R bounded by $y = f(x)$, $y = 0$, $x = a$, and $x = b$ is rotated about the x-axis, then the cross-section of the solid generated in the plane perpendicular to the x-axis at x is a circular disk of radius $|f(x)|$. The area of this cross-section is $A(x) = \pi\big(f(x)\big)^2$, so the volume of the solid of revolution is

$$V = \pi \int_a^b (f(x))^2\,dx.$$

EXAMPLE 1 **(The volume of a ball)** Find the volume of a solid ball having radius a.

Solution The ball can be generated by rotating the half-disk, $0 \le y \le \sqrt{a^2 - x^2}$, $-a \le x \le a$ about the x-axis. See the cutaway view in Figure 7.5(a). Therefore, its volume is

$$V = \pi \int_{-a}^{a} (\sqrt{a^2 - x^2})^2\,dx = 2\pi \int_0^a (a^2 - x^2)\,dx$$
$$= 2\pi \left(a^2 x - \frac{x^3}{3}\right)\Bigg|_0^a = 2\pi \left(a^3 - \frac{1}{3}a^3\right) = \frac{4}{3}\pi a^3 \text{ cubic units.}$$

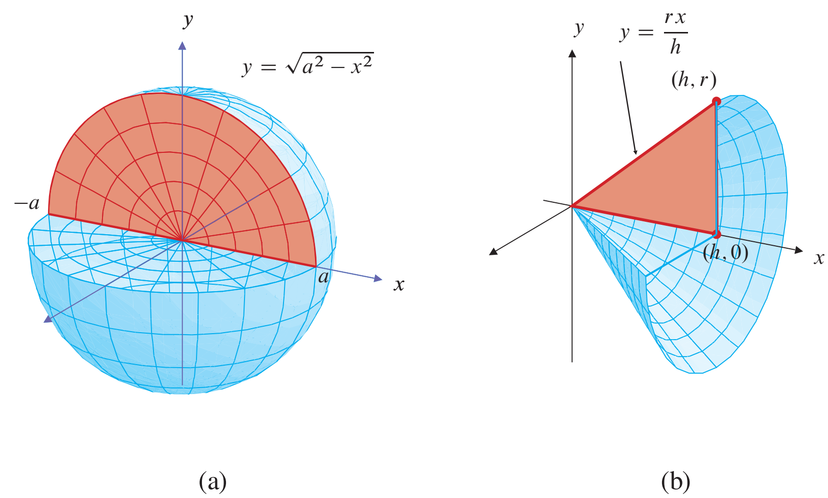


Figure 7.5

(a) The ball is generated by rotating the red half-disk $0 \le y \le \sqrt{a^2 - x^2}$ about the x-axis

(b) The cone of base radius r and height h is generated by rotating the red triangle $0 \le x \le h$, $0 \le y \le rx/h$ about the x-axis

EXAMPLE 2 **(The volume of a right-circular cone)** Find the volume of the right-circular cone of base radius r and height h that is generated by rotating the triangle with vertices $(0, 0)$, $(h, 0)$, and (h, r) about the x-axis.

Solution The line from $(0, 0)$ to (h, r) has equation $y = rx/h$. Thus, the volume of the cone (see the cutaway view in Figure 7.5(b)) is

$$V = \pi \int_0^h \left(\frac{rx}{h}\right)^2 dx = \pi \left(\frac{r}{h}\right)^2 \frac{x^3}{3}\Bigg|_0^h = \frac{1}{3}\pi r^2 h \text{ cubic units.}$$

Improper integrals can represent volumes of unbounded solids. If the improper integral converges, the unbounded solid has a finite volume.

EXAMPLE 3 Find the volume of the infinitely long horn that is generated by rotating the region bounded by $y = 1/x$ and $y = 0$ and lying to the right of $x = 1$ about the x-axis. The horn is illustrated in Figure 7.6.

Solution The volume of the horn is

$$V = \pi \int_1^\infty \left(\frac{1}{x}\right)^2 dx = \pi \lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx$$
$$= -\pi \lim_{R\to\infty} \frac{1}{x}\Bigg|_1^R = -\pi \lim_{R\to\infty} \left(\frac{1}{R} - 1\right) = \pi \text{ cubic units.}$$

It is interesting to note that this finite volume arises from rotating a region that itself has infinite area: $\int_1^\infty dx/x = \infty$. We have a paradox: it takes an infinite amount of paint to paint the region but only a finite amount to fill the horn obtained by rotating the region. (How can you resolve this paradox?)

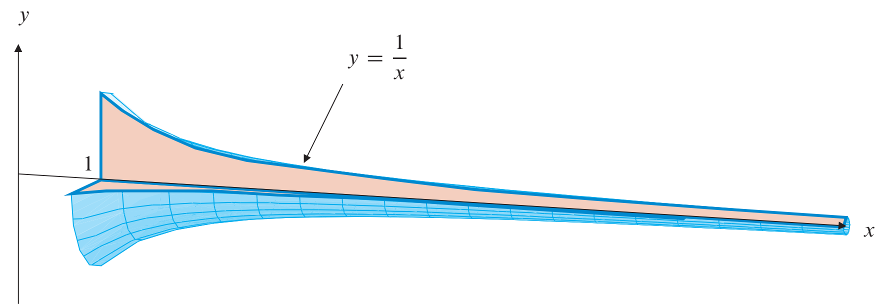


Figure 7.6 Cutaway view of an infinitely long horn

The following example shows how to deal with a problem where the axis of rotation is not the x-axis. Just rotate a suitable area element about the axis to form a volume element.

EXAMPLE 4 A ring-shaped solid is generated by rotating the finite plane region R bounded by the curve $y = x^2$ and the line $y = 1$ about the line $y = 2$. Find its volume.

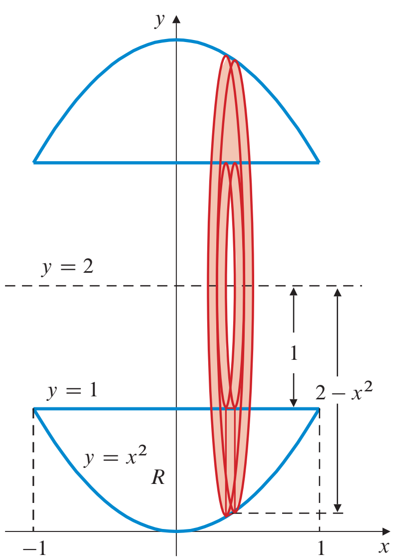


Figure 7.7 The volume element for Example 4

Solution First, we solve the pair of equations $y = x^2$ and $y = 1$ to obtain the intersections at $x = -1$ and $x = 1$. The solid lies between these two values of x. The area element of R at position x is a vertical strip of width dx extending upward from $y = x^2$ to $y = 1$. When R is rotated about the line $y = 2$, this area element sweeps out a thin, washer-shaped volume element of thickness dx and radius $2 - x^2$, having a hole of radius 1 through the middle. (See Figure 7.7.) The cross-sectional area of this element is the area of a circle of radius $2 - x^2$ minus the area of the hole, a circle of radius 1. Thus,

$$dV = \left(\pi(2-x^2)^2 - \pi(1)^2\right) dx = \pi(3 - 4x^2 + x^4)\, dx.$$

Since the solid extends from $x = -1$ to $x = 1$, its volume is

$$\begin{aligned} V &= \pi \int_{-1}^{1} (3 - 4x^2 + x^4)\, dx = 2\pi \int_0^1 (3 - 4x^2 + x^4)\, dx \\ &= 2\pi \left(3x - \frac{4x^3}{3} + \frac{x^5}{5}\right)\Bigg|_0^1 = 2\pi \left(3 - \frac{4}{3} + \frac{1}{5}\right) = \frac{56\pi}{15} \text{ cubic units.} \end{aligned}$$

Sometimes we want to rotate a region bounded by curves with equations of the form $x = g(y)$ about the y-axis. In this case, the roles of x and y are reversed, and we use horizontal slices instead of vertical ones.

EXAMPLE 5 Find the volume of the solid generated by rotating the region to the right of the y-axis and to the left of the curve $x = 2y - y^2$ about the y-axis.

Solution For intersections of $x = 2y - y^2$ and $x = 0$, we have

$$2y - y^2 = 0 \qquad \Longrightarrow \qquad y = 0 \quad \text{or} \quad y = 2.$$

The solid lies between the horizontal planes at $y = 0$ and $y = 2$. A horizontal area element at height y and having thickness dy rotates about the y-axis to generate a thin disk-shaped volume element of radius $2y - y^2$ and thickness dy. (See Figure 7.8.) Its volume is

$$dV = \pi(2y - y^2)^2\, dy = \pi(4y^2 - 4y^3 + y^4)\, dy.$$

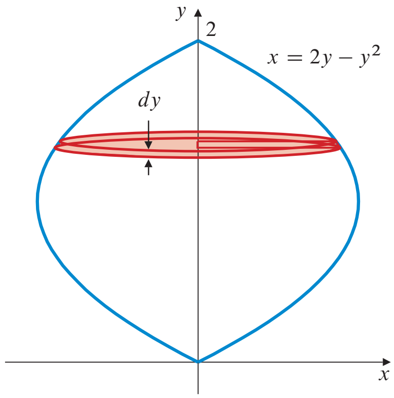


Figure 7.8 The volume element for Example 5

Thus, the volume of the solid is

$$\begin{aligned} V &= \pi \int_0^2 (4y^2 - 4y^3 + y^4)\, dy \\ &= \pi \left(\frac{4y^3}{3} - y^4 + \frac{y^5}{5} \right) \Bigg|_0^2 \\ &= \pi \left(\frac{32}{3} - 16 + \frac{32}{5} \right) = \frac{16\pi}{15} \text{ cubic units.} \end{aligned}$$

Cylindrical Shells

Suppose that the region R bounded by $y = f(x) \geq 0$, $y = 0$, $x = a \geq 0$, and $x = b > a$ is rotated about the y-axis to generate a solid of revolution. In order to find the volume of the solid using (plane) slices, we would need to know the cross-sectional area $A(y)$ in each plane of height y, and this would entail solving the equation $y = f(x)$ for one or more solutions of the form $x = g(y)$. In practice this can be inconvenient or impossible.

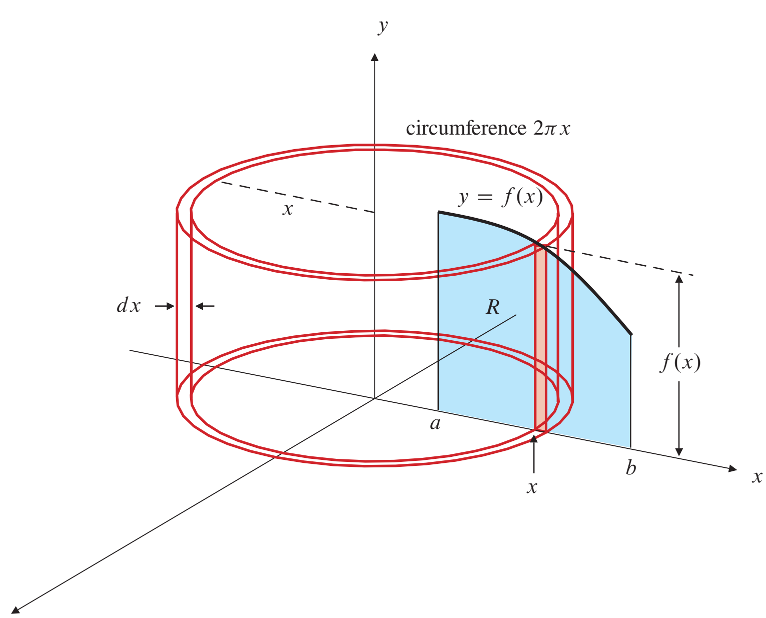


Figure 7.9 When rotated around the y-axis, the area element of width dx under $y = f(x)$ at x generates a cylindrical shell of height $f(x)$, circumference $2\pi x$, and hence volume $dV = 2\pi x\, f(x)\, dx$

The standard area element of R at position x is a vertical strip of width dx, height $f(x)$, and area $dA = f(x)\, dx$. When R is rotated about the y-axis, this strip sweeps out a volume element in the shape of a circular **cylindrical shell** having radius x, height $f(x)$, and thickness dx. (See Figure 7.9.) Regard this shell as a rolled-up rectangular slab with dimensions $2\pi x$, $f(x)$, and dx; evidently, it has volume

$$dV = 2\pi x\, f(x)\, dx.$$

The volume of the solid of revolution is the sum *(integral)* of the volumes of such shells with radii ranging from a to b:

The volume of the solid obtained by rotating the plane region $0 \le y \le f(x)$, $0 \le a < x < b$ about the y-axis is

$$V = 2\pi \int_a^b x\, f(x)\, dx.$$

EXAMPLE 6 **(The volume of a torus)** A disk of radius a has centre at the point $(b, 0)$, where $b > a > 0$. The disk is rotated about the y-axis to generate a **torus** (a doughnut-shaped solid), illustrated in Figure 7.10. Find its volume.

Solution The circle with centre at $(b, 0)$ and having radius a has equation $(x-b)^2 + y^2 = a^2$, so its upper semicircle is the graph of the function

$$f(x) = \sqrt{a^2 - (x-b)^2}.$$

We will double the volume of the upper half of the torus, which is generated by rotating the half-disk $0 \le y \le \sqrt{a^2 - (x-b)^2}$, $b - a \le x \le b + a$ about the y-axis. The volume of the complete torus is

$$\begin{aligned} V &= 2 \times 2\pi \int_{b-a}^{b+a} x\sqrt{a^2 - (x-b)^2}\, dx \qquad \text{Let } u = x - b, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad du = dx \\ &= 4\pi \int_{-a}^{a} (u+b)\sqrt{a^2 - u^2}\, du \\ &= 4\pi \int_{-a}^{a} u\sqrt{a^2 - u^2}\, du + 4\pi b \int_{-a}^{a} \sqrt{a^2 - u^2}\, du \\ &= 0 + 4\pi b \frac{\pi a^2}{2} = 2\pi^2 a^2 b \text{ cubic units.} \end{aligned}$$

(The first of the final two integrals is 0 because the integrand is odd and the interval is symmetric about 0; the second is the area of a semicircle of radius a.) Note that the volume of the torus is $(\pi a^2)(2\pi b)$, that is, the area of the disk being rotated times the distance travelled by the centre of that disk as it rotates about the y-axis. This result will be generalized by Pappus's Theorem in Section 7.5.

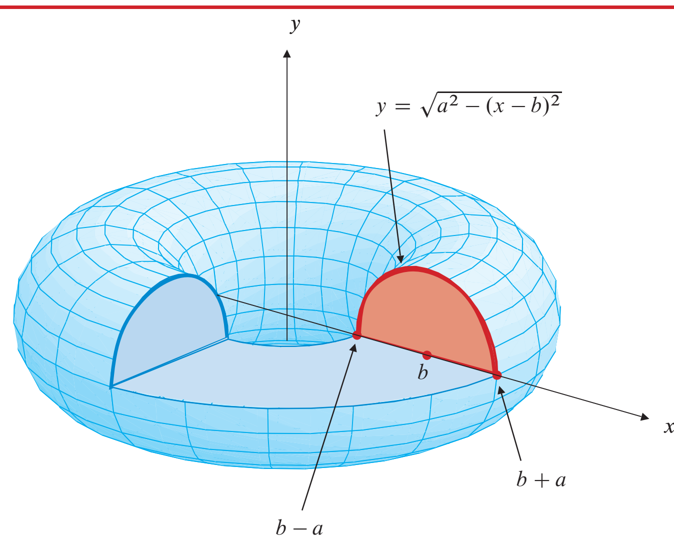


Figure 7.10 Cutaway view of a torus

EXAMPLE 7 Find the volume of a bowl obtained by revolving the parabolic arc $y = x^2$, $0 \le x \le 1$ about the y-axis.

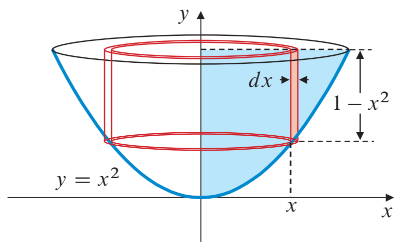


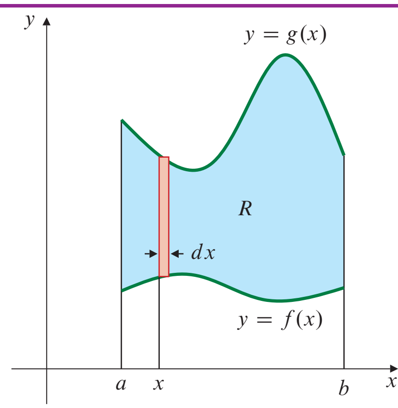
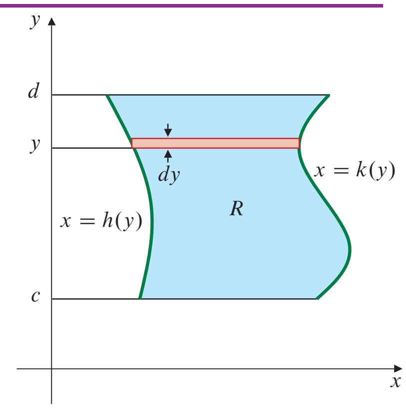
Figure 7.11 A parabolic bowl

Solution The interior of the bowl corresponds to revolving the region given by $x^2 \le y \le 1$, $0 \le x \le 1$ about the y-axis. The area element at position x has height $1-x^2$ and generates a cylindrical shell of volume $dV = 2\pi x(1-x^2)\,dx$. (See Figure 7.11.) Thus, the volume of the bowl is

$$\begin{aligned} V &= 2\pi \int_0^1 x(1-x^2)\,dx \\ &= 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Bigg|_0^1 = \frac{\pi}{2} \text{ cubic units.} \end{aligned}$$

We have described two methods for determining the volume of a solid of revolution, slicing and cylindrical shells. The choice of method for a particular solid is usually dictated by the form of the equations defining the region being rotated and by the axis of rotation. The volume element dV can always be determined by rotating a suitable area element dA about the axis of rotation. If the region is bounded by vertical lines and one or more graphs of the form $y = f(x)$, the appropriate area element is a vertical strip of width dx. If the rotation is about the x-axis or any other horizontal line, this strip generates a disk- or washer-shaped slice of thickness dx. If the rotation is about the y-axis or any other vertical line, the strip generates a cylindrical shell of thickness dx. On the other hand, if the region being rotated is bounded by horizontal lines and one or more graphs of the form $x = g(y)$, it is easier to use a horizontal strip of width dy as the area element, and this generates a slice if the rotation is about a vertical line and a cylindrical shell if the rotation is about a horizontal line. For very simple regions either method can be made to work easily. See the following table.

Table 1. Volumes of solids of revolution

If region R ⟶ is rotated about ↓		
the x-axis	use plane slices $V = \pi \int_a^b \left((g(x))^2 - (f(x))^2\right)dx$	use cylindrical shells $V = 2\pi \int_c^d y\,\big(k(y) - h(y)\big)\,dy$
the y-axis	use cylindrical shells $V = 2\pi \int_a^b x\,\big(g(x) - f(x)\big)\,dx$	use plane slices $V = \pi \int_c^d \left((k(y))^2 - (h(y))^2\right)dy$

Our final example involves rotation about a vertical line other than the y-axis.

EXAMPLE 8 The triangular region bounded by $y = x$, $y = 0$, and $x = a > 0$ is rotated about the line $x = b > a$. (See Figure 7.12.) Find the volume of the solid so generated.

Solution Here the vertical area element at x generates a cylindrical shell of radius $b-x$, height x, and thickness dx. Its volume is $dV = 2\pi(b-x)\,x\,dx$, and the volume of the solid is

$$V = 2\pi \int_0^a (b-x)\,x\,dx = 2\pi \left(\frac{bx^2}{2} - \frac{x^3}{3} \right)\Bigg|_0^a = \pi \left(a^2 b - \frac{2a^3}{3} \right) \text{ cubic units.}$$

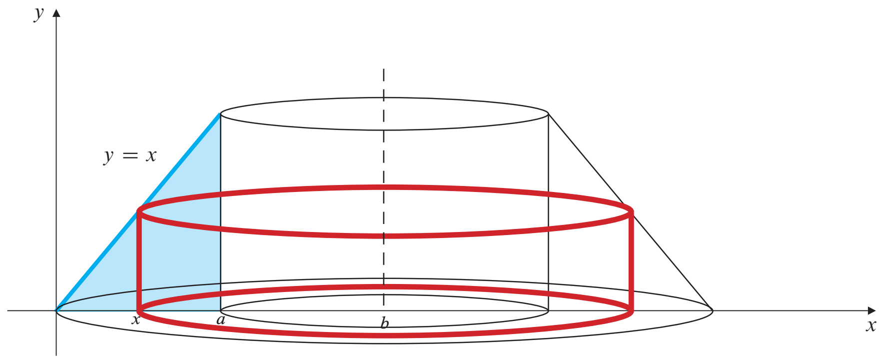


Figure 7.12 The volume element for Example 8

EXERCISES 7.1

Find the volume of each solid S in Exercises 1–4 in two ways, using the method of slicing and the method of cylindrical shells.

1. S is generated by rotating about the x-axis the region bounded by $y = x^2$, $y = 0$, and $x = 1$.
2. S is generated by rotating the region of Exercise 1 about the y-axis.
3. S is generated by rotating about the x-axis the region bounded by $y = x^2$ and $y = \sqrt{x}$ between $x = 0$ and $x = 1$.
4. S is generated by rotating the region of Exercise 3 about the y-axis.

Find the volumes of the solids obtained if the plane regions R described in Exercises 5–10 are rotated about (a) the x-axis and (b) the y-axis.

5. R is bounded by $y = x(2-x)$ and $y = 0$ between $x = 0$ and $x = 2$.
6. R is the finite region bounded by $y = x$ and $y = x^2$.
7. R is the finite region bounded by $y = x$ and $x = 4y - y^2$.
8. R is bounded by $y = 1 + \sin x$ and $y = 1$ from $x = 0$ to $x = \pi$.
9. R is bounded by $y = 1/(1+x^2)$, $y = 2$, $x = 0$, and $x = 1$.
10. R is the finite region bounded by $y = 1/x$ and $3x + 3y = 10$.

11. The triangular region with vertices $(0,-1)$, $(1,0)$, and $(0,1)$ is rotated about the line $x = 2$. Find the volume of the solid so generated.
12. Find the volume of the solid generated by rotating the region $0 \le y \le 1 - x^2$ about the line $y = 1$.
13. What percentage of the volume of a ball of radius 2 is removed if a hole of radius 1 is drilled through the centre of the ball?
14. A cylindrical hole is bored through the centre of a ball of radius R. If the length of the hole is L, show that the volume of the remaining part of the ball depends only on L and not on R.
15. A cylindrical hole of radius a is bored through a solid right-circular cone of height h and base radius $b > a$. If the axis of the hole lies along that of the cone, find the volume of the remaining part of the cone.
16. Find the volume of the solid obtained by rotating a circular disk about one of its tangent lines.
17. A plane slices a ball of radius a into two pieces. If the plane passes b units away from the centre of the ball (where $b < a$), find the volume of the smaller piece.
18. Water partially fills a hemispherical bowl of radius 30 cm so that the maximum depth of the water is 20 cm. What volume of water is in the bowl?
19. Find the volume of the ellipsoid of revolution obtained by rotating the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ about the x-axis.
20. Recalculate the volume of the torus of Example 6 by slicing perpendicular to the y-axis rather than using cylindrical shells.
21. The region R bounded by $y = e^{-x}$ and $y = 0$ and lying to the right of $x = 0$ is rotated (a) about the x-axis and (b) about the y-axis. Find the volume of the solid of revolution generated in each case.
22. The region R bounded by $y = x^{-k}$ and $y = 0$ and lying to the right of $x = 1$ is rotated about the x-axis. Find all real values of k for which the solid so generated has finite volume.
23. Repeat Exercise 22 with rotation about the y-axis.
24. Early editions of this text incorrectly defined a prism or cylinder as being a solid for which cross-sections parallel to the base were congruent to the base. Does this define a larger or smaller set of solids than the definition given in this section? What does the older definition say about the volume of a cylinder or prism having base area A and height h?
25. Continuing Exercise 24, consider the solid S whose cross-section in the plane perpendicular to the x-axis at x is an isosceles right-angled triangle having equal sides of length a cm with one end of the hypotenuse on the x-axis and with hypotenuse making angle x with a fixed direction. Is S a

prism according to the definition given in early editions? Is it a prism according to the definition in this edition? If the height of S is b cm, what is the volume of S?

26. Find the volume of the solid generated by rotating the finite region in the first quadrant bounded by the coordinate axes and the curve $x^{2/3} + y^{2/3} = 4$ about either of the coordinate axes. (Both volumes are the same. Why?)

27. Given that the surface area of a sphere of radius r is kr^2, where k is a constant independent of r, express the volume of a ball of radius R as an integral of volume elements that are the volumes of spherical shells of thickness dr and varying radii r. Hence find k.

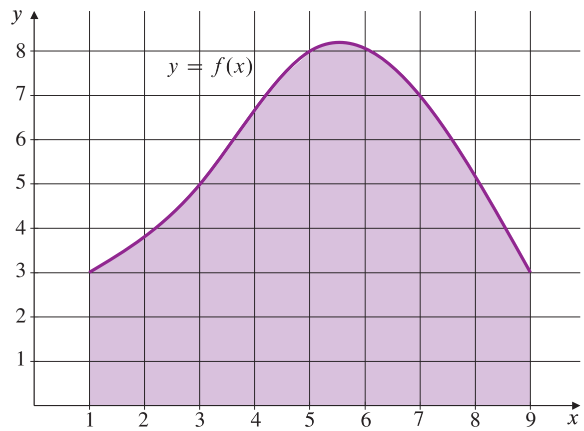


Figure 7.13

28. The region shaded in Figure 7.13 is rotated about the x-axis. Use Simpson's Rule to find the volume of the resulting solid.

29. The region shaded in Figure 7.13 is rotated about the y-axis. Use Simpson's Rule to find the volume of the resulting solid.

30. The region shaded in Figure 7.13 is rotated about the line $x = -1$. Use Simpson's Rule to find the volume of the resulting solid.

The following problems are *very difficult*. You will need some ingenuity and a lot of hard work to solve them by the techniques available to you now.

31. A martini glass in the shape of a right-circular cone of height h and semi-vertical angle α (see Figure 7.14) is filled with liquid. Slowly a ball is lowered into the glass, displacing liquid and causing it to overflow. Find the radius R of the ball that causes the greatest volume of liquid to overflow out of the glass.

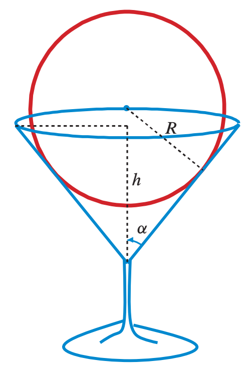


Figure 7.14

32. The finite plane region bounded by the curve $xy = 1$ and the straight line $2x + 2y = 5$ is rotated about that line to generate a solid of revolution. Find the volume of that solid.

7.2 More Volumes by Slicing

The method of slicing introduced in Section 7.1 can be used to determine volumes of solids that are not solids of revolution. All we need to know is the area of cross-section of the solid in every plane perpendicular to some fixed axis. If that axis is the x-axis, if the solid lies between the planes at $x = a$ and $x = b > a$, and if the cross-sectional area in the plane at x is the continuous (or even piecewise continuous) function $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)\,dx.$$

In this section we consider some examples that are not solids of revolution.

Pyramids and **cones** are solids consisting of all points on line segments that join a fixed point, the **vertex**, to all the points in a region lying in a plane not containing the vertex. The region is called the **base** of the pyramid or cone. Some pyramids and cones are shown in Figure 7.15. If the base is bounded by straight lines, the solid is called a pyramid; if the base has a curved boundary the solid is called a cone. All pyramids and cones have volume

$$V = \frac{1}{3}Ah,$$

where A is the area of the base region, and h is the height from the vertex to the plane of the base, measured in the direction perpendicular to that plane. We will give a very simple proof of this fact in Section 16.4. For the time being, we verify it for the case of a rectangular base.

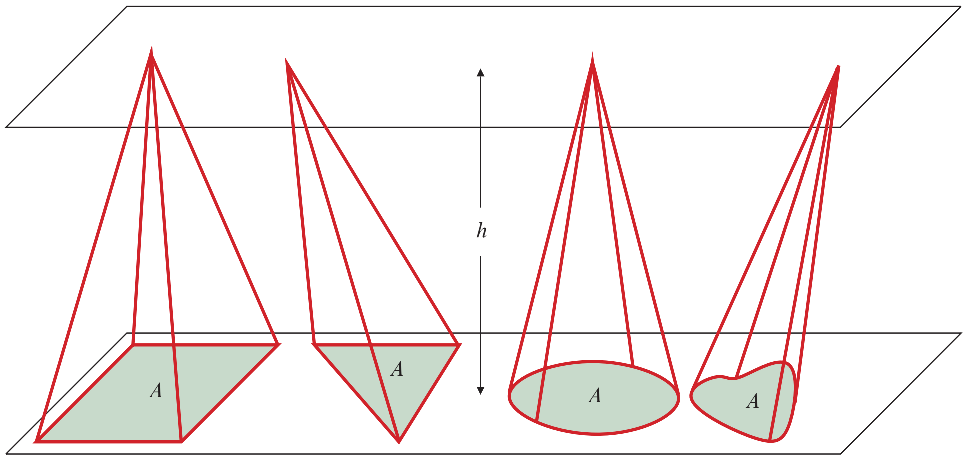


Figure 7.15 Some pyramids and cones. Each has volume $V = \frac{1}{3}Ah$, where A is the area of the base, and h is the height measured perpendicular to the base

EXAMPLE 1 Verify the formula for the volume of a pyramid with rectangular base of area A and height h.

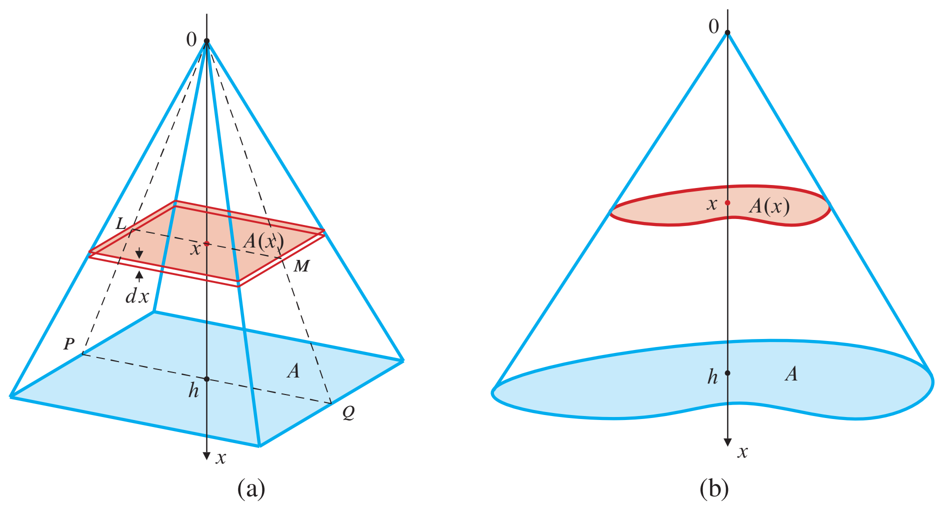


Figure 7.16
(a) A rectangular pyramid
(b) A general cone

Solution Cross-sections of the pyramid in planes parallel to the base are similar rectangles. If the origin is at the vertex of the pyramid and the x-axis is perpendicular to the base, then the cross-section at position x is a rectangle whose dimensions are x/h times the corresponding dimensions of the base. For example, in Figure 7.16(a), the length LM is x/h times the length PQ, as can be seen from the similar triangles OLM and OPQ. Thus, the area of the rectangular cross-section at x is

$$A(x) = \left(\frac{x}{h}\right)^2 A.$$

The volume of the pyramid is therefore

$$V = \int_0^h \left(\frac{x}{h}\right)^2 A\,dx = \frac{A}{h^2}\frac{x^3}{3}\bigg|_0^h = \frac{1}{3}Ah \text{ cubic units.}$$

A similar argument, resulting in the same formula for the volume, holds for a cone, that is, a pyramid with a more general (curved) shape to its base, such as that in Figure 7.16(b). Although it is not as obvious as in the case of the pyramid, the cross-section at x still has area $(x/h)^2$ times that of the base. A proof of this volume formula for an arbitrary cone or pyramid can be found in Example 3 of Section 16.4.

EXAMPLE 2 A tent has a circular base of radius a metres and is supported by a horizontal ridge bar held at height b metres above a diameter of the base by vertical supports at each end of the diameter. The material of the tent is stretched tight so that each cross-section perpendicular to the ridge bar is an isosceles triangle. (See Figure 7.17.) Find the volume of the tent.

Solution Let the x-axis be the diameter of the base under the ridge bar. The cross-section at position x has base length $2\sqrt{a^2 - x^2}$, so its area is

$$A(x) = \frac{1}{2}\left(2\sqrt{a^2 - x^2}\right)b = b\sqrt{a^2 - x^2}.$$

Thus, the volume of the solid is

$$V = \int_{-a}^{a} b\sqrt{a^2 - x^2}\, dx = b\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx = b\frac{\pi a^2}{2} = \frac{\pi}{2}a^2 b \text{ m}^3.$$

Note that we evaluated the last integral by inspection. It is the area of a half-disk of radius a.

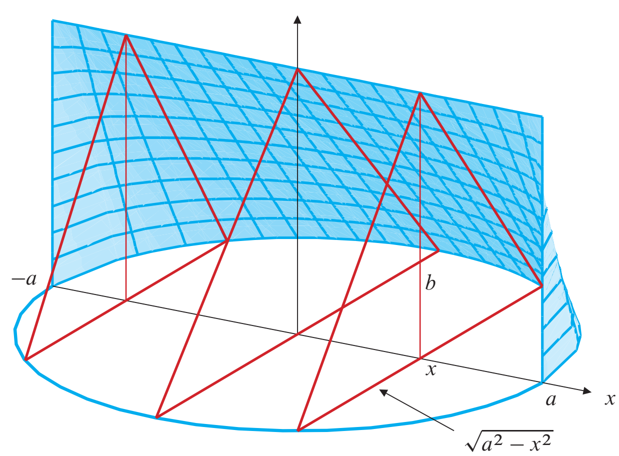


Figure 7.17 The tent of Example 2 with the front covering removed to show the shape more clearly

EXAMPLE 3 Two circular cylinders, each having radius a, intersect so that their axes meet at right angles. Find the volume of the region lying inside both cylinders.

Solution We represent the cylinders in a three-dimensional Cartesian coordinate system where the plane containing the x- and y-axes is horizontal and the z-axis is vertical. One-eighth of the solid is represented in Figure 7.18, that part corresponding to all three coordinates being positive. The two cylinders have axes along the x- and y-axes, respectively. The cylinder with axis along the x-axis intersects the plane of the y- and z-axes in a circle of radius a.

Similarly, the other cylinder meets the plane of the x- and z-axes in a circle of radius a. It follows that if the region lying inside both cylinders (and having $x \geq 0$, $y \geq 0$, and $z \geq 0$) is sliced horizontally, then the slice at height z above the xy-plane is a square of side $\sqrt{a^2 - z^2}$ and has area $A(z) = a^2 - z^2$. The volume V of the whole region, being eight times that of the part shown, is

$$V = 8\int_0^a (a^2 - z^2)\, dz = 8\left(a^2 z - \frac{z^3}{3}\right)\Bigg|_0^a = \frac{16}{3}a^3 \text{ cubic units.}$$

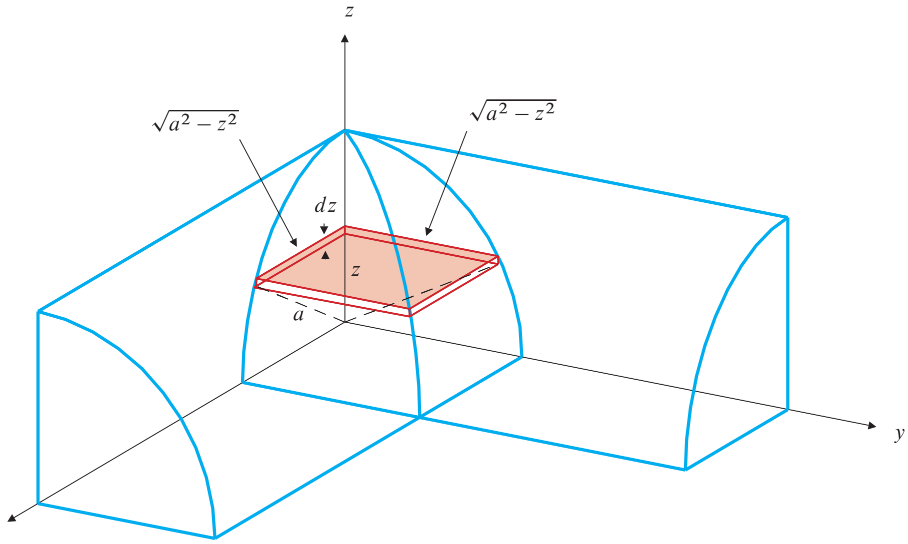


Figure 7.18 One-eighth of the solid lying inside two perpendicular cylindrical pipes. The horizontal slice shown is square

EXERCISES 7.2

1. A solid is 2 m high. The cross-section of the solid at height x above its base has area $3x$ square metres. Find the volume of the solid.
2. The cross-section at height z of a solid of height h is a rectangle with dimensions z and $h - z$. Find the volume of the solid.
3. Find the volume of a solid of height 1 whose cross-section at height z is an ellipse with semi-axes z and $\sqrt{1 - z^2}$.
4. A solid extends from $x = 1$ to $x = 3$. The cross-section of the solid in the plane perpendicular to the x-axis at x is a square of side x. Find the volume of the solid.
5. A solid is 6 ft high. Its horizontal cross-section at height z ft above its base is a rectangle with length $2 + z$ ft and width $8 - z$ ft. Find the volume of the solid.
6. A solid extends along the x-axis from $x = 1$ to $x = 4$. Its cross-section at position x is an equilateral triangle with edge length $\sqrt{x}$. Find the volume of the solid.
7. Find the volume of a solid that is h cm high if its horizontal cross-section at any height y above its base is a circular sector having radius a cm and angle $2\pi\big(1 - (y/h)\big)$ radians.
8. The opposite ends of a solid are at $x = 0$ and $x = 2$. The area of cross-section of the solid in a plane perpendicular to the x-axis at x is kx^3 square units. The volume of the solid is 4 cubic units. Find k.
9. Find the cross-sectional area of a solid in any horizontal plane at height z above its base if the volume of that part of the solid lying below any such plane is z^3 cubic units.
10. All the cross-sections of a solid in horizontal planes are squares. The volume of the part of the solid lying below any plane of height z is $4z$ cubic units, where $0 < z < h$, the height of the solid. Find the edge length of the square cross-section at height z for $0 < z < h$.
11. A solid has a circular base of radius r. All sections of the solid perpendicular to a particular diameter of the base are squares. Find the volume of the solid.
12. Repeat Exercise 11 but with sections that are equilateral triangles instead of squares.
13. The base of a solid is an isosceles right-angled triangle with equal legs measuring 12 cm. Each cross-section perpendicular to one of these legs is half of a circular disk. Find the volume of the solid.
14. **(Cavalieri's Principle)** Two solids have equal cross-sectional areas at equal heights above their bases. If both solids have the same height, show that they both have the same volume.

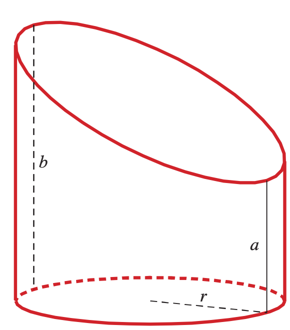


Figure 7.19

15. The top of a circular cylinder of radius r is a plane inclined at an angle to the horizontal. (See Figure 7.19.) If the lowest and highest points on the top are at heights a and b, respectively, above the base, find the volume of the cylinder. (Note that there is an easy geometric way to get the answer, but you should also try to do it by slicing. You can use either rectangular or trapezoidal slices.)

16. **(Volume of an ellipsoid)** Find the volume enclosed by the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Hint: This is not a solid of revolution. As in Example 3, the z-axis is perpendicular to the plane of the x- and y-axes. Each horizontal plane $z=k$ $(-c \le k \le c)$ intersects the ellipsoid in an ellipse $(x/a)^2+(y/b)^2=1-(k/c)^2$. Thus, $dV = dz \times$ the area of this ellipse. The area of the ellipse $(x/a)^2+(y/b)^2=1$ is πab.

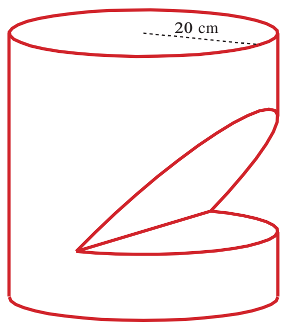


Figure 7.20

17. **(Notching a log)** A 45° notch is cut to the centre of a cylindrical log having radius 20 cm, as shown in Figure 7.20. One plane face of the notch is perpendicular to the axis of the log. What volume of wood was removed from the log by cutting the notch?

18. **(A smaller notch)** Repeat Exercise 17, but assume that the notch penetrates only one quarter way (10 cm) into the log.

19. What volume of wood is removed from a 3-in-thick board if a circular hole of radius 2 in is drilled through it with the axis of the hole tilted at an angle of 45° to board?

20. **(More intersecting cylinders)** The axes of two circular cylinders intersect at right angles. If the radii of the cylinders are a and b $(a>b>0)$, show that the region lying inside both cylinders has volume

$$V=8\int_0^b \sqrt{b^2-z^2}\sqrt{a^2-z^2}\,dz.$$

Hint: Review Example 3. Try to make a similar diagram, showing only one-eighth of the region. The integral is not easily evaluated.

21. A circular hole of radius 2 cm is drilled through the middle of a circular log of radius 4 cm, with the axis of the hole perpendicular to the axis of the log. Find the volume of wood removed from the log. *Hint:* This is very similar to Exercise 20. You will need to use numerical methods or a calculator with a numerical integration function to get the answer.

7.3 Arc Length and Surface Area

In this section we consider how integrals can be used to find the lengths of curves and the areas of the surfaces of solids of revolution.

Arc Length

If A and B are two points in the plane, let $|AB|$ denote the distance between A and B, that is, the length of the straight line segment AB.

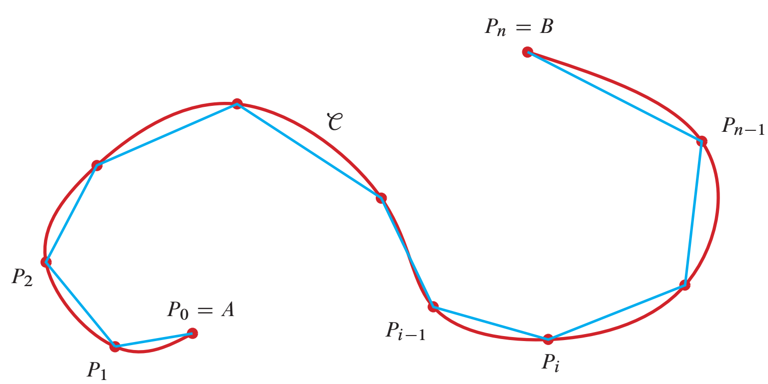


Figure 7.21 A polygonal approximation to a curve $\mathscr{C}$

Given a curve $\mathscr{C}$ joining the two points A and B, we would like to define what is meant by the *length* of the curve $\mathscr{C}$ from A to B. Suppose we choose points $A=P_0$, P_1, P_2, ..., P_{n-1}, and $P_n = B$ in order along the curve, as shown in Figure 7.21. The polygonal line $P_0P_1P_2\ldots P_{n-1}P_n$ constructed by joining adjacent pairs of these points with straight line segments forms a *polygonal approximation* to $\mathcal{C}$, having length

$$L_n = |P_0P_1| + |P_1P_2| + \cdots + |P_{n-1}P_n| = \sum_{i=1}^{n} |P_{i-1}P_i|.$$

Intuition tells us that the shortest curve joining two points is a straight line segment, so the length L_n of any such polygonal approximation to $\mathcal{C}$ cannot exceed the length of $\mathcal{C}$. If we increase n by adding more vertices to the polygonal line between existing vertices, L_n cannot get smaller and may increase. If there exists a finite number K such that $L_n \le K$ for every polygonal approximation to $\mathcal{C}$, then there will be a smallest such number K (by the completeness of the real numbers), and we call this smallest K the arc length of $\mathcal{C}$.

DEFINITION 1

The **arc length** of the curve $\mathcal{C}$ from A to B is the smallest real number s such that the length L_n of every polygonal approximation to $\mathcal{C}$ satisfies $L_n \le s$.

A curve with a finite arc length is said to be **rectifiable**. Its arc length s is the limit of the lengths L_n of polygonal approximations as $n \to \infty$ in such a way that the maximum segment length $|P_{i-1}P_i| \to 0$.

It is possible to construct continuous curves that are bounded (they do not go off to infinity anywhere) but are not rectifiable; they have infinite length. To avoid such pathological examples, we will assume that our curves are **smooth**; they will be defined by functions having continuous derivatives.

The Arc Length of the Graph of a Function

Let f be a function defined on a closed, finite interval $[a, b]$ and having a continuous derivative f' there. If $\mathcal{C}$ is the graph of f, that is, the graph of the equation $y = f(x)$, then any partition of $[a, b]$ provides a polygonal approximation to $\mathcal{C}$. For the partition

$$\{a = x_0 < x_1 < x_2 < \cdots < x_n = b\},$$

let P_i be the point $\big(x_i, f(x_i)\big)$, $(0 \le i \le n)$. The length of the polygonal line $P_0P_1P_2 \ldots P_{n-1}P_n$ is

$$\begin{aligned} L_n &= \sum_{i=1}^{n} |P_{i-1}P_i| = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + \big(f(x_i) - f(x_{i-1})\big)^2} \\ &= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}\,\Delta x_i, \end{aligned}$$

where $\Delta x_i = x_i - x_{i-1}$. By the Mean-Value Theorem there exists a number c_i in the interval $[x_{i-1}, x_i]$ such that

$$\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(c_i),$$

so we have $L_n = \displaystyle\sum_{i=1}^{n} \sqrt{1 + \big(f'(c_i)\big)^2}\,\Delta x_i.$

Thus, L_n is a Riemann sum for $\int_a^b \sqrt{1 + (f'(x))^2}\,dx$. Being the limit of such Riemann sums as $n \to \infty$ in such a way that $\max(\Delta x_i) \to 0$, that integral is the length of the curve $\mathcal{C}$.

The arc length s of the curve $y = f(x)$ from $x = a$ to $x = b$ is given by

$$s = \int_a^b \sqrt{1 + \big(f'(x)\big)^2}\,dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx.$$

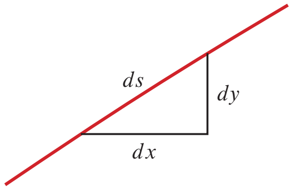


Figure 7.22 A differential triangle

You can regard the integral formula above as giving the arc length s of $\mathcal{C}$ as a "sum" of **arc length elements**:

$$s = \int_{x=a}^{x=b} ds, \qquad \text{where} \qquad ds = \sqrt{1 + \big(f'(x)\big)^2}\, dx.$$

Figure 7.22 provides a convenient way to remember this; it also suggests how we can arrive at similar formulas for arc length elements of other kinds of curves. The *differential triangle* in the figure suggests that

$$(ds)^2 = (dx)^2 + (dy)^2.$$

Dividing this equation by $(dx)^2$ and taking the square root, we get

$$\begin{aligned}
\left(\frac{ds}{dx}\right)^2 &= 1 + \left(\frac{dy}{dx}\right)^2 \\
\frac{ds}{dx} &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \\
ds &= \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \big(f'(x)\big)^2}\, dx.
\end{aligned}$$

A similar argument shows that for a curve specified by an equation of the form $x = g(y)$, $(c \le y \le d)$, the arc length element is

$$ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \sqrt{1 + \big(g'(y)\big)^2}\, dy.$$

EXAMPLE 1 Find the length of the curve $y = x^{2/3}$ from $x = 1$ to $x = 8$.

Solution Since $dy/dx = \frac{2}{3}x^{-1/3}$ is continuous between $x = 1$ and $x = 8$ and $x^{1/3} > 0$ there, the length of the curve is given by

$$\begin{aligned}
s &= \int_1^8 \sqrt{1 + \frac{4}{9}x^{-2/3}}\, dx = \int_1^8 \sqrt{\frac{9x^{2/3} + 4}{9x^{2/3}}}\, dx \\
&= \int_1^8 \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}}\, dx \qquad \begin{array}{l}\text{Let } u = 9x^{2/3} + 4, \\ du = 6x^{-1/3}\, dx\end{array} \\
&= \frac{1}{18}\int_{13}^{40} u^{1/2}\, du = \frac{1}{27}u^{3/2}\bigg|_{13}^{40} = \frac{40\sqrt{40} - 13\sqrt{13}}{27} \text{ units.}
\end{aligned}$$

EXAMPLE 2 Find the length of the curve $y = x^4 + \dfrac{1}{32x^2}$ from $x = 1$ to $x = 2$.

Solution Here $\dfrac{dy}{dx} = 4x^3 - \dfrac{1}{16x^3}$ and

$$\begin{aligned}
1 + \left(\frac{dy}{dx}\right)^2 &= 1 + \left(4x^3 - \frac{1}{16x^3}\right)^2 \\
&= 1 + (4x^3)^2 - \frac{1}{2} + \left(\frac{1}{16x^3}\right)^2 \\
&= (4x^3)^2 + \frac{1}{2} + \left(\frac{1}{16x^3}\right)^2 = \left(4x^3 + \frac{1}{16x^3}\right)^2.
\end{aligned}$$

The expression in the last set of parentheses is positive for $1 \le x \le 2$, so the length of the curve is

$$s = \int_1^2 \left(4x^3 + \frac{1}{16x^3}\right) dx = \left(x^4 - \frac{1}{32x^2}\right)\Bigg|_1^2$$
$$= 16 - \frac{1}{128} - \left(1 - \frac{1}{32}\right) = 15 + \frac{3}{128} \text{ units.}$$

The examples above are deceptively simple; the curves were chosen so that the arc length integrals could be easily evaluated. For instance, the number 32 in the curve in Example 2 was chosen so the expression $1 + (dy/dx)^2$ would turn out to be a perfect square and its square root would cause no problems. Because of the square root in the formula, arc length problems for most curves lead to integrals that are difficult or impossible to evaluate without using numerical techniques.

EXAMPLE 3 **(Manufacturing corrugated panels)** Flat rectangular sheets of metal 2 m wide are to be formed into corrugated roofing panels 2 m wide by bending them into the sinusoidal shape shown in Figure 7.23. The period of the cross-sectional sine curve is 20 cm. Its amplitude is 5 cm, so the panel is 10 cm thick. How long should the flat sheets be cut if the resulting panels must be 5 m long?

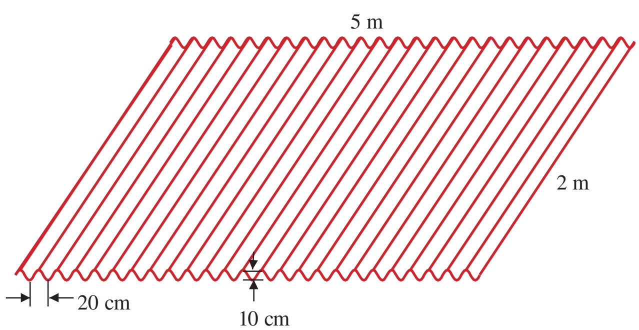


Figure 7.23 A corrugated roofing panel

Solution One period of the sinusoidal cross-section is shown in Figure 7.24. The distances are all in metres; the 5 cm amplitude is shown as 1/20 m, and the 20 cm period is shown as 2/10 m. The curve has equation

$$y = \frac{1}{20}\sin(10\pi x).$$

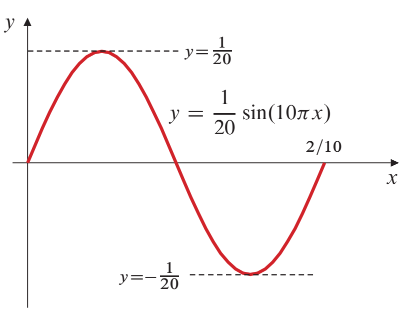


Figure 7.24 One period of the panel's cross-section

Note that 25 periods are required to produce a 5 m long panel. The length of the flat sheet required is 25 times the length of one period of the sine curve:

$$s = 25\int_0^{2/10} \sqrt{1 + \left(\frac{\pi}{2}\cos(10\pi x)\right)^2}\, dx \qquad \text{Let } t = 10\pi x, \quad dt = 10\pi\, dx$$
$$= \frac{5}{2\pi}\int_0^{2\pi} \sqrt{1 + \frac{\pi^2}{4}\cos^2 t}\, dt = \frac{10}{\pi}\int_0^{\pi/2} \sqrt{1 + \frac{\pi^2}{4}\cos^2 t}\, dt.$$

The integral can be evaluated numerically using the techniques of the previous chapter or by using the definite integral function on an advanced scientific calculator or a computer. The value is $s \approx 7.32$. The flat metal sheet should be about 7.32 m long to yield a 5 m long finished panel.

If integrals needed for standard problems such as arc lengths of simple curves cannot be evaluated exactly, they are sometimes used to define new functions whose values are tabulated or built into computer programs. An example of this is the complete elliptic integral function that arises in the next example.

EXAMPLE 4 **(The circumference of an ellipse)** Find the circumference of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

where $a \ge b > 0$. See Figure 7.25.

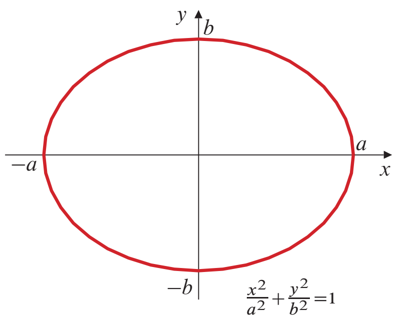


Figure 7.25 The ellipse of Example 4

Solution The upper half of the ellipse has equation $y = b\sqrt{1-\frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2-x^2}$. Hence,

$$\frac{dy}{dx} = -\frac{b}{a}\frac{x}{\sqrt{a^2-x^2}},$$

so

$$\begin{aligned}1+\left(\frac{dy}{dx}\right)^2 &= 1+\frac{b^2}{a^2}\frac{x^2}{a^2-x^2}\\ &= \frac{a^4-(a^2-b^2)x^2}{a^2(a^2-x^2)}.\end{aligned}$$

The circumference of the ellipse is four times the arc length of the part lying in the first quadrant, so

$$\begin{aligned}s &= 4\int_0^a \frac{\sqrt{a^4-(a^2-b^2)x^2}}{a\sqrt{a^2-x^2}}\,dx \qquad \begin{array}{l}\text{Let } x = a\sin t,\\ dx = a\cos t\,dt\end{array}\\ &= 4\int_0^{\pi/2}\frac{\sqrt{a^4-(a^2-b^2)a^2\sin^2 t}}{a(a\cos t)}\,a\cos t\,dt\\ &= 4\int_0^{\pi/2}\sqrt{a^2-(a^2-b^2)\sin^2 t}\,dt\\ &= 4a\int_0^{\pi/2}\sqrt{1-\frac{a^2-b^2}{a^2}\sin^2 t}\,dt\\ &= 4a\int_0^{\pi/2}\sqrt{1-\varepsilon^2\sin^2 t}\,dt \text{ units,}\end{aligned}$$

where $\varepsilon = (\sqrt{a^2-b^2})/a$ is the *eccentricity* of the ellipse. (See Section 8.1 for a discussion of ellipses.) Note that $0 \le \varepsilon < 1$. The function $E(\varepsilon)$, defined by

$$E(\varepsilon) = \int_0^{\pi/2}\sqrt{1-\varepsilon^2\sin^2 t}\,dt,$$

is called the **complete elliptic integral of the second kind**. The integral cannot be evaluated by elementary techniques for general ε, although numerical methods can be applied to find approximate values for any given value of ε. Tables of values of $E(\varepsilon)$ for various values of ε can be found in collections of mathematical tables. As shown above, the circumference of the ellipse is given by $4aE(\varepsilon)$. Note that for $a = b$ we have $\varepsilon = 0$, and the formula returns the circumference of a circle; $s = 4a(\pi/2) = 2\pi a$ units.

Areas of Surfaces of Revolution

When a plane curve is rotated (in three dimensions) about a line in the plane of the curve, it sweeps out a **surface of revolution**. For instance, a sphere of radius a is generated by rotating a semicircle of radius a about the diameter of that semicircle.

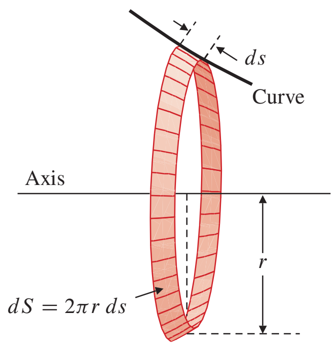


Figure 7.26 The circular band generated by rotating arc length element ds about the axis

The area of a surface of revolution can be found by integrating an area element dS constructed by rotating the arc length element ds of the curve about the given line. If the radius of rotation of the element ds is r, then it generates, on rotation, a circular band of width ds and length (circumference) $2\pi r$. The area of this band is, therefore,

$$dS = 2\pi r\, ds,$$

as shown in Figure 7.26. The areas of surfaces of revolution around various lines can be obtained by integrating dS with appropriate choices of r. Here are some important special cases:

Area of a surface of revolution

If $f'(x)$ is continuous on $[a, b]$ and the curve $y = f(x)$ is rotated about the x-axis, the area of the surface of revolution so generated is

$$S = 2\pi \int_{x=a}^{x=b} |y|\, ds = 2\pi \int_a^b |f(x)|\sqrt{1 + (f'(x))^2}\, dx.$$

If the rotation is about the y-axis, the surface area is

$$S = 2\pi \int_{x=a}^{x=b} |x|\, ds = 2\pi \int_a^b |x|\sqrt{1 + (f'(x))^2}\, dx.$$

If $g'(y)$ is continuous on $[c, d]$ and the curve $x = g(y)$ is rotated about the x-axis, the area of the surface of revolution so generated is

$$S = 2\pi \int_{y=c}^{y=d} |y|\, ds = 2\pi \int_c^d |y|\sqrt{1 + (g'(y))^2}\, dy.$$

If the rotation is about the y-axis, the surface area is

$$S = 2\pi \int_{y=c}^{y=d} |x|\, ds = 2\pi \int_c^d |g(y)|\sqrt{1 + (g'(y))^2}\, dy.$$

Remark Students sometimes wonder whether such complicated formulas are actually necessary. Why not just use $dS = 2\pi|y|\,dx$ for the area element when $y = f(x)$ is rotated about the x-axis instead of the more complicated area element $dS = 2\pi|y|\,ds$? After all, we are regarding dx and ds as both being infinitely small, and we certainly used dx for the width of the disk-shaped volume element when we rotated the region under $y = f(x)$ about the x-axis to generate a solid of revolution. The reason is somewhat subtle. For small thickness Δx, the volume of a slice of the solid of revolution is only approximately $\pi y^2\,\Delta x$, but the error is *small compared to the volume of this slice*. On the other hand, if we use $2\pi|y|\,\Delta x$ as an approximation to the area of a thin band of the surface of revolution corresponding to an x interval of width Δx, the error is *not small compared to the area of that band.* If, for instance, the curve $y = f(x)$ has slope 1 at x, then the width of the band is really $\Delta s = \sqrt{2}\,\Delta x$, so that the area of the band is $\Delta S = 2\pi\sqrt{2}|y|\,\Delta x$, not just $2\pi|y|\,\Delta x$. Always use the appropriate arc length element along the curve when you rotate a curve to find the area of a surface of revolution.

EXAMPLE 5 **(Surface area of a sphere)** Find the area of the surface of a sphere of radius a.

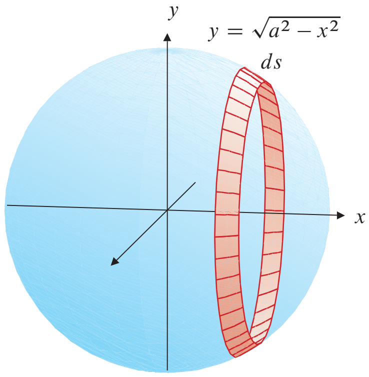


Figure 7.27 An area element on a sphere

Solution Such a sphere can be generated by rotating the semicircle with equation $y = \sqrt{a^2 - x^2}$, $(-a \le x \le a)$, about the x-axis. (See Figure 7.27.) Since

$$\frac{dy}{dx} = -\frac{x}{\sqrt{a^2 - x^2}} = -\frac{x}{y},$$

the area of the sphere is given by

$$\begin{aligned} S &= 2\pi \int_{-a}^{a} y\sqrt{1 + \left(\frac{x}{y}\right)^2}\, dx \\ &= 4\pi \int_0^a \sqrt{y^2 + x^2}\, dx \\ &= 4\pi \int_0^a \sqrt{a^2}\, dx = 4\pi a x\Big|_0^a = 4\pi a^2 \text{ square units.} \end{aligned}$$

EXAMPLE 6 **(Surface area of a parabolic dish)** Find the surface area of a parabolic reflector whose shape is obtained by rotating the parabolic arc $y = x^2$, $(0 \le x \le 1)$, about the y-axis, as illustrated in Figure 7.28.

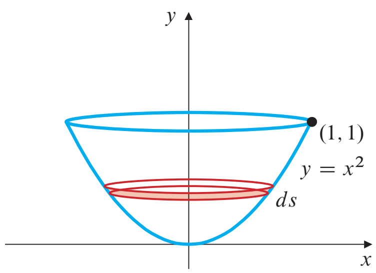


Figure 7.28 The area element is a horizontal band here

Solution The arc length element for the parabola $y = x^2$ is $ds = \sqrt{1 + 4x^2}\, dx$, so the required surface area is

$$\begin{aligned} S &= 2\pi \int_0^1 x\sqrt{1 + 4x^2}\, dx \qquad \text{Let } u = 1 + 4x^2, \\ & \qquad\qquad\qquad\qquad\qquad\quad du = 8x\, dx \\ &= \frac{\pi}{4}\int_1^5 u^{1/2}\, du \\ &= \frac{\pi}{6} u^{3/2}\Big|_1^5 = \frac{\pi}{6}(5\sqrt{5} - 1) \text{ square units.} \end{aligned}$$

EXERCISES 7.3

In Exercises 1–16, find the lengths of the given curves.

1. $y = 2x - 1$ from $x = 1$ to $x = 3$
2. $y = ax + b$ from $x = A$ to $x = B$
3. $y = \dfrac{2}{3}x^{3/2}$ from $x = 0$ to $x = 8$
4. $y^2 = (x - 1)^3$ from $(1, 0)$ to $(2, 1)$
5. $y^3 = x^2$ from $(-1, 1)$ to $(1, 1)$
6. $2(x + 1)^3 = 3(y - 1)^2$ from $(-1, 1)$ to $(0, 1 + \sqrt{2/3})$
7. $y = \dfrac{x^3}{12} + \dfrac{1}{x}$ from $x = 1$ to $x = 4$
8. $y = \dfrac{x^3}{3} + \dfrac{1}{4x}$ from $x = 1$ to $x = 2$
9. $4y = 2\ln x - x^2$ from $x = 1$ to $x = e$
10. $y = x^2 - \dfrac{\ln x}{8}$ from $x = 1$ to $x = 2$
11. $y = \dfrac{e^x + e^{-x}}{2}$ $(= \cosh x)$ from $x = 0$ to $x = a$
12. $y = \ln(1 - x^2)$ from $x = -(1/2)$ to $x = 1/2$
13. $y = \ln\cos x$ from $x = \pi/6$ to $x = \pi/4$
14. $y = x^2$ from $x = 0$ to $x = 2$
15. $y = \ln\dfrac{e^x - 1}{e^x + 1}$ from $x = 2$ to $x = 4$
16. $y = \ln x$ from $x = 1$ to $x = e$
17. Find the circumference of the closed curve $x^{2/3} + y^{2/3} = a^{2/3}$. *Hint:* The curve is symmetric about both coordinate axes (why?), so one-quarter of it lies in the first quadrant.

Use numerical methods (or a calculator with an integration function, or computer software like Maple) to find the lengths of the curves in Exercises 18–21 to 4 decimal places.

18. $y = x^4$ from $x = 0$ to $x = 1$
19. $y = x^{1/3}$ from $x = 1$ to $x = 2$
20. The circumference of the ellipse $3x^2 + y^2 = 3$
21. The shorter arc of the ellipse $x^2 + 2y^2 = 2$ between $(0, 1)$ and $(1, 1/\sqrt{2})$

In Exercises 22–29, find the areas of the surfaces obtained by rotating the given curve about the indicated lines.

22. $y = x^2$, $(0 \le x \le 2)$, about the y-axis

23. $y = x^3$, $(0 \le x \le 1)$, about the x-axis

24. $y = x^{3/2}$, $(0 \le x \le 1)$, about the x-axis

25. $y = x^{3/2}$, $(0 \le x \le 1)$, about the y-axis

26. $y = e^x$, $(0 \le x \le 1)$, about the x-axis

27. $y = \sin x$, $(0 \le x \le \pi)$, about the x-axis

28. $y = \dfrac{x^3}{12} + \dfrac{1}{x}$, $(1 \le x \le 4)$, about the x-axis

29. $y = \dfrac{x^3}{12} + \dfrac{1}{x}$, $(1 \le x \le 4)$, about the y-axis

30. **(Surface area of a cone)** Find the area of the curved surface of a right-circular cone of base radius r and height h by rotating the straight line segment from $(0, 0)$ to (r, h) about the y-axis.

31. **(How much icing on a doughnut?)** Find the surface area of the torus (doughnut) obtained by rotating the circle $(x - b)^2 + y^2 = a^2$ about the y-axis.

32. **(Area of a prolate spheroid)** Find the area of the surface obtained by rotating the ellipse $x^2 + 4y^2 = 4$ about the x-axis.

33. **(Area of an oblate spheroid)** Find the area of the surface obtained by rotating the ellipse $x^2 + 4y^2 = 4$ about the y-axis.

34. The ellipse of Example 4 is rotated about the line $y = c > b$ to generate a doughnut with elliptical cross-sections. Express the surface area of this doughnut in terms of the complete elliptic integral function $E(\varepsilon)$ introduced in that example.

35. Express the integral formula obtained for the length of the metal sheet in Example 3 in terms of the complete elliptic integral function $E(\epsilon)$ introduced in Example 4.

36. **(An interesting property of spheres)** If two parallel planes intersect a sphere, show that the surface area of that part of the sphere lying between the two planes depends only on the radius of the sphere and the distance between the planes, and not on the position of the planes.

37. For what real values of k does the surface generated by rotating the curve $y = x^k$, $(0 < x \le 1)$, about the y-axis have a finite surface area?

38. The curve $y = \ln x$, $(0 < x \le 1)$, is rotated about the y-axis. Find the area of the horn-shaped surface so generated.

39. A hollow container in the shape of an infinitely long horn is generated by rotating the curve $y = 1/x$, $(1 \le x < \infty)$, about the x-axis.

(a) Find the volume of the container.

(b) Show that the container has infinite surface area.

(c) How do you explain the "paradox" that the container can be filled with a finite volume of paint but requires an infinite amount of paint to cover its surface?

7.4 Mass, Moments, and Centre of Mass

Many quantities of interest in physics, mechanics, ecology, finance, and other disciplines are described in terms of densities over regions of space, the plane, or even the real line. To determine the total value of such a quantity we must add up (integrate) the contributions from the various places where the quantity is distributed.

Mass and Density

If a solid object is made of a homogeneous material, we would expect different parts of the solid that have the same volume to have the same mass as well. We express this homogeneity by saying that the object has constant density, that density being the mass divided by the volume for the whole object or for any part of it. Thus, for example, a rectangular brick with dimensions 20 cm, 10 cm, and 8 cm would have volume $V = 20 \times 10 \times 8 = 1{,}600$ cm^3, and if it was made of material having constant density $\rho = 3$ g/cm^3, it would have mass $m = \rho V = 3 \times 1{,}600 = 4{,}800$ g. (We will use the lowercase Greek letter rho (ρ) to represent density.)

By "density at a point P" of a solid object, we mean the limit $\rho(P)$ of mass/volume for the part of the solid lying in small regions containing P (e.g., balls centred at P) as the dimensions of the regions approach zero. Such a density ρ is continuous at P if we can ensure that $|\rho(Q) - \rho(P)|$ is as small as we want by taking Q close enough to P.

If the density of the material constituting a solid object is not constant but varies from point to point in the object, no such simple relationship exists between mass and volume. If the density $\rho = \rho(P)$ is a *continuous* function of position P, we can subdivide the solid into many small volume elements and, by regarding ρ as approximately constant over each such element, determine the masses of all the elements and add them up to get the mass of the solid. The mass Δm of a volume element ΔV containing the point P would satisfy

$$\Delta m \approx \rho(P)\,\Delta V,$$

so the mass m of the solid can be approximated:

$$m = \sum \Delta m \approx \sum \rho(P)\,\Delta V.$$

Such approximations become exact as we pass to the limit of differential mass and volume elements, $dm = \rho(P)\,dV$, so we expect to be able to calculate masses as integrals, that is, as the limits of such sums:

$$m = \int dm = \int \rho(P)\,dV.$$

EXAMPLE 1 The density of a solid vertical cylinder of height H cm and base area A cm^2 is $\rho = \rho_0(1+h)$ g/cm^3, where h is the height in centimetres above the base and ρ_0 is a constant. Find the mass of the cylinder.

Solution See Figure 7.29(a). A slice of the solid at height h above the base and having thickness dh is a circular disk of volume $dV = A\,dh$. Since the density is constant over this disk, the mass of the volume element is

$$dm = \rho\,dV = \rho_0(1+h)\,A\,dh.$$

Therefore, the mass of the whole cylinder is

$$m = \int_0^H \rho_0 A(1+h)\,dh = \rho_0 A\left(H + \frac{H^2}{2}\right) \text{ g.}$$

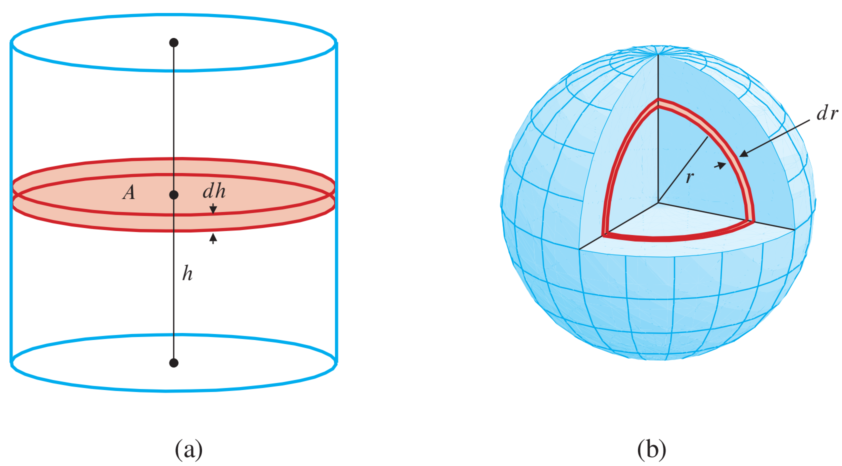


Figure 7.29
(a) A solid cylinder whose density varies with height
(b) Cutaway view of a planet whose density depends on distance from the centre

EXAMPLE 2 **(Using spherical shells)** The density of a certain spherical planet of radius R m varies with distance r from the centre according to the formula

$$\rho = \frac{\rho_0}{1+r^2} \text{ kg/m}^3.$$

Find the mass of the planet.

Solution Recall that the surface area of a sphere of radius r is $4\pi r^2$. The planet can be regarded as being composed of concentric spherical shells having radii between 0 and R. The volume of a shell of radius r and thickness dr (see Figure 7.29(b)) is equal to its surface area times its thickness, and its mass is its volume times its density:

$$dV = 4\pi r^2\,dr; \qquad dm = \rho\,dV = 4\pi\rho_0 \frac{r^2}{1+r^2}\,dr.$$

We add the masses of these shells to find the mass of the whole planet:

$$m = 4\pi\rho_0 \int_0^R \frac{r^2}{1+r^2}\,dr = 4\pi\rho_0 \int_0^R \left(1 - \frac{1}{1+r^2}\right) dr$$
$$= 4\pi\rho_0(r - \tan^{-1} r)\Big|_0^R = 4\pi\rho_0(R - \tan^{-1} R)\text{ kg.}$$

Similar techniques can be applied to find masses of one- and two-dimensional objects, such as wires and thin plates, that have variable densities of the forms mass/unit length (**line density**, which we will usually denote by δ) and σ = mass/unit area (**areal density**, which we will denote by σ).

EXAMPLE 3 A wire of variable composition is stretched along the x-axis from $x = 0$ to $x = L$ cm. Find the mass of the wire if the line density at position x is $\delta(x) = kx$ g/cm, where k is a positive constant.

Solution The mass of a length element dx of the wire located at position x is given by $dm = \delta(x)\,dx = kx\,dx$. Thus, the mass of the wire is

$$m = \int_0^L kx\,dx = \left(\frac{kx^2}{2}\right)\Bigg|_0^L = \frac{kL^2}{2}\text{ g.}$$

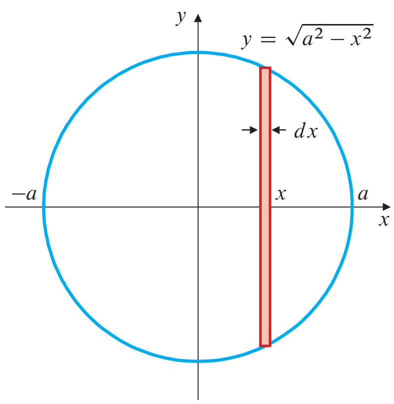


Figure 7.30 The area element of Example 4

EXAMPLE 4 Find the mass of a disk of radius a cm whose centre is at the origin in the xy-plane if the areal density at position (x, y) is $\sigma = k(2a + x)$ g/cm^2. Here k is a constant.

Solution The areal density depends only on the horizontal coordinate x, so it is constant along vertical lines on the disk. This suggests that thin vertical strips should be used as area elements. A vertical strip of thickness dx at x has area $dA = 2\sqrt{a^2 - x^2}\,dx$ (see Figure 7.30); its mass is therefore

$$dm = \sigma\,dA = 2k(2a + x)\sqrt{a^2 - x^2}\,dx.$$

Hence, the mass of the disk is

$$m = \int_{x=-a}^{x=a} dm = 2k \int_{-a}^{a} (2a + x)\sqrt{a^2 - x^2}\,dx$$
$$= 4ak \int_{-a}^{a} \sqrt{a^2 - x^2}\,dx + 2k \int_{-a}^{a} x\sqrt{a^2 - x^2}\,dx$$
$$= 4ak\frac{\pi a^2}{2} + 0 = 2\pi k a^3\text{ g.}$$

We used the area of a semicircle to evaluate the first integral. The second integral is zero because the integrand is odd and the interval is symmetric about $x = 0$.

Distributions of mass along one-dimensional structures (lines or curves) necessarily lead to integrals of functions of one variable, but distributions of mass on a surface or in space can lead to integrals involving functions of more than one variable. Such integrals are studied in multivariable calculus. (See, for example, Section 14.7.) In the examples above, the given densities were functions of only one variable, so these problems, although higher dimensional in nature, led to integrals of functions of only one variable and could be solved by the methods at hand.

Moments and Centres of Mass

The **moment** about the point $x = x_0$ of a mass m located at position x on the x-axis is the product $m(x - x_0)$ of the mass and its (signed) distance from x_0. If the x-axis is a horizontal arm hinged at x_0, the moment about x_0 measures the tendency of the weight of the mass m to cause the arm to rotate. If several masses $m_1, m_2, m_3, \ldots, m_n$ are located at the points $x_1, x_2, x_3, \ldots, x_n$, respectively, then the total moment of the system of masses about the point $x = x_0$ is the sum of the individual moments (see Figure 7.31):

$$M_{x=x_0} = (x_1 - x_0)m_1 + (x_2 - x_0)m_2 + \cdots + (x_n - x_0)m_n = \sum_{j=1}^{n} (x_j - x_0)m_j.$$

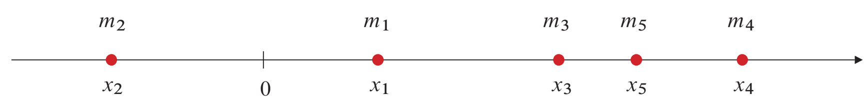


Figure 7.31 A system of discrete masses on a line

The **centre of mass** of the system of masses is the point $\bar{x}$ about which the total moment of the system is zero. Thus,

$$0 = \sum_{j=1}^{n} (x_j - \bar{x})m_j = \sum_{j=1}^{n} x_j m_j - \bar{x} \sum_{j=1}^{n} m_j.$$

The centre of mass of the system is therefore given by

$$\bar{x} = \frac{\displaystyle\sum_{j=1}^{n} x_j m_j}{\displaystyle\sum_{j=1}^{n} m_j} = \frac{M_{x=0}}{m},$$

where m is the total mass of the system and $M_{x=0}$ is the total moment about $x = 0$. If you think of the x-axis as being a weightless wire supporting the masses, then $\bar{x}$ is the point at which the wire could be supported and remain in perfect balance (equilibrium), not tipping either way. Even if the axis represents a nonweightless support, say a seesaw, supported at $x = \bar{x}$, it will remain balanced after the masses are added, provided it was balanced beforehand. For many purposes a system of masses behaves as though its total mass were concentrated at its centre of mass.

Now suppose that a one-dimensional distribution of mass with continuously variable line density $\delta(x)$ lies along the interval $[a, b]$ of the x-axis. An element of length dx at position x contains mass $dm = \delta(x)\,dx$, so its moment is $dM_{x=0} = x\,dm = x\delta(x)\,dx$ about $x = 0$. The total moment about $x = 0$ is the *sum* (integral) of these moment elements:

$$M_{x=0} = \int_a^b x\delta(x)\,dx.$$

Since the total mass is

$$m = \int_a^b \delta(x)\,dx,$$

we obtain the following formula for the centre of mass:

The centre of mass of a distribution of mass with line density $\delta(x)$ on the interval $[a, b]$ is given by

$$\bar{x} = \frac{M_{x=0}}{m} = \frac{\displaystyle\int_a^b x\delta(x)\,dx}{\displaystyle\int_a^b \delta(x)\,dx}.$$

EXAMPLE 5 At what point can the wire of Example 3 be suspended so that it will balance?

Solution In Example 3 we evaluated the mass of the wire to be $kL^2/2$ g. Its moment about $x = 0$ is

$$\begin{aligned} M_{x=0} &= \int_0^L x\delta(x)\,dx \\ &= \int_0^L kx^2\,dx = \left(\frac{kx^3}{3}\right)\bigg|_0^L = \frac{kL^3}{3} \text{ g·cm.} \end{aligned}$$

(Note that the appropriate units for the moment are units of mass times units of distance: in this case gram-centimetres.) The centre of mass of the wire is

$$\bar{x} = \frac{kL^3/3}{kL^2/2} = \frac{2L}{3}.$$

The wire will be balanced if suspended at position $x = 2L/3$ cm.

Two- and Three-Dimensional Examples

The system of mass considered in Example 5 is one-dimensional and lies along a straight line. If mass is distributed in a plane or in space, similar considerations prevail. For a system of masses m_1 at (x_1, y_1), m_2 at (x_2, y_2), $\ldots$, m_n at (x_n, y_n), the **moment about** $x = 0$ is

$$M_{x=0} = x_1m_1 + x_2m_2 + \cdots + x_nm_n = \sum_{j=1}^n x_jm_j,$$

and the **moment about** $y = 0$ is

$$M_{y=0} = y_1m_1 + y_2m_2 + \cdots + y_nm_n = \sum_{j=1}^n y_jm_j.$$

The **centre of mass** is the point $(\bar{x}, \bar{y})$ where

$$\bar{x} = \frac{M_{x=0}}{m} = \frac{\displaystyle\sum_{j=1}^n x_jm_j}{\displaystyle\sum_{j=1}^n m_j} \quad \text{and} \quad \bar{y} = \frac{M_{y=0}}{m} = \frac{\displaystyle\sum_{j=1}^n y_jm_j}{\displaystyle\sum_{j=1}^n m_j}.$$

For continuous distributions of mass, the sums become appropriate integrals.

EXAMPLE 6 Find the centre of mass of a rectangular plate that occupies the region $0 \le x \le a$, $0 \le y \le b$, if the areal density of the material in the plate at position (x, y) is $\sigma = ky$.

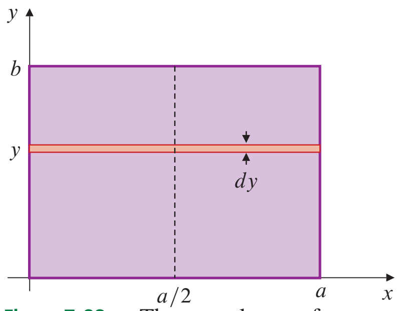


Figure 7.32 The area element for Example 6

Solution Since the areal density is independent of x and the rectangle is symmetric about the line $x = a/2$, the x-coordinate of the centre of mass must be $\bar{x} = a/2$. A thin horizontal strip of width dy at height y (see Figure 7.32) has mass $dm = aky\,dy$. The moment of this strip about $y = 0$ is $dM_{y=0} = y\,dm = kay^2\,dy$. Hence, the mass and moment about $y = 0$ of the whole plate are

$$m = ka\int_0^b y\,dy = \frac{kab^2}{2},$$
$$M_{y=0} = ka\int_0^b y^2\,dy = \frac{kab^3}{3}.$$

Therefore, $\bar{y} = M_{y=0}/m = 2b/3$, and the centre of mass of the plate is $(a/2, 2b/3)$. The plate would be balanced if supported at this point.

For distributions of mass in three-dimensional space one defines, analogously, the moments $M_{x=0}$, $M_{y=0}$, and $M_{z=0}$ of the system of mass about the planes $x = 0$, $y = 0$, and $z = 0$, respectively. The centre of mass is $(\bar{x}, \bar{y}, \bar{z})$ where

$$\bar{x} = \frac{M_{x=0}}{m}, \qquad \bar{y} = \frac{M_{y=0}}{m}, \qquad \text{and} \qquad \bar{z} = \frac{M_{z=0}}{m},$$

m being the total mass: $m = m_1 + m_2 + \cdots + m_n$. Again, the sums are replaced with integrals for continuous distributions of mass.

EXAMPLE 7 Find the centre of mass of a solid hemisphere of radius R ft if its density at height z ft above the base plane of the hemisphere is $\rho_0 z$ lb/ft^3.

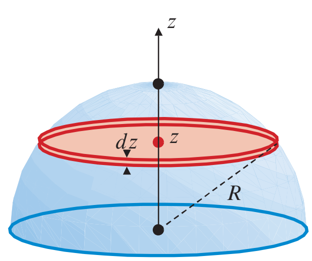


Figure 7.33 Mass element of a solid hemisphere with density depending on height

Solution The solid is symmetric about the vertical axis (let us call it the z-axis), and the density is constant in planes perpendicular to this axis. Therefore, the centre of mass must lie somewhere on this axis. A slice of the solid at height z above the base, and having thickness dz, is a disk of radius $\sqrt{R^2 - z^2}$. (See Figure 7.33.) Its volume is $dV = \pi(R^2 - z^2)\,dz$, and its mass is $dm = \rho_0 z\,dV = \rho_0\pi(R^2 z - z^3)\,dz$. Its moment about the base plane $z = 0$ is $dM_{z=0} = z\,dm = \rho_0\pi(R^2 z^2 - z^4)\,dz$. The mass of the solid is

$$m = \rho_0\pi\int_0^R (R^2 z - z^3)\,dz = \rho_0\pi\left(\frac{R^2 z^2}{2} - \frac{z^4}{4}\right)\bigg|_0^R = \frac{\pi}{4}\rho_0 R^4 \text{ lb}.$$

The moment of the hemisphere about the plane $z = 0$ is

$$M_{z=0} = \rho_0\pi\int_0^R (R^2 z^2 - z^4)\,dz = \rho_0\pi\left(\frac{R^2 z^3}{3} - \frac{z^5}{5}\right)\bigg|_0^R = \frac{2\pi}{15}\rho_0 R^5 \text{ lb·ft}.$$

The centre of mass therefore lies along the axis of symmetry of the hemisphere at height $\bar{z} = M_{z=0}/m = 8R/15$ ft above the base of the hemisphere.

EXAMPLE 8 Find the centre of mass of a plate that occupies the region $a \le x \le b$, $0 \le y \le f(x)$, if the density at any point (x, y) is $\sigma(x)$.

Solution The appropriate area element is shown in Figure 7.34. It has area $f(x)\,dx$, mass

$$dm = \sigma(x)\,f(x)\,dx,$$

and moment about $x = 0$

$$dM_{x=0} = x\sigma(x) f(x)\, dx.$$

Since the density depends only on x, the mass element dm has constant density, so the y-coordinate of *its* centre of mass is at its midpoint: $\bar{y}_{dm} = \frac{1}{2} f(x)$. Therefore, the moment of the mass element dm about $y = 0$ is

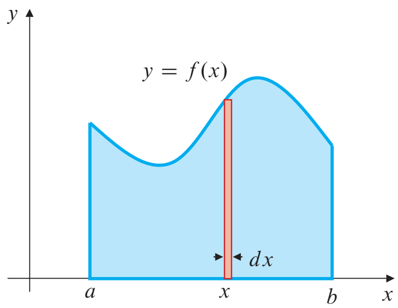


Figure 7.34 Mass element of a plate

$$dM_{y=0} = \bar{y}_{dm}\, dm = \frac{1}{2}\sigma(x)\big(f(x)\big)^2\, dx.$$

The coordinates of the centre of mass of the plate are $\bar{x} = \dfrac{M_{x=0}}{m}$ and $\bar{y} = \dfrac{M_{y=0}}{m}$, where

$$m = \int_a^b \sigma(x) f(x)\, dx,$$
$$M_{x=0} = \int_a^b x\sigma(x) f(x)\, dx,$$
$$M_{y=0} = \frac{1}{2}\int_a^b \sigma(x)\big(f(x)\big)^2\, dx.$$

Remark Similar formulas can be obtained if the density depends on y instead of x, provided that the region admits a suitable horizontal area element (e.g., the region might be specified by $c \le y \le d$, $0 \le x \le g(y)$). Finding centres of mass for plates that occupy regions specified by functions of x, but where the density depends on y, generally requires the use of "double integrals." Such problems are therefore studied in multivariable calculus. (See Section 14.7.)

EXERCISES 7.4

Find the mass and centre of mass for the systems in Exercises 1–16. Be alert for symmetries.

1. A straight wire of length L cm, where the density at distance s cm from one end is $\delta(s) = \sin \pi s/L$ g/cm
2. A straight wire along the x-axis from $x = 0$ to $x = L$ if the density is constant δ_0, but the cross-sectional radius of the wire varies so that its value at x is $a + bx$
3. A quarter-circular plate having radius a, constant areal density σ_0, and occupying the region $x^2 + y^2 \le a^2$, $x \ge 0$, $y \ge 0$
4. A quarter-circular plate of radius a occupying the region $x^2 + y^2 \le a^2$, $x \ge 0$, $y \ge 0$, having areal density $\sigma(x) = \sigma_0 x$
5. A plate occupying the region $0 \le y \le 4 - x^2$ if the areal density at (x, y) is ky
6. A right-triangular plate with legs 2 m and 3 m if the areal density at any point P is $5h$ kg/m^2, h being the distance of P from the shorter leg
7. A square plate of edge a cm if the areal density at P is kx g/cm^2, where x is the distance from P to one edge of the square
8. The plate in Exercise 7, but with areal density kr g/cm^2, where r is the distance (in centimetres) from P to one of the diagonals of the square
9. A plate of areal density $\sigma(x)$ occupying the region $a \le x \le b$, $f(x) \le y \le g(x)$
10. A rectangular brick with dimensions 20 cm, 10 cm, and 5 cm if the density at P is kx g/cm^3, where x is the distance from P to one of the 10×5 faces
11. A solid ball of radius R m if the density at P is z kg/m^3, where z is the distance from P to a plane at distance $2R$ m from the centre of the ball
12. A right-circular cone of base radius a cm and height b cm if the density at point P is kz g/cm^3, where z is the distance of P from the base of the cone
13. ! The solid occupying the quarter of a ball of radius a centred at the origin having as base the region $x^2 + y^2 \le a^2$, $x \ge 0$ in the xy-plane, if the density at height z above the base is $\rho_0 z$
14. ! The cone of Exercise 12, but with density at P equal to kx g/cm^3, where x is the distance of P from the axis of symmetry of the cone. *Hint:* Use a cylindrical shell centred on the axis of symmetry as a volume element. This element has constant density, so its centre of mass is known, and its moment can be determined from its mass.

15. A semicircular plate occupying the region $x^2 + y^2 \le a^2$, $y \ge 0$, if the density at distance s from the origin is ks g/cm^2

16. The wire in Exercise 1 if it is bent in a semicircle

17. It is estimated that the density of matter in the neighbourhood of a gas giant star is given by $\rho(r) = Ce^{-kr^2}$, where C and k are positive constants, and r is the distance from the centre of the star. The radius of the star is indeterminate but can be taken to be infinite since $\rho(r)$ decreases very rapidly for large r. Find the approximate mass of the star in terms of C and k.

18. Find the average distance $\bar{r}$ of matter in the star of Exercise 17 from the centre of the star. $\bar{r}$ is given by $\int_0^\infty r\,dm / \int_0^\infty dm$, where dm is the mass element at distance r from the centre of the star.

7.5 Centroids

If matter is distributed uniformly in a system so that the density δ is constant, then that density cancels out of the numerator and denominator in sum or integral expressions for coordinates of the centre of mass. In such cases the centre of mass depends only on the *shape* of the object, that is, on geometric properties of the region occupied by the object, and we call it the **centroid** of the region.

Centroids are calculated using the same formulas as those used for centres of mass, except that the density (being constant) is taken to be unity, so the mass is just the length, area, or volume of the region, and the moments are referred to as **moments of the region**, rather than of any mass occupying the region. If we set $\sigma(x) = 1$ in the formulas obtained in Example 8 of Section 7.4, we obtain the following result:

The centroid of a standard plane region

The centroid of the plane region $a \le x \le b$, $0 \le y \le f(x)$, is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{M_{x=0}}{A}, \quad \bar{y} = \frac{M_{y=0}}{A}, \quad \text{and)}$$
$$A = \int_a^b f(x)\,dx, \; M_{x=0} = \int_a^b x f(x)\,dx, \; M_{y=0} = \frac{1}{2}\int_a^b \big(f(x)\big)^2\,dx.$$

Thus, for example, $\bar{x}$ is the *average value* of the function x over the region.

The centroids of some regions are obvious by symmetry. The centroid of a circular disk or an elliptical disk is at the centre of the disk. The centroid of a rectangle is at the centre also; the centre is the point of intersection of the diagonals. The centroid of any region lies on any axes of symmetry of the region.

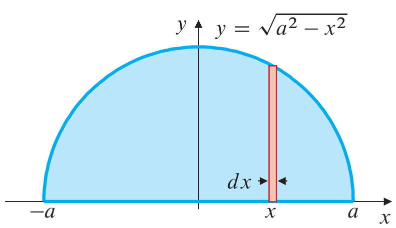


Figure 7.35 The half-disk of Example 1

EXAMPLE 1 What is the average value of y over the half-disk $-a \le x \le a$, $0 \le y \le \sqrt{a^2 - x^2}$? Find the centroid of the half-disk.

Solution By symmetry, the centroid lies on the y-axis, so its x-coordinate is $\bar{x} = 0$. (See Figure 7.35.) Since the area of the half-disk is $A = \frac{1}{2}\pi a^2$, the average value of y over the half-disk is

$$\bar{y} = \frac{M_{y=0}}{A} = \frac{2}{\pi a^2}\,\frac{1}{2}\int_{-a}^{a}(a^2 - x^2)\,dx = \frac{2}{\pi a^2}\,\frac{2a^3}{3} = \frac{4a}{3\pi}.$$

The centroid of the half-disk is $\left(0, \dfrac{4a}{3\pi}\right)$.

EXAMPLE 2 Find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$.

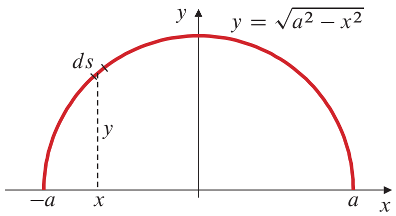


Figure 7.36 The semicircle of Example 2

Solution Here, the "region" is a one-dimensional curve, having length rather than area. Again $\bar{x} = 0$ by symmetry. A short arc of length ds at height y on the semicircle has moment $dM_{y=0} = y\,ds$ about $y = 0$. (See Figure 7.36.) Since

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \sqrt{1 + \frac{x^2}{a^2 - x^2}}\,dx = \frac{a\,dx}{\sqrt{a^2 - x^2}},$$

and since $y = \sqrt{a^2 - x^2}$ on the semicircle, we have

$$M_{y=0} = \int_{-a}^{a} \sqrt{a^2 - x^2}\,\frac{a\,dx}{\sqrt{a^2 - x^2}} = a\int_{-a}^{a} dx = 2a^2.$$

Since the length of the semicircle is πa, we have $\bar{y} = \dfrac{M_{y=0}}{\pi a} = \dfrac{2a}{\pi}$, and the centroid of the semicircle is $\left(0, \dfrac{2a}{\pi}\right)$. Note that the centroid of a semicircle of radius a is not the same as that of half-disk of radius a. Note also that the centroid of the semicircle does not lie on the semicircle itself.

THEOREM 1

The centroid of a triangle

The centroid of a triangle is the point at which all three medians of the triangle intersect.

PROOF Recall that a median of a triangle is a straight line joining one vertex of the triangle to the midpoint of the opposite side. Given any median of a triangle, we will show that the centroid lies on that median. Thus, the centroid must lie on all three medians.

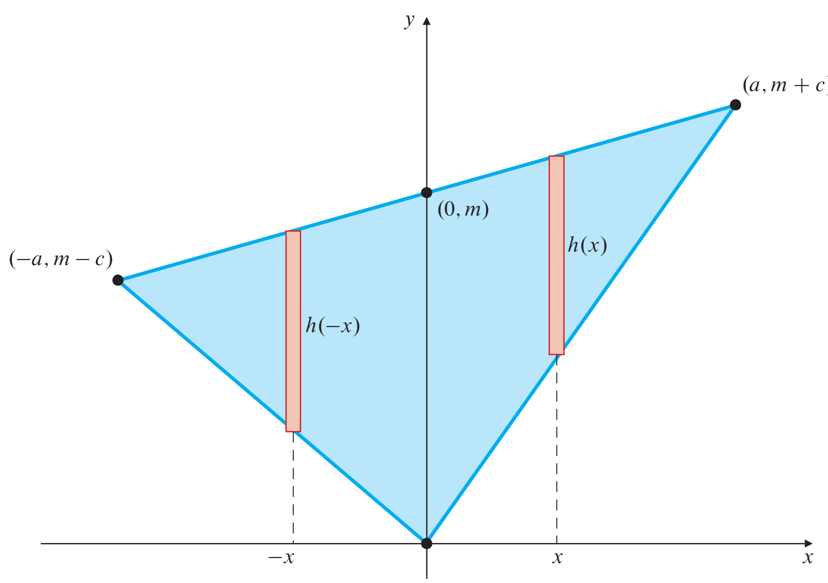


Figure 7.37 The axes of Theorem 1

Adopt a coordinate system where the median in question lies along the y-axis and such that a vertex of the triangle is at the origin. (See Figure 7.37.) Let the midpoint of the opposite side be $(0, m)$. Then the other two vertices of the triangle must have coordinates of the form $(-a, m - c)$ and $(a, m + c)$ so that $(0, m)$ will be the midpoint between them. The two vertical area elements shown in the figure are at the same distance on opposite sides of the y-axis, so they have the same heights $h(-x) = h(x)$ (by similar triangles) and the same area. The sum of the moments about $x = 0$ of these area elements is

$$dM_{x=0} = -xh(-x)\,dx + xh(x)\,dx = 0,$$

so the moment of the whole triangle about $x = 0$ is

$$M_{x=0} = \int_{x=-a}^{x=a} dM_{x=0} = 0.$$

Therefore, the centroid of the triangle lies on the y-axis.

Remark By simultaneously solving the equations of any two medians of a triangle, we can verify the following formula:

Coordinates of the centroid of a triangle

The coordinates of the centroid of a triangle are the averages of the corresponding coordinates of the three vertices of the triangle. The triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) has centroid

$$(\bar{x}, \bar{y}) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3} \right).$$

If a region is a union of nonoverlapping subregions, then any moment of the region is the sum of the corresponding moments of the subregions. This fact enables us to calculate the centroid of the region if we know the centroids and areas of all the subregions.

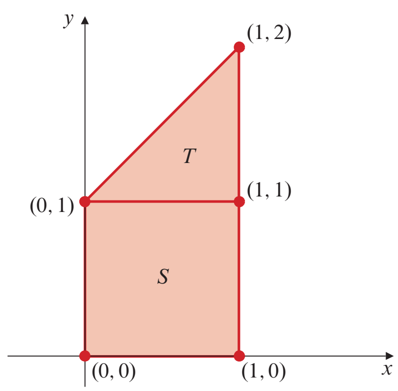


Figure 7.38 The trapezoid of Example 3

EXAMPLE 3 Find the centroid of the trapezoid with vertices $(0, 0)$, $(1, 0)$, $(1, 2)$, and $(0, 1)$.

Solution The trapezoid is the union of a square and a (nonoverlapping) triangle, as shown in Figure 7.38. By symmetry, the square has centroid $(\bar{x}_S, \bar{y}_S) = \left(\frac{1}{2}, \frac{1}{2}\right)$, and its area is $A_S = 1$. The triangle has area $A_T = \frac{1}{2}$, and its centroid is $(\bar{x}_T, \bar{y}_T)$, where

$$\bar{x}_T = \frac{0 + 1 + 1}{3} = \frac{2}{3} \qquad \text{and} \qquad \bar{y}_T = \frac{1 + 1 + 2}{3} = \frac{4}{3}.$$

Continuing to use subscripts S and T to denote the square and triangle, respectively, we calculate

$$M_{x=0} = M_{S;x=0} + M_{T;x=0} = A_S \bar{x}_S + A_T \bar{x}_T = 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} = \frac{5}{6},$$
$$M_{y=0} = M_{S;y=0} + M_{T;y=0} = A_S \bar{y}_S + A_T \bar{y}_T = 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{4}{3} = \frac{7}{6}.$$

Since the area of the trapezoid is $A = A_S + A_T = \frac{3}{2}$, its centroid is

$$(\bar{x}, \bar{y}) = \left(\frac{5}{6} \Big/ \frac{3}{2}, \frac{7}{6} \Big/ \frac{3}{2} \right) = \left(\frac{5}{9}, \frac{7}{9} \right).$$

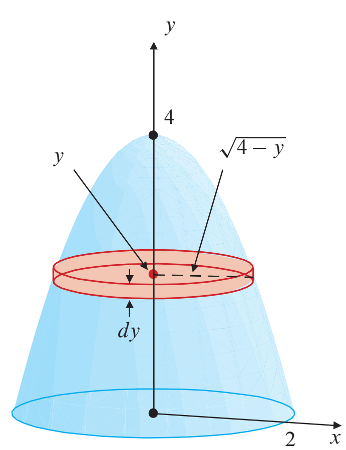


Figure 7.39 A parabolic solid

EXAMPLE 4 Find the centroid of the solid region obtained by rotating about the y-axis the first quadrant region lying between the x-axis and the parabola $y = 4 - x^2$.

Solution By symmetry, the centroid of the parabolic solid will lie on its axis of symmetry, the y-axis. A thin, disk-shaped slice of the solid at height y and having thickness dy (see Figure 7.39) has volume

$$dV = \pi x^2 \, dy = \pi(4 - y) \, dy$$

and moment about the base plane

$$dM_{y=0} = y\,dV = \pi(4y - y^2)\,dy.$$

Hence, the volume of the solid is

$$V = \pi \int_0^4 (4 - y)\,dy = \pi\left(4y - \frac{y^2}{2}\right)\bigg|_0^4 = \pi(16 - 8) = 8\pi,$$

and its moment about $y = 0$ is

$$M_{y=0} = \pi \int_0^4 (4y - y^2)\,dy = \pi\left(2y^2 - \frac{y^3}{3}\right)\bigg|_0^4 = \pi\left(32 - \frac{64}{3}\right) = \frac{32}{3}\pi.$$

Hence, the centroid is located at $\bar{y} = \dfrac{32\pi}{3} \times \dfrac{1}{8\pi} = \dfrac{4}{3}$.

Pappus's Theorem

The following theorem relates volumes or surface areas of revolution to the centroid of the region or curve being rotated.

THEOREM 2

Pappus's Theorem

(a) If a plane region R lies on one side of a line L in that plane and is rotated about L to generate a solid of revolution, then the volume V of that solid is the product of the area of R and the distance travelled by the centroid of R under the rotation; that is,

$$V = 2\pi\bar{r}A,$$

where A is the area of R, and $\bar{r}$ is the perpendicular distance from the centroid of R to L.

(b) If a plane curve $\mathcal{C}$ lies on one side of a line L in that plane and is rotated about that line to generate a surface of revolution, then the area S of that surface is the length of $\mathcal{C}$ times the distance travelled by the centroid of $\mathcal{C}$:

$$S = 2\pi\bar{r}s,$$

where s is the length of the curve $\mathcal{C}$, and $\bar{r}$ is the perpendicular distance from the centroid of $\mathcal{C}$ to the line L.

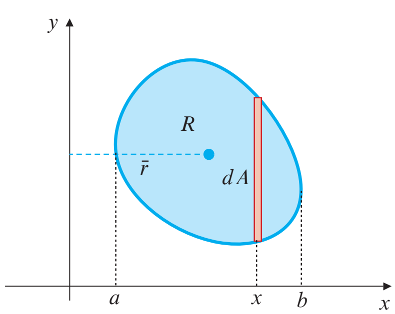


Figure 7.40 Proving Theorem 2(a)

PROOF We prove part (a). The proof of (b) is similar and is left as an exercise.

Let us take L to be the y-axis and suppose that R lies between $x = a$ and $x = b$ where $0 \le a < b$. Thus $\bar{r} = \bar{x}$, the x-coordinate of the centroid of R. Let dA denote the area of a thin strip of R at position x and having width dx. (See Figure 7.40.) This strip generates, on rotation about L, a cylindrical shell of volume $dV = 2\pi x\,dA$, so the volume of the solid of revolution is

$$V = 2\pi \int_{x=a}^{x=b} x\,dA = 2\pi M_{x=0} = 2\pi\bar{x}A = 2\pi\bar{r}A.$$

As the following examples illustrate, Pappus's Theorem can be used in two ways: either the centroid can be determined when the appropriate volume or surface area is known, or the volume or surface area can be determined if the centroid of the rotating region or curve is known.

EXAMPLE 5 Use Pappus's Theorem to find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$.

Solution The centroid of the semicircle lies on its axis of symmetry, the y-axis, so it is located at a point with coordinates $(0, \bar{y})$. Since the semicircle has length πa units and generates, on rotation about the x-axis, a sphere having area $4\pi a^2$ square units, we obtain, using part (b) of Pappus's Theorem,

$$4\pi a^2 = 2\pi(\pi a)\bar{y}.$$

Thus $\bar{y} = 2a/\pi$, as shown previously in Example 2.

EXAMPLE 6 Use Pappus's Theorem to find the volume and surface area of the torus (doughnut) obtained by rotating the disk $(x-b)^2 + y^2 \le a^2$ about the y-axis. Here $0 < a < b$. (See Figure 7.10 in Section 7.1.)

Solution The centroid of the disk is at $(b, 0)$, which is at distance $\bar{r} = b$ units from the axis of rotation. Since the disk has area πa^2 square units, the volume of the torus is

$$V = 2\pi b(\pi a^2) = 2\pi^2 a^2 b \text{ cubic units.}$$

To find the surface area S of the torus (in case you want to have icing on the doughnut), rotate the circular boundary of the disk, which has length $2\pi a$, about the y-axis and obtain

$$S = 2\pi b(2\pi a) = 4\pi^2 ab \text{ square units.}$$

EXERCISES 7.5

Find the centroids of the geometric structures in Exercises 1–21. Be alert for symmetries and opportunities to use Pappus's Theorem.

1. The quarter-disk $x^2 + y^2 \le r^2, x \ge 0, y \ge 0$

2. The region $0 \le y \le 9 - x^2$

3. The region $0 \le x \le 1, 0 \le y \le \dfrac{1}{\sqrt{1+x^2}}$

4. The circular disk sector $x^2 + y^2 \le r^2, 0 \le y \le x$

5. The circular disk segment $0 \le y \le \sqrt{4 - x^2} - 1$

6. The semi-elliptic disk $0 \le y \le b\sqrt{1 - (x/a)^2}$

7. The quadrilateral with vertices (in clockwise order) $(0, 0)$, $(3, 1)$, $(4, 0)$, and $(2, -2)$

8. The region bounded by the semicircle $y = \sqrt{1 - (x-1)^2}$, the y-axis, and the line $y = x - 2$

9. A hemispherical surface of radius r

10. A solid half ball of radius r

11. A solid cone of base radius r and height h

12. A conical surface of base radius r and height h

13. The plane region $0 \le y \le \sin x,\ 0 \le x \le \pi$

14. The plane region $0 \le y \le \cos x,\ 0 \le x \le \pi/2$

15. The quarter-circle arc $x^2 + y^2 = r^2, x \ge 0,\ y \ge 0$

16. The solid obtained by rotating the region in Figure 7.41(a) about the y-axis

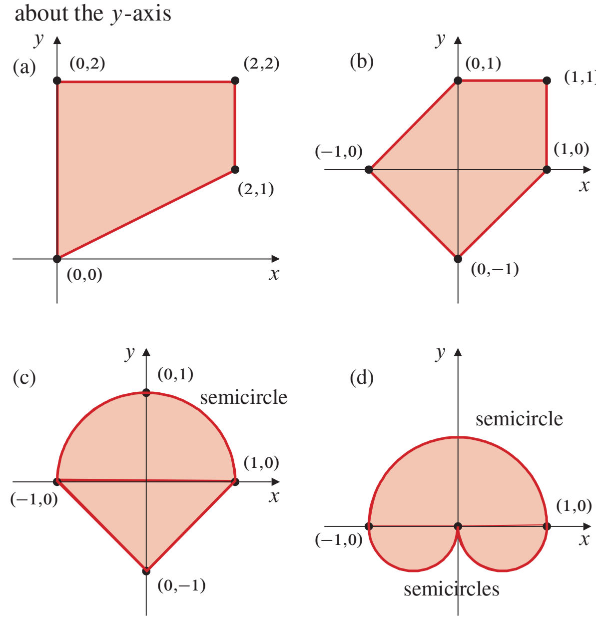


Figure 7.41

17. The region in Figure 7.41(a)

18. The region in Figure 7.41(b)

19. The region in Figure 7.41(c)

20. The region in Figure 7.41(d)

21. The solid obtained by rotating the plane region $0 \le y \le 2x - x^2$ about the line $y = -2$

22. The line segment from $(1, 0)$ to $(0, 1)$ is rotated about the line $x = 2$ to generate part of a conical surface. Find the area of that surface.

23. The triangle with vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$ is rotated about the line $x = 2$ to generate a certain solid. Find the volume of that solid.

24. An equilateral triangle of edge s cm is rotated about one of its edges to generate a solid. Find the volume and surface area of that solid.

25. Find to 5 decimal places the coordinates of the centroid of the region $0 \le x \le \pi/2$, $0 \le y \le \sqrt{x}\cos x$.

26. Find to 5 decimal places the coordinates of the centroid of the region $0 < x \le \pi/2$, $\ln(\sin x) \le y \le 0$.

27. Find the centroid of the infinitely long spike-shaped region lying between the x-axis and the curve $y = (x + 1)^{-3}$ and to the right of the y-axis.

28. Show that the curve $y = e^{-x^2}$ $(-\infty < x < \infty)$ generates a surface of finite area when rotated about the x-axis. What does this imply about the location of the centroid of this infinitely long curve?

29. Obtain formulas for the coordinates of the centroid of the plane region $c \le y \le d$, $0 < f(y) \le x \le g(y)$.

30. Prove part (b) of Pappus's Theorem (Theorem 2).

31. **(Stability of a floating object)** Determining the orientation that a floating object will assume is a problem of critical importance to ship designers. Boats must be designed to float stably in an upright position; if the boat tilts somewhat from upright, the forces on it must be such as to right it again. The two forces on a floating object that need to be taken into account are its weight **W** and the balancing buoyant force **B** $= -$**W**. The weight **W** must be treated for mechanical purposes as being applied at the centre of mass (CM) of the object. The buoyant force, however, acts at the *centre of buoyancy* (CB), which is the centre of mass of the water displaced by the object, and is therefore the centroid of the "hole in the water" made by the object.

For example, consider a channel marker buoy consisting of a hemispherical hull surmounted by a conical tower supporting a navigation light. The buoy has a vertical axis of symmetry. If it is upright, both the CM and the CB lie on this line, as shown in the left half of Figure 7.42.

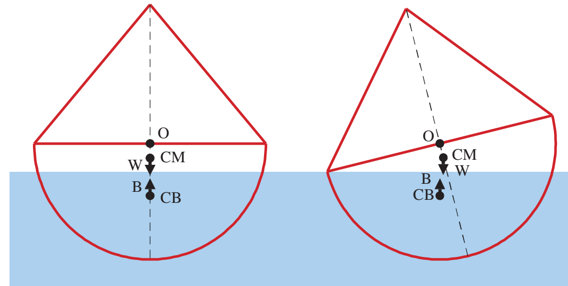


Figure 7.42

Is this upright flotation of the buoy stable? It is if the CM lies below the centre O of the hemispherical hull, as shown in the right half of the figure. To see why, imagine the buoy tilted slightly from the vertical as shown in the right half of the figure. Observe that the CM still lies on the axis of symmetry of the buoy, but the CB lies on the vertical line through O. The forces **W** and **B** no longer act along the same line, but their torques are such as to rotate the buoy back to a vertical upright position. If CM had been above O in the left figure, the torques would have been such as to tip the buoy over once it was displaced even slightly from the vertical.

A wooden beam has a square cross-section and specific gravity 0.5, so that it will float with half of its volume submerged. (See Figure 7.43.) Assuming it will float horizontally in the water, what is the stable orientation of the square cross-section with respect to the surface of the water? In particular, will the beam float with a flat face upward or an edge upward? Prove your assertions. You may find Maple or another symbolic algebra program useful.

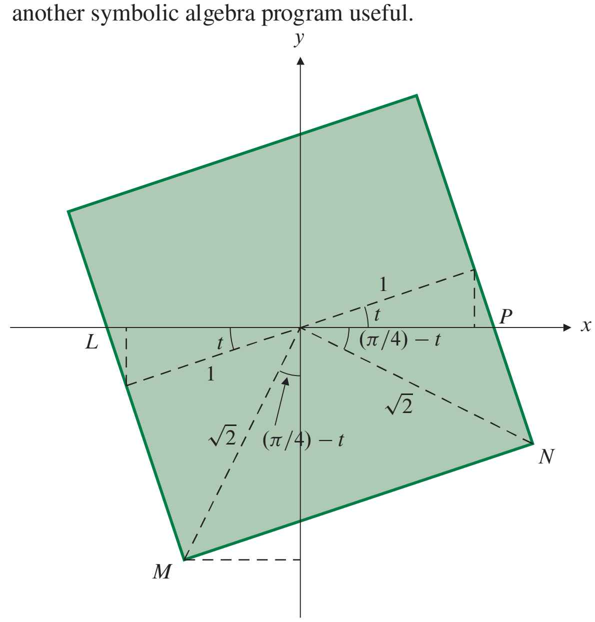


Figure 7.43

7.6 Other Physical Applications

In this section we present some examples of the use of integration to calculate quantities arising in physics and mechanics.

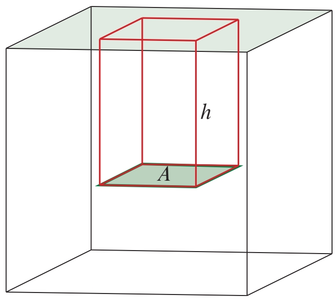


Figure 7.44 The volume of liquid above the area A is $V = Ah$. The weight of this liquid is $\rho V g = \rho g h A$, so the pressure (force per unit area) at depth h is $p = \rho g h$

Hydrostatic Pressure

The **pressure** p at depth h beneath the surface of a liquid is the *force per unit area* exerted on a horizontal plane surface at that depth due to the weight of the liquid above it. Hence, p is given by

$$p = \rho g h,$$

where ρ is the density of the liquid, and g is the acceleration produced by gravity where the fluid is located. (See Figure 7.44.) For water at the surface of the earth we have, approximately, $\rho = 1{,}000$ kg/m^3 and $g = 9.8$ m/s^2, so the pressure at depth h m is

$$p = 9{,}800h \text{ N/m}^2.$$

The unit of force used here is the newton (N); 1 N = 1 kg·m/s^2, the force that imparts an acceleration of 1 m/s^2 to a mass of 1 kg.

The molecules in a liquid interact in such a way that the pressure at any depth acts equally in all directions; the pressure against a vertical surface is the same as that against a horizontal surface at the same depth. This is **Pascal's principle**.

The total force exerted by a liquid on a horizontal surface (say, the bottom of a tank holding the liquid) is found by multiplying the area of that surface by the pressure at the depth of the surface below the top of the liquid. For nonhorizontal surfaces, however, the pressure is not constant over the whole surface, and the total force cannot be determined so easily. In this case we divide the surface into area elements dA, each at some particular depth h, and we then sum (i.e., integrate) the corresponding force elements $dF = \rho g h\, dA$ to find the total force.

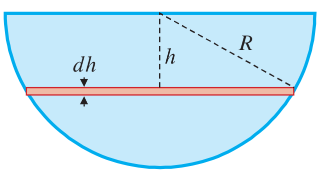


Figure 7.45 An end plate of the water trough

EXAMPLE 1 One vertical wall of a water trough is a semicircular plate of radius R m with curved edge downward. If the trough is full, so that the water comes up to the top of the plate, find the total force of the water on the plate.

Solution A horizontal strip of the surface of the plate at depth h m and having width dh m (see Figure 7.45) has length $2\sqrt{R^2 - h^2}$ m; hence, its area is $dA = 2\sqrt{R^2 - h^2}\, dh$ m^2. The force of the water on this strip is

$$dF = \rho g h\, dA = 2\rho g h\sqrt{R^2 - h^2}\, dh.$$

Thus, the total force on the plate is

$$\begin{aligned} F &= \int_{h=0}^{h=R} dF = 2\rho g \int_0^R h\sqrt{R^2 - h^2}\, dh \qquad \text{Let } u = R^2 - h^2,\ du = -2h\, dh \\ &= \rho g \int_0^{R^2} u^{1/2}\, du = \rho g \left.\frac{2}{3}u^{3/2}\right|_0^{R^2} \\ &\approx \frac{2}{3} \times 9{,}800R^3 \approx 6{,}533R^3 \text{ N.} \end{aligned}$$

EXAMPLE 2 **(Force on a dam)** Find the total force on a section of a dam 100 m long and having a vertical height of 10 m, if the surface holding back the water is inclined at an angle of 30° to the vertical and the water comes up to the top of the dam.

Solution The water in a horizontal layer of thickness dh m at depth h m makes contact with the dam along a slanted strip of width $dh \sec 30^\circ = (2/\sqrt{3})\, dh$ m. (See Figure 7.46.) The area of this strip is $dA = (200/\sqrt{3})\, dh$ m^2, and the force of water against the strip is

$$dF = \rho g h\, dA = \frac{200}{\sqrt{3}} \times 1{,}000 \times 9.8h\, dh \approx 1{,}131{,}600h\, dh \text{ N.}$$

The total force on the dam section is therefore

$$F \approx 1{,}131{,}600 \int_0^{10} h\,dh = 1{,}131{,}600 \times \frac{10^2}{2} \approx 5.658 \times 10^7 \text{ N}.$$

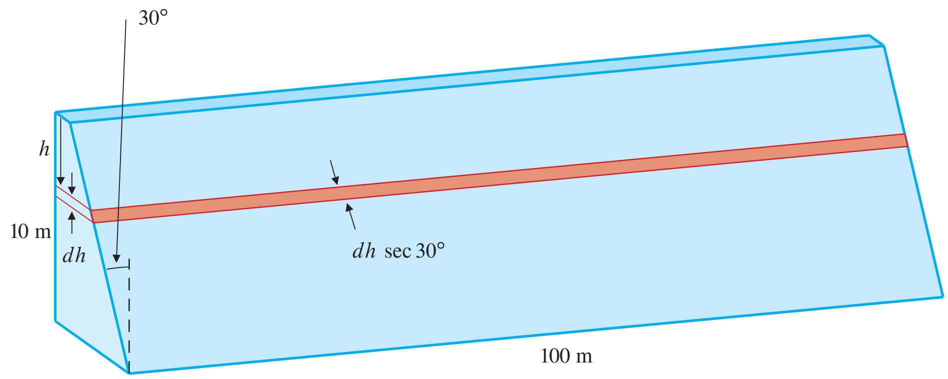


Figure 7.46 The dam of Example 2

Work

When a force acts on an object to move that object, it is said to have done **work** on the object. The amount of work done by a constant force is measured by the product of the force and the distance through which it moves the object. This assumes that the force is in the direction of the motion.

$$\text{work} = \text{force} \times \text{distance}$$

Work is always related to a particular force. If other forces acting on an object cause it to move in a direction opposite to the force F, then work is said to have been done *against* the force F.

Suppose that a force in the direction of the x-axis moves an object from $x = a$ to $x = b$ on that axis and that the force varies continuously with the position x of the object; that is, $F = F(x)$ is a continuous function. The element of work done by the force in moving the object through a very short distance from x to $x + dx$ is $dW = F(x)\,dx$, so the total work done by the force is

$$W = \int_{x=a}^{x=b} dW = \int_a^b F(x)\,dx.$$

EXAMPLE 3 **(Stretching or compressing a spring)** By **Hooke's Law**, the force $F(x)$ required to extend (or compress) an elastic spring to x units longer (or shorter) than its natural length is proportional to x:

$$F(x) = kx,$$

where k is the **spring constant** for the particular spring. If a force of 2,000 N is required to extend a certain spring to 4 cm longer than its natural length, how much work must be done to extend it that far?

Solution Since $F(x) = kx = 2{,}000$ N when $x = 4$ cm, we must have $k = 2{,}000/4 = 500$ N/cm. The work done in extending the spring 4 cm is

$$W = \int_0^4 kx\,dx = k\frac{x^2}{2}\bigg|_0^4 = 500\ \frac{\text{N}}{\text{cm}} \times \frac{4^2\ \text{cm}^2}{2} = 4{,}000 \text{ N·cm} = 40 \text{ N·m}.$$

Forty newton-metres (joules) of work must be done to stretch the spring 4 cm.

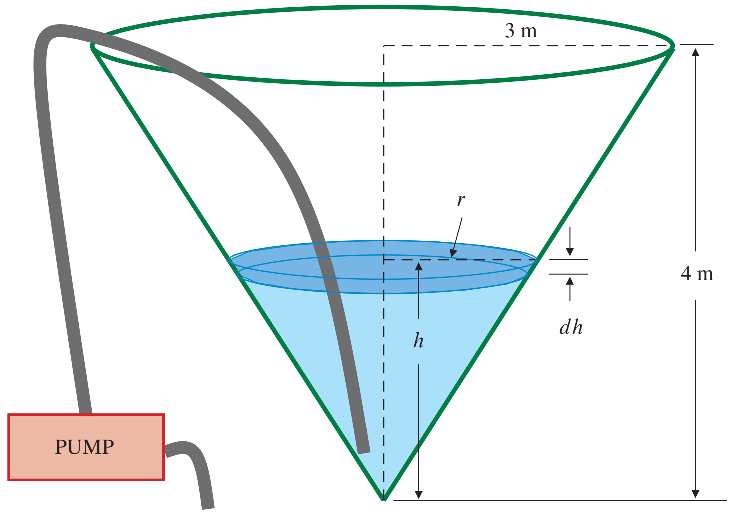


Figure 7.47 Pumping water out of a conical tank

EXAMPLE 4 **(Work done to pump out a tank)** Water fills a tank in the shape of a right-circular cone with top radius 3 m and depth 4 m. How much work must be done (against gravity) to pump all the water out of the tank over the top edge of the tank?

Solution A thin, disk-shaped slice of water at height h above the vertex of the tank has radius r (see Figure 7.47), where $r = \frac{3}{4}h$ by similar triangles. The volume of this slice is

$$dV = \pi r^2\,dh = \frac{9}{16}\pi h^2\,dh,$$

and its *weight* (the force of gravity on the mass of water in the slice) is

$$dF = \rho g\,dV = \frac{9}{16}\rho g\,\pi h^2\,dh.$$

The water in this disk must be raised (against gravity) a distance $(4 - h)$ m by the pump. The work required to do this is

$$dW = \frac{9}{16}\rho g\,\pi(4-h)h^2\,dh.$$

The total work that must be done to empty the tank is the sum (integral) of all these elements of work for disks at depths between 0 and 4 m:

$$\begin{aligned} W &= \int_0^4 \frac{9}{16}\rho g\,\pi(4h^2 - h^3)\,dh \\ &= \frac{9}{16}\rho g\,\pi\left(\frac{4h^3}{3} - \frac{h^4}{4}\right)\Bigg|_0^4 \\ &= \frac{9\pi}{16} \times 1{,}000 \times 9.8 \times \frac{64}{3} \approx 3.69 \times 10^5 \text{ N·m}. \end{aligned}$$

EXAMPLE 5 **(Work to raise material into orbit)** The gravitational force of the earth on a mass m located at height h above the surface of the earth is given by

$$F(h) = \frac{Km}{(R+h)^2},$$

where R is the radius of the earth and K is a constant that is independent of m and h. Determine, in terms of K and R, the work that must be done against gravity to raise an object from the surface of the earth to:

(a) a height H above the surface of the earth, and

(b) an infinite height above the surface of the earth.

Solution The work done to raise the mass m from height h to height $h + dh$ is

$$dW = \frac{Km}{(R+h)^2}\,dh.$$

(a) The total work to raise it from height $h = 0$ to height $h = H$ is

$$W = \int_0^H \frac{Km}{(R+h)^2}\,dh = \left.\frac{-Km}{R+h}\right|_0^H = Km\left(\frac{1}{R} - \frac{1}{R+H}\right).$$

If R and H are measured in metres and F is measured in newtons, then W is measured in newton-metres (N·m), or joules.

(b) The total work necessary to raise the mass m to an infinite height is

$$W = \int_0^\infty \frac{Km}{(R+h)^2}\,dh = \lim_{H\to\infty} Km\left(\frac{1}{R} - \frac{1}{R+H}\right) = \frac{Km}{R}.$$

EXAMPLE 6 One end of a horizontal tank with cross-section a square of edge length L metres is fixed while the other end is a square piston free to travel without friction along the length of the tank. Between the piston and the fixed end there is some water in the tank; its depth depends on the position of the piston. (See Figure 7.48.)

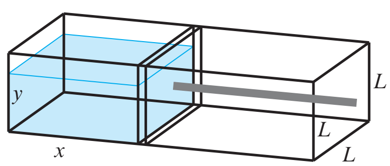


Figure 7.48 The piston in Example 6

(a) When the depth of the water is y metres ($0 \le y \le L$), what force does it exert on the piston?

(b) If the piston is X metres from the fixed end of the tank when the water depth is $L/2$ metres, how much work must be done to force the piston in further to halve that distance and hence cause the water level to increase to fill the available space? Assume no water leaks out but that trapped air can escape from the top of the tank.

Solution

(a) When the depth of water in the tank is y m, a horizontal strip on the face of the piston at depth z below the surface of the water ($0 \le z \le y$) and having height dz has area $dA = L\,dz$. Since the pressure at depth z is $\rho g z = 9{,}800z$ N/m^2, the force of the water on the strip is $dF = 9{,}800\,Lz\,dz$ N. Thus, the force on the piston is

$$F = \int_0^y 9{,}800\,L\,z\,dz = 4{,}900\,L\,y^2 \text{ N}, \quad \text{where } 0 \le y \le L.$$

(b) If the distance from the fixed end of the tank to the piston is x m when the water depth is y m, then the volume of water in the tank is $V = Lxy$ m^3. But we are given that $V = L^2X/2$, so we have $u = LX/2$. Now the work done in moving the piston from x to $x - dx$ is

$$dW = 4{,}900\,Ly^2(-dx) = -4{,}900\,L\,\frac{L^2X^2}{4x^2}\,dx.$$

Thus, the work done to move the piston from position X to position $X/2$ is

$$\begin{aligned} W &= -\int_X^{X/2} 4{,}900\,\frac{L^3X^2}{4}\,\frac{dx}{x^2} \\ &= 4{,}900\,\frac{L^3X^2}{4}\left(\frac{2}{X} - \frac{1}{X}\right) = 1{,}225\,\text{N}\cdot\text{m}. \end{aligned}$$

Potential Energy and Kinetic Energy

The units of energy are the same as those of work (force $\times$ distance). Work done against a force may be regarded as storing up energy for future use or for conversion to other forms. Such stored energy is called **potential energy** (P.E.). For instance, in extending or compressing an elastic spring, we are doing work against the tension in the spring and hence storing energy in the spring. When work is done against a (variable) force $F(x)$ to move an object from $x = a$ to $x = b$, the energy stored is

$$\text{P.E.} = -\int_a^b F(x)\,dx.$$

Since the work is being done against F, the signs of $F(x)$ and $b - a$ are opposite, so the integral is negative; the explicit negative sign is included so that the calculated potential energy will be positive.

One of the forms of energy into which potential energy can be converted is **kinetic energy** (K.E.), the energy of motion. If an object of mass m is moving with velocity v, it has kinetic energy

$$\text{K.E.} = \frac{1}{2}m\,v^2.$$

For example, if an object is raised and then dropped, it accelerates downward under gravity as more and more of the potential energy stored in it when it was raised is converted to kinetic energy.

Consider the change in potential energy stored in a mass m as it moves along the x-axis from a to b under the influence of a force $F(x)$ depending only on x:

$$\text{P.E.}(b) - \text{P.E.}(a) = -\int_a^b F(x)\,dx.$$

(The change in P.E. is negative if m is moving in the direction of F.) According to Newton's Second Law of Motion, the force $F(x)$ causes the mass m to accelerate, with acceleration dv/dt given by

$$F(x) = m\frac{dv}{dt} \qquad \text{(force = mass} \times \text{acceleration).}$$

By the Chain Rule we can rewrite dv/dt in the form

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},$$

so $F(x) = mv\dfrac{dv}{dx}$. Hence,

$$\begin{aligned}\text{P.E.}(b) - \text{P.E.}(a) &= -\int_a^b mv\frac{dv}{dx}\,dx \\ &= -m\int_{x=a}^{x=b} v\,dv \\ &= -\frac{1}{2}mv^2\Big|_{x=a}^{x=b} \\ &= \text{K.E.}(a) - \text{K.E.}(b).\end{aligned}$$

It follows that

$$\text{P.E.}(b) + \text{K.E.}(b) = \text{P.E.}(a) + \text{K.E.}(a).$$

This shows that the total energy (potential + kinetic) remains constant as the mass m moves under the influence of a force F, *depending only on position.* Such a force is said to be **conservative**, and the above result is called the **Law of Conservation of Energy**. Conservative forces will be further discussed in Section 15.2.

EXAMPLE 7 **(Escape velocity)** Use the result of Example 5 together with the following known values,

(a) the radius R of the earth is about 6,400 km, or 6.4×10^6 m,

(b) the acceleration of gravity g at the surface of the earth is about 9.8 m/s^2,

to determine the constant K in the gravitational force formula of Example 5, and use this information to determine the escape velocity for a projectile fired vertically from the surface of the earth. The **escape velocity** is the (minimum) speed that such a projectile must have at firing to ensure that it will continue to move farther and farther away from the earth and not fall back.

Solution According to the formula of Example 5, the force of gravity on a mass m kg at the surface of the earth ($h = 0$) is

$$F = \frac{Km}{(R+0)^2} = \frac{Km}{R^2}.$$

According to Newton's Second Law of Motion, this force is related to the acceleration of gravity (g) there by the equation $F = mg$. Thus,

$$\frac{Km}{R^2} = mg \qquad \text{and} \qquad K = gR^2.$$

According to the Law of Conservation of Energy, the projectile must have sufficient kinetic energy at firing to do the work necessary to raise the mass m to infinite height. By the result of Example 5, this required energy is Km/R. If the initial velocity of the projectile is v, we want

$$\frac{1}{2}mv^2 \ge \frac{Km}{R}.$$

Thus, v must satisfy

$$v \ge \sqrt{\frac{2K}{R}} = \sqrt{2gR} \approx \sqrt{2 \times 9.8 \times 6.4 \times 10^6} \approx 1.12 \times 10^4 \text{ m/s.}$$

Thus, the escape velocity is approximately 11.2 km/s and is independent of the mass m. In this calculation we have neglected any air resistance near the surface of the earth. Such resistance depends on velocity rather than on position, so it is not a conservative force. The effect of such resistance would be to use up (convert to heat) some of the initial kinetic energy and so raise the escape velocity.

EXERCISES 7.6

1. A tank has a square base 2 m on each side and vertical sides 6 m high. If the tank is filled with water, find the total force exerted by the water (a) on the bottom of the tank and (b) on one of the four vertical walls of the tank.

2. A swimming pool 20 m long and 8 m wide has a sloping plane bottom so that the depth of the pool is 1 m at one end and 3 m at the other end. Find the total force exerted on the bottom if the pool is full of water.

3. A dam 200 m long and 24 m high presents a sloping face of 26 m slant height to the water in a reservoir behind the dam (Figure 7.49). If the surface of the water is level with the top of the dam, what is the total force of the water on the dam?

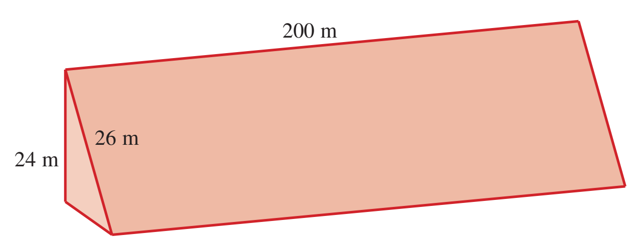


Figure 7.49

4. A pyramid with a square base, 4 m on each side and four equilateral triangular faces, sits on the level bottom of a lake at a place where the lake is 10 m deep. Find the total force of the water on each of the triangular faces.

5. A lock on a canal has a gate in the shape of a vertical rectangle 5 m wide and 20 m high. If the water on one side of the gate comes up to the top of the gate, and the water on the other side comes only 6 m up the gate, find the total force that must be exerted to hold the gate in place.

6. If 100 N·cm of work must be done to compress an elastic spring to 3 cm shorter than its natural length, how much work must be done to compress it 1 cm further?

7. Find the total work that must be done to pump all the water in the tank of Exercise 1 out over the top of the tank.

8. Find the total work that must be done to pump all the water in the swimming pool of Exercise 2 out over the top edge of the pool.

9. Find the work that must be done to pump all the water in a full hemispherical bowl of radius a m to a height h m above the top of the bowl.

! 10. A horizontal cylindrical tank has radius R m. One end of the tank is a fixed disk, but the other end is a circular piston of radius R m free to travel along the length of the tank. There is some water in the tank between the piston and the fixed end; its depth depends on the position of the piston. What force does the water exert on the piston when the surface of the water is y m ($-R \le y \le R$) above the centre of the piston face? (See Figure 7.50.)

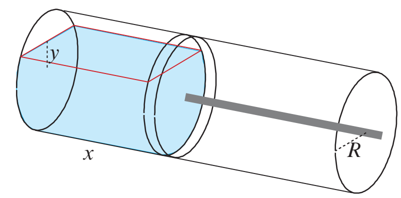


Figure 7.50

! 11. Continuing the previous problem, suppose that when the piston is X m from the fixed end of the tank the water level is at the centre of the piston face. How much work must be done to reduce the distance from the piston to the fixed end to $X/2$ m, and thus cause the water to fill the volume between the piston and the fixed end of the tank? As in Example 6, you can assume the piston can move without friction and that trapped air can escape. *Hint:* The technique used to solve part (b) of Example 6 is very difficult to apply here. Instead, calculate the work done to raise the water in half of the bottom half-cylinder of length X so that it fills the top half-cylinder of length $X/2$.

! 12. A bucket is raised vertically from ground level at a constant speed of 2 m/min by a winch. If the bucket weighs 1 kg and contains 15 kg of water when it starts up but loses water by leakage at a rate of 1 kg/min thereafter, how much work must be done by the winch to raise the bucket to a height of 10 m?

7.7 Applications in Business, Finance, and Ecology

If the rate of change $f'(x)$ of a function $f(x)$ is known, the change in value of the function over an interval from $x = a$ to $x = b$ is just the integral of f' over $[a, b]$:

$$f(b) - f(a) = \int_a^b f'(x)\,dx.$$

For example, if the speed of a moving car at time t is $v(t)$ km/h, then the distance travelled by the car during the time interval $[0, T]$ (hours) is $\int_0^T v(t)\,dt$ km.

Similar situations arise naturally in business and economics, where the rates of change are often called marginals.

EXAMPLE 1 **(Finding total revenue from marginal revenue)** A supplier of calculators realizes a marginal revenue of $\$15 - 5e^{-x/50}$ per calculator when she has sold x calculators. What will be her total revenue from the sale of 100 calculators?

Solution The marginal revenue is the rate of change of revenue with respect to the number of calculators sold. Thus, the revenue from the sale of dx calculators after x have already been sold is

$$dR = (15 - 5e^{-x/50})\,dx \quad \text{dollars.}$$

The total revenue from the sale of the first 100 calculators is $\$R$, where

$$\begin{aligned} R = \int_{x=0}^{x=100} dR &= \int_0^{100} (15 - 5e^{-x/50})\, dx \\ &= \left(15x + 250e^{-x/50}\right)\Big|_0^{100} \\ &= 1,500 + 250e^{-2} - 250 \approx 1,283.83, \end{aligned}$$

that is, about \$1,284.

The Present Value of a Stream of Payments

Suppose that you have a business that generates income continuously at a variable rate $P(t)$ dollars per year at time t and that you expect this income to continue for the next T years. How much is the business worth today?

The answer surely depends on interest rates. One dollar to be received t years from now is worth less than one dollar received today, which could be invested at interest to yield more than one dollar t years from now. The higher the interest rate, the lower the value today of a payment that is not due until sometime in the future.

To analyze this situation, suppose that the nominal interest rate is $r\%$ per annum, but is compounded continuously. Let $\delta = r/100$. As shown in Section 3.4, an investment of \$1 today will grow to

$$\lim_{n\to\infty} \left(1 + \frac{\delta}{n}\right)^{nt} = e^{\delta t} \quad \text{dollars}$$

after t years. Therefore, a payment of \$1 after t years must be worth only $\$e^{-\delta t}$ today. This is called the *present value* of the future payment. When viewed this way, the interest rate δ is frequently called a *discount rate*; it represents the amount by which future payments are discounted.

Returning to the business income problem, in the short time interval from t to $t + dt$, the business produces income $\$P(t)\, dt$, of which the present value is $\$e^{-\delta t} P(t)\, dt$. Therefore, the present value $\$V$ of the income stream over the time interval $[0, T]$ is the "sum" of these contributions:

$$V = \int_0^T e^{-\delta t} P(t)\, dt.$$

EXAMPLE 2 What is the present value of a constant, continual stream of payments at a rate of \$10,000 per year, to continue forever, starting now? Assume an interest rate of 6% per annum, compounded continuously.

Solution The required present value is

$$V = \int_0^\infty e^{-0.06t}\, 10{,}000\, dt = 10{,}000 \lim_{R\to\infty} \frac{e^{-0.06t}}{-0.06}\Bigg|_0^R \approx \$166{,}667.$$

The Economics of Exploiting Renewable Resources

As noted in Section 3.4, the rate of increase of a biological population sometimes conforms to a logistic model[1]

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{L}\right).$$

[1] This example was suggested by Professor C. W. Clark, of the University of British Columbia.

Here, $x = x(t)$ is the size (or biomass) of the population at time t, k is the natural rate at which the population would grow if its food supply were unlimited, and L is the natural limiting size of the population—the carrying capacity of its environment. Such models are thought to apply, for example, to the Antarctic blue whale and to several species of fish and trees. If the resource is harvested (say, the fish are caught) at a rate $h(t)$ units per year at time t, then the population grows at a slower rate:

$$\frac{dx}{dt} = kx\left(1 - \frac{x}{L}\right) - h(t). \tag{$*$}$$

In particular, if we harvest the population at its current rate of growth,

$$h(t) = kx\left(1 - \frac{x}{L}\right),$$

then $dx/dt = 0$, and the population will maintain a constant size. Assume that each unit of harvest produces an income of $\$p$ for the fishing industry. The total annual income from harvesting the resource at its current rate of growth will be

$$T = ph(t) = pkx\left(1 - \frac{x}{L}\right).$$

Considered as a function of x, this total annual income is quadratic and has a maximum value when $x = L/2$, the value that ensures $dT/dx = 0$. The industry can maintain a stable maximum annual income by ensuring that the population level remains at half the maximal size of the population with no harvesting.

The analysis above, however, does not take into account the discounted value of future harvests. If the discount rate is δ, compounded continuously, then the present value of the income $\$ph(t)\,dt$ due between t and $t + dt$ years from now is $e^{-\delta t}ph(t)\,dt$. The total present value of all income from the fishery in future years is

$$T = \int_0^\infty e^{-\delta t}ph(t)\,dt.$$

What fishing strategy will maximize T? If we substitute for $h(t)$ from equation $(*)$ governing the growth rate of the population, we get

$$\begin{aligned} T &= \int_0^\infty pe^{-\delta t}\left[kx\left(1 - \frac{x}{L}\right) - \frac{dx}{dt}\right] dt \\ &= \int_0^\infty kpe^{-\delta t}x\left(1 - \frac{x}{L}\right) dt - \int_0^\infty pe^{-\delta t}\frac{dx}{dt}\,dt. \end{aligned}$$

Integrate by parts in the last integral above, taking $U = pe^{-\delta t}$ and $dV = \dfrac{dx}{dt}\,dt$:

$$\begin{aligned} T &= \int_0^\infty kpe^{-\delta t}x\left(1 - \frac{x}{L}\right) dt - \left[pe^{-\delta t}x\Big|_0^\infty + \int_0^\infty p\delta e^{-\delta t}x\,dt\right] \\ &= px(0) + \int_0^\infty pe^{-\delta t}\left[kx\left(1 - \frac{x}{L}\right) - \delta x\right] dt. \end{aligned}$$

To make this expression as large as possible, we should choose the population size x to maximize the quadratic expression

$$Q(x) = kx\left(1 - \frac{x}{L}\right) - \delta x$$

at as early a time t as possible, and keep the population size constant at that level thereafter. The maximum occurs where $Q'(x) = k - (2kx/L) - \delta = 0$, that is, where

$$x = \frac{L}{2} - \frac{\delta L}{2k} = (k - \delta)\frac{L}{2k}.$$

The maximum present value of the fishery is realized if the population level x is held at this value. Note that this population level is smaller than the optimal level $L/2$ we obtained by ignoring the discount rate. The higher the discount rate δ, the smaller will be the income-maximizing population level. More unfortunately, if $\delta \geq k$, the model predicts greatest income from fishing the species to *extinction* immediately! (See Figure 7.51.)

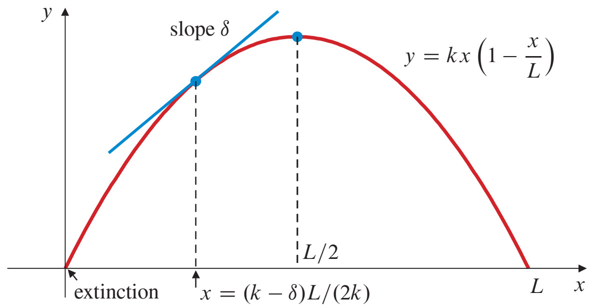


Figure 7.51 The greater the discount rate δ, the smaller the population size x that will maximize the present value of future income from harvesting. If $\delta \geq k$, the model predicts fishing the species to extinction

Of course, this model fails to take into consideration other factors that may affect the fishing strategy, such as the increased cost of harvesting when the population level is small and the effect of competition among various parts of the fishing industry. Nevertheless, it does explain the regrettable fact that, under some circumstances, an industry based on a renewable resource can find it in its best interest to destroy the resource. This is especially likely to happen when the natural growth rate k of the resource is low, as it is for the case of whales and most trees. There is good reason not to allow economics alone to dictate the management of the resource.

EXERCISES 7.7

1. **(Cost of production)** The marginal cost of production in a coal mine is $\$6 - 2 \times 10^{-3}x + 6 \times 10^{-6}x^2$ per ton after the first x tons are produced each day. In addition, there is a fixed cost of \$4,000 per day to open the mine. Find the total cost of production on a day when 1,000 tons are produced.

2. **(Total sales)** The sales of a new computer chip are modelled by $s(t) = te^{-t/10}$, where $s(t)$ is the number of thousands of chips sold per week, t weeks after the chip was introduced to the market. How many chips were sold in the first year?

3. **(Internet connection rates)** An internet service provider charges clients at a continuously decreasing marginal rate of $\$4/(1 + \sqrt{t})$ per hour when the client has already used t hours during a month. How much will be billed to a client who uses x hours in a month? (x need not be an integer.)

4. **(Total revenue from declining sales)** The price per kilogram of maple syrup in a store rises at a constant rate from \$10 at the beginning of the year to \$15 at the end of the year. As the price rises, the quantity sold decreases; the sales rate is $400/(1 + 0.1t)$ kg/year at time t years, $(0 \leq t \leq 1)$. What total revenue does the store obtain from sales of the syrup during the year?

(Stream of payment problems) Find the present value of a continuous stream of payments of \$1,000 per year for the periods and discount rates given in Exercises 5–10. In each case the discount rate is compounded continuously.

5. 10 years at a discount rate of 2%

6. 10 years at a discount rate of 5%

7. 10 years beginning 2 years from now at a discount rate of 8%

8. 25 years beginning 10 years from now at a discount rate of 5%

9. For all future time at a discount rate of 2%

10. Beginning in 10 years and continuing forever after at a discount rate of 5%

11. Find the present value of a continuous stream of payments over a 10-year period beginning at a rate of \$1,000 per year now and increasing steadily at \$100 per year. The discount rate is 5%.

12. Find the present value of a continuous stream of payments over a 10-year period beginning at a rate of \$1,000 per year now and increasing steadily at 10% per year. The discount rate is 5%.

13. Money flows continuously into an account at a rate of \$5,000 per year. If the account earns interest at a rate of 5% compounded continuously, how much will be in the account after 10 years?

14. Money flows continuously into an account beginning at a rate of \$5,000 per year and increasing at 10% per year. Interest causes the account to grow at a real rate of 6% (so that \$1 grows to $\$1.06^t$ in t years). How long will it take for the balance in the account to reach \$1,000,000?

15. If the discount rate δ varies with time, say $\delta = \delta(t)$, show that the present value of a payment of $\$P$ due t years from now is $\$Pe^{-\lambda(t)}$, where

$$\lambda(t) = \int_0^t \delta(\tau)\, d\tau.$$

What is the value of a stream of payments due at a rate $\$P(t)$ at time t, from $t = 0$ to $t = T$?

16. **(Discount rates and population models)** Suppose that the growth rate of a population is a function of the population size: $dx/dt = F(x)$. (For the logistic model, $F(x) = kx(1 - (x/L))$.) If the population is harvested at rate $h(t)$ at time t, then $x(t)$ satisfies

$$\frac{dx}{dt} = F(x) - h(t).$$

Show that the value of x that maximizes the present value of all future harvests satisfies $F'(x) = \delta$, where δ is the (continuously compounded) discount rate. *Hint:* Mimic the argument used above for the logistic case.

17. **(Managing a fishery)** The carrying capacity of a certain lake is $L = 80{,}000$ of a certain species of fish. The natural growth rate of this species is 12% per year ($k = 0.12$). Each fish is worth \$6. The discount rate is 5%. What population of fish should be maintained in the lake to maximize the present value of all future revenue from harvesting the fish? What is the annual revenue resulting from maintaining this population level?

18. **(Blue whales)** It is speculated that the natural growth rate of the Antarctic blue whale population is about 2% per year ($k = 0.02$) and that the carrying capacity of its habitat is about $L = 150{,}000$. One blue whale is worth, on average, \$10,000. Assuming that the blue whale population satisfies a logistic model, and using the data above, find the following:

(a) The maximum sustainable annual harvest of blue whales.

(b) The annual revenue resulting from the maximum annual sustainable harvest.

(c) The annual interest generated if the whale population (assumed to be at the level $L/2$ supporting the maximum sustainable harvest) is exterminated and the proceeds invested at 2%. (d) at 5%.

(e) The total present value of all future revenue if the population is maintained at the level $L/2$ and the discount rate is 5%.

! **19.** The model developed above does not allow for the costs of harvesting. Try to devise a way to alter the model to take this into account. Typically, the cost of catching a fish goes up as the number of fish goes down.

7.8 Probability

Probability theory is a very important field of application of calculus. This subject cannot, of course, be developed thoroughly here—an adequate presentation requires one or more whole courses—but we can give a brief introduction that suggests some of the ways sums and integrals are used in probability theory.

In the context of probability theory the term **experiment** is used to denote a process that can result in different **outcomes**. A particular outcome is also called a **realization**. The set of all possible outcomes is called the **sample space** for the experiment. For example, the process might be the tossing of a coin for which we could have three possible outcomes: H (the coin lands horizontal with "heads" showing on top), T (the coin lands horizontal with "tails" showing on top), or E (the coin lands and remains standing on its edge). Of course, outcome E is not very likely unless the coin is quite thick, but it can happen. So our sample space is $S = \{H, T, E\}$. Suppose we were to toss the coin a great many times, and observe that the outcomes H and T each occur on 49% of the tosses while E occurs only 2% of the time. We would say that on any one toss of the coin the outcomes H and T each have probability 0.49 and E has probability 0.02.

An **event** is any subset of the sample space. The **probability** of an event is a real number between 0 and 1 that measures the proportion of times the outcome of the experiment can be expected to belong to that event if the experiment is repeated many times. If the event is the whole sample space, its occurrence is certain, and its probability is 1; if the event is the empty set $\emptyset = \{\,\}$, it cannot possibly occur, and its probability is 0. For the coin-tossing experiment, there are eight possible events; we record their probabilities as follows:

$$\begin{array}{llll}
\Pr(\emptyset) = 0, & \Pr(\{T\}) = 0.49, & \Pr(\{H,T\}) = 0.98, & \Pr(\{T,E\}) = 0.51, \\
\Pr(\{H\}) = 0.49, & \Pr(\{E\}) = 0.02, & \Pr(\{H,E\}) = 0.51, & \Pr(S) = 1.
\end{array}$$

Given any two events A and B (subsets of sample space S), their **intersection** $A \cap B$ consists of those outcomes belonging to both A and B; it is sometimes called the event "A and B." Two events are **disjoint** if $A \cap B = \emptyset$; no outcome can belong to two disjoint events. For instance, an event A and its **complement**, A^c, consisting of all outcomes in S that don't belong to A, are disjoint. The **union** of two events A and B (also called the event "A or B") consists of all outcomes that belong to at least one of A and B. Note that $A \cup A^c = S$.

We summarize the basic rules governing probability as follows: if S is a sample space, $\emptyset$ is the empty subset of S, and A and B are any events, then

(a) $0 \le \Pr(A) \le 1$,

(b) $\Pr(\emptyset) = 0 \quad \text{and} \quad \Pr(S) = 1$,

(c) $\Pr(A^c) = 1 - \Pr(A)$,

(d) $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.

Note that just adding $\Pr(A) + \Pr(B)$ would count outcomes in $A \cap B$ twice. As an example, in our coin-tossing experiment if $A = \{H, T\}$ and $B = \{H, E\}$, then $A^c = \{E\}$, $A \cup B = \{H, T, E\} = S$, and $A \cap B = \{H\}$. We have

$$\Pr(A^c) = \Pr(\{E\}) = 0.02 = 1 - 0.98 = 1 - \Pr(\{H, T\}) = 1 - \Pr(A)$$
$$\Pr(A \cup B) = \Pr(S) = 1 = 0.51 + 0.51 - 0.02 = \Pr(A) + \Pr(B) - \Pr(A \cap B).$$

Remark The generality of these rules of probability can be misleading. Probability only has meaning in terms of a given sample space or measure. In popular culture probability is sometimes cited in the absence of a sample space at all. Probability theory also has infamous paradoxes and jokes that arise from attempting to compute probabilities across more than one sample space or computing them inadvertently from a different sample space than a user had in mind. Misunderstandings do arise in an overlooked shift in a question about a probability, implying an unnoticed change in the sample space, or lack of precision about what the sample space actually is. Infamous disputes about the "correct" probability have arisen as a result. These are beyond the scope of this section.

Discrete Random Variables

A **random variable** is a function defined on a sample space. We will denote random variables by using uppercase letters such as X and Y. If the sample space contains only discrete outcomes (like the sample space for the coin-tossing experiment), a random variable on it will have only discrete values and will be called a **discrete random variable**. If, on the other hand, the sample space contains all possible measurements of, say, heights of trees, then a random variable equal to that measurement can itself take on a continuum of real values and will be called a **continuous random variable**. We will study both types in this section.

Most discrete random variables have only finitely many values, but some can have infinitely many values if, say, the sample space consisted of the positive integers $\{1, 2, 3, \ldots\}$. A discrete random variable X has an associated **probability function** f defined on the range of X by $f(x) = \Pr(X = x)$ for each possible value x of X. Typically, f is represented by a bar graph; the sum of the heights of all the bars must be 1,

$$\sum_x f(x) = \sum_x \Pr(X = x) = 1,$$

since it is certain that the experiment must produce an outcome, and therefore a value of X.

EXAMPLE 1 A single fair die is rolled so that it will show one of the numbers 1 to 6 on top when it stops. If X denotes the number showing on any roll, then X is a discrete random variable with 6 possible values. Since the die is fair, no one value of X is any more likely than any other, so the probability that the number showing is n must be 1/6 for each possible value of n. If f is the probability function of X, then

$$f(n) = \Pr(X = n) = \frac{1}{6} \qquad \text{for each } n \text{ in } \{1, 2, 3, 4, 5, 6\}.$$

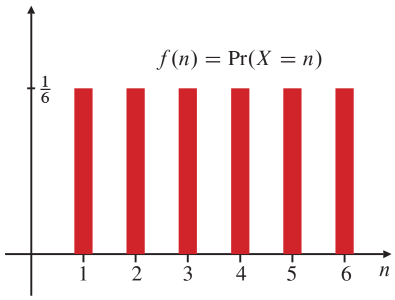


Figure 7.52 The probability function for a single rolled die

The discrete random variable X is therefore said to be distributed **uniformly**. All the bars in the graph of its probability function f have the same height. (See Figure 7.52.) Note that

$$\sum_{n=1}^{6} \Pr(X = n) = 1,$$

reflecting the fact that the rolled die must certainly give one of the six possible outcomes. The probability that a roll will produce a value from 1 to 4 is

$$\Pr(1 \le X \le 4) = \sum_{n=1}^{4} \Pr(X = n) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}.$$

EXAMPLE 2 What is the sample space for the numbers showing on top when two fair dice are rolled? What is the probability that a 4 and a 2 will be showing? Find the probability function for the random variable X that gives the sum of the two numbers showing on the dice. What is the probability that that sum is less than 10?

Solution The sample space consists of all pairs of integers (m, n) satisfying $1 \le m \le 6$ and $1 \le n \le 6$. There are 36 such pairs, so the probability of any one of them is 1/36. Two of the pairs, $(4, 2)$ and $(2, 4)$, correspond to a 4 and a 2 showing, so the probability of that event is $(1/36) + (1/36) = 1/18$. The random variable X defined by $X(m, n) = m + n$ has 11 possible values, the integers from 2 to 12 inclusive. The following table lists the pairs that produce each value k of X and the probability $f(k)$ of that value, that is, the value of the probability function at k.

Table 2. Probability function for the sum of two dice

$k = m + n$	outcomes for which $X = k$	$f(k) = \Pr(X = k)$
2	$(1,1)$	$1/36$
3	$(1,2), (2,1)$	$2/36 = 1/18$
4	$(1,3), (2,2), (3,1)$	$3/36 = 1/12$
5	$(1,4), (2,3), (3,2), (4,1)$	$4/36 = 1/9$
6	$(1,5), (2,4), (3,3), (4,2), (5,1)$	$5/36$
7	$(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)$	$6/36 = 1/6$
8	$(2,6), (3,5), (4,4), (5,3), (6,2)$	$5/36$
9	$(3,6), (4,5), (5,4), (6,3)$	$4/36 = 1/9$
10	$(4,6), (5,5), (6,4)$	$3/36 = 1/12$
11	$(5,6), (6,5)$	$2/36 = 1/18$
12	$(6,6)$	$1/36$

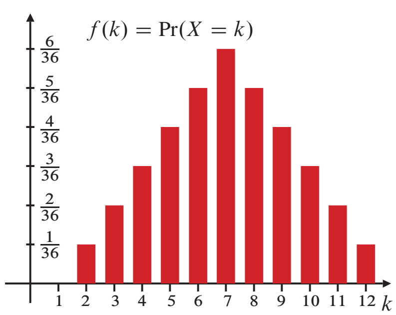


Figure 7.53 The probability function for the sum of two dice

The bar graph of the probability function f is shown in Figure 7.53. We have

$$\Pr(X < 10) = 1 - \Pr(X \ge 10) = 1 - \left(\frac{1}{12} + \frac{1}{18} + \frac{1}{36}\right) = \frac{5}{6}.$$

Expectation, Mean, Variance, and Standard Deviation

Consider a simple gambling game in which the player pays the house C dollars for the privilege of rolling a single die and in which he wins X dollars, where X is the number showing on top of the rolled die. In each game the possible winnings are 1, 2, 3, 4, 5, or 6 dollars, each with probability 1/6. In n games the player can expect to win about $n/6 + 2n/6 + 3n/6 + 4n/6 + 5n/6 + 6n/6 = 21n/6 = 7n/2$ dollars, so that his expected *average winnings per game* are 7/2 dollars, that is, \$3.50. If $C > 3.5$, the player can expect, on average, to lose money. The amount 3.5 is called the **expectation**, or **mean**, of the discrete random variable X. The mean is usually denoted by μ, the Greek letter "mu" (pronounced "mew").

DEFINITION 2

Mean or expectation

If X is a discrete random variable with range of values R and probability function f, then the **mean** (denoted μ), or **expectation** of X (denoted $E(X)$), is

$$\mu = E(X) = \sum_{x \in R} x\, f(x).$$

Also, the **expectation** of any function $g(X)$ of the random variable X is

$$E(g(X)) = \sum_{x \in R} g(x)\, f(x).$$

Note that in this usage $E(X)$ does not define a function of X but a constant (parameter) associated with the random variable X. Note also that if $f(x)$ were a mass density such as that studied in Section 7.4, then μ would be the moment of the mass about 0 and, since the total mass would be $\sum_{x \in R} f(x) = 1$, μ would in fact be the centre of mass.

Another parameter used to describe the way probability is distributed for a random variable is the variable's standard deviation.

DEFINITION 3

Variance and standard deviation

The **variance** of a random variable X with range R and probability function f is the expectation of the square of the distance of X from its mean μ. The variance is denoted σ^2 or Var(X).

$$\sigma^2 = \mathrm{Var}(X) = E\big((X-\mu)^2\big) = \sum_{x \in R} (x-\mu)^2\, f(x).$$

The **standard deviation** of X is the square root of the variance and therefore is denoted σ.

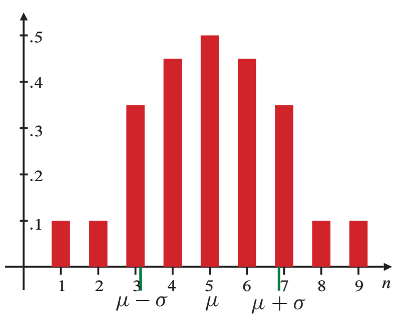


Figure 7.54 A probability function with mean $\mu = 5$ and standard deviation $\sigma = 1.86$

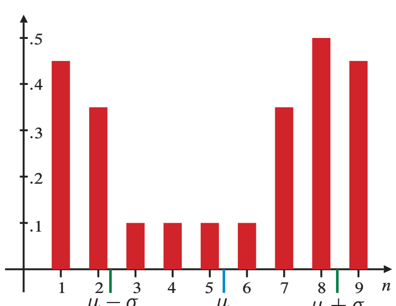


Figure 7.55 A probability function with mean $\mu = 5.38$ and standard deviation $\sigma = 3.05$

The symbol σ is the lowercase Greek letter "sigma." (The symbol Σ used for summation is an uppercase sigma.) The standard deviation gives a measure of how spread out the probability distribution of X is. The smaller the standard deviation, the more the probability is concentrated at values of X close to the mean. Figure 7.54 and Figure 7.55 illustrate the probability functions of two random variables with sample space $\{1, 2, \ldots, 9\}$, one having small σ and one with large σ. Note how a significant fraction of the total probability lies between $\mu - \sigma$ and $\mu + \sigma$ in each case. Note also that the distribution of probability in Figure 7.54 is symmetric, resulting in $\mu = 5$, the midpoint of the sample space, while the distribution in Figure 7.55 is skewed a bit to the right, resulting in $\mu > 5$.

Since $\sum_{x \in R} f(x) = 1$, the expression given in the definition of variance can be rewritten as follows:

$$\begin{aligned}
\sigma^2 = \mathrm{Var}(X) &= \sum_{x \in R} (x^2 - 2\mu x + \mu^2)\, f(x) \\
&= \sum_{x \in R} x^2 f(x) - 2\mu \sum_{x \in R} x f(x) + \mu^2 \sum_{x \in R} f(x) \\
&= \sum_{x \in R} x^2 f(x) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2,
\end{aligned}$$

that is,

$$\sigma^2 = \mathrm{Var}(X) = E(X^2) - \mu^2 = E(X^2) - (E(X))^2.$$

Therefore, the standard deviation of X is given by

$$\sigma = \sqrt{E(X^2) - \mu^2}.$$

EXAMPLE 3 Find the mean of the random variable X of Example 2. Also find the expectation of X^2 and the standard deviation of X.

Solution We have

$$\mu = E(X) = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + 6 \times \frac{5}{36} + 7 \times \frac{6}{36} + 8 \times \frac{5}{36} + 9 \times \frac{4}{36} + 10 \times \frac{3}{36} + 11 \times \frac{2}{36} + 12 \times \frac{1}{36} = 7,$$

a fact that is fairly obvious from the symmetry of the graph of the probability function in Figure 7.53. Also,

$$E(X^2) = 2^2 \times \frac{1}{36} + 3^2 \times \frac{2}{36} + 4^2 \times \frac{3}{36} + 5^2 \times \frac{4}{36} + 6^2 \times \frac{5}{36} + 7^2 \times \frac{6}{36} + 8^2 \times \frac{5}{36} + 9^2 \times \frac{4}{36} + 10^2 \times \frac{3}{36} + 11^2 \times \frac{2}{36} + 12^2 \times \frac{1}{36} = \frac{1{,}974}{36} \approx 54.8333.$$

The variance of X is $\sigma^2 = E(X^2) - \mu^2 \approx 54.8333 - 49 = 5.8333$, so the standard deviation of X is $\sigma \approx 2.4152$.

Continuous Random Variables

Now we consider an example with a continuous range of possible outcomes.

EXAMPLE 4 Suppose that a needle is dropped at random on a flat table with a straight line drawn on it. For each drop, let X be the acute angle, measured in degrees, that the needle makes with the line. (See Figure 7.56(a).) Evidently, X can take any real value in the interval $[0, 90]$; therefore, X is called a **continuous random variable**. The probability that X takes on any particular real value is 0. (There are infinitely many real numbers in $[0, 90]$, and none is more likely than any other.) However, the probability that X lies in some interval, say $[10, 20]$, is the same as the probability that it lies in any other interval of the same length. Since the interval has length 10 and the interval of all possible values of X has length 90, this probability is

$$\Pr(10 \le X \le 20) = \frac{10}{90} = \frac{1}{9}.$$

More generally, if $0 \le x_1 \le x_2 \le 90$, then

$$\Pr(x_1 \le X \le x_2) = \frac{1}{90}(x_2 - x_1).$$

This situation can be conveniently represented as follows: Let $f(x)$ be defined on the interval $[0, 90]$, taking at each point the constant value $1/90$:

$$f(x) = \frac{1}{90}, \qquad 0 \le x \le 90.$$

The area under the graph of f is 1, and $\Pr(x_1 \le X \le x_2)$ is equal to the area under that part of the graph lying over the interval $[x_1, x_2]$. (See Figure 7.56(b).) The function $f(x)$ is called the **probability density function** for the random variable X. Since $f(x)$ is constant on its domain, X is said to be **uniformly distributed**.

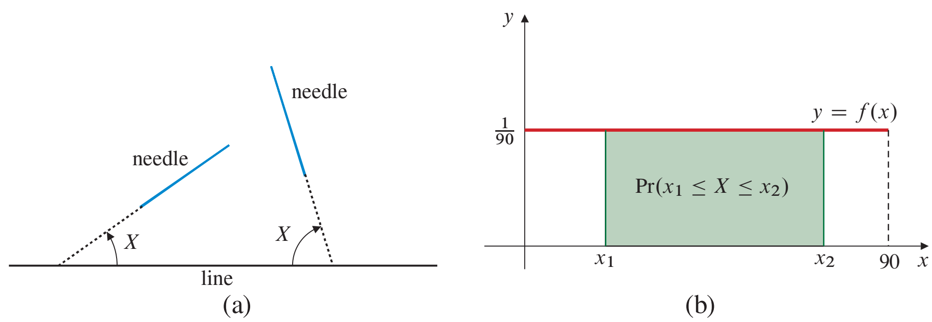


Figure 7.56
(a) X is the acute angle, measured in degrees, that the needle makes with the line
(b) The probability density function f of the random variable X

DEFINITION 4

Probability density functions

A function defined on an interval $[a, b]$ is a probability density function for a continuous random variable X distributed on $[a, b]$ if, whenever x_1 and x_2 satisfy $a \le x_1 \le x_2 \le b$, we have

$$\Pr(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x)\,dx,$$

which is the area above the interval $[x_1, x_2]$ and under the graph of f, provided $f(x) \ge 0$. In order to be such a probability density function, f must satisfy two conditions:

(a) $f(x) \ge 0$ on $[a, b]$ (probability cannot be negative) and
(b) $\int_a^b f(x)\,dx = 1$ ($\Pr(a \le X \le b) = 1$).

Note that this definition of probability density function generalizes the probability function used in the discrete case if we regard the bar graphs there as the graphs of step functions with unit base lengths.

These ideas extend to random variables distributed on semi-infinite or infinite intervals, but the integrals appearing will be improper in those cases. In any event, the role played by sums in the analysis of discrete random variables is taken over by integrals for continuous random variables.

In the example of the dropping needle, the probability density function has a horizontal straight line graph, and we termed such a probability distribution uniform. The uniform probability density function on the interval $[a, b]$ is

$$f(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$$

Many other functions are commonly encountered as density functions for continuous random variables.

EXAMPLE 5 **(The exponential distribution)** The length of time T that any particular atom in a radioactive sample survives before decaying is a random variable taking values in $[0, \infty)$. It has been observed that the proportion of atoms that survive to time t becomes small exponentially as t increases; thus,

$$\Pr(T \ge t) = Ce^{-kt}.$$

Let f be the probability density function for the random variable T. Then

$$\int_t^\infty f(x)\,dx = \Pr(T \ge t) = Ce^{-kt}.$$

Differentiating this equation with respect to t (using the Fundamental Theorem of Calculus), we obtain $-f(t) = -Cke^{-kt}$, so $f(t) = Cke^{-kt}$. C is determined by the requirement that $\int_0^\infty f(t)\,dt = 1$. We have

$$1 = Ck\int_0^\infty e^{-kt}\,dt = \lim_{R\to\infty} Ck\int_0^R e^{-kt}\,dt = -C\lim_{R\to\infty}(e^{-kR} - 1) = C.$$

Thus, $C = 1$ and $f(t) = ke^{-kt}$. Note that $\Pr(T \ge (\ln 2)/k) = e^{-k(\ln 2)/k} = 1/2$, reflecting the fact that the half-life of such a radioactive sample is $(\ln 2)/k$.

EXAMPLE 6 For what value of C is $f(x) = C(1-x^2)$ a probability density function on $[-1,1]$? If X is a random variable with this density, what is the probability that $X \le 1/2$?

Solution Observe that $f(x) \ge 0$ on $[-1,1]$ if $C \ge 0$. Since

$$\int_{-1}^{1} f(x)\,dx = C\int_{-1}^{1}(1-x^2)\,dx = 2C\left(x - \frac{x^3}{3}\right)\bigg|_0^1 = \frac{4C}{3},$$

$f(x)$ will be a probability density function if $C = 3/4$. In this case

$$\begin{aligned}\Pr\left(X \le \frac{1}{2}\right) &= \frac{3}{4}\int_{-1}^{1/2}(1-x^2)\,dx = \frac{3}{4}\left(x - \frac{x^3}{3}\right)\bigg|_{-1}^{1/2}\\ &= \frac{3}{4}\left(\frac{1}{2} - \frac{1}{24} - (-1) + \frac{-1}{3}\right) = \frac{27}{32}.\end{aligned}$$

By analogy with the discrete case, we formulate definitions for the mean (or expectation), variance, and standard deviation of a continuous random variable as follows:

DEFINITIONS 5

If X is a continuous random variable on $[a,b]$ with probability density function $f(x)$, the **mean** μ, (or **expectation** $E(X)$) of X is

$$\mu = E(X) = \int_a^b x f(x)\,dx.$$

The expectation of a function g of X is

$$E\big(g(X)\big) = \int_a^b g(x)\, f(x)\,dx.$$

Similarly, the **variance** σ^2 of X is the mean of the squared deviation of X from its mean:

$$\sigma^2 = \mathrm{Var}(X) = E((X-\mu)^2) = \int_a^b (x-\mu)^2 f(x)\,dx,$$

and the **standard deviation** is the square root of the variance.

As was the case for a discrete random variable, it is easily shown that

$$\sigma^2 = E(X^2) - \mu^2, \qquad \sigma = \sqrt{E(X^2) - \mu^2}.$$

Again the standard deviation gives a measure of how spread out the probability distribution of X is. The smaller the standard deviation, the more concentrated is the area under the density curve around the mean, and so the smaller is the probability that a value of X will be far away from the mean. (See Figure 7.57.)

EXAMPLE 7 Find the mean μ and the standard deviation σ of a random variable X distributed uniformly on the interval $[a,b]$. Find $\Pr(\mu - \sigma \le X \le \mu + \sigma)$.

Solution The probability density function is $f(x) = 1/(b-a)$ on $[a,b]$, so the mean is given by

$$\mu = E(X) = \int_a^b \frac{x}{b-a}\,dx = \frac{1}{b-a}\,\frac{x^2}{2}\bigg|_a^b = \frac{1}{2}\,\frac{b^2-a^2}{b-a} = \frac{b+a}{2}.$$

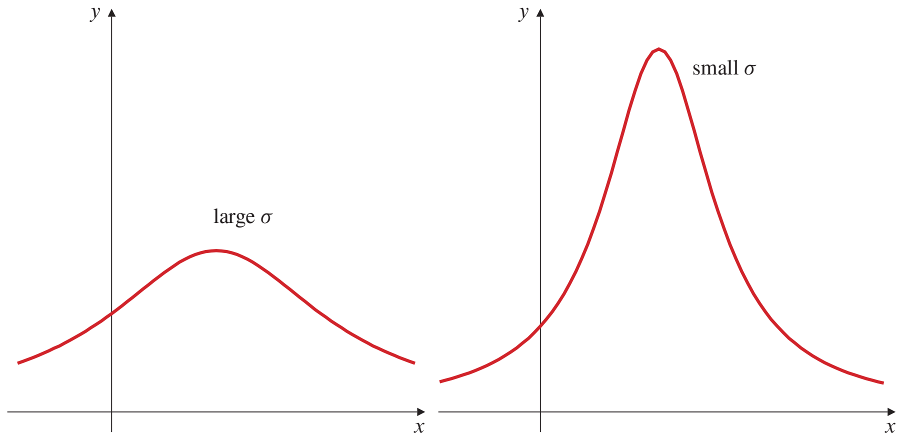


Figure 7.57 Densities with large and small standard deviations

Hence, the mean is, as might have been anticipated, the midpoint of $[a, b]$. The expectation of X^2 is given by

$$E(X^2) = \int_a^b \frac{x^2}{b-a}\,dx = \frac{1}{b-a}\frac{x^3}{3}\bigg|_a^b = \frac{1}{3}\frac{b^3-a^3}{b-a} = \frac{b^2+ab+a^2}{3}.$$

Hence, the variance is

$$\sigma^2 = E(X^2) - \mu^2 = \frac{b^2+ab+a^2}{3} - \frac{b^2+2ab+a^2}{4} = \frac{(b-a)^2}{12},$$

and the standard deviation is

$$\sigma = \frac{b-a}{2\sqrt{3}} \approx 0.29(b-a).$$

Finally,

$$\Pr(\mu-\sigma \le X \le \mu+\sigma) = \int_{\mu-\sigma}^{\mu+\sigma} \frac{dx}{b-a} = \frac{1}{b-a}\frac{2(b-a)}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \approx 0.577.$$

EXAMPLE 8 Find the mean μ and the standard deviation σ of a random variable X distributed exponentially with density function $f(x) = ke^{-kx}$ on the interval $[0, \infty)$. Find $\Pr(\mu-\sigma \le X \le \mu+\sigma)$.

Solution We use integration by parts to find the mean:

$$\begin{aligned}
\mu = E(X) &= k\int_0^\infty xe^{-kx}\,dx \\
&= \lim_{R\to\infty} k\int_0^R xe^{-kx}\,dx \qquad \begin{array}{lll} \text{Let} & U = x, & dV = e^{-kx}\,dx. \\ \text{Then} & dU = dx, & V = -e^{-kx}/k. \end{array} \\
&= \lim_{R\to\infty}\left(-xe^{-kx}\bigg|_0^R + \int_0^R e^{-kx}\,dx\right) \\
&= \lim_{R\to\infty}\left(-Re^{-kR} - \frac{1}{k}(e^{-kR}-1)\right) = \frac{1}{k}, \qquad \text{since } k > 0.
\end{aligned}$$

Thus, the mean of the exponential distribution is $1/k$. This fact can be quite useful in determining the value of k for an exponentially distributed random variable. A similar integration by parts enables us to evaluate

$$E(X^2) = k\int_0^\infty x^2e^{-kx}\,dx = 2\int_0^\infty xe^{-kx}\,dx = \frac{2}{k^2},$$

so the variance of the exponential distribution is

$$\sigma^2 = E(X^2) - \mu^2 = \frac{1}{k^2},$$

and the standard deviation is equal to the mean

$$\sigma = \mu = \frac{1}{k}.$$

Now we have

$$\begin{aligned}\Pr(\mu - \sigma \le X \le \mu + \sigma) &= \Pr(0 \le X \le 2/k)\\ &= k \int_0^{2/k} e^{-kx}\,dx\\ &= -e^{-kx}\Big|_0^{2/k}\\ &= 1 - e^{-2} \approx 0.86,\end{aligned}$$

which is independent of the value of k. Exponential densities for small and large values of k are graphed in Figure 7.58.

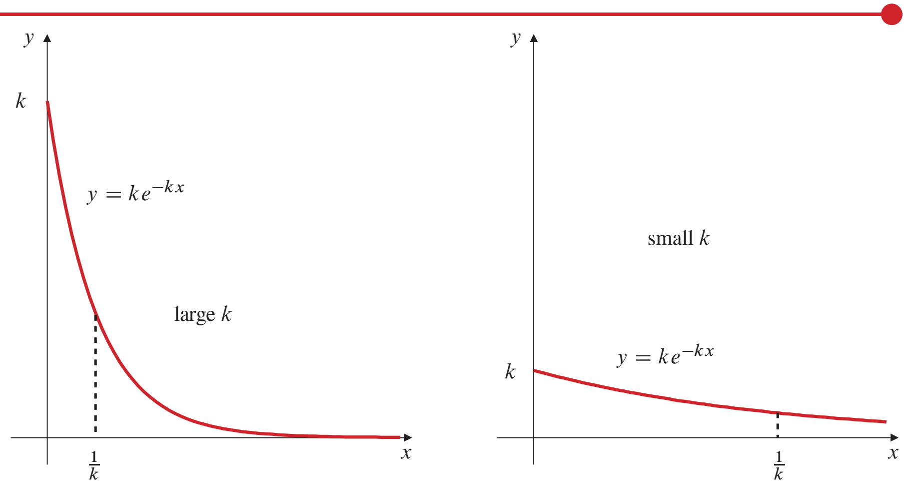


Figure 7.58 Exponential density functions

The Normal Distribution

The most important probability distributions are the so-called **normal** or **Gaussian** distributions. Such distributions govern the behaviour of many interesting random variables, in particular, those associated with random errors in measurements. There is a family of normal distributions, all related to the particular normal distribution called the **standard normal distribution**, which has the following probability density function:

DEFINITION 6

The standard normal probability density

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad -\infty < z < \infty.$$

It is common to use z to denote the random variable in the standard normal distribution; the other normal distributions are obtained from this one by a change of variable. The graph of the standard normal density has a pleasant bell shape, as shown in Figure 7.59.

As we have noted previously, the function e^{-z^2} has no elementary antiderivative, so the improper integral

$$I = \int_{-\infty}^{\infty} e^{-z^2/2}\,dz$$

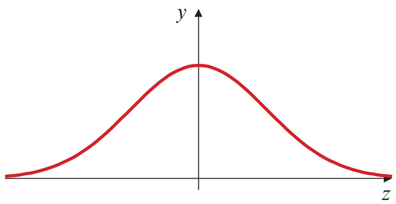


Figure 7.59 The standard normal density function $f(z) = \dfrac{1}{\sqrt{2\pi}} e^{-z^2/2}$

cannot be evaluated using the Fundamental Theorem of Calculus, although it is a convergent improper integral. The integral can be evaluated using techniques of multivariable calculus involving double integrals of functions of two variables. (We do so in Example 4 of Section 14.4.) The value is $I = \sqrt{2\pi}$, which ensures that the above-defined standard normal density $f(z)$ is indeed a probability density function:

$$\int_{-\infty}^{\infty} f(z)\,dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\,dz = 1.$$

Since $ze^{-z^2/2}$ is an odd function of z and its integral on $(-\infty, \infty)$ converges, the mean of the standard normal distribution is 0:

$$\mu = E(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ze^{-z^2/2}\,dz = 0.$$

We calculate the variance of the standard normal distribution using integration by parts as follows:

$$\begin{aligned}
\sigma^2 &= E(Z^2) \\
&= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2}\,dz \\
&= \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \int_{-R}^{R} z^2 e^{-z^2/2}\,dz \qquad \begin{array}{l} \text{Let} \quad U = z, \quad dV = ze^{-z^2/2}\,dz. \\ \text{Then } dU = dz, \quad V = -e^{-z^2/2}. \end{array} \\
&= \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} \left(-ze^{-z^2/2} \Big|_{-R}^{R} + \int_{-R}^{R} e^{-z^2/2}\,dz \right) \\
&= \frac{1}{\sqrt{2\pi}} \lim_{R\to\infty} (-2Re^{-R^2/2}) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2}\,dz \\
&= 0 + 1 = 1.
\end{aligned}$$

Hence, the standard deviation of the standard normal distribution is 1.

Other normal distributions are obtained from the standard normal distribution by a change of variable.

DEFINITION 7

The general normal distribution

A random variable X on $(-\infty, \infty)$ is said to be *normally distributed with mean* μ and *standard deviation* σ (where μ is any real number and $\sigma > 0$) if its probability density function $f_{\mu,\sigma}$ is given in terms of the standard normal density f by

$$f_{\mu,\sigma}(x) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.$$

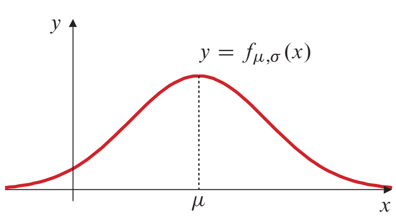


Figure 7.60 A general normal density with mean μ

(See Figure 7.60.) Using the change of variable $z = (x-\mu)/\sigma$, $dz = dx/\sigma$, we can verify that

$$\int_{-\infty}^{\infty} f_{\mu,\sigma}(x)\,dx = \int_{-\infty}^{\infty} f(z)\,dz = 1,$$

so $f_{\mu,\sigma}(x)$ is indeed a probability density function. Using the same change of variable, we can show that

$$E(X) = \mu \qquad \text{and} \qquad E((X-\mu)^2) = \sigma^2.$$

Hence, the density $f_{\mu,\sigma}$ does indeed have mean μ and standard deviation σ.

To find the probability $\Pr(Z \leq z)$ we compute what is called the **cumulative distribution function** of a random variable with standard normal distribution,

$$F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2}\, dx = \Pr(Z \leq z),$$

which represents the area under the standard normal density function from $-\infty$ up to z, as shown in Figure 7.61. According to the definition of the error function in Section 6.4, an antiderivative of $e^{-z^2/2}$ is $\sqrt{2/\pi}\,\operatorname{erf}(z/\sqrt{2})$. Thus,

$$F(z) = \frac{1}{2}\left(\operatorname{erf}\left(\frac{z}{\sqrt{2}}\right) + 1\right).$$

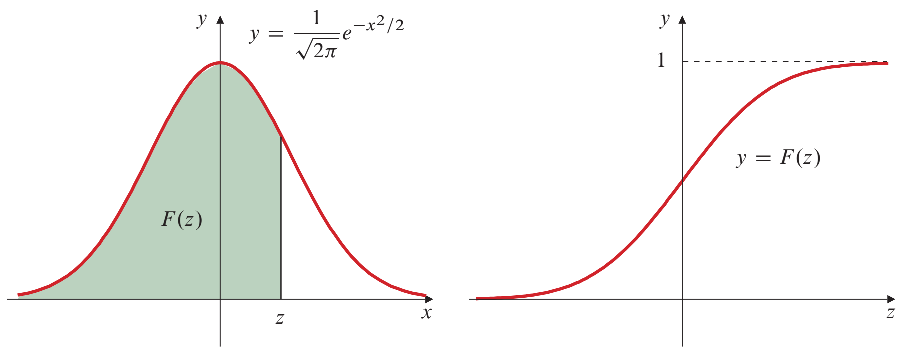


Figure 7.61 The cumulative distribution function $F(z)$ for the standard normal distribution is the area under the standard normal density function from $-\infty$ to z

For convenience in the following examples and exercises, we include an abbreviated lookup table for this expression. Alternatively, $F(z)$ is easily defined in Maple to calculate to any desired number of decimal places, say 10, using the known error function:

```
> F := x -> (1/2)*(erf(z/sqrt(2)) + 1);
```

which can then be used to calculate values of F. See the following examples.

Table 3. Values of the standard normal distribution function $F(z)$ (rounded to 3 decimal places)

z	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
−3.0	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
−2.0	0.023	0.018	0.014	0.011	0.008	0.006	0.005	0.003	0.003	0.002
−1.0	0.159	0.136	0.115	0.097	0.081	0.067	0.055	0.045	0.036	0.029
−0.0	0.500	0.460	0.421	0.382	0.345	0.309	0.274	0.242	0.212	0.184
0.0	0.500	0.540	0.579	0.618	0.655	0.691	0.726	0.758	0.788	0.816
1.0	0.841	0.864	0.885	0.903	0.919	0.933	0.945	0.955	0.964	0.971
2.0	0.977	0.982	0.986	0.989	0.992	0.994	0.995	0.997	0.997	0.998
3.0	0.999	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000

EXAMPLE 9 If Z is a standard normal random variable, find (a) $\Pr(-1.2 \leq Z \leq 2.0)$, and (b) $\Pr(Z \geq 1.5)$, using the table to three decimal places or using Maple to 10 decimal places.

Solution Using values from the table, we obtain

$$\begin{aligned}\Pr(-1.2 \leq Z \leq 2.0) &= \Pr(Z \leq 2.0) - \Pr(Z < -1.2)\\ &= F(2.0) - F(-1.2) \approx 0.977 - 0.115\\ &= 0.862\end{aligned}$$

$$\begin{aligned}\Pr(Z \geq 1.5) &= 1 - \Pr(Z < 1.5)\\ &= 1 - F(1.5) \approx 1 - 0.933 = 0.067.\end{aligned}$$

After making the Maple definition shown above, we calculate $\Pr(-1.2 \le Z \le 2.0)$ to 10 decimal places using

```
> evalf(F(2) - F(-1.2), 10)
```

$$0.8621801977$$

and for $\Pr(Z \ge 1.5)$

```
> evalf(1 -(F(1.5), 10)
```

$$0.0668072012$$

EXAMPLE 10 A certain random variable X is distributed normally with mean 2 and standard deviation 0.4. Find (a) $\Pr(1.8 \le X \le 2.4)$, and (b) $\Pr(X > 2.4)$, using the table to three decimal places or using Maple to 10 decimal places.

Solution Since X is distributed normally with mean 2 and standard deviation 0.4, $Z = (X - 2)/0.4$ is distributed according to the standard normal distribution (with mean 0 and standard deviation 1). Accordingly,

$$\begin{aligned}\Pr(1.8 \le X \le 2.4) &= \Pr(-0.5 \le Z \le 1)\\ &= F(1) - F(-0.5) \approx 0.841 - 0.309 = 0.532,\\ \Pr(X > 2.4) &= \Pr(Z > 1) = 1 - \Pr(Z \le 1)\\ &= 1 - F(1) \approx 1 - 0.841 = 0.159.\end{aligned}$$

Alternatively, using Maple with F defined as above, $\Pr(1.8 \le X \le 2.4)$ is

```
> evalf(F(1) - F(-0.5), 10)
```

$$0.5328072072$$

For $\Pr(X > 2.4)$

```
> evalf(1 - F(1), 10)
```

$$0.1586552540$$

Heavy Tails

With continuous random variables over an infinite domain, complications of improper integrals arise for certain established probability density functions that do not satisfy the conditions needed for the normal distribution to hold. For an important class of these functions the integrals for the mean or variance do not exist. For example, a common nonnormal probability density function arising in physics is the **Cauchy distribution**:

DEFINITION 8

The Cauchy probability density

$$C(x) = \frac{1}{\pi}\frac{\gamma}{(x-\mu)^2+\gamma^2} \qquad -\infty < x < \infty.$$

Here the constants μ and γ play roles similar to those of the mean and standard deviation in the normal distribution. The graph of $C(x)$ is symmetric about the line $x = \mu$, and γ is a measure of the width of the single peak. However, μ is not really a mean and γ is certainly not a standard deviation, as neither $\int_{-\infty}^{\infty} xC(x)\,dx$ nor $\int_{-\infty}^{\infty} x^2C(x)\,dx$ is a convergent improper integral. The Cauchy density function is known in spectroscopy as the Lorentz profile for spectral lines, while in nuclear physics it is known as the probability density function of the Breit-Wigner distribution.

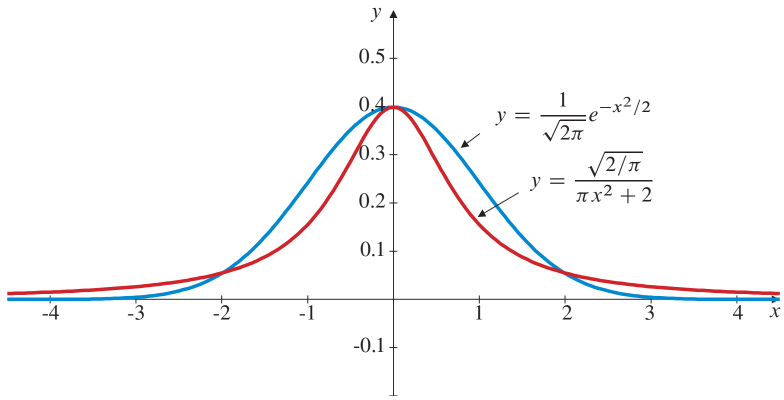


Figure 7.62 The standard normal density (blue) and the Cauchy density with $\mu = 0$ and $\gamma = \sqrt{2/\pi}$ (red)

Figure 7.62 shows the graphs of the standard Normal density and the Cauchy density with $\mu = 0$ and $\gamma = \sqrt{2/\pi}$, the latter value being chosen so that both curves would peak at the same height. Observe the tails of these curves (i.e., the parts where $|x| > 2$, say). While the normal curve is higher than the Cauchy one for small $|x|$, the exponential factor in the normal density decreases very rapidly for large $|x|$; it is $O(|x|^{-n})$ for every positive integer n as $|x| \to \infty$ while the Cauchy density is only $O(|x|^{-2})$. Because of this polynomial asymptotic behaviour as $|x| \to \infty$, the Cauchy density is said to have **fat tails** or **heavy tails**. To understand the significance of this, we use direct integration to find the tail probability from some x to infinity for the Cauchy distribution,

$$\Pr(X > x) = \frac{1}{2} - \frac{1}{\pi} \tan^{-1} x,$$

and divide it by the corresponding tail probability, $\Pr(X > x) = 1 - F(x)$, for the standard normal distribution. This ratio is plotted in Figure 7.63 for $1 \le x \le 8$. From about $x = 7$ on, the ratio grows extremely rapidly! This ratio shows that the amount of probability in the tail of $C(x)$ is very "heavy" relative to the normal.

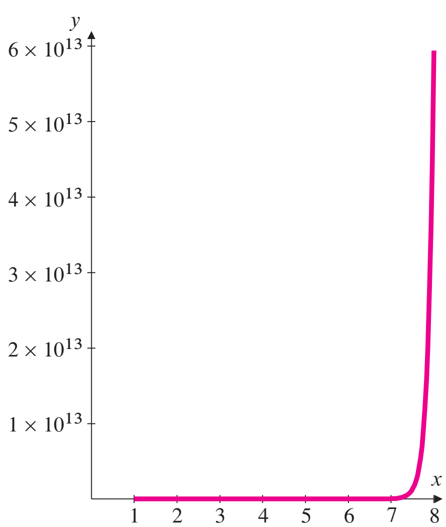


Figure 7.63 Plot of the ratio

$$y = \frac{\text{Cauchy } \Pr(x < X < \infty)}{\text{Normal } \Pr(x < X < \infty)}$$

$C(x)$ is far from the only heavy-tailed probability density function. One important class of heavy-tailed probability distributions are the **Lévy stable distributions** with densities $S_\alpha(x)$ for $0 < \alpha \le 2$. Except for $\alpha = 1$ (the Cauchy case) and $\alpha = 2$ (the normal case), the densities $S_\alpha(x)$ are not elementary functions, and providing exact descriptions of them is beyond the scope of this section. They can be represented explicitly as integral transforms (see Section 18.7), but must be computed numerically to get specific values, as is the case for any other nonalgebraic function. The graphs of the symmetric versions of $S_\alpha(x)$ are similar to those of the normal and Cauchy densities. The definitive differences are found in the specifics of the tail behaviours, which are discussed in Exercises 23–24.

Remark Poincaré's telling remark, in the quotations at the beginning of this chapter, humorously warns us that a presumption of normality is common. However, in the standard physics examples given, normality does not hold. $C(x)$ arises theoretically in them. Without a theoretical basis, any presumption of normality can only be confirmed with data alone. But the tail represents the least probable events, so tail data are least likely to be observed over any finite time. Thus, the greatest deviations from normal distributions are the least likely to be observed, making empirical demonstrations of heavy tails difficult.

Remark Suppose that, because of a naïve expectation of the universality of normal distributions, we mistakenly assume that the outcomes of a heavy tail process are distributed normally. Since the largest disagreement between the supposed and actual distributions only occur for the rarest of events, finite samples of outcomes will not likely expose our error; the samples won't have any outcomes from far out in the tail. But, being the most extreme, the rarest of events can also be the most consequential. The sharp growth of the ratio in Figure 7.63 shows that the probability of such events can be seriously underestimated in this circumstance. People adhering to the assumption of normality can thus experience costly surprises. A surprise of this type is named a **Black Swan** by Nassim Taleb, who authored a well-known book on this topic, entitled *The Black Swan: The Impact of the Highly Improbable.*

Remark Measured data from a continuous process are always finite in number and discretely sampled. While the mean and variance may not exist for a continuous heavy tail process, they will, nonetheless, always exist for finite data. Thus statistical uncertainty based on normality may be moot. Caution should be taken for questions that depend heavily on normality.

EXERCISES 7.8

1. How much should you be willing to pay to play a game where you toss the coin discussed at the beginning of this section and win \$1 if it comes up heads, \$2 if it comes up tails, and \$50 if it remains standing on its edge? Assume you will play the game many times and would like to at least break even.

2. A die is weighted so that if X represents the number showing on top when the die is rolled, then $\Pr(X = n) = Kn$ for $n \in \{1, 2, 3, 4, 5, 6\}$.

(a) Find the value of the constant K.

(b) Find the probability that $X \le 3$ on any roll of the die.

3. Find the standard deviation of your winings on a roll of the die in Exercise 1.

4. Find the mean and standard deviation of the random variable X in Exercise 2.

5. A die is weighted so that the probability of rolling each of the numbers 2, 3, 4, and 5 is still 1/6, but the probability of rolling 1 is 9/60 and the probability of rolling 6 is 11/60. What are the mean and standard deviation of the number X rolled using this die? What is the probability that $X \le 3$?

6. Two dice, each weighted like the one in Exercise 5, are thrown. Let X be the random variable giving the sum of the numbers showing on top of the two dice.

(a) Find the probability function for X.

(b) Determine the mean and standard deviation of X. Compare them with those found for unweighted dice in Example 3.

7. A thin but biased coin has probability 0.55 of landing heads and 0.45 of landing tails. (Standing on its edge is not possible for this coin.) The coin is tossed three times. (Determine all numerical answers to the following questions to six decimal places.)

(a) What is the sample space of possible outcomes of the three tosses?

(b) What is the probability of each of these possible outcomes?

(c) Find the probability function for the number X of times heads comes up during the three tosses.

(d) What is the probability that the number of heads is at least 1?

(e) What is the expectation of X?

8. A sack contains 20 balls all the same size; some are red and the rest are blue. If you reach in and pull out a ball at random, the probability that it is red is 0.6.

(a) If you reach in and pull out two balls, what is the probability they are both blue?

(b) Suppose you reach in the bag of 20 balls and pull out three balls. Describe the sample space of possible outcomes of this experiment. What is the expectation of the number of red balls among the three balls you pulled out?

For each function $f(x)$ in Exercises 9–15, find the following:

(a) the value of C for which f is a probability density on the given interval,

(b) the mean μ, variance σ^2, and standard deviation σ of the probability density f, and

(c) $\Pr(\mu - \sigma \le X \le \mu + \sigma)$, that is, the probability that the random variable X is no further than one standard deviation away from its mean.

9. $f(x) = Cx$ on $[0, 3]$

10. $f(x) = Cx$ on $[1, 2]$

11. $f(x) = Cx^2$ on $[0, 1]$

12. $f(x) = C \sin x$ on $[0, \pi]$

13. $f(x) = C(x - x^2)$ on $[0, 1]$

14. $f(x) = C\,xe^{-kx}$ on $[0, \infty)$, $(k > 0)$

15. $f(x) = C\,e^{-x^2}$ on $[0, \infty)$. *Hint:* Use properties of the standard normal density to show that $\int_0^\infty e^{-x^2}\,dx = \sqrt{\pi}/2$.

16. Is it possible for a random variable to be uniformly distributed on the whole real line? Explain why.

17. Carry out the calculations to show that the normal density $f_{\mu,\sigma}(x)$ defined in the text is a probability density function and has mean μ and standard deviation σ.

18. Show that $f(x) = \dfrac{2}{\pi(1 + x^2)}$ is a probability density on $[0, \infty)$. Find the expectation of a random variable X having this density. If a machine generates values this random variable X, how much would you be willing to pay, per game, to play a game in which you operate the machine to produce a value of X and win X dollars? Explain.

19. Calculate $\Pr(|X - \mu| \ge 2\sigma)$ for

(a) the uniform distribution on $[a, b]$,

(b) the exponential distribution with density $f(x) = ke^{-kx}$ on $[0, \infty)$, and

(c) the normal distribution with density $f_{\mu,\sigma}(x)$.

20. The length of time T (in hours) between malfunctions of a computer system is an exponentially distributed random variable. If the average length of time between successive malfunctions is 20 hours, find the probability that the system, having just had a malfunction corrected, will operate without malfunction for at least 12 hours.

21. The number X of metres of cable produced any day by a cable-making company is a normally distributed random variable with mean 5,000 and standard deviation 200. On what fraction of the days the company operates will the number of metres of cable produced exceed 5,500?

22. A spinner is made with a scale from 0 to 1. Over time it suffers from wear and tends to stick at the number 1/4. Suppose it sticks at 1/4 half of the time and the rest of the time it gives values uniformly distributed in the interval $[0, 1]$. What is the mean and standard deviation of the spinner's values? (Note: the random variable giving the spinner's value has a distribution that is partially discrete and partially continuous.)

23. Lévy stable probability densities are known to have the following asymptotic behaviour as $x \to \infty$

$$S_\alpha(x) = c_\alpha x^{-(1+\alpha)} + O\left(x^{-(1+2\alpha)}\right)$$

for $0 < \alpha < 2$, and for simplicity $S_\alpha(x)$ is assumed symmetric about $x = 0$. Note that the normal case, $\alpha = 2$, is excluded.

(a) Under what conditions can moments (i.e., $\int_{-\infty}^{\infty} x^p S_\alpha(x)\,dx$ for some $p \ge 0$) exist?

(b) For what values of α do the mean ($p = 1$) and the variance ($p = 2$) not exist?

24. Use the asymptotic behaviour from the previous exercise to find the probability in the tail of symmetric Lévy stable distributions valid for large x.

7.9 First-Order Differential Equations

This final section on applications of integration concentrates on application of the indefinite integral rather than of the definite integral. We can use the techniques of integration developed in Chapters 5 and 6 to solve certain kinds of first-order differential equations that arise in a variety of modelling situations. We have already seen some examples of applications of differential equations to modelling growth and decay phenomena in Section 3.4.

Separable Equations

Consider the logistic equation introduced in Section 3.4 to model the growth of an animal population with a limited food supply:

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right),$$

where $y(t)$ is the size of the population at time t, k is a positive constant related to the fertility of the population, and L is the steady-state population size that can be sustained by the available food supply. This equation has two particular solutions, $y = 0$ and $y = L$, that are constant functions of time.

The logistic equation is an example of a class of first-order differential equations called **separable equations** because when they are written in terms of differentials, they can be separated with only the dependent variable on one side of the equation and only the independent variable on the other. The logistic equation can be written in the form

$$\frac{L\,dy}{y(L - y)} = k\,dt$$

and solved by integrating both sides. Expanding the left side in partial fractions and integrating, we get

$$\int \left(\frac{1}{y} + \frac{1}{L - y}\right) dy = kt + C.$$

Assuming that $0 < y < L$, we therefore obtain

$$\ln y - \ln(L - y) = kt + C,$$
$$\ln\left(\frac{y}{L - y}\right) = kt + C.$$

We can solve this equation for y by taking exponentials of both sides:

$$\frac{y}{L-y} = e^{kt+C} = C_1 e^{kt}$$
$$y = (L-y)C_1 e^{kt}$$
$$y = \frac{C_1 L e^{kt}}{1 + C_1 e^{kt}},$$

where $C_1 = e^C$.

Generally, separable equations are of the form

$$\frac{dy}{dx} = f(x)g(y).$$

We solve them by rewriting them in the form

$$\frac{dy}{g(y)} = f(x)\,dx$$

and integrating both sides. Note that the separable equation above will have a constant solution $y(x) = C$ for any constant C satisfying $g(C) = 0$.

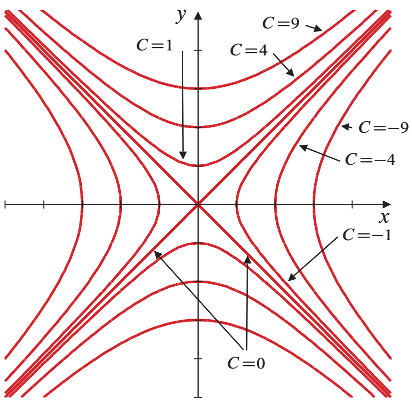


Figure 7.64 Some curves of the family $y^2 - x^2 = C$

EXAMPLE 1 Solve the equation $\dfrac{dy}{dx} = \dfrac{x}{y}$.

Solution We rewrite the equation in the form $y\,dy = x\,dx$ and integrate both sides to get

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1,$$

or $y^2 - x^2 = C$, where $C = 2C_1$ is an arbitrary constant. The solution curves are rectangular hyperbolas. (See Figure 7.64.) Their asymptotes $y = x$ and $y = -x$ are also solutions corresponding to $C = 0$.

EXAMPLE 2 Solve the initial-value problem

$$\begin{cases} \dfrac{dy}{dx} = x^2 y^3 \\ y(1) = 3. \end{cases}$$

Solution Separating the differential equation gives $\dfrac{dy}{y^3} = x^2\,dx$. Thus,

$$\int \frac{dy}{y^3} = \int x^2\,dx, \qquad \text{so} \qquad \frac{-1}{2y^2} = \frac{x^3}{3} + C.$$

Since $y = 3$ when $x = 1$, we have $-\frac{1}{18} = \frac{1}{3} + C$ and $C = -\frac{7}{18}$. Substituting this value into the above solution and solving for y, we obtain

$$y(x) = \frac{3}{\sqrt{7 - 6x^3}}. \quad \text{(Only the positive square root of } y^2 \text{ satisfies } y(1) = 3.)$$

This solution is valid for $x < \left(\frac{7}{6}\right)^{1/3}$. (See Figure 7.65.)

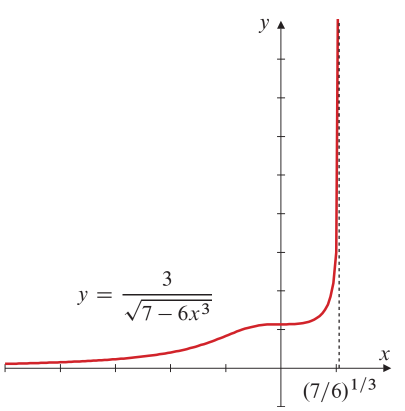


Figure 7.65 The solution of $\dfrac{dy}{dx} = x^2 y^3$ satisfying $y(1) = 3$

EXAMPLE 3 Solve the **integral equation** $y(x) = 3 + 2\int_1^x ty(t)\,dt$.

Solution Differentiating the integral equation with respect to x gives

$$\frac{dy}{dx} = 2x\,y(x) \qquad \text{or} \qquad \frac{dy}{y} = 2x\,dx.$$

Thus, $\ln|y(x)| = x^2 + C$, and solving for y, $y(x) = C_1 e^{x^2}$. Putting $x = 1$ in the integral equation provides an initial value: $y(1) = 3 + 0 = 3$, so $C_1 = 3/e$ and

$$y(x) = 3e^{x^2-1}.$$

EXAMPLE 4 **(A solution concentration problem)** Initially a tank contains 1,000 L of brine with 50 kg of dissolved salt. Brine containing 10 g of salt per litre is flowing into the tank at a constant rate of 10 L/min. If the contents of the tank are kept thoroughly mixed at all times, and if the solution also flows out at 10 L/min, how much salt remains in the tank at the end of 40 min?

Solution Let $x(t)$ be the number of kilograms of salt in solution in the tank after t min. Thus, $x(0) = 50$. Salt is coming into the tank at a rate of 10 g/L × 10 L/min = 100 g/min = 1/10 kg/min. At all times the tank contains 1,000 L of liquid, so the concentration of salt in the tank at time t is $x/1{,}000$ kg/L. Since the contents flow out at 10 L/min, salt is being removed at a rate of $10x/1{,}000 = x/100$ kg/min. Therefore,

$$\frac{dx}{dt} = \text{rate in} - \text{rate out} = \frac{1}{10} - \frac{x}{100} = \frac{10 - x}{100}.$$

Although $x(t) = 10$ is a constant solution of the differential equation, it does not satisfy the initial condition $x(0) = 50$, so we will find other solutions by separating variables:

$$\frac{dx}{10 - x} = \frac{dt}{100}.$$

Integrating both sides of this equation, we obtain

$$-\ln|10 - x| = \frac{t}{100} + C.$$

Observe that $x(t) \neq 10$ for any finite time t (since $\ln 0$ is not defined). Since $x(0) = 50 > 10$, it follows that $x(t) > 10$ for all $t > 0$. ($x(t)$ is necessarily continuous, so it cannot take any value less than 10 without somewhere taking the value 10 by the Intermediate-Value Theorem.) Hence, we can drop the absolute value from the solution above and obtain

$$\ln(x - 10) = -\frac{t}{100} - C.$$

Since $x(0) = 50$, we have $-C = \ln 40$ and

$$x = x(t) = 10 + 40e^{-t/100}.$$

After 40 min there will be $10 + 40e^{-0.4} \approx 36.8$ kg of salt in the tank.

EXAMPLE 5 **(A rate of reaction problem)** In a chemical reaction that goes to completion in solution, one molecule of each of two reactants, A and B, combine to form each molecule of the product C. According to the law of mass action, the reaction proceeds at a rate proportional to the product of the concentrations of A and B in the solution. Thus, if there were initially present $a > 0$ molecules/cm^3 of A and $b > 0$ molecules/cm^3 of B, then the number $x(t)$ of molecules/cm^3 of C present at time t thereafter is determined by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x).$$

This equation has constant solutions $x(t) = a$ and $x(t) = b$, neither of which satisfies the initial condition $x(0) = 0$. We find other solutions for this equation by separation of variables and the technique of partial fraction decomposition under the assumption that $b \neq a$:

$$\int \frac{dx}{(a - x)(b - x)} = k \int dt = kt + C.$$

Since

$$\frac{1}{(a - x)(b - x)} = \frac{1}{b - a}\left(\frac{1}{a - x} - \frac{1}{b - x}\right),$$

and since necessarily $x \le a$ and $x \le b$, we have

$$\frac{1}{b - a}\big(-\ln(a - x) + \ln(b - x)\big) = kt + C,$$

or

$$\ln\left(\frac{b - x}{a - x}\right) = (b - a)\,kt + C_1, \quad \text{where } C_1 = (b - a)C.$$

By assumption, $x(0) = 0$, so $C_1 = \ln(b/a)$ and

$$\ln \frac{a(b - x)}{b(a - x)} = (b - a)\,kt.$$

This equation can be solved for x to yield $x = x(t) = \dfrac{ab(e^{(b-a)kt} - 1)}{be^{(b-a)kt} - a}$.

EXAMPLE 6 Find a family of curves, each of which intersects every parabola with equation of the form $y = Cx^2$ at right angles.

Solution The family of parabolas $y = Cx^2$ satisfies the differential equation

$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{d}{dx}C = 0;$$

that is,

$$x^2\frac{dy}{dx} - 2xy = 0 \qquad \text{or} \qquad \frac{dy}{dx} = \frac{2y}{x}.$$

Any curve that meets the parabolas $y = Cx^2$ at right angles must, at any point (x, y) on it, have slope equal to the negative reciprocal of the slope of the particular parabola passing through that point. Thus, such a curve must satisfy

$$\frac{dy}{dx} = -\frac{x}{2y}.$$

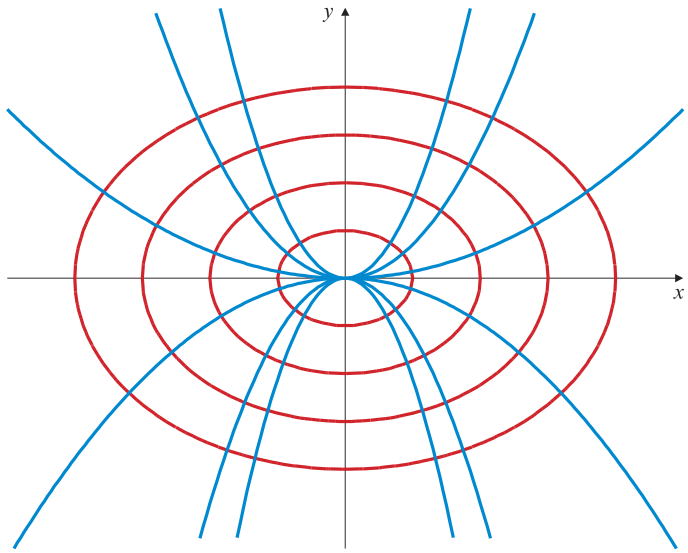


Figure 7.66 The parabolas $y = C_1 x^2$ (blue) and the ellipses $x^2 + 2y^2 = C_2$ (red) intersect at right angles

Separation of the variables leads to $2y\,dy = -x\,dx$, and integration of both sides then yields $y^2 = -\frac{1}{2}x^2 + C_1$ or $x^2 + 2y^2 = C$, where $C = 2C_1$. This equation represents a family of ellipses centred at the origin. Each ellipse meets each parabola at right angles, as shown in Figure 7.66. When the curves of one family intersect the curves of a second family at right angles, each family is called the family of **orthogonal trajectories** of the other family.

First-Order Linear Equations

A first-order **linear** differential equation is one of the type

$$\frac{dy}{dx} + p(x)y = q(x), \qquad (*)$$

where $p(x)$ and $q(x)$ are given functions, which we assume to be continuous. The equation is called **nonhomogeneous** unless $q(x)$ is identically zero. The corresponding **homogeneous** equation,

$$\frac{dy}{dx} + p(x)y = 0,$$

is separable and so is easily solved to give $y = K\,e^{-\mu(x)}$, where K is any constant and $\mu(x)$ is any antiderivative of $p(x)$:

$$\mu(x) = \int p(x)\,dx \qquad \text{and} \qquad \frac{d\mu}{dx} = p(x).$$

There are two methods for solving the nonhomogeneous equation $(*)$. Both involve the function $\mu(x)$ defined above.

METHOD I. Using an Integrating Factor. Multiply equation $(*)$ by $e^{\mu(x)}$ (which is called an **integrating factor** for the equation) and observe that the left side is just the derivative of $e^{\mu(x)}y$; by the Product Rule

$$\begin{aligned}\frac{d}{dx}\big(e^{\mu(x)}y(x)\big) &= e^{\mu(x)}\frac{dy}{dx} + e^{\mu(x)}\frac{d\mu}{dx}\,y(x)\\ &= e^{\mu(x)}\left(\frac{dy}{dx} + p(x)y\right) = e^{\mu(x)}\,q(x).\end{aligned}$$

Therefore, $e^{\mu(x)}\,y(x) = \displaystyle\int e^{\mu(x)}\,q(x)\,dx$, or

$$y(x) = e^{-\mu(x)} \int e^{\mu(x)} q(x)\, dx.$$

METHOD II. Variation of the Parameter. Start with the solution of the corresponding homogeneous equation, namely $y = K\, e^{-\mu(x)}$, and replace the constant (i.e., parameter) K by an as yet unknown function $k(x)$ of the independent variable. Then substitute this expression for y into the differential equation $(*)$ and simplify:

$$\frac{d}{dx}\Big(k(x)e^{-\mu(x)}\Big) + p(x)k(x)e^{-\mu(x)} = q(x)$$
$$k'(x)e^{-\mu(x)} - \mu'(x)k(x)e^{-\mu(x)} + p(x)k(x)e^{-\mu(x)} = q(x),$$

which, since $\mu'(x) = p(x)$, reduces to

$$k'(x) = e^{\mu(x)}q(x).$$

Integrating the right side leads to the solution for $k(x)$ and thereby to the solution y for $(*)$.

EXAMPLE 7 Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = 1$ for $x > 0$. Use both methods for comparison.

Solution Here, $p(x) = 1/x$, so $\mu(x) = \int p(x)\, dx = \ln x$ (for $x > 0$).
METHOD I. The integrating factor is $e^{\mu(x)} = x$. We calculate

$$\frac{d}{dx}(xy) = x\frac{dy}{dx} + y = x\left(\frac{dy}{dx} + \frac{y}{x}\right) = x,$$

and so

$$xy = \int x\, dx = \frac{1}{2}x^2 + C.$$

Finally,

$$y = \frac{1}{x}\left(\frac{1}{2}x^2 + C\right) = \frac{x}{2} + \frac{C}{x}.$$

This is a solution of the given equation for any value of the constant C.

METHOD II. The corresponding homogeneous equation, $\dfrac{dy}{dx} + \dfrac{y}{x} = 0$, has solution $y = Ke^{-\mu(x)} = \dfrac{K}{x}$. Replacing the constant K with the function $k(x)$ and substituting into the given differential equation we obtain

$$\frac{1}{x}k'(x) - \frac{1}{x^2}k(x) + \frac{1}{x^2}k(x) = 1,$$

so that $k'(x) = x$ and $k(x) = \dfrac{x^2}{2} + C$, where C is any constant. Therefore,

$$y = \frac{k(x)}{x} = \frac{x}{2} + \frac{C}{x},$$

the same solution obtained by METHOD I.

Remark Both methods really amount to the same calculations expressed in different ways. Use whichever one you think is easiest to understand. The remaining examples in this section will be done by using integrating factors, but variation of parameters will prove useful later on (Section 18.6) to deal with nonhomogeneous linear differential equations of second or higher order.

EXAMPLE 8 Solve $\dfrac{dy}{dx} + xy = x^3$.

Solution Here, $p(x) = x$, so $\mu(x) = x^2/2$ and $e^{\mu(x)} = e^{x^2/2}$. We calculate

$$\frac{d}{dx}(e^{x^2/2}y) = e^{x^2/2}\frac{dy}{dx} + e^{x^2/2}xy = e^{x^2/2}\left(\frac{dy}{dx} + xy\right) = x^3 e^{x^2/2}.$$

Thus,

$$\begin{aligned} e^{x^2/2}\,y &= \int x^3\,e^{x^2/2}\,dx \qquad \begin{array}{ll} \text{Let} \quad U = x^2, & dV = x\,e^{x^2/2}\,dx. \\ \text{Then } dU = 2x\,dx, & V = e^{x^2/2}. \end{array} \\ &= x^2\,e^{x^2/2} - 2\int x\,e^{x^2/2}\,dx \\ &= x^2\,e^{x^2/2} - 2\,e^{x^2/2} + C, \end{aligned}$$

and, finally, $y = x^2 - 2 + Ce^{-x^2/2}$.

EXAMPLE 9 **(An inductance-resistance circuit)** An electric circuit (see Figure 7.67) contains a constant DC voltage source of V volts, a switch, a resistor of size R ohms, and an inductor of size L henrys. The circuit has no capacitance. The switch, initially open so that no current is flowing, is closed at time $t = 0$ so that current begins to flow at that time. If the inductance L were zero, the current would suddenly jump from 0 amperes when $t < 0$ to $I = V/R$ amperes when $t > 0$. However, if $L > 0$ the current cannot change instantaneously; it will depend on time t. Let the current t seconds after the switch is closed be $I(t)$ amperes. It is known that $I(t)$ satisfies the initial-value problem

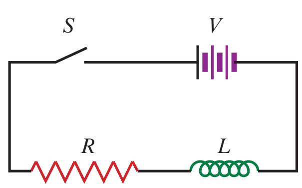


Figure 7.67 An inductance-resistance circuit

$$\begin{cases} L\dfrac{dI}{dt} + RI = V \\ \qquad I(0) = 0. \end{cases}$$

Find $I(t)$. What is $\lim_{t\to\infty} I(t)$? How long does it take after the switch is closed for the current to rise to 90% of its limiting value?

Solution The DE can be written in the form $\dfrac{dI}{dt} + \dfrac{R}{L}I = \dfrac{V}{L}$. It is linear and has integrating factor $e^{\mu(t)}$, where

$$\mu(t) = \int \frac{R}{L}\,dt = \frac{Rt}{L}.$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\left(e^{Rt/L}I\right) &= e^{Rt/L}\left(\frac{dI}{dt} + \frac{R}{L}I\right) = e^{Rt/L}\frac{V}{L} \\ e^{Rt/L}I &= \frac{V}{L}\int e^{Rt/L}\,dt = \frac{V}{R}e^{Rt/L} + C \\ I(t) &= \frac{V}{R} + Ce^{-Rt/L}. \end{aligned}$$

Since $I(0) = 0$, we have $0 = (V/R) + C$, so $C = -V/R$. Thus, the current flowing at any time $t > 0$ is

$$I(t) = \frac{V}{R}\left(1 - e^{-Rt/L}\right).$$

It is clear from this solution that $\lim_{t\to\infty} I(t) = V/R$; the *steady-state* current is the current that would flow if the inductance were zero.

$I(t)$ will be 90% of this limiting value when

$$\frac{V}{R}\left(1 - e^{-Rt/L}\right) = \frac{90}{100}\frac{V}{R}.$$

This equation implies that $e^{-Rt/L} = 1/10$, or $t = (L \ln 10)/R$. The current will grow to 90% of its limiting value in $(L \ln 10)/R$ seconds.

Our final example reviews a typical *stream of payments* problem of the sort considered in Section 7.7. This time we treat the problem as an initial-value problem for a differential equation.

EXAMPLE 10 A savings account is opened with a deposit of A dollars. At any time t years thereafter, money is being continually deposited into the account at a rate of $(C + Dt)$ dollars per year. If interest is also being paid into the account at a nominal rate of $100R$ percent per year, compounded continuously, find the balance $B(t)$ dollars in the account after t years. Illustrate the solution for the data $A = 5{,}000$, $C = 1{,}000$, $D = 200$, $R = 0.13$, and $t = 5$.

Solution As noted in Section 3.4, continuous compounding of interest at a nominal rate of $100R$ percent causes \$1.00 to grow to e^{Rt} dollars in t years. Without subsequent deposits, the balance in the account would grow according to the differential equation of exponential growth:

$$\frac{dB}{dt} = RB.$$

Allowing for additional growth due to the continual deposits, we observe that B must satisfy the differential equation

$$\frac{dB}{dt} = RB + (C + Dt)$$

or, equivalently, $dB/dt - RB = C + Dt$. This is a linear equation for B having $p(t) = -R$. Hence, we may take $\mu(t) = -Rt$ and $e^{\mu(t)} = e^{-Rt}$. We now calculate

$$\frac{d}{dt}\left(e^{-Rt}B(t)\right) = e^{-Rt}\frac{dB}{dt} - Re^{-Rt}B(t) = (C + Dt)\,e^{-Rt}$$

and

$$\begin{aligned}
e^{-Rt}B(t) &= \int (C + Dt)e^{-Rt}\,dt \qquad \text{Let } U = C + Dt, \quad dV = e^{-Rt}\,dt. \\
&\qquad\qquad\qquad\qquad\quad\ \text{Then } dU = D\,dt, \quad V = -e^{-Rt}/R. \\
&= -\frac{C + Dt}{R}e^{-Rt} + \frac{D}{R}\int e^{-Rt}\,dt \\
&= -\frac{C + Dt}{R}e^{-Rt} - \frac{D}{R^2}e^{-Rt} + K, \qquad (K = \text{constant}).
\end{aligned}$$

Hence,

$$B(t) = -\frac{C + Dt}{R} - \frac{D}{R^2} + Ke^{Rt}.$$

Since $A = B(0) = -\frac{C}{R} - \frac{D}{R^2} + K$, we have $K = A + \frac{C}{R} + \frac{D}{R^2}$ and

$$B(t) = \left(A + \frac{C}{R} + \frac{D}{R^2}\right)e^{Rt} - \frac{C + Dt}{R} - \frac{D}{R^2}.$$

For the illustration $A = 5{,}000$, $C = 1{,}000$, $D = 200$, $R = 0.13$, and $t = 5$, we obtain, using a calculator, $B(5) = 19{,}762.82$. The account will contain \$19,762.82, after five years, under these circumstances.

EXERCISES 7.9

Solve the separable equations in Exercises 1–10.

1. $\frac{dy}{dx} = \frac{y}{2x}$ **2.** $\frac{dy}{dx} = \frac{3y - 1}{x}$

3. $\frac{dy}{dx} = \frac{x^2}{y^2}$ **4.** $\frac{dy}{dx} = x^2y^2$

5. $\frac{dY}{dt} = tY$ **6.** $\frac{dx}{dt} = e^x \sin t$

7. $\frac{dy}{dx} = 1 - y^2$ **8.** $\frac{dy}{dx} = 1 + y^2$

9. $\frac{dy}{dt} = 2 + e^y$ **10.** $\frac{dy}{dx} = y^2(1 - y)$

Solve the linear equations in Exercises 11–16.

11. $\frac{dy}{dx} - \frac{2y}{x} = x^2$ **12.** $\frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^2}$

13. $\frac{dy}{dx} + 2y = 3$ **14.** $\frac{dy}{dx} + y = e^x$

15. $\frac{dy}{dx} + y = x$ **16.** $\frac{dy}{dx} + 2e^x y = e^x$

Solve the initial-value problems in Exercises 17–20.

17. $\begin{cases} \frac{dy}{dt} + 10y = 1 \\ y(1/10) = 2/10 \end{cases}$ **18.** $\begin{cases} \frac{dy}{dx} + 3x^2y = x^2 \\ y(0) = 1 \end{cases}$

19. $\begin{cases} x^2y' + y = x^2e^{1/x} \\ y(1) = 3e \end{cases}$ **20.** $\begin{cases} y' + (\cos x)y = 2xe^{-\sin x} \\ y(\pi) = 0 \end{cases}$

Solve the integral equations in Exercises 21–24.

21. $y(x) = 2 + \int_0^x \frac{t}{y(t)}\,dt$ **22.** $y(x) = 1 + \int_0^x \frac{\left(y(t)\right)^2}{1 + t^2}\,dt$

23. $y(x) = 1 + \int_1^x \frac{y(t)\,dt}{t(t + 1)}$ **24.** $y(x) = 3 + \int_0^x e^{-y(t)}\,dt$

25. If $a > b > 0$ in Example 5, find $\lim_{t\to\infty} x(t)$.

26. If $b > a > 0$ in Example 5, find $\lim_{t\to\infty} x(t)$.

27. Why is the solution given in Example 5 not valid for $a = b$? Find the solution for the case $a = b$.

28. An object of mass m falling near the surface of the earth is retarded by air resistance proportional to its velocity so that, according to Newton's Second Law of Motion,

$$m\frac{dv}{dt} = mg - kv,$$

where $v = v(t)$ is the velocity of the object at time t, and g is the acceleration of gravity near the surface of the earth. Assuming that the object falls from rest at time $t = 0$, that is, $v(0) = 0$, find the velocity $v(t)$ for any $t > 0$ (up until the object strikes the ground). Show $v(t)$ approaches a limit as $t \to \infty$. Do you need the explicit formula for $v(t)$ to determine this limiting velocity?

29. Repeat Exercise 28 except assume that the air resistance is proportional to the square of the velocity so that the equation of motion is

$$m\frac{dv}{dt} = mg - kv^2.$$

30. Find the amount in a savings account after one year if the initial balance in the account was \$1,000, if the interest is paid continuously into the account at a nominal rate of 10% per annum, compounded continuously, and if the account is being continuously depleted (by taxes, say) at a rate of $y^2/1{,}000{,}000$ dollars per year, where $y = y(t)$ is the balance in the account after t years. How large can the account grow? How long will it take the account to grow to half this balance?

31. Find the family of curves each of which intersects all of the hyperbolas $xy = C$ at right angles.

32. Repeat the solution concentration problem in Example 4, changing the rate of inflow of brine into the tank to 12 L/min but leaving all the other data as they were in that example. Note that the volume of liquid in the tank is no longer constant as time increases.

CHAPTER REVIEW

Key Ideas

- **What do the following phrases mean?**
  - ◇ a solid of revolution
  - ◇ a volume element
  - ◇ the arc length of a curve
  - ◇ the moment of a point mass m about $x = 0$

◇ the centre of mass of a distribution of mass

◇ the centroid of a plane region

◇ a first-order separable differential equation

◇ a first-order linear differential equation

- **Let D be the plane region $0 \le y \le f(x)$, $a \le x \le b$. Use integrals to represent the following:**

◇ the volume generated by revolving D about the x-axis

◇ the volume generated by revolving D about the y-axis

◇ the moment of D about the y-axis

◇ the moment of D about the x-axis

◇ the centroid of D

- **Let C be the curve $y = f(x)$, $a \le x \le b$. Use integrals to represent the following:**

◇ the length of C

◇ the area of the surface generated by revolving C about the x-axis

◇ the area of the surface generated by revolving C about the y-axis

Review Exercises

1. Figure 7.68 shows cross-sections along the axes of two circular spools. The left spool will hold 1,000 metres of thread if wound full with no bulging. How many metres of thread of the same size will the right spool hold?

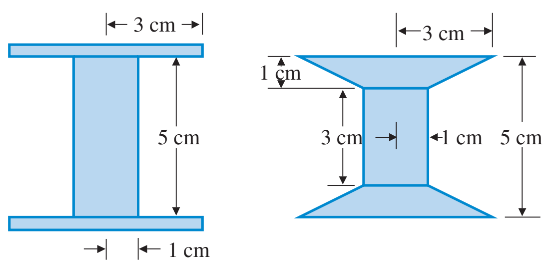


Figure 7.68

2. Water sitting in a bowl evaporates at a rate proportional to its surface area. Show that the depth of water in the bowl decreases at a constant rate, regardless of the shape of the bowl.

3. A barrel is 4 ft high and its volume is 16 cubic feet. Its top and bottom are circular disks of radius 1 ft, and its side wall is obtained by rotating the part of the parabola $x = a - by^2$ between $y = -2$ and $y = 2$ about the y-axis. Find, approximately, the values of the positive constants a and b.

4. The solid in Figure 7.69 is cut from a vertical cylinder of radius 10 cm by two planes making angles of 60° with the horizontal. Find its volume.

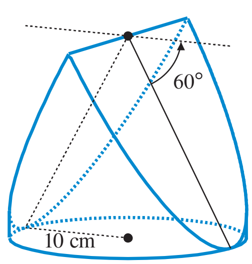


Figure 7.69

5. Find to 4 decimal places the value of the positive constant a for which the curve $y = (1/a)\cosh ax$ has arc length 2 units between $x = 0$ and $x = 1$.

6. Find the area of the surface obtained by rotating the curve $y = \sqrt{x}$, $(0 \le x \le 6)$, about the x-axis.

7. Find the centroid of the plane region $x \ge 0$, $y \ge 0$, $x^2 + 4y^2 \le 4$.

8. A thin plate in the shape of a circular disk has radius 3 ft and constant areal density. A circular hole of radius 1 ft is cut out of the disk, centred 1 ft from the centre of the disk. Find the centre of mass of the remaining part of the disk.

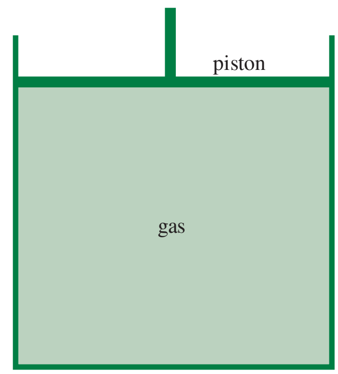


Figure 7.70

9. According to Boyle's Law, the product of the pressure and volume of a gas remains constant if the gas expands or is compressed isothermally. The cylinder in Figure 7.70 is filled with a gas that exerts a force of 1,000 N on the piston when the piston is 20 cm above the base of the cylinder. How much work is done by the piston if it compresses the gas isothermally by descending to a height of 5 cm above the base?

10. Suppose two functions f and g have the following property: for any $a > 0$, the solid produced by revolving the region of the xy-plane bounded by $y = f(x)$, $y = g(x)$, $x = 0$, and $x = a$ about the x-axis has the same volume as the solid produced by revolving the same region about the y-axis. What can you say about f and g?

11. Find the equation of a curve that passes through the point $(2, 4)$ and has slope $3y/(x - 1)$ at any point (x, y) on it.

12. Find a family of curves that intersect every ellipse of the form $3x^2 + 4y^2 = C$ at right angles.

13. The income and expenses of a seasonal business result in deposits and withdrawals from its bank account that correspond to a flow rate into the account of $\$P(t)$/year at time t years, where $P(t) = 10{,}000 \sin(2\pi t)$. If the account earns interest at an instantaneous rate of 4% per year and has \$8,000 in it at time $t = 0$, how much is in the account two years later?

Challenging Problems

1. The curve $y = e^{-kx} \sin x$, $(x \ge 0)$, is revolved about the x-axis to generate a string of "beads" whose volumes decrease to the right if $k > 0$.
   (a) Show that the ratio of the volume of the $(n + 1)$st bead to that of the nth bead depends on k, but not on n.
   (b) For what value of k is the ratio in part (a) equal to 1/2?
   (c) Find the total volume of all the beads as a function of the positive number k.

2. **(Conservation of earth)** A landscaper wants to create on level ground a ring-shaped pool having an outside radius of 10 m and a maximum depth of 1 m surrounding a hill that will be built up using all the earth excavated from the pool. (See Figure 7.71.) She decides to use a fourth-degree polynomial to determine the cross-sectional shape of the hill and pool bottom: at distance r metres from the centre of the development the height above or below normal ground level will be

$$h(r) = a(r^2 - 100)(r^2 - k^2) \text{ metres},$$

for some $a > 0$, where k is the inner radius of the pool. Find k and a so that the requirements given above are all satisfied. How much earth must be moved from the pool to build the hill?

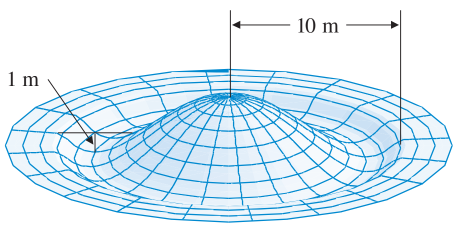


Figure 7.71

3. **(Rocket design)** The nose of a rocket is a solid of revolution of base radius r and height h that must join smoothly to the cylindrical body of the rocket. (See Figure 7.72.) Taking the origin at the tip of the nose and the x-axis along the central axis of the rocket, various nose shapes can be obtained by revolving the cubic curve

$$y = f(x) = ax + bx^2 + cx^3$$

about the x-axis. The cubic curve must have slope 0 at $x = h$, and its slope must be positive for $0 < x < h$. Find the particular cubic curve that maximizes the volume of the nose. Also show that this choice of the cubic makes the slope dy/dx at the origin as large as possible and, hence, corresponds to the bluntest nose.

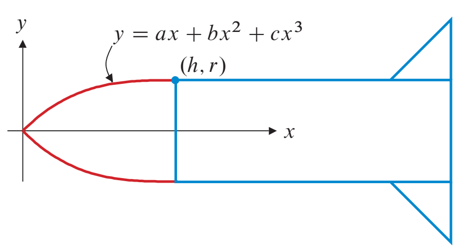


Figure 7.72

4. **(Quadratic splines)** Let $A = (x_1, y_1)$, $B = (x_2, y_2)$, and $C = (x_3, y_3)$ be three points with $x_1 < x_2 < x_3$. A function $f(x)$ whose graph passes through the three points is a *quadratic spline* if $f(x)$ is a quadratic function on $[x_1, x_2]$ and a possibly different quadratic function on $[x_2, x_3]$, and the two quadratics have the same slope at x_2. For this problem, take $A = (0, 1)$, $B = (1, 2)$, and $C = (3, 0)$.

(a) Find a one-parameter family $f(x, m)$ of quadratic splines through A, B, and C, having slope m at B.

(b) Find the value of m for which the length of the graph $y = f(x, m)$ between $x = 0$ and $x = 3$ is minimum. What is this minimum length? Compare it with the length of the polygonal line ABC.

5. A concrete wall in the shape of a circular ring must be built to have maximum height 2 m, inner radius 15 m, and width 1 m at ground level, so that its outer radius is 16 m. (See Figure 7.73.) Built on level ground, the wall will have a curved top with height at distance $15 + x$ metres from the centre of the ring given by the cubic function

$$f(x) = x(1 - x)(ax + b) \text{ m},$$

which must not vanish anywhere in the open interval $(0, 1)$. Find the values of a and b that minimize the total volume of concrete needed to build the wall.

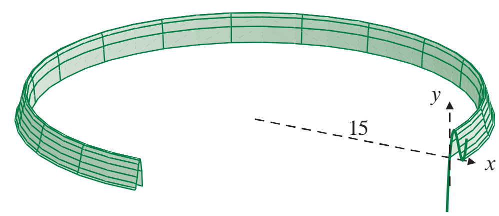


Figure 7.73

6. **(The volume of an *n*-dimensional ball)** Euclidean n-dimensional space consists of *points* $(x_1, x_2, \dots, x_n)$ with n real coordinates. By analogy with the 3-dimensional case, we call the set of such points that satisfy the inequality $x_1^2 + x_2^2 + \cdots + x_n^2 \le r^2$ the n-dimensional *ball* centred at the origin. For example, the 1-dimensional ball is the interval $-r \le x_1 \le r$, which has *volume* (i.e., *length*) $V_1(r) = 2r$. The 2-dimensional ball is the disk $x_1^2 + x_2^2 \le r^2$, which has *volume* (i.e., *area*)

$$\begin{aligned} V_2(r) = \pi r^2 &= \int_{-r}^{r} 2\sqrt{r^2 - x^2}\, dx \\ &= \int_{-r}^{r} V_1\left(\sqrt{r^2 - x^2}\right) dx. \end{aligned}$$

The 3-dimensional ball $x_1^2 + x_2^2 + x_3^2 \le r^2$ has volume

$$\begin{aligned} V_3(r) = \frac{4}{3}\pi r^3 &= \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx \\ &= \int_{-r}^{r} V_2\left(\sqrt{r^2 - x^2}\right) dx. \end{aligned}$$

By analogy with these formulas, the volume $V_n(r)$ of the n-dimensional ball of radius r is the integral of the volume of the $(n - 1)$-dimensional ball of radius $\sqrt{r^2 - x^2}$ from $x = -r$ to $x = r$:

$$V_n(r) = \int_{-r}^{r} V_{n-1}\left(\sqrt{r^2 - x^2}\right) dx.$$

Using a computer algebra program, calculate $V_4(r), V_5(r), \dots, V_{10}(r)$, and guess formulas for $V_{2n}(r)$ (the even-dimensional balls) and $V_{2n+1}(r)$ (the odd-dimensional balls). If your computer algebra software is sufficiently powerful, you may be able to verify your guesses by induction. Otherwise, use them to predict $V_{11}(r)$ and $V_{12}(r)$, then check your predictions by starting from $V_{10}(r)$.

7. **(Buffon's needle problem)** A horizontal flat surface is ruled with parallel lines 10 cm apart, as shown in Figure 7.74. A needle 5 cm long is dropped at random onto the surface. Find the probability that the needle intersects one of the lines. *Hint:* Let the "lower" end of the needle (the end further down the page in the figure) be considered the reference point. (If both ends are the same height, use the left end.) Let y be the distance from the reference point to the nearest line above it, and let θ be the angle between the needle and the line extending to the right of the reference point in the figure. What are the possible values of y and θ? In a plane with Cartesian coordinates θ and y, sketch the region consisting of all points (θ, y) corresponding to possible positions of the needle. Also sketch the region corresponding to those positions for which the needle crosses one of the parallel lines. The required probability is the area of the second region divided by the area of the first.

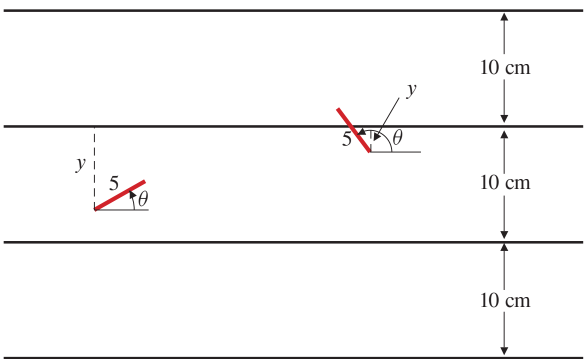


Figure 7.74

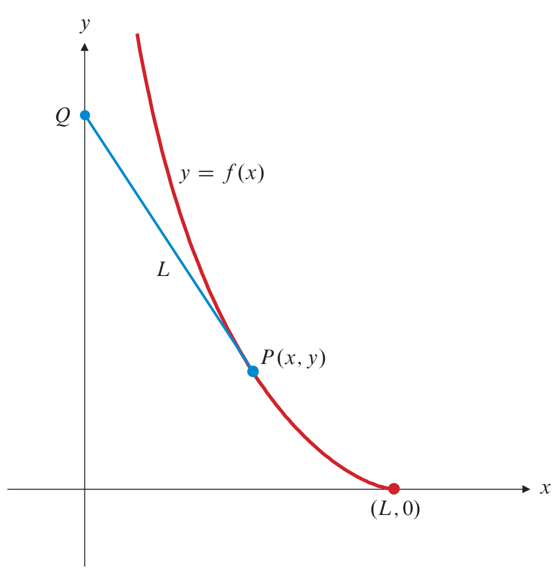


Figure 7.75

8. **(The path of a trailer)** Find the equation $y = f(x)$ of a curve in the first quadrant of the xy-plane, starting from the point $(L, 0)$, and having the property that if the tangent line to the curve at P meets the y-axis at Q, then the length of PQ is the constant L. (See Figure 7.75. This curve is called a **tractrix** after the Latin participle *tractus*, meaning *dragged*. It is the path of the rear end P of a trailer of length L, originally lying along the x-axis, as the trailer is pulled (dragged) by a tractor Q moving along the y-axis away from the origin.)

9. **(Approximating the surface area of an ellipsoid)** A physical geographer studying the flow of streams around oval stones needed to calculate the surface areas of many such stones that he modelled as ellipsoids:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

He wanted a simple formula for the surface area so that he could implement it in a spreadsheet containing the measurements a, b, and c of the stones. Unfortunately, there is no exact formula for the area of a general ellipsoid in terms of elementary functions. However, there are such formulas for ellipsoids of revolution, where two of the three semi-axes are equal. These ellipsoids are called spheroids; an *oblate spheroid* (like the earth) has its two longer semi-axes equal; a *prolate spheroid* (like an American football) has its two shorter semi-axes equal. A reasonable approximation to the area of a general ellipsoid can be obtained by linear interpolation between these two.

To be specific, assume the semi-axes are arranged in decreasing order $a \ge b \ge c$, and let the surface area be $S(a, b, c)$.

(a) Calculate $S(a, a, c)$, the area of an oblate spheroid.

(b) Calculate $S(a, c, c)$, the area of a prolate spheroid.

(c) Construct an approximation for $S(a, b, c)$ that divides the interval from $S(a, a, c)$ to $S(a, c, c)$ in the same ratio that b divides the interval from a to c.

(d) Approximate the area of the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$$

using the above method.