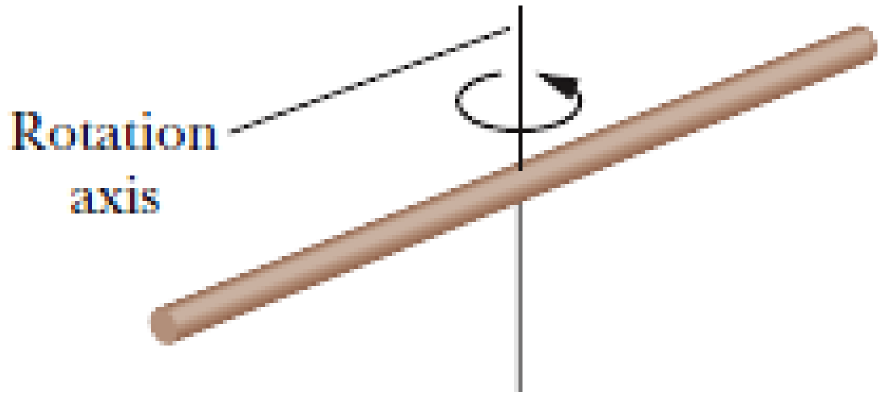
Chapter 10: Rotation

- ✓ Angular Position, Velocity and Acceleration
- ✓ Rotational Kinematics
- ✓ Kinetic Energy of Rotation
- ✓ Rotational Inertia
- ✓ Torque
- ✓ Energy Consideration in Rotational Motion

Session 21:

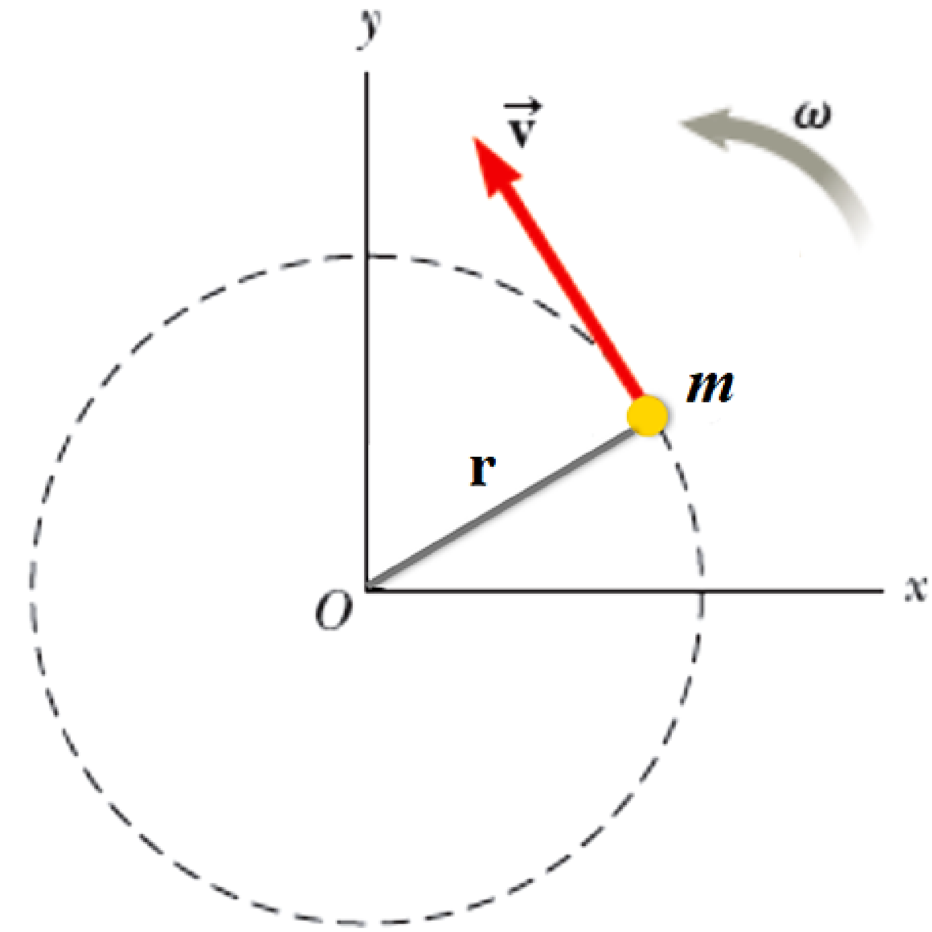
- ✓ Kinetic Energy of Rotation
- ✓ Rotational Inertia
- ✓ Examples

Kinetic Energy of Rotation



$$\boxed{k = \frac{1}{2} M v_{com}^{2} = 0}$$

$$\boxed{k = \frac{1}{2} m v^2 = \frac{1}{2} m (r\omega)^2 = \frac{1}{2}\left(m r^2\right)\omega^2}$$



For a collection of rotating objects:

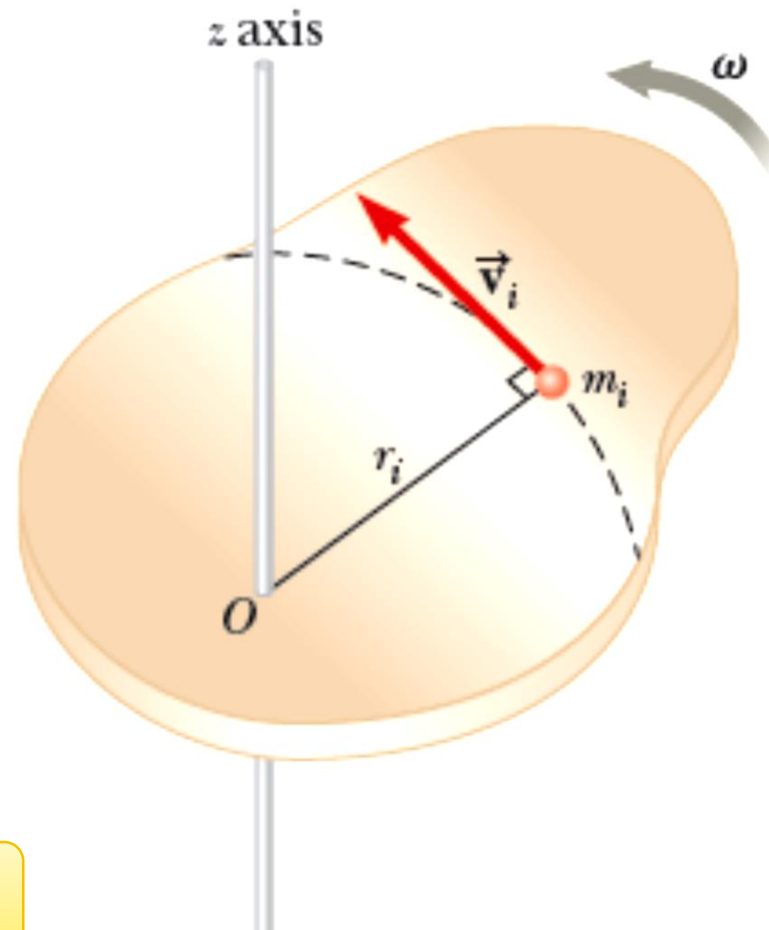
$$k = \sum_{i=1}^{N} k_i = \sum_{i=1}^{N} \frac{1}{2}\left(m_i r_i^2\right)\omega_i^2 = \frac{1}{2}\sum_{i=1}^{N}\left(m_i r_i^2\right)\omega_i^2$$

if $\omega =$ constant

$$\boxed{k = \frac{1}{2}\left(\sum_{i=1}^{N} m_i r_i^2\right)\omega^2 = \frac{1}{2} I \omega^2}$$

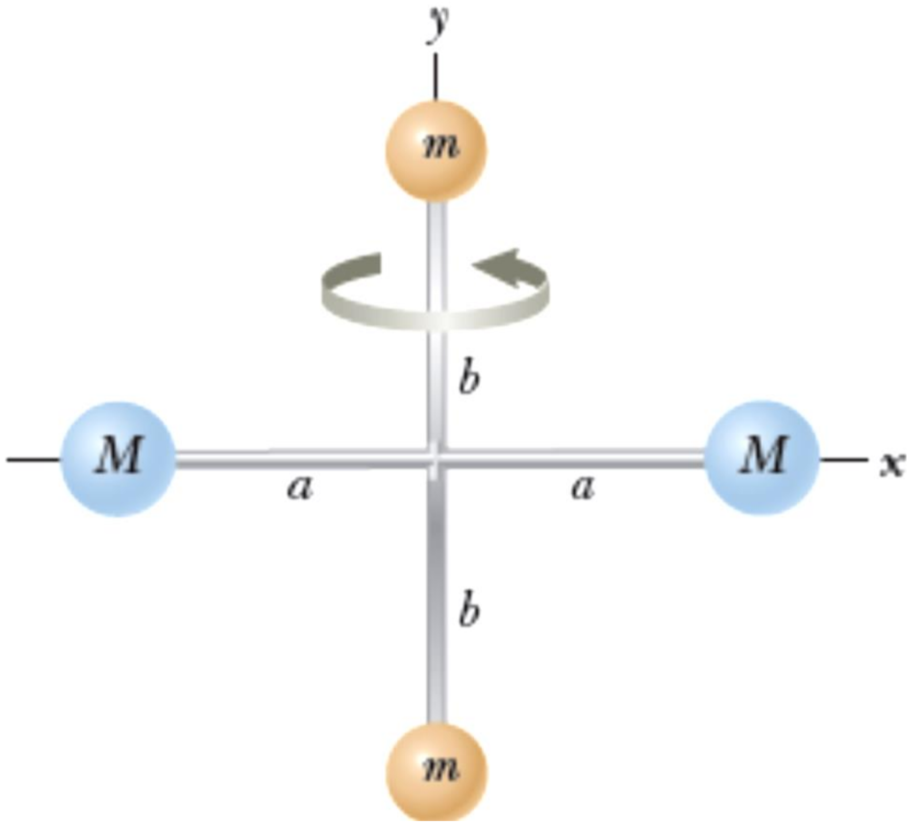
$$\boxed{I = \sum_{i=1}^{N} m_i r_i^2}$$

Rotational Inertial (kg.m²)



Kinetic Energy of Rotation

$$k = \frac{1}{2}\left(\sum_{i=1}^{N} m_i r_i^2\right)\omega^2 = \frac{1}{2} I\omega^2$$

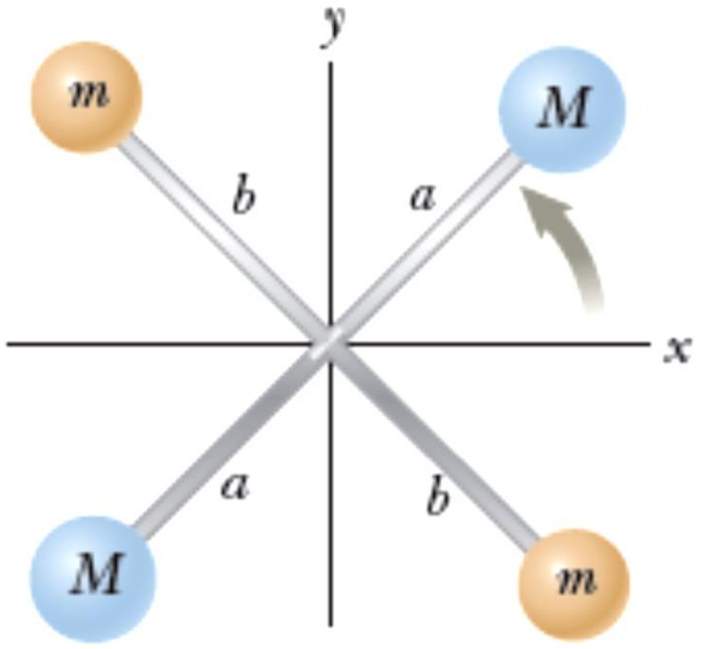


For rotation about *y* axis:

$$I_y = \sum_{i=1}^{4} m_i r_i^2 = Ma^2 + Ma^2 + m(0) + m(0) = 2Ma^2$$

$$\boxed{k_1 = \frac{1}{2}(2Ma^2)\omega^2 = Ma^2\omega^2}$$

For rotation about *z* axis:



$$I_z = \sum_{i=1}^{4} m_i r_i^2 = Ma^2 + Ma^2 + mb^2 + mb^2 = 2Ma^2 + 2mb^2$$

$$\boxed{k_2 = \frac{1}{2}(2Ma^2 + 2mb^2)\omega^2 = (Ma^2 + mb^2)\omega^2}$$

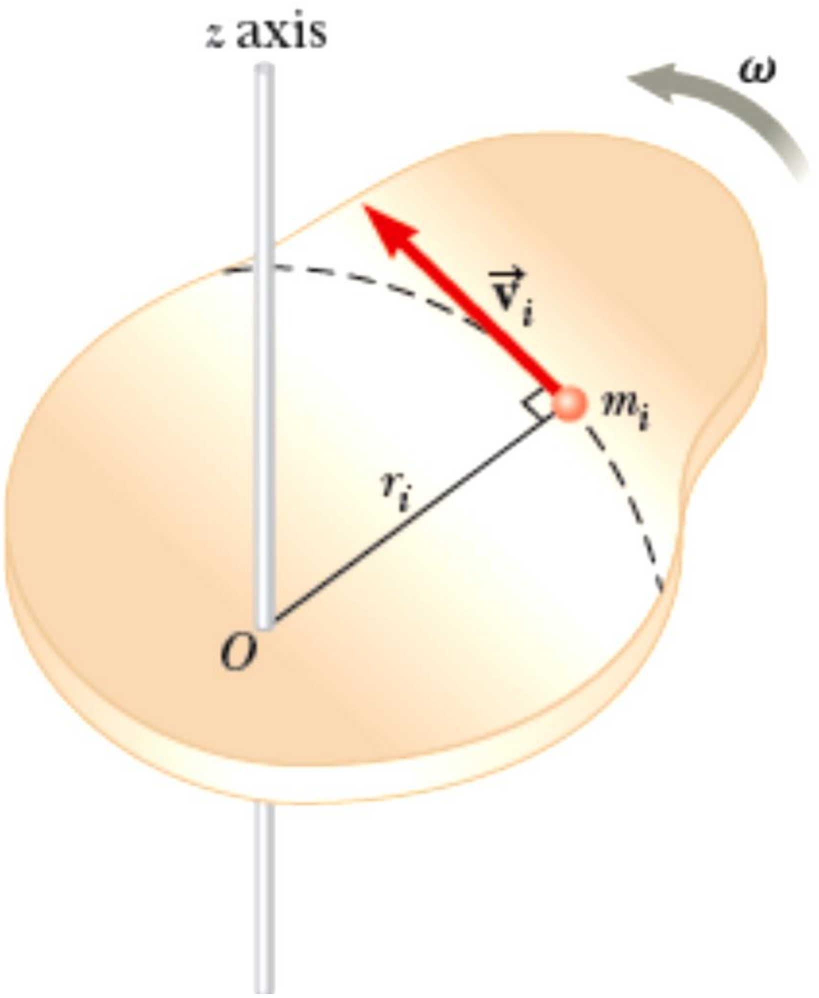
$$\boxed{k_2 > k_1}$$

Rotational Inertia of A Rigid Body

$$I = \sum_{i=1}^{N} m_i r_i^2$$

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i$$

$$\boxed{I = \int r^2 dm}$$



1) **Linear Mass Density** → **mass per unit length :** $\lambda = \dfrac{m}{L}$ ⇒ $dm = \lambda\, dL$

2) **Surface Mass Density** → **mass per unit surface, :** $\sigma = \dfrac{m}{A}$ ⇒ $dm = \sigma\, dA$

3) **Volumetric Mass Density** → **mass per unit volume:** $\rho = \dfrac{m}{V}$ ⇒ $dm = \rho\, dV$

Ex 6: Calculate the moment of inertia of a uniform thin rod of length **L** and mass **M** about an axis perpendicular to the rod and passing through its center of mass.

$$\boxed{I = \int r^2 dm}$$

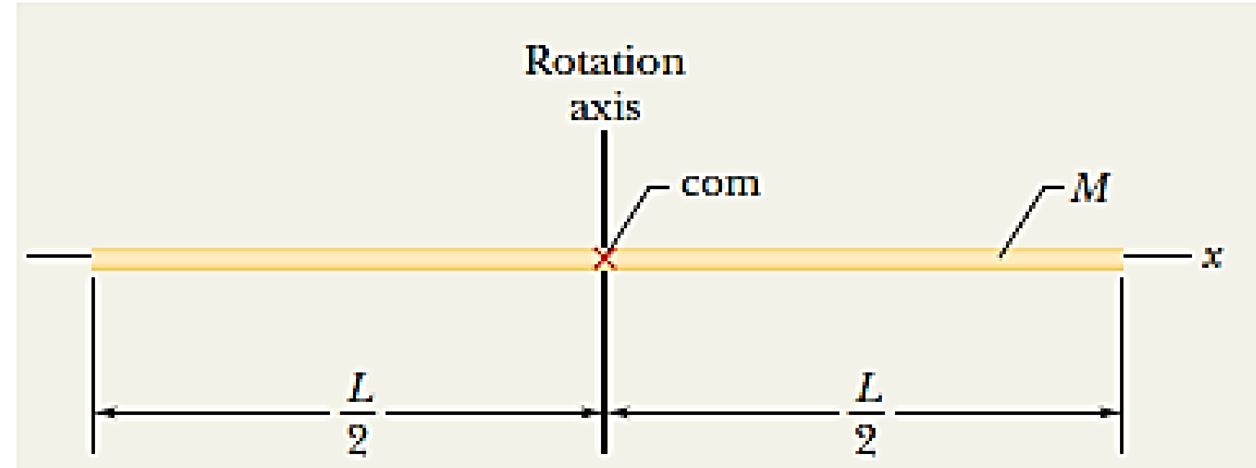
$$\lambda = \frac{M}{L} \Rightarrow dm = \lambda\, dx = \frac{M}{L} dx$$

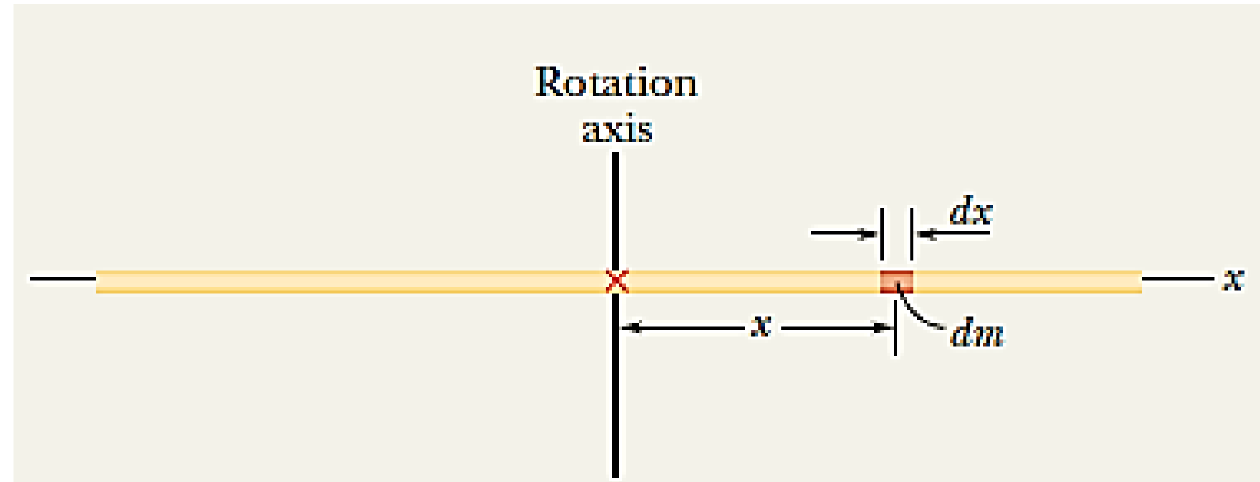
$$I_y = \int r^2 dm = \int_{-L/2}^{L/2} x^2 \left(\frac{M}{L}\, dx\right)$$

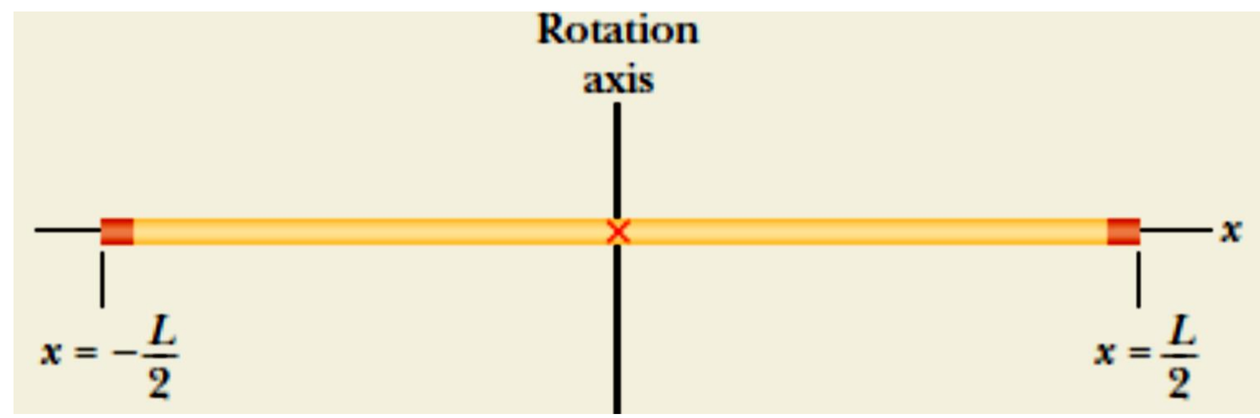
$$\boxed{I_y = 2\left(\frac{M}{L}\right)\frac{x^3}{3}\bigg|_0^{\frac{L}{2}} = 2\left(\frac{M}{L}\right)\frac{L^3}{24} = \frac{1}{12} M L^2}$$

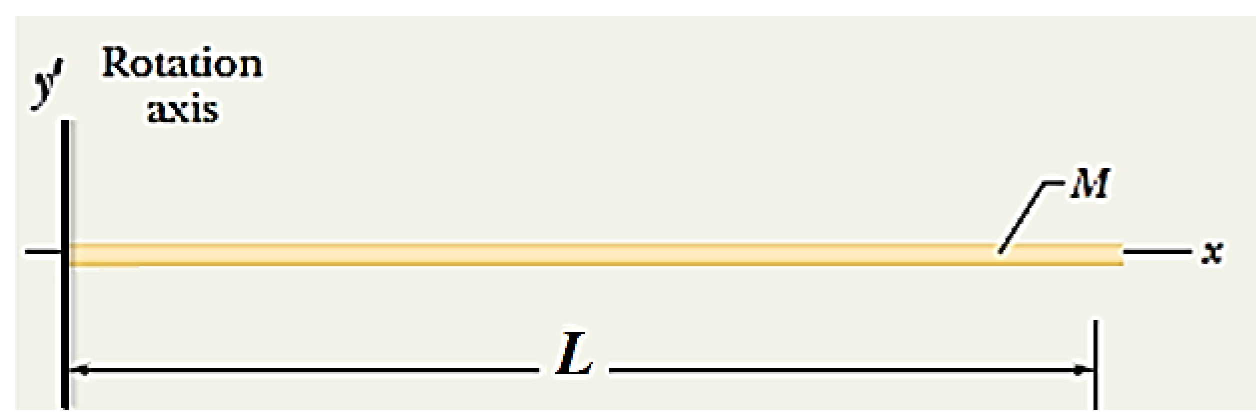
$$\boxed{I_{y'} = \int r^2 dm = \int_0^L x^2 \left(\frac{M}{L}\, dx\right) = \frac{1}{3} M L^2}$$

$$\boxed{I_{y'} > I_y}$$

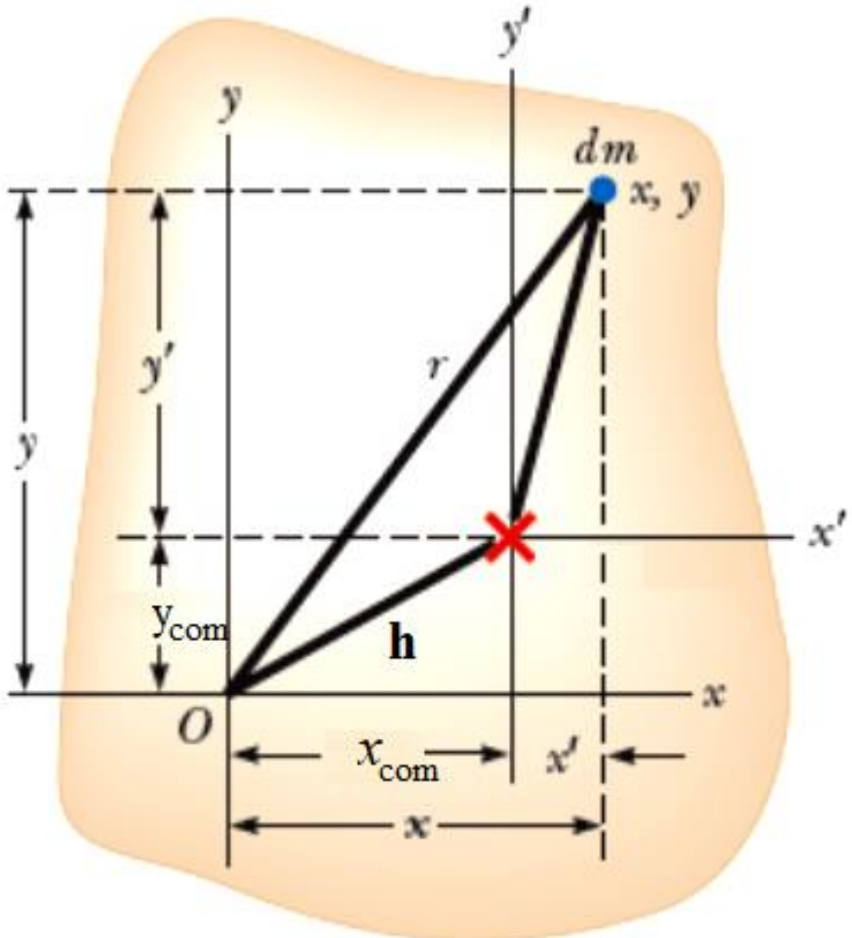




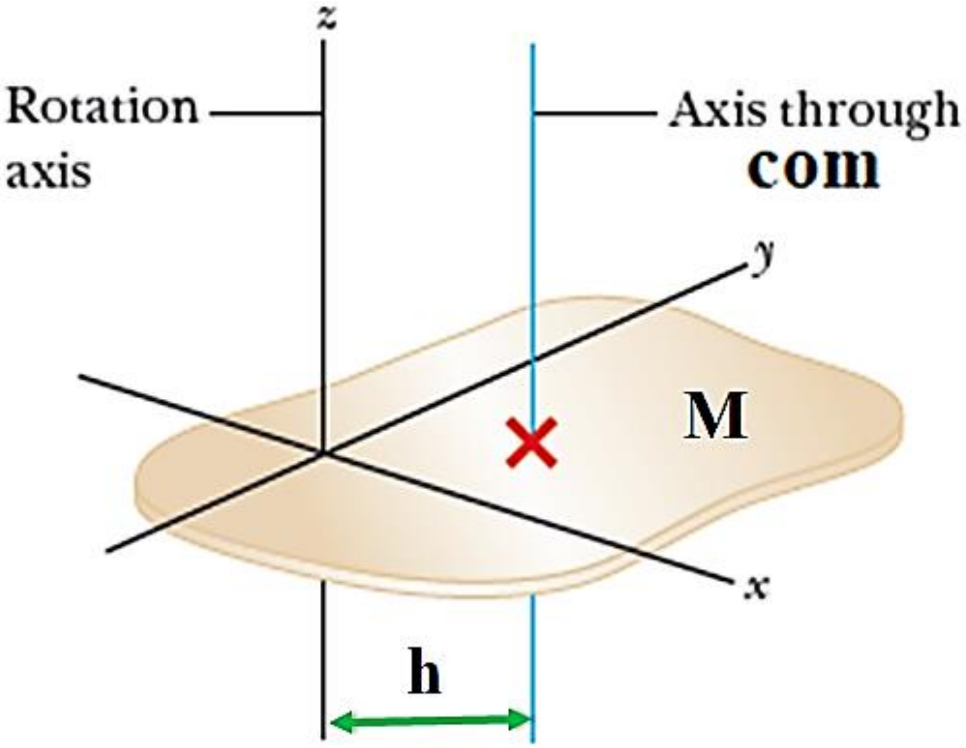




Parallel-Axis Theorem



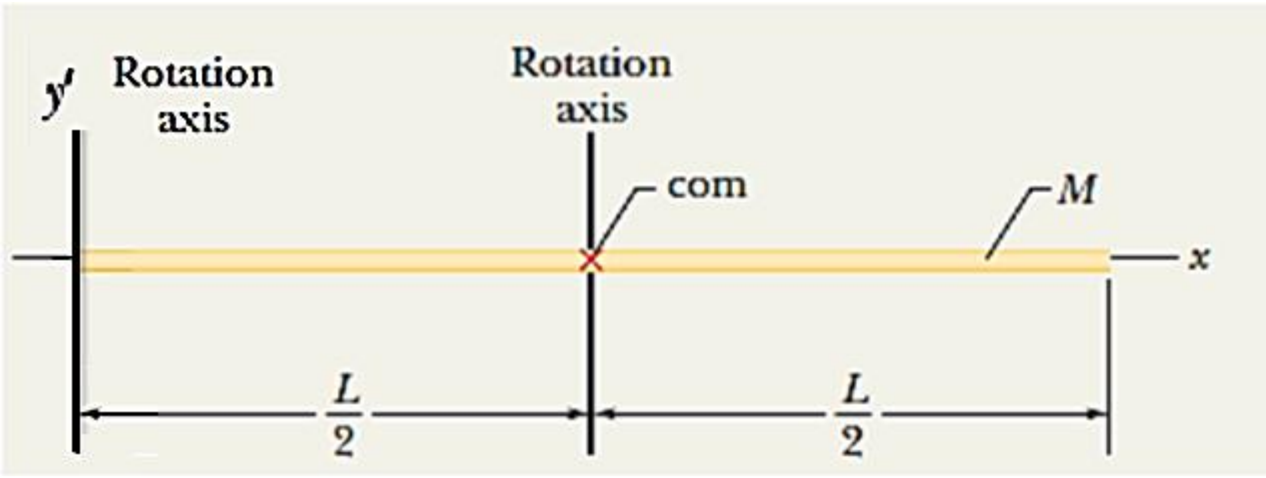
$$\boxed{I_z = I_{com} + M h^2}$$



$$I_z = \int r^2 dm = \int (x^2 + y^2) dm = \int \left[(x' + x_{com})^2 + (y' + y_{com})^2 \right] dm$$

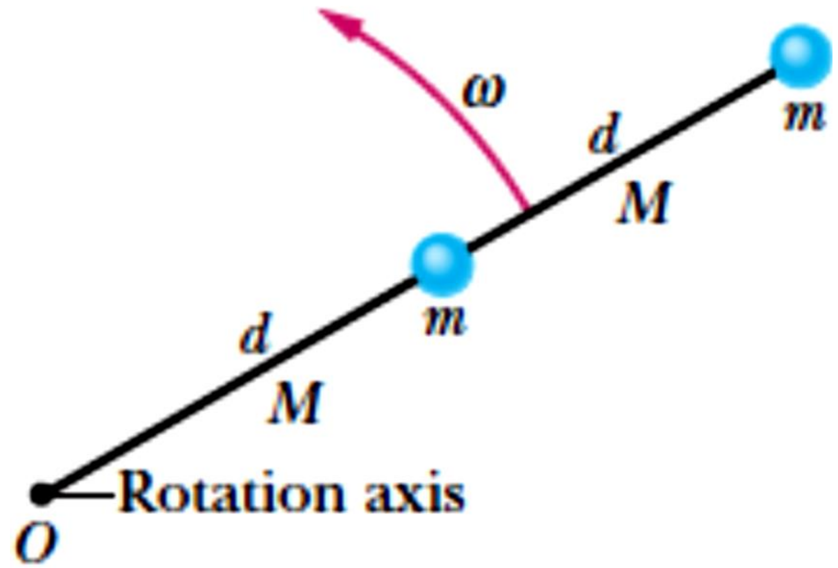
$$I_z = \underbrace{\int (x'^2 + y'^2) dm}_{I_{com}} + 2x_{com} \underbrace{\int x' dm}_{0} + 2y_{com} \underbrace{\int y' dm}_{0} + (\underbrace{x_{com}{}^2 + y_{com}{}^2}_{h^2}) \underbrace{\int dm}_{M}$$

$$\boxed{I_{y'} = I_{com} + M h^2 = \frac{1}{12} M L^2 + M \left(\frac{L}{2} \right)^2 = \frac{1}{3} M L^2}$$



Ex 7: (Problem 10. 41 Halliday)

Two particles, each with mass **m = 0.85 kg**, are fastened to each other, and to a rotation axis at O, by two thin rods, each with length **d = 5.6 cm** and mass **M = 1.2 kg**. The combination rotates around the rotation axis with the angular speed **ω = 0.30 rad/s**. Measured bout O, what are the combination's (a) rotational inertia and (b) kinetic energy?



Two particles:

$$I_1 = md^2 + m(2d)^2 = 5md^2$$

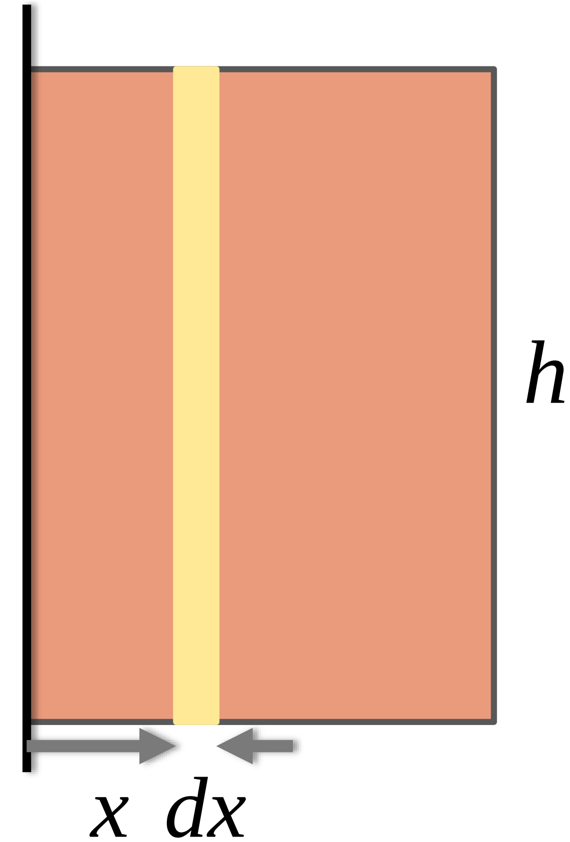
Two rods:

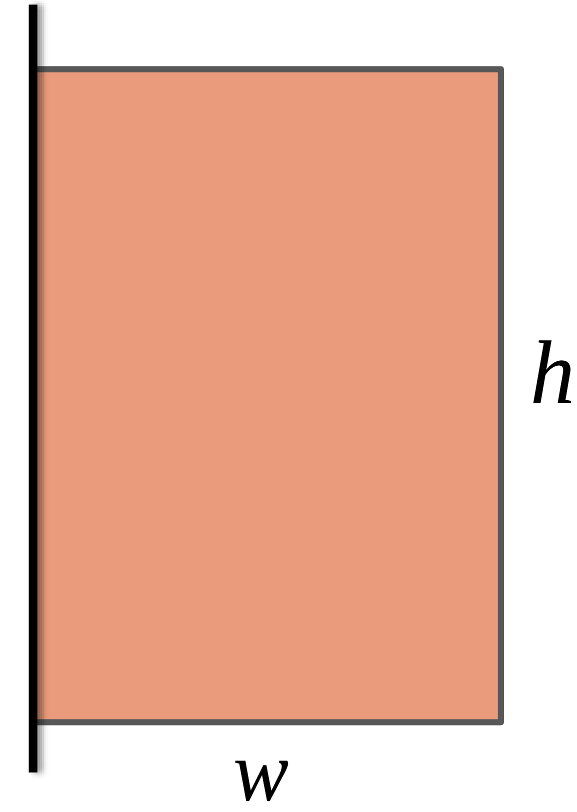
$$I_2 = \left[\frac{1}{12}Md^2 + M\left(\frac{d}{2}\right)^2\right] + \left[\frac{1}{12}Md^2 + M\left(\frac{3d}{2}\right)^2\right] = \frac{8}{3}Md^2$$

$$I_T = I_1 + I_2 = 5md^2 + \frac{8}{3}Md^2 = \left(5\times 0.85 + \frac{8}{3}\times 1.2\right)(0.056)^2 = 0.023\ kg.m^2$$

$$k = \frac{1}{2}I_T\omega^2 = \frac{1}{2}(0.023)(0.30)^2 = 1.03\times 10^{-3}\ J$$

Ex 8: A uniform, thin, solid door has height **2.20 m**, width **0.870 m**, and mass **23 kg**. Find its **rotational inertia** for rotation on its **hinges**.





$$\boxed{I = \int r^2 dm}$$

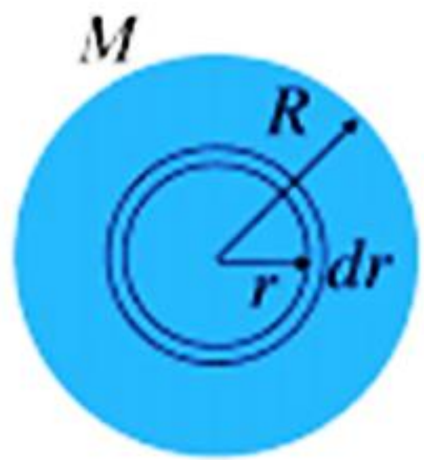
$$\sigma = \frac{M}{A} \Rightarrow dm = \sigma\, dA = \frac{M}{A} h\, dx$$

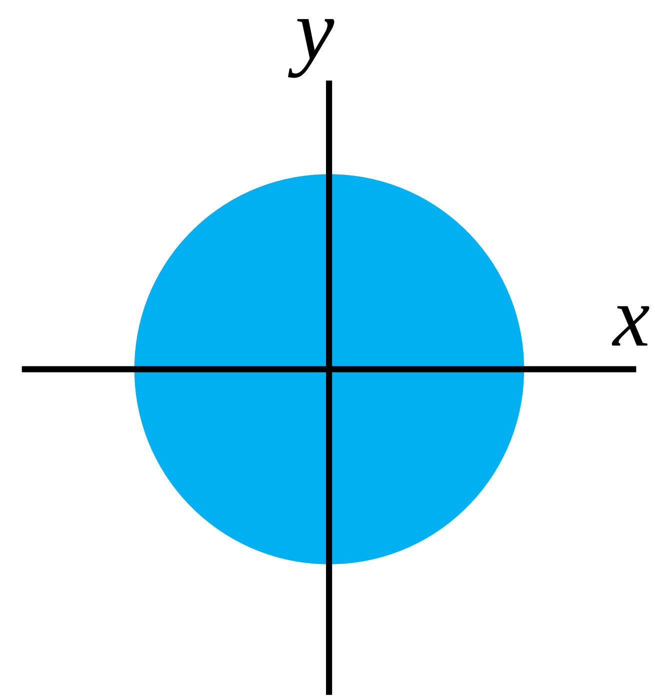
$$I = \int r^2 dm = \int_0^w x^2 \left(\frac{M}{A} h\, dx\right) = \frac{M}{A} h \int_0^w x^2 dx = \frac{M}{A} h \frac{w^3}{3}$$

$$A = wh$$

$$\boxed{I = \frac{1}{3} M w^2 = \frac{1}{3}(23)(0.87)^2 = 5.80\ kg.m^2}$$

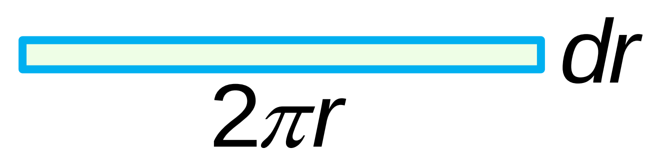
Ex 9: Find the moment of inertia of a uniform thin **disc** of radius **R** and mass **M** about an axis perpendicular to the disc and passing through its center of mass.





$$\boxed{I = \int r^2 dm}$$

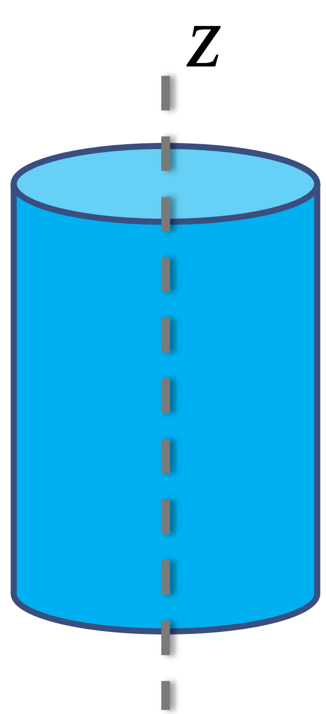
$$\sigma = \frac{M}{A} \Rightarrow dm = \sigma\, dA = \frac{M}{A}(2\pi r\, dr)$$



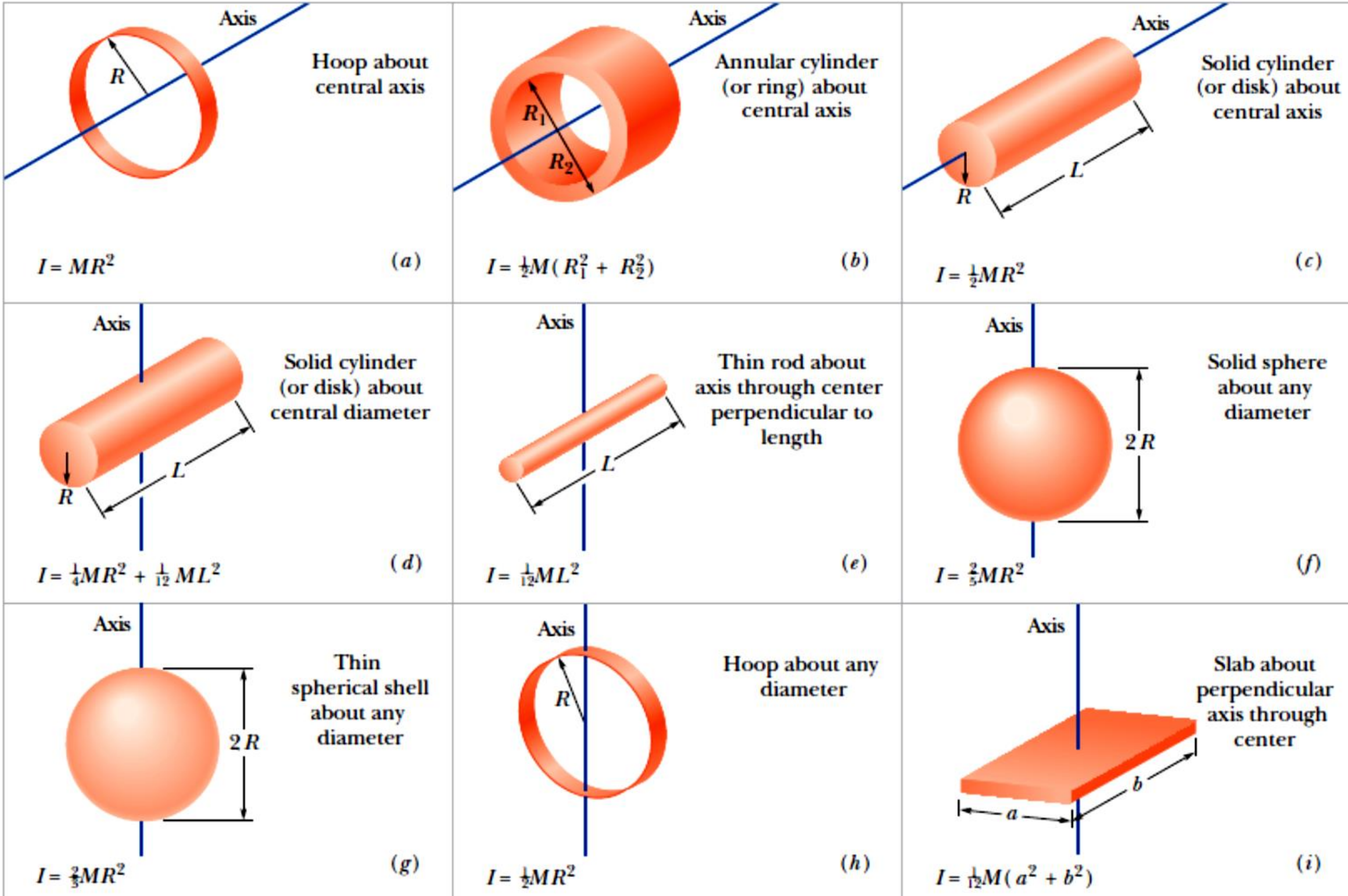
$$I_z = \int_0^R r^2 \left[\frac{M}{A}(2\pi r\, dr)\right] = 2\pi \frac{M}{A}\int_0^R r^3 dr = 2\pi \frac{M}{A}\frac{R^4}{4} \qquad A = \pi R^2$$

$$\boxed{I_z = \frac{1}{2}MR^2}$$

$$I_z = I_x + I_y = 2I_x = 2I_y \Rightarrow \boxed{I_x = I_y = \frac{1}{4}MR^2}$$

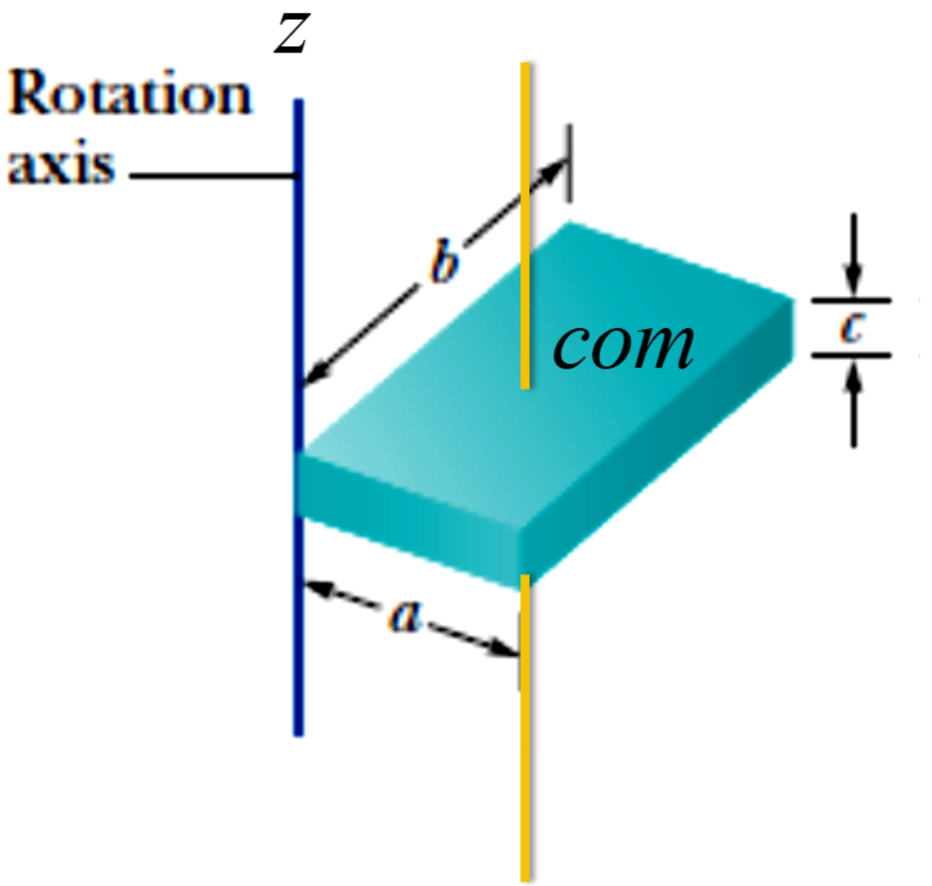


Some Rotational Inertias



Ex 10: **(Problem 10. 43 Halliday)**

The uniform solid block has mass **0.172 kg** and edge lengths **a = 3.5** cm, **b = 8.4 cm**, and **c = 1.4 cm**. Calculate its rotational inertia about an axis through one corner and perpendicular to the large faces.



$$\boxed{I_z = I_{com} + M h^2}$$

$$I_z = \frac{1}{12} M(a^2 + b^2) + M\left(\sqrt{\frac{a^2}{4} + \frac{b^2}{4}}\right)^2 = \frac{1}{3} M(a^2 + b^2)$$

$$\boxed{I_z = \frac{1}{3}(0.172)\left[(0.35)^2 + (0.84)^2\right] = 0.048\ kg.m^2}$$

$$\boxed{I = \int r^2 dm}$$

$$\rho = \frac{M}{V} \Rightarrow dm = \rho dV = \frac{M}{V}(c\, dx\, dy)$$

$$I_z = \int_0^a\int_0^b (x^2 + y^2)\left[\frac{M}{V}(c\, dx\, dy)\right] = \frac{M}{V} c \left(\int_0^a\int_0^b x^2\, dx\, dy + \int_0^a\int_0^b y^2\, dx\, dy\right) = \frac{M}{V} c \left(\frac{ba^3}{3} + \frac{ab^3}{3}\right)$$

$$V = abc \Rightarrow \boxed{I_z = \frac{1}{3} M\left(a^2 + b^2\right)}$$