Applications of Lévy processes

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Matthias Winkel

Departmental lecturer

(Institute of Actuaries and Aon lecturer in Statistics)

6. Poisson point processes in fluctuation theory
7. Lévy processes and population models
8. Lévy processes in mathematical finance

Summary of Introduction to Lévy processes

We've defined Lévy processes via stationary independent increments.

We've seen how Brownian motion, stable processes and Poisson processes arise as limits of random walks, indicated more general results.

We've analysed the structure of general Lévy processes and given representations in terms of compound Poisson processes and Brownian motion with drift.

We've simulated Lévy processes from their marginal distributions and from their Lévy measure.

6. Poisson point proc. in fluctuation theory

Fluctuation theory studies the extremes of the sample paths:

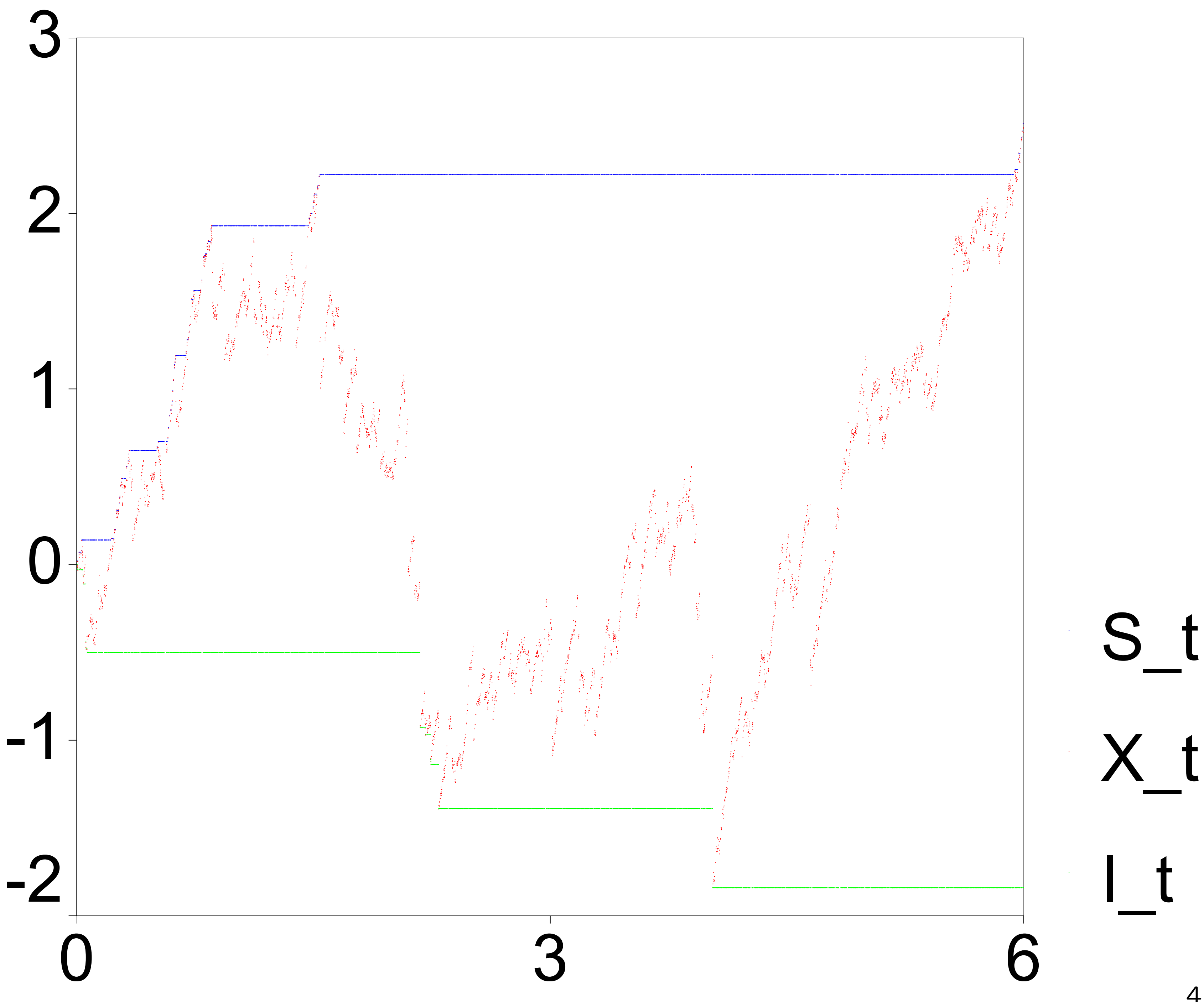
$$S_t = \sup_{s\le t} X_s \quad \text{and} \quad I_t = \inf_{s\le t} X_s, \quad t \ge 0.$$

This also includes level passages and overshoots

$$T_x = \inf\{t \ge 0 : X_t > x\}, \quad K_x = X_{T_x} - x,$$

and the set of times that X spends at its supremum

$$\mathcal{R} = \{t \ge 0 : X_t = S_t\}^{cl} = \{T_x : x \ge 0\}^{cl}$$


3
2
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-1
-2
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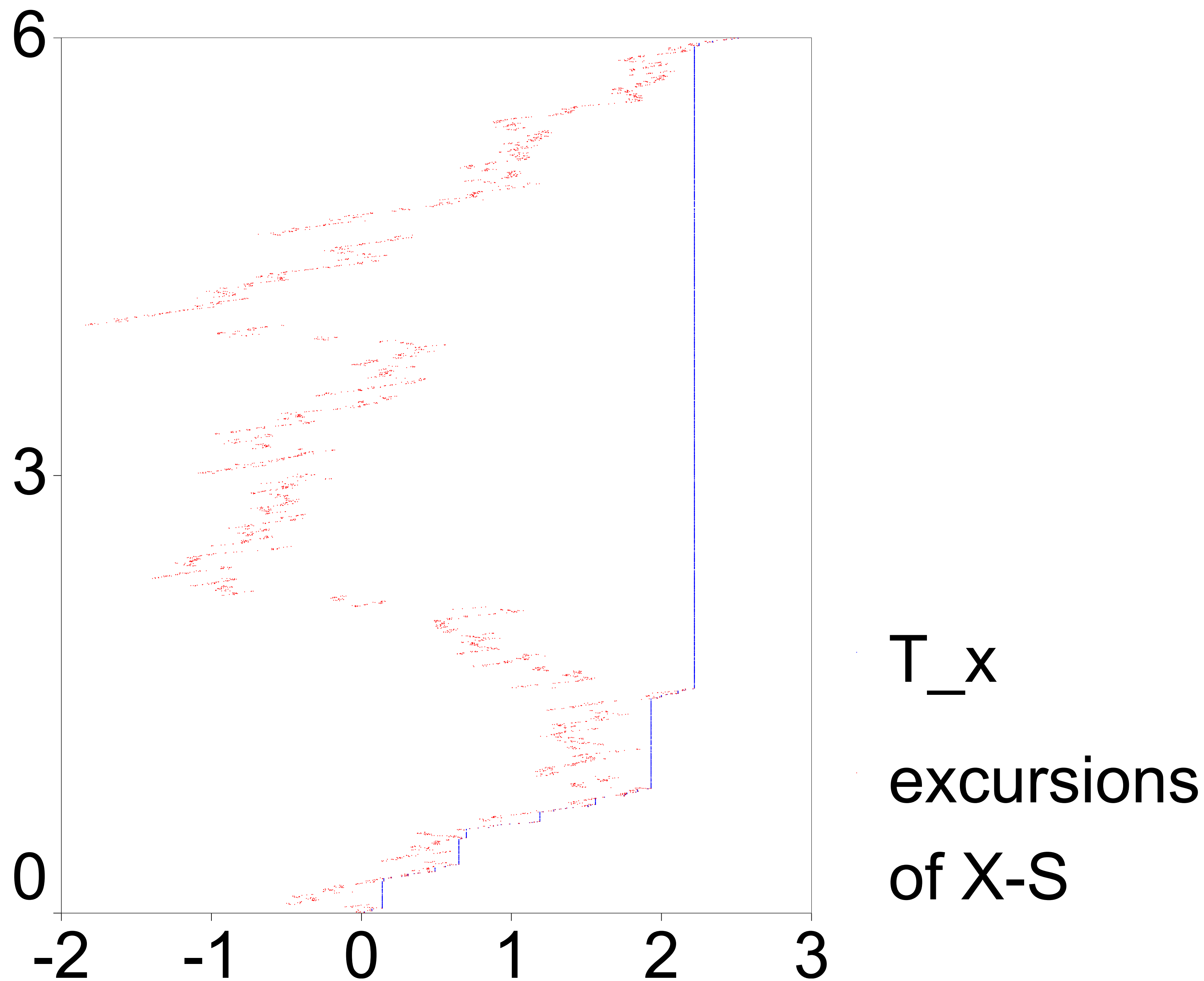
3
6
S_t
X_t
I_t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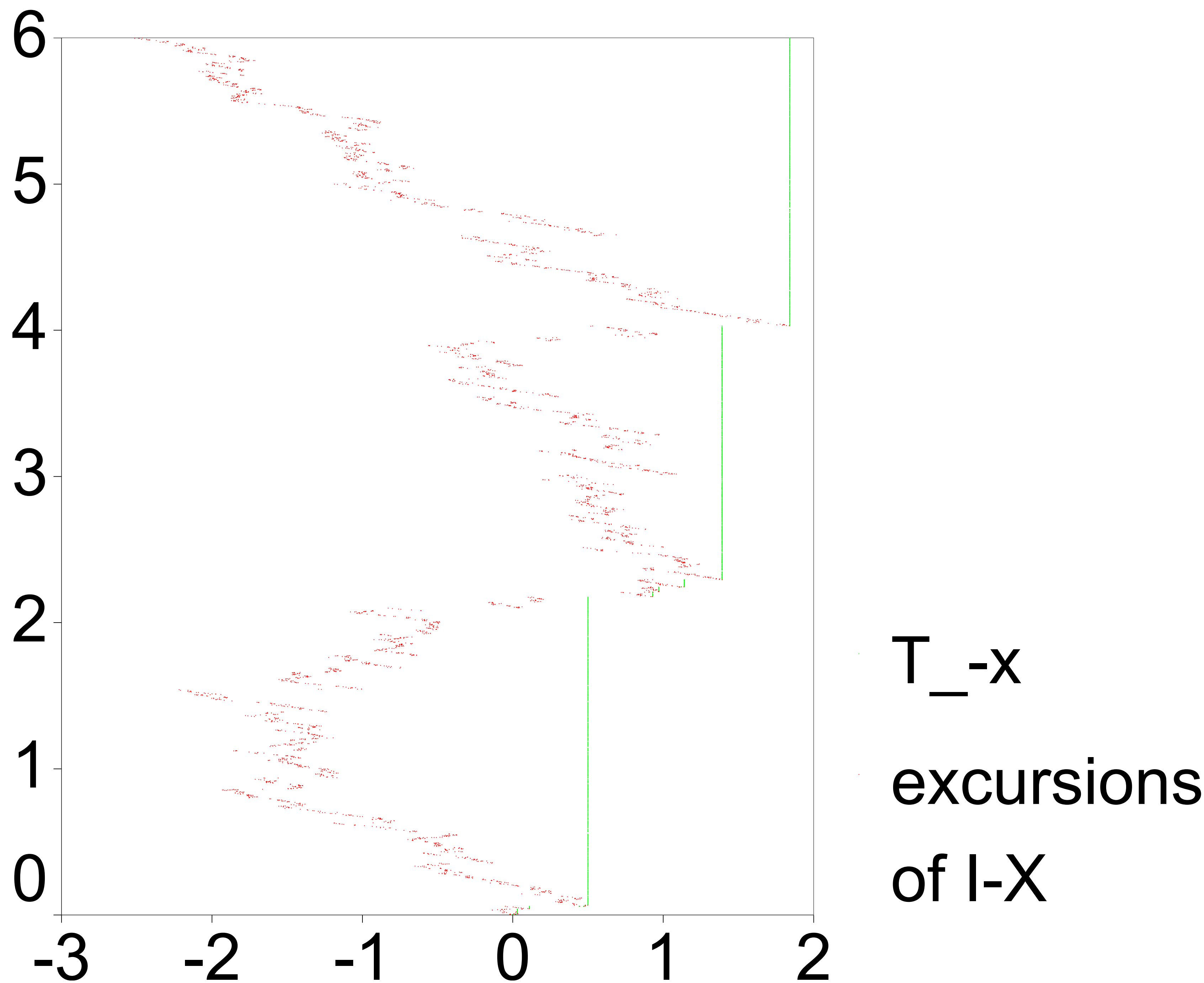
T_x
excursions
of X-S



6
5
4
3
2
1
0
-3
-2
-1
0
1
2
T_-x
excursions
of I-X

Markov property

Theorem 10 (Bingham) *Lévy proc. are strong Markov processes, i.e.* $(X_{T+s}-X_T)_{s\geq 0}\sim X$ *and is indep. of* $(X_r)_{r\leq T}$.

The independence of $(X_{T+s}-X_T)_{s\geq 0}$ and X_T is called spatial homogeneity (in addition to temporal homogeneity).

Proof of simple Markov property: $T=t$

Choose $s>0$, $0\leq r\leq t$, then $X_{t+s}-X_t$ and X_r (and X_t-X_r) are independent, and similarly for $0=s_0<\ldots<s_m$, $0=r_0<\ldots<r_k\leq t$ finite-dimensional subfamilies are independent. Their distributions determine the distribution of $(X_{t+s}-X_t)_{s\geq 0}$ and $(X_r)_{r\leq t}$.

T_x, $x \geq 0$, for spectrally negative processes, $\mathbb{E}(X_1) \geq 0$

Theorem 11 (Zolotarev) *T_x, $x \geq 0$, is a Lévy process.*

Proof: X has no positive jumps. Therefore $X_{T_x} = x$ a.s. By the strong Markov property $\tilde{X} = (X_{T_x+s} - x)_{s\geq 0} \sim X$ is independent of $(X_r)_{r\leq T_x}$, in particular of T_x. Also, $\tilde{T}_y \sim T_y$ and $T_x + \tilde{T}_y = T_{x+y}$, i.e. $\tilde{T}_y = T_{x+y} - T_x$.

In particular ΔT_x, $x \geq 0$, is a Poisson point process. In fact, also $(X_{T_{x-}+t} - X_{T_{x-}})_{0\leq t\leq \Delta T_x}$, $x \geq 0$, is a Poisson point process, a so-called excursion process.

Example: X Brownian motion $\Rightarrow$ T 1/2-stable.

Results from fluctuation theory for general X

Theorem 12 *For fixed $t > 0$, $(S_t, X_t - S_t) \sim (X_t - I_t, I_t)$.*

Theorem 13 (Rogozin) *For $\tau \sim Exp(q)$ and all $\beta > 0$*

$$\mathbb{E}(e^{-\beta S_\tau}) = \exp\left(\int_0^\infty \int_{[0,\infty)} (e^{-\beta x} - 1) t^{-1} e^{-qt} \mathbb{P}(X_t \in dx)\right).$$

Theorem 14 $\mathcal{R} = \{T_x : x \geq 0\}^{cl} = \{U_s : s \geq 0\}^{cl}$ *is the range of an increasing Lévy process U, and also $(U_s, X_{U_s})_{s\geq 0}$ is a bivariate Lévy process, the so-called ladder process.*

Theorem 15 (Wiener-Hopf factorisation) *If* $\mathbb{E}(e^{i\lambda X_1}) = e^{-\psi(\lambda)}$, $\mathbb{E}(e^{-\alpha U_1 - \beta S_{U_1}}) = e^{-\kappa(\alpha,\beta)}$, $\mathbb{E}(e^{-\alpha V_1 + \beta I_{V_1}}) = e^{-\widehat{\kappa}(\alpha,\beta)}$,

$$\frac{q}{q + \psi(\lambda)} = \frac{\kappa(q, 0)}{\kappa(q, -i\lambda)} \frac{\widehat{\kappa}(q, 0)}{\widehat{\kappa}(q, i\lambda)}.$$

Subordination and time change

The operation $Z_t = X_{A_t}$ with a subordinator (increasing Lévy process) is called subordination, or time change.

Example in fluctuation theory: ladder height process X_{U_t}.

Bochner's subordination, Bochner(57), A independent. Conditional distributions $\mathcal{L}(A|Z)$, also more gen. A in W(02b)

Subordination in the wide sense, Huff(59), Monroe(78), Bertoin (97), Simon(99), W(WIP), A suitably dependent on X.

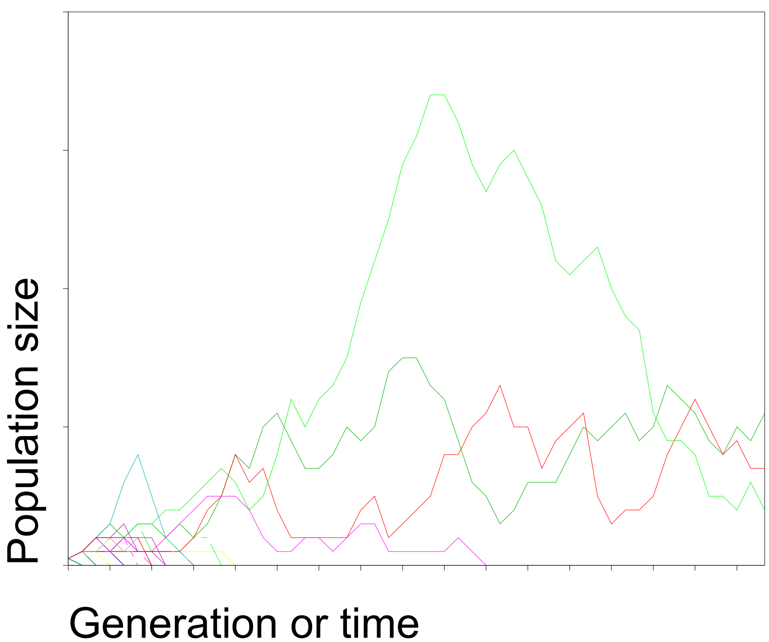
Right inverses, Evans(00), W(02a), $X_{A_t} = t$.

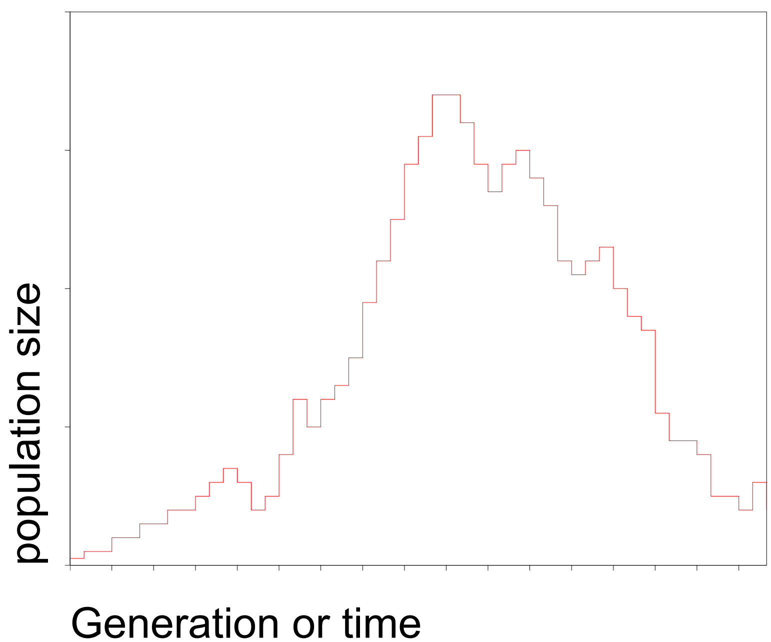
7. Lévy processes and population models

Galton-Watson branching processes: each individual either doubles or dies at the end of each time unit, independently.

Centered case: populations die out

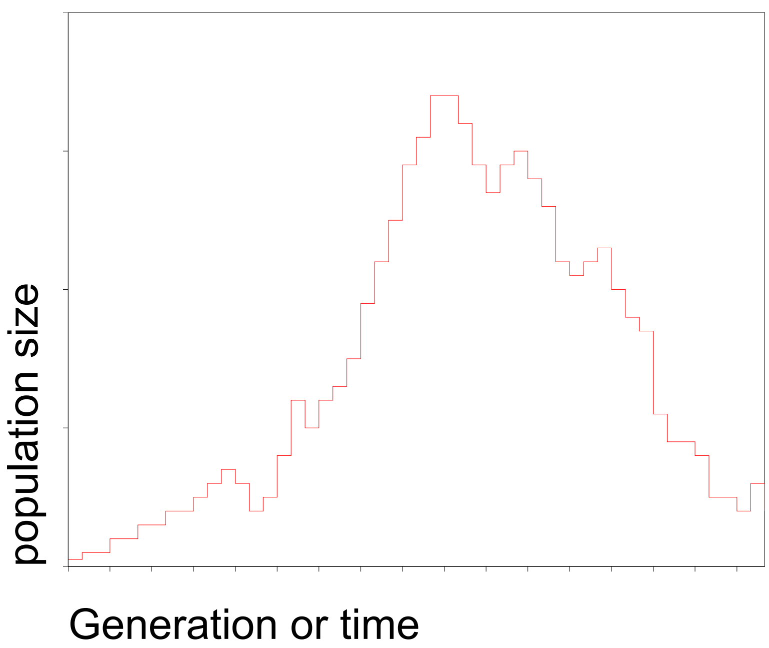
Note higher fluctuations at higher pop. sizes.

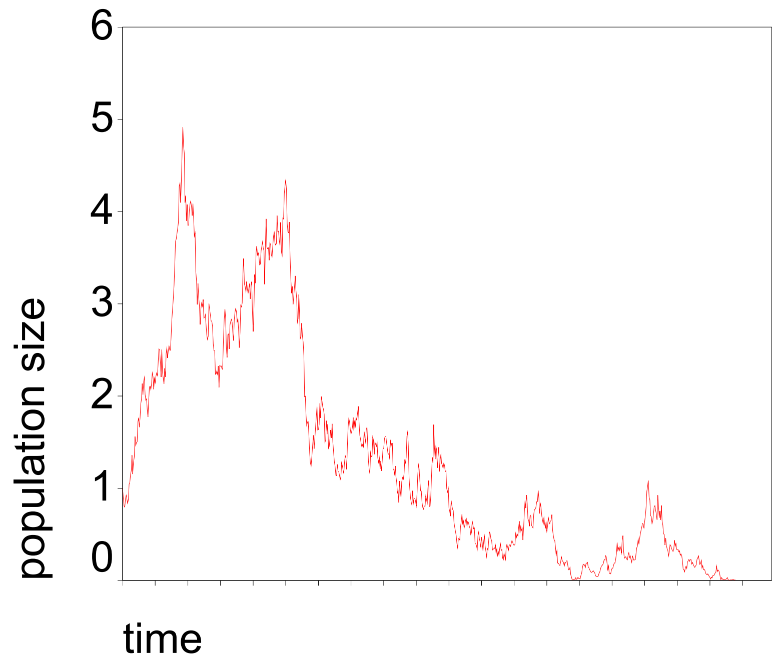




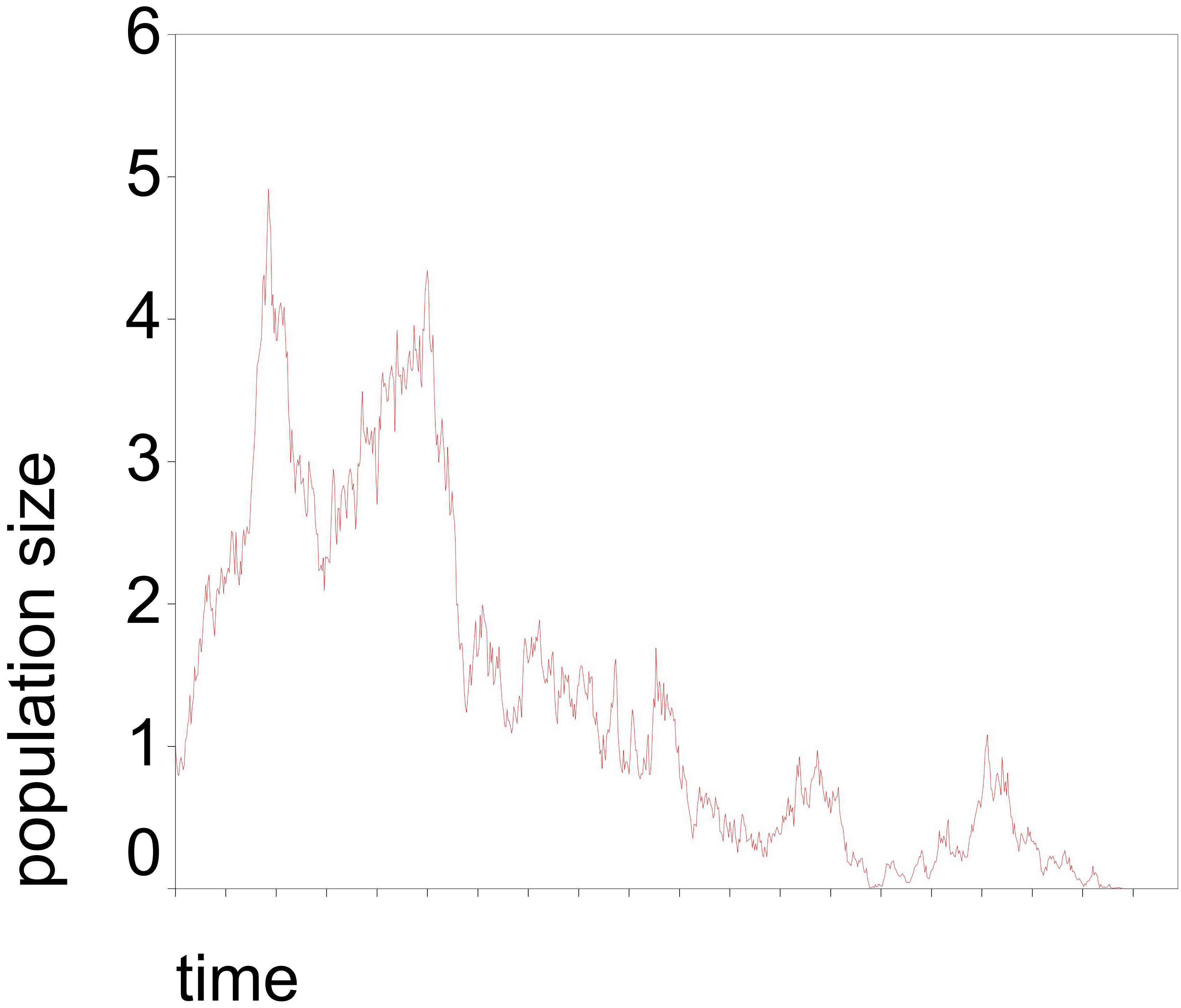
Continuous limits of Galton-Watson processes

Scaling limits give so-called Feller's diffusion, which is not Brownian motion: $\sigma(x) = cx$, x population size.

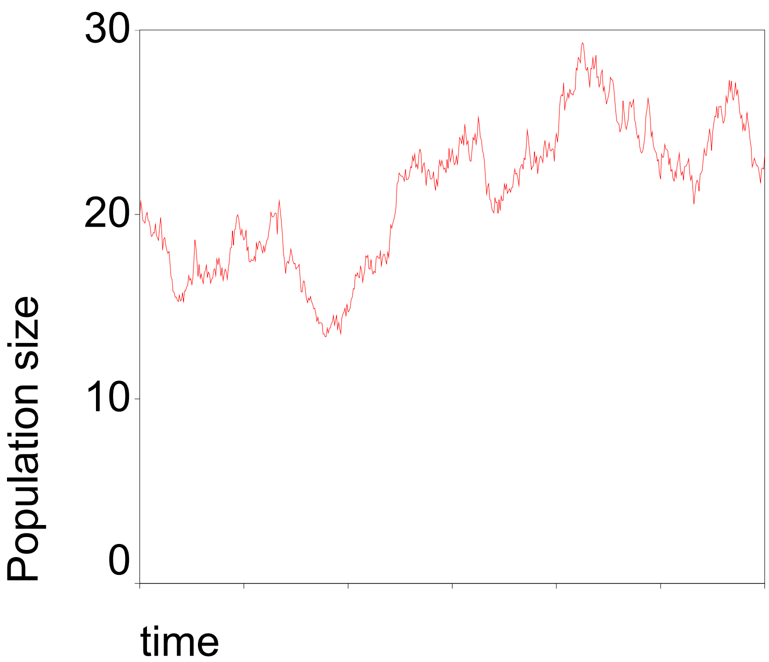


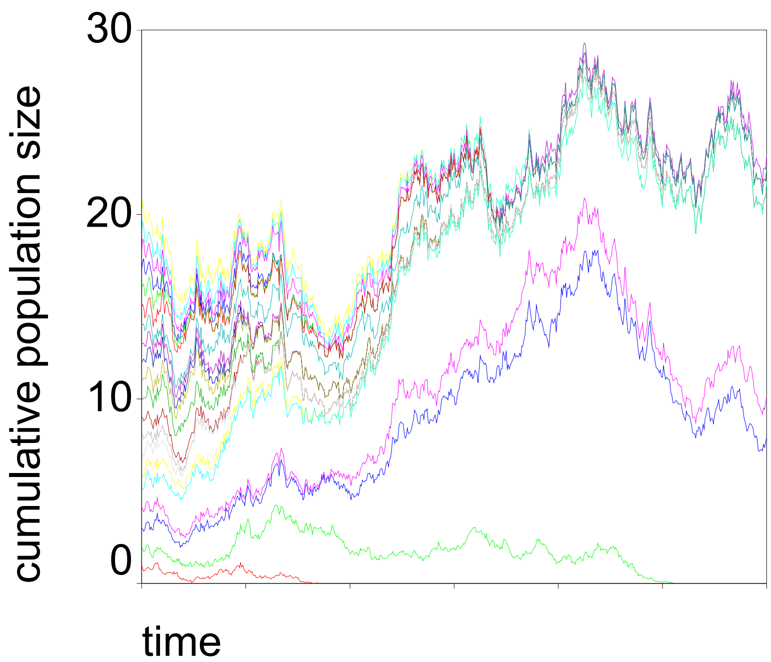


As for random walk limits, there are generalisations to stable and infinitely divisible branching mechanisms.

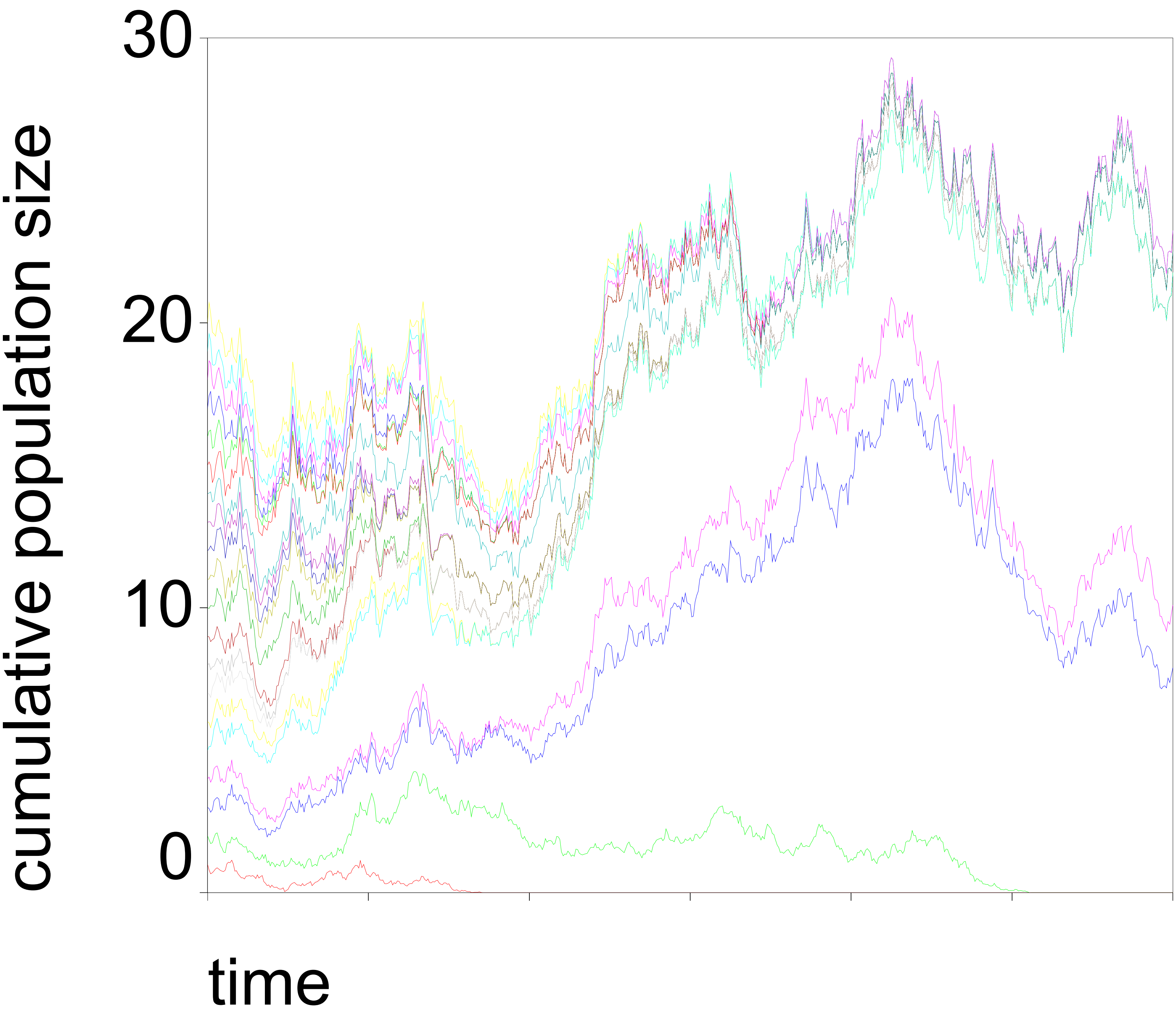

6
5
4
3
2
1
0
population size
time

Genealogy of populations





Split population into n parts and look at the evolution of their descendants (here $n = 20$). Let $n \to \infty$ to get full genealogy.


30
20
10
0
cumulative population size
time

Links to subordination and random trees

At $t = 0$ infinitely many unrelated ancestors, at large $t > 0$, most individuals descendants of a single ancestor. Study evolution of families, can be expressed by a family of subordinators $S^{(s,t)}$ with subordination $S^{(r,s)}_{S^{(s,t)}_x} = S^{(r,t)}_x$, $0 \leq r \leq s \leq t$, expressing that the descendants of a time-r-individual at time t are the time-t-descendants of all his time-s-descendants. Cf. Bertoin-LeGall-LeJan (1997)

Describe continually branching family trees as stochastic objects. Literature: Aldous, Le Gall, Evans-Winter, Pitman-W(03), Duquesne-W(WIP).

8. Lévy processes in mathematical finance

The Black-Scholes model

Two assets: Risk-free bank account $A_t = \exp\{rt\}$ and a risky stock at prices

$$Z_t = Z_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\}, \quad t \geq 0,$$

where r interest rate, B Brownian motion, σ volatility and μ drift parameter.

Data: non-Normality, semi-heavy tails, non-constant σ

Therefore: need more flexible models: Lévy-based models

A trading strategy is a (bounded predictable) process U_t to signify the number of stock units that we hold at time $t \geq 0$. All money invested is taken from or borrowed on the bank account. Given an initial wealth W_0, this determines the (random) terminal wealth W_T at time T.

Theorem 16 (Predictable representation property) *For every square-integrable T-measurable random variable H, there is a trading strategy U and a unique 0-measurable initial wealth W_0 s.th. $W_T = H$.*

As a consequence, we have a unique price W_0 for all contingent claims H, e.g. $H = (Z_T - q)^+$ European call option.

Example: The Predictable representation property is easier to believe in discrete time, say in a 2-step model

$$A_0 = 10 \qquad \nearrow A_1 = 12 \qquad \nearrow A_2 = 16$$

$$\begin{array}{ccccc}
 & & & \nearrow & Z_2 = 22 \\
 & \nearrow & Z_1 = 15 & & \\
 & & & \searrow & Z_2 = 12 \\
Z_0 = 10 & & & \nearrow & Z_2 = 8 \\
 & \searrow & Z_1 = 6 & & \\
 & & & \searrow & Z_2 = 5
\end{array}$$

$$\begin{array}{ccccc}
 & & & \nearrow & W_2 = 30 \\
 & \nearrow & W_1 = 18, \boxed{U_1 = 3} & & \\
\boxed{W_0 = 10} & & & \searrow & W_2 = 0 \\
\boxed{U_0 = 2} & & & \nearrow & W_2 = 0 \\
 & \searrow & W_1 = 0, \boxed{U_1 = 0} & & \\
 & & & \searrow & W_2 = 0
\end{array}$$

$$\begin{array}{ccccc}
 & & & \nearrow & (66, -36) \\
 & \nearrow & (30, -12) \to (45, -27) & & \\
 & & & \searrow & (36, -36) \\
(20, -10) & & & \nearrow & (0, 0) \\
 & \searrow & (12, -12) \to (0, 0) & & \\
 & & & \searrow & (0, 0)
\end{array}$$

Given W_2, W_0 (and W_1) are independent of the transition probabilities.

Calculations are quite heavy, in many-step or continuous models.

However, there is a unique probability measure Q, s.th. the wealth can be calculated as conditional expectations of $H = W_T = g(Z)$, for all H. Q is called a martingale measure since $(A_t^{-1} W_t)_{0 \le t \le T}$ is a martingale under Q. In particular

$$A_0^{-1} W_0 = A_T^{-1} \mathbb{E}_Q(W_T).$$

Exponential Lévy processes as stock prices

The Predictable representation property fails,
hence no uniqueness of arbitrage free prices,
different ways to choose a martingale measure.

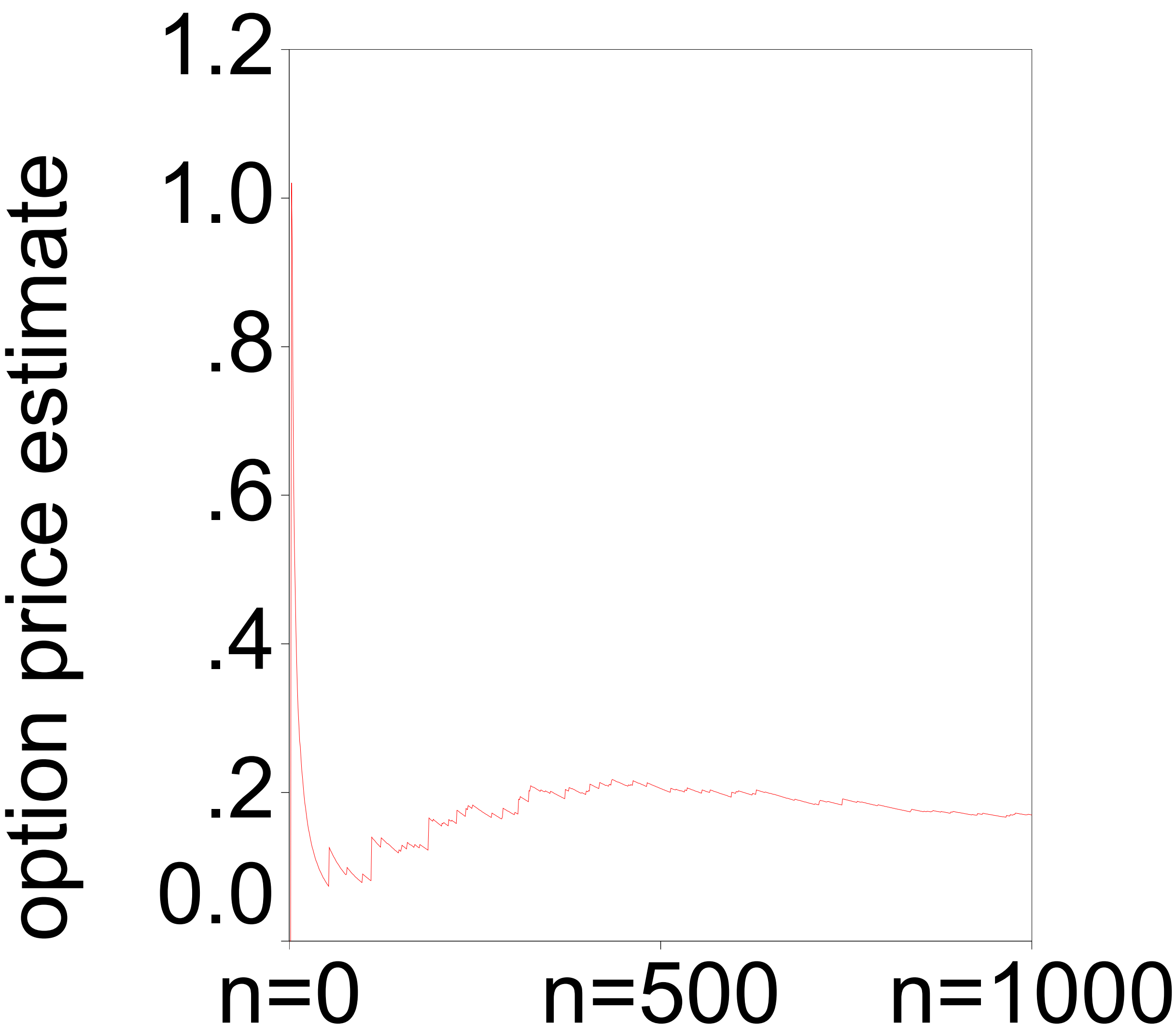
Once martingale measure chosen (changes parameters of the Lévy process), options can be priced by simulation:

Option described by contingent claim $H = g(Z)$. Price

$$\frac{1}{n}\sum_{k=1}^{n} g(Z^{(k)}) \to \mathbb{E}_Q(g(Z)).$$

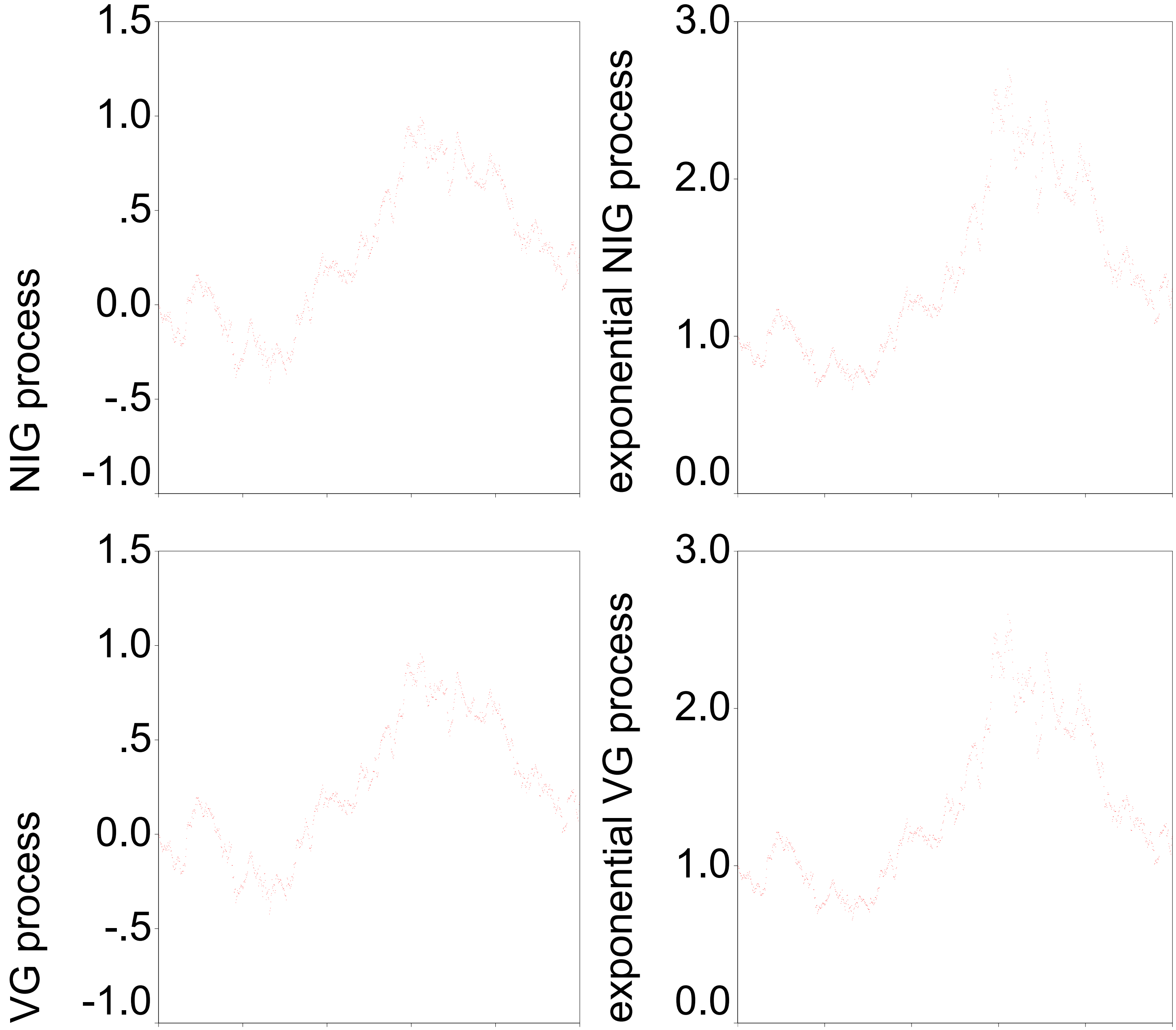
g may depend on the path of Z, not just Z_T (barriers etc.).

Exmpl: Black-Scholes, $r = 0$, $\sigma = 1$, $t = 1$, $H = (Z_1 - 2)^+$.



Popular models

$Z_t = B_{X_t}$, X inverse Gaussian ($X_t = T_t^{\tilde{B}}$ where $\tilde{B}$ is a Brownian motion with drift), or Gamma subord.




NIG process
1.5
1.0
.5
0.0
-.5
-1.0
exponential NIG process
3.0
2.0
1.0
0.0
VG minus NIG
.1
0.0
-.1
-.2
exp VG minus exp NIG
.1
0.0
-.1
-.2

Parametric families are useful to facilitate model fitting

Stochastic volatility

Stochastic volatility: Time-change by an integrated stationary volatility process, e.g. OU processes driven by subordinators X_t:

$$
\begin{aligned}
Y_t &= \exp\{-\lambda t\}Y_0 + \int_0^t \exp\{-\lambda(t-s)\}dX_{\lambda s} \\
I_t &= \int_0^t Y_s ds \\
Z_t &= B_{I_t}
\end{aligned}
$$

This model is by Barndorff-Nielsen and Shephard. This and others can be simulated and used for option pricing.

Summary

We've studied the extremes of Lévy processes. Ladder processes are two-dimensional Lévy processes.

We've studied subordination to construct and relate Lévy processes.

Limits of branching processes can be studied like limits of random walks, giving continuous processes. We've indicated how their genealogy can be expressed by subordination. In some sense, the genealogy of branching processes is an infinite-dimensional Lévy process.

In mathematical finance, stock prices can be modelled using specific Lévy processes, often constructed by subordination. This can be used, e.g., for option pricing.