Algorithmic Complexity of Proper Labeling Problems

Ali Dehghan[a], Mohammad-Reza Sadeghi[a], Arash Ahadi[b]

[a] *Faculty of Mathematics and Computer Science, Amirkabir University of Technology, Tehran, Iran*

[b] *Department of Mathematical Sciences, Sharif University of Technology, Tehran, Iran* *

Abstract

A proper labeling of a graph is an assignment of integers to some elements of a graph, which may be the vertices, the edges, or both of them, such that subject to some conditions we obtain a proper vertex coloring via the labeling. The problem of proper labeling offers many variants and received a great interest during these last years. In this work, we consider the computational complexity of some variants of the proper labeling problems such as: *multiplicative vertex-coloring*, *fictional coloring* and *gap coloring*. For instance, we show that, for a given bipartite graph G, determining whether G has a *vertex-labeling by gap* from $\{1, 2\}$ is **NP**-complete. Also, we prove that there is a polynomial time algorithm for determining whether a given planar bipartite graph G has a *vertex-labeling by gap* from $\{1, 2\}$. In sharp contrast, it is **NP**-complete to decide whether a given planar 3-colorable graph G has a *vertex-labeling by gap* from $\{1, 2\}$.

Key words: Proper Labeling; Multiplicative vertex-coloring weightings; Gap vertex-distinguishing edge colorings ; Fictional Coloring; Computational Complexity.
Subject classification: 05C15, 05C20, 68Q25

1 Introduction

A proper labeling of a graph is an assignment of integers to some elements of a graph, which may be the vertices, the edges, or both of them, such that subject to some conditions we obtain a proper vertex coloring via the labeling. Karoński, Łuczak and Thomason initiated the study of proper-labelings [16]. They introduced an edge-labeling which is additive vertex-coloring that means for every edge uv, the sum of labels of the edges incident to u

* *E-mail addresses*: ali_dehghan16@aut.ac.ir, msadeghi@aut.ac.ir, arash_ahadi@mehr.sharif.edu.

is different than the sum of labels of the edges incident to v [16]. The problem of proper labeling offers many variants and received a great interest during these last years, for instance see [1, 7, 8, 15, 16, 20]. First, consider the following two famous variants.

(P1) Edge-labeling by sum.
An edge-labeling f is *edge-labeling by sum* if $c(v) = \sum_{e \ni v} f(e)$, $\forall v \in V$ is a proper vertex coloring. This parameter was introduced by Karoński et al. and it is conjectured that three integer labels $\{1, 2, 3\}$ are sufficient for every connected graph, except K_2 (1, 2, 3-Conjecture, see [16]). This labeling have been studied extensively by several authors, for instance see [1, 2, 6, 17, 20]. Currently, we know that every connected graph has an *edge-labeling by sum*, using the labels from $\{1, 2, 3, 4, 5\}$ [15]. Also, it is shown that determining whether a given graph has a *edge-labeling by sum* from $\{1, 2\}$ is **NP**-complete [12].

(P2) Vertex-labeling by sum (Lucky labling and sigma coloring).
A vertex-labeling f is *vertex-labeling by sum* if $c(v) = \sum_{u \sim v} f(u)$, $\forall v \in V$ is a proper vertex coloring. *vertex-labeling by sum* is a vertex versions of the above problem, which was introduced recently by Czerwiński et al. [8]. It was conjectured that every graph G has a *vertex-labeling by sum*, using the labels $\{1, 2, \cdots, \chi(G)\}$ [8] and it was shown that every graph G with $\Delta(G) \geq 2$, has a *vertex-labeling by sum*, using the labels $\{1, 2, \cdots, \Delta^2 - \Delta + 1\}$ [4], also, it was shown that, it is **NP**-complete to decide for a given planar 3-colorable graph G, whether G has a *vertex-labeling by sum* from $\{1, 2\}$ [3]. Furthermore, it is **NP**-complete to determine for a given 3-regular graph G, whether G has a *vertex-labeling by sum* from $\{1, 2\}$ [10]. A similar version of this labeling was introduced by Chartrand et al. [7].

In this work, we consider the algorithmic complexity of the following proper labeling problems.

(P3) Edge-labeling by product. (Multiplicative vertex-coloring)
An edge-labeling f is *edge-labeling by product* if $c(v) = \prod_{e \ni v} f(e)$, $\forall v \in V$ is a proper vertex coloring. This variant was introduced by Skowronek-Kaziów and it is conjectured that every non-trivial graph G has an *edge-labeling by product*, using the labels from $\{1, 2, 3\}$ (Multiplicative 1, 2, 3-Conjecture, see [21]). Currently, we know that every non-trivial graph has an *edge-labeling by product*, using the labels from $\{1, 2, 3, 4\}$ [21]. Also, every non-trivial, 3-colorable graph G permits an *edge-labeling by product* from $\{1, 2, 3\}$ [21]. We will prove that determining whether a given planar 3-colorable graph has an *edge-labeling by product* from $\{1, 2\}$ is **NP**-complete.

(P4) Vertex-labeling by product.
A vertex-labeling f is *vertex-labeling by product* if $c(v) = \prod_{u \sim v} f(u)$, $\forall v \in V$ is a proper vertex coloring. For a given graph G, let $\{V_1, V_2, \cdots, V_k\}$ be the color classes of a proper

vertex coloring of G. Label the set of vertices of V_1 by 1; also, for each i, $1 < i \leq k$ label the set of vertices of V_i by the $(i-1)$-th prime number; this labeling is a *vertex-labeling by product*. In number theory, the prime number theorem describes the asymptotic distribution of the prime numbers. The prime number theorem implies estimates for the size of the n-th prime number p_n (i.e., $p_1 = 2$, $p_2 = 3$, etc.): up to a bounded factor, p_n grows like $n \log(n)$. As a consequence of the prime number theorem we have the following bound: $p_n < n \ln n + n \ln \ln n$, for $n \geq 6$ (see [5] p. 233). So, every graph G has a *vertex-labeling by product*, from $\{1, 2, \cdots, \chi \ln \chi + \chi \ln \ln \chi + 2\}$. Here, we ask the following question.

Problem 1. *Does every graph G have a* vertex-labeling by product*, using the labels* $\{1, 2, \cdots, \chi(G)\}$*?*

We shown that, every planar graph G has a *vertex-labeling by product* from $\{1, 2, \cdots, 5\}$. We will prove that determining whether a given planar 3-colorable graph has a *vertex-labeling by product* from $\{1, 2\}$ is **NP**-complete. Furthermore, for every k, $k \geq 3$ we show that determining whether a given graph has a *vertex-labeling by product* from $\{1, 2, \cdots, k\}$ is **NP**-complete.

(P5) Edge-labeling by gap.

An edge-labeling f is *edge-labeling by gap* if

$$c(v) = \begin{cases} f(e)_{e \ni v} & \text{if } d(v) = 1, \\ \max_{e \ni v} f(e) - \min_{e \ni v} f(e) & \text{otherwise,} \end{cases}$$

is a proper vertex coloring. Every graph G has an *edge-labeling by gap* if and only if it has no connected component isomorphic to K_1 or K_2 (put the different powers of two $(1, 2, \cdots, 2^{|E(G)|-1})$ on the edges of G; this labeling is a vertex-labeling by gap). A similar definition was introduced by Tahraoui et al. [22]. They introduced the following variant: Let G be a graph, k be a positive integer and f be a mapping from $E(G)$ to the set $\{1, 2, \cdots, k\}$. For each vertex v of G, the label of v is defined as

$$c(v) = \begin{cases} f(e)_{e \ni v} & \text{if } d(v) = 1, \\ \max_{e \ni v} f(e) - \min_{e \ni v} f(e) & \text{otherwise,} \end{cases}$$

The mapping f is called *gap vertex-distinguishing labeling* if distinct vertices have distinct labels. Such a coloring is called a *gap-k-coloring* and is denoted by *gap*(G) [22]. It was conjectured that for a connected graph G of order n with $n > 2$, *gap*$(G) \in \{n-1, n, n+1\}$ [22]. They purpose study of the variant of the gap coloring problem that distinguishes the adjacent vertices only.

Let f be an *edge-labeling by gap* form $\{1, 2, \cdots, k\}$ for a graph G, we have $k \geq \chi(G) - 1$. First, consider the following example.

Remark 1 Every complete graph K_n of order n with $n > 2$, has an *edge-labeling f_n by gap* form $\{1, 2, \cdots, \chi(K_n) + 1\}$. Suppose that $K_3 = v_1v_2v_3$ and let f_3 be the following function: $f_3(v_1v_2) = 4$, $f_3(v_1v_3) = 1$ and $f_3(v_2v_3) = 2$. Define f_n recursively.

$$f_n(v_iv_j) = \begin{cases} f_{n-1}(v_iv_j) + 1 & \text{if } i, j < n, \\ 1 & \text{if } i = n \text{ and } j \neq 2, \\ 2 & \text{otherwise,} \end{cases}$$

Now, we state the following problem:

Problem 2. *Does every connected graph G of order n with $n > 2$, have an* edge-labeling by gap *form $\{1, 2, \cdots, \chi(G) + 1\}$?*

We will prove that determining whether a given planar bipartite graph has an *edge-labeling by gap* from $\{1, 2\}$ is **NP**-complete. Also, we show that for every k, $k \geq 3$, it is **NP**-complete to determine whether a given graph has an *edge-labeling by gap* from $\{1, 2, \cdots, k\}$.

(P6) Vertex-labeling by gap.

A vertex-labeling f is *vertex-labeling by gap* if

$$c(v) = \begin{cases} f(u)_{u \sim v} & \text{if } d(v) = 1, \\ \max_{u \sim v} f(u) - \min_{u \sim v} f(u) & \text{otherwise,} \end{cases}$$

is a proper vertex coloring. A graph may lack any *vertex-labeling by gap*. Here we ask the following:

Problem 3. *Does there is a polynomial time algorithm to determine whether a given graph has a* vertex-labeling by gap*?*

We show that, for a given bipartite graph G, determining whether G has a *vertex-labeling by gap* from $\{1, 2\}$ is **NP**-complete. Also, we prove that there is a polynomial time algorithm for determining whether a given planar bipartite graph G has a *vertex-labeling by gap* from $\{1, 2\}$. In sharp contrast, it is **NP**-complete to decide whether a given planar 3-colorable graph G has a *vertex-labeling by gap* from $\{1, 2\}$.

Every bipartite graph $G = [X, Y]$ has a *vertex-labeling by gap*, label the set of vertices X by 1 and label the set of vertices of Y by different powers of two $(2^1, \cdots, 2^{|Y|})$. Here we ask the following:

Problem 4. *Does there is a constant k such that every bipartite graph G, have a* vertex-labeling by gap *form $\{1, 2, \cdots, k\}$?*

It was shown by Thomassen [23] that, for any k-uniform and k-regular hypergraph H, if $k \geq 4$, then H is 2-colorable. For every r-regular bipartite graph $G = [X, Y]$ with $r \geq 4$, label the set of vertices of one of the color classes in part X by 1 and label other vertices by 2. This Labeling is a *vertex-labeling by gap* from $\{1, 2\}$ for G.

(P7) Vertex-labeling by degree. (Fictional coloring)
A vertex-labeling f is *vertex-labeling by degree* if $c(v) = f(v)d(v)$, where $d(v)$ is the degree of vertex v is a proper vertex coloring. This parameter was introduced by Bosek, Grytczuk, Matecki and Żelazny [26]. They conjecture that every graph G has a *vertex-labeling by degree* from $\{1, 2, \cdots, \chi(G)\}$. Let p be a prime number and let G be a graph such that $\chi(G) \leq p-1$, they proved that G has a *vertex-labeling by degree* from $\{1, 2, \cdots, p-1\}$. For every k greater than two it is clear that determining whether a given graph has a *vertex-labeling by degree* from $\{1, 2, \cdots, k\}$ is **NP**-complete. We will prove that determining whether a given graph has a *vertex-labeling by degree* from $\{1, 2\}$ is in **P**.

(P8) Vertex-labeling by maximum.
A vertex-labeling f is *vertex-labeling by maximum* if $c(v) = \max_{u \sim v} f(u), \ \forall v \in V$ is a proper vertex coloring. A graph G may lack any *vertex-labeling by maximum* and it has a *vertex-labeling by maximum* from $\{1, 2\}$ if and only if G is bipartite. We present a nontrivial necessary condition that can be checked in polynomial time for a graph to have a *vertex-labeling by maximum.*

Remark 2 Let k be the minimum number such that G has a *vertex-labeling by maximum* from the set $\{1, 2, \cdots, k\}$, then $\chi(G) - k$ can be arbitrary large. For instance, for a given $t > 3$ consider the graph G with vertex set $V(G) = \{a_i : 1 \leq i \leq t\} \cup \{b_j : 1 \leq j \leq t-2\}$ and edge set $E(G) = \{a_i a_{i+1} : 1 \leq i \leq t-1\} \cup \{a_j b_j, b_j a_{j+1} : 1 \leq j \leq t-2\}$. Clearly $k - \chi(G) = t - 3$.

We will show that determining whether a given 3-regular graph has a *vertex-labeling by maximum* from $\{1, 2, 3\}$ is **NP**-complete.

Throughout this paper all graphs are finite and simple. We follow [13, 25] for terminology and notation not defined here, and we consider finite undirected simple graphs $G = (V, E)$. We denote the induced subgraph G on S by $G[S]$. Also, for every $v \in V(G)$ and $S \subseteq V(G)$, $N(v)$ and $N(S)$ denote the neighbor set of v and the set of vertices of G which has a neighbor in S, respectively. A *proper vertex coloring* of $G = (V, E)$ is a function $c : V(G) \longrightarrow L$, such that if $u, v \in V(G)$ are adjacent, then $c(u)$ and $c(v)$ are different. A *proper vertex k-coloring* is a proper vertex coloring with $|L| = k$. The smallest integer k such that G has a proper vertex k-coloring is called the *chromatic number* of G

Table 1: Graph Labeling Results

Edge-labeling by	$\{1,2\}$	$\{1,2,3\}$	Current Upper Bound	Conjecture
Sum	**NP**-c	-	$\{1,2,3,4,5\}$	$\{1,2,3\}$
Product	**NP**-c	-	$\{1,2,3,4\}$	$\{1,2,3\}$
Gap	**NP**-c	**NP**-c	$\{1,2,\cdots,2^{\lvert E(G)\rvert-1}\}$	$\{1,2,\cdots,\chi+1\}$
Vertex-labeling by				
Sum	**NP**-c	**NP**-c	$\Delta^2-\Delta+1$	$\{1,2,\cdots,\chi\}$
Product	**NP**-c	**NP**-c	$\{1,\cdots,\chi\ln\chi+\chi\ln\ln\chi+2\}$	$\{1,2,\cdots,\chi\}$
Degree	**P**	**NP**-c	$\{1,2,\cdots,2\chi\}$	$\{1,2,\cdots,\chi\}$
Maximum	**P**	**NP**-c	-	-
Gap	**NP**-c	**NP**-c	-	-

and denoted by $\chi(G)$. Similarly, for $k\in\mathbb{N}$, a *proper edge k-coloring* of G is a function $c: E(G)\longrightarrow\{1,\ldots,k\}$, such that if $e,e'\in E(G)$ share a common endpoint, then $c(e)$ and $c(e')$ are different. The smallest integer k such that G has a proper edge k-coloring is called the *edge chromatic number* of G and denoted by $\chi'(G)$. By Vizing's theorem [24], the edge chromatic number of a graph G is equal to either $\Delta(G)$ or $\Delta(G)+1$. Those graphs G for which $\chi'(G)=\Delta(G)$ are said to belong to *Class* 1, and the others to *Class* 2.

2 Results

2.1 Edge-labeling by product

Theorem 1 *For a given planar 3-colorable graph G, determining whether G has an* edge-labeling by product *from* $\{1,2\}$ *is* **NP***-complete.*

Proof Clearly, the problem is in NP. We reduced *Cubic Planar 1-In-3 3-Sat* to our problem. Moore and Robson [18] proved that the following problem is **NP**-complete.
Cubic Planar 1-In-3 3-Sat.
INSTANCE: Set X of variables, collection C of clauses over X such that each clause $c\in C$ has $\mid c\mid=3$ and every variable appears in exactly three clauses, there is no negation in the formula, and the bipartite graph obtained by linking a variable and a clause if and only if the variable appears in the clause, is planar.
QUESTION: Is there a truth assignment for X such that each clause in C has exactly one

true literal?

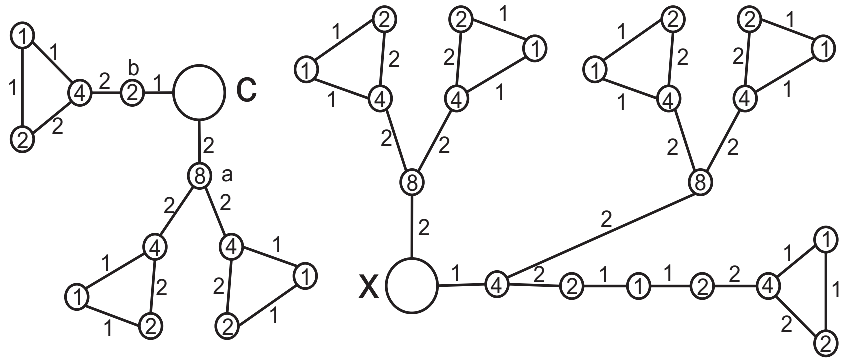

Figure 1: The two gadgets H_x and I_c. I_c is on the left hand side of the figure.

Consider an instance Φ, we transform this into a graph G_Φ such that G_Φ has an *edge-labeling by product* from $\{1, 2\}$ if and only if Φ has a 1-in-3 assignment. We use two gadgets H_x and I_c which are shown in Figure 1. The graph G_Φ has a copy of H_x for each variable $x \in X$ and a copy of I_c for each clause $c \in C$. Also, for each clause $c = y \vee z \vee w$ add the edges cy, cz and cw. First, suppose that G_Φ has a *edge-labeling by product* from $\{1, 2\}$. In every copy of H_x and I_c the label of every edge is determined uniquely. See Figure 1 (the label of each edge is written on the edge and the color of each vertex induced by edge labels is written on the vertex). Every variable x appears in exactly three clauses, suppose that x appears in c_i, c_j and c_k. By attention to the structure of H_x the set of labels of edges $c_i x$, $c_j x$ and $c_k x$ are $\{1, 1, 1\}$ or $\{2, 2, 2\}$. Furthermore, by attention to the H_x and I_c, for every clause $c = x \vee y \vee z$, the set of labels of edges cx, cy and cz is $\{2, 1, 1\}$. Now, for every variable x, which is appeared in c_i, c_j and c_k put $\Gamma(x) = True$ if and only if the set of labels of edges $c_i x$, $c_j x$ and $c_k x$ is $\{2, 2, 2\}$. Clearly, Γ is an 1-in-3 satisfying assignment. Next, suppose that Φ has an 1-in-3 satisfying assignment $\Gamma : X \to \{true, false\}$, for every variable x, which is appeared in c_i, c_j and c_k, label $c_i x$, $c_j x$ and $c_k x$ by 2 if and only if $\Gamma(x) = True$. The labels of other vertices are determined uniquely and it is clear the this labeling is an *edge-labeling by product* from $\{1, 2\}$. $\square$

2.2 Vertex-labeling by product

In the next, we consider the computational complexity of *vertex-labeling by product*.

Theorem 2 *For a given planar 3-colorable graph G, determining whether G has a* vertex-labeling by product *from $\{1,2\}$ is* **NP***-complete.*

Proof Clearly, the problem is in NP. We reduced *Cubic Planar 1-In-3 3-Sat* to our problem. First, we construct an auxiliary graph H_i^c. Put a copy of triangle $K_3 = z_1^c z_2^c z_3^c$. For every vertex z_j^c, $1 \leq j \leq 2$, put $2i$ new isolated vertices $t_1^j, t_2^j, \cdots, t_{2i}^j$ and join z_j^c to all of them. Also, add the edges $t_1^j t_2^j$, $t_3^j t_4^j, \cdots, t_{2i-1}^j t_{2i}^j$. Next, put $2i-2$ new isolated vertices $t_1^3, t_2^3, \cdots, t_{2i-2}^3$ and join z_3^c to all of them. Finally, add the edges $t_1^3 t_2^3$, $t_3^3 t_4^3, \cdots, t_{2i-3}^3 t_{2i-2}^3$. Call the resulting graph H_i^c. Now, consider an instance Ψ, we transform this into a graph G_Ψ such that G_Ψ has a *vertex-labeling by product* from $\{1,2\}$ if and only if Ψ has a 1-in-3 assignment. Our construction consists of three steps.

Step 1. For each clause $c \in C$ put a vertex c and a copy of H_3^c, H_5^c and H_6^c. Connect the vertex z_3^c of H_3^c to c, also, join the vertex z_3^c of H_5^c to c and finally, connect the vertex z_3^c of H_6^c to c.

Step 2. For each variable $x \in X$ put a vertex x.

Step 3. For each clause $c = x \vee y \vee w$ add the edges cx, cy and cw.

First, suppose that G_Ψ has a *vertex-labeling f by product* from $\{1,2\}$ and let ℓ be the induced coloring by f. In every copy of H_3^c the label of vertex z_3^c is 2. We have the similar property for H_5^c and H_6^c. By attention to the structure of H_3^c, we have $f(c) = 1$ and $\ell(z_3^c) = 8$; similarly for H_5^c, we have $\ell(z_3^c) = 32$ and for H_6^c, we have $\ell(z_3^c) = 64$. So for every clause vertex c we have $\ell(c) = 16$. Now, for every variable x, put $\Gamma(x) = True$ if and only if $f(x) = 2$. Since for every clause c, $\ell(c) = 16$, Γ is an 1-in-3 satisfying assignment. Next, suppose that Ψ is 1-in-3 satisfiable with the satisfying assignment $\Gamma : X \to \{true, false\}$, for every variable x, label the vertex x by 2 if and only if $\Gamma(x) = True$. The labels of other vertices are determined uniquely and it is clear the this labeling is a *vertex-labeling by product* from $\{1,2\}$.

□

Theorem 3 *For every k, $k \geq 3$, it is* **NP***-complete to determine whether a given graph has a* vertex-labeling by product *from $\{1,2,\cdots,k\}$.*

Proof We present a polynomial time reduction from *3-colorability* to our problem.
3-Colorability: Given a graph G; is $\chi(G) \leq 3$?
First define the following sets: $\mathcal{A}_k = \{mn : m,n \in \mathbb{N}_k\}$, $\mathcal{B}_k = \{\frac{m}{n} : m,n \in \mathbb{N}_k\}$, where $\mathbb{N}_k = \{1,2,\cdots,k\}$. Also, define $\alpha(k) = \max_{\mathcal{D}_k \in \mathcal{C}_k} \mid \mathcal{D}_k \mid$, where $\mathcal{C}_k$ is the set of sets such that for every set $\mathcal{D}_k \in \mathcal{C}_k$, we have $\mathcal{D}_k \subseteq \mathcal{A}_k$ and $\{\frac{d}{d'} : d,d' \in \mathcal{D}_k\} \cap \mathcal{B}_k = \emptyset$. Since k is constant, so we can compute $\alpha(k)$ in $O(1)$. Now, for a given graph G with n

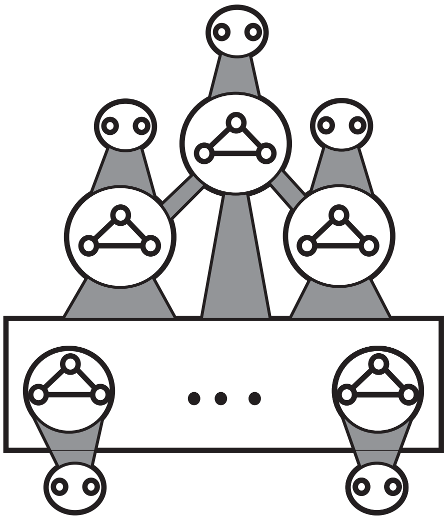

Figure 2: The graph $\widetilde{G}$ derived from $G = P_3$ for $k = 3$.

vertices $v_1, v_2, \cdots, v_n$, join all vertices of G to the all vertices of complete graph $K_{\alpha(k)-3}$ with vertices $v_{n+1}, \cdots, v_{n+\alpha(k)-3}$. Call the resulting graph G^*. Now consider the graph G^{**} with the vertex set $\{v_i^j : i \in \mathbb{N}_{n+\alpha(k)-3}, j \in \mathbb{N}_k\}$ such that v_x^y is joined to v_z^w if and only if $x = z$ or $v_x v_z \in E(G^*)$. Finally, consider a copy of graph G^{**}, for every i, $1 \leq i \leq n + \alpha(k) - 3$, put two new isolated vertices v'_i and v''_i and join them to the set of vertices $\{v_i^1, \cdots, v_i^k\}$. Call the resulting graph $\widetilde{G}$ (see Figure 2). We show that $\widetilde{G}$ has a *vertex-labeling by product* from $\{1, 2, \cdots, k\}$ if and only if G is 3-colorable. Let f be a *vertex-labeling by product* for $\widetilde{G}$. Clearly, $f(v_1^1), \cdots, f(v_1^k)$ should be different numbers. For every i, $i \in \mathbb{N}_{n+\alpha(k)-3}$, we have: $\{f(v_i^j) : j \in \mathbb{N}_k\} = \mathbb{N}_k$. Furthermore, for every i_1, i_2, $1 \leq i_1 < i_2 \leq n + \alpha(k) - 3$, we have: $f(v'_{i_1})f(v''_{i_1}), f(v'_{i_2})f(v''_{i_2}) \in \mathcal{A}_k$. Also, for every i_1 and i_2, if $v_{i_1}v_{i_2} \in E(G)$, then

$$\frac{f(v'_{i_1})f(v''_{i_1})}{f(v'_{i_2})f(v''_{i_2})} \notin \mathcal{B}_k.$$

Therefore, $| \{f(v'_i)f(v''_i) : 1 \leq i \leq n + \alpha(k) - 3\} | \geq \alpha(k) - 3 + \chi(G)$. So, $\widetilde{G}$ has a *vertex-labeling by product* from $\{1, 2, \cdots, k\}$ if and only if $\chi(G) \leq 3$. The proof is complete.

□

2.3 Edge-labeling by gap

Theorem 4 *For a given planar bipartite graph G, determining whether G has an edge-labeling by gap from $\{1,2\}$ is* **NP***-complete.*

Proof Let Φ be a 3-SAT formula with clauses $C = \{c_1, \cdots, c_k\}$ and variables $X = \{x_1, \cdots, x_n\}$. Let $G(\Phi)$ be a graph with the vertices $C \cup X \cup (\neg X)$, where $\neg X = \{\neg x_1, \cdots, \neg x_n\}$, such that for each clause $c_j = y \vee z \vee w$, c_j is adjacent to y, z and w, also every $x_i \in X$ is adjacent to $\neg x_i$. Φ is called planar 3-SAT type 2 formula if $G(\Phi)$ is a planar graph. It was shown that the problem of satisfiability of planar 3-SAT type 2 is **NP**-complete [11].
Planar 3-*SAT type* 2.
INSTANCE: A 3-SAT type 2 formula Φ.
QUESTION: Is there a truth assignment for Φ that satisfies all the clauses?
We reduce *planar* 3-*SAT type* 2 problem to our problem. In *planar* 3-*SAT type* 2, if we only consider the set of formulas such that the bipartite graph G obtained by linking a variable and a clause if and only if the variable appears in the clause, is connected and it does not have any vertex of degree one, the problem remains **NP**-complete. We reduce this version to our problem. Consider an instance Φ, we transform this into a graph G_Φ such that G_Φ has an *edge-labeling by gap* from $\{1,2\}$ if and only if Φ has a satisfying assignment. For each variable $x \in X$ put a copy of path $P_3 = xt_x\neg x$, also, for each clause $c \in C$ put a copy of gadget $P_4 = cc'c''c'''$. Now, put a copy C_6. Also, for each clause $c = y \vee z \vee w$ add the edges cy, cz and cw. Finally, let x be an arbitrary literal, connect x to one of the vertices of C_6. G_Φ is connected, bipartite and planar. First, suppose that G_Φ has an *edge-labeling f by gap* from $\{1,2\}$ and l is the induced proper coloring by f. Since for every variable x the degrees of vertices x and $\neg x$ are greater than one, also for every clause c the degree of vertex c is 4 and G_Φ is connected, hence in the induced coloring l by f, for the set of variables $\{x_1, \cdots, x_n\}$ and the set of clauses $\{c_1, \cdots, c_m\}$ we have $l(x_1) = l(\neg x_1) = \cdots = l(x_n) \neq l(c_1) = l(\neg c_1) = \cdots = l(c_m)$ and $l(x_1) \neq 2 \neq l(c_1)$. First, suppose that $l(x) = 1$. Since x is adjacent to one of the vertices of C_6, in this situation G_Φ does not have any *edge-labeling f by gap* from $\{1,2\}$. So $l(x) = 0$ and $l(c) = 1$. Hence, the labels of all edges incident with x_1 are same. Also, for every variable x, because of t_x, the labels of all edges incident with x are different from the labels of all edges incident with $\neg x$. Now, for every variable x, which is appeared in $c_i, c_j, \cdots, c_k$ put $\Gamma(x) = True$ if and only if the labels of edge $c_i x$ is 2. For every clause $c = x \vee y \vee w$, $l(c) = 1$, if the set of labels of edges $\{cx, cy, cw\}$ is $\{1\}$, then since $l(c) = 1$ and by attention to the gadget $cc'c''c'''$, G does not have any *edge-labeling f by gap* from $\{1,2\}$. So, $2 \in \{f(cx), f(cy), f(cw)\}$. Therefore, Γ is an satisfying assignment. Now, let Γ be an satisfying assignment for Φ.

For every variable x, label all the edges incident with x by 2 if and only if $\Gamma(x) = True$. It is easy to extend this labeling to an *edge-labeling f by gap* from $\{1, 2\}$. This completes the proof. $\square$

Theorem 5 *For every k, $k \geq 3$, it is* **NP***-complete to determine whether a given graph has an* edge-labeling by gap *from $\{1, 2, \cdots, k\}$.*

Proof We present a polynomial time reduction from *k-colorability*, to our problem.
k-Colorability: Given a graph G; is $\chi(G) \leq k$?
For a given graph G, we construct a graph G^* such that $\chi(G) \leq k$ if and only if G^* has an *edge-labeling by gap* from $\{1, 2, \cdots, k\}$. Let G be a graph, for every vertex $v \in V(G)$, put a copy $P_3 = vv'v''$ and join v to u if and only if $uv \in E(G)$. Call the resulting G^*. First, suppose that G^* has an *edge-labeling f by gap* from $\{1, 2, \cdots, k\}$ and ℓ is the induced coloring by f. for every vertex v, $v \in V(G^*)$ of degree more then one, we have $\ell(v) \in \{0, 1, cdots, k-1\}$, so ℓ is also a proper vertex coloring for G. Now, let c be a proper vertex coloring for G. For every vertex $v\ inV(G^*)$, label all edges incident with v except vv' by 1 and label vv' by $c(v)$. Finally for every edge $v'v''$, label $v'v''$ by 1 if $c(v) \neq 1$, otherwise label $v'v''$ by k. This labeling is an *edge-labeling by gap* from $\{1, 2, \cdots, k\}$. $\square$

2.4 Vertex-labeling by gap

Theorem 6 *For a given bipartite graph G, determining whether G has a vertex-labeling by gap from $\{1, 2\}$ is* **NP***-complete.*

Proof We reduce *Not-All-Equal 3-Sat* to our problem in polynomial time. It is shown that the following problem is **NP**-complete [13].
Not-All-Equal 3-Sat .
INSTANCE: Set X of variables, collection C of clauses over X such that each clause $c \in C$ has $\mid c \mid = 3$.
QUESTION: Is there a truth assignment for X such that each clause in C has at least one true literal and at least one false literal?
For a given Φ, we transform Φ into a graph G_Φ such that G_Φ has a *vertex-labeling by gap* from $\{1, 2\}$ if and only if Φ has a satisfying assignment. Construction of G_Φ is similar to the proof Theorem 4, except the gadget $P_4 = cc'c''c'''$. For each clause $c \in C$ instead of $P_4 = cc'c''c'''$, put a isolated vertex c. First, suppose that G_Φ has an *edge-labeling f by gap* from $\{1, 2\}$ and l is the induced proper coloring by f. By an argument similar to argument of proof of Theorem 4, for every clause $c = x \vee y \vee w$, $l(c) = 1$. So

$\{f(x), f(y), f(w)\} = \{1, 2\}$, therefore Γ is a NAE satisfying assignment. Now, let Γ be an satisfying assignment for Φ. For every variable x, label the vertex x by 2 if and only if $\Gamma(x) = True$. This completes the proof. $\square$

Theorem 7 *For a given planar bipartite graph G, determining whether G has a* vertex-labeling by gap *from $\{1, 2\}$ is in* **P**.

Proof First we show that every tree T with more than two vertex has a *vertex-labeling by gap* from $\{1, 2\}$. Let T be a tree with more than two vertex and $v \in V(T)$ be an arbitrary vertex, define:

$$f(u) = \begin{cases} 1 & \text{if } d(u, v) \equiv 0 \pmod 4, \\ 2 & \text{otherwise,} \end{cases}$$

We call this kind of labeling as *good labeling by center v*. It is easy to see that *good labeling by center v* is a *vertex-labeling by gap* from $\{1, 2\}$. Now, consider the following problem.
Planar Not-All-Equal 3-Sat.
INSTANCE: Set X of variables, collection C of clauses over X such that each clause $c \in C$ has $| c |= 3$ and the following graph obtained from 3-Sat is planar. The graph has one vertex for each variable, one vertex for each clause; all variable vertices are connected in a simple cycle and each clause vertex is connected by an edge to variable vertices corresponding to the literals present in the clause.
QUESTION: Is there a Not-All-Equal truth assignment for X?
It was proved in [19] that *Planar Not-All-Equal 3-Sat* is in **P** by a reduction to a known problem in **P**, namely Planar(Simple) MaxCut. By a simple argument it was shown that the following problem is in **P** (for more information see [10]).
Planar Not-All-Equal 3-Sat Type 2.
INSTANCE: Set X of variables, collection C of clauses over X such that each clause $c \in C$ has $| c |= 3$ and the bipartite graph obtained by linking a variable and a clause if and only if the variable appears in the clause, is planar.
QUESTION: Is there a Not-All-Equal truth assignment for X?
Now, consider the following:
Planar Not-All-Equal Sat Type 2.
INSTANCE: Set X of variables, collection C of clauses over X such that each clause $c \in C$ has $| c |\geq 2$ and the bipartite graph obtained by linking a variable and a clause if and only if the variable appears in the clause, is planar.
QUESTION: Is there a Not-All-Equal truth assignment for X?
We can transform any instance of Φ *Planar Not-All-Equal Sat Type 2* to an instance Ψ of *Planar Not-All-Equal 3-Sat Type 2* in polynomial time. For a given instance Φ, for each clause with exactly two literals like $c = (x \vee y)$, put two clauses $x \vee y \vee t$ and

$x \vee y \vee \neg t$ in Ψ, where t is a new variable. And for each clause with exactly four literals like $c = (x \vee y \vee w \vee z)$, put two clauses $x \vee y \vee t$ and $w \vee z \vee \neg t$ in Ψ, where t is a new variable. For clauses with more than five variable we have a similar argument.

Let $G = [X, Y]$ be a planar bipartite graph, remove all vertices of degree one, repeat this procedure to obtain a graph $G' = [X', Y']$ such that G' does not have a vertex of degree one. For every vertex $v \in X'$, consider a variable v in Φ and for every vertex $u \in Y'$ with $d_G(u) = d_{G'}(u)$ put a clause $(\vee_{v \sim u} v)$ in Φ. Now determine whether Φ has a Not-All-Equal truth assignment. If Φ has a Not-All-Equal truth assignment Γ, for every vertex v, $v \in X'$ label v by 1 if and only if $\Gamma(v) = False$. Label other vertices of G' by 2, call this labeling by f. The induced graph on $V(G) \setminus V(G')$ is a forest, call this forest by F. Suppose that $F = T_1 \cup \cdots \cup T_k$, where T_i is a tree. For every i, $1 \leq i \leq k$ let v_i, $v_i \in V(G')$ be a vertex with minimum distance from T_i. Now for every T_i four cases can be considered:

Case 1: $v_i \in Y'$ and $\{\bigcup_{v \sim u} f(u)\} = \{1, 2\}$. Let $z \in N_{G'}(v_i)$ such that $f(z) = 1$ and $T'_i = T_i \cup v_i \cup z$. Suppose that f_i is a *good labeling by center* z for T'_i.

Case 2: $v_i \in Y'$ and $\{\bigcup_{v \sim u} f(u)\} = \{2\}$. Let $z \in N_{T'_i}(v_i)$. Suppose that f_i is a *good labeling by center* z for T_i.

Case 3: $v_i \in Y'$ and $\{\bigcup_{v \sim u} f(u)\} = \{1\}$. Let $z \in N_{G'}(v_i)$ such that $f(z) = 1$ and $T'_i = T_i \cup v_i \cup z$. Suppose that f_i is a *good labeling by center* z for T'_i.

Case 4: $v_i \in X'$ and $\{\bigcup_{v \sim u} f(u)\} = \{2\}$. Let $T'_i = T_i \cup v_i \cup t$, where t is anew vertex and t is joined to v_i in T'_i. Suppose that f_i is a *good labeling by center* t for T'_i.

It is easy to see that the union of good labelings $f, f_1, f_2, \cdots, f_k$ is a *vertex-labeling by gap* from $\{1, 2\}$ for G. If Φ does not have a Not-All-Equal truth assignment. Then, for every vertex $v \in Y'$, consider a variable v in Ψ and for every vertex $u \in X'$ with $d_G(u) = d_{G'}(u)$ put a clause $(\vee_{v \sim u} v)$ in Ψ. Now determine whether Ψ has a Not-All-Equal truth assignment. If Φ has a Not-All-Equal truth assignment Γ by a similar method we can find *vertex-labeling by gap* from $\{1, 2\}$ for G. Otherwise, G does not have any *vertex-labeling by gap* from $\{1, 2\}$.

□

Theorem 8 *For every k, $k \geq 3$, it is* **NP***-complete to determine whether a given graph has a* vertex-labeling by gap *from* $\{1, 2, \cdots, k\}$.

Proof The proof is similar to the proof of Theorem 5. □

It was shown that 3-colorability of planar 4-regular graphs is NP-complete [9]. So we have the following:

Theorem 9 *It is* **NP**-*complete to decide whether a given planar 3-colorable graph* G *has a* vertex-labeling by gap *from* $\{1,2\}$.

2.5 Vertex-labeling by degree

For every k greater than three it is clear that determining whether a given graph has a *vertex-labeling by degree* from $\{1, 2, \cdots, k\}$ is **NP**-complete.

Theorem 10 *Determining whether a given graph has a* vertex-labeling by degree *from* $\{1,2\}$ *is in* **P**.

Proof We reduce our problem to 2-*SAT* problem in polynomial time.
2-*SAT*.
INSTANCE: A 2-SAT formula Φ.
QUESTION: Is there a truth assignment for Φ that satisfies all the clauses?
For a given graph G of order n we construct a 2-SAT formula Φ with n variables $v_1, v_2 \cdots, v_n$ such that G has a *vertex-labeling by degree* from $\{1,2\}$ if and only if there is a truth assignment for Φ. For every edge $e = v_iv_j$, if $d(v_i) = d(v_j)$, add the clauses $v_i \vee v_j$ and $\neg v_i \vee \neg v_j$ and if $d(v_i) = 2d(v_j)$, add the clause $v_i \vee \neg v_j$, otherwise if $2d(v_i) = d(v_j)$, add the clause $\neg v_i \vee v_j$. First, suppose that Γ is satisfying assignment for Φ. For every vertex v_i, label v_i by 2 if and only if $\Gamma(v_i) = true$. It is easy to see that this labeling is a *vertex-labeling by degree* from $\{1,2\}$. Next, let f be a *vertex-labeling by degree* from $\{1,2\}$, for every variable v_i, put $\Gamma(v_i) = true$ if and only if $f(v_i) = 2$. As we know 2-*SAT* problem is in **P** [13]. This completes the proof. □

2.6 Vertex-labeling by maximum

A graph may lack any *vertex-labeling by maximum*, in the next we consider the complexity of *vertex-labeling by maximum*; also, we present a necessary condition that can be checked in polynomial time for a graph to have a *vertex-labeling by maximum*.

Theorem 11 *For a given 3-regular graph* G, *determining whether* G *has a* vertex-labeling by maximum *from* $\{1,2,3\}$ *is* **NP**-*complete.*

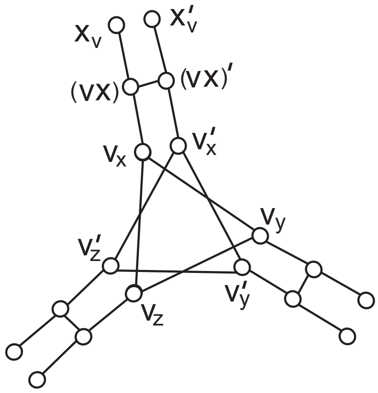

Figure 3: Transformation in constructing G'.

Proof Clearly, the problem is in NP. It was shown that it is **NP**-hard to determine the edge chromatic number of a cubic graph [14]. Let G be a 3-regular graph. We construct a 3-regular graph G' from G such that G' has a *vertex-labeling by maximum* from $\{1,2,3\}$ if and only if G belongs to *Class* 1. In order to construct G', for every vertex $v \in V(G)$ with the neighbors x, y and z consider two disjoint triangles $v_x v_y v_z$ and $v'_x v'_y v'_z$ in G'. Also, for every edge $e \in E(G)$, consider two vertices e and e' in G'. Finally, for every edge $e = uv \in E(G)$, join e to v_u and u_v; also join e' to v'_u and u'_v. Name the constructed graph G' (see Figure 3). Since G' has triangles, so every *vertex-labeling by maximum* needs at least 3 distinct labels. First suppose that G' has a *vertex-labeling* f *by maximum* from $\{1,2,3\}$ and let ℓ be the induced vertex coloring by f. For every vertex $v \in V(G)$ with the neighbors x, y and z in G, we have $\{\ell(v_x), \ell(v_y), \ell(v_z)\} = \{1,2,3\} = \{\ell(v'_x), \ell(v'_y), \ell(v'_z)\}$. Suppose that there are u and v such that $\ell(v_u) = \ell(v'_u) = 3$, then $f(vu) = f((vu)') = 3$. Since f can not assign 3 to the vertices in a triangle, hence $\ell(vu) = \ell((vu)') = 3$ and this is a contradiction. so we have the following fact:
There are no u and v such that $\ell(v_u) = \ell(v'_u) = 3$ (Fact 1).
Now, consider the following proper 3-edge coloring for G: $g : E(G) \longrightarrow \{1,2,3\}$,

$$g(uv) = \begin{cases} 1 & if\ f(uv) = 3, \\ 2 & if\ f((uv)') = 3, \\ 3 & otherwise. \end{cases}$$

By Fact 1, g is well-defined and G belongs to *Class* 1. On the other hand, assume that $g : E(G) \longrightarrow \{1,2,3\}$ is a proper 3-edge coloring. Define $f : V(G') \longrightarrow \{1,2,3\}$ such that for every edge $uv \in E(G)$, $f(v_u) = f(v'_u) = 1$, $f(uv) = g(uv)$ and $f((uv)') \equiv g(uv) + 1($ mod 3$)$. It is easy to see that f is a *vertex-labeling by maximum*. □

For a given graph G, put a new vertex v and join it to the all vertices of G, next put a new vertex u and join it to v. Name the constructed graph G'. We can construct G' in polynomial time and G has a *vertex-labeling by maximum* from $\{1, 2, \cdots, k\}$ if and only if G' has a *vertex-labeling by maximum* from $\{1, 2, \cdots, k+1\}$,so we have the following:

Theorem 12 *For every $k \geq 3$, it is* **NP***-complete to decide whether G has a* vertex-labeling by maximum *from $\{1, 2, \cdots, k\}$ for a given k-colorable graph G.*

Every triangle-free graph has a *vertex-labeling by maximum* (put different numbers on vertices) and if G is graph such that every vertex appears in some triangles then G does not have *vertex-labeling by maximum.* Here, we present a nontrivial necessary condition for a graph to have a *vertex-labeling by maximum.* First consider the following definition.

Definition 1*For a given graph G the subset S of vertices is called kernel if every $v \in S$ appears in a triangle in $G[S]$ and for every two adjacent vertices v and u, where $v \in S$ and $u \in N(S) \setminus S$, there exists a vertex $z \in S$ such that z is adjacent to v and u.*

Let S be a kernel for G. To the contrary, assume that f is a *vertex-labeling by maximum* for G and $v \in S \cup N(S)$ is a vertex that gets the maximum of $\{f(u) : u \in S \cup N(S)\}$. Then v has two neighbors x and y in S with $\max_{u \sim x} f(u) = \max_{u \sim y} f(u) = f(v)$. This is a contradiction. Therefore, if G has a kernel, then G does not have a *vertex-labeling by maximum.* Now, consider Algorithm 1.

When Algorithm 1 terminates, if it returns "G has the kernel S", then S is a kernel, so G does not have *vertex-labeling by maximum.* Suppose that Algorithm 1 returns "G has no kernel", but G has a kernel S'. In the lines $2-3$ of algorithm, the set of vertices S' are added to S. Now, consider the line 5 of algorithm and let $v \in S'$ be the first vertex form the set S' that is eliminated from S. When Algorithm 1 chooses the vertex v, v is in a triangle in $G[S']$, so is in a triangle in $G[S]$. Therefore, there is a vertex u such that $uv \in E(G)$, $v \in S'$, $u \in N(S) \setminus S$ and there is no vertex $z \in S$ such that z is adjacent to v and u. So S' is not kernel. It is a contradiction. So when Algorithm 1 returns "G has no kernel", G does not have any kernel. Here, we ask the following question: Is the necessary condition, sufficient for a given graph to have a *vertex-labeling by maximum*?

Problem 5. *Does there is a polynomial time algorithm to determine whether a given graph has a* vertex-labeling by maximum*?*

Algorithm 1 (Kernel)

$S = \emptyset$
for (Every vertex u in a triangle) **do**
  $S \leftarrow S \cup \{u\}$
end for
while (There are two adjacent vertices u and v such that $v \in S$, $u \in N(S) \setminus S$ and there is no vertex $z \in S$ such that z is adjacent to v and u.) or (v is not in any triangle in $G[S]$) **do**
  $S \leftarrow S \setminus \{v\}$
end while
if ($S \neq \emptyset$) **then**
  Return "G has the kernel S."
else
  Return "G has no kernel."
end if

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