Elementary Mathematics International Contest

Individual Contest

1. For any two numbers a and b, $a*b$ means $a+b-\dfrac{2011}{2}$.

   Calculate: $1*2*3*\cdots*2010*2011$.

【Solution】

$a*b*c=(a+b-\dfrac{2011}{2})*c=a+b+c-2011$.

$a_1*a_2*\cdots*a_n=a_1+a_2+\cdots+a_n-\dfrac{2011}{2}\times(n-1)$, （$n$=2, 3, 4, $\cdots$）

$$1*2*3*\cdots*2010*2011=(1+2+3+\cdots+2011)-\frac{2011}{2}\times 2010$$

$$=\frac{2011\times 2012}{2}-2011\times 1005=2011$$

ANS:2011

2. Suppose 11 coconuts have the same cost as 14 pineapples, 22 mango have the same cost as 21 pineapples, 10 mango have the same cost as 3 bananas, and 5 oranges have the same cost as 2 bananas. How many coconuts have the same cost as 13 oranges?

【Solution】

An orange is worth $\dfrac{2}{5}$ banana. A banana is worth $\dfrac{10}{3}$ mango. A mango is worth $\dfrac{21}{22}$ pineapple. A pineapple is worth $\dfrac{11}{14}$ coconut. Then an orange is worth $\dfrac{2}{5}\times\dfrac{10}{3}\times\dfrac{21}{22}\times\dfrac{11}{14}=1$ coconut. So 13 oranges have the same worth as 13 coconuts.

ANS:13

3. A girl calculates $\dfrac{1+2}{3}+\dfrac{4+5}{6}+\cdots+\dfrac{2011+2012}{2013}$ and a boy calculates $1+\dfrac{1}{2}+\dfrac{1}{3}+\cdots+\dfrac{1}{671}$. What is the sum of their answers?

【Solution】

Notice that sum of the corresponding summands of a girl and a boy always equals 2. So, the sum of all the fractions is 2×671 = 1342.

ANS: 1342

4. What is the first time between 4:00 and 5:00 that the hour hand and the minute hand are exactly 10° apart?

【Solution】

At 4:00, the hour hand and minute hand form 120°.

The rate of minute hand is 6°/min (360°/ 60 mins) and

The rate of hour hand is $\frac{1}{2}$°/min (30°/ 60 mins).

The rate of change = 6°/min $-\ \frac{1}{2}$°/min = 5.5°/min.

Now the number of minutes the two hands move a difference of 110° (120° − 10°)

$= \frac{110^\circ}{5.5^\circ/\text{min}}$ = 20 mins. Thus, the answer = 4:20

ANS: 4:20

5. Two squirrels, Tim and Kim, are dividing a pile of hazelnuts. Tim starts by taking 5 hazelnuts. Thereafter, they take alternate turns, each time taking 1 more hazelnut than the other in the preceding turn. If the number of hazelnuts to be taken is larger than what remains in the pile, then all remaining hazelnuts are taken. At the end, Tim has taken 101 hazelnuts. What is the exact number of hazelnuts at the beginning?

【Solution】

Note that 101=5+7+9+11+13+15+17+19+5. Hence Kim has taken 6+8+10+12+14 +16+18+20=104, and there are 101+104=205 hazelnuts at the beginning.

ANS: 205

6. In how many ways can we pay a bill of \$500 by a combination of \$10, \$20 and \$50 notes?

【Solution】

Suppose we use x \$10 notes, y \$20 notes and z \$50 notes. Then $10x+20y+50z=500$ or $x+2y+5z=50$, where $x, y, z \geq 0$.

a) When $z=0$, $x+2y=50 \Rightarrow x = 50-2y$ and $0 \leq y \leq 25$. So we have 26 solutions.

b) When $z=1$, $x+2y=45 \Rightarrow x = 45-2y$ and $0 \leq y \leq 22\frac{1}{2}$. So we have 23 solutions.

c) When $z=2$, $x+2y=40 \Rightarrow x = 40-2y$ and $0 \leq y \leq 20$. So we have 21 solutions.

d) When $z=3$, $x+2y=35 \Rightarrow x = 35-2y$ and $0 \leq y \leq 17\frac{1}{2}$. So we have 18 solutions.

Continuing in a similar manner, we find that the total number of combinations is 26+23+21+18+16+13+11+8+6+3+1=146.

ANS: 146

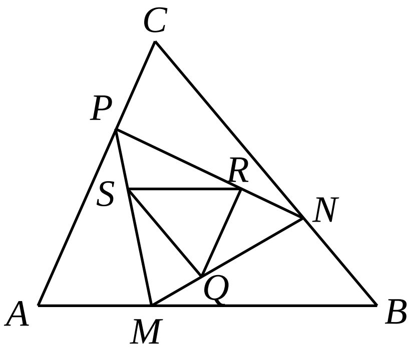
7. The least common multiple of the numbers 16, 50 and A is 1200. How many positive integers A have this property?

【Solution】

Note that $16 = 2^4$ and $50 = 2 \times 5^2$. Hence the least common multiple of 16 and 50 is 400, and 1200÷400 = 3. It follows that the prime factorization of A consists of exactly one 3, up to four 2s and up to two 5s. Hence the number of possible values of A is (4+1)(2+1)=15.

ANS: 15

8. In the figure below, $\frac{AM}{MB}=\frac{BN}{NC}=\frac{CP}{PA}=\frac{1}{2}$ and $\frac{MQ}{QN}=\frac{NR}{RP}=\frac{PS}{SM}=\frac{1}{2}$. If the area of $\triangle ABC$ is 360 cm^2, what is the area of $\triangle QRS$, in cm^2?



【Solution】

Connect point M and point C,

then $S_{\Delta AMC}=\frac{1}{3}S_{\Delta ABC}$, and $S_{\Delta AMP}=\frac{2}{3}S_{\Delta AMC}$,

so $S_{\Delta AMP}=\frac{1}{3}\cdot\frac{2}{3}S_{\Delta ABC}=\frac{2}{9}S_{\Delta ABC}$.

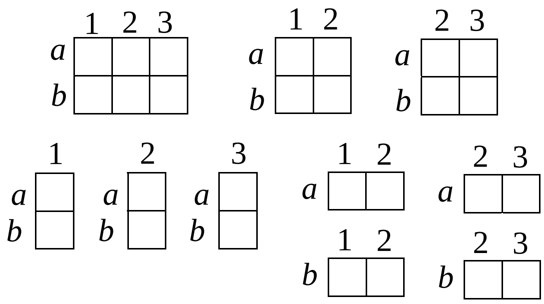
Similarly, $S_{\Delta BNM}=\frac{2}{9}S_{\Delta ABC}$, $S_{\Delta CPN}=\frac{2}{9}S_{\Delta ABC}$.

Hence $S_{\Delta PMN}=(1-\frac{2}{9}\times 3)S_{\Delta ABC}=\frac{1}{3}S_{\Delta ABC}$.

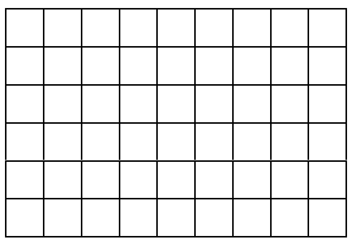
Similarly, $S_{\Delta QRS}=\frac{1}{3}S_{\Delta PMN}$. So $S_{\Delta QRS}=\frac{1}{9}S_{\Delta ABC}=40$ cm^2.

ANS: 40 cm^2

9. In a 2×3 table, there are 10 rectangles which consist of an even number of unit squares.



How many rectangles are there in a 6×9 table which consist of an even number of unit squares?



【Solution 1】

At least one dimension of the rectangle is even. Suppose only the vertical dimension is even. Then there are (5+3+1)×(9+8+⋯+2+1)=405 such rectangles. Suppose only the horizontal dimension is even. Then there are (6+5+4+3+2+1)×(8+6+4+2)=420 such rectangles. Suppose both dimensions are even. Then there are (5+3+1)×(8+6+4+2)=180 such rectangles. It follows that the desired number is 405+420-180=645.

【Solution 2】

In the figure, there are (6+5+4+3+2+1)x(9+8+⋯+2+1)=945 rectangles. Observe that the length and width of a rectangle which is formed by odd unit squares are both odd, so there are (6+4+2)x(9+7+5+3+1) =300 such rectangles. Hence there are 945－600 =645 rectangles which consist of an even number of unit squares.

ANS: 645

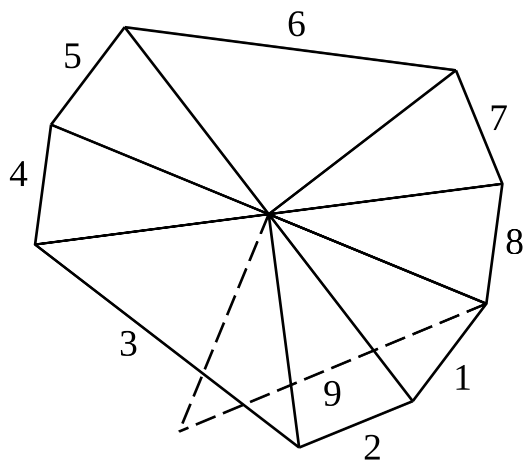
10. Find the smallest positive common multiple of 4 and 6 such that each digit is either 4 or 6, there is at least one 4 and there is at least one 6.

【Solution】

The answer is 4464. The number formed of the last two digits must be 44, 46, 64 or 66. If the number is divisible by 4, the number formed of the last two digits must be divisible by 4. This eliminates 46 and 66, so that the last digit is 4. We now need the number to be divisible by 3, which means that its digit sum must be divisible by 3. Since we must use at least one 4, we must use at least three 4s. Since we must use at least one 6, the smallest number we seek is 4464.

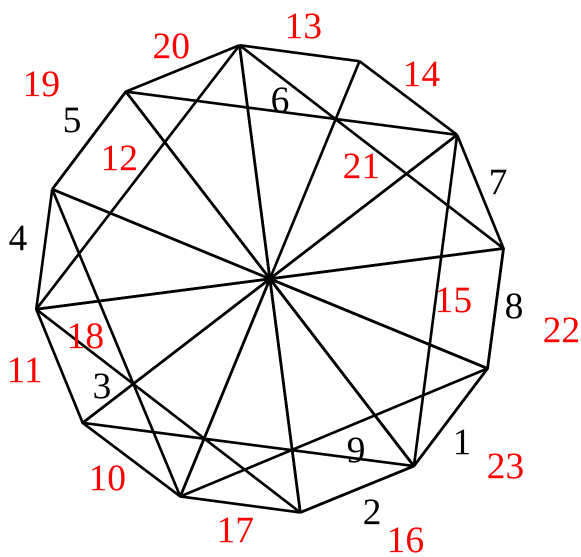
ANS: 4464

11. We have two kinds of isosceles triangles each with two sides of length 1. The acute triangle has a 30° angle between the two equal sides, and the right triangle has a right angle between the two equal sides. We place a sequence of isosceles triangles around a point according to the following rules. The *n*-th isosceles triangle is a right isosceles triangle if *n* is a multiple of 3, and an acute isosceles triangle if it is not. Moreover, the *n*-th and (*n*+1)-st isosceles triangles share a common side, as shown in the diagram below. What is the smallest value of *n*>1such that the *n*-th isosceles triangle coincides with the 1-st one?



【Solution】

The right diagram shows that *n*=23.



ANS: 23

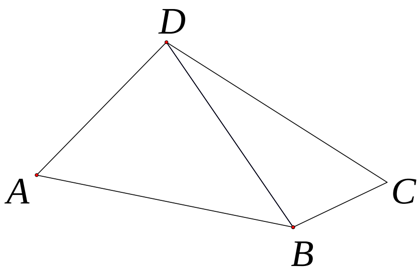
12. When the digits of a two-digit number are reversed, the new number is at least 3 times as large as the original number. How many such two-digit numbers are there?

【Solution】

Let the number be $\overline{ab}=10a+b$, then $10b+a\geq 3(10a+b)$, so that $7b\geq 29a$. Since a and b are digits, we must have $a\leq 2$. When a=2, then $7b\geq 58$ or $b\geq \frac{58}{7}=8\frac{2}{7}$, i.e. b=9. When a=1, then $7b\geq 29$ or $b\geq \frac{29}{7}=4\frac{1}{7}$, i.e. b=9, 8, 7, 6 or 5. So there are only six numbers: 29, 15, 16, 17, 18, 19.

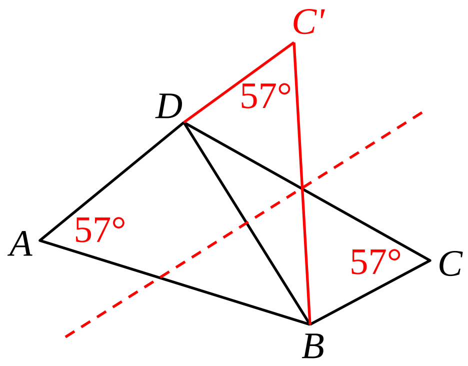
ANS: 6

13. In the quadrilateral *ABCD*, *AB*=*CD*, $\angle BCD$=57°, and $\angle ADB+\angle CBD$ = 180°. Find the value of $\angle BAD$.



【Solution】

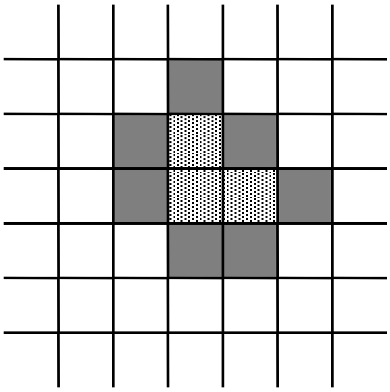
Reflect Triangle *BCD* on perpendicular bisector of *BD*. $\angle ADB+\angle C'DB$= 180°, so *ADC'* is a straight line. Triangle *ABC*' is isosceles. Hence $\angle BAD$= 57°.



ANS: 57°

14. Squares on an infinite chessboard are being painted. As shown in the diagram below, three squares (lightly shaded) are initially painted. In the first step, we paint all squares (darkly shaded) which share at least one edge with squares already painted. The same rule applies in all subsequent steps. Find the number of painted squares after one hundred steps.

First step



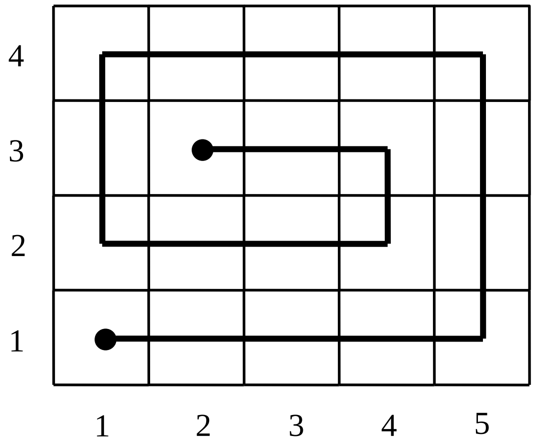
【Solution】

Initially, there are 2 rows containing painted cells. In each move, this number increases by 2, so that at the end, there are 202 rows containing painted cells.

On each move, we paint one cell at each end of each row containing painted cells. Moreover, we add a new row on top containing one painted cell and a new row at the bottom containing two painted cells. Hence the total amount of painted cells is 3 + 3×100 + 2×(2+4+…+200) = 20503.

ANS: 20503

15. The rows of a 2011×4024 chessboard are numbered from 1 to 2011 from bottom to top, and the columns from 1 to 4024 from left to right. A snail starts crawling from the cell on row 1 and column 1 along row 1. Whenever it is about to crawl off the chessboard or onto a cell which it has already visited, it will make a left turn and then crawl forwards in a straight line. Thus it follows a spiraling path until it has visited every cell. Find the sum of the row number and the column number of the cell where the path ends. (The answer is 3+2=5 for a 4×5 table.)



【Solution】

By symmetry, the snail will end up on row 1006. It will start there on column 1005 and end on column 4024－1005=3019. Thus the path ends on the cell on row 1006 and column 3019.

ANS: 4025