Arrangements and Duality

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Outline

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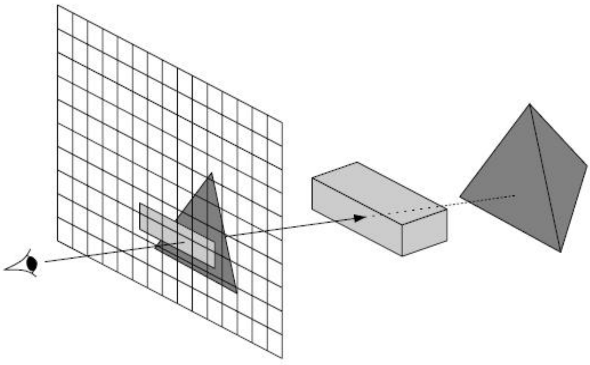
1. Supersampling in Ray Tracing
2. Computing the Discrepancy
3. Duality
4. Arrangements of Lines
5. Levels and Discrepancy

Ray Tracing:



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Rendering the scene:

Generating a 2-dimensional image of a 3-dimensional scene that amounts to:

- determining the visible object at each pixel on the screen,
- determining how bright the object is.

Ray tracing:

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Note:

Ray tracing can also determine how bright the object is.

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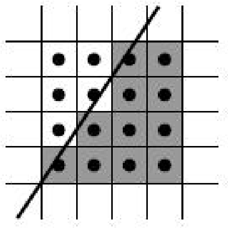
- A pixel is not a point, but a small square area.
- Shooting a ray through each pixel center results in the well-known jaggies in the image.
- The solution is to shoot more than one ray per pixel.



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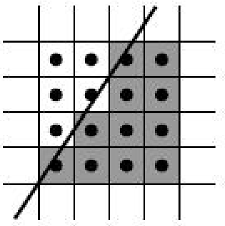

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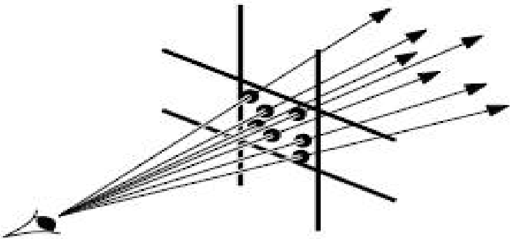
How should we distribute the rays over the pixel :

- Distributing rays regularly isn't such a good idea. Small per-pixel error, but regularity in error across rows and columns. (which triggers the human visual system.)
- It's better to choose the sample points in a somewhat random fashion.
- We want the sample points to be distributed in such a way that the number of hits is closed to the percentage of covered area.



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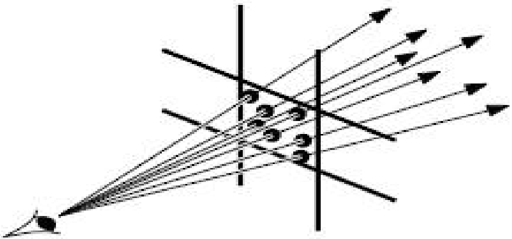

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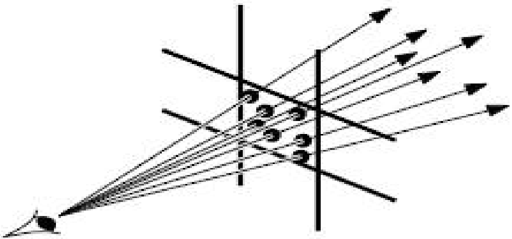

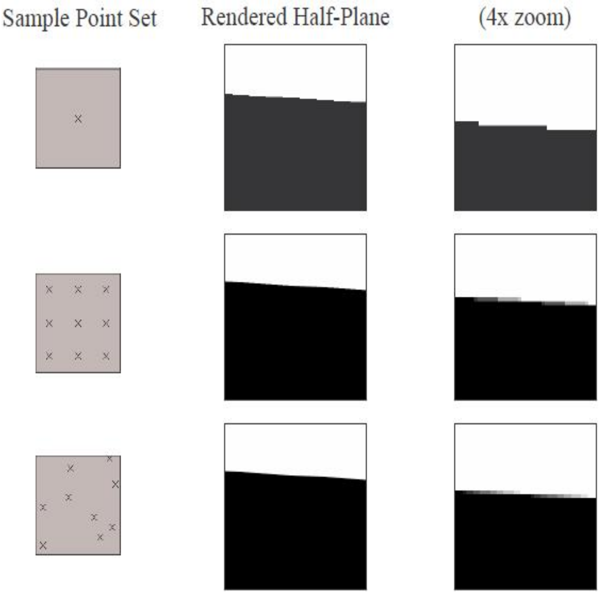
Sample Point Set
Rendered Half-Plane
(4x zoom)

Discrepancy:

Discrepancy of sample set with respect to object:

The difference between the percentage of hits for an object and the percentage of the pixel area where that object is visible.

Note:

we don't know in advance which objects will be visible in the pixel.

Discrepancy of the sample set:

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How discrepancy can be useful?

Based on the discrepancy of given set of sample points we can decide if it is good enough: if the discrepancy is low enough we decide to keep it, and otherwise we generate a new random set.

- For this we need an algorithm that computes the discrepancy of a given point set.

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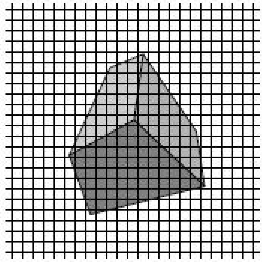
- Assume that curved objects are approximated using polygonal meshes.
- So the 2-dimensional objects that we must consider are the projections of the facet of polyhedra.



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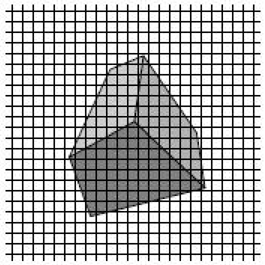

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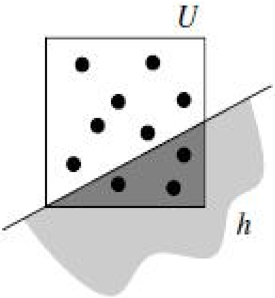
- $U = [0:1] \times [0:1]$: The unit square (pixel)
- $H =$ The (infinite) set of all possible half-planes (scene)
- $S =$ A set of n sample points in U
- Continuous measure : $\mu(h) =$ area of $h \cap U$
- Discrete measure : $\mu_S(h) = card(S \cap h)/card(S)$
- Discrepancy of h wrt S : $\Delta_S(h) = |\mu(h) - \mu_S(h)|$
- Half-plane discrepancy of S : $\Delta_H(S) = \sup_{h \in H} \Delta_S(h)$



- We first identify a finit set of condidate half-planes.
- The half-plane of maximum discrepancy must pass through at least one sample point.
- Let it pass through exactly one point.
- The maximum discrepancy must be at a local extremum of the continuous measure.
- There are an infinite number of h through each point p, but only $O(1)$ of them are local extrema.

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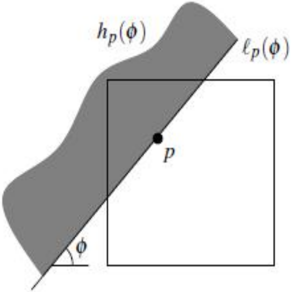
Let:

- $p := (p_x, p_y)$ be a point in S,
- $l_p(\phi)$ be the line through p that makes an angle ϕ with the positive x-axis for $0 \leq \phi < 2\pi$,
- $h_p(\phi)$ be the half-plane initially lying above $l_p(\phi)$.



- We are interested in the local extrema of the function $\phi \to \mu(h_p(\phi))$.

- There is a constant number of local extrema per point $p \in S$.
- Thus the total number of condidate half-planes with one point on their boundary is $O(n)$.
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Lemma 8.1

Let S be a set of n points in the unit square U. A half-plane h that achieves the maximum discrepancy with respect to S is of one of the following types:

(i) h contains one point $p \in S$ on its boundary,

(ii) h contains two or more points of S on its boundary.

The number of type (i) condicates is $O(n)$, and they can be found in $O(n)$ time.

- The number of type (*ii*) condidates is quadratic.
- Because the number of type (*i*) condidates is linear, we treat them in a brute-force way: for each of the $O(n)$ half-planes we compute their continuous measure in constant time, and their discrete measure in $O(n)$ time. This way the muximum of the discrepancies of this half-planes can be computed in $O(n^2)$ time.
- For the type (*ii*) candidates we need some new techniques.

Theorem 8.2

The half-plane discrepancy of a set S of n points in the unit square can be computed in $O(n^2)$ time.

Duality:

- A point in the plane has two parameters: its x-coordinate and its y-coordinate.
- A (non-vertical) line in the plane also has two parameters: its slope and its intersection with the y-axis.
- Therefore we can map a set of points to a set of lines, and vice versa, in a one-to-one manner.

Duality transform:

One-to-one mapping of a set of points to a set of lines such that certain properties are preserved.

- The image of an object under a duality transform is called the dual of the object.

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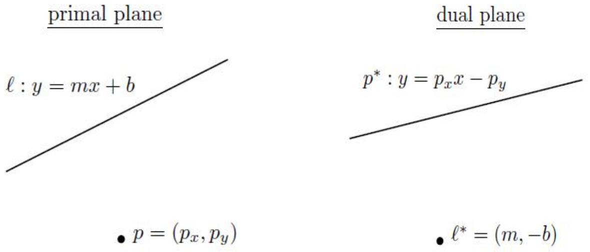
One possible and simple duality tranform:

- point $p : (p_x, p_y) \iff$ line $p^* : y = p_x x - p_y$
- line $l : y = mx + b \iff$ point $l^* : (m, -b)$

Note:

The duality transform is not defined for vertical lines.



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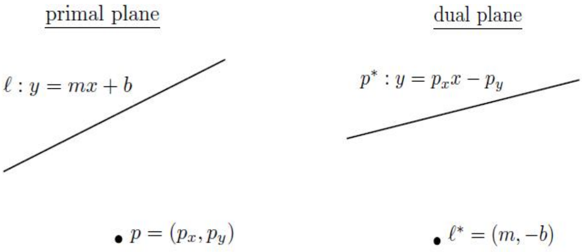
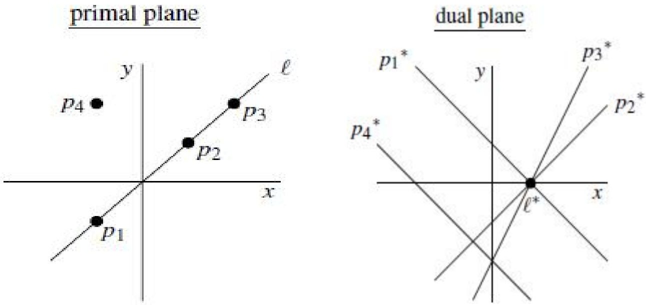


Observation 8.3

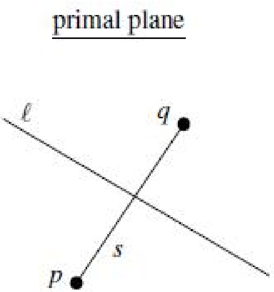
Let p be a point in the plane and let l be a non-vertical line in the plane. The duality transform $o \mapsto o^*$ has the following properties.

- It is incidence preserving: $p \in l$ if and only if $l^* \in p^*$.
- It is order preserving: p lies above l if and only if l^* lies above p^*.



Duality can be applied to other objects, e.g. segments:

- Let $s := \bar{pq}$ be a line segment



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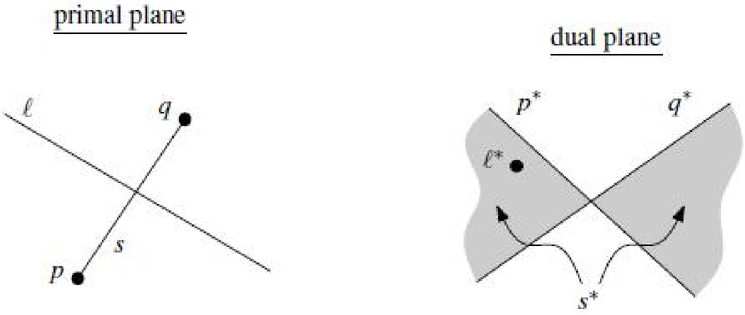

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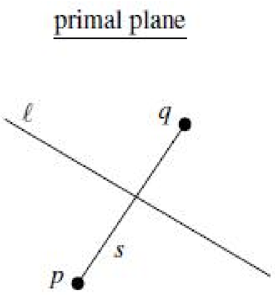

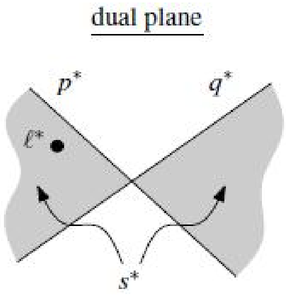

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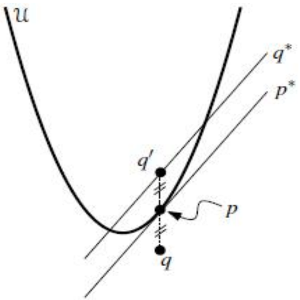
Duality can be applied to other objects, e.g. parabola:

- parabola $\mathcal{U} : y = x^2/2$
- point $p = (p_x, p_y)$ on $\mathcal{U}$
- derivative of $\mathcal{U}$ at p is p_x, i.e., p^* has same slope as tangent line
- tangent line intersects y-axis at $(0, -p_x^2/2)$
- $\Rightarrow p^*$ is tangent line at p
- if q lies directly above or below p, then q^* is the line parallel to p^*



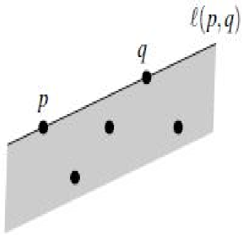
How duality can be useful?

- If you can solve a problem in the dual plane, you could solved it in the primal plane as well by mimicking the solution to the dual problem in the primal problem.
- Looking at things on the dual plane provides new perspectives.

Back to the discrepancy problem:

To determine our discrete measure, we need to:

- Determine how many sample points lie below a given line(in the primal plane).



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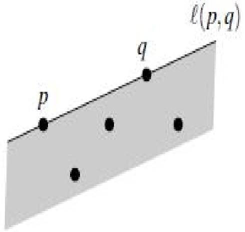

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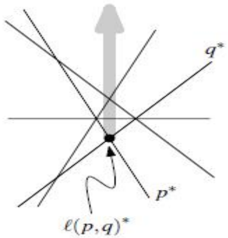
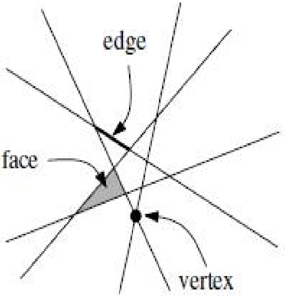


Arrangements:

Arrangement $A(L)$:

Let L be a set of n lines in the plane. L induces a subdivision of the plane that consists of vertices,edges, and faces.This is called the arrangement induced by L, denoted $A(L)$.



Simple arrangment:

An arrangement is called simple if no three lines pass through the same point and no two lines are parallel.

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The complexity of an arrangement is defined as the total number of vertices, edges, and faces of the arrangement.

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Theorem 8.4

Let L be a set of n lines in the plane, and let $A(L)$ be the arrangement induced by L.

(i) The number of vertices of $A(L)$ is at most $n(n-1)/2$.

(ii) The number of edges of $A(L)$ is at most n^2.

(iii) The number of faces of $A(L)$ is at most $n^2/2 + n/2 + 1$.

Equality holds in these three statements if and only if $A(L)$ is simple.

- Total complexity of an arrangement is $O(n^2)$.

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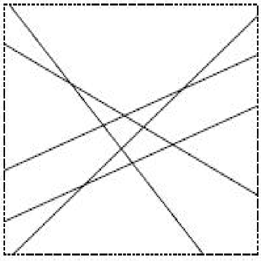
Constructing Arrangements:

- We place a bounding box $B(L)$ that contains all the vertices of $A(L)$ in its interior.



- The subdivision defined by the bounding box plus the part of the arrangement inside it has bounded edges only and can be stored in a doubly-connected edge list.

Constructing Arrangements:

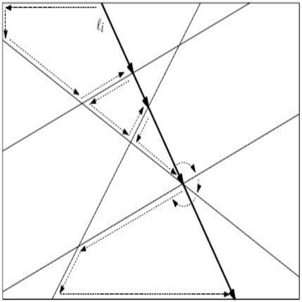
Goal:

Compute A(L) in bounding box in DCEL representation

- A plane sweep algorithm would run in $O(n^2 \log n)$ time.
- faster: Incremental algorithm ($O(n^2)$)

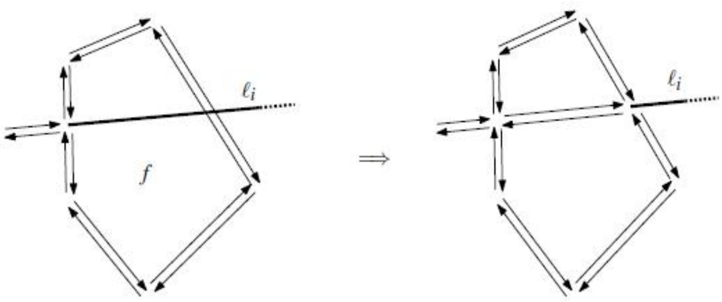
Incremental Algorithm:

- Compute a bounding box $B(L)$ that contains all vertices of $A(L)$ in its iterior and initialize the DCEL.
- Incrementally add each line l_i to A_{i-1} and update DCEL.
  - Find the edge e on $B(L)$ that contains the leftmost intersection point of l_i and A_i
  - Split face bounded by e
  - Move on to next intersected face



Incremental Algorithm:

- Splitting a face f intersected by l_i:



Incremental Algorithm:

- Splitting a face f intersected by l_i:
  - Assume that the face intersected by l_i to the left of f has already been split.
  - Find the edge e' where l_i leaves f and its twin.
  - Create two new records for new faces f' and f'' created by l_i.
  - Create a new vertex record for vertex v' where l_i intersects e' ($l_i \cap e'$).
  - Create two new records for half-edges created by v'.
  - Create half-edge record for the edge $l_i \cap f$.
  - Delete records for e' and f.
- Move to face bounded by twin(e').

Incremental Algorithm:

Algorithm CONSTRUCTARRANGEMENT(L)
Input. A set L of n lines in the plane.
Output. The doubly-connected edge list for the subdivision induced by $\mathcal{B}(L)$ and the part of $\mathcal{A}(L)$ inside $\mathcal{B}(L)$, where $\mathcal{B}(L)$ is a bounding box containing all vertices of $\mathcal{A}(L)$ in its interior.

1. Compute a bounding box $\mathcal{B}(L)$ that contains all vertices of $\mathcal{A}(L)$ in its interior.
2. Construct the doubly-connected edge list for the subdivision induced by $\mathcal{B}(L)$.
3. **for** $i \leftarrow 1$ **to** n
4. **do** Find the edge e on $\mathcal{B}(L)$ that contains the leftmost intersection point of ℓ_i and $\mathcal{A}_i$.
5. $f \leftarrow$ the bounded face incident to e
6. **while** f is not the unbounded face, that is, the face outside $\mathcal{B}(L)$
7. **do** Split f, and set f to be the next intersected face.

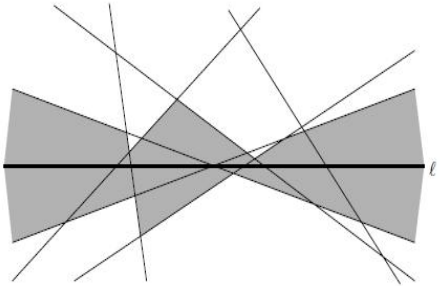
Running time analysis:

- Step 1, computing $B(L)$, can be done in $O(n^2)$ time.
- Step 2, constructing DCEL for $B(L)$, takes only constant time.
- Step 4, Finding the first face split by l_i takes $O(n)$ time.
- We now bound the time it takes to split the faces intersected by l_i (step 7).
- The edges we encounter are on the boundary of faces whose closure is intersected by l_i. This leads us to the concept of zones.

Zones:

Zone of a line l in an arrangement:

The zone of a line l in an arrangement $A(L)$ is the set of faces of $A(L)$ whose closure intersects l.



Zones:

Complexity:

The complexity of a zone is defined as the total complexity of all the faces it consists of, i.e. the sum of the number of edges and vertices of these faces.

- The time we need to insert line l_i is linear in the complexity of the zone of l_i in $A(l_1, \ldots, l_i)$.
- The Zone Theorem tells us that his quantity is linear.

Zones:

Complexity:

The complexity of a zone is defined as the total complexity of all the faces it consists of, i.e. the sum of the number of edges and vertices of these faces.

- The time we need to insert line l_i is linear in the complexity of the zone of l_i in $A(l_1, \ldots, l_i)$.
- The Zone Theorem tells us that his quantity is linear.

Zone Theorem:

Theorem 8.5 (Zone Theorem)

The complexity of the zone of a line in an arrangement of m lines in the plane is $O(m)$.

Proof of Zone Theorem:

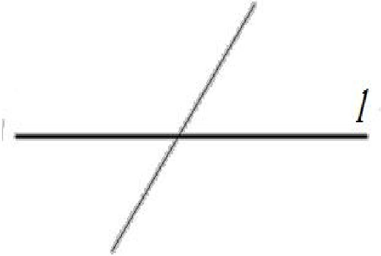
- Given an arrangement of m lines, $A(L)$, and a line l.
- Without loss of generality we assume that l coincides with the x-axis.
- An edge is a left bounding edge for the face lying to the right of it and a right bounding edge for the face lying to the left of it.
- Claim: the number of left bounding edges of the faces in the zone of l is at most $5m$.(Same for number of right bounding edges)

Proof of Zone Theorem:

(i) Assume first that no line of L is horizontal.

Claim: the number of left bounding edges of the faces in the zone of l is at most $5m$.(Same for number of right bounding edges)

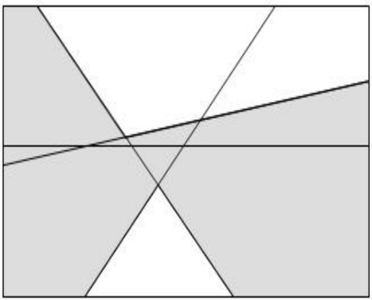
- By induction on m.
- For $m = 1$: Trivial.
  (1 left bounding edge ≤ 5)
- For $m > 1$:



Proof of Zone Theorem:

(1) Let l_1 be the rightmost line intersecting l (assume it's unique).

- The zone of l in $A(L \setminus l_1)$ has at most $5(m-1)$ left bounding edges.
- When adding l_1, the number of such edges increases:
  - One new left bounding edge on l_1.
  - Two old left bounding edges split by l_1.
- Hence, the total number of left bounding edged in this case is at most $5(m-1)+3 < 5m$.



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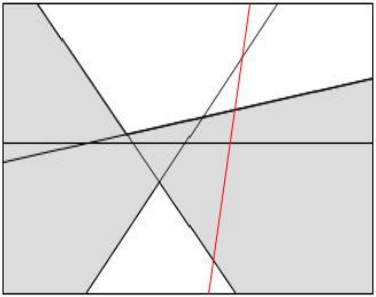

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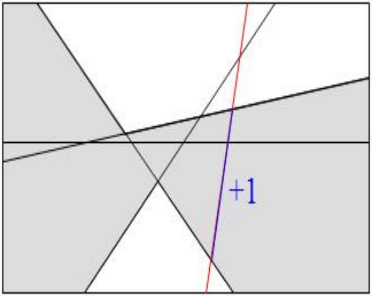

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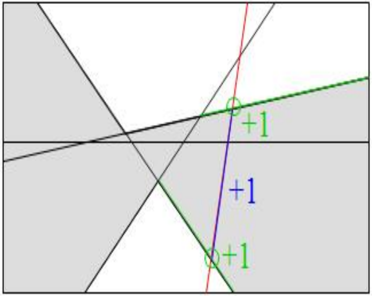

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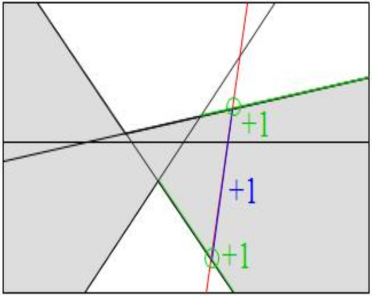
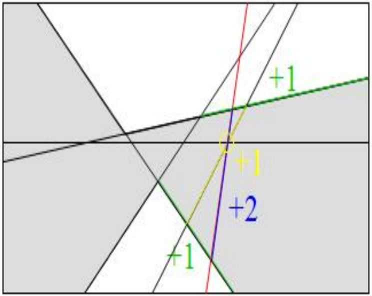


Proof of Zone Theorem:

(2) If exactly two lines intersect l in the rightmost intersection point:

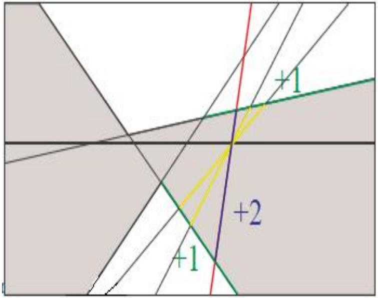
- Denote these lines by l_1 ,l_2.
- The zone of l in $A(L \setminus l_1)$ has at most 5(m-1) left bounding edges.
- l_1 has two left bounding edges
- l_2 is split into two left bounding edges
- l_1 splits two other left bounding edges
- Hence, the new zone complexity is at most $5(m-1) + 5 = 5m$.



Proof of Zone Theorem:

(3) If several lines (> 2) intersect l in the rightmost intersection point:

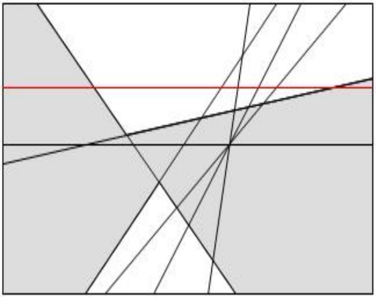
- Pick l_1 randomly out of these lines.
- The zone of l in $A(L \setminus l_1)$ has at most $5(m-1)$ left bounding edges.
- When adding l_1, the number of such edges increases:
  - Two new edges on l_1.
  - Two old edges split by l_1.
- Hence, the new zone complexity is at most $5(m-1) + 4 < 5m$.



Proof of Zone Theorem:

(ii) And what if there are horizontal lines?

- A horizontal line that does'nt coincide with l, introduces less complexity into A(L) than a non-horizontal line.
- If L contains a line l_i that coincide with l, the addition of l_i to $A(L \setminus l_i)$ increases the number of left bounding edges by at most $4m - 2$
- This concludes the proof of the Zone Theorem.



Theorem 8.6

A doubly-connected edge list for the arrangement induced by a set of n lines in the plane can be constructed in $O(n^2)$ time.

Run time analysis:

1. $O(n^2)$
2. constant
3. $\sum_{i=1}^{n} O(i) = O(n^2)$

in total $O(n^2)$

Algorithm CONSTRUCTARRANGEMENT(L)
Input. Set L of n lines.
Output. DCEL for $\mathcal{A}(L)$ in $\mathcal{B}(L)$.
1. Compute bounding box $\mathcal{B}(L)$.
2. Construct DCEL for subdivision induced by $\mathcal{B}(L)$.
3. **for** $i \leftarrow 1$ **to** n
4. **do** insert ℓ_i.

Back to Discrepancy (Again):

- For every line between two sample points, we want to determine how many sample points lie below that line.
- For every vertex in the dual plane, we want to determine how many sample lines lie above it.
- We build the arrangement $A(S^*)$ and use that to determine, for each vertex, how many lines lie above it.

Levels and Discrepancy:

level of a point:

The level of a point in an arrangement of lines is the number of lines strictly above it.

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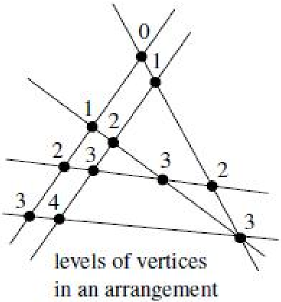
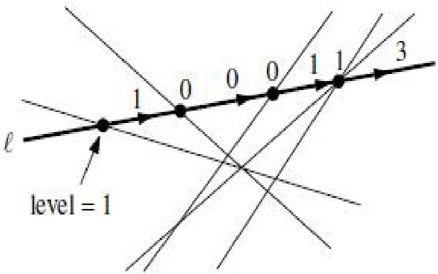


Computing the Levels:

- For each line l in S^*:
  - Compute the level of the leftmost vertex. $O(n)$
  - Walk along l from left to right to visit the other vertices on l, using the DCEL. The level only changes at a vertex, and the change can be computed by inspecting the edges incident to the vertex that is encountered. $O(1)$
- The levels of all vertices of $A(S^*)$ can be computed in $O(n^2)$ time.



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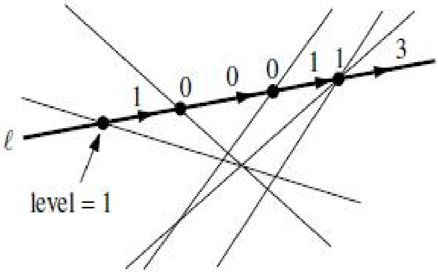

Summary:

- Problem regarding points S in ray-tracing
- Dualize to a problem of lines L.
- Compute arrangement of lines $A(L)$.
- Compute level of each vertex in $A(L)$.
- Use this to compute discrete measures in primal space.
- We can determine how good a distribution of sample points is in $O(n^2)$ time.

END.