2009 Durban Invitational World Youth Mathematics Intercity Competition

10th IWYMIC

Solutions of Individual Contest

Time limit: 120 minutes 8th July 2009 Durban, South Africa

Section A.

1. If a, b and c are three consecutive odd numbers in increasing order, find the value of $a^2 - 2b^2 + c^2$.

Solution

Note that $a = b - 2$ and $c = b + 2$.

Hence, $a^2 - 2b^2 + c^2 = b^2 - 4b + 4 - 2b^2 + b^2 + 4b + 4 = 8$.

2. When the positive integer n is put into a machine, the positive integer $\frac{n(n+1)}{2}$ is produced. If we put 5 into this machine, and then put the produced number into the machine, what number will be produced?

Solution

The first number produced is $\frac{5(5+1)}{2} = 15$. The second number produced is $\frac{15(15+1)}{2} = 120$.

3. Children A, B and C collect mangos.

   A and B together collect 6 mangos less than C.

   B and C together collect 16 mangos more than A.

   C and A together collect 8 mangos more than B.

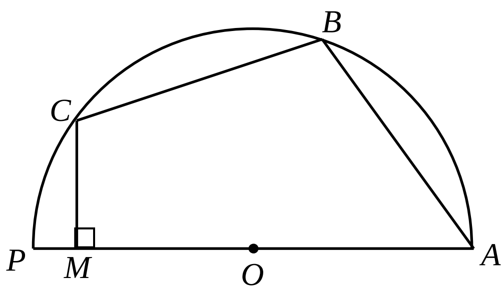
   What is the product of the number of mangos that A, B and C collect individually?

Solution

$$a + b - c = -6 \quad \cdots\cdots(1)$$
$$b + c - a = 16 \quad \cdots\cdots(2)$$
$$c + a - b = 8 \quad \cdots\cdots(3)$$

From (1)+(2)+(3), we have $a + b + c = 18$, so that c = 12, b = 5, a = 1. The product is $1 \times 5 \times 12 = 60$.

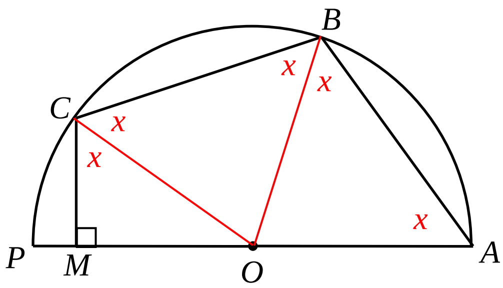
4. The diagram shows a semicircle with centre O. A beam of light leaves the point M in a direction perpendicular to the diameter PA, bounces off the semicircle at C in such a way that *angle MCO* = *angle OCB* and then bounces off the semicircle at B in a similar way, hitting A. Determine *angle COM*, in degrees.



Solution

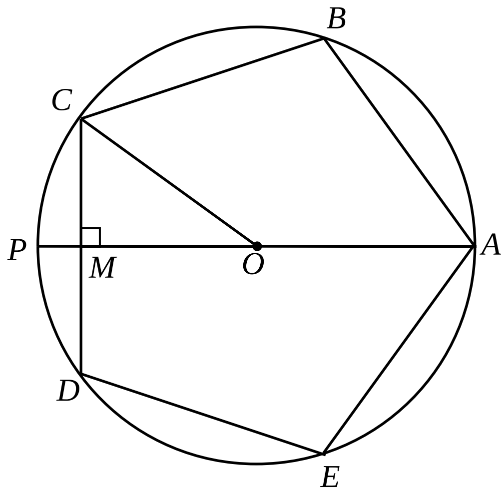
Let $\angle OCB = x$ and draw OB. Then $\angle OCB = x = \angle OCM = \angle OBC = \angle OBA = \angle OAB$. From the angles in quadrilateral $MCBA$, it follows that $\angle OCM = \dfrac{270°}{5} = 54°$.

Hence, $\angle COM = 36°$.



Alternate Solution

Reflection across the diameter PA produces a circle and the path of the light ray is a regular pentagon $ABCDE$. Hence $\angle DCB = 108°$, $\angle OCM = 54°$ and $\angle COM = 36°$.



5. Nineteen children, aged 1 to 19 respectively, are standing in a circle. The difference between the ages of each pair of adjacent children is recorded. What is the maximum value of the sum of these 19 positive integers?

Solution

The nineteen differences come from nineteen subtractions, involving each number exactly twice. To maximize their sum, the big numbers 11 to 19 should appear as the first number of subtraction while the small numbers 1 to 9 should appear as the second number of a subtraction. The number 10 appears once in each term. Thus the maximum value of the sum is

$$2(11+12+\cdots+19)-2(1+2+\cdots+9)+10-10=180.$$

(This can be attained by arranging the big and small numbers alternately, and putting number 10 into the circle anywhere you like.)

6. Simplify as a fraction in lowest terms

$$\frac{(2^4+2^2+1)(4^4+4^2+1)(6^4+6^2+1)(8^4+8^2+1)(10^4+10^2+1)}{(3^4+3^2+1)(5^4+5^2+1)(7^4+7^2+1)(9^4+9^2+1)(11^4+11^2+1)}.$$

Solution

Note that $x^4+x^2+1=(x^2+1)^2-x^2=(x^2+x+1)(x^2-x+1)$ and $(x+1)^2-(x+1)+1=x^2+x+1$. Hence the given expression is equal to

$$\frac{1^2+1+1}{3^2+3+1}\times\frac{3^2+3+1}{5^2+5+1}\times\frac{5^2+5+1}{7^2+7+1}\times\frac{7^2+7+1}{9^2+9+1}\times\frac{9^2+9+1}{11^2+11+1}=\frac{1^2+1+1}{11^2+11+1}=\frac{3}{133}$$

7. Given a quadrilateral *ABCD* not inscribed in a circle with *E*, *F*, *G* and *H* the circumcentres of triangles *ABD*, *ADC*, *BCD* and *ABC* respectively. If *I* is the intersection of *EG* and *FH*, and *AI* = 4 and *BI* = 3, find *CI*.

Solution

$FA = FC, HA = HC$ implies *FH* is the perpendicular bisector of *AC*. But *I* is on *FH*, thus *CI* = *AI* = 4.

8. To pass a certain test, 65 out of 100 is needed. The class average is 66. The average score of the students who pass the test is 71, and the average score of the students who fail the test is 56. It is decided to add 5 to every score, so that a few more students pass the test. Now the average score of the students who pass the test is 75, and the average score of the students who fail the test is 59. How many students are in this class, given that the number of students is between 15 and 30?

Solution

Let *n* be the number of students in this class. If *x* students pass the test before the scores are raised, we have $71x + 56(n - x) = 66n$, which simplifies to $x = \frac{2n}{3}$. Hence *n* is a multiple of 3. If *y* students pass the test after the scores are raised, we have $75y + 59(n - y) = 71n$, which simplifies to $y = \frac{3n}{4}$. Hence *n* is a multiple of 4 and hence a multiple of 12. Since 15 < *n* < 30, we must have *n* = 24.

9. How many right angled triangles are there, all the sides of which are integers, having 2009^{12} as one of its shorter sides?
Note that a triangle with sides *a*, *b*, *c* is the same as a triangle with sides *b*, *a*, c; where *c* is the hypotenuse.

Solution:

Let *a*, 2009^{12}, *c* be the sides, where *c* is the hypotenuse. Then $(c - a)(c + a) = 2009^{24} = 41^{24}7^{48}$. Now $c + a > c - a$ and both these numbers are odd, so each pair of factors leads a solution. Finally the number on the right has 25×49 different factors, one of which, namely, 2009^{12}, is invalid since it gives *a*=0. So there are $\frac{49 \times 25 - 1}{2} = 612$ right angled triangles having 2009^{12} as a shorter side.

10. Find the smallest six-digit number such that the sum of its digits is divisible by 26, and the sum of the digits of the next positive number is also divisible by 26.

Solution

Clearly, some carry-overs are involved. It takes at least three carry-overs to account for a difference of at least 26. Hence the last three digits are all 9s, and the sum of the first three digits must be 25. The smallest such number is 898999, because we must not have more than three 9s at the end.

11. On a circle, there are 2009 blue points and 1 red point. Jordan counts the number of convex polygons that can be drawn by joining only blue vertices. Kiril counts the number of convex polygons which must include the red point among its vertices. What is the difference between Jordan's number and Kiril's number?

Solution

For each polygon Jordan counts, Kiril counts the corresponding polygon with the red point as an additional vertex. This correspondence is reversible except when Kiril's polygon is a triangle, since the removal of the red vertex will leave a line segment. Hence Kiril's number exceeds Jordan's number by $\frac{2008\times 2009}{2}$, which is 2017036.

12. Musa sold drinks at a sports match. He sold bottles of spring water at R4 each, and bottles of cold drinks at R7 each. He started with a total of 350 bottles. Not all were sold and his total income was R2009. What was the minimum number of bottles of cold drink that Musa could have sold?

Solution

Let s and c be the number of bottles of spring water and the number of bottles of cold drink that sells, respectively.
Then

$$\begin{cases} 4s+7c=2009 \\ s+c<350 \end{cases}$$

It follows that

$$s=\frac{2009-7c}{4}=\frac{2008-8c+1+c}{4}=(502-2c)+\frac{1+c}{4},$$

where $\frac{1+c}{4}$ must be an integer, and

$$2009=4(s+c)+3c<4\times 350+3c.$$

Hence $c>203$, and $c_{\min}=207$.

Section B.

1. In a chess tournament, each of the 10 players plays each other player exactly once. After some games have been played, it is noticed that among any three players, there are at least two of them who have not yet played each other. What is the maximum number of games played so far?

Solution

Suppose there are 5 boys and 5 girls, and the only games played so far are between a boy and a girl. [5 points]
For any three players, there must either be at least two boys or at least two girls, and these two have not played each other. Hence as many as 25 games may have been played. [5 points]
To show that this is indeed the maximum, let k be the highest number of games

played by one player so far. No two of the k opponents may play each other, so that each has played at most $10-k$ games. The other $10-k$ players have played at most k games. [5 points]

Hence the total number of games played so far is at most

$$\frac{k(10-k)+(10-k)k}{2}=-k^2+10k=25-(k-5)^2\le 25.$$ [5 points]

2. P is a point inside triangle ABC such that *angle PBC* = 30°, *angle PBA* = 8° and *angle PAB* = *angle PAC* = 22°. Find *angle APC*, in degrees.

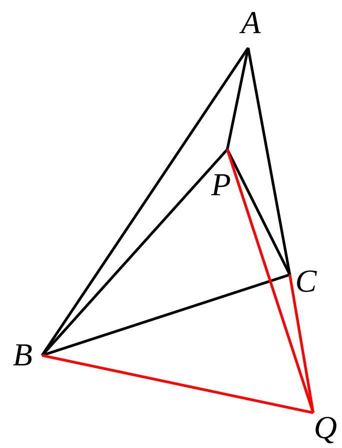
Solution

Extend AC to Q such that $AB=AQ$. Then triangle BAP and QAP are congruent. [5 points]

Since $\angle APB=180°-\angle PBA-\angle PAB=150°$, we have $\angle BPQ=360°-\angle APB-\angle APQ=60°$. Hence BPQ is an equilateral triangle. [5 points]

Now BC is the perpendicular bisector of PQ. [5 points]

It follows that $\angle CPQ=\angle CQP=\angle PBA=8°$ and $\angle APC=\angle APQ-\angle CPQ=142°$. [5 points]



3. Find the smallest positive integer which can be expressed as the sum of four positive squares, not necessarily different, and divides 2^n+15 for some positive integer n.

Solution

The smallest five integers that can be expressed as the sum of four positive squares are: 4=1+1+1+1, 7=4+1+1+1, 10=4+4+1+1, 12=9+1+1+1 and 13=4+4+4+1. [5 points]

Since 2^n+15 is an odd integer, this implies the smallest positive integers can not be 4, 10 and 12. [5 points]

When $n=3k$, $2^n+15=2^{3k}+15\equiv 1+15\equiv 2 \pmod 7$, where $k\ge 0$ and k is an integer ;

When $n=3k+1$, $2^n+15=2\cdot 2^{3k}+15\equiv 2+15\equiv 3 \pmod 7$;

When $n=3k+2$, $2^n+15=4\cdot 2^{3k}+15\equiv 4+15\equiv 5 \pmod 7$;

Therefore 2^n+15 is not a multiple of 7, [5 points]

but $2^7+15=143=11\times 13$, thus 13 is the smallest positive integer that satisfy the condition. [5 points]