International Mathematics Competition 2008
(IMC 2008)

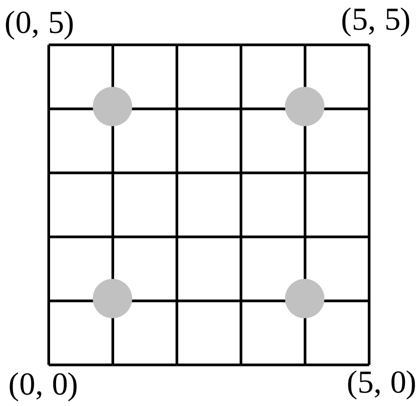
World Youth Mathematics Intercity Competition

Individual Contest Time limit: 120 minutes 2008/10/28

Section A.

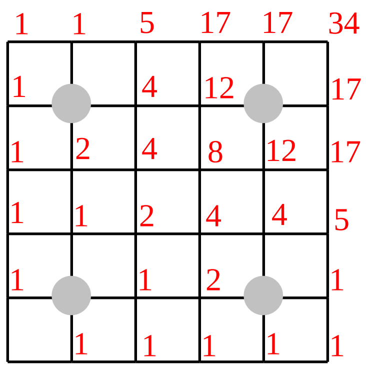
In this section, there are 12 questions, fill in the correct answers in the spaces provided at the end of each question. Each correct answer is worth 5 points.

1. Starting from the southwest corner (0,0) of a 5×5 net, an ant crawls along the lines towards the northeast corner (5,5). It can only go east or north, but cannot get pass the four broken intersections at (1,1), (1,4), (4,1) and (4,4). What is the total number of different paths?



Solution:

In the diagram below, we represent each intersection by a box, containing a number which indicates the number of ways this intersection can be reached. The answer, as indicated by the box at the northeast corner, is **34**.



Answer : **34**

2. The positive integer a - 2 is a divisor of $3a^2$ - $2a$+10. What is the sum of all possible values of a?

Solution:

Dividing $3a^2$ - $2a$+10 by a - 2, we obtain a quotient of $3a$+4 and a remainder of 18.

Then $a-2$ is a divisor of $3a^2$ - $2a$+10 if and only if it is a divisor of 18. Now the divisors of 18 are 1, 2, 3, 6, 9 and 18. The corresponding values of a are 3, 4, 5, 8, 11 and 20, and their sum is **51**.

Answer : **51**

3. Let a, b and c be real numbers such that $a + b + c = 11$ and $\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}=\frac{13}{17}$. What is the value of $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$?

Solution:

We have

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}=\frac{11-(b+c)}{b+c}+\frac{11-(c+a)}{c+a}+\frac{11-(a+b)}{a+b}$$

$$=11\left(\frac{1}{b+c}+\frac{1}{c+a}+\frac{1}{a+b}\right)-3$$

$$=\mathbf{\frac{92}{17}}$$

Answer : $\mathbf{\frac{92}{17}}$ or $5\frac{7}{17}$ or 5.4117

4. Let x be any real number. What is the maximum real value of $\sqrt{2008-x}+\sqrt{x-2000}$?

Solution:

We have $\left(\sqrt{2008-x}-\sqrt{x-2000}\right)^2=8+2\sqrt{(2008-x)(x-2000)}$. Now $2008-x$ and $x-2000$ are two positive numbers with constant sum 8. Hence the maximum value of their product occurs when they are equal. In other words, each is 4 and the maximum value of the product is 16. It follows that the maximum value of $\sqrt{2008-x}+\sqrt{x-2000}$ is $\sqrt{8+2\sqrt{16}}=4$.

Answer : **4**

5. How many ten-digit numbers are there in which every digit is either 2 or 3, and no two 3s are adjacent?

Solution #1:

Using the the given condition, we can deduce the number of digits in increasing order as follow: there are 2 different one-digit numbers (they are 2 or 3); there are 3 different two-digit numbers (they are 22, 23 or 32); there are 5 different three-digit numbers (they are 222, 223, 232, 322, 323); there are 8 different four-digit numbers (they are 2222, 2223, 2232, 2322, 2323, 3222, 3232, 3222), as we observe there is a pattern 2, 3, 5, 8, …, where this just follow the Fibonnaci Pattern, Hence, we have 13 different five-digit numbers; 21 different six-digit numbers; 34 different seven-digit numbers; 55 different eight-digit numbers; 89 different nine-digit numbers and 144 different ten-digit numbers.

Therefore, there are 144 ten-digit numbers satisfy the given condition in the problem.

Solution #2:

From the the given condition of this ten-digit number, we have the following cases:

(a) Suppose, none of the digit is 3, then there is only 1 ten-digit numbers satisfy the given condition.

(b) Suppose the digit 3 appear only once, then the remaining nine digits are 2 and we can insert the digit 3 in ten different places. C_1^{10} = 10 different ways.

(c) Suppose the digit 3 appear twice in the ten-digit number, then the remaining eight digits are 2 and we can intepret as insert the digit 3 in nine different places. C_2^{9} = 36 different ways.

(d) Suppose the digit 3 will appear three times in the ten-digit number, then we have C_3^{8} = 56 different ways.

(e) Suppose the digit 3 will appear four times in the ten-digit number, then we have C_4^{7} = 35 different ways.

(f) Suppose the digit 3 will appear five times in the ten-digit number, then we have C_5^{6} = 6 different ways.

(g) When the digit 3 will appear six times or more in the ten-digit number, then there the digit 3 will be in adjacent position.

Therefore, we have a total of 1 + 10 + 36 + 56 + 35 + 6 = **144** different ten-digit numbers in which every digit is either 2 or 3, and no two 3s are adjacent.

ANS: **144**

6. On a circle there are n (n> 3) integers with a total sum 94, such that each number is equal to the absolute value of the difference between the two numbers which follow it in clockwise order. What is the possible value of n?

Solution:

Among the n integers, there is one with the maximum value m. Since it is the absolute value of the difference between two numbers, one of these two numbers is also m and the other is 0. Therefore, the n numbers consist of several 3-cycles (m, m, 0), so that n=3k for some integer k. Now 2km=94 or km=47. Since 47 is prime, either k=1 and m=47 or k=47 and m=1. The sum of all possible values of n is therefore 3 × 47=**141**.

Answer：<u>**141**</u>

7. If the thousands digit of a four-digit perfect square is decreased by 3 and its units digit is increased by 3, the result is another four-digit perfect square. What is the original number?

Solution:

Let $A^2 = \overline{abcd}$, then

$$\begin{cases} A^2 = 1000a + 100b + 10c + d \\ B^2 = 1000(a-3) + 100b + 10c + (d+3) \end{cases}$$

We have $A^2 - B^2 = 2997$, hence $(A-B)(A+B) = 3^4 \times 37$.

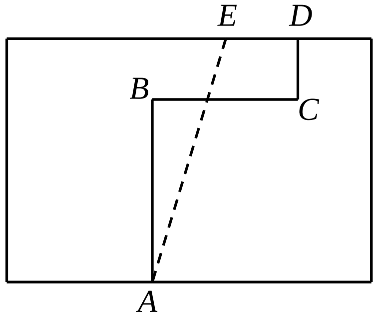
Since $A + B \le 2 \times 99 = 198$, hence

$$\begin{cases} A - B = 3^3, 37, \\ A + B = 3 \times 37, 3^4, \end{cases}$$

We get $A = 69, B = 42, M = 4761$, or $A = 59, B = 22, M = 3481$(not our answer).

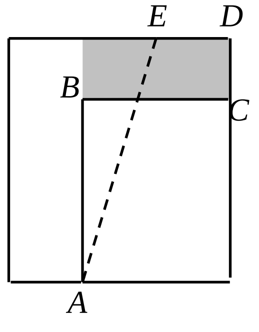
Answer：<u>**4761**</u>

8. Each segment of the broken line *A*-*B*-*C*-*D* is parallel to an edge of the rectangle, and it bisects the area of the rectangle. *E* is a point on the perimeter of the rectangle such that *AE* also bisects the area of the rectangle. If *AB*=30, *BC*=24 and *CD*=10, what is the length of *DE*?



Solution:

Move the two vertical edges of the rectangle inwards by an equal amount, until the right edge contains *CD*. The broken line *A*-*B*-*C* still bisects the area of the reduced rectangle. The area of the right half is 30×24=720, and so is the area of the left half. The shaded part of the left half has area 10×24=240, so that the area of the unshaded part of the left half is 720 - 240=480. This is a rectangle of height 30+10=40, so that its width is 12. Now *A* is at a distance of 12 from the right edge, and *AE* bisects the area of the reduced rectangle also. Hence the length of *DE*, which is the distance from *E* to the right edge, is also **12**.



Answer：_______**12**_______

9. Let $f(x) = ax^2 - c$, where *a* and *c* are real numbers satisfying $-4 \le f(1) \le -1$ and $-1 \le f(2) \le 2$. What is the maximum value of $f(8)$?

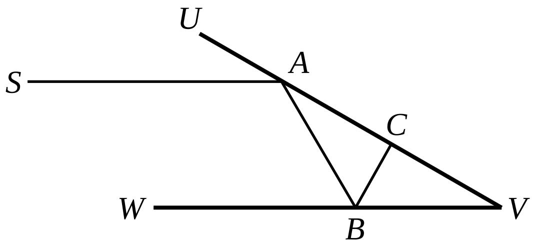
Solution:

From $f(1)=a-c$ and $f(2)=4a-c$, we have $a=\dfrac{f(2)-f(1)}{3}$ and $c=\dfrac{f(2)-4f(1)}{3}$. It follows that we have $f(8)=64a-c=21f(2)-20f(1)$.

The maximum value of $f(8)$ is $21\times(2)-20\times(-4)=122$

Answer：**122**

10. Two vertical mirrors facing each other form a 30° angle. A horizontal light beam from source *S* parallel to the mirror *WV* strikes the mirror *UV* at *A*, reflects to strike the mirror *WV* at *B*, and reflects to strike the mirror *UV* at *C*. After that, it goes back to *S*. If *SA*=*AV*=1, what is the total distance covered by the light beam?



Solution:

Let the mirrors be *UV* and *VW*, and let the light beam be parallel to *VW* initially. Then it strikes *UV* at some point *A*, reflects off to strike *VW* at *B*, and off to strike *UV* again at *C*, and so on. At *A*, the angle of incidence ∠*UAS*=∠*UVW*=30°. Hence the angle of reflection ∠*BAV*=30°. At *B*, the angle of incidence ∠*ABW*=∠*BAV*+∠*AVB*=60°. Hence the angle of reflection ∠*CBV*=60°. At *C*, the angle of incidence ∠*BCV*=180° - ∠*UVW* - ∠*CBV*=90°. It follows that the light beam will retrace its path back through *B*, *A* to *S*. Since *AV*=1, $AC=\dfrac{1}{2}$. Hence $AB=\dfrac{\sqrt{3}}{3}$ and $BC=\dfrac{\sqrt{3}}{6}$. The total length of the path is 2(*SA*+*AB*+*BC*)=$2+\sqrt{3}$.

Answer：________ or **3.73205**

11. Let *n* be a positive integer such that n^2-n+11 is the product of four prime numbers, some of which may be the same. What is the minimum value of *n*?

Solution:

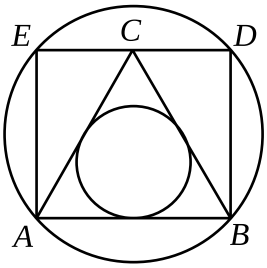
The chart below shows that n^2-n+11 is not divisible by any of 2, 3, 5 and 7. The smallest number with four prime divisors which are divisible by these numbers is 11^4. From $n^2-n+11=11^4$, we have $n(n-1)=2\times5\times7\times11\times19$.

This does yield any integral value for n. The next smallest number is 13×11^3. From $n^2-n+11=13\times 11^3$, we have $n(n-1)=2\times 2\times 3\times 11\times 131$. This yields the minimal integral value n=**132**.

	mod 2		mod 3			mod 5					mod 7						
n	0	1	0	1	2	0	1	2	3	4	0	1	2	3	4	5	6
n^2	0	1	0	1	1	0	1	4	4	1	0	1	4	2	2	4	1
n^2-n+11	1	1	2	2	1	1	1	3	2	3	4	4	6	3	2	3	6

Answer：**132**

12. ABC is an equilateral triangle, and $ABDE$ is a rectangle with DE passing through C. If the circle touching all three sides of ABC has radius 1, what is the diameter of the circle passing through A, B, D and E?



Solution:

In an equilateral triangle, the incentre coincides with the centroid. Hence its altitude is three times its inradius. It follows that AE=3. Since AC=2CE, $CE=\sqrt{3}$ and $DE=2\sqrt{3}$. Finally, $AD^2=AE^2+DE^2=21$, so that the diameter of the circle passing through A, B, D and E is $\sqrt{\mathbf{21}}$.

Answer：$\sqrt{\mathbf{21}}$ or **4.5825**

Section B.

Answer the following 3 questions, and show your detailed solution in the space provided after each question. Each question is worth 20 points.

1. In the expression $\left[\sqrt{2008+\sqrt{2008+\sqrt{2008+\ldots+\sqrt{2008}}}}\right]$, the number 2008 appears 2008 times, and $[x]$ stands for the greatest integer not exceeding x. What is the value of this expression?

Solution:

Let $a_1=\sqrt{2008}$ and for any positive integer n, let $a_{n+1}=\sqrt{2008+a_n}$. Then our expression is a_{2008}. We have $44^2=1936<2008<2025=45^2$, so that $44<a_1<45$.**(5pt)** We claim that $45<a_n<46$ for all $n\geq 2$. Since

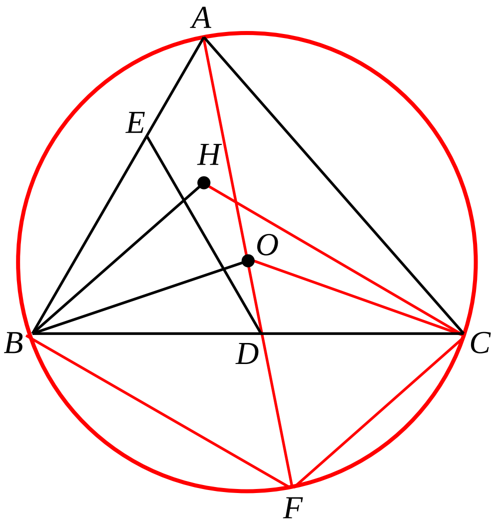
$45^2 = 2025 < 2052 = 2008 + 44 < 2008 + a_1 < 2008 + 45 = 2053 \le 2116 = 46^2$, the claim holds for n=2. **(10pt)**

Suppose its holds for some $n \ge 2$. Then $45 < \sqrt{2053} < a_{n+1} < \sqrt{2054} < 46$. This justifies the claim. In particular, $[a_{2008}] = \mathbf{45}$.**(5pt)**

2. In the triangle *ABC*, ∠*ABC*=60°. *O* is its circumcentre and *H* is its orthocentre. *D* is a point on *BC* such that *BD*=*BH*. *E* is a point on *AB* such that *BE*=*BO*. If *BO*=1, what is the area of the triangle *BDE*? (The orthocenter is the intersection of the lines from each vertex of the triangle making a perpendicular with its opposite sides. The circumcenter is the center of the circle passing through each vertex of the triangle.)

Solution:

Let *AF* be a diameter of the circumcircle. Then ∠*AFC*=∠*ABC*=60°. Since *OF*=*OC*, *COF* is an equilateral triangle. **(5pt)** Moreover, *CF* is perpendicular to *AC*, and therefore parallel to *BH*. Similarly, BF is parallel to *CH*, so that *BFCH* is a parallelogram. **(5pt)** It follows that *BD*=*BH*=*CF*=*CO*=*BO*=*BE*, so that *BED* is also an equilateral triangle. **(5pt)** Since *BO*=1, its area is $\frac{\sqrt{3}}{4}$ (or **0.433**).**(5pt)**



3. Let *t* be a positive integer such that $2^t = a^b \pm 1$ for some integers *a* and *b*, each greater than 1. What are all the possible values of *t*?

Solution:

Clearly, *a* is odd. Suppose *b* is also odd. **(5pt)** Then $2^t = (a \pm 1)(a^{b-1} \mp a^{b-2} + a^{b-3} \mp \cdots \mp a + 1)$. The second factor is an odd number, so that it must be 1. However, this means that $a^b \pm 1 = a \pm 1$, but this contradicts $b \ge 2$.**(5pt)** Hence *b*=2*m* for some positive integer *m*. Then $a^{2m} \equiv 1 \pmod 4$. If $2^t = a^b + 1$, then $2^t = a^{2m} + 1 \equiv 2 \pmod 4$. Hence *t*=1, but this contradicts $a \ge 2$.**(5pt)** It follows that we must have $2^t = a^b - 1 = (a^m + 1)(a^m - 1)$. The only two consecutive even numbers both of which are powers of 2 are 2 and 4. Hence *a*=3, *b*=2 and the only possible value for *t* is **3**. **(5pt)**