FRACTIONAL POISSON PROCESS IN TERMS OF ALPHA-STABLE DENSITIES

by

DEXTER ODCHIGUE CAHOY

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Dissertation Advisor: Dr. Wojbor A. Woyczynski

Department of Statistics
CASE WESTERN RESERVE UNIVERSITY

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CASE WESTERN RESERVE UNIVERSITY

SCHOOL OF GRADUATE STUDIES

We hereby approve the dissertation of

DEXTER ODCHIGUE CAHOY

candidate for the Doctor of Philosophy degree *

Committee Chair: ______________________________
Dr. Wojbor A. Woyczynski
Dissertation Advisor
Professor
Department of Statistics

Committee: ______________________________
Dr. Joe M. Sedransk
Professor
Department of Statistics

Committee: ______________________________
Dr. David Gurarie
Professor
Department of Mathematics

Committee: ______________________________
Dr. Matthew J. Sobel
Professor
Department of Operations

August 2007

*We also certify that written approval has been obtained for any proprietary material contained therein.

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Fractional Poisson Process in Terms of Alpha-Stable Densities

Abstract

by

Dexter Odchigue Cahoy

The link between fractional Poisson process (fPp) and α-stable density is established by solving an integral equation. The result is then used to study the properties of fPp such as asymptotical n-th arrival time, number of events distributions, covariance structure, stationarity and dependence of increments, self-similarity, and intermittency property. Asymptotically normal parameter estimators and their variants are derived; their properties are studied and compared using synthetic data. An alternative fPp model is also proposed.Finally, the asymptotic distribution of a scaled fPp random variable is shown to be free of some parameters; formulae for integer-order, non-central moments are also derived.

Keywords: fractional Poisson process, α-stable, intermittency, scaled fPp, self-similarity

Chapter 1

Motivation and Introduction

1.1 Motivation

For almost two centuries, Poisson process served as the simplest, and yet one of the most important stochastic models. Its main properties, namely, absence of memory and jump-shaped increments model a large number of processes in several scientific fields such as epidemiology, industry, biology, queueing theory, traffic flow, and commerce (see *Haight* (1967, chap. 7)). On the other hand, there are many processes that exhibit long memory (e.g., network traffic and other complex systems) as well. It would be useful if one could generalize the standard Poisson process to include systems or processes that don't have rapid memory loss in the long run. It is largely this appealing feature that drives this thesis to investigate further the statistical properties of a particular generalization of a Poisson process called fractional Poisson process (fPp).

Moreover, the generalization has some parameters that need to be estimated in order for the model to be applicable to a wide variety of interesting counting phenomena. This problem also motivates us to find "good" parameter estimators for would-be end users.

We begin by summarizing the properties of Poisson distribution, Poisson process and $\alpha-$stable distribution.

1.2 Poisson Distribution

The distribution is due to Simeon Denis Poisson (1781-1840). The characteristic and probability mass functions of a Poisson distribution are

$$\phi(k) = \exp[\mu(e^{ik} - 1)] \quad \text{and} \quad \mathsf{P}\{X = n\} = \frac{\mu^n}{n!}e^{-\mu}, \qquad n = 0, 1, 2, \dots .$$

Some of the properties are:

(1) $\mathsf{E}X = \mu,\ \mathrm{var}X = \mu.$

(2) The factorial moment of the nth order:

$$\mathsf{E}X(X-1)\dots(X-n+1) = \mu^n.$$

(3) Sum of independent Poisson random variables $X_1, X_2, ..., X_m$, with means $\mu_1, \mu_2, ..., \mu_m$, is a Poisson random variable X, with a mean $\mu = \mu_1 + \mu_2 + ... + \mu_m$. When $m = 2$,

$$\begin{aligned}
\mathsf{P}(X = n) &= \mathsf{P}(X_1 + X_2 = n) \\
&= \sum_{j=0}^{n} \mathsf{P}(X_1 + X_2 = n | X_1 = j)\mathsf{P}(X_1 = j) \\
&= \sum_{j=0}^{n} \mathsf{P}(X_2 = n - j)\mathsf{P}(X_1 = j) \\
&= \sum_{j=0}^{n} \frac{\mu_2^{n-j}}{(n-j)!}e^{-\mu_2}\frac{\mu_1^j}{j!}e^{-\mu_1} \\
&= \frac{1}{n!}\left(\sum_{j=0}^{n} \frac{n!}{j!(n-j)!}\mu_1^j\mu_2^{n-j}\right)e^{-(\mu_1+\mu_2)} = \frac{(\mu_1+\mu_2)^n}{n!}e^{-(\mu_1+\mu_2)}.
\end{aligned}$$

(4) Let $\{X_j\}, \quad j = 1, 2, \dots, m$, be independent and have Poisson distribution with means $\{\mu_j\}, \quad j = 1, 2, \dots, m$. If $n_1 + n_2 + \dots + n_m = s$ then

$$\mathsf{P}\left(X_1 = n_1, X_2 = n_2, \dots, X_m = n_m \middle| S = s\right) = \mathsf{M}\left(s; p_1, \dots, p_m)\right)$$

where $p_j = \mu_j / \sum_{j=1}^{m} \mu_j$, and M stands for the multinomial distribution. When $m = 2$, the conditional distribution of X_1 given $X_1 + X_2 = n$, is $\mathsf{B}(n, \mu_1/(\mu_1 + \mu_2))$, where B denotes the binomial distribution.

(5) Let $X_j, j = 1, 2, 3, ...$, be independent random variables taking the values 0 and 1 with probability $q = 1 - p$, and p, respectively. If M is a Poisson random variable with mean μ, independent of $\{X_j\}$, then

$$S = X_1 + X_2 + \cdots + X_M$$

is a Poisson random variable with mean $p\mu$. Additionally,

$$\begin{aligned}
\mathsf{P}(X_1 + X_2 + \cdots + X_M = n) &= \sum_{m=n}^{\infty} \mathsf{P}(X_1 + X_2 + \cdots + X_M = n | M = m)\mathsf{P}(M = m) \\
&= \sum_{m=n}^{\infty} \binom{m}{n} p^n q^{m-n} \frac{\mu^m}{m!} e^{-\mu} \\
&= \frac{(p\mu)^n}{n!} \sum_{m=n}^{\infty} \frac{(q\mu)^{m-n}}{(m-n)!} e^{-\mu} \\
&= \frac{(p\mu)^n}{n!} e^{-p\mu}.
\end{aligned}$$

Note that the conditional probability

$$\mathsf{P}(X_1 + X_2 + \cdots + X_M = n | M = m) = \binom{m}{n} p^n q^{m-n},$$

i.e., B(m;p) (see *Feller* (1950, chap. 6)).

(6) Suppose we play heads and tails for a large number of turns m with a coin such that $\mathsf{P}(X_j = 1) = \theta/m$. The number of tails S_m you observe is distributed according to the *binomial* distribution with sample size n and parameter $\theta \in (0, 1)$:

$$p_m(n) \equiv \mathsf{P}(S_m = n) = \binom{m}{n} \left(\frac{\theta}{m}\right)^n \left(1 - \frac{\theta}{m}\right)^{m-n}.$$

The limiting distribution as $m \to \infty$ when θ is constant can be easily derived as follows: If $n = 0$, we have

$$p_\infty(0) = \lim_{m\to\infty} p_m(0) = \lim_{m\to\infty} \left(1 - \frac{\theta}{m}\right)^m = e^{-\theta}.$$

Also,

$$\frac{p_m(n+1)}{p_m(n)} = \frac{\frac{m-n}{n+1}\frac{\theta}{m}}{1-\frac{\theta}{m}} \to \frac{\theta}{n+1}, \ m \to \infty.$$

Therefore, for all $n \geq 0$,

$$p_\infty(n) = \frac{\theta^n}{n!}e^{-\theta}.$$

This phenomenon is called the *Poisson law of rare events*, because, as $m \to \infty$, tail events are becoming rare with probability θ/m.

1.3 Poisson Process

Recall that a continuous-time stochastic process $\{N(t), t \geq 0\}$ is said to be a *counting process* if it satisfies:

(a) $N(t) \geq 0$,

(b) $N(t)$ is integer-valued, and

(c) $N(t_1) \leq N(t_2)$, if $t_1 < t_2$.

Usually N is associated with the number of random events in the interval $(t_1, t_2]$.

The random times $T_j : 0 < T_1 < T_2 < T_3 < \cdots < T_n < \ldots$ at which the function $N(t)$ changes its value are called the *arrival or event times*. Thus, $T_{N(t)}$ denotes the arrival time of the last event before t, while $T_{N(t)+1}$ is the first arrival time after t. Alternatively, $N(t)$ can be determined as the largest value of n for which the nth event occurs before or at time t:

$$N(t) = \max\{n : T_n \leq t\}.$$

The time from fixed t since the last event

$$A(t) = t - T_{N(t)}$$

is called the *age* at t, and the time from t until the next event

$$R(t) = T_{N(t)+1} - t$$

is called the *residual life*, or *excess*, at time t. The random variables

$$\Delta T_j \equiv \begin{cases} T_1, & j = 1; \\ T_j - T_{j-1}, & j > 1 \end{cases}$$

are called *interarrival* or *waiting times*.

There exists an important relationship between $N(t)$ and T_j: the number of events by time t is greater than or equal to n if, and only if, the nth event occurs before or at time T:

$$N(t) \geq n \Longleftrightarrow T_n \leq t.$$

A counting process $\{N(t), t \geq 0\}$ is said to be a *Poisson process* if:

(a) $N(0) = 0$, and

(b) for every $0 \leq s < t < \infty$, and $h > 0$, $N(t) - N(s)$ is a Poisson random variable with mean $\mu(t-s)$, i.e.,

$$\begin{aligned} \mathsf{P}\left(N(t) - N(s) = n\right) &= \mathsf{P}\left(N(t+h) - N(s+h) = n\right) \\ &= e^{-\mu(t-s)} \frac{\left(\mu(t-s)\right)^n}{n!}, \qquad n = 0, 1, 2, \ldots, \end{aligned}$$

and, for every $t_0, t_1, \ldots, t_l,\ 0 \leq t_0 < t_1 < \ldots < t_l < \infty$, the increments

$$\{N(t_0); N(t_k) - N(t_{k-1}), \quad k = 1, \ldots, l\}$$

form a set of independent random variables. Under the above conditions, the waiting times ΔT_j have an exponential distribution: $\mathsf{P}(\Delta T_j > t) = e^{-\mu t}, \mu > 0$. The positive constant μ is called the *rate* or the *intensity* of the Poisson process. An equivalent definition can be found in *Feller* (1950) and *Ross* (1996).

What follows summarizes some important properties of a Poisson process.

(1) The Poisson process has stationary and independent increments. It follows that the Poisson distribution belongs to the class of infinitely divisible distributions (see (*Feller*, 1966, pp. 173-179)).

(2) The probability distribution of the n-th arrival time is given by the Erlang density (n-fold convolution of the exponential density $f(t) = \mu \exp(-\mu t),\ t \geq 0$),

$$f_n(t) = f^{n\star}(t) = \frac{(\mu t)^{n-1}}{(n-1)!} \mu e^{-\mu t}, \qquad t \geq 0,$$

with mean $\mathsf{E}T_n = n\mu_0$, and variance $\mathrm{Var}T_n = n\mu_0^2$, where $\mu_0 = 1/\mu$.

(3) The probability distribution of the number of events $N(t)$ which occurred up to time t is given by Poisson's law

$$P(N(t) = n) = \frac{(\mu t)^n}{n!} e^{-\mu t}, \qquad n = 0, 1, 2, \ldots,$$

with mean and variance $\mathsf{E}N(t) = \mathrm{Var}N(t) = \mu t = t/\mu_0$.

(4) The finite-dimensional probability distribution of the Poisson process is given by the formula

$$\begin{aligned} &P(N(t_1) = n_1, N(t_2) = n_2, \ldots, N(t_k) = n_k) \\ &= P(N(t_1) = n_1, N(t_2) - N(t_1) = n_2 - n_1, \ldots, N(t_k) - N(t_{k-1}) = n_k - n_{k_1}) \\ &= P(N(t_1) = n_1) P(N(t_2) - N(t_1) = n_2 - n_1) \ldots P(N(t_k) - N(t_{k-1}) = n_k - n_{k_1}) \\ &= P(N(t_1) = n_1) P(N(t_2 - t_1) = n_2 - n_1) \ldots P(N(t_k - t_{k-1}) = n_k - n_{k_1}) \\ &= \frac{t_1^{n_1}(t_2 - t_1)^{n_2 - n_1} \ldots \mu(t_k - t_{k-1})^{n_k - n_{k-1}}}{n_1!(n_2 - n_1)! \ldots (n_k - n_{k-1})!} \mu^{n_k} e^{-\mu t_k}, \\ &0 < t_1 < t_2 < \cdots < t_k,\ 0 \leq n_1 \leq n_2 \leq \cdots \leq n_k. \end{aligned}$$

(5) The conditional probability distributions of the number of events are given by

$$P(N(t_2) = n_2 | N(t_1) = n_1) = \frac{[\mu(t_2 - t_1)]^{n_2 - n_1}}{(n_2 - n_1)!} e^{-\mu(t_2 - t_1)}, \qquad n_2 = 0, 1, 2, \ldots,$$

and

$$P(N(t_1) = n_1 | N(t_2) = n_2) = \binom{n_2}{n_1} \left(\frac{t_1}{t_2}\right) \left(1 - \frac{t_1}{t_2}\right)^{n_2 - n_1}, \qquad n_1 = 0, 1, \ldots, n_2,$$

where $t_1 < t_2$.

(6) If $t_1 < t_2$ then the covariance function is given by

$$\begin{aligned} \mathrm{Cov}(N(t_1), N(t_2)) &= \mathsf{E}N(t_1)N(t_2) - \mathsf{E}N(t_1)\mathsf{E}N(t_2) \\ &= \mathsf{E}\{N(t_1)[N(t_1) + N(t_2) - N(t_1)]\} - \mathsf{E}N(t_1)\mathsf{E}N(t_2) \\ &= \mathsf{E}N^2(t_1) + \mathsf{E}\{N(t_1)[N(t_2) - N(t_1)]\} - \mathsf{E}N(t_1)\mathsf{E}N(t_2) \\ &= \mu t_1 + (\mu t_1)^2 + (\mu t_1)(\mu(t_2 - t_1)) - \mu t_1(\mu t_2) \\ &= \mu t_1. \end{aligned}$$

In the general case,

$$\mathrm{Cov}(N(t_1), N(t_2)) = \mu \min\{t_1, t_2\} = [\mu/2](t_1 + t_2 - |t_1 - t_2|).$$

(7) The above conditional distribution of arrival times $P(T_1 \in dt_1, T_2 \in dt_2, \dots, T_n \in dt_n | N(t) = n)$ is uniform in the simplex $0 < t_1 < t_2 < \cdots < t_n < t$:

$$P(T_1 \in dt_1, T_2 \in dt_2, \dots, T_n \in dt_n | N(t) = n) = \frac{n!}{t^n} dt_1 \dots dt_n,$$

$$0 < t_1 < t_2 < \cdots < t_n < t.$$

Taking into account the corresponding theorem from the theory of order statistics, we can say, that under the condition that n events have occurred in $(0, t)$, the unordered random times $T_j,\ j = 1, 2, \dots, n$ at which events occur, are distributed independently and uniformly in the interval $(0, t)$.

(9) Lack of memory is an intrinsic property of the exponential distribution. If X is a random variable with the exponential distribution,

$$P(T > t) = e^{-\mu t}$$

then

$$P(T > t + \Delta t | T > t) = \frac{P(T > t, T > t + \Delta t)}{P(T > t)} = e^{-\mu \Delta t}.$$

(10) Suppose that $\{N_i(t), t \geq 0\},\ i = 1, 2, \dots, m$ are independent Poisson processes with rates $\mu_1, \mu_2, \dots, \mu_m$, then $\{\sum_{i=1}^m N_i(t), t \geq 0\}$ is a Poisson process with rate $\mu_1 + \mu_2 + \cdots + \mu_m$.

(11) The probabilities $P_n(t) \equiv P(N(t) = n)$ obey the system of differential equations

$$\frac{dP_0(t)}{dt} = -\mu P_0(t), \qquad p_0(0) = 1;$$

$$\frac{dP_n(t)}{dt} = -\mu P_n(t) + \mu P_{n-1}(t), \qquad P_n(0) = \delta_{n0},\ n = 1, 2, 3, \dots.$$

(12) The characteristic function $\widetilde{P}(k, t) = \mathsf{E} e^{ikN(t)}$ of the Poisson process has the form

$$\widetilde{P}(k, t) = \exp[-\mu t(1 - e^{ik})], \qquad -\infty < k < \infty,$$

and obeys the differential equation

$$\frac{d\widetilde{P}(k,t)}{dt} = -\mu(1-e^{ik})\widetilde{P}(k,t), \qquad \widetilde{P}(k,0) = 1.$$

More properties of the Poisson process can be found in *Haight* (1967), *Kingman* (1993), *Ross* (1996). In addition, *Karr* (1991) and *Grandell* (1997) provide generalizations and modifications of the Poisson process.

1.4 α-stable Distribution

The distribution has gained popularity since the 1960's when Mandelbrot used stable laws in modeling economic phenomena. *Zolotarev* (1986) has three representations of an $\alpha-$stable distribution in terms of characteristic functions. In this section, we base our definitions from *Gnedenko and Kolmogorov* (1968), *Samorodnitsky and Taqqu* (1994), and *Uchaikin and Zolotarev* (1999).

Definition 1 *The common distribution F_X of independent random variables $X_1, X_2, \ldots, X_n$ belongs to the* domain of attraction *of the distribution F if there exist normalizing sequences, $a_n, b_n > 0$, such that*

$$b_n^{-1}\left(\sum_{i=1}^{n} X_i - a_n\right) \overset{n\to\infty}{\longrightarrow} X$$

in distribution.

The non-degenerate limit laws X are called *stable* laws. The above definition can also be restated as follows: The probability distribution F has a *domain of attraction* if and only if it is stable. Furthermore, the distribution of X_i is then said to be in the *domain of attraction* of the distribution of X.

If $a_n = 0$ then X is said to have a strictly stable distribution.

A characteristic function representation of stable distributions that serves as an equivalent but is a more rigorous definition is given below.

Definition 2 *A random variable X is said to have a stable distribution if there exist parameters $\alpha, \beta, \gamma, 0 < \alpha \leq 2, -1 \leq \beta \leq 1, \gamma > 0$, and $\eta \in \mathbb{R}$, such that the* *characteristic function has the following form:*

$$\phi(t) = \begin{cases} \exp\Big\{ -\gamma^\alpha |t|^\alpha \left(1 - i\beta \,(\text{sign } t) \tan \frac{\pi\alpha}{2}\right) + it\eta \Big\}, & \alpha \neq 1, \\ \\ \exp\Big\{ -\gamma |t| \left(1 + i\beta \frac{2}{\pi} \,(\text{sign } t) \ln |t|\right) + it\eta \Big\}, & \alpha = 1. \end{cases}$$

The parameter α is called the stability index (tail parameter), and

$$\text{sign } t \overset{df}{=} \begin{cases} 1, & t > 0, \\ 0, & t = 0, \\ -1, & t < 0. \end{cases}$$

Adopting a notation from *Samorodnitsky and Taqqu* (1994), we write $X \sim S_\alpha(\gamma, \beta, \eta)$ to say that X has a stable distribution $S_\alpha(\gamma, \beta, \eta)$, where γ is the scale parameter, β is the skewness parameter, and η is the location parameter. If $X \sim S_\alpha(\gamma, 0, 0)$ then we have a symmetric α-stable distribution. When $X \sim S_\alpha(\gamma, \beta, 0)$, with $\alpha \neq 1$, we get strictly stable distributions. For our current purposes, we are interested only in the one-sided α-stable distributions with location parameter η equal to zero, scale parameter γ equal to one, skewness parameter β equal to one, and the stability index $0 < \alpha < 1$. We denote this class of distributions as $g^{(\alpha)}(x)$, and simply refer to it as α^+-stable distributions. Please note that α-stable densities have heavy, Pareto-type tails, i.e.,

$$P(|X| > x) \sim \textit{constant} \cdot x^{-\alpha}.$$

Moreover, it is well-known that the probability densities of α-stable random variables can be expressed in terms of elementary functions only in the following cases:

(1) *Gaussian* distribution $S_2(\gamma, \beta = 0, \eta) = N(\mu, 2\gamma^2)$ with probability density function

$$f(x) = \left(2\gamma\sqrt{\pi}\right)^{-1} e^{-(x-\eta)^2/4\gamma^2},$$

(2) *Lévy* distribution $S_{1/2}(\gamma, 1, \eta)$, whose probability density function is

$$f(x) = \left(\frac{\gamma}{2\pi}\right)^2 \frac{e^{-\gamma/2(x-\eta)}}{(x-\eta)^{3/2}},$$

(3) and *Cauchy* distribution $S_1(\gamma, 0, \eta)$ with density function

$$f(x) = \frac{\gamma}{\pi\left((x-\eta)^2+\gamma^2\right)}.$$

Woyczynski (2001) studies α-stable distributions and processes as natural models in many physical, biological, and economical phenomena. For other applications of α-stable distributions, please see *Willinger et al.* (1998) and *Crovella et al.* (1998). Multidimensional (and even ∞-dimensional) α-stable distributions were also studied in *Marcus and Woyczynski* (1979).

1.4.1 Parameter Estimation

Numerous methods for estimating the stable index in various settings exist in the available literature. For instance, *Piryatinska* (2005) and *Piryatinska et al.* (2006) provide estimators of the stable index in tempered-alpha stable distributions. It was *Fama and Roll* (1968, 1971) who constructed some of the first estimators for symmetric stable distributions. Other estimators then followed based on different criteria. In this subsection, we briefly review a few popular consistent estimators of the stable index α.

Press Estimator

Press (1972) constructed a method-of-moments estimator based on a characteristic function. Given symmetric $X_1, X_2, \ldots, X_n$, the empirical characteristic function can be defined as

$$\widehat{\phi}(t) = \frac{1}{n}\sum_{j=1}^{n}\exp\left(itX_j\right),$$

for every given t. With the preceding representation of the characteristic function, and, for all $0 < \alpha < 2$,

$$|\phi(t)| = \exp\left(-\kappa|t|^{\alpha}\right),$$

where $\kappa = \gamma^{\alpha}$ and is sometimes called the scale parameter. Given two nonzero values of t, t_1 and t_2 say, such that $t_1 \neq t_2$, we can get two equations:

$$\kappa|t_1|^{\alpha} = -\ln(|\phi(t_1)|),$$

and

$$\kappa|t_2|^\alpha = -\ln(|\phi(t_2)|).$$

Assume $\alpha \neq 1$. Replacing $\phi(t)$ by its estimated values $\widehat{\phi}(t)$ and solving these two equations simultaneously for κ and α, gives the asymptotically normal estimator

$$\widehat{\alpha}_P = \frac{\ln\left(\left|\frac{\ln(|\widehat{\phi}(t_1)|)}{\ln(|\widehat{\phi}(t_2)|)}\right|\right)}{\ln\left(\left|t_1/t_2\right|\right)}.$$

Zolotarev's Estimator

Zolotarev (1986) derived a method-of-moments estimator using the transformed random variables. Let $X_1, X_2, \ldots, X_n$ be independently and identically distributed (IID) according to a strictly stable distribution. In addition, define $U_j = \text{sign}(X_j)$, and $V_i = \ln(|X_j|),\ j = 1, 2, \ldots, n$. With the equality

$$\frac{1}{\alpha^2} = \frac{6}{\pi^2}\text{var}(V) - \frac{3}{2}\text{var}(U) + 1,$$

we can obtain an estimator of $1/\alpha^2$,

$$\widehat{1/\alpha^2} = \frac{6}{\pi^2}S_V^2 - \frac{3}{2}S_U^2 + 1, \tag{1.1}$$

where S_U^2 and S_V^2 are the unbiased sample variances of $U_1, U_2, \ldots, U_n$, and $V_1, V_2, \ldots, V_n$, respectively. Equation (1.1) gives the unbiased and consistent estimator $\widehat{\alpha^2}$.

Quadratic Estimators

The following consistent and asymptotically unbiased estimator of $1/\alpha = \gamma$ was proposed in *Meerschaert and Scheffler* (1998):

$$\widehat{\gamma}\left([X_j]_{j=1,\ldots,n}\right) = \frac{\ln_+ \sum_{j=1}^n \left(X_j - \overline{X}\right)^2}{2\ln n},$$

where $\ln_+(x) = max\{\ln(x), 0\}$. The above estimator is shift-invariant but not scale-invariant. A scale-invariant correction is introduced in *Bianchi and Meerschaert*

(2000), and their corresponding estimator of γ is

$$\widehat{\phi} = \widehat{\gamma}\left([X_{0,j}]_{j=1,\ldots,n}\right),$$

where $[X_{0,j}] = [(X_j - \widetilde{M}_n])/M_n$, $\widetilde{M}_n$ is the sample median of $[X_j]$, and M_n is the sample median of $[|X_j - \widetilde{M}_n|]$. They have further shown that

(1) $\widehat{\phi}_n \xrightarrow{p} 1/\alpha$, as $n \to \infty$, and that

(2) there exist some $\tilde{c}_n$, $n = 1, 2, \ldots$, and an $\alpha/2$ stable random variable Y, with $\mathsf{E}[\ln Y] = 0$ such that, as $n \to \infty$,

$$2 \ln n \left(\widehat{\phi}_n - 1/\alpha - \tilde{c}_n\right) \xrightarrow{d} \ln Y.$$

Sampling-Based Estimators

Fan (2004) introduced an estimator using a U-statistic. Recall the following property of strictly stable random variables:

$$\frac{X_1 + X_2 + \cdots + X_n}{n^{1/\alpha}} \xrightarrow{d} X_1.$$

This implies that

$$\ln n \left(\frac{\ln |\sum_{j=1}^{n} X_j|}{\ln n} - \frac{1}{\alpha}\right) \xrightarrow{d} \ln |X_1|.$$

A natural estimator for α would then be

$$\widehat{\alpha} = \frac{\ln n}{\ln |\sum_{j=1}^{n} X_j|}.$$

With the kernel

$$k(x_1, x_2, \ldots, x_m) = \frac{\ln |\sum_{j=1}^{m} X_j|}{\ln m}, \quad m \leq n,$$

of a U-statistic

$$\widehat{\alpha_F^{-1}} = U_m(k) = \begin{pmatrix} n \\ m \end{pmatrix}^{-1} \sum_{1 \leq j_1 \leq j_2, \ldots, j_m} k\left(X_{j_1}, X_{j_2}, \ldots, X_{j_m}\right), \tag{1.2}$$

he further proved that for IID strictly stable random variables, $X_1, X_2, \ldots, X_n$,

$$\sqrt{n}S_n^{-1}\left(\widehat{\alpha_F^{-1}} - \alpha^{-1}\right) \xrightarrow{d} N(0,\ 1),\ \text{as}\ n \to \infty,\ m = o(n^{1/2}),$$

where

$$S_n^2 = \frac{\sum\limits_{j=1}^{n}\left(U_{n-1}^{(-j)} = \overline{U_{n-1}^{(-j)}}\right)^2}{n-1} \xrightarrow{p} m^2\zeta_1,$$

$U_{n-1}^{(-j)}$ is the jacknifed U-statistic (1.2), and

$$\zeta_1 = \text{var}\left\{\mathsf{E}\left[k(x_1, x_2, \ldots, x_m)\middle|x_1\right] - \mathsf{E}\left[k(x_1, x_2, \ldots, x_m)\right]\right\}.$$

He also modified his method by dividing the data (with $n = lm$ observations) into l independent sub-samples, each having m observations. For every sub-sample, we get an estimator $\widehat{\alpha_F^{-1}}_j$, and then the average

$$\widehat{\alpha_{FI}^{-1}} = \frac{\sum\limits_{j=1}^{m}\widehat{\alpha_F^{-1}}_j}{m}$$

is what he called the *incomplete* U-statistic estimator. He even proposed another estimator based on a randomized resampling procedure which gives the estimator

$$\widehat{\alpha_{FR}^{-1}} = \frac{\ln\left|\sum_{j=1}^{m} X_j Y_j\right|}{\ln np},$$

where $Y_j \sim$Bernoulli(p).

Other Estimators

Given IID positive random variables, $X_1, X_2, \ldots, X_n$, with distribution $F(x)$ satisfying

$$1 - F(x) \sim Cx^{-\alpha}, \qquad \text{as}\ \ x \to \infty,$$

and let $Z_1 \geq Z_2 \geq \cdots Z_n$ be the associated order statistics. *Hill* (1975) suggested the following estimator:

$$\widehat{\alpha^{-1}} = \frac{1}{m}\sum_{j=1}^{m}\log\left(\frac{Z_i}{Z_{m+1}}\right),$$

where m is chosen such that

$$m(n) \to \infty, \quad m(n) = o(n), \qquad \text{as} \quad n \to \infty,$$

to achieve consistency of $\widehat{\alpha^{-1}}$.

Additionally, *Pickands* (1975) proposed the estimator

$$\widehat{\alpha^{-1}} = \big(\log(2)\big)^{-1} \log\left(\frac{Z_m - Z_{2m}}{Z_{2m} - Z_{4m}}\right),$$

where m is chosen appropriately. Similarly, *Haan and Resnick* (1980) derived the estimator

$$\widehat{\alpha^{-1}} = \big(\log(m)\big)^{-1} \log\left(\frac{Z_1}{Z_m}\right),$$

where

$$m \to \infty, \;\; m/n \to 0, \qquad \text{as } n \to \infty,$$

to attain asymptotic normality.

A description of the relationship between stable distributions and fractional calculus through generalized diffusion equations can be found in *Gorenflo and Mainardi* (1998). For other estimators of the tail index, please see *DuMouchel* (1983) and *Fan* (2001). *DuMouchel* (1973) and *Nolan* (2001) also consider maximum-likelihood estimation for stable distributions.

1.5 Outline of The Remaining Chapters

In Chapter 2, we cite relevant literature for the generalizations of the standard Poisson process including fractional compound Poisson process. More specifically, we clearly derive the transition from standard Poisson process to its fractional generalizations.

In Chapter 3, we restate known characteristics, and derive new properties of fractional Poisson process (fPp). We also establish the link between fPp and α-stable densities by solving an integral equation. The link then leads to an algorithm for generating fPp that eventually paves the way to discovering more interesting properties (e.g., limiting scaled nth arrival time distribution, dependence and nonstationarity of increments, intermittency, etc). We also derive the limiting distribution of a scaled fPp random variable and its integer-order, non-central moments.

In Chapter 4, we derive method-of-moments estimators for the intensity rate μ and fractional order ν. We show asymptotic normality of the estimators. We also propose alternative estimators of μ. We then compare and test our estimators using synthetic data.

In Chapter 5, we recapitulate the main points of this thesis, give some conclusions, and enumerate research directions which we plan to pursue in the future.

Chapter 2

Generalizations of the Standard Poisson Process

A few generalizations of the ordinary or standard Poisson process exist (*Repin and Saichev*, 2000; *Jumarie*, 2001; *Laskin*, 2003; *Mainardi et al.*, 2004, 2005). These generalizations add a parameter $\nu \in (0, 1]$, and is called the fractional exponent of the process. In this chapter, we review some of the key concepts concerning the extensions of the standard Poisson process. In addition, a work on Poisson fractional processes, which is independent from our current investigation can be found in *Wang and Wen* (2003), *Wang et al.* (2006), and *Wang et al.* (2007). A closely-related fractional model for anomalous sub-diffusive processes is studied by *Piryatinska et al.* (2005), and the relation between fractional calculus and multifractality is established in *Frisch and Matsumoto* (2002).

2.1 Standard Poisson Process

For a standard Poisson process with intensity rate μ, we have

$$P_0(t + \Delta t) = P_0(t)P_0(\Delta t) = P_0(t)\big\{1 - P_1(\Delta t) - P_{\geq 2}(\Delta t)\big\}.$$

Also, it can be simply shown that

$$P_1(\Delta t) = \mu \Delta t + o(\Delta t), \qquad \Delta t \to 0,$$

and

$$P_{\geq 2}(\Delta t) = o(\Delta t), \qquad \Delta t \to 0.$$

That is, the probability that there's a single event over a short time range Δt is $\mu\Delta t$, and that the chance of having two or more events during Δt is negligible. Hence,

$$P_0(t + \Delta t) = P_0(t)\big\{1 - \mu\Delta t + o(\Delta t)\big\}.$$

Rearranging, and as $\Delta t \to 0$, we obtain the equation

$$\frac{dP_0(t)}{dt} = -\mu P_0(t), \qquad P_0(0) = 1,$$

with the solution

$$P_0(t) = e^{-\mu t}.$$

Similarly, it can be straightforwardly shown that

$$P_n(t + \Delta t) = P_n(t)(1 - \mu\Delta t) + P_{n-1}(t)\mu\Delta t, \qquad n \geq 1.$$

Putting terms together, and letting $\Delta t \to 0$, we get

$$\frac{dP_n(t)}{dt} = -\mu P_n(t) + \mu P_{n-1}(t),\ P_n(0) = \delta_{n0}, \qquad n \geq 1.$$

Thus, the standard Poisson process satisfies the following recursive family of equations:

$$\frac{dP_n(t)}{dt} = \mu[P_{n-1}(t) - P_n(t)] + \delta_{n0}\delta(t), \qquad 0 \leq t < \infty, P_{-1}(t) = 0, \tag{2.1}$$

$n = 1, 2, \ldots$, where $\delta(t)$ is the Dirac delta function. We can use mathematical induction to solve the system (2.1) of difference equations; however, the method of generating function is more convenient to use. Introducing the generating function

$$G(u, t) = \sum_{n=0}^{\infty} u^n P_n(t),$$

it is easy to verify that

$$P_n(t) = \frac{1}{n!}\frac{\partial^n G(u, t)}{\partial u^n}\bigg|_{u=0} \tag{2.2}$$

and

$$\frac{\partial G(u,t)}{\partial u} = \mu(u-1)G(s,t). \tag{2.3}$$

Equation (2.3) yields the solution

$$G(u,t) = \exp[\mu(u-1)t],$$

and, using equation (2.2), we obtain the well-known probability of having n events by time t

$$P_n(t) = \frac{(\mu t)^n}{n!}e^{-\mu t}. \tag{2.4}$$

It is easy to see that $P_n(t)$ satisfies the normalizing condition $\sum\limits_{n=0}^{\infty} P_n(t) = 1$. Consequently, the waiting time density function $\psi(t)$ is the probability density of an event that occurred at time $t_k = t_{k-1} + t$ after the previous event that happened at time t_{k-1}. Now,

$$P(T > t) = \int\limits_t^{\infty} \psi(t')dt' = 1 - \int\limits_0^t \psi(t')dt',$$

or,

$$\psi(t) = -\frac{d}{dt}P(T > t).$$

It is clear that $\int\limits_0^t \psi(t')dt'$ is the probability of at least one event occurring during the time interval $[0,t]$. Hence,

$$\int\limits_0^t \psi(t')dt' = \sum_{n=1}^{\infty} P_n(t) = 1 - e^{-\mu t}.$$

Thus,

$$\psi(t) = \mu e^{-\mu t}.$$

2.2 Standard Fractional Generalization I

Repin and Saichev (2000) generalize the standard Poisson process by defining the Laplace transform of the waiting time density $\psi_\nu(t)$ via the formula

$$\{\mathsf{L}\psi_\nu(t)\}(\lambda) \equiv \int_0^\infty e^{-\lambda t}\psi_\nu(t)dt \equiv \widetilde{\psi}_\nu(\lambda) = \frac{\mu}{\mu + \lambda^\nu}. \tag{2.5}$$

When $\nu = 1$, the above transformation coincides with the Laplace transform of the exponential waiting time density corresponding to the ordinary Poisson process, i.e.,

$$\widetilde{\psi}_\nu(\lambda) = \frac{\mu}{\mu + \lambda}.$$

They also derived the succeeding fractional integral and differential (*Saichev and Woyczynski*, 1997) equation based on the inverse Laplace transform of $\left\{\widetilde{\psi}_\nu(\lambda)\right\}\left(\frac{\mu}{\mu+\lambda^\nu}\right)^{-1} = 1$:

$$\psi_\nu(t) + \frac{\mu}{\Gamma(\nu)}\int_0^t [\mu(t-\tau)]^{\nu-1}\psi_\nu(\tau)d\tau = \frac{\mu^\nu}{\Gamma(\nu)}t^{\nu-1}.$$

Notice that the preceding equation involving the waiting time density is equivalent to

$${}_0D_t^\nu \psi_\nu(t) + \mu\psi_\nu(t) = \delta(t),$$

where the Liouville derivative (*Samko et al.*, 1993; *Podlubny*, 1999) operator ${}_0D_t^\nu = d^\nu/dt^\nu$ is defined as

$${}_0D_t^\nu \psi_\nu(\tau) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_0^t \frac{\psi_\nu(\tau)d\tau}{[\mu(t-\tau)]^{1-\nu}}.$$

Additionally, they represented their solution (waiting time density $\psi_\nu(t)$) in two different forms:

$$(i) \quad \psi_\nu(t) = -\frac{d}{dt}\text{Prob}(T > t), \quad \text{Prob}(T > t) = E_\nu(-\mu t^\nu), \tag{2.6}$$

where

$$E_\nu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu n + 1)}$$

is the Mittag-Leffler (*Saxena et al.*, 2002) function, and

$$(ii) \quad \psi_\nu(t) = \frac{1}{t}\int_0^\infty e^{-x}\phi_\nu(\mu t/x)dx, \tag{2.7}$$

where

$$\phi_\nu(\xi) = \frac{\sin(\nu\pi)}{\pi[\xi^\nu + \xi^{-\nu} + 2\cos(\nu\pi)]}.$$

Observe that the Mittag-Leffler function is a fractional generalization of the exponential function $\exp(z)$. For instance, at $\nu = 1$, $E_\nu(z) = E_1(z) = \exp(z)$. This Mittag-Leffler type of a waiting time density has been widely used in finance and economics (high-frequency), semiconductor (transport of charge carriers), and optics (light propagation through random media). For more details of the above applications, please see *Scalas et al.* (2000a), *Scalas et al.* (2000b), *Raberto et al.* (2002), *Sabatelli et al.* (2002), *Uchaikin* (2004), *Uchaikin* (2006), and *Scalas* (2006). On the other hand, if we generalize the order of differentiation in (2.1) as follows:

$${}_0D_t^\nu P_n^\nu(t) = \mu[P_{n-1}^\nu(t) - P_n^\nu(t)] + \delta_{n0}\delta(t), \qquad 0 \le t < \infty,\ P_{-1}^\nu(t) = 0, \tag{2.8}$$

we get the equality

$${}_0D_t^\nu \sum_{n=0}^{\infty} P_n^\nu(t) = 0, \qquad \forall\, t > 0, \tag{2.9}$$

where ${}_0D_t^\nu = d^\nu/dt^\nu$ is the Riemann-Liouville (*Oldham and Spanier*, 1974; *Hilfer*, 2000; *Kilbas et al.*, 2006) fractional derivative operator and is defined as

$${}_aD_t^\nu f(t) = \frac{1}{\Gamma(1-\nu)}\frac{d}{dt}\int_a^t (t-\tau)^{-\nu} f(\tau)d\tau.$$

For $\nu = 1$, equation (2.9) satisfies the normalizing condition

$$\sum_{n=0}^{\infty} P_n^{\nu}(t) = 1,$$

i.e.,

$${}_0D_t^1 1 = 0.$$

But, for $0 < \nu < 1$, the left-hand side of equation (2.9) becomes

$${}_0D_t^{\nu} 1 = {}_0D_t^{\nu} H(t) = \frac{d^{\nu} 1}{dt^{\nu}} = \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t (t-\tau)^{-\nu} 1 d\tau, \qquad t > 0$$

where $H(t)$ is the Heaviside unit step function. Letting $u = t - \tau$, we get

$$\begin{aligned} {}_0D_t^{\nu} 1 &= \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \int_0^t u^{-\nu} du \\ &= \frac{1}{\Gamma(1-\nu)} \frac{d}{dt} \left(\left. \frac{u^{1-\nu}}{1-\nu} \right|_0^t \right) \\ &= \frac{t^{-\nu}}{\Gamma(1-\nu)}. \end{aligned}$$

This suggests that equation (2.8) does not meet the normalization condition. However, the normalization can be met by simply tweaking (2.8) as follows:

$${}_0D_t^{\nu} P_n^{\nu}(t) = \mu[P_{n-1}^{\nu}(t) - P_n^{\nu}(t)] + \delta_{n0} \left[{}_0D_t^{\nu} H(t) \right], \tag{2.10}$$

where $0 \le t < \infty$, and $0 < \nu \le 1$. Hence, a fractional generalization of the ordinary Poisson process satisfies the equation

$${}_0D_t^{\nu} P_n^{\nu}(t) = \mu[P_{n-1}^{\nu}(t) - P_n^{\nu}(t)] + \delta_{n0} \frac{t^{-\nu}}{\Gamma(1-\nu)}, \tag{2.11}$$

where $0 \le t < \infty$, and $0 < \nu \le 1$. Expression (2.11) is what *Laskin* (2003) called the fractional Kolmogorov-Feller (*Metzler and Klafter*, 2000; *Zaslavsky*, 2002) equation.

The system of equations (2.11) has been solved by *Jumarie* (2001) (see also *El-Wakil and Zahran* (1999)) using the generating function method. Notice that multiplying equation (2.11) by u^n and summing over n, we get

$$\begin{aligned} {}_0D_t^{\nu} G_\nu(u,t) &= \mu\left[\sum_{n=0}^{\infty} u^n P_{n-1}^{\nu}(t) - \sum_{n=0}^{\infty} u^n P_n^{\nu}(t)\right] + \frac{t^{-\nu}}{\Gamma(1-\nu)} \\ &= \mu(u-1)G_\nu(u,t) + \frac{t^{-\nu}}{\Gamma(1-\nu)}. \end{aligned}$$

This has a solution of the form

$$G_\nu(u,t) = E_\nu\left(\mu t^{\nu}(u-1)\right), \tag{2.12}$$

where $E_\nu(z)$ is the Mittag-Leffler function given by its series representation

$$E_\nu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu n + 1)}.$$

To check that (2.12) is indeed the solution of (2.11), we utilize the known Laplace transform of the Riemann-Liouville fractional derivative operator (*Miller and Ross*, 1993; *West et al.*, 2003). Therefore,

$$\begin{aligned} &\mathsf{L}\left\{ {}_0D_t^{\nu} G_\nu(u,t) = \mu(u-1)G_\nu(u,t) + \frac{t^{-\nu}}{\Gamma(1-\nu)} \right\} \\ \Rightarrow \quad &\lambda^{\nu}\widetilde{G}_\nu(u,\lambda) = \mu(u-1)\widetilde{G}_\nu(u,\lambda) + \lambda^{\nu-1}. \end{aligned}$$

This further implies that

$$\widetilde{G}_\nu(u,\lambda) = \frac{\lambda^{\nu-1}}{\lambda^{\nu} - \mu(u-1)} = \frac{1}{\lambda\left[1 - \mu(u-1)\lambda^{-\nu}\right]}. \tag{2.13}$$

Taking the Laplace transform of (2.12),

$$\mathsf{L}\Big\{E_\nu\left(\mu t^\nu(u-1)\right)\Big\} = \int_0^\infty e^{-\lambda t}\sum_{k=0}^\infty \frac{\left[\mu(u-1)t^\nu\right]^k}{\Gamma(\nu k+1)}dt$$

$$= \sum_{k=0}^\infty \frac{\left[\mu(u-1)\right]^k}{\Gamma(\nu k+1)}\int_0^\infty e^{-\lambda t}t^{\nu k}dt$$

$$= \sum_{k=0}^\infty \left[\mu(u-1)\right]^k \lambda^{-\nu k-1}$$

$$= \sum_{k=0}^\infty \frac{1}{\lambda}\left[\frac{\mu(u-1)}{\lambda^\nu}\right]^k = \frac{1}{\lambda}\left[\frac{1}{1-\mu(u-1)\lambda^{-\nu}}\right]. \tag{2.14}$$

Note that (2.14) is exactly (2.13), and that

$$G_\nu(u,t) = \sum_{n=0}^\infty u^n P_n^\nu(t). \tag{2.15}$$

Hence, the fractional generalization of the probability mass function (2.4) can be shown (expanding over u, and rearranging (2.12) in the fashion of (2.15)) to be

$$P_n^\nu(t) = \frac{(-z)^n}{n!}\frac{d^n}{dz^n}E_\nu(z)\Big|_{z=-\mu t^\nu} = \frac{(\mu t^\nu)^n}{n!}\sum_{k=0}^\infty \frac{(k+n)!}{k!}\frac{(-\mu t^\nu)^k}{\Gamma(\nu(k+n)+1)}. \tag{2.16}$$

We verify that (2.16) satisfies the normalization condition as follows:

$$\sum_{n=0}^\infty P_n^\nu(t) = \sum_{n=0}^\infty \frac{(\mu t^\nu)^n}{n!}\sum_{k=0}^\infty \frac{(k+n)!}{k!}\frac{(-\mu t^\nu)^k}{\Gamma(\nu(k+n)+1)}$$

$$= \sum_{n=0}^\infty \frac{(\mu t^\nu)^n}{n!}\sum_{k=n}^\infty \frac{k!}{(k-n)!}\frac{(-\mu t^\nu)^{k-n}}{\Gamma(\nu k+1)}$$

$$= \sum_{k=0}^\infty \frac{k!}{\Gamma(\nu k+1)}\sum_{n=0}^k \frac{(\mu t^\nu)^n(-\mu t^\nu)^{k-n}}{n!(k-n)!}$$

$$= \sum_{k=0}^\infty \frac{(\mu t^\nu)^k(1-1)^k}{\Gamma(\nu k+1)} = 1.$$

Furthermore, *Laskin* (2003) showed the moment generating function (MGF) of the fractional Poisson process to be

$$M_\nu(s,t) = \sum_{n=0}^{\infty} e^{-sn} P_n^\nu(t) = \sum_{m=0}^{\infty} \frac{\left[\mu t^\nu \left(e^{-s}-1\right)\right]^m}{\Gamma(m\nu+1)}, \tag{2.17}$$

where

$$\mathsf{E}\left[N_\nu(t)\right]^k = (-1)^k \frac{\partial^k}{\partial s^k} M_\nu(s,t)\Big|_{s=0}.$$

Using the MGF, the first two moments of $N_\nu(t)$ can be easily computed as

$$\mathsf{E}\left[N_\nu(t)\right] = \mu_{N_\nu(t)} = \frac{\mu t^\nu}{\Gamma(\nu+1)}$$

and

$$\mathsf{E}\left[N_\nu(t)\right]^2 = \mu_{N_\nu(t)} + \mu_{N_\nu(t)}^2 \frac{\sqrt{\pi}\,\Gamma(1+\nu)}{2^{2\nu-1}\Gamma(\nu+\frac{1}{2})}.$$

The second order moment becomes trivial by using gamma's duplication formula (*Abramowitz and Stegun*, 1964, p. 256):

$$\Gamma(2\nu) = (2\pi)^{-\frac{1}{2}} 2^{2\nu-\frac{1}{2}} \Gamma(\nu)\Gamma\left(\nu+\frac{1}{2}\right).$$

This further indicates that the variance of the fractional Poisson process is

$$\sigma_{N_\nu(t)}^2 = \frac{\mu t^\nu}{\Gamma(\nu+1)} \left\{1 + \frac{\mu t^\nu}{\Gamma(\nu+1)} \left[\frac{\nu B(\nu,1/2)}{2^{2\nu-1}} - 1\right]\right\}, \tag{2.18}$$

where

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

It is also apparent that, as $\nu \to 1$, the mean and variance tend to the mean (and variance) of the ordinary Poisson process.

Consequently, the waiting time density for the fractional Poisson process is

$$\psi_\nu(t) = -\frac{d}{dt} P_\nu(T>t) = -\frac{d}{dt} P_0^\nu(t) = -\frac{d}{dt} E_\nu(-\mu t^\nu). \tag{2.19}$$

Details on calculating $P_0^{\nu}(t)$ can be found in *Jumarie* (2001). The density (2.19) above can be easily shown to be

$$\psi_{\nu}(t) = \mu t^{\nu-1} E_{\nu,\nu}(-\mu t^{\nu}), \tag{2.20}$$

where

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}$$

is the generalized two-parameter Mittag-Leffler function. In particular,

$$\psi_{1/2}(t) = \mu t^{1/2-1} E_{1/2,1/2}\left(-\mu t^{1/2}\right),$$

where

$$\begin{aligned} E_{1/2,1/2}(-z) &= \sum_{n=0}^{\infty} \frac{(-z)^n}{\Gamma\left(\frac{n}{2} + \frac{1}{2}\right)} \\ &= \frac{1}{\sqrt{\pi}} - z E_{1/2,1}(-z). \end{aligned} \tag{2.21}$$

Using the identity,

$$E_{1/2,1}(-z) = e^{z^2} \mathrm{Erfc}(z),$$

where $\mathrm{Erfc}(z)$ is the complementary error function:

$$\mathrm{Erfc}(t) = \frac{2}{\sqrt{\pi}} \int_z^{\infty} e^{-u^2} du,$$

we finally obtain,

$$\begin{aligned} \psi_{1/2}(t) &= \mu t^{-1/2} \left(\frac{1}{\sqrt{\pi}} - \mu t^{1/2} e^{\left(\mu t^{1/2}\right)^2} \mathrm{Erfc}(\mu\sqrt{t}) \right) \\ &= \frac{\mu}{\sqrt{\pi t}} - \mu^2 e^{\mu^2 t} \mathrm{Erfc}(\mu\sqrt{t}). \end{aligned} \tag{2.22}$$

In addition, we can directly verify that the density (2.22) has the Laplace transform (2.5) using the formula (*Saxena et al.*, 2002)

$$\int_0^{\infty} e^{-\lambda t} E_{\nu_1,\nu_2}\left(pat^{\nu_1}\right) t^{\nu_2-1} dt = \frac{\lambda^{-\nu_2}}{(1 - a\lambda^{-\nu_1})}.$$

2.3 Standard Fractional Generalization II

From the preceding section, we see that the fractional generalization of the ordinary Poisson process is not unique. *Jumarie* (2001) defines a function $y(t) : \mathbb{R} \to \mathbb{R}$ to be continuous of order ν, $0 < \nu < 1$ when

$$y(t + \Delta t) - y(t) = o\big[(\Delta t)^\nu\big],$$

and the finite fractional derivative of order ν of $y(t)$ to be

$$\frac{dy}{dt^\nu} = \lim_{\Delta t \downarrow 0} \frac{y(t + \Delta t) - y(t)}{(\Delta t)^\nu}.$$

The above definition leads to the next extension of the standard Poisson process: Let $Q_n^\nu(\Delta t)$ be the probability that there are n arrivals in the small time interval Δt.

$$\begin{aligned} Q_1^\nu(\Delta t) &= \mu(\Delta t)^\nu, \qquad 0 < \nu < 1, \\ Q_n^\nu(\Delta t) &= (\Delta t)^\nu O(\Delta t), \qquad n \geq 2. \end{aligned}$$

This shows that

$$Q_0(\Delta t) = 1 - \mu(\Delta t)^\nu,$$

and the equation of the process is given by

$$\begin{aligned} P_n^\nu(t + \Delta t) &= P_n^\nu(t) Q_0^\nu(\Delta t) + P_{n-1}^\nu(t) Q_1^\nu(\Delta t) \\ &= P_n^\nu(t) - \mu(\Delta t)^\nu \left(P_n^\nu(t) - P_{n-1}^\nu(t)\right). \end{aligned}$$

Passing to the limit yields

$$\frac{dP_n^\nu(t)}{dt^\nu} = -\mu \left(P_n^\nu(t) - P_{n-1}^\nu(t)\right), \qquad n \geq 1,$$

where

$$\frac{dP_0^\nu(t)}{dt^\nu} = -\mu P_0^\nu(t).$$

Jumarie (2001) uses the operator $d/(dt)^\nu$ instead of the fractional derivative operator $(d/dt)^\nu$. The above fractional difference-differential equation has the following solution, which is claimed to be the probability that there are n arrivals or events by time t (see *Jumarie* (2001) for the proof):

$$\hat{P}_n^{\nu}(t) = \mu^n \left(\frac{t^n}{n!}\right)^{\nu} e^{-\mu t^{\nu}}, \qquad n = 0, 1, 2, \ldots.$$

But it is clear that this function does not meet the normalization condition, i.e.,

$$\sum_{n=0}^{\infty} \mu^n \left(\frac{t^n}{n!}\right)^{\nu} e^{-\mu t^{\nu}} \neq 1,$$

if $\nu \neq 1$, and cannot be used to represent a probability distribution. This makes the above generalization not a viable model of real-life random processes.

2.4 Non-Standard Fractional Generalization

Suppose we define a fractional Poisson process of order ν as a process in which there is only at most one event or arrival in a small time interval Δt with probabilities

$$Q_0^{\nu}(\Delta t) = 1 - \frac{\mu}{\Gamma(\nu+1)}(\Delta t)^{\nu},$$

and

$$Q_1^{\nu}(\Delta t) \cong \frac{\mu}{\Gamma(\nu+1)}(\Delta t)^{\nu}.$$

Notice that $Q_1^{\nu}(\Delta t)$ exactly corresponds to $n = 1$ in equation (2.16). Then according to the previous section, we shall get the equations

$$\frac{dP_n^{\nu}(t)}{dt^{\nu}} = -\frac{\mu}{\Gamma(\nu+1)}\left(P_n^{\nu}(t) - P_{n-1}^{\nu}(t)\right),$$

and

$$\frac{dP_0^{\nu}(t)}{dt^{\nu}} = -\frac{\mu}{\Gamma(\nu+1)}P_0^{\nu}(t),$$

where the solutions are

$$\hat{P}_n^{\nu}(t) = \frac{(\mu t^{\nu})^n}{(n!)^{\nu}\,\Gamma^n(1+\nu)} \sum_{j=0}^{\infty} \frac{(-\mu t^{\nu})^j}{\Gamma^n(1+\nu) j!}, \qquad n = 0, 1, 2, \ldots,$$

and

$$\hat{P}_0^{\nu}(t) = \sum_{j=0}^{\infty} \frac{(-\mu t^{\nu})^j}{\Gamma^n(1+\nu)j!}, \qquad n = 0, 1, 2, \ldots,$$

correspondingly. Jumarie (2001) considers this approach as a non-standard fractional generalization of the standard Poisson process.

2.5 Fractional Compound Poisson Process

A stochastic process $\{X(t), t \geq 0\}$ is called a fractional compound Poisson process if it can be represented as

$$X(t) = \sum_{j=1}^{N(t)} Y_j,$$

where $\{Y_j, j = 1, 2 \ldots\}$ is a family of independent and identically distributed random variables with probability distribution $p(Y)$, and $\{N(t), t \geq 0\}$ is a fractional Poisson process. If we assume independence of $\{N(t), t \geq 0\}$ and $\{Y_j, j = 1, 2 \ldots\}$ then we can calculate the moment generating function $J_{\nu}(s, t)$ of the fractional compound Poisson process

$$\begin{aligned}
J_{\nu}(s,t) &= \mathsf{E}\big[\exp\{sX(t)\}\big]_{Y_j, N(t)} \\
&= \sum_{n=0}^{\infty} \mathsf{E}\big[\exp\{sX(t)\big|N(t) = n\}\big]_{Y_j} \times P_n^{\nu}(t) \\
&= \sum_{n=0}^{\infty} \mathsf{E}\big[\exp\{s(Y_1 + Y_2 + \cdots + Y_n)\big|N(t) = n\}\big]_{Y_j} \\
&\times \frac{(\mu t^{\nu})^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\mu t^{\nu})^k}{\Gamma(\nu(k+n)+1)} \\
&= \sum_{n=0}^{\infty} \mathsf{E}\big[\exp\{sY_1\big|N(t) = n\}\big]_{Y_j}^n \times \frac{(\mu t^{\nu})^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\mu t^{\nu})^k}{\Gamma(\nu(k+n)+1)}.
\end{aligned}$$

Moreover, if we let

$$g(s) = \mathsf{E}\big[\exp\{sY\}\big]_Y$$

be the moment generating function of the random variables Y_j then we can easily show that

$$\begin{aligned}
J_\nu(s,t) &= \sum_{n=0}^{\infty} g(s)^n \times \frac{(\mu t^\nu)^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\mu t^\nu)^k}{\Gamma(\nu(k+n)+1)} \\
&= \sum_{n=0}^{\infty} \frac{[g(s)\mu t^\nu]^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\mu t^\nu)^k}{\Gamma(\nu(k+n)+1)} \\
&= \sum_{n=0}^{\infty} \frac{[g(s)\mu t^\nu]^n}{n!} \sum_{k=n}^{\infty} \frac{k!}{(k-n)!} \frac{(-\mu t^\nu)^{k-n}}{\Gamma(\nu k+1)} \\
&= \sum_{k=0}^{\infty} \frac{k!}{\Gamma(\nu k+1)} \sum_{n=0}^{k} \frac{[g(s)\mu t^\nu]^n(-\mu t^\nu)^{k-n}}{n!(k-n)!} \\
&= E_\nu(\mu t^\nu(g(s)-1)).
\end{aligned}$$

We can see that the kth order moment of $X(t)$ can be obtained by

$$\mathsf{E}\big[X(t)^k\big]_{Y_j,N(t)} = \frac{\partial^k}{\partial s^k} J_\nu(s,t)\bigg|_{s=0}.$$

When $k=1$, we get

$$\begin{aligned}
\mathsf{E}\big[X(t)\big]_{Y_j,N(t)} &= \frac{\partial}{\partial s} J_\nu(s,t)\bigg|_{s=0} \\
&= (\mathsf{E}Y)\left(\frac{\mu t^\nu}{\Gamma(\nu+1)}\right).
\end{aligned}$$

More details can be found in *Laskin* (2003).

2.6 Alternative Fractional Generalization

A fractional generalization of the ordinary Poisson process using Caputo's definition can be found in *Mainardi et al.* (2004) and *Mainardi et al.* (2005). Observe that the survival probability function ($P(T > t) = \Theta(t)$) for the standard Poisson process (with parameter μ) satisfies the ordinary differential equation

$$\frac{d}{dt}\Theta(t) = -\mu\Theta(t), \qquad t \geq 0,\ \Theta(0^+) = 1.$$

The alternative generalization comes in by replacing the first derivative operator by the fractional derivative (in Caputo's sense) of order ν . Hence, we have now the new ordinary fractional differential equation,

$$ {}_0D_t^{*\nu}\Theta(t) = -\mu\Theta(t), \qquad t \geq 0,\ 0 < \nu \leq 1,\ \Theta(0^+) = 1, \tag{2.23} $$

where Caputo's derivative of a well-behaved function $f(t) \in \mathbb{R}^+$ is defined as

$$ {}_0D_t^{*\nu} f(t) = \begin{cases} \frac{1}{\Gamma(1-\nu)} \int_0^t \frac{f^{(1)}(\tau)}{(t-\tau)^\nu} d\tau, & 0 < \nu < 1; \\ \frac{d}{dt} f(t), & \nu = 1. \end{cases} $$

Its Laplace transform happens to be

$$ \mathsf{L}\big\{ {}_0D_t^{*\nu} f(t) \big\} = \lambda^\nu \widetilde{f}(\lambda) - \lambda^{\nu-1} f(0^+). $$

For more information on the theory and applications of Caputo derivative of order $\nu > 0$, please see *Carpinteri and Mainardi* (1997), and *Caputo* (2001). Now, solving equation (2.23) using Laplace transform gives

$$ \begin{aligned} \mathsf{L}\big\{ {}_0D_t^{*\nu}\Theta(t) \big\} &= -\mu \mathsf{L}\big\{\Theta(t)\big\} \\ \Longrightarrow \lambda^\nu \widetilde{\Theta}(\lambda) - \lambda^{\nu-1}\Theta(0^+) &= -\mu\widetilde{\Theta}(\lambda) \\ \Longrightarrow \widetilde{\Theta}(\lambda) &= \frac{\lambda^{\nu-1}}{\mu + \lambda^\nu}. \end{aligned} \tag{2.24} $$

Note that

$$ \mathsf{L}\big\{ E_\nu(-\mu t^\nu) \big\} = \widetilde{\Theta}(\lambda) = \frac{\lambda^{\nu-1}}{\mu + \lambda^\nu}. $$

By simple inspection, we can see that equation (2.24) automatically yields the solution $\Theta(t)$, which is the Mittag-Leffler function

$$ E_\nu(-\mu t^\nu) = \sum_{n=0}^{\infty} \frac{\big(-\mu t^\nu\big)^n}{\Gamma(1+\nu n)} $$

as defined previously. A more rigorous solution to a more general class of problems that includes the above ordinary fractional differential equation can be found in *Carpinteri and Mainardi* (1997).

Furthermore, the Poisson and Erlang distributions (corresponding to the nth arrival or event time, $n \in \mathbb{N}$) are generalized in what follows: It can be shown that

$$\mathsf{L}\Big\{\frac{(\mu t)^n}{n!}e^{-\mu t}\Big\} = \frac{\mu^n}{(\mu+\lambda)^{n+1}},$$

and

$$\mathsf{L}\Big\{\mu\frac{(\mu t)^{n-1}}{(n-1)!}e^{-\mu t}\Big\} = \frac{\mu^n}{(\mu+\lambda)^{n}}.$$

From (1.80) of *Podlubny* (1999),

$$\mathsf{L}\Big\{t^{\nu_1 n+\nu_2-1}E^{(n)}_{\nu_1,\nu_2}\big(\pm\mu t^{\nu_1}\big)\Big\} = \frac{n!\lambda^{\nu_1-\nu_2}}{(\lambda^{\nu_1}\mp\mu)^{n+1}}, \qquad \nu_1>0,\ \nu_2>0,$$

where

$$E^{(n)}_{\nu_1,\nu_2}(y) = \frac{d^n}{dy^n}E_{\nu_1,\nu_2}(y).$$

When $\nu_1=\nu$, and $\nu_2=1$, we get

$$\mathsf{L}\Big\{\mu^n t^{\nu n}E^{(n)}_{\nu,1}\big(-\mu t^{\nu}\big)\Big\} = \mathsf{L}\Big\{\mu^n t^{\nu n}E^{(n)}_{\nu}\big(-\mu t^{\nu}\big)\Big\} = \frac{n!\lambda^{\nu-1}\mu^n}{(\lambda^{\nu}+\mu)^{n+1}}, \qquad \nu>0.$$

This implies that a generalization of the Poisson distribution is given by

$$P^{\nu}_n(t) = P(N_\nu(t)=n) = \frac{t^{\nu n}}{n!}E^{(n)}_{\nu}\big(-\mu t^{\nu}\big), \tag{2.25}$$

where the Laplace transform of the probability mass function is

$$\mathsf{L}\big\{P^{\nu}_n(t)\big\} = \frac{\lambda^{\nu-1}\mu^n}{(\mu+\lambda^{\nu})^{n+1}}.$$

Accordingly, a generalization of the Erlang distribution is shown to be

$$f(T=T_1+T_2+\cdots+T_n) = f^{\nu}_n(t) = \mu^n\nu\frac{t^{\nu n-1}}{(n-1)!}E^{(n)}_{\nu}\big(-\mu t^{\nu}\big), \tag{2.26}$$

where the Laplace transform of the probability density function is

$$\mathsf{L}\big\{f^{\nu}_n(t)\big\} = \frac{\mu^n}{(\mu+\lambda^{\nu})^{n}}.$$

When $\nu\to1$, the distributions (2.25) and (2.26) converge to Poisson Distribution, and Erlang distribution, respectively.

Chapter 3

Fractional Poisson Process

From this chapter on, we adopt the first fractional generalization as it provides a natural extension of the ordinary Poisson process, and refer to it as the fractional Poisson process (fPp). When $\nu = 1$, fPp becomes the standard (memoryless) Poisson process. But when $\nu < 1$, fPp physically exhibits a long-run memory property, that is, events in non-overlapping time intervals are correlated. This makes the memory "length" a function of the parameter ν, which is very attractive for further exploration. In the subsequent discussion, we establish the link between fPp and α-stable densities by solving an integral equation, reformulate known and uncover new properties, show that the asymptotic distribution of a scaled fPp random variable is independent of some parameters and derive formulas for integral order, non-central moments, and propose an alternative fractional generalization of the standard Poisson process worthy of exploration.

3.1 Some Known Properties of fPp

As shown in *Repin and Saichev* (2000), and *Laskin* (2003), the waiting time density varies as the ordinary Poisson process, with $\psi(t) = \mu e^{-\mu t}$ transitions to fPp, with $\psi_\nu(t) = \mu t^{\nu-1} E_{\nu,\nu}(-\mu t^\nu), \nu < 1$. The above transition removes the characteristics of the ordinary Poisson process (see section 3.2). Table 3.1 below compares some known properties of fPp with those of the standard Poisson process.

Table 3.1: *Properties of fPp compared with those of the ordinary Poisson process.*

	Poisson process ($\nu = 1$)	Fractional Poisson Process ($\nu < 1$)
$P_0(t)$	$e^{-\mu t}$	$E_\nu(-\mu t^\nu)$
$\psi(t)$	$\mu e^{-\mu t}$	$\mu t^{\nu-1} E_{\nu,\nu}(-\mu t^\nu)$
$P_n(t)$	$\frac{(\mu t)^n}{n!} e^{-\mu t}$	$\frac{(\mu t^\nu)^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\mu t^\nu)^k}{\Gamma(\nu(k+n)+1)}$
$\mu_{N(t)}$	μt	$\frac{\mu t^\nu}{\Gamma(\nu+1)}$
$\sigma^2_{N(t)}$	μt	$\frac{\mu t^\nu}{\Gamma(\nu+1)} \left\{ 1 + \frac{\mu t^\nu}{\Gamma(\nu+1)} \left[\frac{\nu B(\nu,1/2)}{2^{2\nu-1}} - 1 \right] \right\}$, $B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$
$\mathsf{E}\left[N(t)\right]^k$	$\frac{\partial^k}{\partial s^k} s^k \exp\left[\mu(s-1)t\right] \big\vert_{s=0}$	$(-1)^k \frac{\partial^k}{\partial s^k} \sum_{m=0}^{\infty} \frac{\left[\mu t^\nu \left(e^{-s}-1\right)\right]^m}{\Gamma(m\nu+1)} \Big\vert_{s=0}$

We also plot the mean and variance as a function of ν and time t below (Figures (3.1) and (3.2)). As $\nu \to 1$, the mean and variance become that of the ordinary Poisson process, that is, they both equal μt_1. When $\nu \to 0$, the mean and variance become constant μ. This suggests that fPp's corresponding to small fractional orders (close to $\nu = 0$) have slowly varying mean and variance that depend little on time.

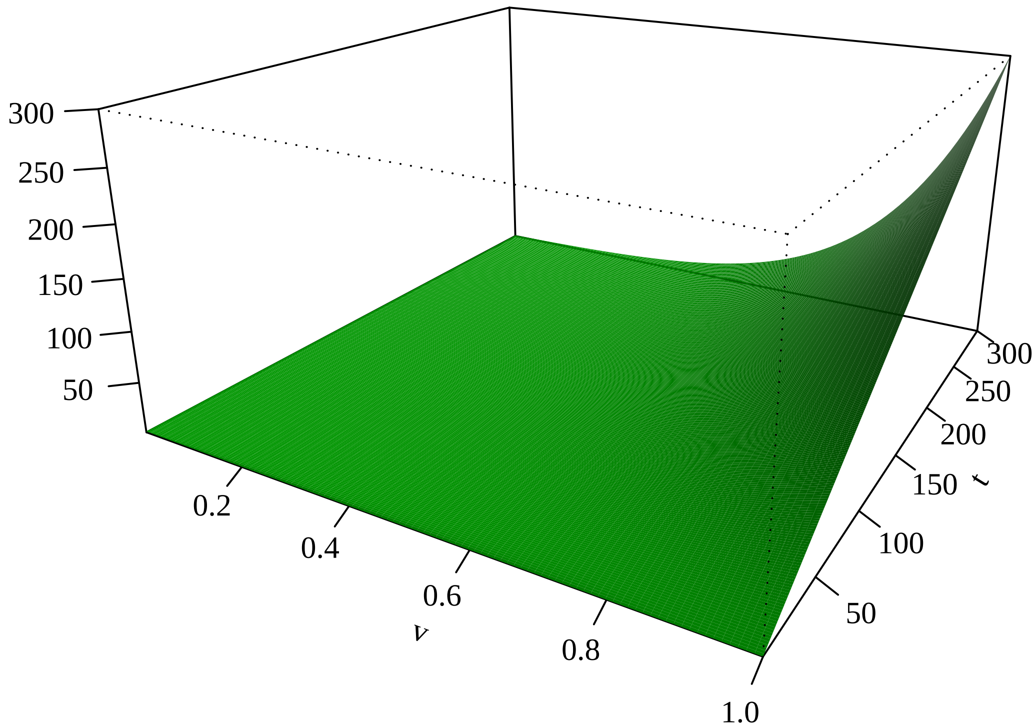

Figure 3.1: *The mean of fPp as a function of time t and fractional order ν.*

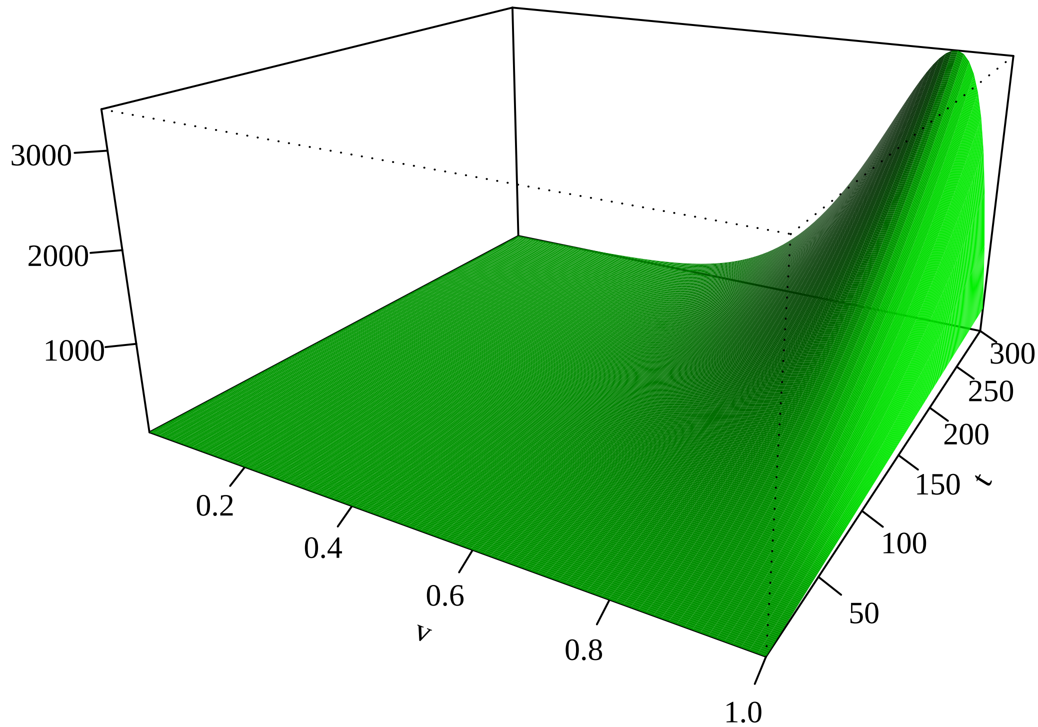

Figure 3.2: *The variance of fPp as a function of time t and fractional order ν.*

Figure 3.2 seems to indicate that, for large t's the variance achieves the maximum for a certain $\nu = \nu(t) < 1$. It would be interesting to investigate the properties of this "maximum" in the future.

3.2 Asymptotic Behavior of the Waiting Time Density

The waiting (interarrival) time T with density $\psi(t)$ plays a crucial role in renewal theory. In the ordinary Poisson case, it has the exponential function $\psi(t) = \mu e^{-\mu t}$. Recall that *Repin and Saichev* (2000) came up with the density

$$\psi_\nu(t) = \frac{1}{t}\int_0^\infty e^{-x}\phi_\nu(\mu t/x)dx, \tag{3.1}$$

where

$$\phi_\nu(\xi) = \frac{\sin(\nu\pi)}{\pi[\xi^\nu + \xi^{-\nu} + 2\cos(\nu\pi)]}.$$

The above formula allows us to find the waiting time density behavior for small and large times. For instance, as $t \to \infty$,

$$\begin{aligned}
\psi_\nu(t) &= \frac{\sin(\nu\pi)}{\pi t}\int_0^\infty \frac{\exp(-x)dx}{(\mu t/x)^\nu + (\mu t/x)^{-\nu} + 2\cos(\pi\nu)} \\
&= \frac{\sin(\nu\pi)}{\pi\mu^\nu t^{\nu+1}}\int_0^\infty \frac{\exp(-x)dx}{x^{-\nu} + x^\nu\,(\mu t)^{-2\nu} + 2\cos(\pi\nu)\,(\mu t)^{-\nu}}. \qquad (3.2)
\end{aligned}$$

From (3.2), we see that

$$\int_0^\infty \frac{\exp(-x)dx}{x^{-\nu} + x^\nu\,(\mu t)^{-2\nu} + 2\cos(\pi\nu)\,(\mu t)^{-\nu}} \xrightarrow{t\to\infty} \int_0^\infty x^\nu \exp(-x)dx = \Gamma(1+\nu).$$

Thus, equation (3.2) becomes

$$\psi_\nu(t) \sim \frac{\sin(\nu\pi)}{\pi\mu^\nu t^{\nu+1}}\Gamma(1+\nu), \qquad t \to \infty. \qquad (3.3)$$

Substituting the identities $\Gamma(1+\nu) = \nu\Gamma(\nu)$ and $\pi/\sin(\pi\nu) = \Gamma(1-\nu)\Gamma(\nu)$ into (3.3), and simplifying the resulting equation, we observe that

$$\psi_\nu(t) \sim \frac{\nu t^{-\nu-1}}{\mu^\nu\Gamma(1-\nu)}, \qquad t \to \infty.$$

Similarly, as $t \to 0$,

$$\begin{aligned}
\psi_\nu(t) &= \frac{\sin(\nu\pi)}{\pi\mu^{-\nu}t^{1-\nu}}\int_0^\infty \frac{\exp(-x)dx}{(\mu t)^{2\nu}\,x^{-\nu} + x^\nu + 2\cos(\pi\nu)\,(\mu t)^\nu} \\
&\sim \frac{\sin(\nu\pi)}{\pi\mu^{-\nu}t^{1-\nu}}\Gamma(1-\nu), \qquad (3.4)
\end{aligned}$$

as

$$\int_0^\infty \frac{\exp(-x)dx}{(\mu t)^{2\nu}\,x^{-\nu} + x^\nu + 2\cos(\pi\nu)\,(\mu t)^\nu} \xrightarrow{t\to 0} \Gamma(1-\nu).$$

Using the previous identity involving the sine function, we can obtain the small-time behavior of the probability density function:

$$\psi_\nu(t) \sim \frac{t^{\nu-1}}{\mu^{-\nu}\Gamma(\nu)}, \qquad t \to 0.$$

Expressing the above results in compact form, we get

$$\psi_\nu(t) \sim \begin{cases} \frac{\mu^\nu}{\Gamma(\nu)} t^{\nu-1}, & t \to 0, \\ \\ \frac{\nu\mu^{-\nu}}{\Gamma(1-\nu)} t^{-\nu-1}, & t \to \infty. \end{cases}$$

Additionally, expression (3.1) provides a useful formula for plotting the interarrival time densities. Figure 3.3 below shows the log-log plot of the fPp waiting time densities for different ν's.

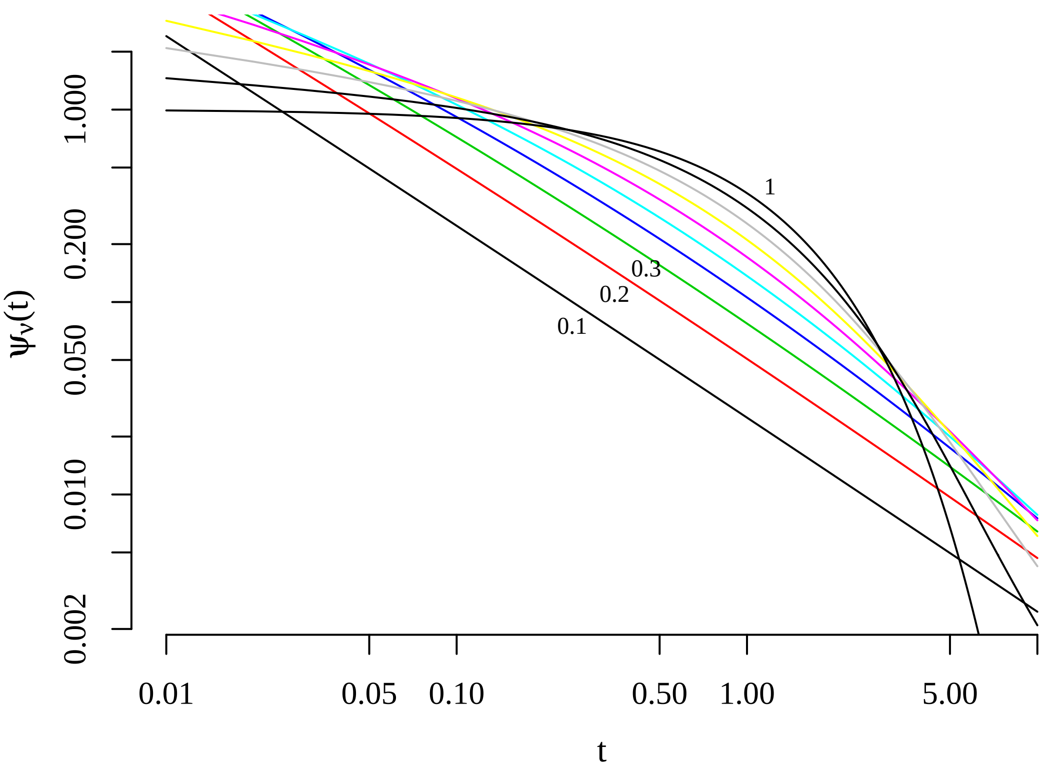

Figure 3.3: *Waiting time densities of fPp (3.1) using $\mu = 1$, and $\nu = 0.1(0.1)1$ (log-log scale).*

3.3 Simulation of Waiting Time

A more thorough investigation of the properties of fPp may also be achieved by Monte Carlo simulation. This aim convinces us to use the more convenient representation of the interarrival time density, which is of the form (2.19). We now introduce a lemma below.

Lemma. *The three-parameter Mittag-Leffler function*

$$E_\nu(-\mu(\rho t)^\nu) = \sum_{k=0}^{\infty} \frac{[-\mu(\rho t)^\nu]^k}{\Gamma(1+\nu k)}$$

can be expressed as

$$\int_0^\infty e^{-\mu(\rho t)^\nu/\tau^\nu} g^{(\nu)}(\tau) d\tau, \qquad 0 < \nu \le 1, \tag{3.5}$$

where $g^{(\nu)}(\tau)$ is the one-sided α-stable density (see Appendix), $\mu > 0$, and $\rho > 0$.

Proof. Expanding the exponential function in equation (3.5)

$$e^{-\mu(\rho t)^\nu/\tau^\nu} = \sum_{k=0}^{\infty} \frac{1}{k!}[-\mu(\rho t)^\nu/\tau^\nu]^k$$

and using formula (A.2) for calculating negative order moments of the α-stable density

$$\int_0^\infty g^{(\nu)}(\tau)\tau^{-\nu k} d\tau = \frac{k!}{\Gamma(1+\nu k)},$$

we obtain

$$P(T > t) = \sum_{k=0}^{\infty} \frac{1}{k!} \int_0^\infty [-\mu(\rho t)^\nu/\tau^\nu]^k g^{(\nu)}(\tau) d\tau$$

$$= \sum_{k=0}^{\infty} \frac{[-\mu(\rho t)^\nu]^k}{k!} \int_0^\infty \tau^{-\nu k} g^{(\nu)}(\tau) d\tau = \sum_{k=0}^{\infty} \frac{[-\mu(\rho t)^\nu]^k}{\Gamma(1+\nu k)} = E_\nu[-\mu(\rho t)^\nu].$$

The Lemma is proved. ■

When $\rho = 1$, we state the following direct corollary of the above lemma without proof.

Theorem. *The complementary cumulative distribution function,*

$$P(T > t) = E_\nu(-\mu t^\nu),$$

can be represented in the form

$$P(T > t) = \int_0^\infty e^{-\mu t^\nu/\tau^\nu} g^{(\nu)}(\tau) d\tau, \tag{3.6}$$

where $g^{(\nu)}(\tau)$ is the one-sided α-stable density (see Appendix).

We now state a theorem that alternatively describes the distribution of the fPp interarrival times, and that provides a tool for their simulation.

Theorem. *The random variable T determined above has the same distribution as*

$$T' \stackrel{d}{=} \frac{|lnU|^{1/\nu}}{\mu^{1/\nu}} S(\nu),$$

where $S(\nu)$ is a random variable distributed according to $g^{(\nu)}(\tau)$, U is uniformly distributed in $[0,1]$, and U is independent of $S(\nu)$.

Proof. Using the formula of total probability, we can represent equality (3.6) in the form

$$P(T > t) = \int_0^\infty P(T > t|\tau) g^{(\nu)}(\tau) d\tau,$$

where

$$P(T > t|\tau) = e^{-\mu t^\nu/\tau^\nu}$$

is the conditional distribution. This means that

$$P(T > t|\tau) = P(U < e^{-\mu t^\nu/\tau^\nu}) = P\left(\frac{|\ln U|^{1/\nu}}{\mu^{1/\nu}} \tau > t\right),$$

or

$$T|_\tau \stackrel{d}{=} \frac{|\ln U|^{1/\nu}}{\mu^{1/\nu}} \tau.$$

Because τ is a fixed possible value of $S(\nu)$, we obtain the following equivalence (in distribution) for the unconditional interarrival time:

$$T \stackrel{d}{=} \frac{|\ln U|^{1/\nu}}{\mu^{1/\nu}} S(\nu). \quad \blacksquare$$

We now cite the succeeding consequence that highlights the formula for generating fPp waiting times.

Corollary. *The random variable*

$$T \stackrel{d}{=} \frac{|\ln U_1|^{1/\nu}}{\mu^{1/\nu}} \frac{\sin(\nu\pi U_2)[\sin((1-\nu)\pi U_2)]^{1/\nu-1}}{[\sin(\pi U_2)]^{1/\nu}|\ln U_3|^{1/\nu-1}}, \tag{3.7}$$

where U_1, U_2 *and* U_3 *are independently and uniformly distributed in* $[0,1]$.

This result follows from Kanter's algorithm of simulating $S(\nu)$ (*Kanter*, 1975). It is worth mentioning that one can use the algorithm of *Chambers et al.* (1976) in simulating stable random variables as well (see also *Devroye* (1986) and *Janicki and Weron* (1994)). Below is the algorithm for generating n fPp interarrival times, which will also be used in the subsequent calculations.

Algorithm

I. Generate U_1, U_2, U_3 from $U(0,1)$.

II. Compute

$$T = \frac{|\ln U_1|^{1/\nu}}{\mu^{1/\nu}} \frac{\sin(\nu\pi U_2)[\sin((1-\nu)\pi U_2)]^{1/\nu-1}}{[\sin(\pi U_2)]^{1/\nu}|\ln U_3|^{1/\nu-1}}.$$

III. Repeat I and II n times.

When $\nu \to 1$, the above algorithm reduces to the well known formula of generating random numbers from an exponential distribution, i.e.,

$$T \stackrel{d}{=} \frac{|\ln U|}{\mu}.$$

Below are the goodness-of-fit test statistics (see Table 3.2), which signify favorable results.

Table 3.2: χ^2 *Goodness-of-fit Test Statistics with* $\mu = 1$.

$\psi_\nu(t)$	χ^2 Test Statistic Values	Critical Values
$\psi_{1/2}(t)$	31.0	$\chi^2_{29,0.05} = 42.5$
$\psi_1(t)$	27.5	$\chi^2_{29,0.1} = 39.1$

3.4 The Limiting Scaled nth Arrival Time Distribution

Let $S_n = T_1 + T_2 + \cdots + T_n,\ n = 1, 2, 3, \ldots,$ be the nth arrival time, and

$$\psi^{*n}(t) = \underbrace{\psi * \psi * \cdots * \psi}_{n \text{ times}}(t)$$

be its probability density. Here, the T_j's are mutually independent copies of the interarrival random time T, and the symbol $*$ denotes the convolution operation

$$\psi * \psi(t) \equiv \int_0^t \psi(t-\tau)\psi(\tau)d\tau.$$

For the standard Poisson process, the interarrival time T_j is distributed according to $\exp(1/\mu)$, and that the nth arrival time S_n is Erlang $(n, 1/\mu)$ distributed. The n-fold convolution is then

$$\psi^{*n}(t) = \mu\frac{(\mu t)^{n-1}}{(n-1)!}e^{-\mu t}.$$

A generalization of the preceding distribution exists and is given by (2.26). Now, let

$$Z_n = \frac{S_n - n/\mu}{\sqrt{n}/\mu}.$$

It can be shown without difficulty that

$$f_{Z_n}(x) = f_{S_n}(n/\mu + \left(\sqrt{n}/\mu\right) x)\frac{\sqrt{n}}{\mu}.$$

According to the Central Limit Theorem (CLT),

$$\Psi^{(n)}(t) \equiv f_{Z_n}(t) = (\sqrt{n}/\mu)\psi^{*n}(n/\mu + t\sqrt{n}/\mu) \Rightarrow \frac{1}{\sqrt{2\pi}}e^{-z^2/2}, \qquad n \to \infty. \tag{3.8}$$

Figure 3.4 below shows that $\Psi^{(n)}(t)$ already reaches its limit curve by $n = 10$.

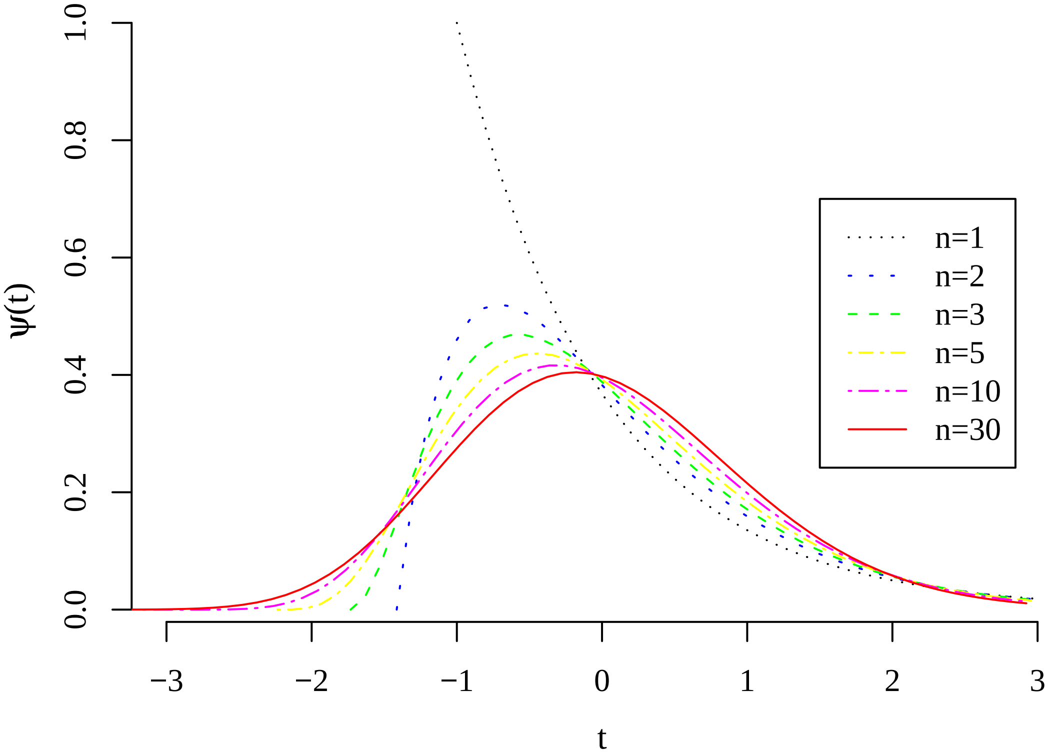

Figure 3.4: *Scaled nth arrival time distributions for standard Poisson process (3.8) with $n = 1, 2, 3, 5, 10, 30$, and $\mu = 1$.*

In the case of fPp,

$$\mathsf{E}T = \int_0^\infty \psi_\nu(t) t dt = \infty,$$

and the CLT is no longer valid. Recall the Generalized Central Limit Theorem (GCLT) (*Gnedenko and Kolmogorov*, 1968; *Uchaikin and Zolotarev*, 1999; *Rachev*, 2003) which states that the only possible distributions with a domain of attraction are stable. Let

$$Z_n^\nu = \frac{\sum_{i=1}^n T_i}{bn^{1/\nu}}.$$

By a simple algebraic manipulation, it can be straightforwardly shown that

$$f_{Z_n^\nu}(x) = f_{S_n^\nu}\left(bn^{1/\nu}x\right) bn^{1/\nu}.$$

Our goal now reduces to finding the constant b in the relation

$$bn^{1/\nu}\psi_\nu^{*n}(tbn^{1/\nu}) \to g^{(\nu)}(t), \qquad n \to \infty, \tag{3.9}$$

where $g^{(\nu)}(t)$ is a one-sided $\nu-$stable probability density or ν^+-stable probability density.

Using formula (2.5) in getting the Laplace transform of the left-hand side of expression (3.9) gives

$$\begin{aligned}
\mathsf{L}\{(bn^{1/\nu})\psi_\nu^{*n}(tbn^{1/\nu})\} &= \left[\widetilde{\psi_\nu}(\lambda/(bn^{1/\nu}))\right]^n \\
&= \left[\frac{\mu}{\mu + [\lambda/(bn^{1/\nu})]^\nu}\right]^n \\
&= \left[1 - \frac{\left[\lambda/bn^{1/\nu}\right]^\nu}{\mu + \left[\lambda/bn^{1/\nu}\right]^\nu}\right]^n \\
&= \left[1 - \frac{\lambda^\nu/\mu b^\nu}{n + \lambda^\nu/\mu b^\nu}\right]^n \\
&\overset{n\to\infty}{\longrightarrow} \exp(-\lambda^\nu/\mu b^\nu) \\
&\sim 1 - \frac{\lambda^\nu}{\mu b^\nu}, \qquad \lambda \to \infty.
\end{aligned}$$

Similarly, applying result (A.1) on the right hand side of expression (3.9), we obtain

$$1 - \lambda^\nu/\mu b^\nu \sim 1 - \lambda^\nu, \qquad \lambda \to \infty.$$

This yields the scaling constant

$$b = \mu^{-1/\nu}.$$

Hence, we can deduce that

$$\begin{aligned}
\Psi_\nu^{(n)}(t) \equiv f_{Z_n^\nu}(t) &= \left(\frac{n}{\mu}\right)^{1/\nu} \psi_\nu^{*n}\left(t\left(\frac{n}{\mu}\right)^{1/\nu}\right) \\
&= n^{1/\nu} \overset{\circ}{\psi}{}_\nu^{*n} (tn^{1/\nu}) \\
&\overset{n\to\infty}{\longrightarrow} g^{(\nu)}(t),
\end{aligned}$$

where

$$\overset{\circ}{\psi}_\nu(t) = \psi_\nu(t)|_{\mu=1} = t^{\nu-1}E_{\nu,\nu}(-t^\nu).$$

The multiple integrals involved in the convolution can be computed by a Monte Carlo technique. When $\mu = 1$, $\Psi_\nu^{(n)}(t)$ is the probability density of the renormalized sum $(T_1+T_2+\cdots+T_n)/n^{1/\nu}$, where T_j's are distributed according to $\overset{\circ}{\psi}_\nu(t)$. We could directly simulate this sum using the above algorithm, and construct the corresponding histogram, but the steep left tail of the waiting time densities makes observations or points (from the left tail) less likely to be sampled. For the above reason, we find the regular histogram as an unfavorable and inappropriate estimator. This leads us to consider the Monte Carlo approach.

With $n = 2$, $\Psi_\nu^{(2)}(t)$ becomes the probability density of the scaled sum of two IID random variables $\big((T_1+T_2)/2^{1/\nu} = T_1/2^{1/\nu} + T_2/2^{1/\nu}\big)$, and can be represented as the convolution of their densities $p_{T_j/2^{1/\nu}}(t)$:

$$\Psi_\nu^{(2)}(t) = p_{T_2/2^{1/\nu}} * p_{T_1/2^{1/\nu}}(t) = \int_0^t p_{T_2/2^{1/\nu}}(t-t')p_{T_1/2^{1/\nu}}(t')dt'.$$

Taking into account the relation

$$p_{T_j/2^{1/\nu}}(t) = 2^{1/\nu}\,\overset{\circ}{\psi}_\nu(2^{1/\nu}t),$$

and changing the variable of integration, we get

$$\Psi_\nu^{(2)}(t) = \int_0^{2^{1/\nu}t} 2^{1/\nu}\,\overset{\circ}{\psi}_\nu(2^{1/\nu}t-\tau)\,\overset{\circ}{\psi}_\nu(\tau)d\tau = \mathsf{E}2^{1/\nu}\,\overset{\circ}{\psi}_\nu(2^{1/\nu}t-T_1).$$

Observe that the density

$$\overset{\circ}{\psi}_\nu(2^{1/\nu}t-T_1) = \frac{1}{2^{1/\nu}t-T_1}\int_0^\infty \phi_\nu((2^{1/\nu}t-T_1)/x)e^{-x}dx$$

can be expressed as

$$\overset{\circ}{\psi}_\nu(2^{1/\nu}t-T_1) = \mathsf{E}\left\{\frac{\mathbf{1}(2^{1/\nu}t-T_1)}{(2^{1/\nu}t-T_1)}\phi_\nu((2^{1/\nu}t-T_1)/E)\right\},$$

where $E = |\ln U|$ and

$$\mathbf{1}(x) = \begin{cases} 0, & x \leq 0; \\ 1, & x > 0. \end{cases}$$

Consequently, the value

$$\Psi_\nu^{(2)}(t) = \mathsf{E}\left\{ \frac{\mathbf{1}(2^{1/\nu}t - T_1)}{(2^{1/\nu}t - T_1)} 2^{1/\nu} \phi_\nu((2^{1/\nu}t - T_1)/E) \right\}$$

can be estimated by

$$\widehat{\Psi}_\nu^{(2)}(t) = \frac{1}{N} \sum_{j=1}^{N} \left\{ \frac{\mathbf{1}(2^{1/\nu}t - T_1)}{(2^{1/\nu}t - T_1)} 2^{1/\nu} \phi_\nu((2^{1/\nu}t - T_1)/E) \right\}_j$$

where T_1 and E are independent random variables.

Continuing in a similar fashion for an arbitrary integer n, yields

$$\Psi_\nu^{(n)}(t) = \mathsf{E}\left\{ \frac{\mathbf{1}(n^{1/\nu}t - S_{n-1})}{(n^{1/\nu}t - S_{n-1})} n^{1/\nu} \phi_\nu(n^{1/\nu}t - S_{n-1})/E) \right\}, \tag{3.10}$$

where $S_{n-1} = T_1 + T_2 + \cdots + T_{n-1}$. Thus, the corresponding estimator takes the form

$$\widehat{\Psi}_\nu^{(n)}(t) = \frac{1}{N} \sum_{j=1}^{N} \left\{ \frac{\mathbf{1}(n^{1/\nu}t - S_{n-1})}{(n^{1/\nu}t - S_{n-1})} n^{1/\nu} \phi_\nu((n^{1/\nu}t - S_{n-1})/E) \right\}_j. \tag{3.11}$$

Using estimator (3.11), we computed the distributions $\Psi_\nu^{(n)}(t)$ for various n, with $\nu = 0.5$. Figure 3.5 presents the corresponding estimates (in log-log scale) of the limiting distribution of the scaled nth arrival time. The figure also illustrates that for $n = 30$, the simulated scaled nth arrival time distribution approaches the true limiting stable distribution corresponding to $\nu = 0.5$. Observe that the roughness of the curve estimates is caused by the infinite variance of the random times.

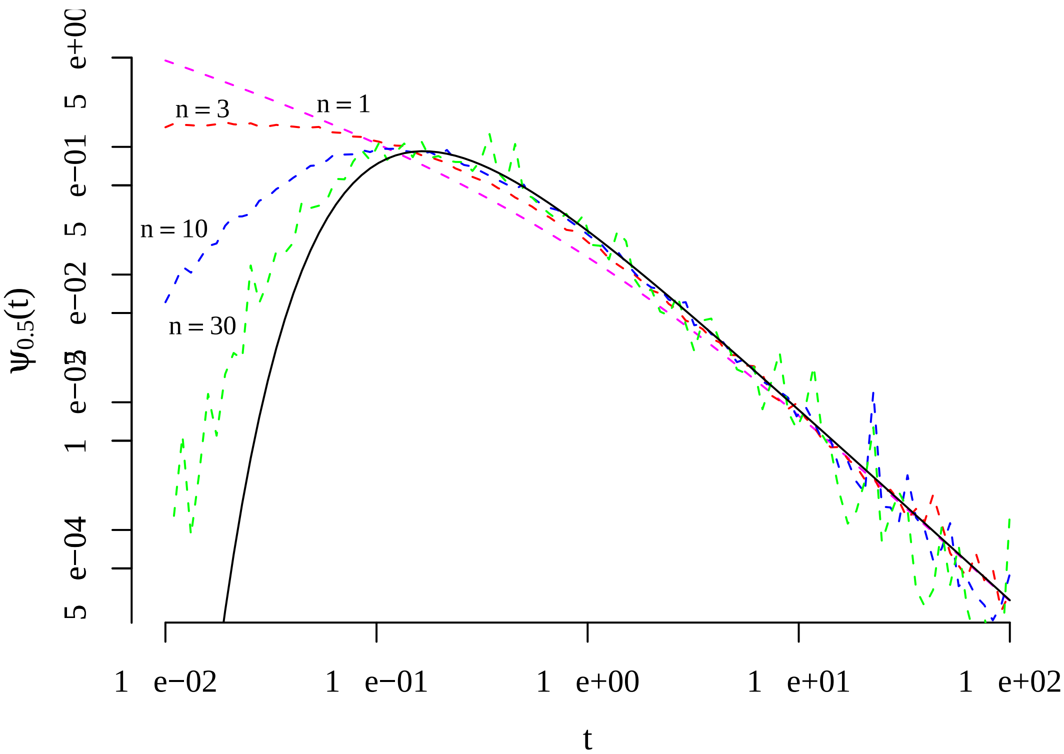

Figure 3.5: *Scaled nth fPp arrival time distributions (3.11) corresponding to* $\nu = 0.5, n = 1, 3, 10, 30,$ *and* $\mu = 1$ *(log-log scale).*

3.5 Intermittency

Having the algorithm for simulating interarrival times, we can study special properties of the fractional Poisson process that are difficult to analyze analytically.

Let us consider the unit-length time interval (0, 1), divide it into B bins, each of length τ, and present the distribution of events or arrivals using a histogram. In this particular investigation, we consider 50 bins only. With a standard Poisson process having intensity rate μ, we observe some bins to be empty if $\mu\tau$ is of order 1, and all bins to be completely filled if $\mu\tau \gg 1$ (Figure 3.6, leftmost panel). Also, the coefficient of variation (CV) or relative fluctuation (ratio of the standard deviation to the first moment) of the random number of events is given by $1/\sqrt{\mu\tau}$, which tends to 0 as $\mu\tau \to \infty$, i.e., the distribution of events over the bins looks almost uniform.

Additionally, simulating fPp reveals its more significant property: the proportion of empty bins does not vanish with μ, but tends to a finite limit depending on the order ν (Figure 3.6, center and rightmost panel). In other words, fractional Poisson processes have empty bins at all scales even if the total number of events on the interval under consideration becomes very large. These events form clusters on the time axis with noticeable voids between them. Such behavior known as intermittency can be studied using different techniques (*Botet and Ploszajczak*, 2002). In this manuscript, we propose a simple measure to describe intermittency of fPp. We introduce $R(\nu)$, which measures the proportion of empty bins. The proportion $R(\nu)$ is specifically defined as the ratio of mean number of empty bins ($\mathsf{E}N_B(\nu)$) to the total number of bins (B), i.e., $R(\nu) = \mathsf{E}N_B(\nu)/B$. Figure 3.7 shows that $R(\nu)$ is a decreasing function of ν, and smoothly falls between zero (almost all bins are filled when fPp is close to the standard Poisson process) and one (almost all bins are empty when ν is close to 0).

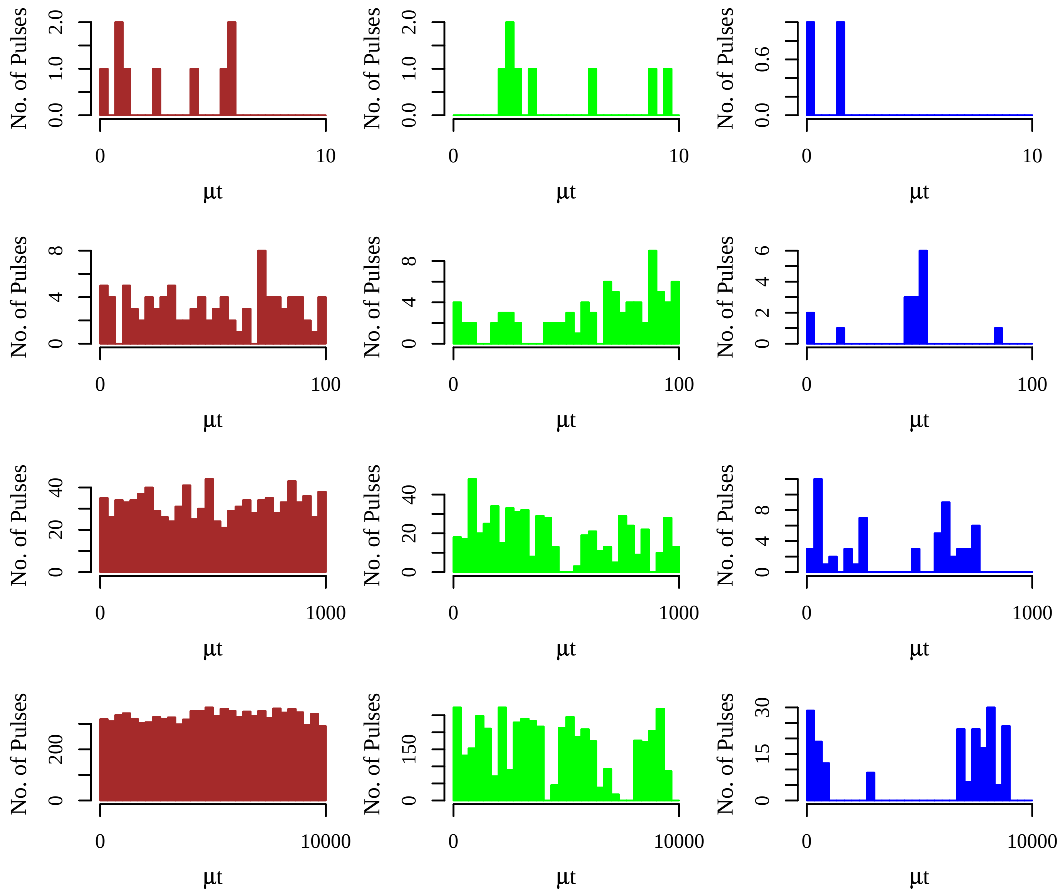

Figure 3.6: *Histograms for standard Poisson process (leftmost panel) and fractional Poisson processes of orders* $\nu = 0.9$ & 0.5 *(middle and rightmost panels).*

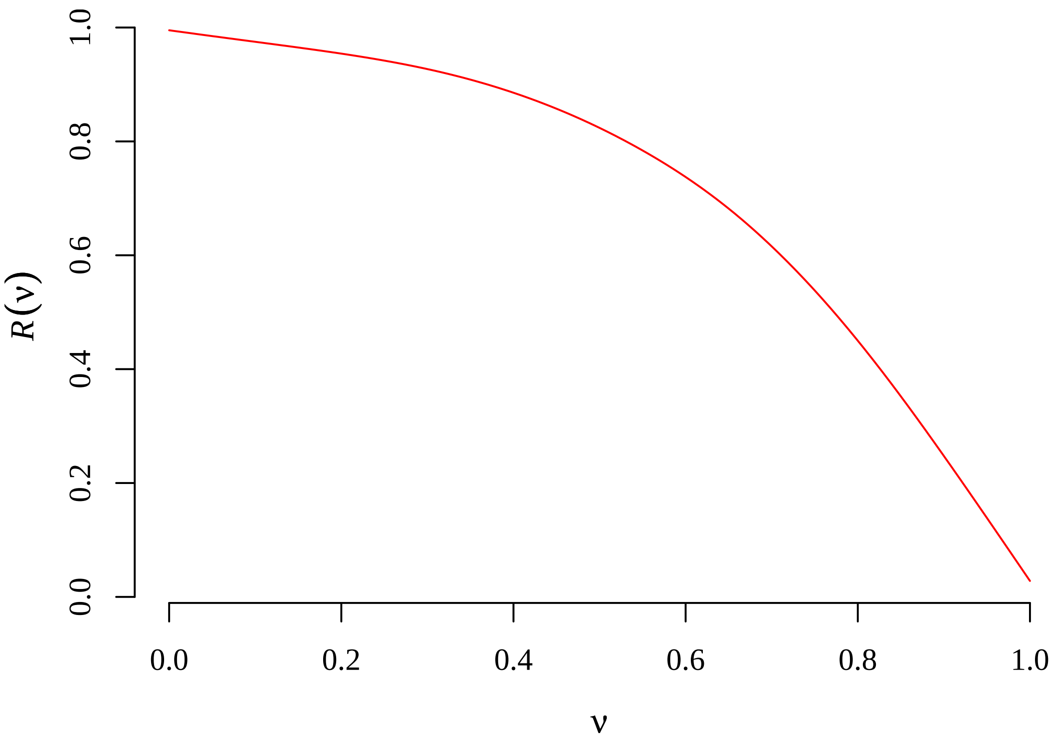

Figure 3.7: *The limit proportion of empty bins $R(\nu)$ using a total of B=50 bins.*

3.6 Stationarity and Dependence of Increments

In this section, we investigate the stationarity and dependence structure of the fPp increments. Consider $t_1 < t_2$,

$$\begin{aligned}\sigma_{\Delta_1,\Delta_2} = cov\Big\{N_\nu(t_1) - N_\nu(0), N_\nu(t_2) - N_\nu(t_1)\Big\} &= cov\Big\{N_\nu(t_1), N_\nu(t_2) - N_\nu(t_1)\Big\} \\ &= \mathsf{E}\big[N_\nu(t_1)\big(N_\nu(t_2) - N_\nu(t_1)\big)\big] \\ &- \mathsf{E}N_\nu(t_1)\mathsf{E}\big(N_\nu(t_2) - N_\nu(t_1)\big) \\ &= cov\big\{N_\nu(t_1), N_\nu(t_2)\big\} - \sigma^2_{N_\nu(t_1)} \\ &= \sigma_{12} - \sigma^2_{N_\nu(t_1)}.\end{aligned}$$

It is clear that when $\nu = 1$ (corresponding to the ordinary Poisson process), the covariance above is zero. Figure 3.8 below demonstrates the dependence structure of the fPp increments as $t_1 \to t_2$, where $t_2 = 300$, and $\nu < 1$.

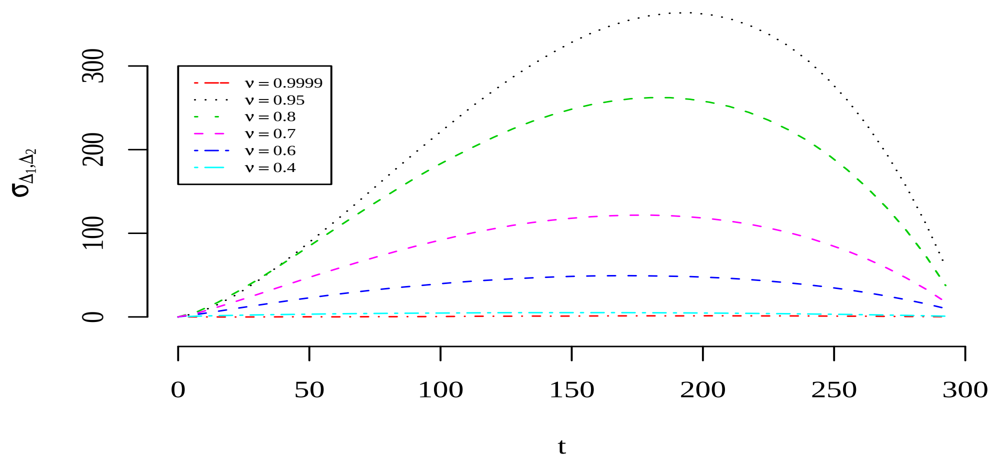

Figure 3.8: *Dependence structure of the fPp increments for fractional orders* $\nu = 0.4, 0.6, 0.7, 0.8, 0.95,$ *and* 0.99999*, with* $\mu = 1$*.*

Furthermore, recall (*Samorodnitsky and Taqqu* (1994)) that a real-valued process $\{N_\nu(t), t \in \mathbb{S}\}$ has stationary increments if

$$\big\{N_\nu(t) - N_\nu(0), t \in \mathbb{S}\big\} \stackrel{d}{=} \big\{N_\nu(t+\Delta t) - N_\nu(\Delta t), t \in \mathbb{S}\big\}, \qquad \text{for all } \ \Delta t \in \mathbb{S}.$$

Considering the sampling times $t_1 = 600, t_2 = 1200$, $t_3 = 1800$, and $t_4 = 2400$, we estimate the distribution of the increments

$$\big\{N_\nu(t_1) - N_\nu(0)\big\}, \big\{N_\nu(t_2) - N_\nu(t_1)\big\}, \big\{N_\nu(t_3) - N_\nu(t_2)\big\}, \text{ and } \big\{N_\nu(t_4) - N_\nu(t_3)\big\}$$

using a histogram, where

$$N_\nu(t_{j+1}) - N_\nu(t_j) = \sum_{n=1}^{\infty} \boldsymbol{I}_{[S_n,\infty)}(t_{j+1}) \boldsymbol{I}_{[0,S_n)}(t_j), \quad 0 \le t_j < t_{j+1} < \infty.$$

Clearly, $N_\nu(t_{j+1}) - N_\nu(t_j)$ counts the random times S_n that occur between fixed times t_{j+1} and t_j. This further implies that $\Delta t = t_1 = 600$. We now compare the simulated distributions of the increments visually as most goodness-of-fit tests assume independence, and are dependent on the binning scheme. Please note also that the existence of the zero frequencies highly depend on the binning procedure, and their positions over the bins are not fixed a priori. These make *chi*-square and Anderson-Darling tests (see *Scholz and Stephens* (1987); *Best* (1994)) trickier to use. Below are the estimated distributions of the above four increments for $\nu = 0.6, 0.8$, and 0.99999 using a sample of size $n = 5000$. For ν close to one, we see a process that has nearly stationary increments (see Figure 3.9). In contrast, Figures 3.10 and 3.11 (corresponding to $\nu = 0.6, 0.8$) indicate nonstationarity of the fPp increments. Notice also that the distribution of the increments seems to converge to $\delta(N_\nu(t + \Delta t) - N_\nu(\Delta t))$ as the sampling interval $(t_j, , t_{j+1}]$ shifts away from the origin $t_0 = 0$.

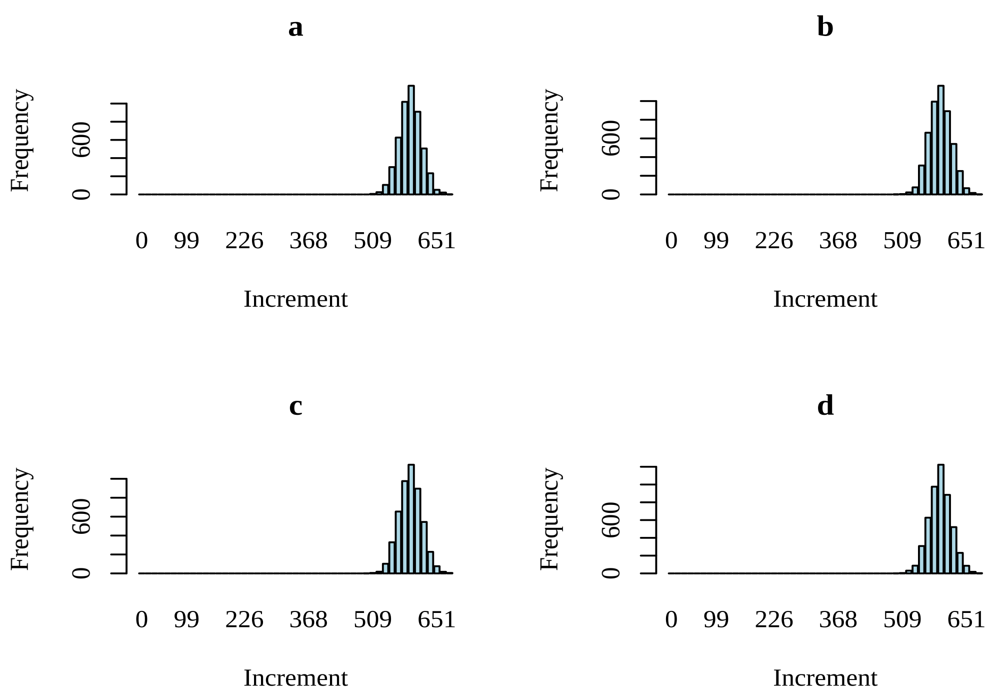

Figure 3.9: *Distribution of the fPp increments on the sampling intervals a)*$[0, 600]$*, b)*$(600, 1200]$*, c)* $(1200, 1800]$*, and* $(1800, 2400]$ *corresponding to fractional order* $\nu = 0.99999$*, and* $\mu = 1$*.*

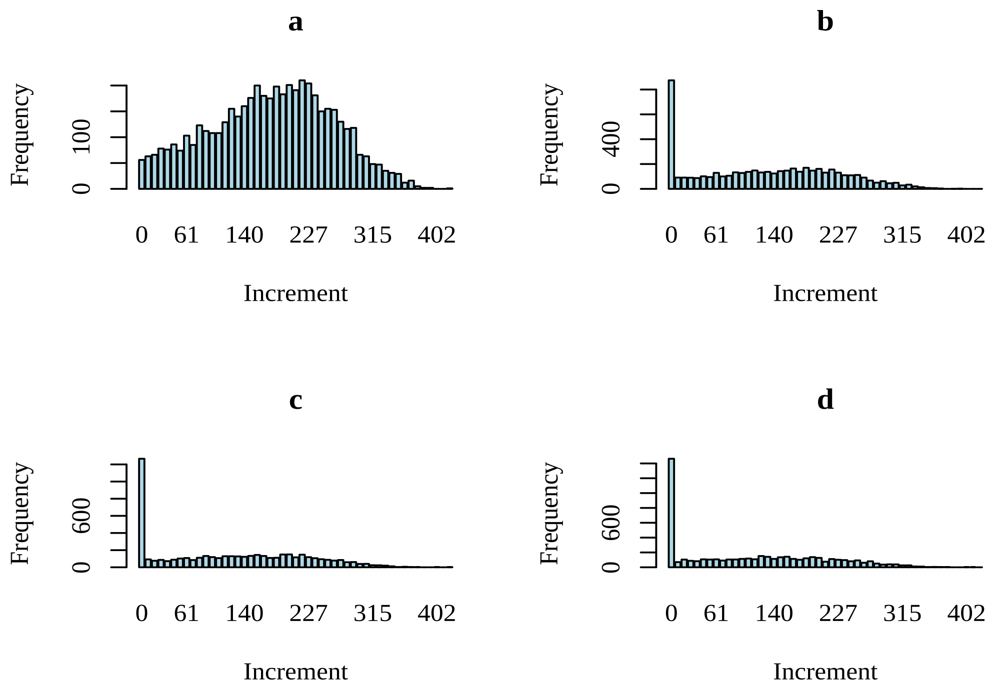

Figure 3.10: *Distribution of the fPp increments on the sampling intervals a)*$[0, 600]$*, b)*$(600, 1200]$*, c)* $(1200, 1800]$*, and* $(1800, 2400]$ *corresponding to fractional order* $\nu = 0.8$*, and* $\mu = 1$*.*

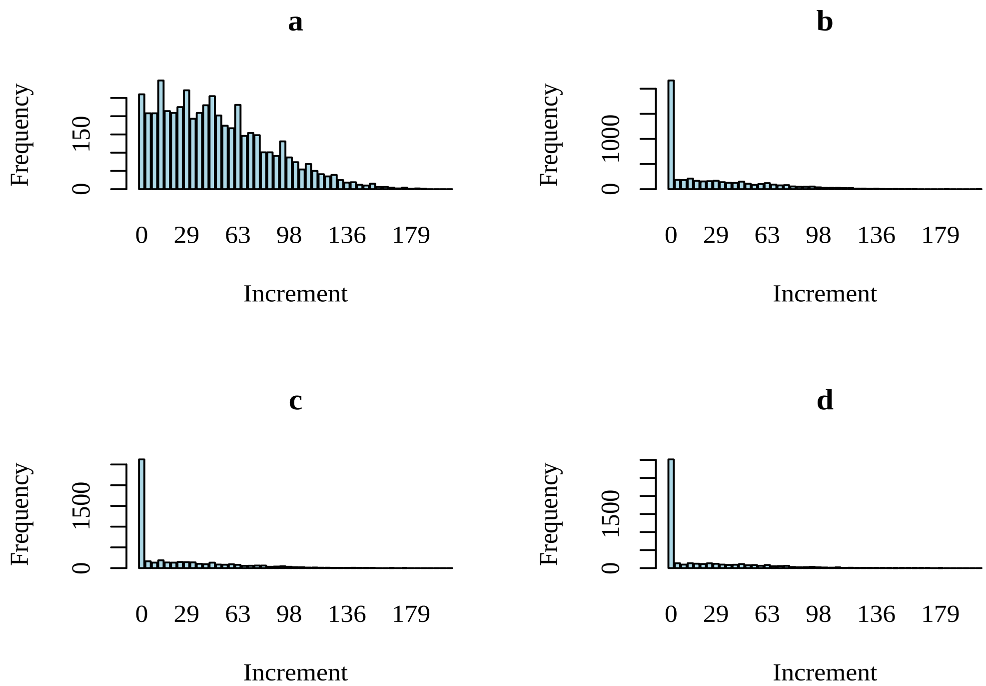

Figure 3.11: *Distribution of the fPp increments on the sampling intervals a)*$[0, 600]$*, b)*$(600, 1200]$*, c)* $(1200, 1800]$*, and* $(1800, 2400]$ *corresponding to fractional order* $\nu = 0.6$*, and* $\mu = 1$*.*

3.7 Covariance Structure and Self-Similarity

The covariance between two random variables $N_\nu(t_1)$ and $N_\nu(t_2)$ is defined as

$$\sigma_{12} = cov\big\{N_\nu(t_1), N_\nu(t_2)\big\} = \mathsf{E}\left[N_\nu(t_1)N_\nu(t_2)\right] - \mathsf{E}\left[N_\nu(t_1)\right]\mathsf{E}\left[N_\nu(t_2)\right].$$

A natural estimator of σ_{12} would then be

$$\begin{aligned}\widehat{\sigma}_{12} &= \frac{1}{n}\sum_{j=1}^{n} N_{\nu j}(t_1)N_{\nu j}(t_2) - \left(\frac{\sum_{j=1}^{n} N_{\nu j}(t_1)}{n}\right)\left(\frac{\sum_{k=1}^{n} N_{\nu k}(t_2)}{n}\right)\\ &= \frac{1}{n}\sum_{j=1}^{n} N_{\nu j}(t_1)N_{\nu j}(t_2) - \widehat{\mu}_{N_\nu(t_1)}\widehat{\mu}_{N_\nu(t_2)}.\end{aligned}$$

In addition, we model the covariance structure by fitting the function at^b, where parameters a and b are estimated using simple linear regression. Table 3.3 displays the parameter estimates for different ν's while Figure 3.12 illustrates b as a function of ν, with $\mu = 1$. Finding the theoretical functional dependence of b on ν would require further study. Moreover, Figure 3.13 below presents the simulated covariance (log-log scale) with the fitted model as t_1 approaches $t_2 = 300$, where $t_1 < t_2$. We generally see good fits corresponding to various ν's for the particular time interval $(0, 300)$.

Table 3.3: *Parameter estimates of the fitted model at^b, $\mu = 1$.*

ν	fitted b	fitted a
0.05	0.0777	2.0580
0.10	0.1595	2.1042
0.20	0.3298	2.1194
0.30	0.5061	2.0820
0.40	0.68412	1.9842
0.50	0.8534	1.9082
0.60	1.0264	1.7234
0.70	1.1857	1.5346
0.80	1.3331	3.7929
0.90	1.4560	0.9243
0.95	1.4761	4.3757
0.9999	1.0106	2.7471

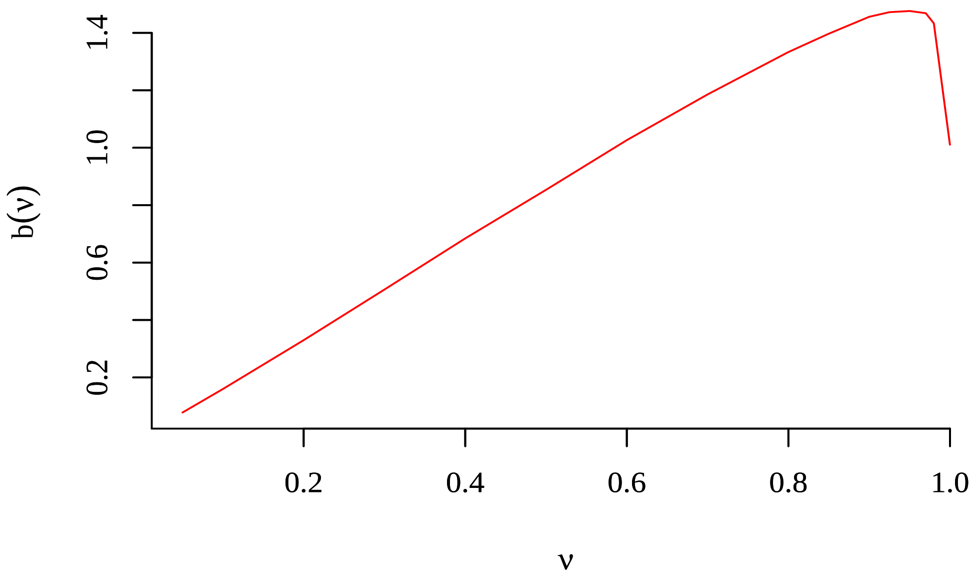

Figure 3.12: *The parameter estimate b as a function of ν, with $\mu = 1$.*

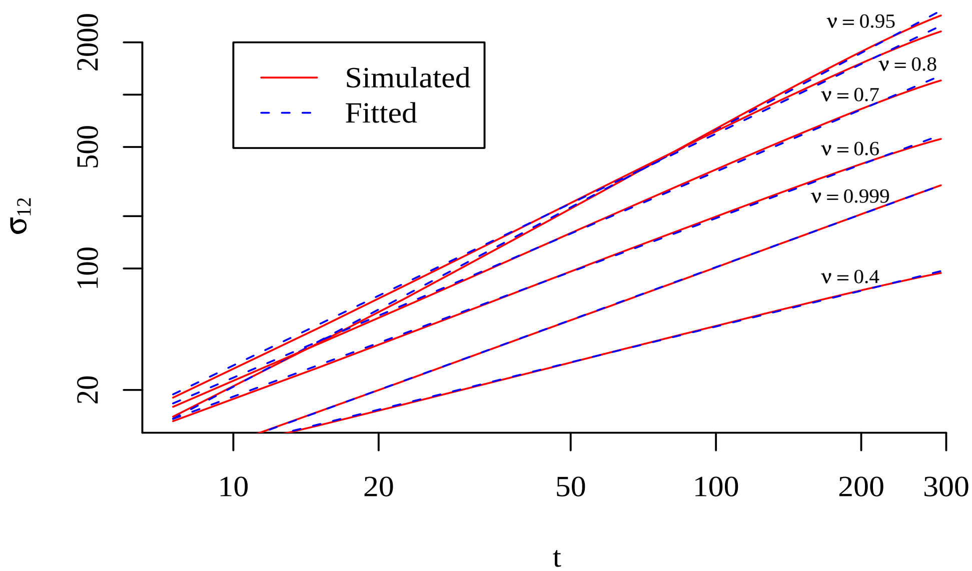

Figure 3.13: *The function at^b fitted to the simulated covariance of fPp for different fractional order ν, with $\mu = 1$.*

Below are the simulated two-dimensional covariance structures of fPp corresponding to several ν's.

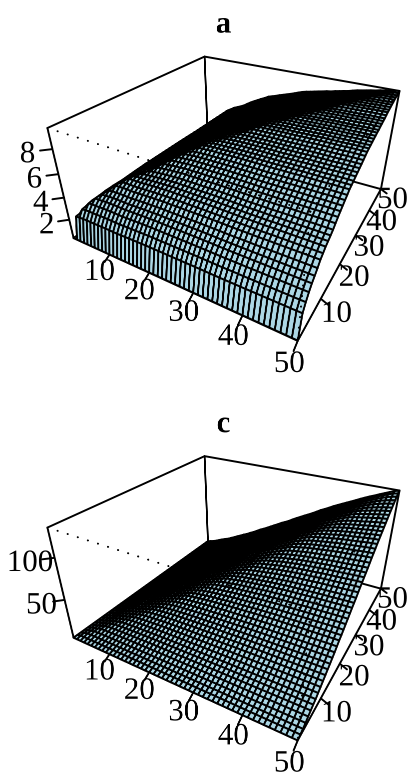

Figure 3.14: *Two-dimensional covariance structure of fPp for fractional orders a) $\nu = 0.25$, b) $\nu = 0.5$, c) $\nu = 0.75$, and d) $\nu = 1$, with $\mu = 1$.*

On the other hand, there are several but non-equivalent definitions of self-similarity in the probabilistic sense. In this subsection, we try to explore the kind of self-similarity property that fPp possesses. The standard definition says that a continuous-time process $\{N_\nu(t), t \geq 0\}$ is self-similar if

$$N_\nu(at) \stackrel{d}{=} a^D N_\nu(t), \qquad \text{for all } a \geq 0,$$

where $D, 0 < D < 1$ is the self-similarity index, and "$\stackrel{d}{=}$" refers to the equality of finite-dimensional distributions. Estimating the self-similarity exponent is going to be difficult in this case as fPp has nonstationary and dependent increments. *Leland et al.* (1994), *Weron et al.* (2005), and *Beran* (1994) have more details on estimating the self-similarity exponent in a time series setting. Nonetheless, we clearly see that fPp has second-order self-similarity property, i.e.,

$$\mathsf{E}\big[N_\nu(at)\big] = a^\nu \mathsf{E}\big[N_\nu(t)\big],$$

and

$$\mathsf{E}\left[\left(N_\nu(at) - \mu_{N_\nu(at)}\right)^2\right] = a^{2\nu}\mathsf{E}\left[\left(N_\nu(t) - \mu_{N_\nu(t)}\right)^2\right].$$

This property follows from Corollary 3.2 of *Houdré and Kawai* (2005).

3.8 Limiting Scaled Fractional Poisson Distribution

In the case of the standard Poisson process, the probability distribution for the random number $N(t)$ of events by time t obeys the Poisson law with $\mathsf{E}N(t) \equiv \overline{n} = \mu t$, that approaches to a normal law at large $\overline{n}$. That is,

$$P_n(t) = \frac{(\mu t)^n}{n!}e^{-\mu t} = \frac{(\overline{n})^n}{n!}e^{-\overline{n}} \quad \stackrel{\overline{n}\to\infty}{\longrightarrow} \quad (2\pi\overline{n})^{-1/2}\exp\left\{-\frac{(n-\overline{n})^2}{2\overline{n}}\right\}.$$

Recall that a generalization of the Poisson distribution is given by (2.25). Introducing the quasi-continuous random variable $Z = N(t)/\overline{n}$, we can easily deduce that

$$\begin{aligned} f(z;\overline{n}) &= \overline{n}\frac{\overline{n}^{\overline{n}z}}{\Gamma(\overline{n}z+1)}e^{-\overline{n}} \\ &\sim \sqrt{\frac{\overline{n}}{2\pi}}\exp\left\{-\frac{(z-1)^2}{2/\overline{n}}\right\} \\ &\to \delta(z-1), \qquad \overline{n}\to\infty. \end{aligned}$$

Lemma. *Let $Z = N_\nu(t)/\overline{n}_\nu$ be the scaled fPp random variable where $\mathsf{E}N_\nu(t) \equiv \overline{n}_\nu = \frac{\mu t^\nu}{\Gamma(\nu+1)}$. Then*

$$\mathsf{E}s^{N_\nu(t)} = \mathsf{E}e^{-\lambda Z} \sim E_\nu(-\lambda'), \qquad \lambda' = \lambda\Gamma(\nu+1),$$

where $\lambda = -\overline{n}\ln s$, and $E_\nu(z)$ is given by the Mittag-Leffler function

$$E_\nu(z) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\nu n+1)}.$$

Proof. Consider the distribution of $N_\nu(t)$ for fPp (see Table 3.1). The corresponding generating function has the form

$$\begin{aligned}
G_\nu(s,t) \equiv \mathsf{E}s^{N_\nu(t)} &= \sum_{n=0}^{\infty} s^n P_n^\nu(t) \\
&= \sum_{n=0}^{\infty} s^n \times \frac{(\mu t^\nu)^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\mu t^\nu)^k}{\Gamma(\nu(k+n)+1)} \\
&= \sum_{n=0}^{\infty} \frac{(s\mu t^\nu)^n}{n!} \sum_{k=0}^{\infty} \frac{(k+n)!}{k!} \frac{(-\mu t^\nu)^k}{\Gamma(\nu(k+n)+1)} \\
&= \sum_{k=0}^{\infty} \frac{k!}{\Gamma(\nu k+1)} \sum_{n=0}^{k} \frac{[s\mu t^\nu]^n(-\mu t^\nu)^{k-n}}{n!(k-n)!} \\
&= E_\nu(\mu t^\nu(s-1)) \\
&= E_\nu((s-1)\Gamma(\nu+1)\overline{n}_\nu),
\end{aligned}$$

as $\overline{n}_\nu = \mu t^\nu/\Gamma(1+\nu)$. Introducing the scaled random variable $Z = N_\nu(t)/\overline{n}_\nu$, and new parameter $\lambda = -\overline{n}\ln s$, we get the generating function

$$\mathsf{E}s^{N_\nu(t)} = \mathsf{E}e^{-\lambda Z} = E_\nu((e^{-\lambda/\overline{n}_\nu}-1)\Gamma(\nu+1)\overline{n}_\nu).$$

At large $\overline{n}_\nu$ (pertaining to large time t),

$$\begin{aligned}
(e^{-\lambda/\overline{n}_\nu}-1)\Gamma(\nu+1)\overline{n}_\nu &= e^{-\lambda/\overline{n}_\nu}\Gamma(\nu+1)\overline{n}_\nu - \Gamma(\nu+1)\overline{n}_\nu \\
&= \left(1 - \frac{\lambda}{\overline{n}_\nu} + \frac{(\lambda/\overline{n}_\nu)^2}{2!} - \cdots\right)\Gamma(\nu+1)\overline{n}_\nu - \Gamma(\nu+1)\overline{n}_\nu \\
&= -\lambda\Gamma(\nu+1) + \left[\frac{(\lambda/\overline{n}_\nu)^2}{2!} - \cdots\right]\Gamma(\nu+1)\overline{n}_\nu \\
&\sim -\lambda\Gamma(\nu+1).
\end{aligned}$$

Thus,

$$\mathsf{E}e^{-\lambda Z} \sim E_\nu(-\lambda'), \qquad \lambda' = \lambda\Gamma(\nu+1). \ \blacksquare$$

Theorem. *Let* $Z = N_\nu(t)/\overline{n}_\nu$ *be the scaled fPp random variable where* $\overline{n}_\nu = \frac{\mu t^\nu}{\Gamma(\nu+1)}$. *Then*

$$f_\nu(z;\overline{n}_\nu) \overset{\overline{n}_\nu\to\infty}{\longrightarrow} f_\nu(z) = \frac{[\Gamma(\nu+1)]^{1/\nu}}{\nu} g^{(\nu)}\left(\left(\frac{z}{\Gamma(\nu+1)}\right)^{-1/\nu}\right) z^{-1-1/\nu}. \tag{3.12}$$

Proof. From the lemma above, we have

$$\mathsf{E}e^{-\lambda Z} \equiv \int_0^\infty e^{-\lambda z} f_\nu(z)dz \sim E_\nu(-\lambda'), \qquad \lambda' = \lambda\Gamma(\nu+1). \tag{3.13}$$

Comparing equation (3.13) with formula (6.9.8) of *Uchaikin and Zolotarev* (1999),

$$\begin{aligned} E_\nu(-\lambda') &= \nu^{-1}\int_0^\infty \exp(-\lambda' x) g^{(\nu)}(x^{-1/\nu})x^{-1-1/\nu}dx \\ &= \nu^{-1}\int_0^\infty \exp(-\lambda\Gamma(\nu+1)x) g^{(\nu)}(x^{-1/\nu})x^{-1-1/\nu}dx. \end{aligned}$$

Letting $z = \Gamma(\nu+1)x$, we obtain

$$E_\nu(-\lambda') = \int_0^\infty e^{-\lambda z}\left\{\frac{[\Gamma(\nu+1)]^{1/\nu}}{\nu} g^{(\nu)}\left(\left(\frac{z}{\Gamma(\nu+1)}\right)^{-1/\nu}\right) z^{-1-1/\nu}\right\} dz,$$

which shows that the random variable Z has a non-degenerate limit distribution (3.12) as $\overline{n}_\nu \to \infty$ (see also *Uchaikin* (1999)). Hence,

$$f_\nu(z;\overline{n}_\nu) \overset{\overline{n}_\nu\to\infty}{\longrightarrow} f_\nu(z) = \frac{[\Gamma(\nu+1)]^{1/\nu}}{\nu} g^{(\nu)}\left(\left(\frac{z}{\Gamma(\nu+1)}\right)^{-1/\nu}\right) z^{-1-1/\nu}. \ \blacksquare$$

The moments can be calculated (using (A.2)) as

$$\mathsf{E}Z^k = \frac{1}{\nu}\int_0^\infty z^k \left\{\frac{[\Gamma(\nu+1)]^{1/\nu}}{\nu} g^{(\nu)}\left(\left(\frac{z}{\Gamma(\nu+1)}\right)^{-1/\nu}\right) z^{-1-1/\nu}\right\} dz.$$

Letting $y = (z/\Gamma(\nu+1))^{-1/v}$, we get

$$\begin{aligned}\mathsf{E}Z^k &= [\Gamma(\nu+1)]^k \int\limits_0^\infty y^{-\nu k} g^{(\nu)}(y)dy \\ &= \frac{[\Gamma(1+\nu)]^k \Gamma(1+k)}{\Gamma(1+k\nu)}.\end{aligned}$$

Moreover, to verify the formula

$$f_\nu(z) = \sum_{k=0}^\infty \frac{(-z)^k}{k!\Gamma(1-(k+1)\nu)[\Gamma(\nu+1)]^{k+1}}, \tag{3.14}$$

we can apply the asymptotic formula of the Mittag-Leffler function, which is given by

$$E_\nu(-x) \sim -\sum_{n=0}^\infty \frac{(-x)^n}{\Gamma(1-n\nu)}.$$

Getting the Laplace transform of the density (3.14),

$$\begin{aligned}\int_0^\infty e^{-\lambda z} f_\nu(z)dz &= \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(1-(k+1)\nu)[\Gamma(\nu+1)]^{k+1}} \int_0^\infty z^k e^{-\lambda z} dz \\ &= \sum_{k=0}^\infty \frac{(-1)^k}{k!\Gamma(1-(k+1)\nu)[\Gamma(\nu+1)]^{k+1}} \left(\frac{1}{\lambda} k! \frac{1}{\lambda^k}\right) \\ &\overset{n=k+1}{=} \sum_{n=1}^\infty \frac{(-1)^{n-1}}{\Gamma(1-n\nu)\,[\Gamma(1+\nu)\lambda]^n} \\ &= -\sum_{n=1}^\infty \frac{[-\lambda\Gamma(\nu+1)]^{-n}}{\Gamma(1-n\nu)} \\ &= E_\nu(-\lambda\Gamma(\nu+1)), \qquad (3.15)\end{aligned}$$

we arrive at equation (3.13).

As $z \to 0$ (using (A.4)), we obtain

$$
\begin{aligned}
f_\nu(z) &= \frac{[\Gamma(\nu+1)]^{1/\nu}}{\nu} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \frac{n\nu}{\Gamma(1-n\nu)} \left[\left(\frac{z}{\Gamma(\nu+1)} \right)^{-1/\nu} \right]^{-n\nu-1} z^{-1-1/\nu} \\
&= \frac{1}{\Gamma(\nu+1)} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{(n-1)!} \frac{1}{\Gamma(1-n\nu)} \left[\frac{z}{\Gamma(\nu+1)} \right]^{n-1} \\
&\overset{k=n-1}{=} \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(1-(k+1)\nu)[\Gamma(\nu+1)]^{k+1}} \\
&\to f_\nu(0) = \frac{1}{\Gamma(1+\nu)\Gamma(1-\nu)} = \frac{\sin(\nu\pi)}{\nu\pi}.
\end{aligned}
\tag{3.16}
$$

We also see that, $\mathsf{E}Z = 1$, and $\mathsf{E}Z^2 = 2\nu \mathrm{B}(\nu, 1+\nu)$, so that the relative fluctuation, or CV of the random variable Z can be computed as

$$
\delta_\nu \equiv \sigma_Z/\mathsf{E}Z = \sqrt{2\nu \mathrm{B}(\nu, 1+\nu) - 1} = \begin{cases} 1, \ \ \nu = 0, \\ \sqrt{\pi/2} - 1, \ \ \nu = 1/2 \\ 0, \ \ \nu = 1. \end{cases}
$$

If $\nu = 1/2$ then we can obtain an explicit expression for $f_{1/2}(z)$. Using equation (3.12), we have

$$
\begin{aligned}
f_{1/2}(z) &= \frac{\left\{\Gamma\left(\frac{1}{2}+1\right)\right\}^2}{\frac{1}{2}} g^{(1/2)} \left(\left(\frac{z}{\Gamma(\frac{1}{2}+1)} \right)^{-2} \right) z^{-3}. \\
&= \frac{\pi}{2} g^{(1/2)} \left(\left(\frac{z}{\sqrt{\pi}/2} \right)^{-2} \right) z^{-3}.
\end{aligned}
$$

From equation (A.3),

$$
g^{(1/2)} \left(\left(\frac{z}{\sqrt{\pi}/2} \right)^{-2} \right) = 4 \frac{z^3}{\pi^2} e^{-z^2/\pi}.
$$

Thus, we get

$$
f_{1/2}(z) = \frac{2}{\pi} e^{-z^2/\pi}, \ \ z \geq 0.
$$

The family of these limit distributions (in log-log scale) is plotted in Figure 3.15. The values of density (3.12) for different ν's, and z's can be found in Appendix B.

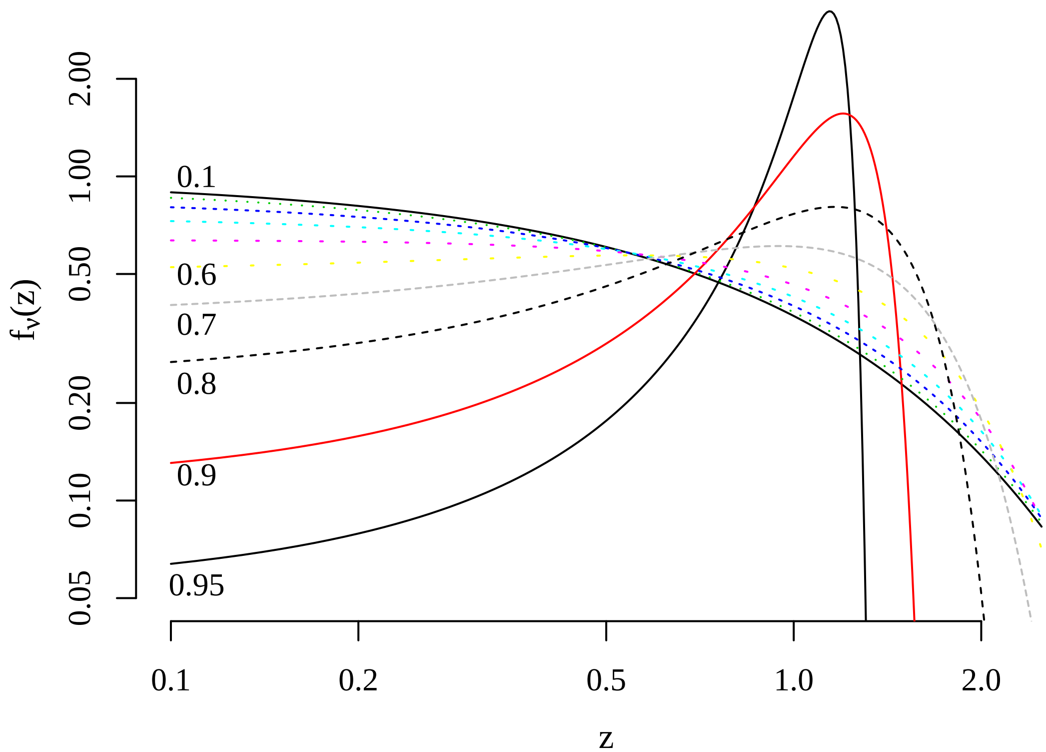

Figure 3.15: *Limiting distribution (3.12) for* ν*= 0.1(0.1)0.9 and 0.95, with* μ *=1.*

3.9 Alternative fPp

Another generalization based on the analogy with fractional Brownian motion can be formulated as follows: instead of the stochastic differential equation

$$\frac{d^\nu B_\nu}{dt^\nu} = W(t),$$

where $W(t)$ is a white Gaussian noise, we consider the equation

$$\frac{d^\nu Y_\nu}{dt^\nu} = X(t). \tag{3.17}$$

The random function $X(t)$ denotes the standard Poisson flow

$$X(t) = \sum_{j=1}^{\infty} \delta(t - T^{(j)}),$$

where $T^{(j)} = T_1 + T_2 + \dots T_j$, and $T_1, T_2, \dots T_j$ are independent random variables with common density

$$\psi(t) = \mu e^{-\mu t}, \qquad t \geq 0,\ \mu > 0.$$

Integrating the stochastic fractional differential equation (3.17) yields (*Kilbas et al.*, 2006)

$$\begin{aligned}
Y_\nu(t) &= \frac{1}{\Gamma(\nu)} \int_0^t \frac{X(\tau)d\tau}{(t-\tau)^{1-\nu}} \\
&= \frac{1}{\Gamma(\nu)} \sum_{j=1}^{N(t)} \int_0^t \frac{\delta(\tau - T^{(j)})d\tau}{(t-\tau)^{1-\nu}} \\
&= \sum_{j=1}^{N(t)} \frac{1}{\Gamma(\nu)} \frac{1}{(t-T^{(j)})_+^{1-\nu}}.
\end{aligned}$$

It is easy to see that, for $\nu = 1$, the process becomes the standard Poisson process. The stochastic process (3.17) can be interpreted as a resulting signal generated by the Poisson flow of pulses, each of which giving the contribution

$$A(t - T^{(j)}) = \frac{1}{\Gamma(\nu)(t - T^{(j)})_+^{1-\nu}}. \tag{3.18}$$

It is also well known that, when $N(t) = n$ (see *Ross* (1996)), the unordered random times $T^{(1)}, T^{(2)}, \dots, T^{(n)}$ at which events occur, are distributed independently and uniformly in the interval $(0,\ t)$. Therefore,

$$Y_\nu(t)|_{N(t)=n} = \sum_{j=1}^{n} A_j,$$

where A_j is determined by equation (3.18). Now,

$$\begin{aligned}
P(A_j > y) &= P\left(\Gamma(\nu)(t - T^{(j)})^{1-\nu} < y^{-1}\right) \\
&= P\left(t - T^{(j)} < [\Gamma(\nu)y]^{-1/(1-\nu)}\right) \\
&= P\left(T^{(j)} > t - [\Gamma(\nu)y]^{-1/(1-\nu)}\right) \\
&= P\left(T^{(j)} < [\Gamma(\nu)y]^{-1/(1-\nu)}\right) \\
&= \frac{1}{t\,[\Gamma(\nu)y]^{1/(1-\nu)}}.
\end{aligned}$$

Because $\nu > 0$, the expectation of A_j exists, and according to the law of large numbers, the limit distribution of the scaled random variable Z has the degenerate limit distribution $f_\nu(z) = \delta(z-1)$. Figure 3.16 illustrates the sample paths of standard Poisson process, fPp, and the alternative fPp.

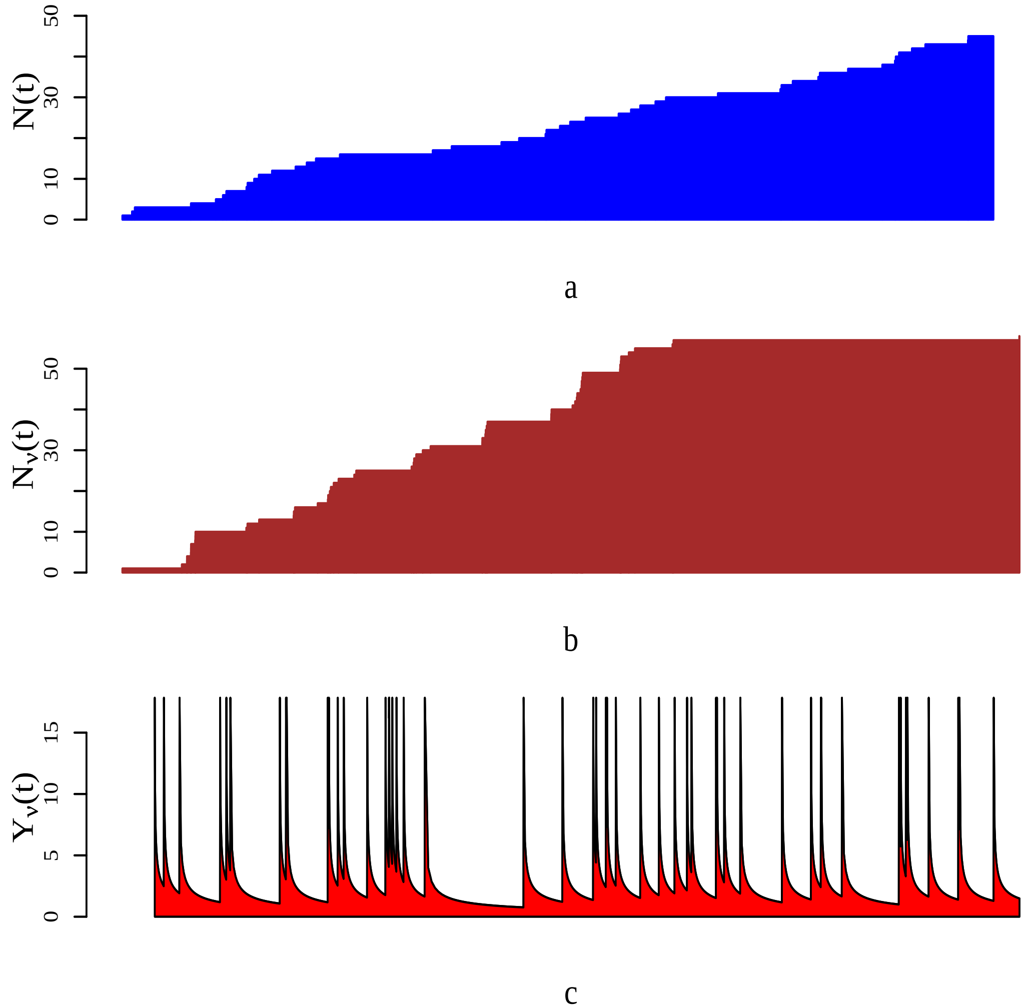

Figure 3.16: *Sample trajectories of (a) standard Poisson process, (b) fPp, and (c) the alternative fPp generated by stochastic fractional differential equation (3.17), with* $\nu = 0.5$.

Chapter 4

Estimation

In the succeeding discussion, we propose parameter estimators and establish some important statistical properties. We also construct alternative parameter estimators, which are hoped to be improved versions (in some sense) of the original estimators. We test and compare these estimators using synthetic data.

4.1 Method of Moments

We derive method-of-moments estimators for parameters ν, and μ, based on the first two moments of a transformed random variable T. It is important to emphasize that the *Hill* (1975), *Pickands* (1975), and *Haan and Resnick* (1980) estimators can be used to estimate these parameters as well. However, the above estimators are using only a portion of the data making these estimators statistically questionable. It is this drawback that motivates us to look for estimators that utilize, or even optimize the use, of all available data or information.

Recall that

$$T \stackrel{d}{=} \frac{|\ln U|^{1/\nu}}{\mu^{1/\nu}} S(\nu), \tag{4.1}$$

where U has $U(0,\ 1)$ distribution, $S(\nu)$ is one-sided $\alpha-$stable, and the random variables U and $S(\nu)$ are statistically independent. Since the first moment doesn't exist, we consider the log-transformation of the absolute value of the original random variable T. But $T > 0$, hence the absolute sign can be omitted.

The formulation (4.1) above implies that

$$\ln(T) \stackrel{d}{=} \ln\left(\frac{|\ln U|^{1/\nu}}{\mu^{1/\nu}} S(\nu)\right). \tag{4.2}$$

Simplifying (4.2), we get the equivalent expression

$$\ln(T) \stackrel{d}{=} \frac{1}{\nu}\ln\left(\frac{|\ln U|}{\mu}\right) + \ln(S(\nu)). \tag{4.3}$$

Taking the expectation of (4.3), we obtain the equality

$$\mathsf{E}\ln(T) = \frac{1}{\nu}\big[\mathsf{E}\ln(|\ln U|) - \ln(\mu)\big] + \mathsf{E}\ln(S(\nu)). \tag{4.4}$$

Our task now is to obtain the first moments of the random variables $\ln(|\ln U|)$ and $\ln(S(\nu))$. In doing so, we start by finding the distribution of the former random variable. Let $Y = |\ln U| = -\ln U$. The random variable Y is known to have the distribution e^{-y}, $y > 0$. Then using a standard statistical technique in finding the distribution of the monotone transformation $X = \ln Y$, we can easily show that X has the probability density function

$$f_X(x) = e^{x-e^x}, \qquad x \in \mathbb{R}.$$

Thus, the first moment of $\ln(|\ln U|)$ can be calculated now as

$$\mathsf{E}X = \int_{\mathbb{R}} x e^{x-e^x} dx = \int_{\mathbb{R}+} \ln(y) e^{-y} dy = -\mathbb{C}. \tag{4.5}$$

If we let $y = e^x$ then the above equality becomes the well-known integral formula involving Euler's constant $\mathbb{C} \cong 0.57721566490153286$. We omit the proof here as it can be found in many related sources (see *Boros and Moll* (2004)).

The next step is to find the expectation of $\ln(S(\nu))$. *Zolotarev* (1986, p. 213-220) shows that

$$\mathsf{E}\ln(S(\nu)) = \mathbb{C}\left(\frac{1}{\nu} - 1\right). \tag{4.6}$$

When (4.5) and (4.6) are substituted into (4.4), the equality becomes

$$\mathsf{E}\ln(T) = \frac{1}{\nu}\left((-\mathbb{C}) - \ln(\mu)\right) + \mathbb{C}\left(\frac{1}{\nu} - 1\right) = -\frac{\ln(\mu)}{\nu} - \mathbb{C}. \tag{4.7}$$

From equation (4.7), we obtain

$$\mu = \exp(-\nu[\mathsf{E}\ln(T) + \mathbb{C}]). \tag{4.8}$$

Alternatively, the second moment of the log-transformed random variable T is given by

$$\begin{aligned}
\mathsf{E}\left[\ln(T)\right]^2 &= \mathsf{E}\left[\ln\left(\left(\frac{|\ln U|}{\mu}\right)^{1/\nu} S(\nu)\right)\right]^2 \\
&= \mathsf{E}\left[\frac{1}{\nu}\ln\left(\frac{|\ln U|}{\mu}\right) + \ln(S(\nu))\right]^2.
\end{aligned} \tag{4.9}$$

Expanding the right hand side (RHS) of (4.9), we obtain the equality

$$\begin{aligned}
\mathsf{E}\left[\ln(T)\right]^2 &= \mathsf{E}\left[\frac{1}{\nu^2}\left(\ln(|\ln U|) - \ln(\mu)\right)^2 + \frac{2}{\nu}\ln\left(\frac{|\ln U|}{\mu}\right)\ln(S(\nu)) + \ln(S(\nu))^2\right] \\
&= \mathsf{E}\Bigg[\frac{1}{\nu^2}\left(\ln(|\ln U|) - \ln(\mu)\right)^2 + \frac{2}{\nu}\ln(|\ln U|)\ln(S(\nu)) \\
&\qquad - \frac{2}{\nu}\ln(\mu)\ln(S(\nu)) + \ln(S(\nu))^2\Bigg] \\
&= \mathsf{E}\Bigg(\frac{1}{\nu^2}\left\{\left[\ln(|\ln U|)\right]^2 - 2\ln(\mu)\ln(|\ln U|) + \ln(\mu)^2\right\} \\
&\qquad + \frac{2}{\nu}\ln(|\ln U|)\ln(S(\nu)) - \frac{2}{\nu}\ln(\mu)\ln(S(\nu)) + \ln(S(\nu))^2\Bigg).
\end{aligned} \tag{4.10}$$

From another integral formula involving the Euler constant, we can easily obtain

$$\mathsf{E}\left[\ln(|\ln U|)\right]^2 = \mathsf{E}X^2 = \int_{\mathbb{R}} x^2 e^{x-e^x}dx = \int_{\mathbb{R}+} \ln(y)^2 e^{-y}dy = \mathbb{C}^2 + \frac{\pi^2}{6}. \tag{4.11}$$

Note that $\pi^2/6 = \zeta(2)$ is the value of the Riemann zeta function at the point 2. Furthermore, *Bening et al.* (2004) reveals that

$$\mathsf{E}\left[\ln(S(\nu))\right]^2 = \left(\frac{1}{\nu} - 1\right)^2 \mathbb{C}^2 + \frac{\pi^2}{6}\left(\frac{1}{\nu^2} - 1\right). \tag{4.12}$$

Using equation (4.11), equation (4.12), and the independence between U and $S(\nu)$, equation (4.10) becomes

$$\mathsf{E}\left[\ln(T)\right]^2 = \frac{\pi^2}{3\nu^2} + \frac{\left(\ln(\mu)\right)^2}{\nu^2} + \mathbb{C}^2 - \frac{\pi^2}{6} + \frac{2\mathbb{C}\ln(\mu)}{\nu}. \tag{4.13}$$

From (4.8),

$$\ln(\mu) = -\nu[\mathsf{E}\ln(T) + \mathbb{C}]. \tag{4.14}$$

Substituting (4.14) into (4.13) and simplifying the resulting expression, we can come up with

$$\mathsf{E}\left[\ln(T)\right]^2 - \left[\mathsf{E}\ln(T)\right]^2 + \frac{\pi^2}{6} = \frac{\pi^2}{3\nu^2}.$$

This implies that

$$\nu^2 = \frac{\pi^2}{3\left(\sigma^2_{\ln T} + \pi^2/6\right)}.$$

Thus, an estimator for ν is

$$\widehat{\nu} = \frac{\pi}{\sqrt{3\left(\widehat{\sigma^2_{\ln T}} + \pi^2/6\right)}} \tag{4.15}$$

and from (4.8),

$$\widehat{\mu} = \exp\left(-\widehat{\nu}\left(\widehat{\mathsf{E}\ln(T)} + \mathbb{C}\right)\right) = \exp\left(-\widehat{\nu}\left(\widehat{\mu_{\ln T}} + \mathbb{C}\right)\right) \tag{4.16}$$

is an estimator for μ.

In the sequel, we construct alternative estimators of μ. Recall that the first moment of the random variable $N_\nu(t)$ is $\mathsf{E}N_\nu(t) = \mu t^\nu / \Gamma(\nu+1)$. This directly suggests that

$$\widehat{\mu} = \frac{\widehat{\mu_{N_\nu(t)}}\Gamma(1+\widehat{\nu})}{t^{\widehat{\nu}}} \tag{4.17}$$

is an estimator of μ, where $t = \sum_{j=1}^{n} t_j$. Note that the number of jumps by time t is a possible estimator of $\mu_{N_\nu(t)}$.

Moreover, rearranging terms in (4.4) yields

$$\begin{aligned}\frac{1}{\nu}\ln(\mu) &= \frac{1}{\nu}\big[\mathsf{E}\ln(|\ln U|)\big] + \mathsf{E}\ln(S(\nu)) - \mathsf{E}\ln(T) \\ &= \mathsf{E}\ln(|\ln U|) + \nu\,\mathsf{E}\ln(S(\nu)) - \nu\,\mathsf{E}\ln(T). \qquad (4.18)\end{aligned}$$

Therefore, we can easily deduce that

$$\widehat{\mu} = \exp\left(\mathsf{E}\,\widehat{\ln(|\ln U|)} + \widehat{\nu}\,\mathsf{E}\,\widehat{\ln(S(\nu))} - \widehat{\nu}\,\mathsf{E}\,\widehat{\ln(T)}\right). \qquad (4.19)$$

Observe that we know time t, $\widehat{\nu}$ is given by (4.15), and U and $S(\widehat{\nu})$ can be generated using our algorithm described earlier. We plan to explore the maximum likelihood estimation and other estimation techniques in the future.

4.2 Asymptotic Normality of Estimators

We show asymptotic normality of our estimators for ν and μ. From the preceding section, we observe that

$$\mathsf{E}\ln(|\ln U|) = -\mathbb{C} \quad \text{and} \quad \mathsf{E}\left[\ln(|\ln U|)\right]^2 = \mathbb{C}^2 + \frac{\pi^2}{6}.$$

A further calculation using Mathematica shows that

$$\mathsf{E}\left[\ln(|\ln U|)\right]^3 = -\mathbb{C}^3 - \frac{\mathbb{C}\pi^2}{2} - 2\zeta(3)$$

and

$$\mathsf{E}\left[\ln(|\ln U|)\right]^4 = \mathbb{C}^2\left(\mathbb{C}^2 + \pi^2\right) + \frac{3\pi^4}{20} + 8\mathbb{C}\zeta(3).$$

Additionally, we have

$$\mathsf{E}\ln(S(\nu)) = \mathbb{C}\left(\frac{1}{\nu} - 1\right),$$

and

$$\mathsf{E}\left[\ln(S(\nu))\right]^2 = \left(\frac{1}{\nu} - 1\right)^2 \mathbb{C}^2 + \frac{\pi^2}{6}\left(\frac{1}{\nu^2} - 1\right).$$

Zolotarev (1986) provides the following formula for finding the higher log-moments of $S(\nu)$:

$$\mathsf{E}\left(\ln|S(\nu)|\right)^k = \left(d^k w_\nu(s)/ds^k\right)\big|_{s=0},$$

where

$$w_\nu(s) = \frac{\Gamma(1 - s/\nu)}{\Gamma(1-s)}.$$

To calculate these moments, we need to find the power series expansion of $w_\nu(s)$. This turns out to be easier if we first expand

$$\ln w_\nu(s) = \ln\Gamma(1 - s/\nu) - \ln\Gamma(1-s)$$

into a power series (*Bening et al.*, 2004). Using the log-gamma expansion

$$\ln\Gamma(1-\theta) = \mathbb{C}\theta + \sum_{k=2}^{\infty}\frac{\zeta(k)}{k}\theta^k,$$

we get

$$\ln w_\nu(s) = \mathbb{C}\left(\frac{1}{\nu} - 1\right)s + \frac{\pi^2}{12}\left(\frac{1}{\nu^2} - 1\right)s^2 + \frac{1}{3}\zeta(3)\left(\frac{1}{\nu^3} - 1\right)s^3$$
$$+ \frac{1}{4}\zeta(4)\left(\frac{1}{\nu^4} - 1\right)s^4 + \frac{1}{5}\zeta(5)\left(\frac{1}{\nu^5} - 1\right)s^5 + O(s^6)$$

and, hence,

$$w_\nu(s) = 1 + \mathbb{C}\left(\frac{1}{\nu} - 1\right)s + \left[\frac{\pi^2}{12}\left(\frac{1}{\nu^2} - 1\right) + \frac{1}{2}\mathbb{C}^2\left(\frac{1}{\nu^2} - 1\right)^2\right]s^2$$
$$+ \left[\frac{1}{3}\zeta(3)\left(\frac{1}{\nu^3} - 1\right) + \frac{1}{6}\mathbb{C}^3\left(\frac{1}{\nu} - 1\right)^3 + \mathbb{C}\left(\frac{1}{\nu} - 1\right)\left(\frac{1}{\nu^2} - 1\right)\frac{\pi^2}{12}\right]s^3$$
$$+ \frac{1}{1440}\Bigg[\left(\frac{1}{\nu^3} - \frac{1}{\nu^4}\right)\Big(60\mathbb{C}^4(\nu-1)^3 - 60\mathbb{C}^2\pi^2(\nu-1)^2(1+\nu)$$
$$+ \pi^4(\nu-3)(1+\nu)(3+\nu) + 480\mathbb{C}(\nu^3-1)\zeta(3)\Big)\Bigg]s^4 + O(s^5).$$

The kth log-moment of $S(\nu)$ is simply the coefficient of the term $s^k/k!$ in the above power series expansion (can also be obtained via $\left(d^k w_\nu(s)/ds^k\right)\big|_{s=0}$). In particular, the third and fourth log-moments can be shown to be

$$\mathsf{E}\left[\ln(S(\nu))\right]^3 = \frac{-2(\nu-1)^3\mathbb{C}^3 + \mathbb{C}\pi^2(\nu-1)^2(1+\nu) - 4(\nu^3-1)\zeta(3)}{2\nu^3}$$

and

$$\mathsf{E}\left[\ln(S(\nu))\right]^4 = \frac{1}{60}\left[\left(\frac{1}{\nu^3} - \frac{1}{\nu^4}\right)\left(60\mathbb{C}^4(\nu-1)^3 - 60\mathbb{C}^2\pi^2(\nu-1)^2(1+\nu)\right.\right.$$
$$\left.\left. + \pi^4(\nu-3)(1+\nu)(3+\nu) + 480\mathbb{C}(\nu^3-1)\zeta(3)\right)\right],$$

respectively. In addition, our derivations above show that

$$\mu_{\ln T} = -\left(\frac{\ln(\mu)}{\nu} + \mathbb{C}\right) \quad \text{and} \quad \sigma^2_{\ln T} = \frac{\pi^2}{3}\left(\frac{1}{\nu^2} - \frac{1}{2}\right).$$

The second, third, and fourth order moments of $\ln T$ are

$$\mathsf{E}\left(\ln T\right)^2 = \mathbb{C}^2 - \frac{\pi^2\left(\nu^2-2\right)}{6\nu^2} + \frac{\ln(\mu)\left[2\mathbb{C}\nu + \ln(\mu)\right]}{\nu^2},$$

$$\mathsf{E}\left(\ln T\right)^3 = -\frac{[\mathbb{C}\nu + \ln(\mu)][2\mathbb{C}^2\nu^2 - \pi^2(\nu^2-2) + 2\ln(\mu)(2\mathbb{C}\nu + \ln(\mu))]}{2\nu^3} - 2\zeta(3),$$

and

$$\mathsf{E}\left(\ln T\right)^4 = \frac{1}{60\nu^4}\left\{60\mathbb{C}^4\nu^4 - 60\mathbb{C}^2\nu^2(\nu^2-2) + \pi^4(28 - 20\nu^2 + \nu^4)\right.$$
$$+ 60\ln(\mu)[2\mathbb{C}\nu + \ln(\mu)]\left(2\mathbb{C}^2\nu^2 - \pi^2(\nu^2-2) + 2\mathbb{C}\nu\ln(\mu) + [\ln(\mu)]^2\right)$$
$$\left. + 480\nu^3[\mathbb{C}\nu + \ln(\mu)]\zeta(3)\right\},$$

correspondingly. We now calculate the higher-order central moments of the random variable $\ln T$. After a tedious algebraic manipulation, we get

$$\begin{aligned}
\mu_3 &= \mathsf{E}\left(\ln T - \mu_{\ln T}\right)^3 \\
&= \mathsf{E}\left\{\frac{1}{\nu}\ln\left(\frac{|\ln U|}{\mu}\right) + \ln(S(\nu)) - \left[-\left(\frac{\ln(\mu)}{\nu} + \mathbb{C}\right)\right]\right\}^3 \\
&= -2\zeta(3)
\end{aligned}$$

and

$$\mu_4 = \mathsf{E}\left(\ln T - \mu_{\ln T}\right)^4 = \frac{\pi^4(28 - 20\nu^2 + \nu^4)}{60\nu^4}.$$

If we let

$$\overline{\ln T} = \frac{\sum_{j=1}^{n} \ln T_j}{n} \quad \text{and} \quad \widehat{\sigma^2_{\ln T}} = \frac{\sum_{j=1}^{n}\left(\ln T_j - \overline{\ln T}\right)^2}{n}$$

then

$$\sqrt{n}\begin{pmatrix} \overline{\ln T}_n - \mu_{\ln T} \\ \widehat{\sigma^2_{\ln T}} - \sigma^2_{\ln T} \end{pmatrix} \xrightarrow{d} N\left[\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_{\ln T} & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4_{\ln T} \end{pmatrix}\right],$$

where μ_3, μ_4, and $\sigma^2_{\ln T}$ are defined above. We state Cramer's theorem (see (*Ferguson*, 1996; *Lehmann*, 1999)) below without proof.

Theorem (Cramer). *Let* $\mathbf{g}$ *be a mapping* $\mathbf{g}\colon \mathbb{R}^d \to \mathbb{R}^k$ *such that* $\dot{\mathbf{g}}(\mathbf{x})$ *is continuous in a neighborhood of* $\boldsymbol{\theta} \in \mathbb{R}^d$. *If* $\boldsymbol{X}_n$ *is a sequence of d-dimensional random vectors such that* $\sqrt{n}(\boldsymbol{X}_n - \boldsymbol{\theta}) \xrightarrow{d} \boldsymbol{X}$, *then*

$$\sqrt{n}\big(\boldsymbol{g}(\boldsymbol{X}_n) - \boldsymbol{g}(\boldsymbol{\theta})\big) \xrightarrow{d} \dot{\boldsymbol{g}}(\boldsymbol{\theta})\boldsymbol{X}.$$

In particular, if $\sqrt{n}(\boldsymbol{X}_n - \boldsymbol{\theta}) \xrightarrow{d} N(\mathbf{0}, \boldsymbol{\Sigma})$ *where* $\boldsymbol{\Sigma}$ *is a* $d \times d$ *covariance matrix, then*

$$\sqrt{n}\big(\boldsymbol{g}(\boldsymbol{X}_n) - \boldsymbol{g}(\boldsymbol{\theta})\big) \xrightarrow{d} N(\mathbf{0},\ \dot{\boldsymbol{g}}(\boldsymbol{\theta})\boldsymbol{\Sigma}\dot{\boldsymbol{g}}(\boldsymbol{\theta})^T).$$

For $\sigma^2_{\ln T} > 0$, we can now use Cramer's theorem to show asymptotic normality of our parameter estimators. Thus,

$$\begin{aligned}
\sqrt{n}\,(\widehat{\nu} - \nu) &\xrightarrow{d} N\left[0,\ \frac{18\pi^2}{\left(6\sigma^2_{\ln T} + \pi^2\right)^3}\left(\mu_4 - \sigma^4_{\ln T}\right)\right] \\
&\xrightarrow{d} N\left[0,\ \frac{18\pi^2\left(\frac{\pi^4\left(32 - 20\nu^2 - \nu^4\right)}{90\nu^4}\right)}{\left(6\sigma^2_{\ln T} + \pi^2\right)^3}\right] \\
&\xrightarrow{d} N\left[0,\ \frac{\pi^6\left(32 - 20\nu^2 - \nu^4\right)}{5\left(6\sigma^2_{\ln T} + \pi^2\right)^3 \nu^4}\right] \\
&\xrightarrow{d} N\left[0,\ \frac{\nu^2\left(32 - 20\nu^2 - \nu^4\right)}{40}\right].
\end{aligned}$$

The last line of the preceding simplification is attained by substituting $\sigma^2_{\ln T} = \frac{\pi^2}{3}\left(\frac{1}{\nu^2} - \frac{1}{2}\right)$.

Similarly, the estimator $\widehat{\mu}$ can be rewritten as

$$\widehat{\mu} = \exp\Big(- \widehat{\nu}\,(\widehat{\mu_{\ln T}} + \mathbb{C})\Big) = \exp\left(- \frac{\pi}{\sqrt{3(\widehat{\sigma^2_{\ln T}} + \pi^2/6)}}(\widehat{\mu_{\ln T}} + \mathbb{C})\right).$$

Let

$$\boldsymbol{g}(\mu_{\ln T}, \sigma^2_{\ln T}) = \exp\left(- \frac{\pi}{\sqrt{3(\sigma^2_{\ln T} + \pi^2/6)}}(\mu_{\ln T} + \mathbb{C})\right).$$

The gradient then becomes

$$\dot{\boldsymbol{g}}(\mu_{\ln T}, \sigma^2_{\ln T}) = \begin{pmatrix} \frac{-\sqrt{2}\pi}{\sqrt{\pi^2 + 6\sigma^2_{\ln T}}} \exp\left(\frac{-\sqrt{2}\pi(\mu_{\ln T} + \mathbb{C})}{\sqrt{\pi^2 + 6\sigma^2_{\ln T}}}\right) \\ \frac{3\sqrt{2}\pi(\mu_{\ln T} + \mathbb{C})}{\left(\pi^2 + 6\sigma^2_{\ln T}\right)^{3/2}} \exp\left(\frac{-\sqrt{2}\pi(\mu_{\ln T} + \mathbb{C})}{\sqrt{\pi^2 + 6\sigma^2_{\ln T}}}\right) \end{pmatrix}.$$

By Cramer's theorem,

$$\sqrt{n}\big(\widehat{\mu} - \mu\big) \xrightarrow{d} N\left[0,\ \sigma^2_a\right],$$

where

$$\sigma_a^2 = \dot{\boldsymbol{g}}(\mu_{\ln T}, \sigma^2_{\ln T})^T \begin{pmatrix} \sigma^2_{\ln T} & \mu_3 \\ \mu_3 & \mu_4 - \sigma^4_{\ln T} \end{pmatrix} \dot{\boldsymbol{g}}(\mu_{\ln T}, \sigma^2_{\ln T})$$

$$= \frac{\mu^2 \Big[20\pi^4(2-\nu^2) - 3\pi^2(\nu^4 + 20\nu^2 - 32)(\ln\mu)^2}{120\pi^2}$$

$$- \frac{720\nu^3(\ln\mu)\zeta(3) \Big]}{120\pi^2}. \tag{4.20}$$

Therefore, we have shown that our method-of-moments estimators are asymptotically normal (asymptotically unbiased). We can now approximate the $(1-\varepsilon)100\%$ confidence interval for μ, and ν as

$$\widehat{\mu} \pm z_{\varepsilon/2} \sqrt{\frac{\widehat{\mu}^2 \Big[20\pi^4(2-\widehat{\nu}^2) - 3\pi^2(\widehat{\nu}^4 + 20\widehat{\nu}^2 - 32)(\ln\widehat{\mu})^2 - 720\widehat{\nu}^3(\ln\widehat{\mu})\zeta(3) \Big]}{120\pi^2}},$$

and

$$\widehat{\nu} \pm z_{\varepsilon/2} \sqrt{\frac{\widehat{\nu}^2 \left(32 - 20\widehat{\nu}^2 - \widehat{\nu}^4\right)}{40}},$$

correspondingly, where $z_{\varepsilon/2}$ satisfies $2P(Z > z_{\varepsilon/2}) = \varepsilon$.

4.3 Numerical Experiment

We computationally compare and test our estimators using the mean absolute deviation (MAD) and the square root of mean squared error ($\sqrt{\text{MSE}}$) as our criteria. Recall our method-of-moments estimators for the fractional order ν (4.15) and the intensity rate μ (4.16):

$$\widehat{\nu}_{mm} = \frac{\pi}{\sqrt{3\left(\widehat{\sigma^2_{\ln T}} + \pi^2/6\right)}}$$

and

$$\widehat{\mu}_{mm} = \exp\left(-\widehat{\nu}\left(\widehat{\mathsf{E}\ln(T)} + \mathbb{C}\right)\right) = \exp\left(-\widehat{\nu}\left(\widehat{\mu_{\ln T}} + \mathbb{C}\right)\right).$$

Moreover, we try to improve the above estimators by bootstrapping, and avoid the computationally demanding algorithms based on U-statistics. Hence, we can directly obtain estimators $\widehat{\nu}_b$ and $\widehat{\mu}_b$. Finally, we consider estimators (4.17) and (4.19) as well:

$$\widehat{\mu}_3 = \frac{\widehat{\mu_{N(t)}}\Gamma(1+\widehat{\nu})}{t^{\widehat{\nu}}},$$

and

$$\widehat{\mu}_4 = \exp\left(\widehat{\mathsf{E}\ln(|\ln U|)} + \widehat{\nu}\,\widehat{\mathsf{E}\ln(S(\nu))} - \widehat{\nu}\,\widehat{\mathsf{E}\ln(T)}\right).$$

4.3.1 Simulated fPp Data

We generate $n = 100$ samples of fPp jump times with sample sizes N=50, 200, and 1000. We then estimate the parameters, and average them over the 100 samples. The tables below reveal the sample means $\overline{\widehat{\nu}}$ and $\overline{\widehat{\mu}}$, MAD and $\sqrt{\text{MSE}}$ using simulated fPp data for various μ's and ν's.

Tables 4.1 and 4.2 indicate that there is no significant gain in bootstrapping the method-of-moments estimators $\widehat{\nu}_{mm}$ and $\widehat{\mu}_{mm}$. Note that we generate 500 bootstrap samples, each of which has a sample size equal to the number of jumps (N) considered. In addition, Tables 4.3 and 4.4 disclose that the errors for all estimators of ν are getting under 5%, indicating a reputable performance. Observe that $\widehat{\mu}_3$ is still inadequate for μ even at N= 1000, and the errors are worse for larger μ values. But $\widehat{\mu}_3$ could be improved if we have a better estimator of $\mu_{N(t)}$.

Furthermore, Tables 4.1-4.4 empirically confirm that the method-of-moments estimators are asymptotically unbiased, and could be regarded as reasonable starting values (except $\widehat{\mu}_3$ maybe) for better iterative estimation procedures. Overall, the method-of-moments estimators did fairly well for various ν and μ values.

Table 4.1: *Test statistics for comparing parameter* $(\nu,\ \mu) = (0.9,\ 10)$ *estimators using a simulated fPp data.*

	N=50			N=200			N= 1000		
	Mean	MAD	$\sqrt{\text{MSE}}$	Mean	MAD	$\sqrt{\text{MSE}}$	Mean	MAD	$\sqrt{\text{MSE}}$
$\widehat{\nu}_{mm}$	0.9117	0.0620	0.0762	0.8962	0.0347	0.0424	0.9008	0.0128	0.0141
$\widehat{\nu}_b$	0.9244	0.0625	0.0762	0.9001	0.0340	0.0424	0.9016	0.0127	0.0141
$\widehat{\mu}_{mm}$	10.4699	1.8899	2.2788	10.0410	0.8682	1.0814	10.0550	0.4130	0.5130
$\widehat{\mu}_b$	10.9976	2.0310	2.5017	10.1843	0.8727	1.1008	10.0842	0.4153	0.5192
$\widehat{\mu}_3$	10.2452	2.5923	3.4187	9.2606	2.7007	3.4624	9.6752	2.3714	3.0529
$\widehat{\mu}_4$	10.4673	2.3887	3.1407	9.9801	1.3064	1.6491	10.1417	0.6004	0.7405

Table 4.2: *Test statistics for comparing parameter* $(\nu,\ \mu) = (0.3,\ 1)$ *estimators using a simulated fPp data.*

	N=50			N=200			N= 1000		
	Mean	MAD	$\sqrt{\text{MSE}}$	Mean	MAD	$\sqrt{\text{MSE}}$	Mean	MAD	$\sqrt{\text{MSE}}$
$\widehat{\nu}_{mm}$	0.3079	0.0268	0.0346	0.3012	0.0142	0.0173	0.3001	0.0059	0.0073
$\widehat{\nu}_b$	0.3169	0.0298	0.0374	0.3036	0.0145	0.0173	0.3006	0.0059	0.0073
$\widehat{\mu}_{mm}$	1.0358	0.2109	0.2600	1.0015	0.0945	0.1187	1.0096	0.0473	0.0616
$\widehat{\mu}_b$	1.0728	0.2286	0.2832	1.0088	0.0964	0.1216	1.0110	0.0473	0.0624
$\widehat{\mu}_3$	0.9275	0.5806	0.7505	0.8846	0.5718	0.7481	0.9403	0.6383	0.8226
$\widehat{\mu}_4$	1.0722	0.2861	0.3735	1.0104	0.1567	0.1972	1.0138	0.0668	0.0849

Table 4.3: *Test statistics for comparing parameter* $(\nu,\ \mu) = (0.2,\ 100)$ *estimators using a simulated fPp data.*

	N=50			N=200			N= 1000		
	Mean	MAD	$\sqrt{\text{MSE}}$	Mean	MAD	$\sqrt{\text{MSE}}$	Mean	MAD	$\sqrt{\text{MSE}}$
$\widehat{\nu}_{mm}$	0.2062	0.0202	0.0264	0.2008	0.0090	0.0100	0.2008	0.0041	0.0054
$\widehat{\nu}_b$	0.2128	0.0226	0.0283	0.2024	0.0092	0.0100	0.2012	0.0042	0.0055
$\widehat{\mu}_{mm}$	140.68	65.65	108.79	107.83	24.26	31.88	102.36	10.13	13.42
$\widehat{\mu}_b$	172.49	89.13	150.84	112.96	26.51	35.03	103.35	10.29	13.77
$\widehat{\mu}_3$	126.69	86.16	124.96	100.74	71.86	107.60	108.75	74.64	91.43
$\widehat{\mu}_4$	140.07	70.57	109.80	107.52	26.20	33.66	102.23	11.74	15.04

Table 4.4: *Test statistics for comparing parameter* $(\nu,\ \mu) = (0.6,\ 1000)$ *estimators using a simulated fPp data.*

	N=50			N=200			N= 1000		
	Mean	MAD	$\sqrt{\text{MSE}}$	Mean	MAD	$\sqrt{\text{MSE}}$	Mean	MAD	$\sqrt{\text{MSE}}$
$\widehat{\nu}_{mm}$	0.6041	0.0552	0.0693	0.6052	0.0306	0.0374	0.5999	0.0119	0.0141
$\widehat{\nu}_{b}$	0.6178	0.0555	0.0707	0.6091	0.0310	0.0387	0.6008	0.0119	0.0141
$\widehat{\mu}_{mm}$	1540	962	2042	1160	398	536	1019	143	189
$\widehat{\mu}_{b}$	1838	1160	2459	1220	423	580	1030	143	189
$\widehat{\mu}_{3}$	1400	999	2051	1033	676	920	1128	567	737
$\widehat{\mu}_{4}$	1601	1005	2098	1171	420	580	1017	147	187

Chapter 5

Summary, Conclusions, and Future Research Directions

In this chapter, we provide a synopsis of the preceding discussions. We also detail our conclusions, and outline possible research extensions.

5.1 Summary

At the outset, we were able to provide an algorithm to generate the fractional Poisson process. We also presented typical sample paths of fPp, standard Poisson process, and the alternative fPp (see Figure 3.16). These paths indicated that realizations of the standard Poisson process have generally shorter waiting times than fPp. We have also computed the limiting distributions of the scaled nth arrival or event time for fPp using the algorithm. We have shown that fPp's have empty bins between clusters of events for all time scales using simple intermittency measures.

Secondly, we proposed an alternative fPp generated by the stochastic differential equation

$$\frac{d^{\nu}Y_{\nu}}{dt^{\nu}} = \sum_{j=1}^{\infty}\delta(t - T^{(j)}),$$

whose solution is

$$Y_\nu(t) = \sum_{j=1}^{N(t)} \frac{1}{\Gamma(\nu)} \frac{1}{(t - T^{(j)})_+^{1-\nu}}.$$

We also showcased the covariance structure of fPp with attempts to describe it in a closed form; its increments are dependent and nonstationary. This indicates that the properties of fPp are quite different from the properties of the standard Poisson process. We also described the limiting distribution of $Z = N(t)/\mathsf{E}\big[N(t)\big]$ for all the three processes above, and the second-order self-similarity property of fPp has been established.

Lastly, we were able to find asymptotically normal estimators of the parameters of the fractional Poisson process.

5.2 Conclusions

Results generally showed appealing and promising features of fPp for real-life applications. Additionally, substantial results are already obtained, albeit the study still calls for more in-depth explorations. Nevertheless, we can conclude that α-stable densities are useful in analyzing the theoretical and numerical properties of an important fractional stochastic process called fractional Poisson process. Finally, we have implemented fractional Poisson process (fPp).

5.3 Future Research Directions

Intensive numerical and theoretical investigations have been done, but a number of matters are still left undone which could be considered as possible research extensions of the current exploration. These may include: construction of a larger class of counting models by extending the fractional order to $1 < \nu < 2$; expansion of fPp to fractional Poisson fields; expression of fPp in terms of tempered-stable densities; derivation of a model based on nonconstant intensity rate corresponding to $0 < \nu < 2$; investigation of the multiscaling property and long-range dependence of fPp; enlarging the class of counting process to include waiting times that are heavy-tailed for small time magnitudes but exponential for large times; and lastly, applying this theory to model real physical phenomena, such as network traffic, particle streams, economic "events", etc.

Appendix

Appendix A. Some Properties of α^+-Stable Densities

The α^+-stable density, or one-sided alpha-stable distribution, denoted by $g^{(\alpha)}(t)$ is determined by its Laplace transform as follows (*Samorodnitsky and Taqqu*, 1994; *Uchaikin and Zolotarev*, 1999):

$$\{\mathsf{L}g^{(\alpha)}(t)\}(\lambda) \equiv \widetilde{g}^{(\alpha)}(\lambda) \equiv \int_0^\infty g^{(\alpha)}(t)e^{-\lambda t}dt = e^{-\lambda^\alpha}. \tag{A.1}$$

It is equal to 0 on the negative semiaxis including the origin, positive on the positive semiaxis and satisfies the normalization condition

$$\int_0^\infty g^{(\alpha)}(t)dt = 1.$$

The term "stable" means that these densities belong to the class of the *Lévy stable laws*: the convolution of two α^+-densities is again the α^+-density (up to a scale factor):

$$\int_0^t g^{(\alpha)}(t-t')g^{(\alpha)}(t')dt' = 2^{-1/\alpha}g^{(\alpha)}(2^{-1/\alpha}t).$$

This is easily seen in terms of Laplace transforms:

$$\widetilde{g}^{(\alpha)}(\lambda)\widetilde{g}^{(\alpha)}(\lambda) = \widetilde{g}^{(\alpha)}(2^{1/\alpha}\lambda).$$

The main property of the densities is that they play the role of limit distributions beyond the central limit theorem. Namely, if $T_1, T_2, \ldots, T_n$ are independent and identically distributed random variables with $P(T_j > t) \sim at^{-\alpha}$, $t \to \infty$, then the probability density of their sum

$$f_{\sum T_j}(t) \sim [a\Gamma(1-\alpha)]^{1/\alpha} g^{(\alpha)}\left([a\Gamma(1-\alpha)]^{1/\alpha} t\right)$$

Let us give some other important properties of these densities:

(i) when $\alpha \to 1,\ g^{(\alpha)}(t) \to \delta(t-1)$;

(ii) moments of the densities (Mellin transform):

$$\int_0^\infty g^{(\alpha)}(t) t^\nu dt = \begin{cases} \Gamma(1-\nu/\alpha)/\Gamma(1-\nu), & -\infty < \nu < \alpha; \\ \infty, & \nu \geq \alpha, \end{cases} \tag{A.2}$$

(iii) only one of the densities is expressed through elementary functions:

$$g^{(1/2)}(t) = \frac{1}{2\sqrt{\pi}} t^{-3/2} \exp[-1/(4t)],\ t > 0; \tag{A.3}$$

(iv) the densities can be represented in the form of a convergent series as $t \to 0$

$$g^{(\alpha)}(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n!} \frac{n\alpha}{\Gamma(1-n\alpha)} t^{-n\alpha-1}; \tag{A.4}$$

(v) for numerical calculations, the following integral formula is more convenient:

$$g^{(\alpha)}(t) = \frac{\alpha t^{1/(\alpha-1)}}{\pi(1-\alpha)} \int_{-\pi/2}^{\pi/2} \exp\left\{-t^{\alpha/(\alpha-1)} U(\phi;\alpha)\right\} U(\phi;\alpha) d\phi, \tag{A.5}$$

where

$$U(\phi;\alpha) = \left[\frac{\sin(\alpha(\phi+\pi/2))}{\cos\phi}\right]^{\alpha/(\alpha-1)} \frac{\cos\left((\alpha-1)\phi + \alpha\pi/2\right)}{\cos\phi};$$

(*vi*) the following asymptotical approximation obtained by saddle-point method is useful:

$$g^{(\alpha)}(t) \sim \frac{1}{\sqrt{2\pi(1-\alpha)\alpha}}(t/\alpha)^{(\alpha-2)/(2-2\alpha)} \exp[-(1-\alpha)(t/\alpha)^{-\alpha/(1-\alpha)}], \quad t \to 0. \quad (A.6)$$

Results of numerical calculations according to (A.3) for $\alpha = 1/2$ and (A.5) for all other values of α are represented in Figure A below. The detailed description of the Levy stable distributions and their applications can also be found in *Samorodnitsky and Taqqu* (1994) and *Uchaikin and Zolotarev* (1999).

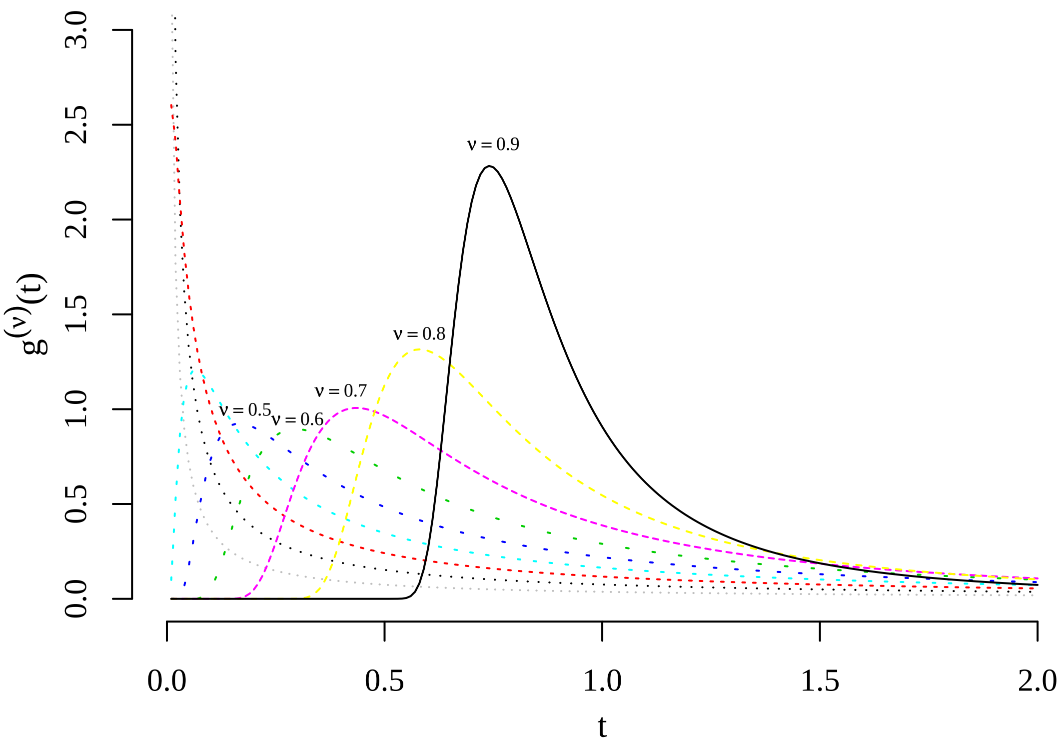

Figure A : α^{+}-*Stable Densities.*

Appendix B. Scaled fPp Density (3.12)Values

Table 5.1: *Probability density (3.12) values for* ν = 0.05(0.05)0.50 *and* z = 0.0(0.1)3.0.

$z \setminus \nu$	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
0.0	0.996	0.984	0.963	0.936	0.900	0.858	0.810	0.757	0.699	0.637
0.1	0.902	0.893	0.879	0.859	0.833	0.803	0.768	0.728	0.683	0.635
0.2	0.817	0.811	0.801	0.787	0.770	0.750	0.725	0.697	0.665	0.629
0.3	0.740	0.736	0.730	0.721	0.711	0.698	0.683	0.665	0.643	0.619
0.4	0.670	0.668	0.664	0.660	0.655	0.648	0.641	0.631	0.619	0.605
0.5	0.606	0.606	0.605	0.604	0.603	0.601	0.599	0.597	0.593	0.588
0.6	0.549	0.549	0.550	0.552	0.553	0.556	0.559	0.562	0.565	0.568
0.7	0.497	0.498	0.500	0.503	0.508	0.513	0.519	0.527	0.535	0.545
0.8	0.450	0.452	0.455	0.459	0.465	0.472	0.481	0.492	0.505	0.519
0.9	0.407	0.409	0.413	0.418	0.425	0.434	0.445	0.458	0.474	0.492
1.0	0.369	0.371	0.375	0.381	0.389	0.398	0.410	0.425	0.442	0.463
1.1	0.334	0.336	0.340	0.347	0.355	0.365	0.377	0.392	0.411	0.433
1.2	0.302	0.305	0.309	0.315	0.323	0.333	0.346	0.361	0.380	0.403
1.3	0.273	0.276	0.280	0.286	0.294	0.304	0.316	0.331	0.349	0.372
1.4	0.247	0.250	0.254	0.260	0.267	0.277	0.288	0.302	0.320	0.341
1.5	0.224	0.226	0.230	0.236	0.243	0.251	0.262	0.275	0.291	0.311
1.6	0.203	0.205	0.208	0.213	0.220	0.228	0.238	0.250	0.264	0.282
1.7	0.183	0.185	0.189	0.193	0.199	0.206	0.215	0.226	0.239	0.254
1.8	0.166	0.168	0.171	0.175	0.180	0.187	0.194	0.204	0.214	0.227
1.9	0.150	0.152	0.155	0.158	0.163	0.168	0.175	0.183	0.192	0.202
2.0	0.136	0.137	0.140	0.143	0.147	0.152	0.157	0.164	0.171	0.178
2.1	0.123	0.124	0.126	0.129	0.133	0.137	0.141	0.146	0.151	0.156
2.2	0.111	0.112	0.114	0.117	0.119	0.123	0.126	0.130	0.134	0.136
2.3	0.101	0.102	0.103	0.105	0.107	0.110	0.113	0.115	0.117	0.118
2.4	0.091	0.092	0.093	0.095	0.097	0.099	0.100	0.102	0.103	0.102
2.5	0.082	0.083	0.084	0.085	0.087	0.088	0.089	0.090	0.089	0.087
2.6	0.074	0.075	0.076	0.077	0.078	0.079	0.079	0.079	0.077	0.074
2.7	0.067	0.068	0.068	0.069	0.070	0.070	0.070	0.069	0.067	0.063
2.8	0.061	0.061	0.062	0.062	0.062	0.062	0.062	0.060	0.057	0.052
2.9	0.055	0.055	0.056	0.056	0.056	0.056	0.055	0.053	0.049	0.044
3.0	0.050	0.050	0.050	0.050	0.050	0.049	0.048	0.046	0.042	0.036

Table 5.2: *Probability density (3.12) values for* $\nu = 0.05(0.05)0.50$ *and* $z = 3.1(0.1)5.0$.

$z \setminus \nu$	0.05	0.10	0.15	0.20	0.25	0.30	0.35	0.40	0.45	0.50
3.1	0.045	0.045	0.045	0.045	0.045	0.044	0.042	0.040	0.036	0.030
3.2	0.041	0.041	0.041	0.040	0.040	0.039	0.037	0.034	0.030	0.024
3.3	0.037	0.037	0.037	0.036	0.035	0.034	0.032	0.029	0.025	0.020
3.4	0.033	0.033	0.033	0.032	0.032	0.030	0.028	0.025	0.021	0.016
3.5	0.030	0.030	0.030	0.029	0.028	0.027	0.025	0.022	0.018	0.013
3.6	0.027	0.027	0.027	0.026	0.025	0.023	0.021	0.018	0.015	0.010
3.7	0.025	0.024	0.024	0.023	0.022	0.021	0.018	0.016	0.012	0.008
3.8	0.022	0.022	0.022	0.021	0.020	0.018	0.016	0.013	0.010	0.006
3.9	0.020	0.020	0.019	0.019	0.017	0.016	0.014	0.011	0.008	0.005
4.0	0.018	0.018	0.017	0.017	0.015	0.014	0.012	0.009	0.007	0.004
4.1	0.016	0.016	0.016	0.015	0.014	0.012	0.010	0.008	0.005	0.003
4.2	0.015	0.015	0.014	0.013	0.012	0.011	0.009	0.007	0.004	0.002
4.3	0.013	0.013	0.013	0.012	0.011	0.009	0.008	0.006	0.004	0.002
4.4	0.012	0.012	0.011	0.011	0.009	0.008	0.007	0.005	0.003	0.001
4.5	0.011	0.011	0.010	0.009	0.008	0.007	0.006	0.004	0.002	0.001
4.6	0.010	0.010	0.009	0.008	0.007	0.006	0.005	0.003	0.002	0.001
4.7	0.009	0.009	0.008	0.007	0.007	0.005	0.004	0.003	0.001	0.001
4.8	0.008	0.008	0.007	0.007	0.006	0.005	0.003	0.002	0.001	
4.9	0.007	0.007	0.007	0.006	0.005	0.004	0.003	0.002	0.001	
5.0	0.007	0.006	0.006	0.005	0.004	0.003	0.002	0.002	0.001	

Table 5.3: *Probability density ([3.12]) values for* $\nu = 0.55(0.05)0.95$ *and* $z = 0.0(0.1)3.0$.

$z \setminus \nu$	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
0.0	0.572	0.504	0.436	0.368	0.300	0.234	0.170	0.109	0.052
0.1	0.582	0.525	0.464	0.401	0.335	0.267	0.199	0.131	0.064
0.2	0.588	0.542	0.491	0.435	0.373	0.306	0.234	0.158	0.079
0.3	0.590	0.556	0.516	0.469	0.414	0.351	0.278	0.194	0.100
0.4	0.587	0.565	0.537	0.502	0.458	0.402	0.331	0.241	0.131
0.5	0.581	0.570	0.555	0.533	0.502	0.458	0.395	0.305	0.176
0.6	0.570	0.570	0.568	0.561	0.547	0.520	0.474	0.392	0.248
0.7	0.555	0.565	0.575	0.584	0.589	0.586	0.567	0.510	0.367
0.8	0.536	0.555	0.576	0.600	0.626	0.652	0.674	0.673	0.578
0.9	0.514	0.540	0.571	0.609	0.656	0.715	0.791	0.888	0.981
1.0	0.489	0.520	0.559	0.608	0.674	0.766	0.906	1.154	1.768
1.1	0.461	0.495	0.539	0.598	0.678	0.799	1.001	1.425	2.970
1.2	0.431	0.467	0.514	0.576	0.666	0.803	1.043	1.563	2.470
1.3	0.400	0.435	0.482	0.545	0.634	0.772	1.001	1.354	0.068
1.4	0.368	0.401	0.445	0.504	0.585	0.701	0.852	0.753	
1.5	0.335	0.366	0.405	0.455	0.519	0.595	0.614	0.190	
1.6	0.303	0.329	0.361	0.400	0.442	0.464	0.353	0.012	
1.7	0.272	0.293	0.317	0.342	0.358	0.327	0.150		
1.8	0.241	0.257	0.273	0.284	0.275	0.205	0.043		
1.9	0.212	0.223	0.231	0.228	0.199	0.112	0.007		
2.0	0.185	0.191	0.191	0.177	0.135	0.052	0.001		
2.1	0.160	0.161	0.154	0.133	0.085	0.020			
2.2	0.137	0.134	0.122	0.096	0.049	0.006			
2.3	0.116	0.110	0.095	0.066	0.026	0.001			
2.4	0.098	0.089	0.072	0.044	0.013				
2.5	0.082	0.071	0.053	0.028	0.006				
2.6	0.067	0.056	0.038	0.017	0.002				
2.7	0.055	0.043	0.027	0.010	0.001				
2.8	0.044	0.033	0.018	0.005					
2.9	0.036	0.025	0.012	0.003					
3.0	0.028	0.018	0.008	0.001					

Table 5.4: *Probability density (3.12) values for* $\nu = 0.55(0.05)0.95$ *and* $z = 3.1(0.1)5.0$.

$z \setminus \nu$	0.55	0.60	0.65	0.70	0.75	0.80	0.85	0.90	0.95
3.1	0.022	0.013	0.005	0.001					
3.2	0.017	0.009	0.003						
3.3	0.013	0.007	0.002						
3.4	0.010	0.005	0.001						
3.5	0.008	0.003	0.001						
3.6	0.006	0.002							
3.7	0.004	0.001							
3.8	0.003	0.001							
3.9	0.002	0.001							
4.0	0.002								
4.1	0.001								
4.2	0.001								
4.3	0.001								

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