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Differential Equation , Ebrahim Shah Ebrahimi , Teacher Assistant June 2, 2018

Question 1

Use Laplace transforms to solve each of the following ODE's.

(a) $y''(x) = e^x + 2e^x \int_0^x \frac{y'(t)}{e^t} dt \qquad y(0) = y'(0) = 0$

(b) $y = sin(t) + \int_0^t y(\theta)\cos(t-\theta)\, d\theta$

Question 2

evaluate the given definite integral.

(c) $\int_0^\infty e^{-2t}.\delta(t-\frac{\pi}{2}).t.sin(t).[L^{-1}(ln\frac{s}{s-1})]dt$ **Hint :** At first Find the inverse Laplace transforms.

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ans :

$(a)\ L(y''(x)) = L(e^x) + 2L\left(\int_0^x e^{x-t}\, y'(t)dt\right)$

$\xrightarrow{L(y)=F(s)} s^2F(s) - sf(0) - f'(0) = \frac{1}{s-1} + 2\frac{sF(s)-f(0)}{s-1} \rightarrow s^2F(s) - 2\frac{sF(s)}{s-1} = \frac{1}{s-1}$

$\rightarrow \left(s^2 - \frac{2s}{s-1}\right)F(s) = \frac{1}{s-1} \rightarrow \left(\frac{s^3-s^2-2s}{s-1}\right)F(s) = \frac{1}{s-1}$

$\rightarrow F(s) = \frac{1}{(s^3-s^2-2s)} = \frac{1}{s(s-2)(s+1)}$

$\rightarrow F(s) = \frac{\frac{-1}{2}}{s} + \frac{\frac{1}{6}}{(s-2)} + \frac{\frac{1}{3}}{(s+1)} \xrightarrow{L^{-1}} \boxed{y = \frac{-1}{2} + \frac{1}{6}e^{2x} + \frac{1}{3}e^{-x}}$

$(b)\ L(y) = L(sin(t)) + L\left(\int_0^t y(\theta)\cos(t-\theta)\,d\theta\right)$

$\xrightarrow{L(y)=F(s)} F(s) = \frac{1}{s^2+1} + F(s)\frac{s}{s^2+1} \rightarrow F(s) - F(s)\frac{s}{s^2+1} = \frac{1}{s^2+1}$

$\rightarrow \left(1 - \frac{s}{s^2+1}\right)F(s) = \frac{1}{s^2+1} \rightarrow \left(\frac{s^2-s+1}{s^2+1}\right)F(s) = \frac{1}{s^2+1}$

$\rightarrow F(s) = \frac{1}{s^2-s+1} = \frac{1}{(s-\frac{1}{2})^2+\frac{3}{4}} \xrightarrow{L^{-1}} \boxed{y = \frac{2}{\sqrt{3}}e^{\frac{t}{2}}.sin(\frac{\sqrt{3}}{2}t)}$

$(c)\ \int_0^\infty e^{-2t}.\delta(t-\frac{\pi}{2}).t.sin(t).[L^{-1}(ln\frac{s}{s-1})]dt \xrightarrow{L(f(t))=\int_0^\infty e^{-st}f(t)dt} \begin{cases} L\left(\delta(t-\frac{\pi}{2}).t.sin(t).[L^{-1}(ln\frac{s}{s-1})]^*\right) \\ s=2 \end{cases}$

$\xrightarrow{*} \left[L^{-1}\left(ln\frac{s}{s-1}\right)\right] = \left[(\frac{-1}{t})L^{-1}\left(ln\frac{s}{s-1}\right)'\right] = (\frac{-1}{t})L^{-1}\left(\frac{1}{s} - \frac{1}{s-1}\right) = (\frac{-1}{t}(1-e^t))$

$L\left(\delta(t-\frac{\pi}{2}).t.sin(t).[L^{-1}(ln\frac{s}{s-1})]^*\right) = L\left(\delta(t-\frac{\pi}{2}).sin(t).(e^t-1)\right) = e^{-\frac{\pi}{2}s}.sin\left(\frac{\pi}{2}\right).\left(e^{\frac{\pi}{2}}-1\right) \xrightarrow{s=2} = \boxed{e^{-\pi}(e^{\frac{\pi}{2}}-1)}$