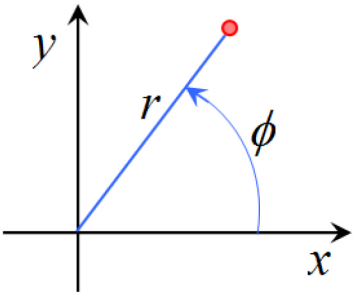
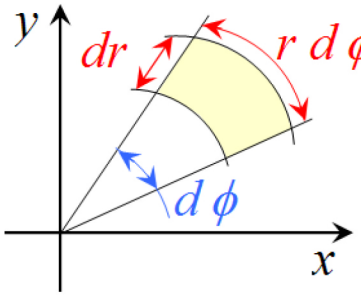
5.1.1. Volume Element in Cylindrical and Spherical Polar Coordinates

(1) Polar coordinates (r, ϕ): the area element





$x = r \cos\phi$
$y = r \sin\phi$

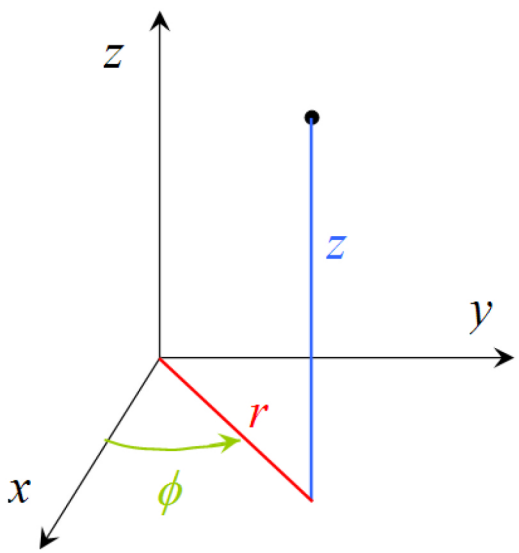
$dS = r\, dr\, d\phi$,
the area element

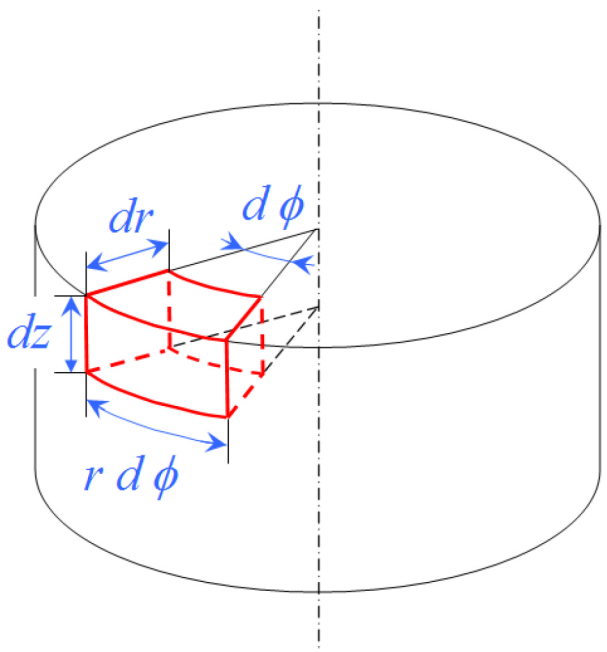
Change of variables in the double integral:

$$\iint_R f\, dx\, dy = \iint_R f\, r\, dr\, d\phi$$

---

(2) Cylindrical polar coordinates (r, ϕ, z)





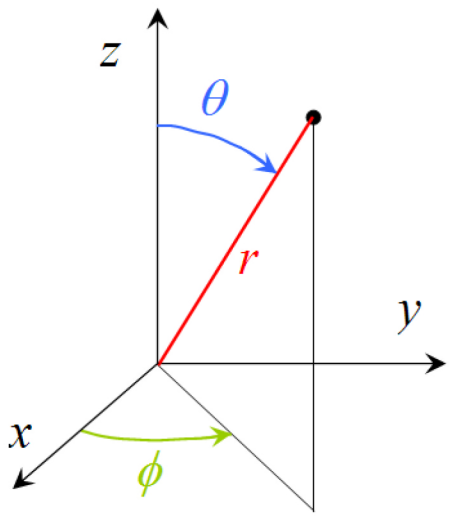
$x = r\cos\phi\ ,\ y = r\sin\phi\ ,\ z = z$

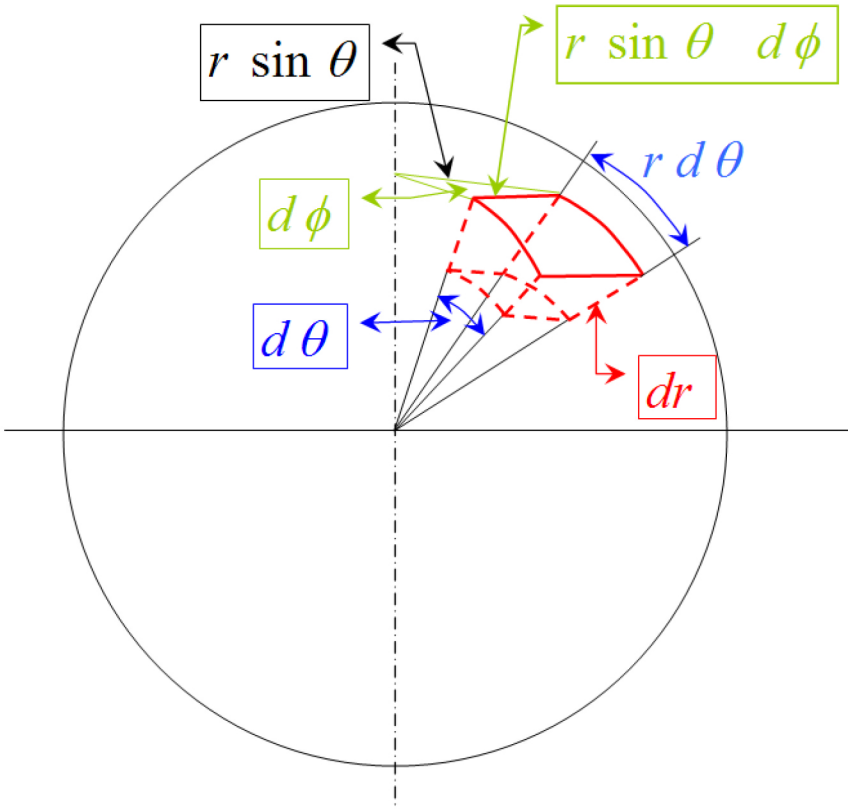
Volume element: $dV = r\, dr\, d\phi\, dz$

Change of variables in the volume (triple) integral:

$$\iiint_V f\, dx\, dy\, dz = \iiint_V f\, r\, dr\, d\phi\, dz$$

(3) Spherical polar coordinates (r, θ, ϕ)





$$x = r\sin\theta\cos\phi\ ,\ \ y = r\sin\theta\sin\phi\ ,\ \ z = r\cos\theta$$

Volume element:

$$\begin{aligned} dV &= dr \cdot r\,d\theta \cdot r\sin\theta\,d\phi \\ &= r^2\sin\theta\,dr\,d\theta\,d\phi \end{aligned}$$

Change of variables in the volume (triple) integral:

$$\iiint_V f\,dx\,dy\,dz = \iiint_V f\,r^2\sin\theta\,dr\,d\theta\,d\phi$$