Divide-and-Conquer Algorithms

Part One

Announcements

- Problem Set One completely due right now. Solutions distributed at the end of lecture.
- Programming section today in Gates B08 from from 3:45PM – 5:00PM.
    - Resumes at normal Thursday schedule (4:15PM – 5:05PM) next week.

Where We've Been

- We have just finished discussing fundamental algorithms on graphs.
- These algorithms are indispensable and show up *everywhere*.
- You can now solve a large class of problems by recognizing that they *reduce* to a problem you already know how to solve.

Where We're Going

- We are about to explore the **divide-and-conquer** paradigm, which gives a useful framework for thinking about problems.
- We will explore several major techniques:
  - Solving problems recursively.
  - Intuitively understanding how the structure of recursive algorithms influences runtime.
  - Recognizing when a problem can be solved by reducing it to a simpler case.

Outline for Today

- **Recurrence Relations**
  - Representing an algorithm's runtime in terms of a simple recurrence.
- **Solving Recurrences**
  - Determining the runtime of a recursive function from a recurrence relation.
- **Sampler of Divide-and-Conquer**
  - A few illustrative problems.

Insertion Sort

- As we saw in Lecture 00, insertion sort can be used to sort an array in time $\Omega(n)$ and $O(n^2)$.
  - It's $\Theta(n^2)$ in the average case.
- Can we do better?

A Better Sorting Algorithm: **Mergesort**

Thinking About O(n^2)

14	6	3	9	7	16	2	15	5	10	8	11	1	13	12	4

Thinking About O(n^2)

$$\underbrace{\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 14 & 6 & 3 & 9 & 7 & 16 & 2 & 15 & 5 & 10 & 8 & 11 & 1 & 13 & 12 & 4 \\ \hline \end{array}}_{\mathrm{T}(n)}$$

Thinking About O(n^2)

14	6	3	9	7	16	2	15	5	10	8	11	1	13	12	4

T(n)

14	6	3	9	7	16	2	15

5	10	8	11	1	13	12	4

Thinking About $O(n^2)$

$$\underbrace{\begin{array}{|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|}\hline 14 & 6 & 3 & 9 & 7 & 16 & 2 & 15 & 5 & 10 & 8 & 11 & 1 & 13 & 12 & 4 \\ \hline \end{array}}_{T(n)}$$

$$\underbrace{\begin{array}{|c|c|c|c|c|c|c|c|}\hline 14 & 6 & 3 & 9 & 7 & 16 & 2 & 15 \\ \hline \end{array}}_{T(\frac{1}{2}n) \approx \frac{1}{4}T(n)} \qquad \underbrace{\begin{array}{|c|c|c|c|c|c|c|c|}\hline 5 & 10 & 8 & 11 & 1 & 13 & 12 & 4 \\ \hline \end{array}}_{T(\frac{1}{2}n) \approx \frac{1}{4}T(n)}$$

Thinking About O(n^2)

14	6	3	9	7	16	2	15	5	10	8	11	1	13	12	4

T(n)

2	3	6	7	9	14	15	16

T(½n) ≈ ¼T(n)

1	4	5	8	10	11	12	13

T(½n) ≈ ¼T(n)

The Key Insight: **Merge**

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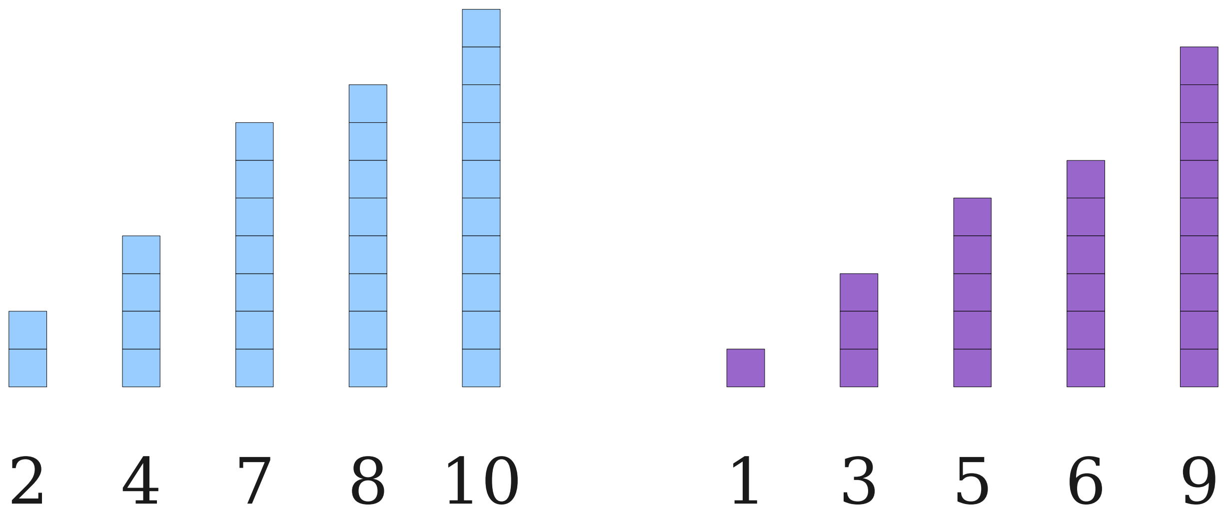

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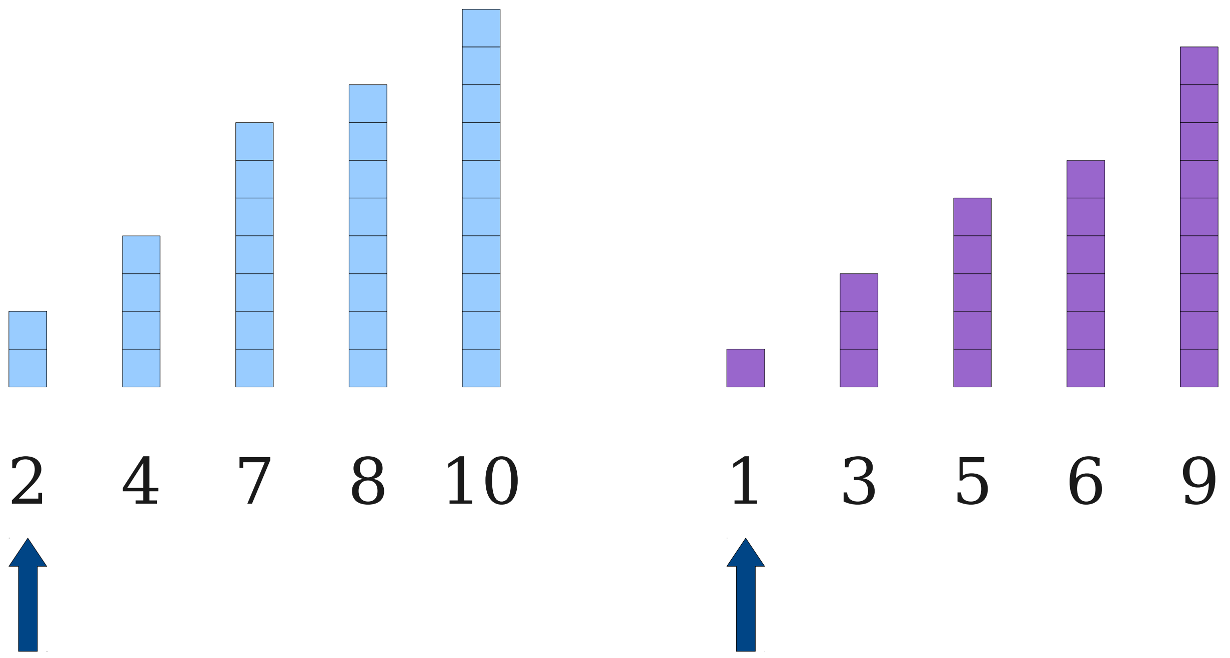

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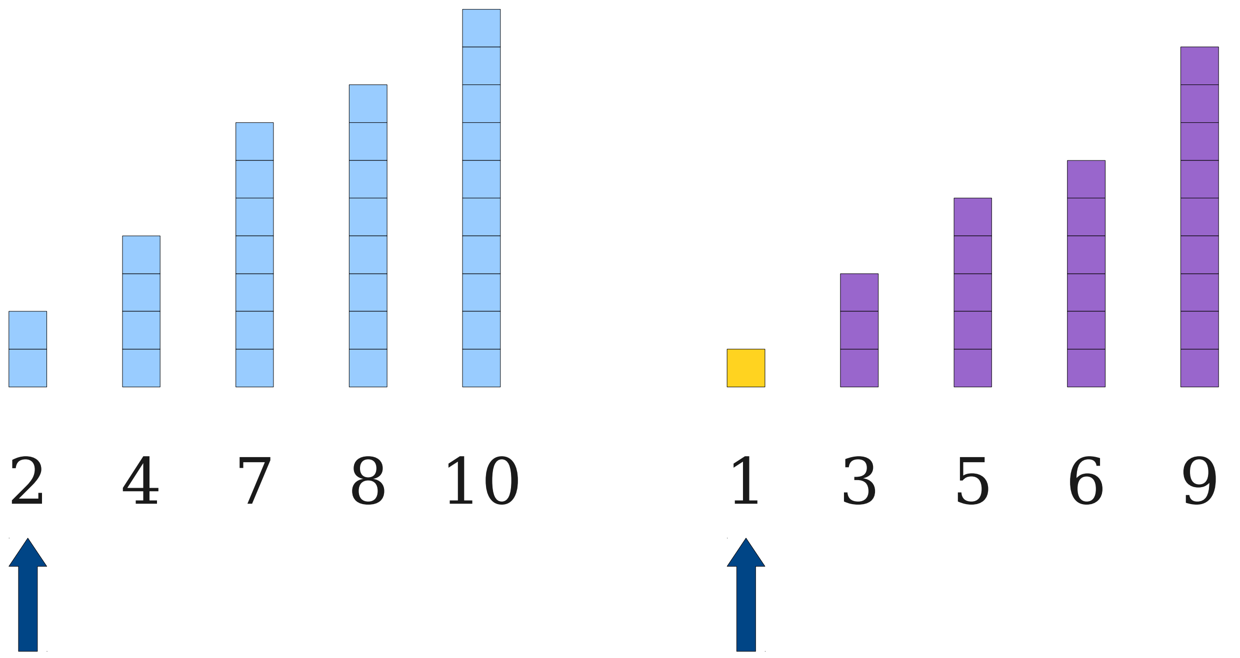

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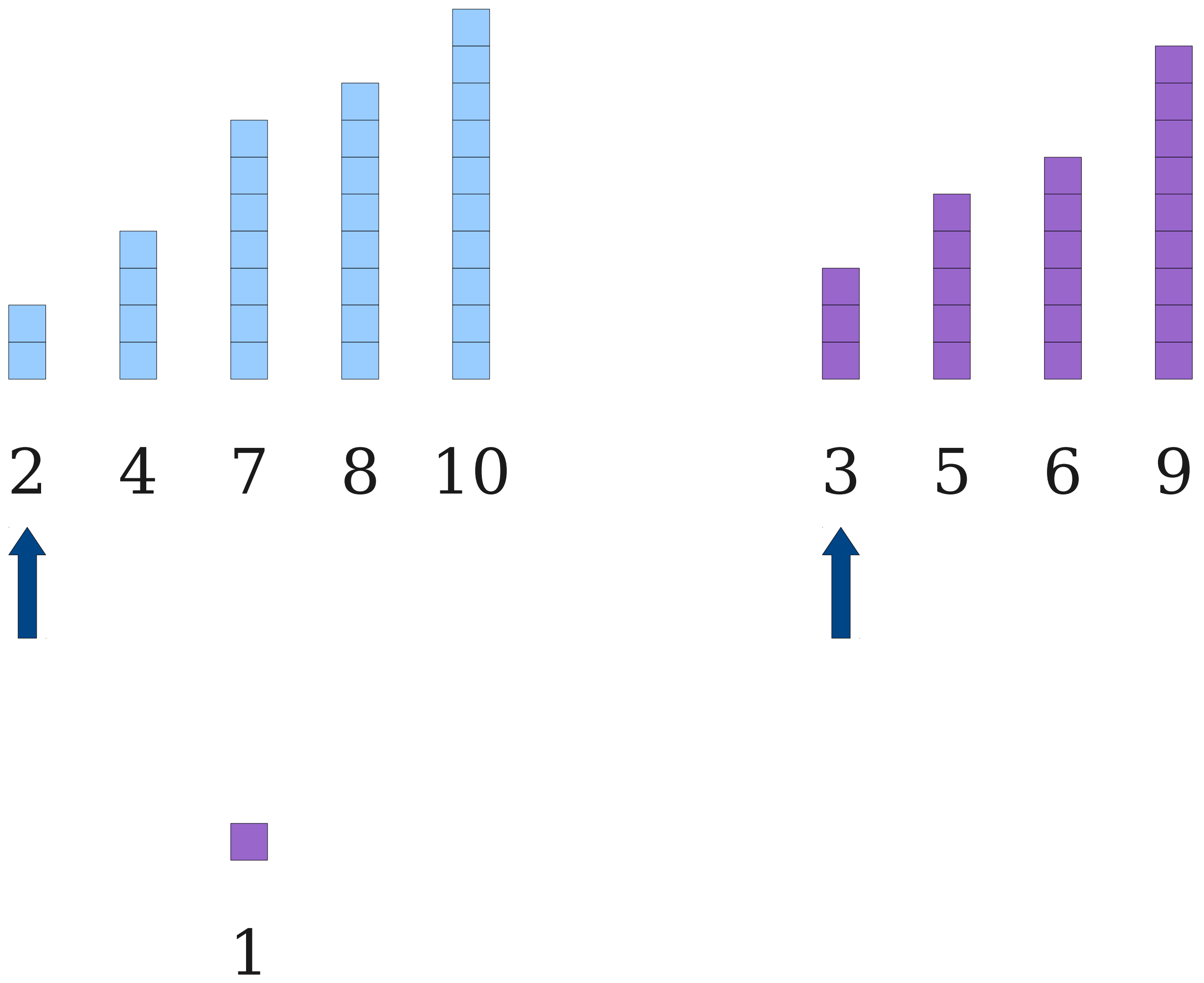

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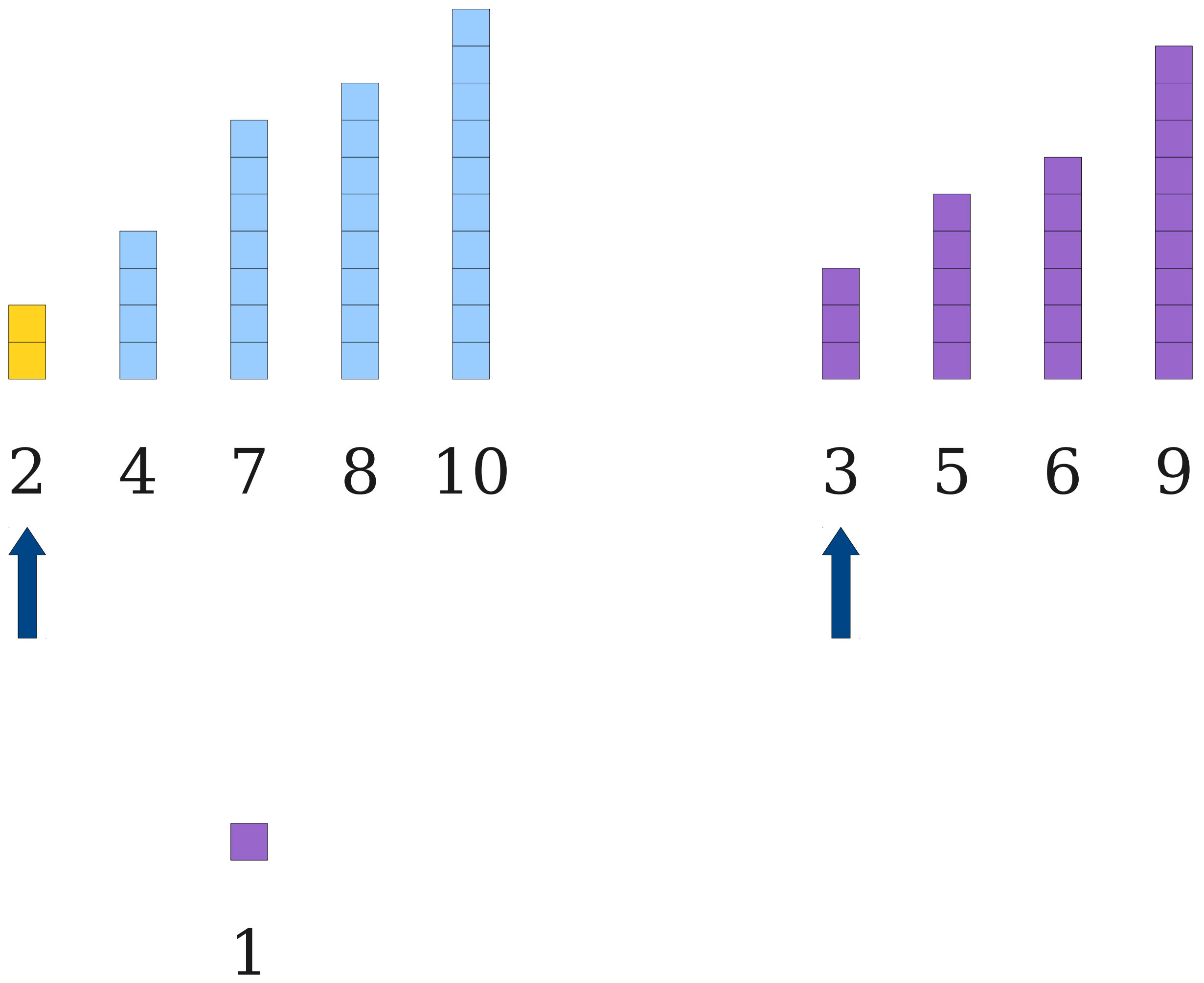

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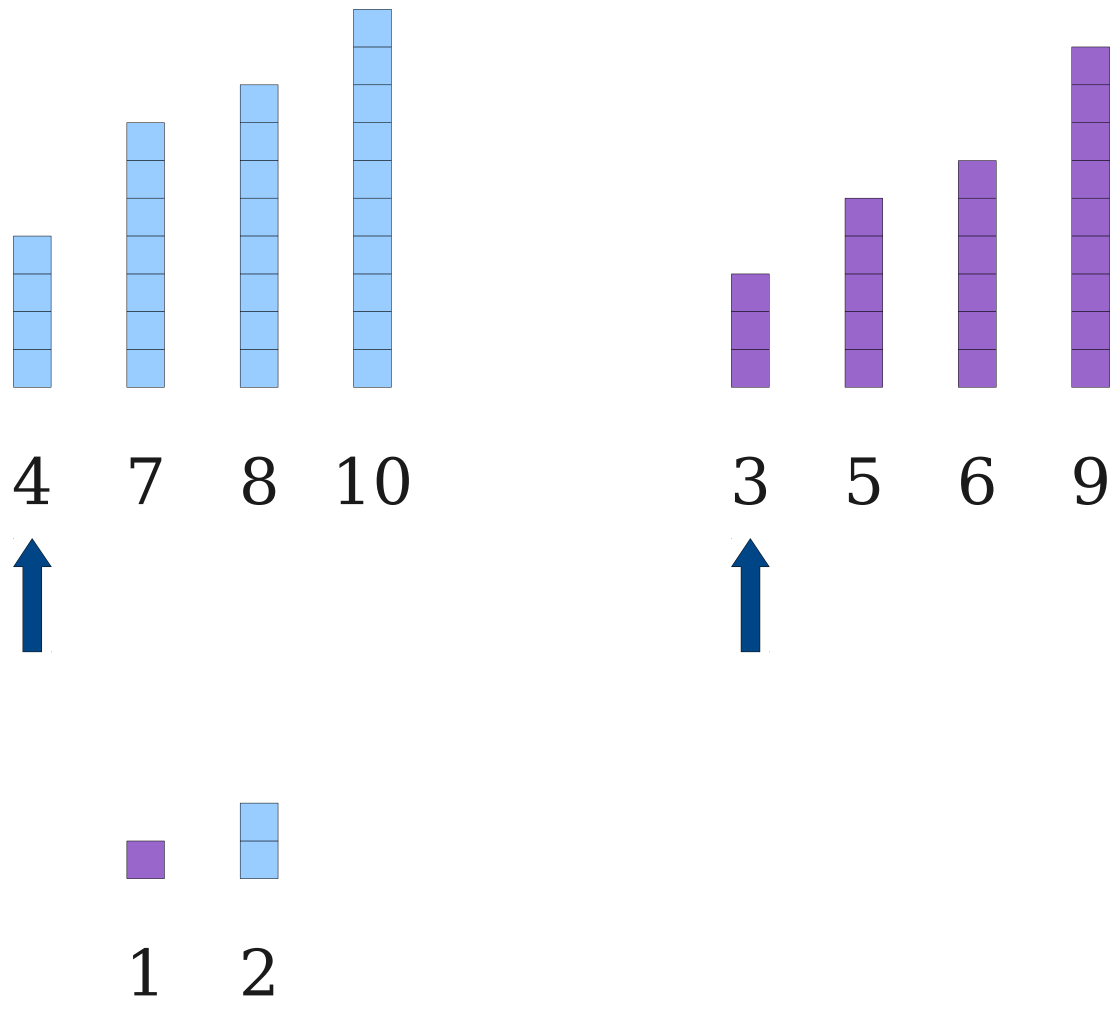

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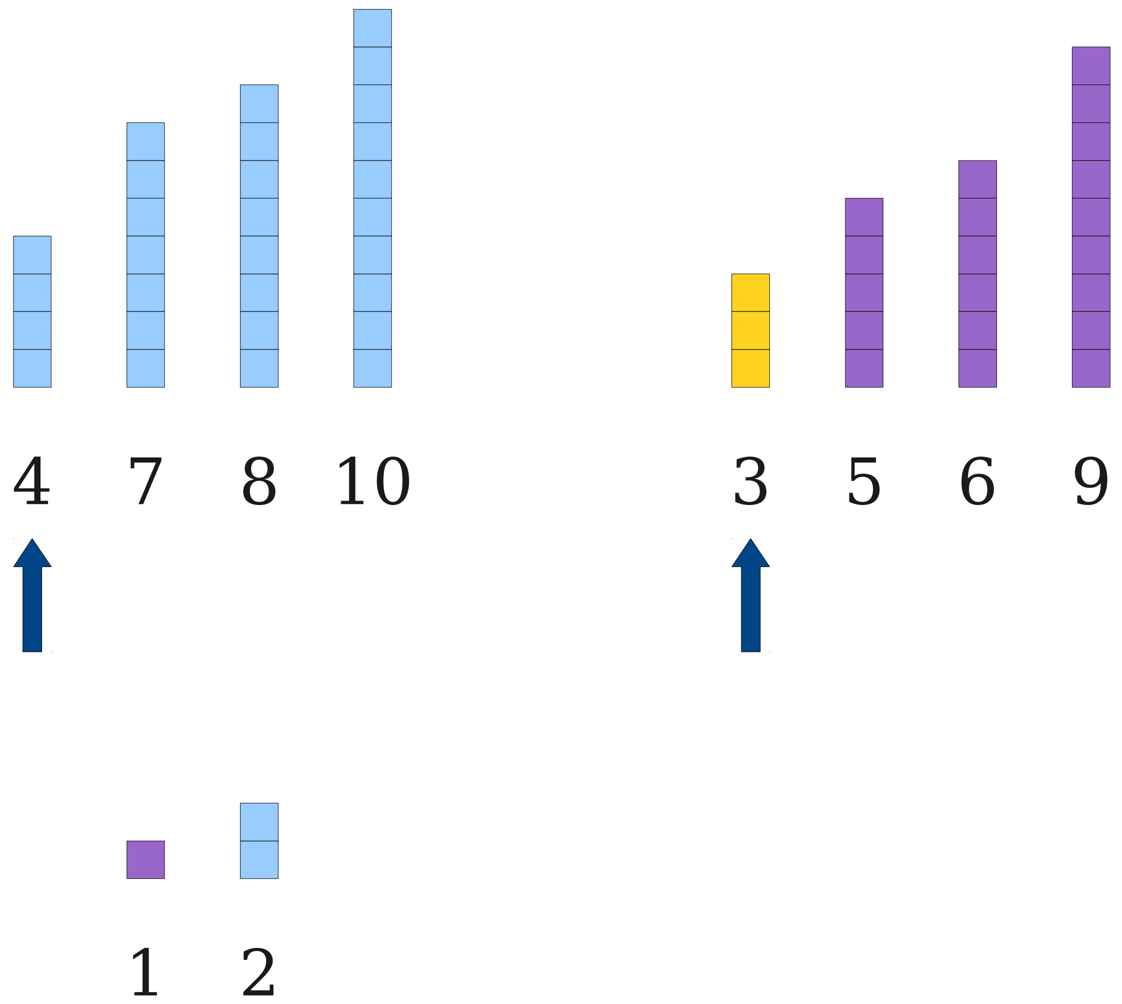

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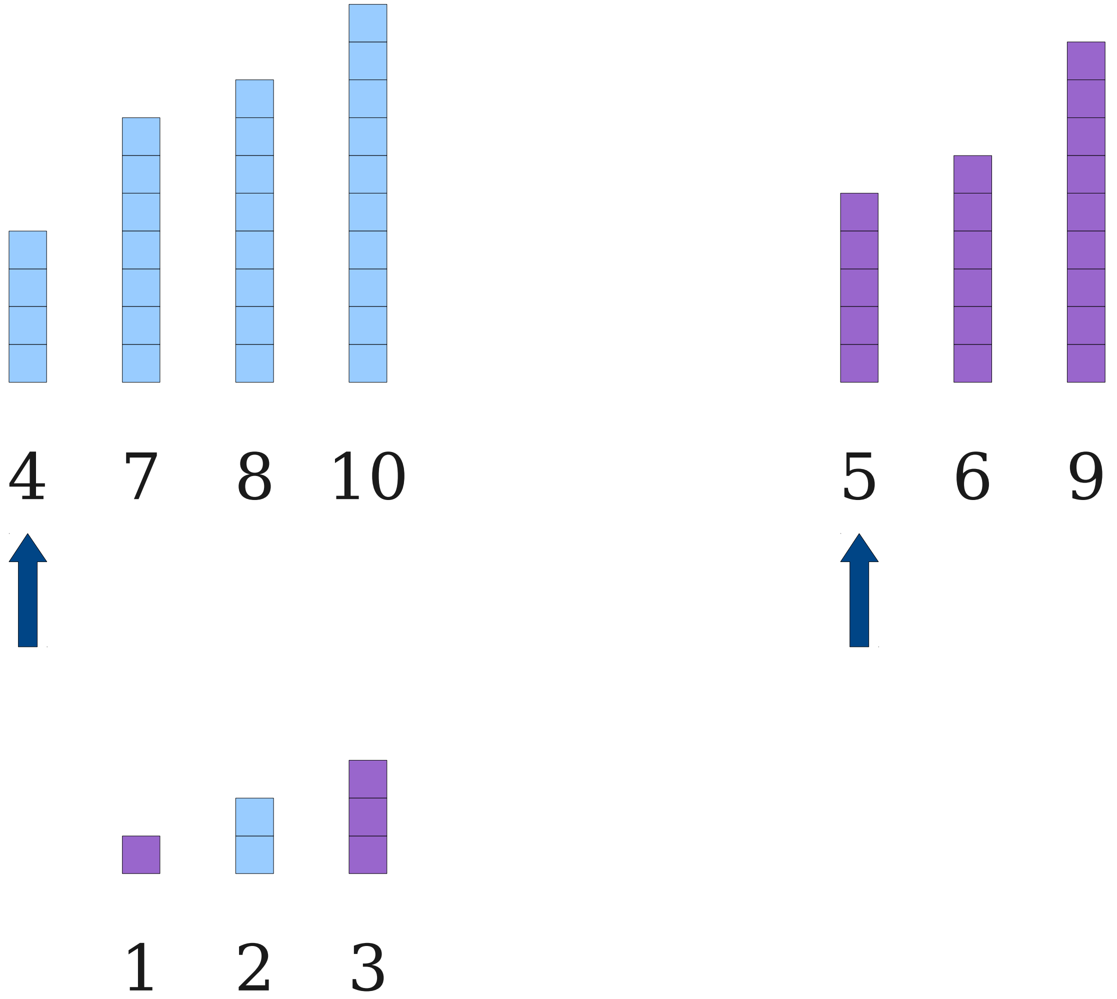

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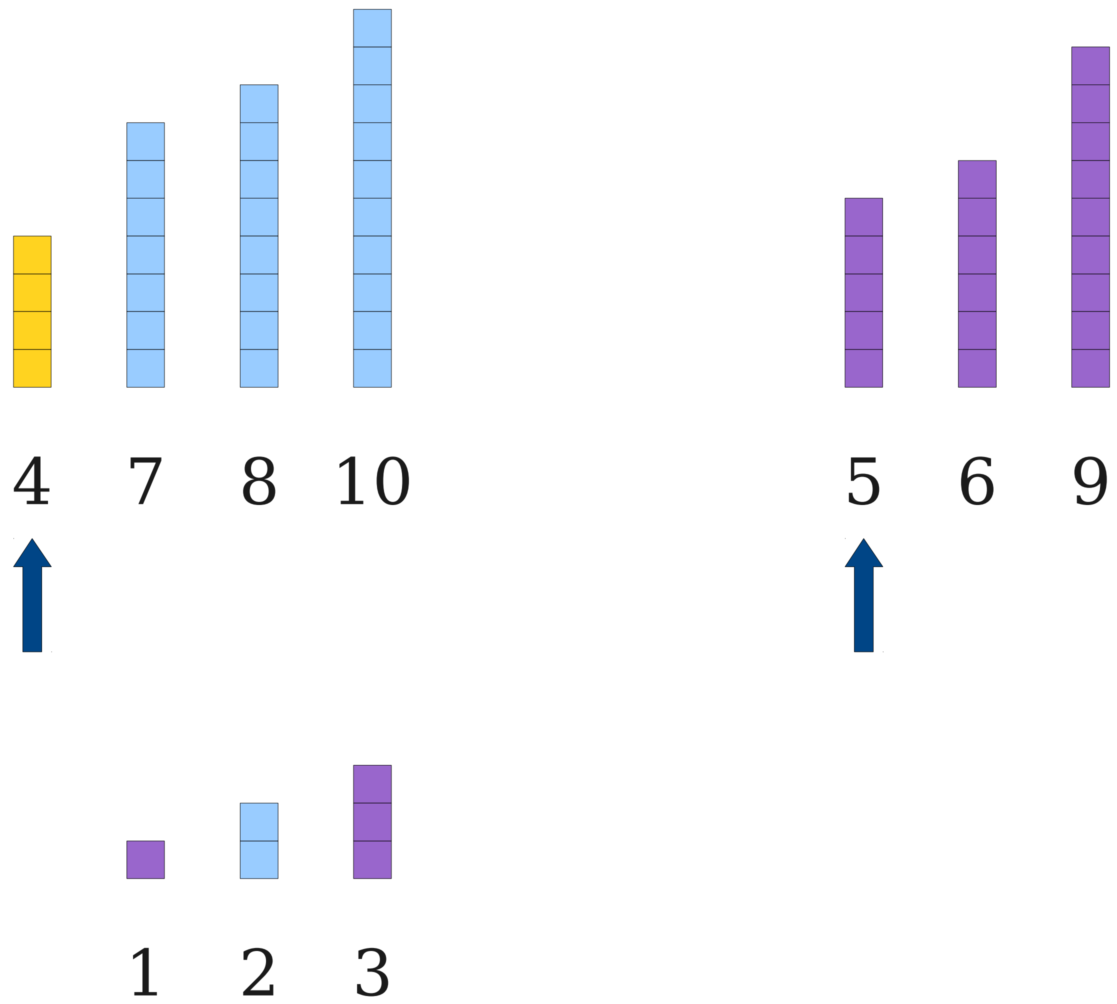

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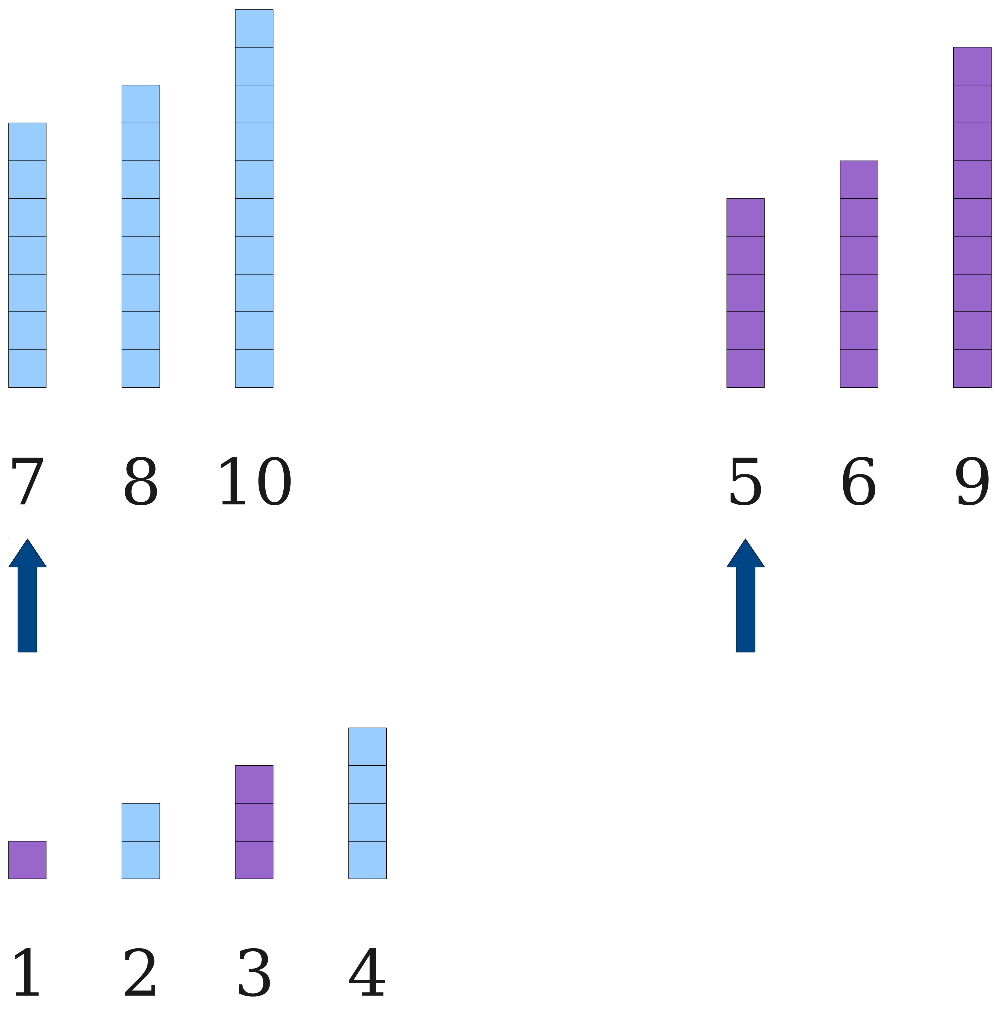

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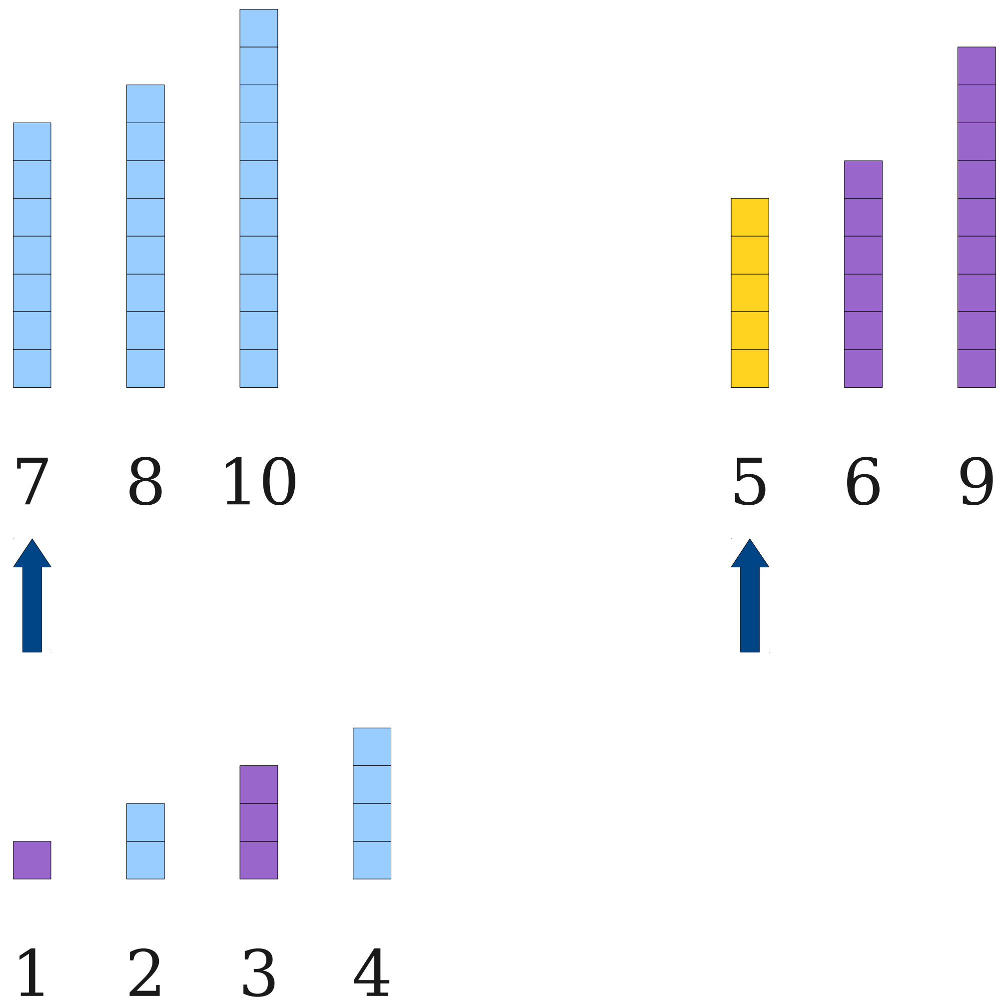

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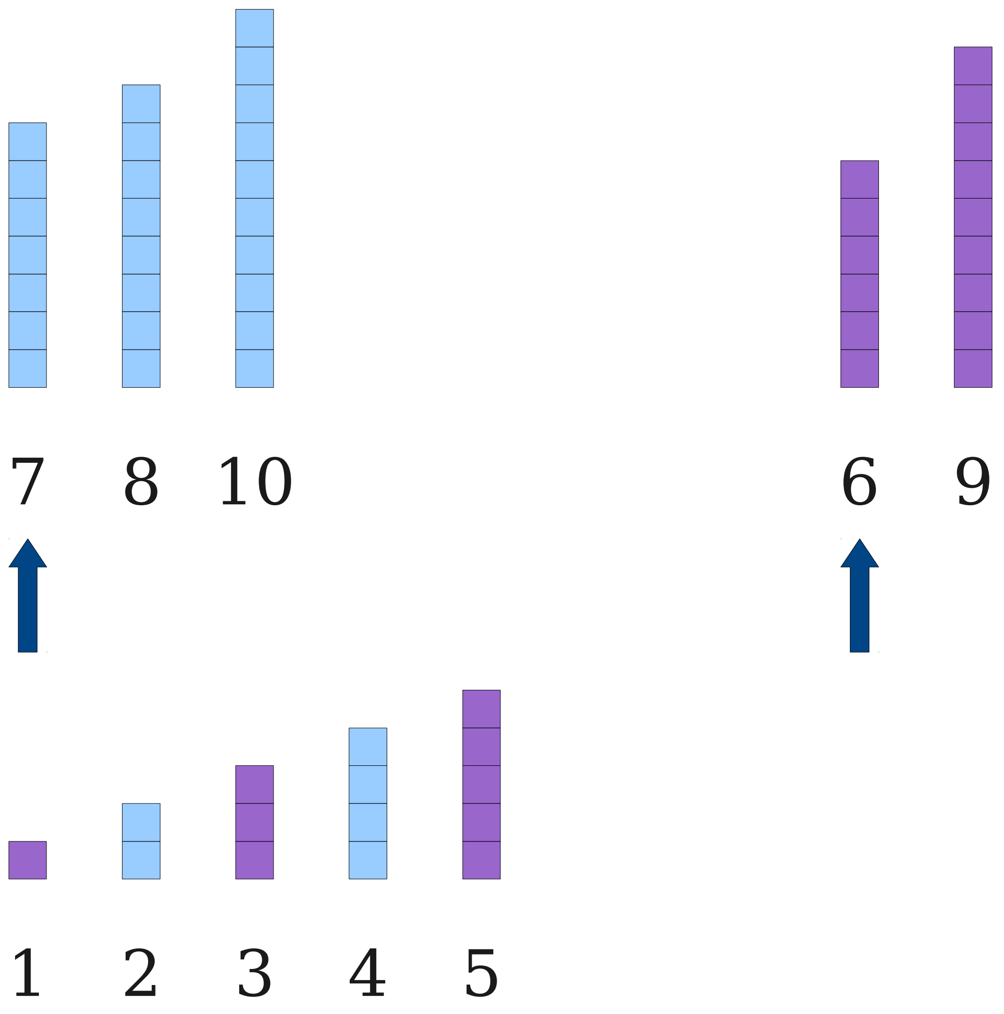

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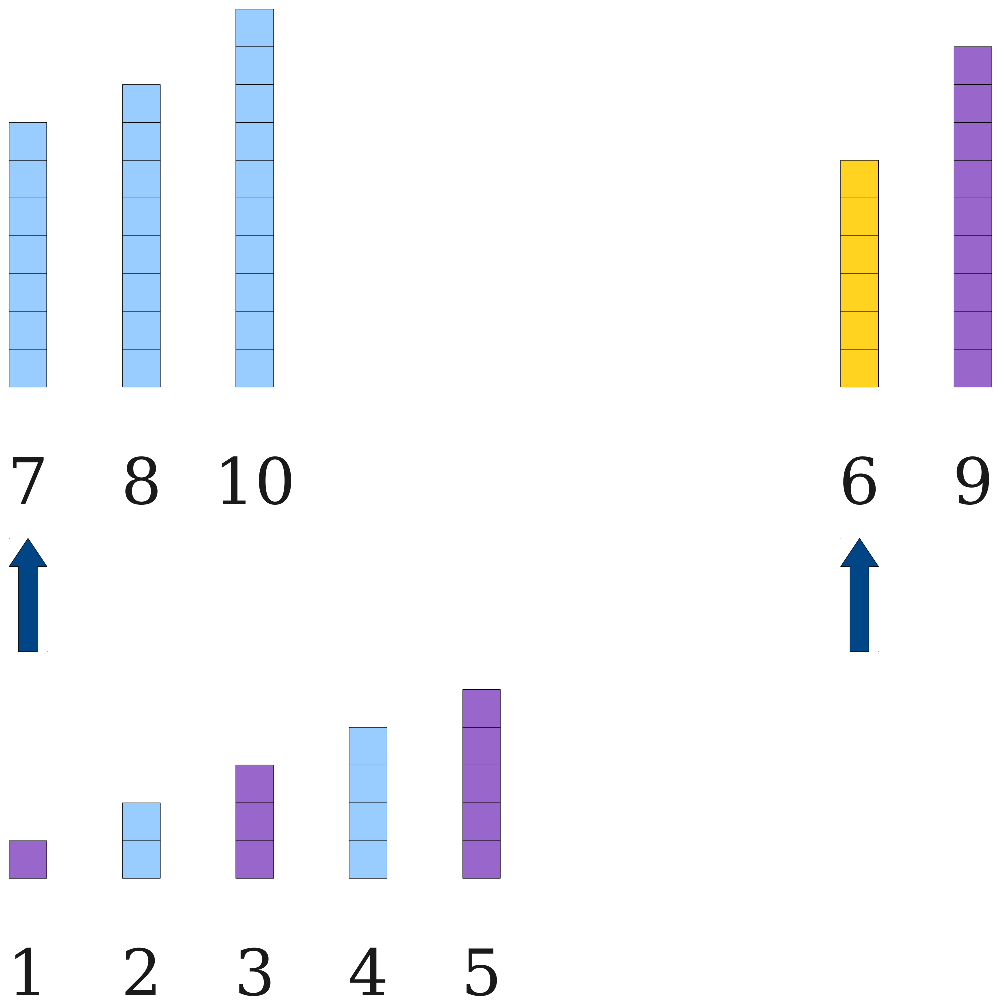

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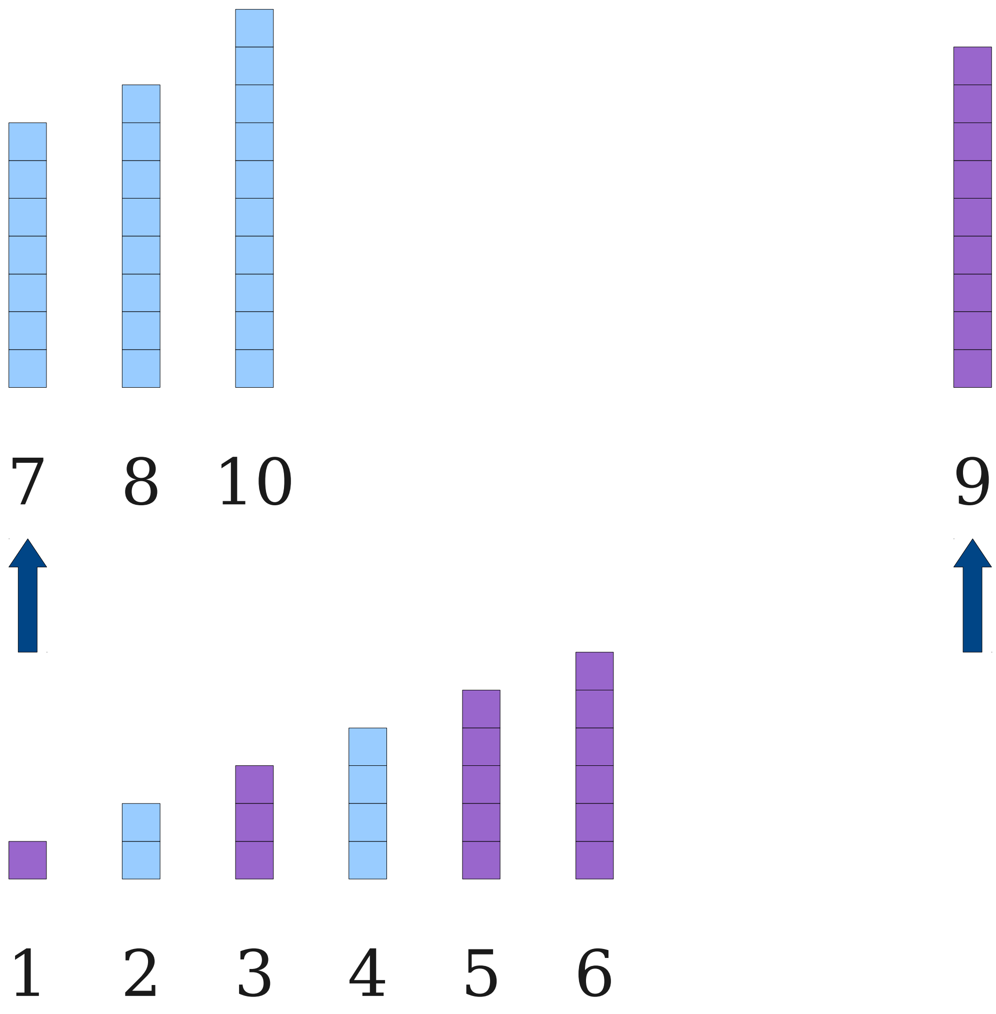

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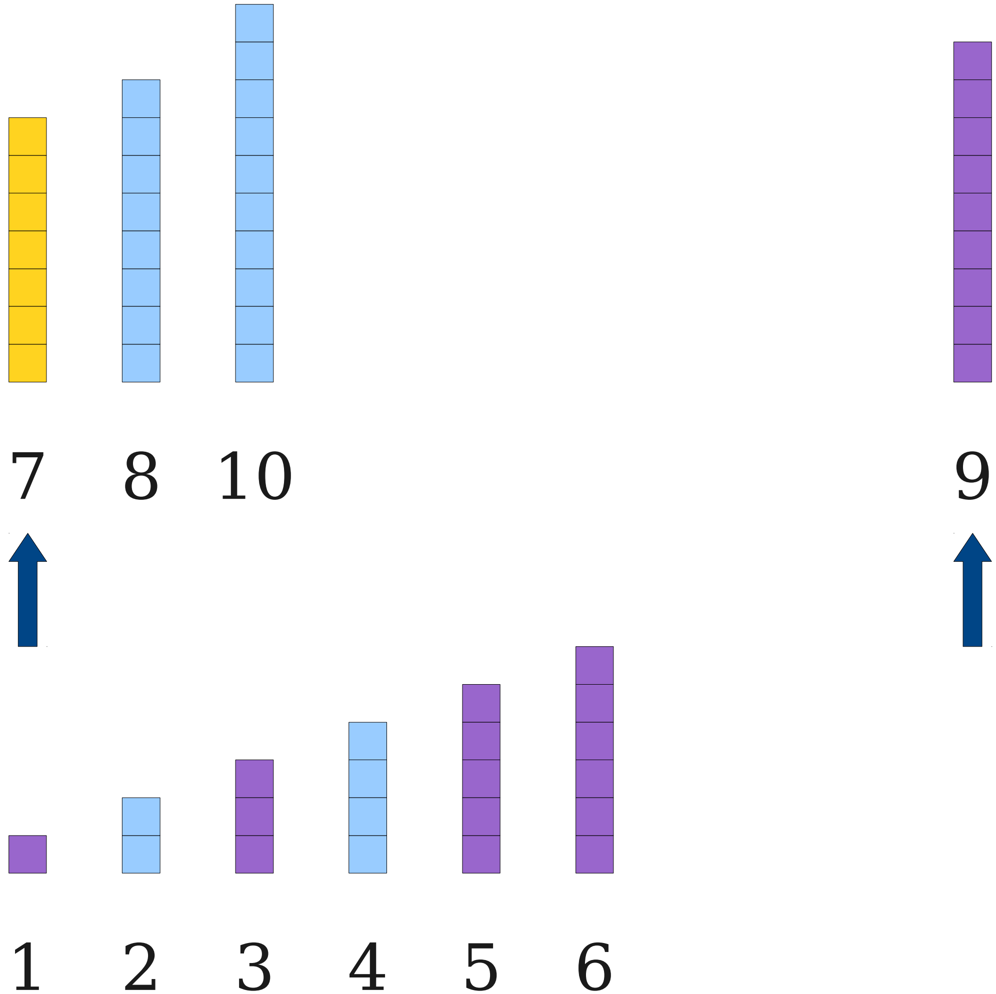

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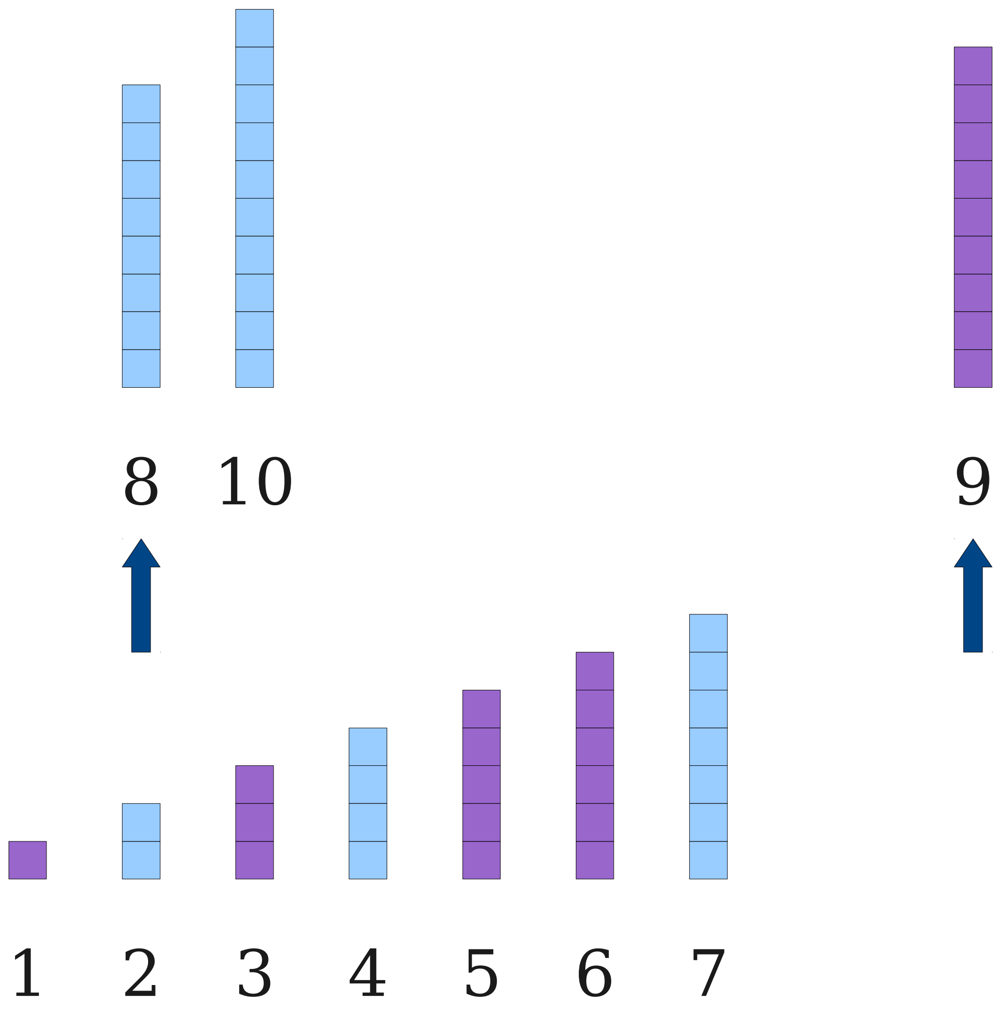

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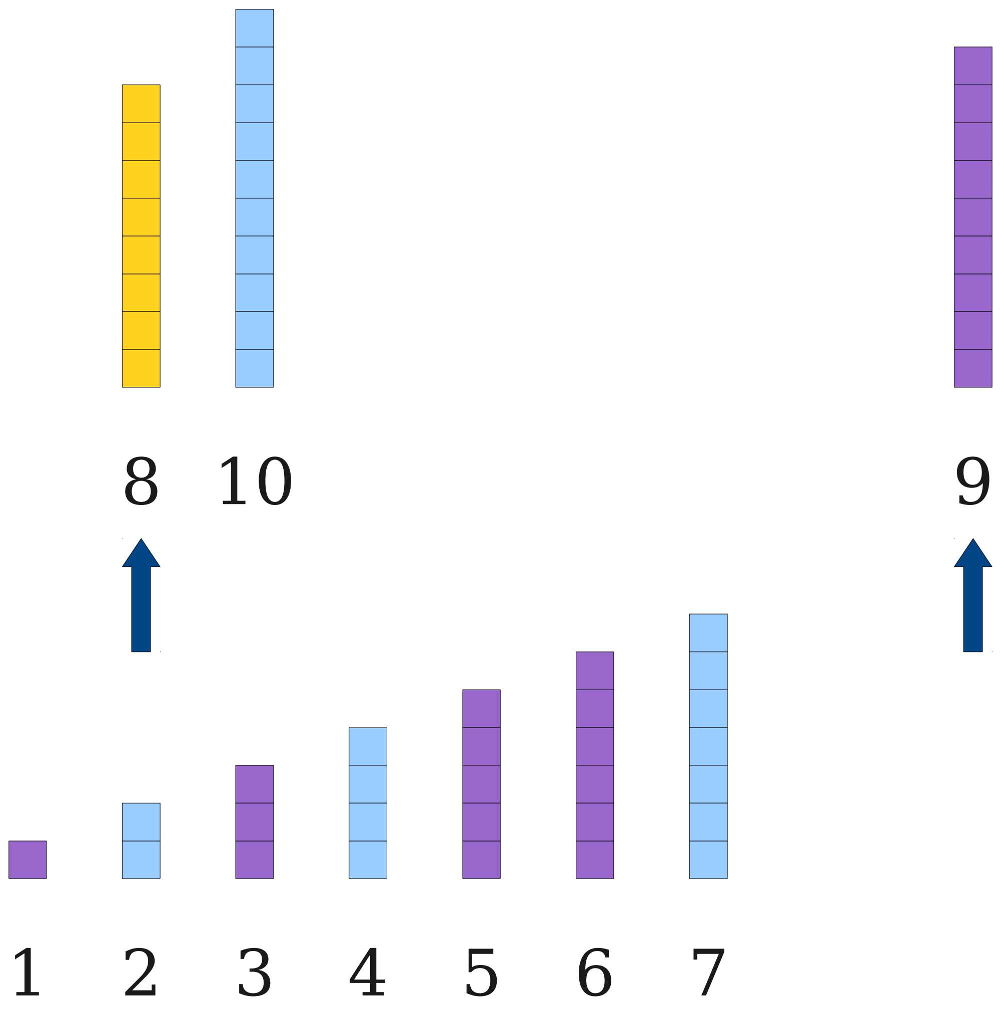

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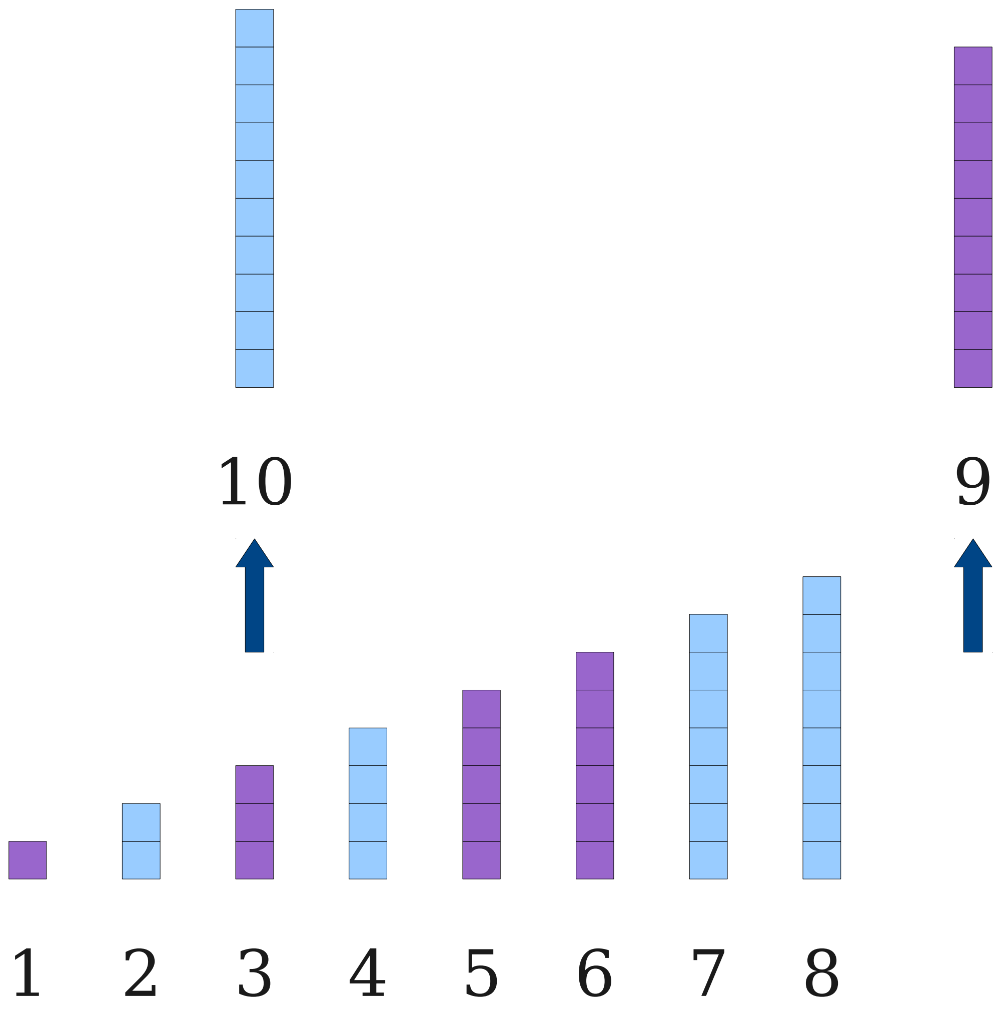

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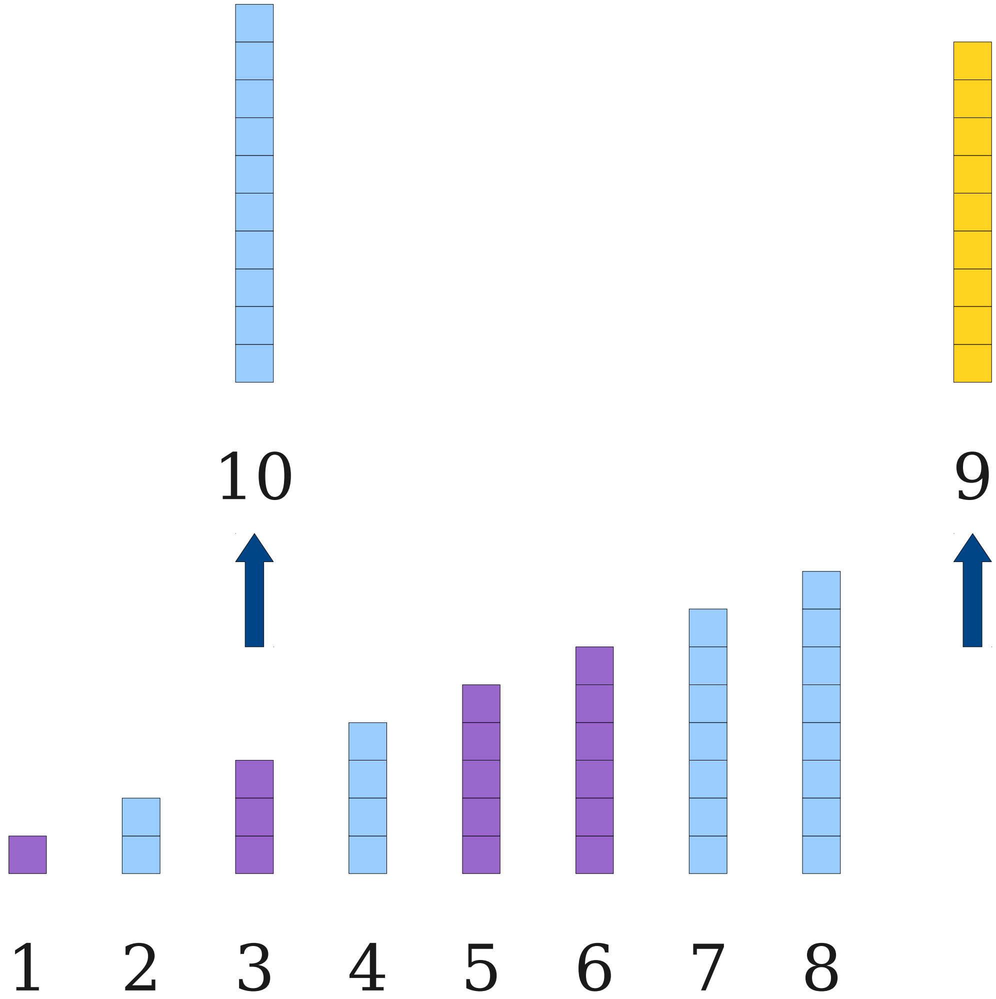

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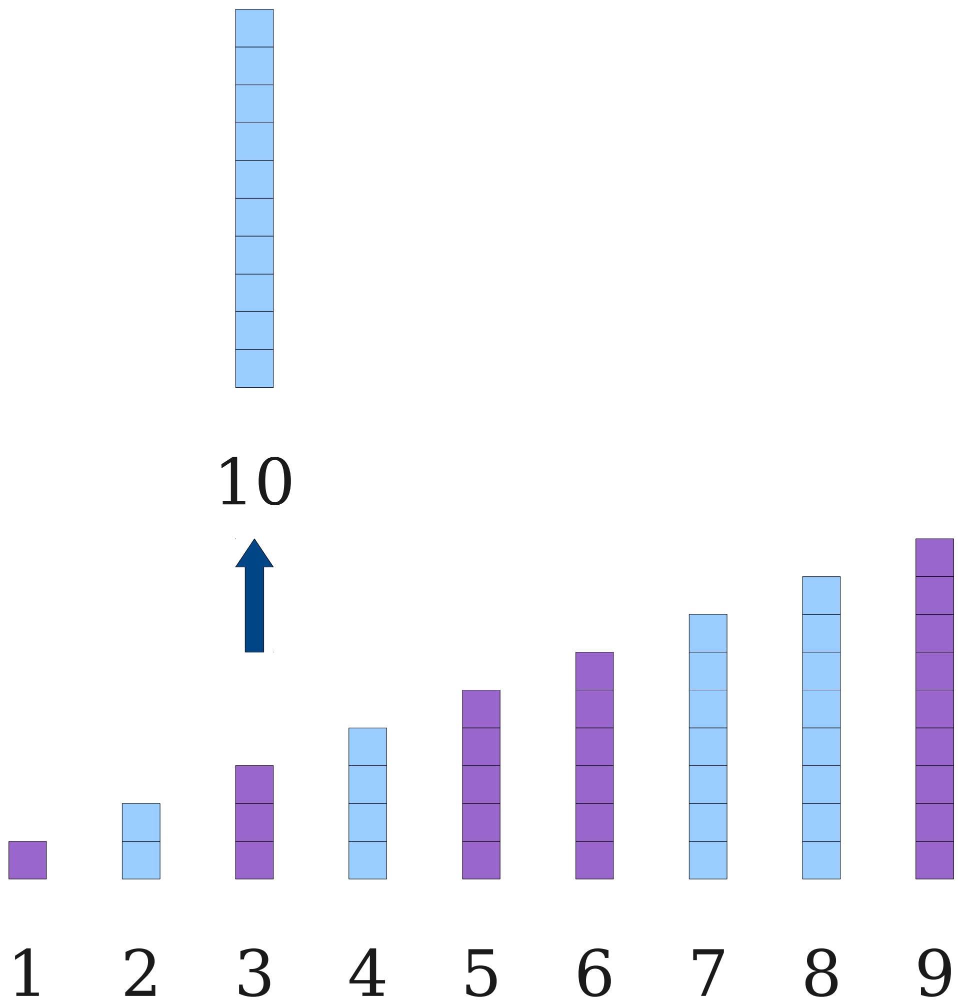

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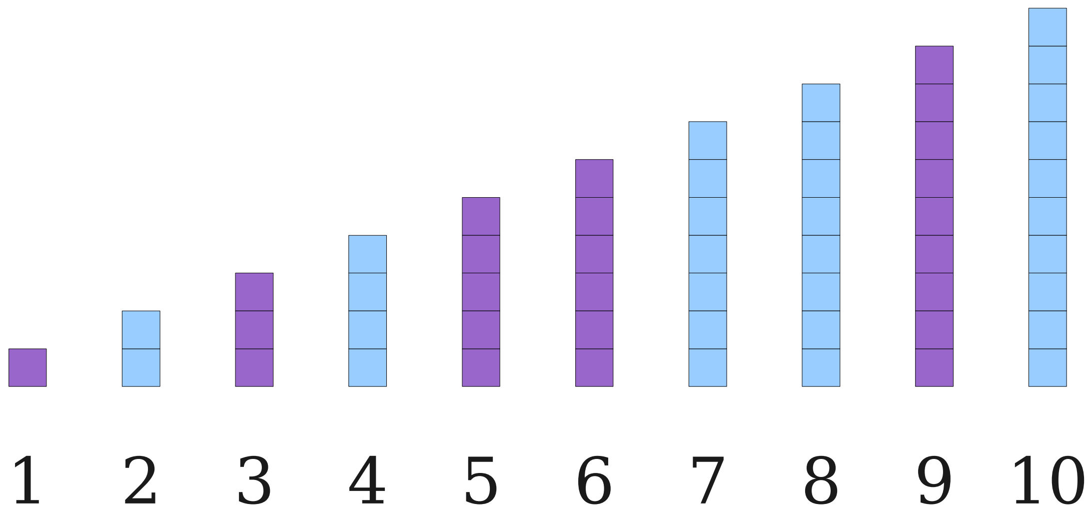
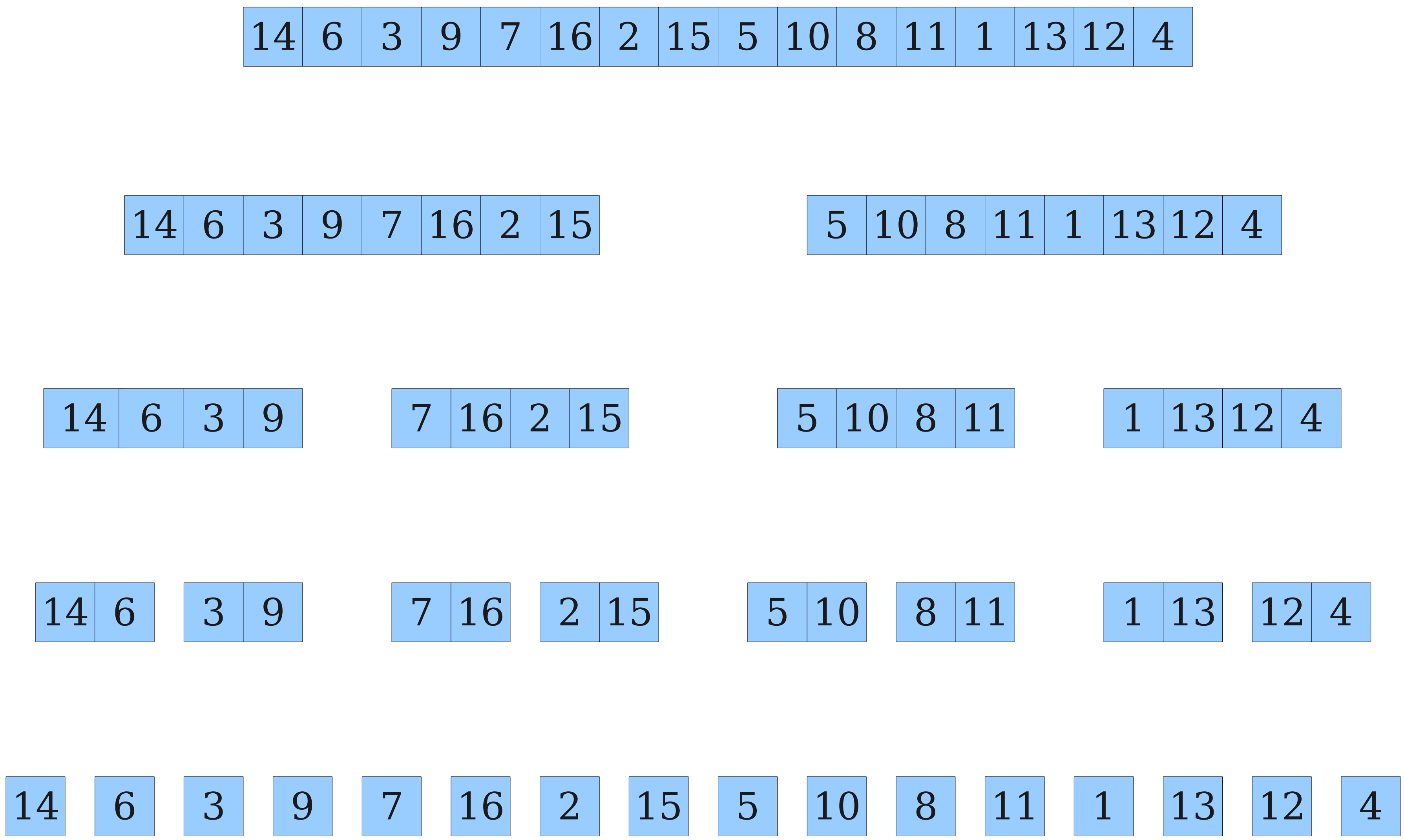


```
procedure merge(list A, list B):
   let result be an empty list.
   while both A and B are nonempty:
      if head(A) < head(B):
         append head(A) to result
         remove head(A) from A
      else:
         append head(B) to result
         remove head(B) from B

   append all elements remaining in A to result
   append all elements remaining in B to result

   return result
```

Complexity: $\boldsymbol{\Theta(m + n)}$,
where m and n are the lengths of the input lists.

Motivating Mergesort

- Splitting the input array in half, sorting each half, and merging them back together will take roughly half as long as soring the original array.
- So why not split the array into fourths? Or eighths?
- **Question**: What happens if we *never stop splitting?*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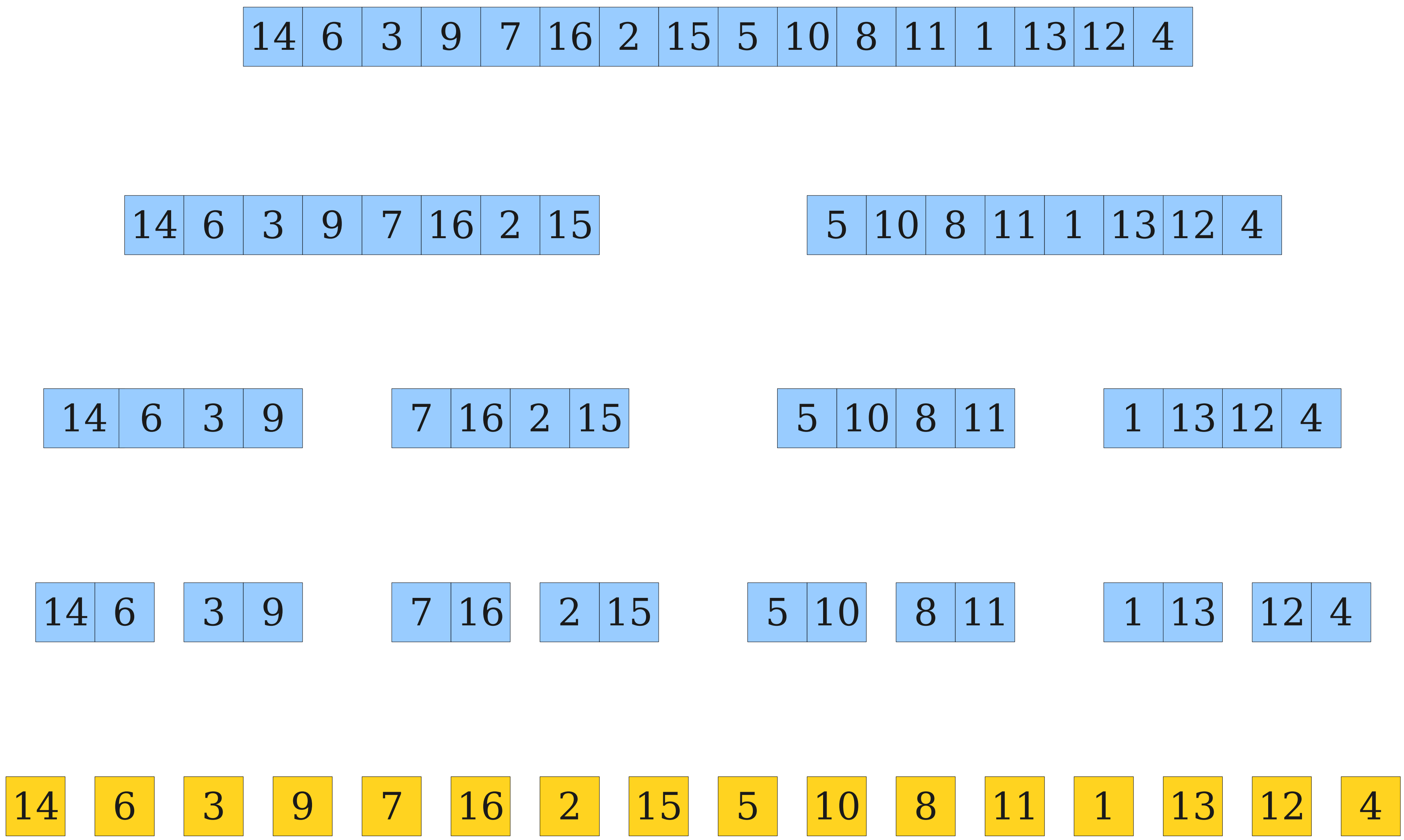
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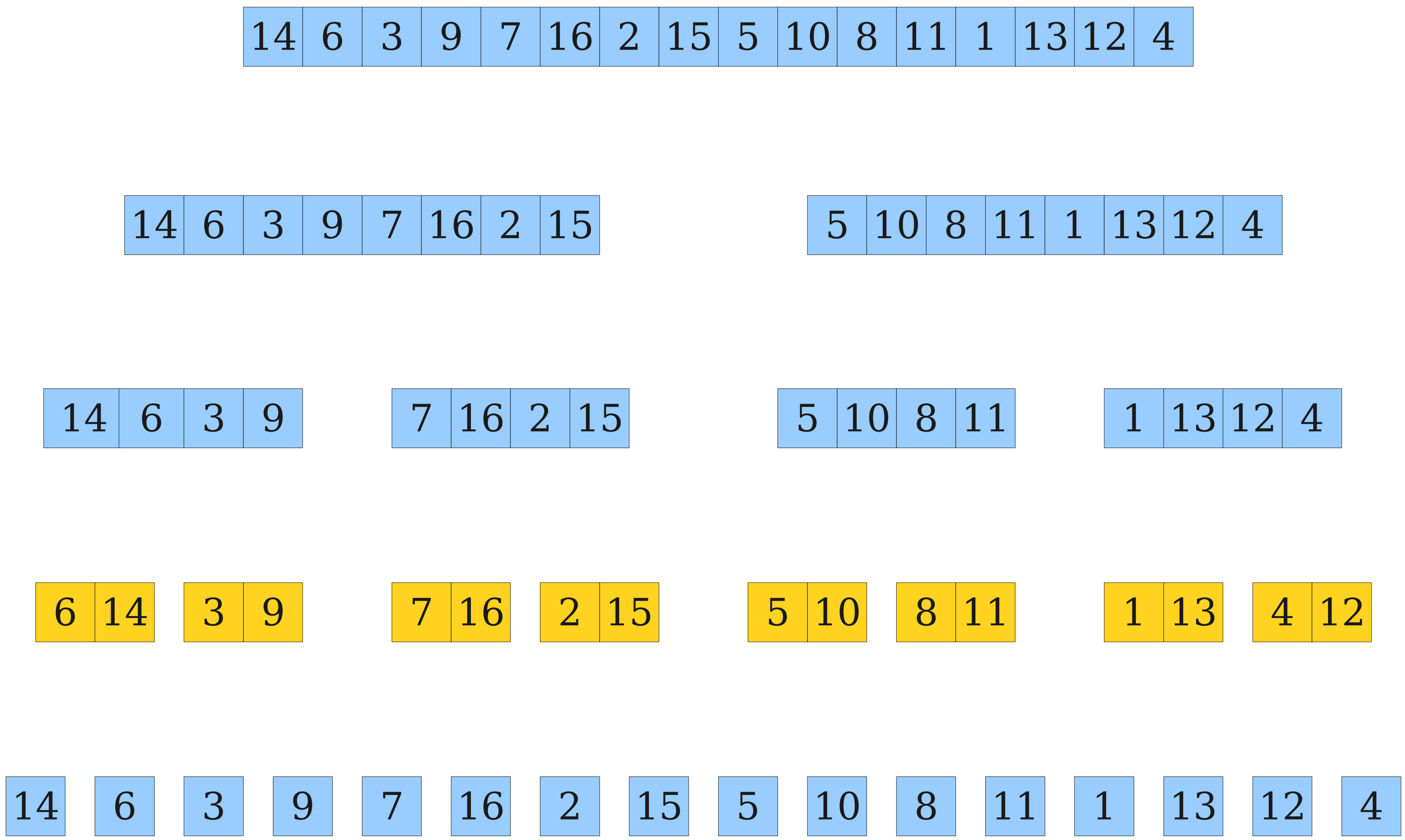
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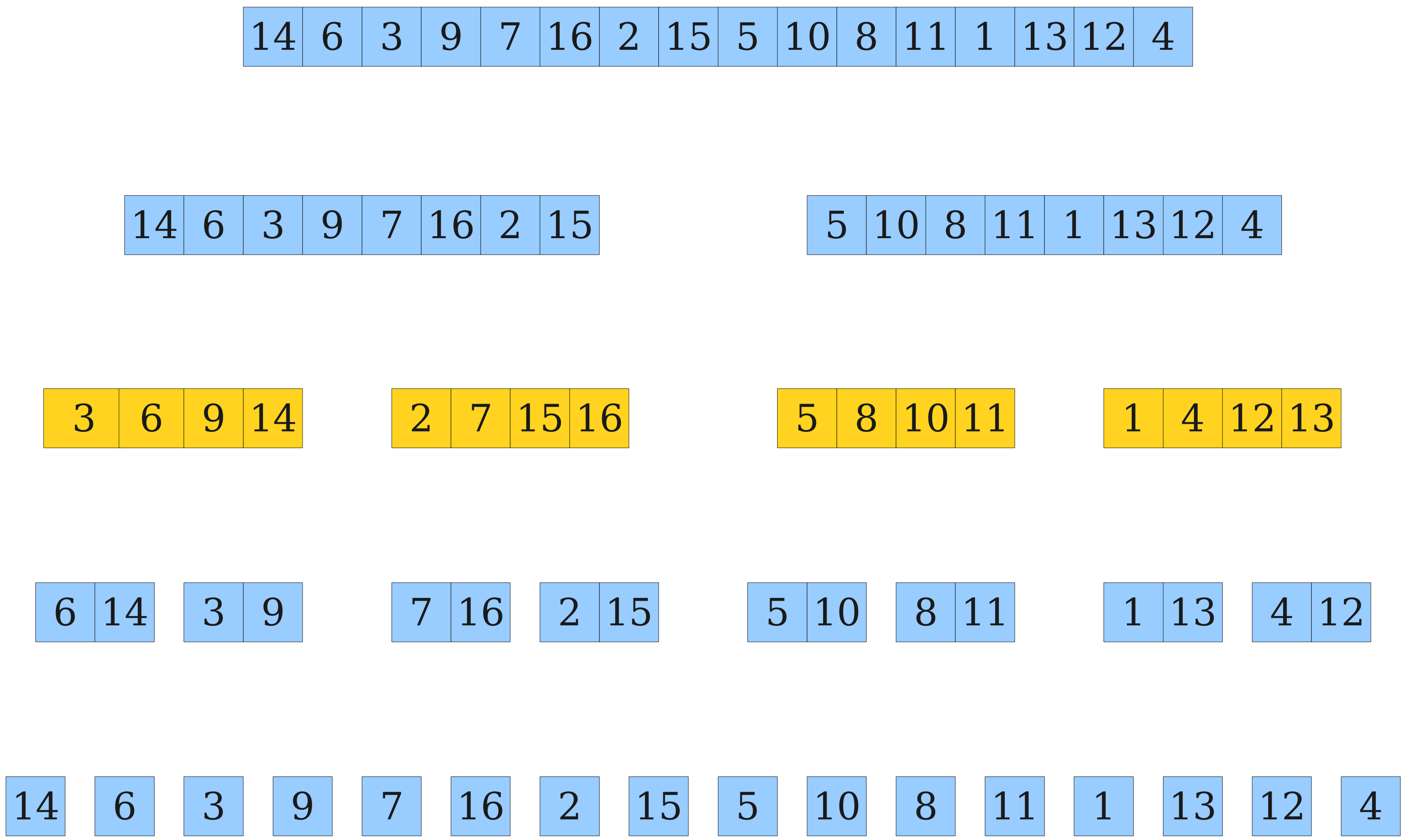
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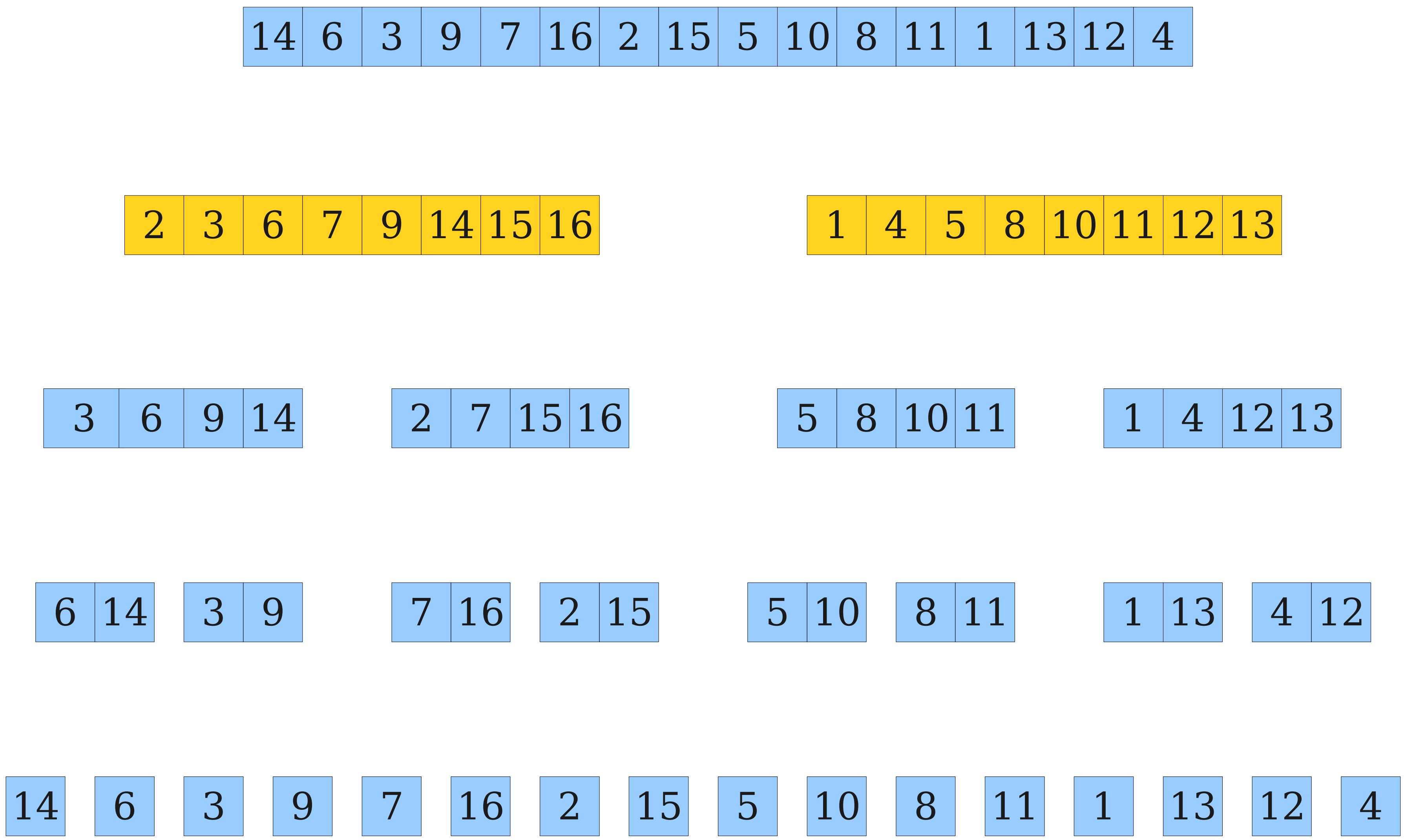
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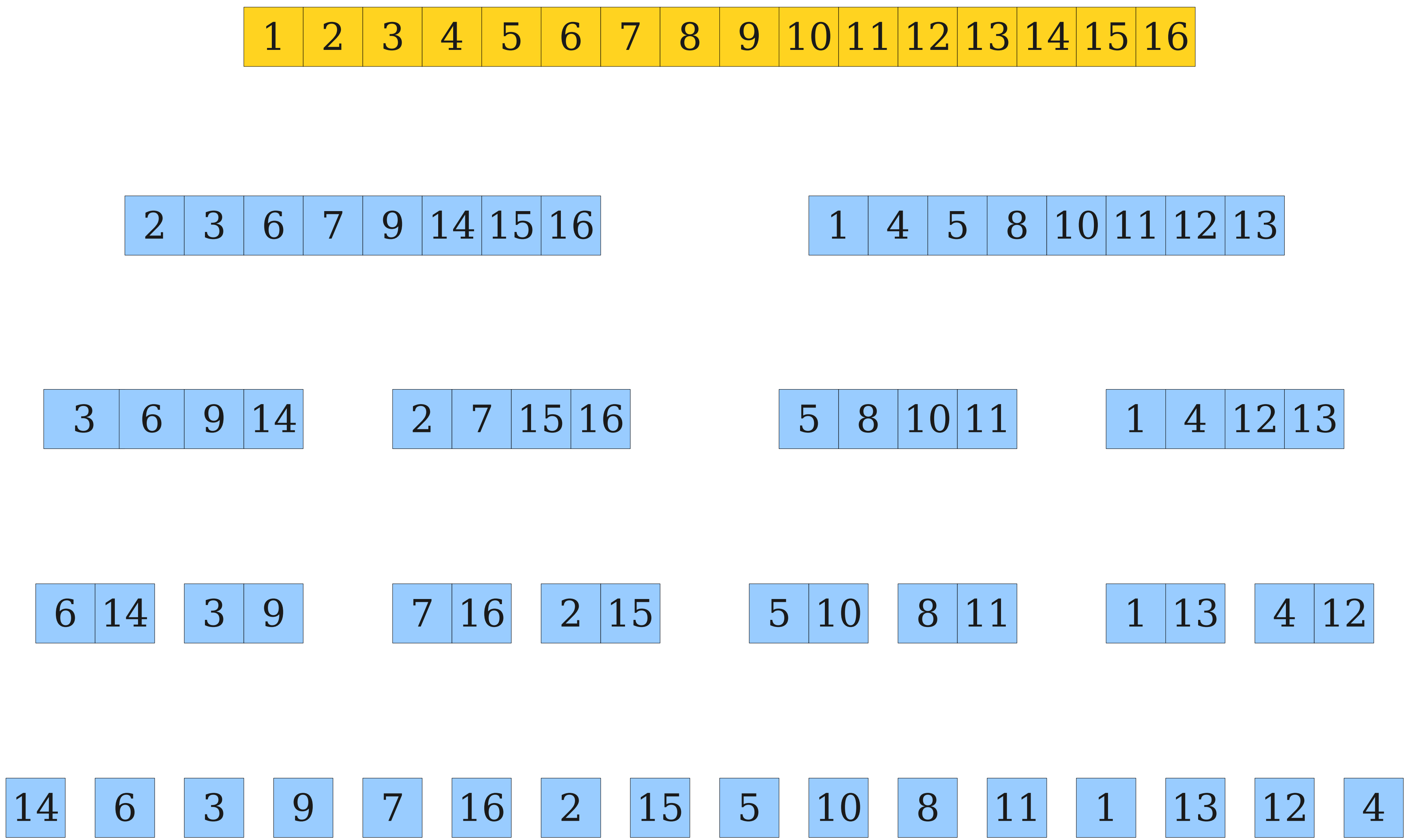
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High-Level Idea

- A recursive sorting algorithm!
- **Base Case**:
  - An empty or single-element list is already sorted.
- **Recursive step**:
  - Break the list in half and recursively sort each part.
  - Merge the sorted halves back together.
- This algorithm is called *mergesort*.

```
procedure mergesort(list A):
   if length(A) ≤ 1:
      return A

   let left  be the first  half of the elements of A
   let right be the second half of the elements of A

   return merge(mergesort(left), mergesort(right))
```

What is the complexity of mergesort?

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$$T(0) = \Theta(1)$$
$$T(1) = \Theta(1)$$
$$T(n) = T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n)$$

Recurrence Relations

- A **recurrence relation** is a function or sequence whose values are defined in terms of earlier values.
- In our case, we get this recurrence for the runtime of mergesort:

$$\boxed{\begin{aligned}&T(0) = \Theta(1)\\&T(1) = \Theta(1)\\&T(n) = T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n)\end{aligned}}$$

- We can **solve** a recurrence by finding an explicit expression for its terms, or by finding an asymptotic bound on its growth rate.
- How do we solve this recurrence?

Simplifying our Recurrence

- It is often difficult to solve recurrences involving floors and ceilings, as ours does.

$$\boxed{\begin{array}{l} T(0) = \Theta(1) \\ T(1) = \Theta(1) \\ T(n) = T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n) \end{array}}$$

- Note that if we only consider $n = 1, 2, 4, 8, 16, \ldots$, then the floors and ceilings are always equivalent to standard division.
- **Simplifying Assumption 1:** We will only consider the recurrence as applied to powers of two.
- We need to justify why this is safe, which we'll do later.

Simplifying our Recurrence

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$$T(n) = T(n / 2) + T(n / 2) + \Theta(n)$$

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Simplifying our Recurrence

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$$T(1) = \Theta(1)$$
$$T(n) = 2T(n / 2) + \Theta(n)$$

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Simplifying our Recurrence

- Without knowing the actual functions hidden by the Θ notation, we cannot get an exact value for the terms in this recurrence.

$$
\boxed{\begin{aligned} T(1) &= \Theta(1) \\ T(n) &= 2T(n / 2) + \Theta(n) \end{aligned}}
$$

- If the $\Theta(1)$ just hides a constant and $\Theta(n)$ just hides a multiple of n, this would be a lot easier to manipulate!

- **Simplifying Assumption 2:** We will pretend that $\Theta(1)$ hides some constant and $\Theta(n)$ hides a multiple of n.
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Simplifying our Recurrence

- Without knowing the actual functions hidden by the Θ notation, we cannot get an exact value for the terms in this recurrence.

$$T(1) = c_1$$
$$T(n) = 2T(n / 2) + c_2 n$$

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Simplifying our Recurrence

- Working with two constants c_1 and c_2 is most accurate, but it makes the math a *lot* harder.

$$T(1) = c_1$$
$$T(n) = 2T(n / 2) + c_2 n$$

- If all we care about is getting an asymptotic bound, these constants are unlikely to make a noticeable difference.
- **Simplifying Assumption 3:** Set $c = \max\{c_1, c_2\}$ and replace the equality with an upper bound.
- We need to justify why this is safe, which we'll do later.

Simplifying our Recurrence

- Working with two constants c_1 and c_2 is most accurate, but it makes the math a *lot* harder.

$$T(1) \le c$$
$$T(n) \le 2T(n / 2) + cn$$

- If all we care about is getting an asymptotic bound, these constants are unlikely to make a noticeable difference.
- **Simplifying Assumption 3:** Set $c = \max\{c_1, c_2\}$ and replace the equality with an upper bound.
- This is less exact, but is easier to manipulate.

The Final Recurrence

- Here is the final version of the recurrence we'll be working with:

$$\mathbf{T(1)} \leq c$$
$$\mathbf{T}(n) \leq 2\mathbf{T}(n / 2) + cn$$

- As before, we will justify why all of these simplifications are safe later on.
- The analysis we're about to do (without justifying the simplifications) is at the level we will expect for most of our discussion of divide-and-conquer algorithms.

Getting an Intuition

- Simple recurrence relations often give rise to surprising results.
- It is often useful to build up an intuition for what the recursion solves to before trying to formally prove it.
- We will explore two methods for doing this:
  - The *iteration method*.
  - The *recursion-tree method*.

Getting an Intuition

Simple recurrence relations often give rise to surprising results.

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We will explore two methods for doing this:

- The *iteration method*.
- The *recursion-tree method*.

$$T(1) \leq c$$
$$T(n) \leq 2T(n / 2) + cn$$

$$T(n) \leq 2T\left(\frac{n}{2}\right) + c\,n$$

$$\boxed{\begin{aligned} &T(1) \leq c \\ &T(n) \leq 2T(n/2) + cn \end{aligned}}$$

$$
\begin{aligned}
\mathrm{T}(n) &\leq 2\mathrm{T}\left(\frac{n}{2}\right) + c\,n \\
&\leq 2\left(2\mathrm{T}\left(\frac{n}{4}\right) + \frac{c\,n}{2}\right) + c\,n
\end{aligned}
$$

$$\boxed{\begin{aligned} &\mathrm{T}(1) \leq c \\ &\mathrm{T}(n) \leq 2\mathrm{T}(n / 2) + cn \end{aligned}}$$

$$
\begin{aligned}
T(n) &\leq 2T\left(\frac{n}{2}\right) + cn \\
&\leq 2\left(2T\left(\frac{n}{4}\right) + \frac{cn}{2}\right) + cn \\
&= 4T\left(\frac{n}{4}\right) + cn + cn
\end{aligned}
$$

$$
\boxed{\begin{array}{l} T(1) \leq c \\ T(n) \leq 2T(n / 2) + cn \end{array}}
$$

$$T(1) \le c$$
$$T(n) \le 2T(n/2) + cn$$

$$\begin{aligned} T(n) &\le 2T\left(\frac{n}{2}\right) + cn \\ &\le 2\left(2T\left(\frac{n}{4}\right) + \frac{cn}{2}\right) + cn \\ &= 4T\left(\frac{n}{4}\right) + cn + cn \\ &= 4T\left(\frac{n}{4}\right) + 2cn \end{aligned}$$

$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le 2T(n/2) + cn \end{array}}$$

$$\begin{aligned}
T(n) &\le 2T\left(\frac{n}{2}\right) + cn \\
&\le 2\left(2T\left(\frac{n}{4}\right) + \frac{cn}{2}\right) + cn \\
&= 4T\left(\frac{n}{4}\right) + cn + cn \\
&= 4T\left(\frac{n}{4}\right) + 2cn \\
&\le 4\left(2T\left(\frac{n}{8}\right) + \frac{cn}{4}\right) + 2cn
\end{aligned}$$

$$T(1) \le c$$
$$T(n) \le 2T(n/2) + cn$$

$$\begin{aligned}
T(n) &\le 2T\left(\frac{n}{2}\right) + cn \\
&\le 2\left(2T\left(\frac{n}{4}\right) + \frac{cn}{2}\right) + cn \\
&= 4T\left(\frac{n}{4}\right) + cn + cn \\
&= 4T\left(\frac{n}{4}\right) + 2cn \\
&\le 4\left(2T\left(\frac{n}{8}\right) + \frac{cn}{4}\right) + 2cn \\
&= 8T\left(\frac{n}{8}\right) + cn + 2cn
\end{aligned}$$

$$\mathrm{T}(1) \leq c$$
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$$\begin{aligned}
\mathrm{T}(n) &\leq 2\mathrm{T}\left(\frac{n}{2}\right) + cn \\
&\leq 2\left(2\mathrm{T}\left(\frac{n}{4}\right) + \frac{cn}{2}\right) + cn \\
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&= 4\mathrm{T}\left(\frac{n}{4}\right) + 2cn \\
&\leq 4\left(2\mathrm{T}\left(\frac{n}{8}\right) + \frac{cn}{4}\right) + 2cn \\
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$T(1) \le c$
$T(n) \le 2T(n / 2) + cn$

$$
\begin{aligned}
T(n) &\le 2T\left(\frac{n}{2}\right)+cn \\
&\le 2\left(2T\left(\frac{n}{4}\right)+\frac{cn}{2}\right)+cn \\
&= 4T\left(\frac{n}{4}\right)+cn+cn \\
&= 4T\left(\frac{n}{4}\right)+2cn \\
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&\dots \\
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\end{aligned}
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$n / 2^k = 1$

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$n / 2^k = 1$
$n = 2^k$

$$
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&\leq 4\left(2T\left(\frac{n}{8}\right) + \frac{cn}{4}\right) + 2cn \\
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$$

$$\boxed{\begin{array}{l} T(1) \leq c \\ T(n) \leq 2T(n/2) + cn \end{array}}$$

$$\boxed{\begin{array}{c} n / 2^k = 1 \\ n = 2^k \\ \log_2 n = k \end{array}}$$

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$$\boxed{\begin{array}{l} \mathrm{T}(1) \le c \\ \mathrm{T}(n) \le 2\mathrm{T}(n / 2) + cn \end{array}}$$

$$\mathrm{T}(n) \quad \le \quad 2^k \, \mathrm{T}\left(\frac{n}{2^k}\right) + k\, c\, n$$

$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le 2T(n / 2) + cn \end{array}}$$

$$\begin{aligned} T(n) &\le 2^k T\left(\frac{n}{2^k}\right) + kcn \\ &= 2^{\log_2 n} T(1) + cn\log_2 n \end{aligned}$$

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$$\begin{aligned} T(n) &\leq 2^k T\left(\frac{n}{2^k}\right) + k c n \\ &= 2^{\log_2 n} T(1) + c n \log_2 n \\ &= n T(1) + c n \log_2 n \end{aligned}$$

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$$\begin{aligned}
T(n) &\le 2^k T\left(\frac{n}{2^k}\right) + k c n \\
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&= n T(1) + c n \log_2 n \\
&\le c n + c n \log_2 n
\end{aligned}$$

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$$\begin{aligned}
T(n) &\leq 2^k T\left(\frac{n}{2^k}\right) + kcn \\
&= 2^{\log_2 n} T(1) + cn\log_2 n \\
&= nT(1) + cn\log_2 n \\
&\leq cn + cn\log_2 n \\
&= O(n\log n)
\end{aligned}$$

The Iteration Method

- What we just saw is an example of the *iteration method*.
- Keep plugging the recurrence into itself until you spot a pattern, then try to simplify.
- Doesn't always give an exact answer, but useful for building up an intuition.

Getting an Intuition

- Simple recurrence relations often give rise to surprising results.
- It is often useful to build up an intuition for what the recursion solves to before trying to formally prove it.
- We will explore two methods for doing this:
  - The *iteration method*.
  - The *recursion-tree method*.

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Simple recurrence relations often give rise to surprising results.

It is often useful to build up an intuition for what the recursion solves to before trying to formally prove it.

We will explore two methods for doing this:

- The *iteration method*.
- The *recursion-tree method*.

$$\boxed{\begin{aligned} &T(1) \le c \\ &T(n) \le 2T(n / 2) + cn \end{aligned}}$$

$\mathrm{T}(1) \leq c$
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cn

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cn

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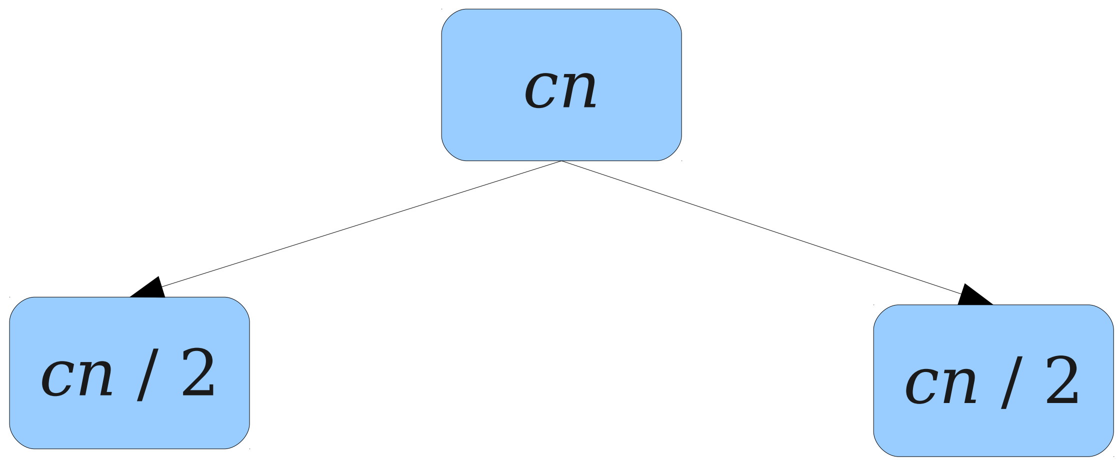

cn

$T(1) \leq c$

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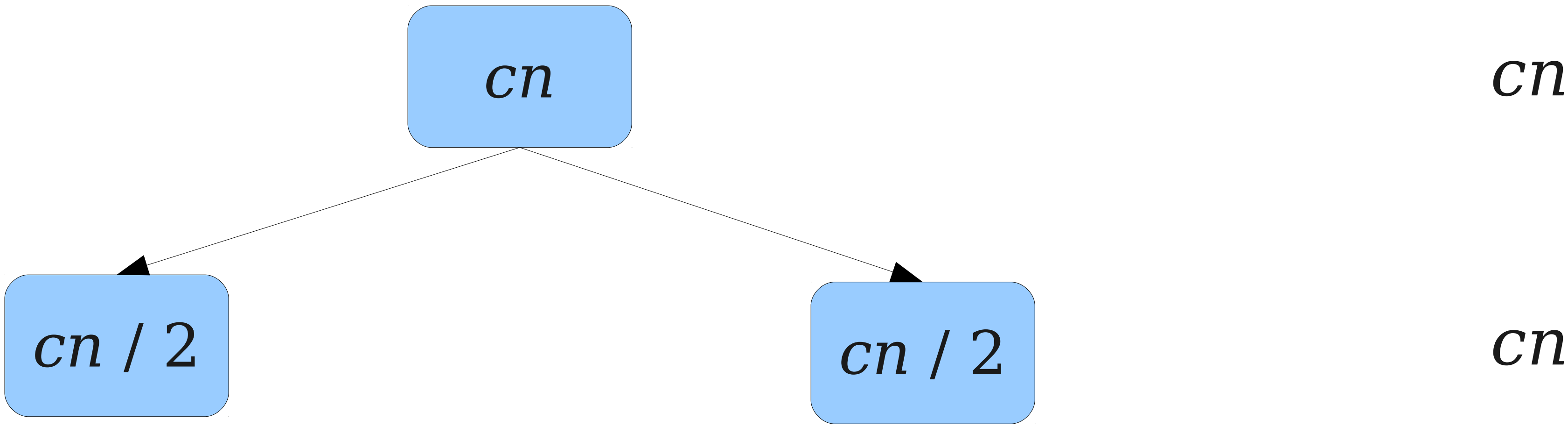

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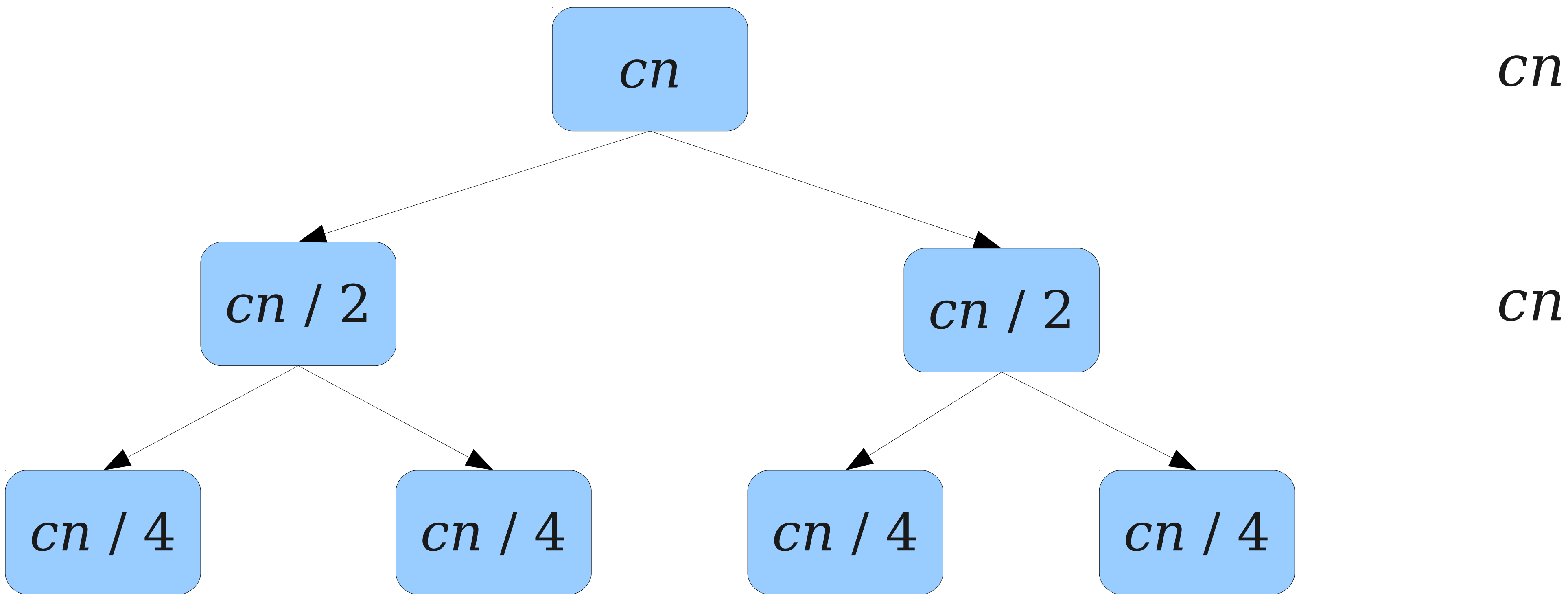

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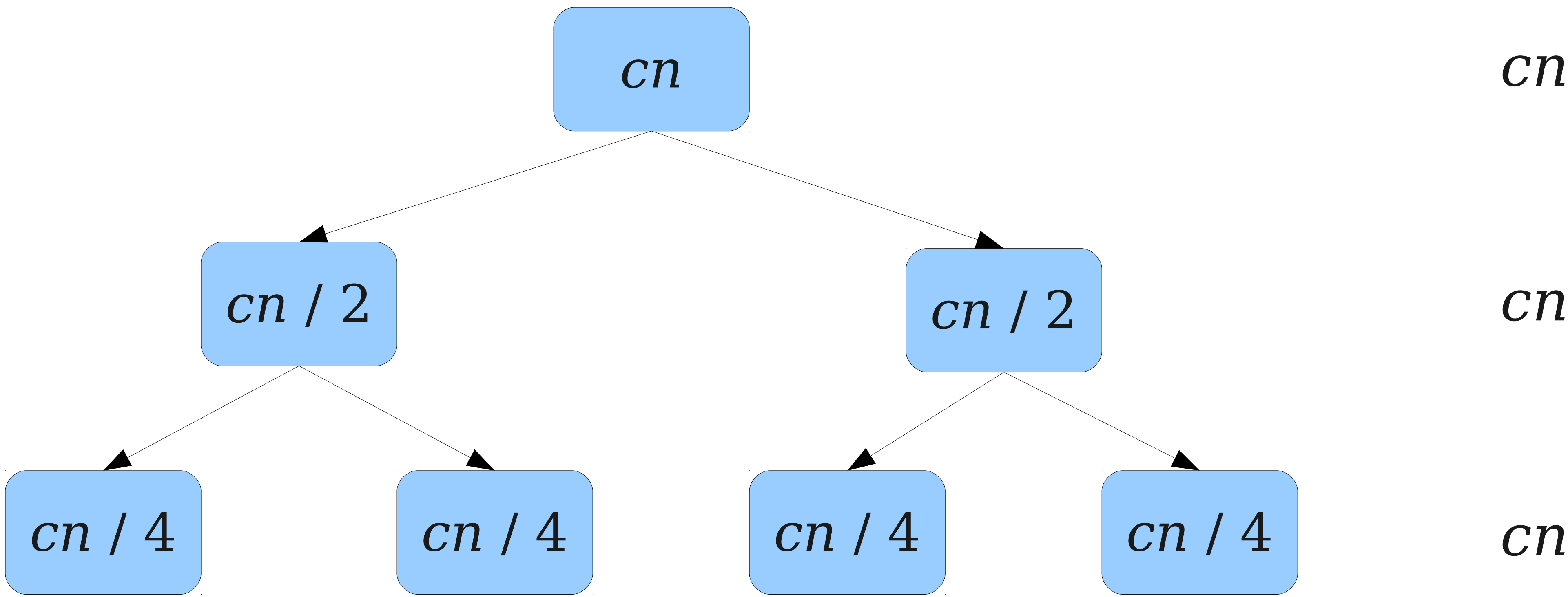

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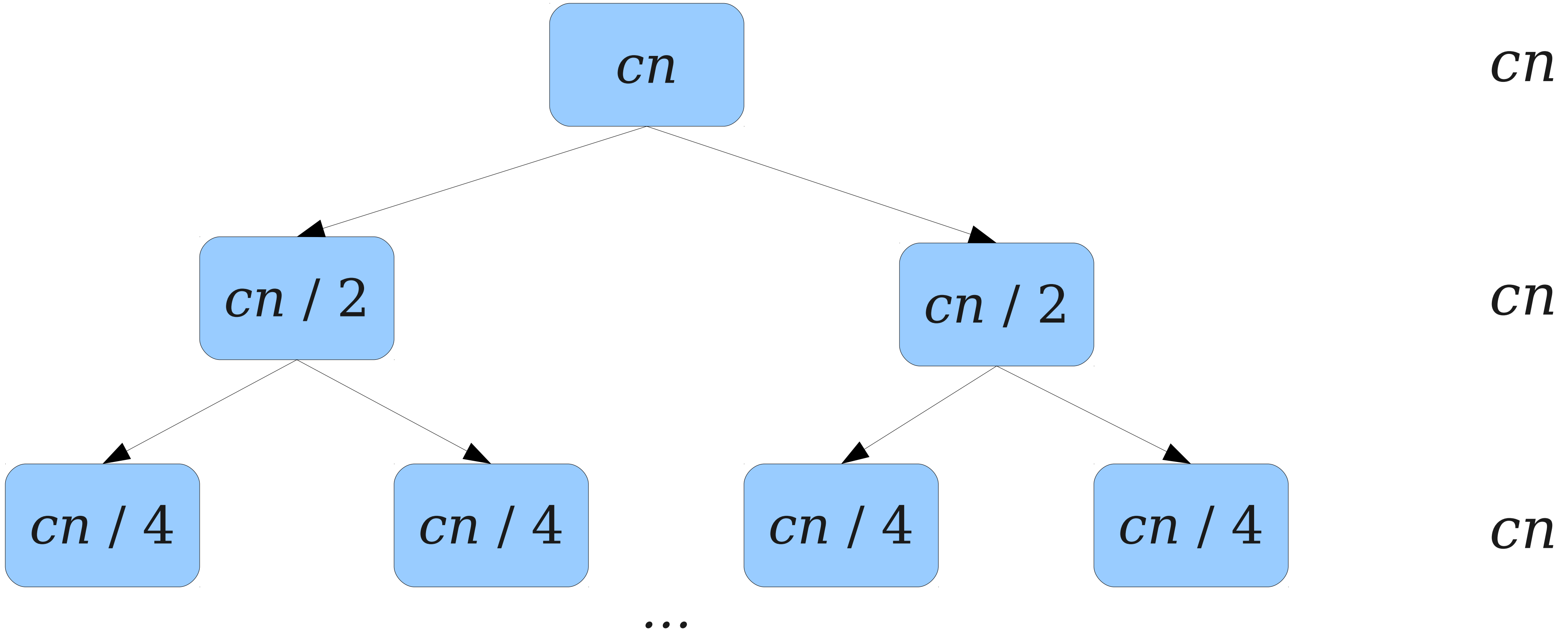

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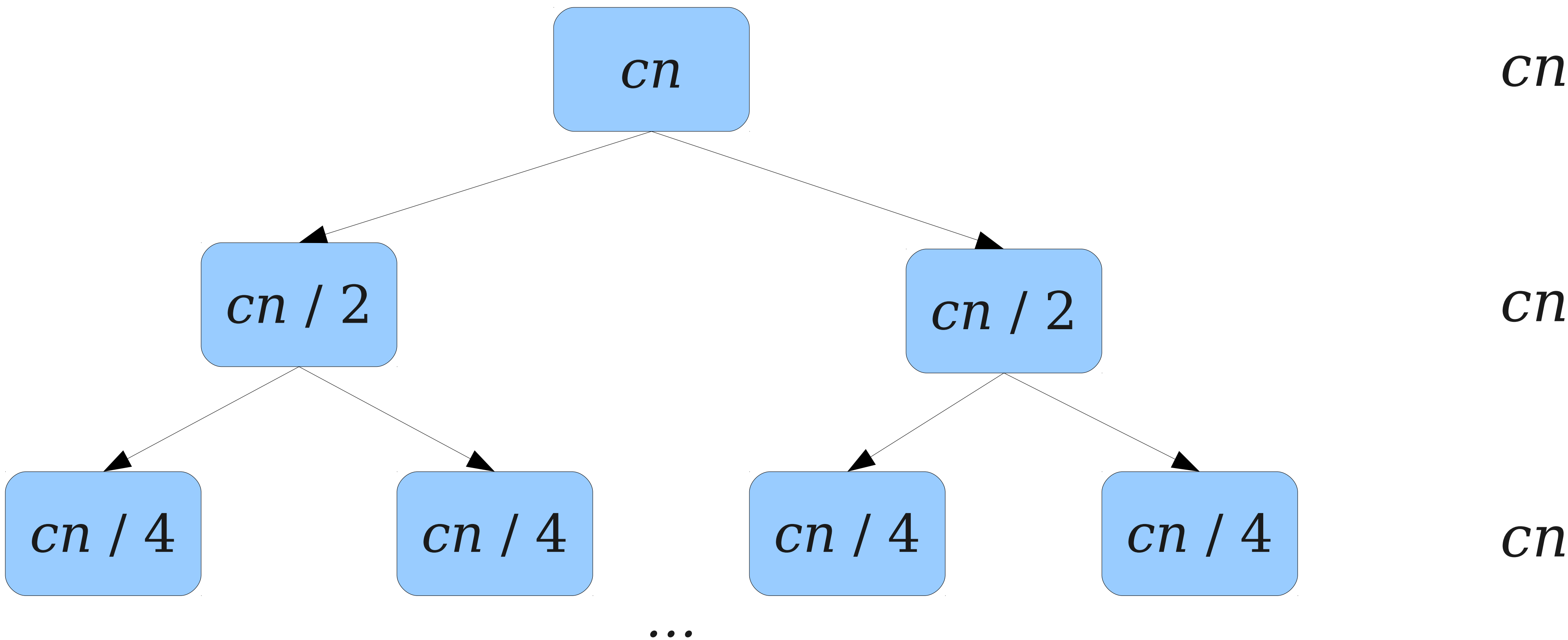
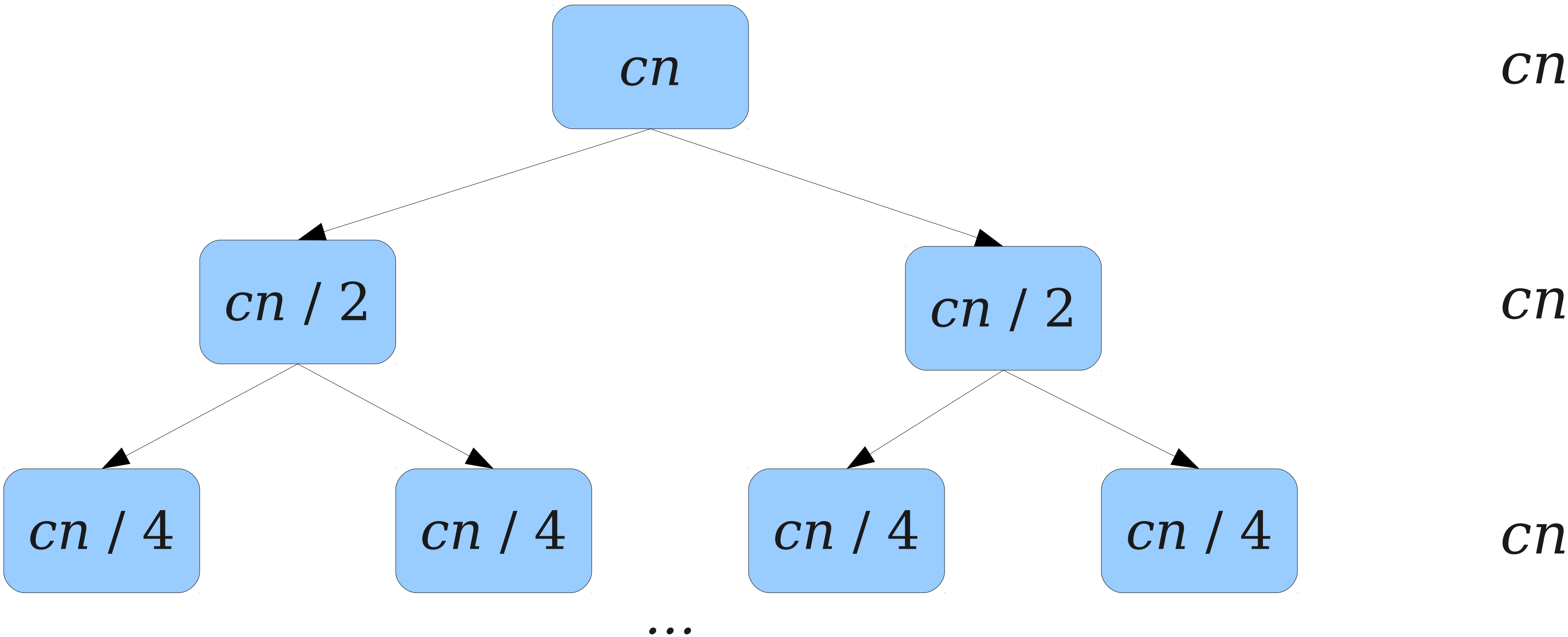


There are **$\log_2 n + 1$** layers in the tree (numbered 0, 1, 2, …, $\log_2 n$).

The first $\log_2 n$ of them are the recursive case. The last one consists purely of base cases.

$$T(1) \le c$$
$$T(n) \le 2T(n / 2) + cn$$



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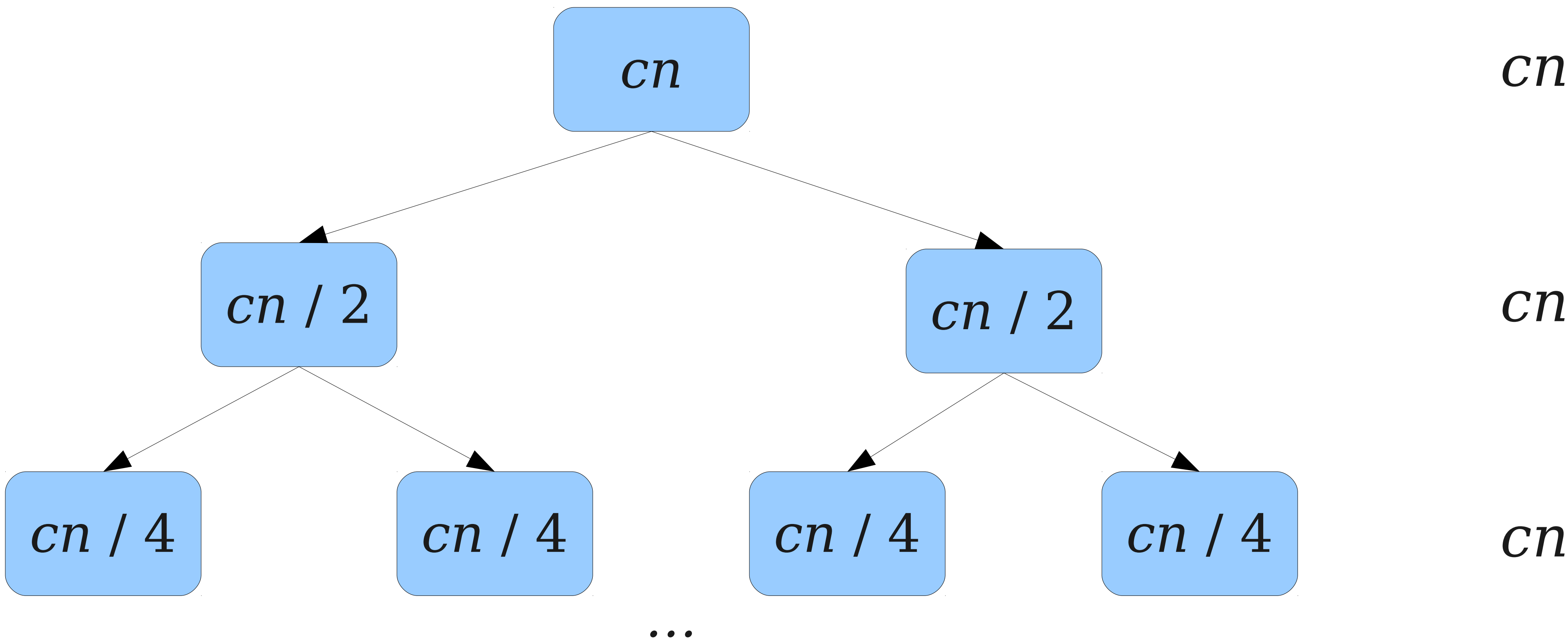

$$cn \log_2 n$$

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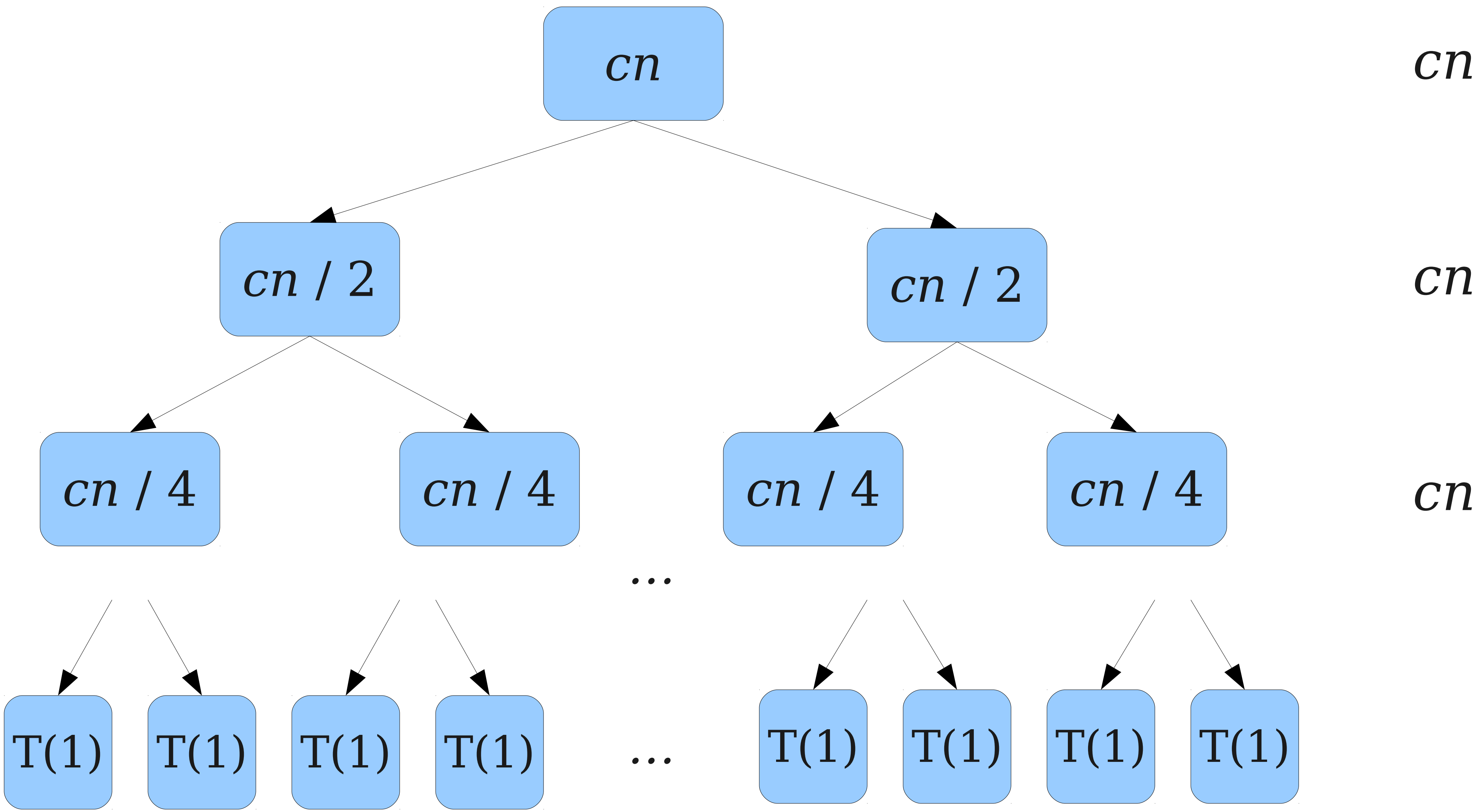

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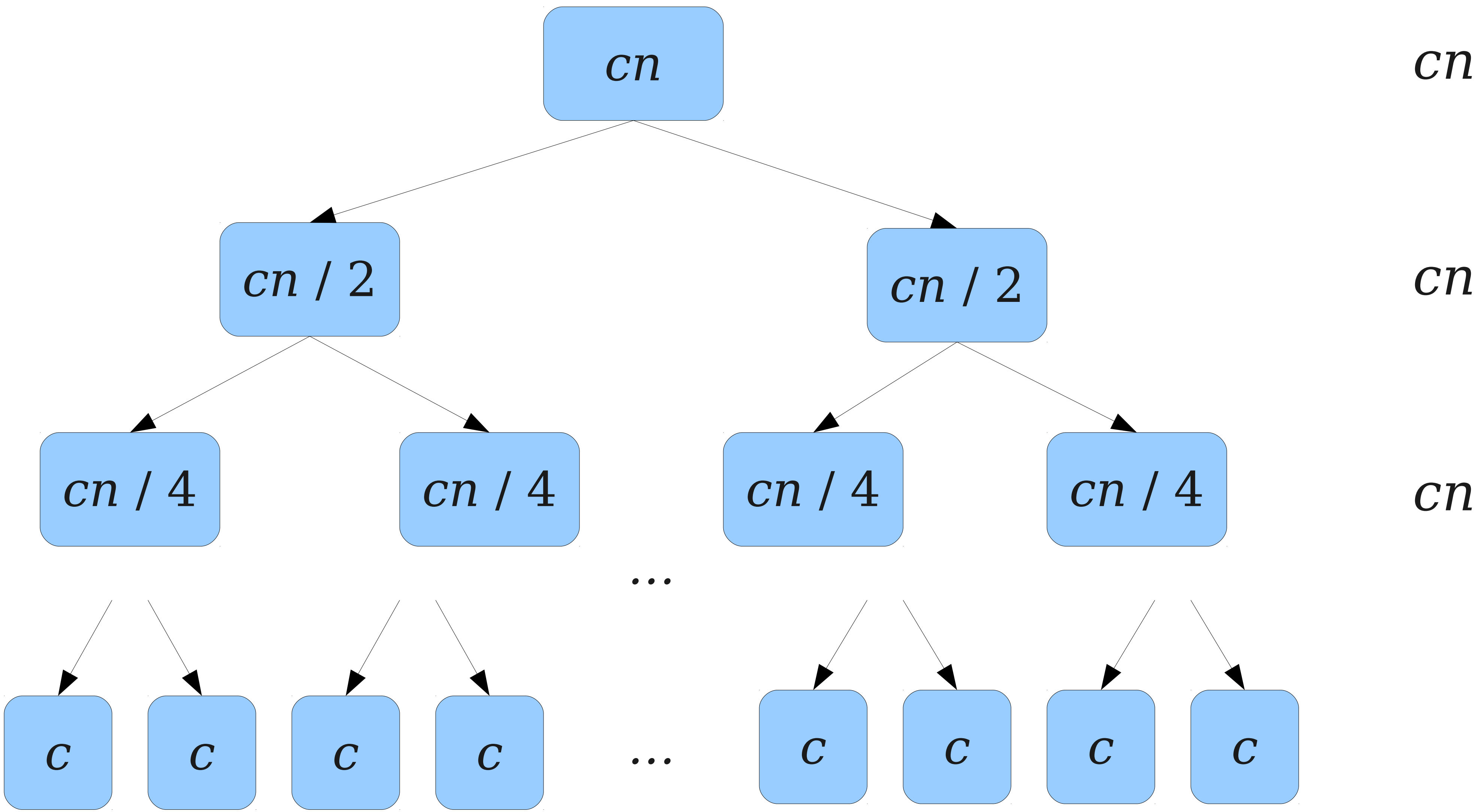

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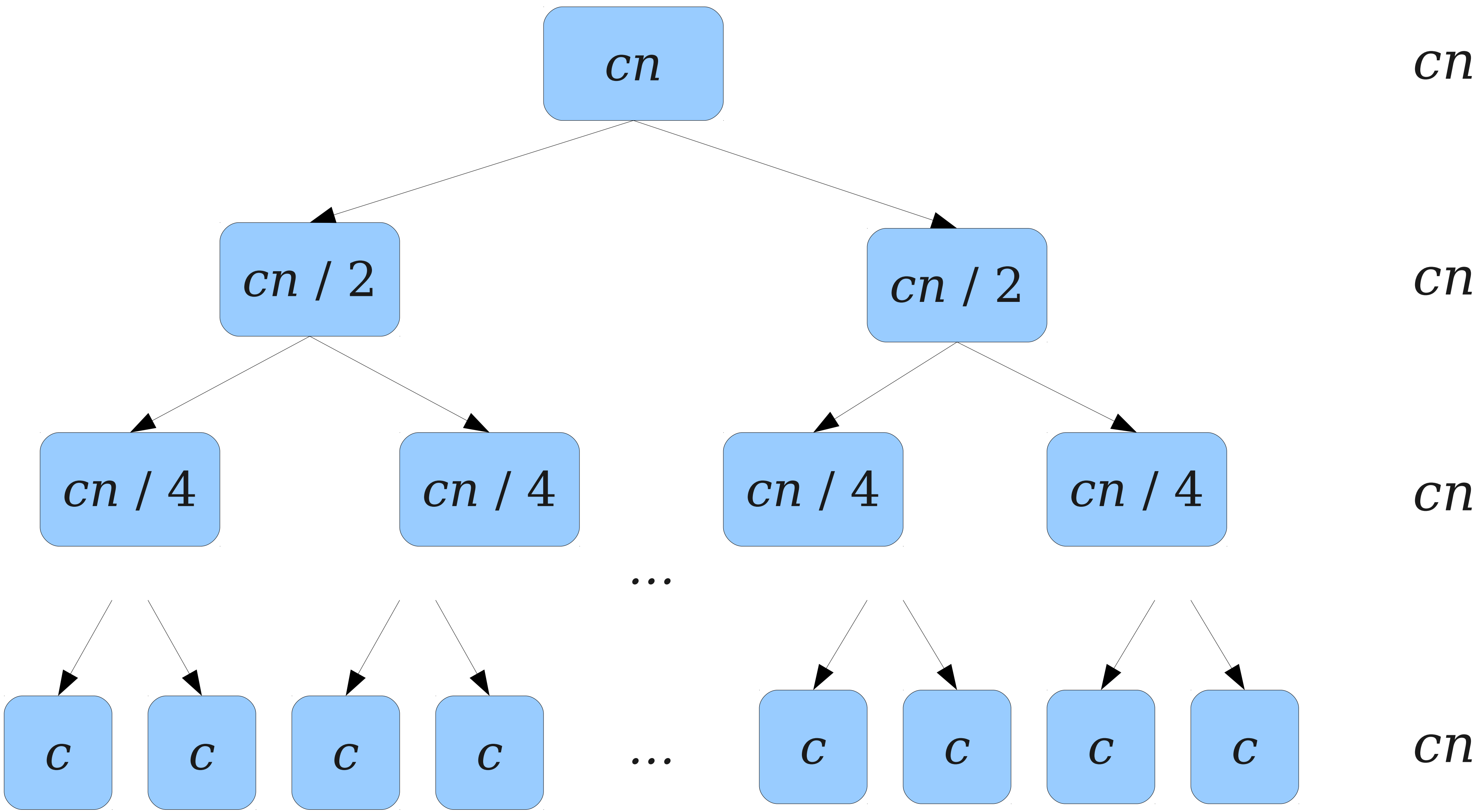

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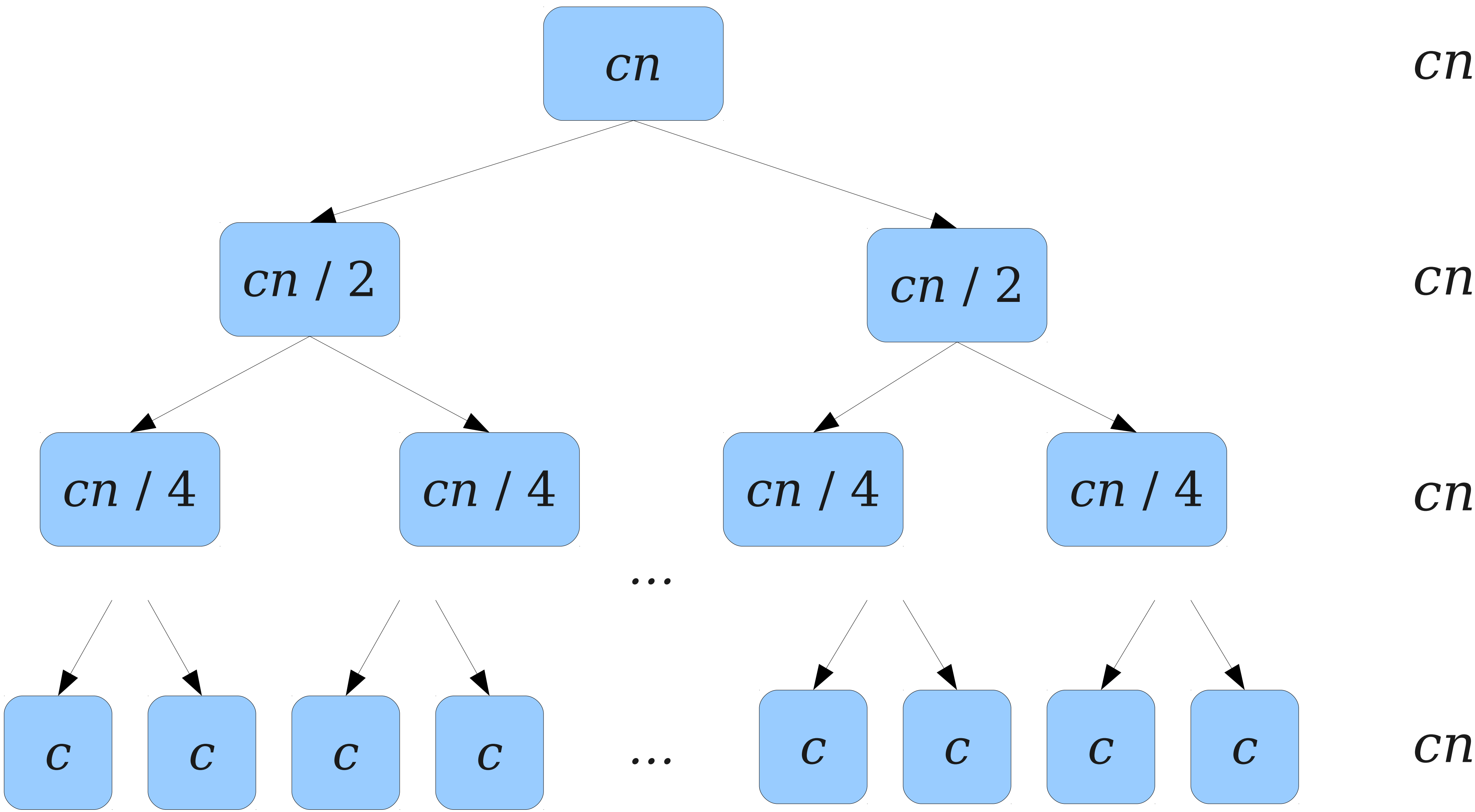

$$cn \log_2 n + cn$$

The Recursion Tree Method

- This diagram is called a **recursion tree** and accounts for how much total work each recursive call makes.
- Often useful to sum up the work across the layers of the tree.

A Formal Proof

- Both the iteration and recursion tree methods suggest that the runtime is at most

$$cn \log_2 n + cn$$

- Neither of these lines of reasoning are perfectly rigorous; how could we formalize this?
- **Induction!**

Theorem: **If n is a power of 2, $T(n) \leq cn \log_2 n + cn$**

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Proof: By induction.

***Theorem:* If n is a power of 2, $T(n) \leq cn \log_2 n + cn$**

Proof: By induction. As a base case, if $n = 2^0 = 1$, then

$$T(n) = T(1)$$

Theorem: If n is a power of 2, $T(n) \le cn \log_2 n + cn$

Proof: By induction. As a base case, if $n = 2^0 = 1$, then

$$
\begin{aligned}
T(n) &= T(1) \\
&\le c
\end{aligned}
$$

***Theorem:* If n is a power of 2, $T(n) \leq cn \log_2 n + cn$**

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For the inductive step, assume the claim holds for all $n' < n$ that are powers of two.

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$$\begin{aligned} T(n) &\leq 2T(n / 2) + cn \\ &= 2((cn / 2) \log_2 (n / 2) + cn / 2) + cn \end{aligned}$$

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&= cn (\log_2 n - 1) + cn + cn \\
&= cn \log_2 n - cn + cn + cn \\
&= cn \log_2 n + cn \quad \blacksquare
\end{aligned}
$$

What This Means

- We have shown that as long as we *only* look at powers of two, the runtime for mergesort is bounded from above by $cn \log_2 n + cn$.

In most cases, it's perfectly safe to stop here and claim we have a working bound. Mergesort is indeed O(*n* log *n*).

- For completeness, let's take some time to see why it is safe to stop here.
- In the future, we won't go into this level of detail.

Replacing Θ

- Our original recurrence was

$$\boxed{\begin{aligned} &T(0) = \Theta(1) \\ &T(1) = \Theta(1) \\ &T(n) \le T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n) \end{aligned}}$$

- We claimed it was safe to remove the Θ notation and rewrite it as

$$\boxed{\begin{aligned} &T(0) \le c \\ &T(1) \le c \\ &T(n) \le T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + cn \end{aligned}}$$

- Why can we do this?

Fat Base Cases

- When $n \geq n_0$, we can replace $\Theta(n)$ by cn for some constant c.
- Our simplification in the previous step assumed that $n_0 = 0$. What if this isn't the case?
- Can always rewrite the recurrence to use a "fat base case:"

$$\boxed{\begin{aligned} &T(0) = \Theta(1) \\ &T(1) = \Theta(1) \\ &T(n) \leq T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n) \end{aligned}}$$

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- Can always rewrite the recurrence to use a "fat base case:"

$$\boxed{\begin{array}{ll} T(n) \leq T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + cn & \text{(if } n \geq n_0\text{)} \\ T(n) \leq c & \text{(otherwise)} \end{array}}$$

- Makes the induction a *lot* harder to do, but the result would come out the same.

Non Powers of Two

- Consider this recurrence:

$$\boxed{\begin{aligned}&T(0) \leq c\\ &T(1) \leq c\\ &T(n) \leq T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + cn\end{aligned}}$$

- We know that for powers of two, this is upper bounded by $cn \log_2 n + cn$.
- Does that upper bound still hold for values other than powers of two?
- If not, is our bound even useful?

Non Powers of Two

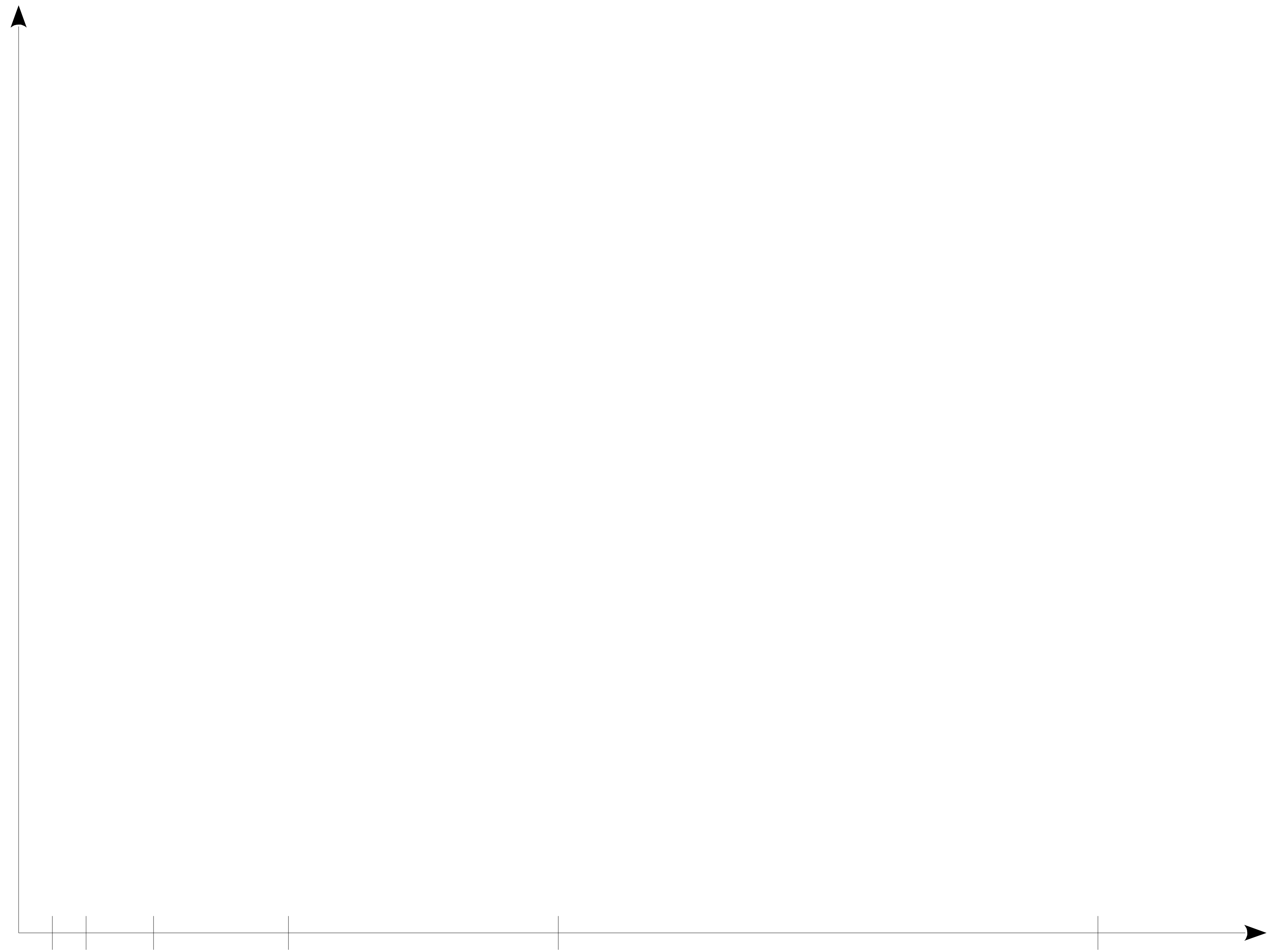
- Can we claim that since $T(n) \leq cn \log_2 n + cn$ when n is a power of two, that $T(n) = O(n \log n)$?

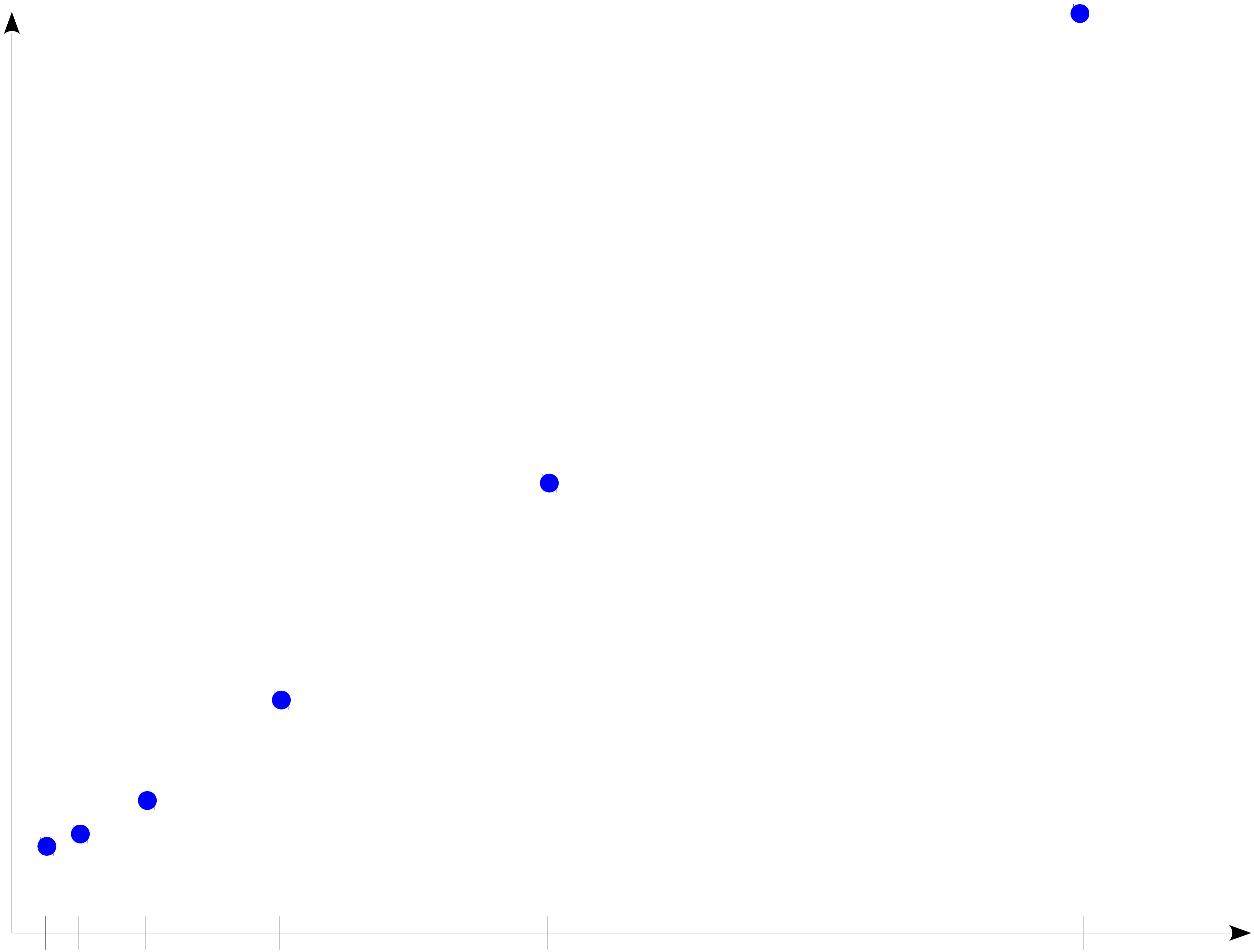
Non Powers of Two

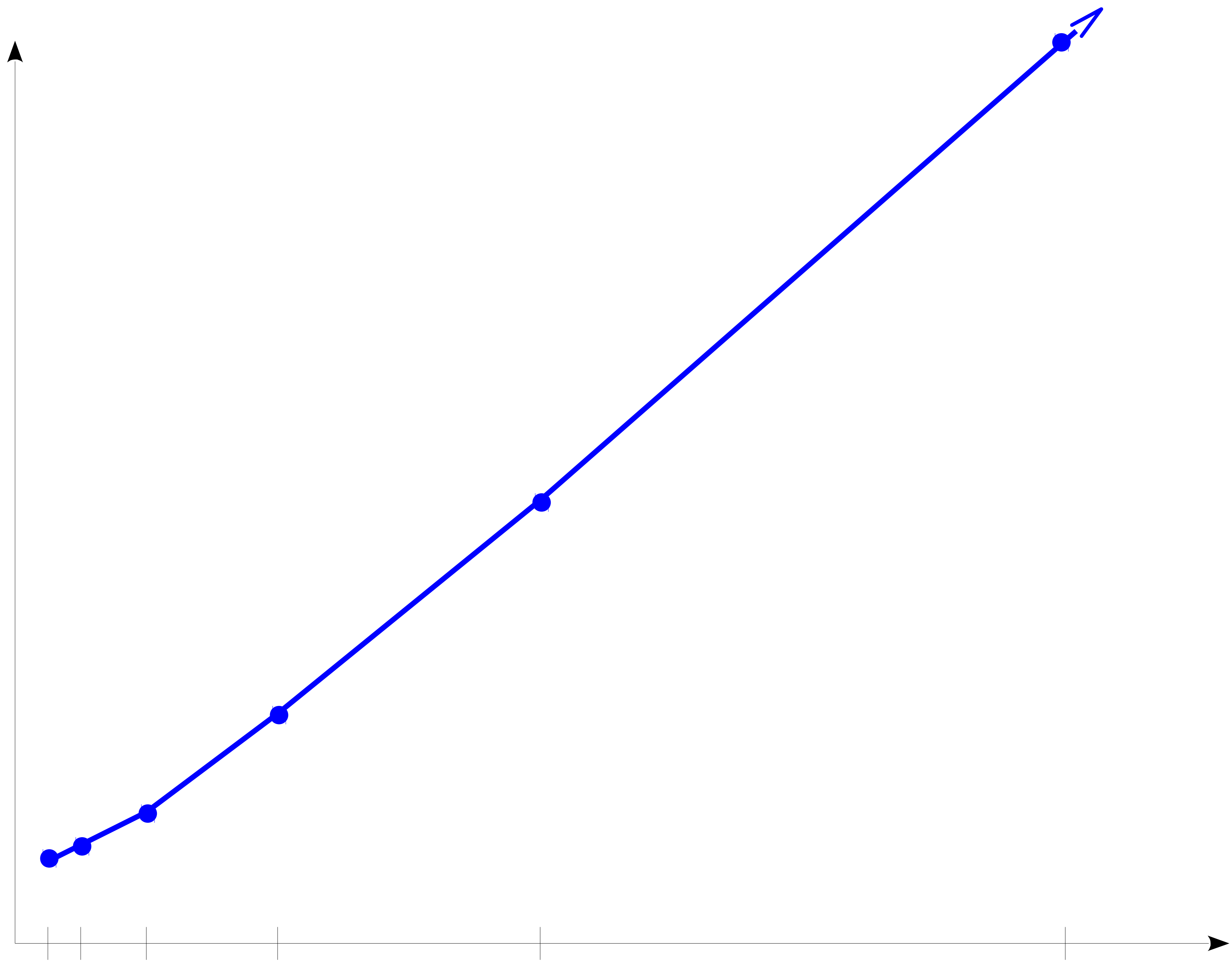
- Can we claim that since $T(n) \leq cn \log_2 n + cn$ when n is a power of two, that $T(n) = O(n \log n)$?
- Without more work, **no**. Consider this function:

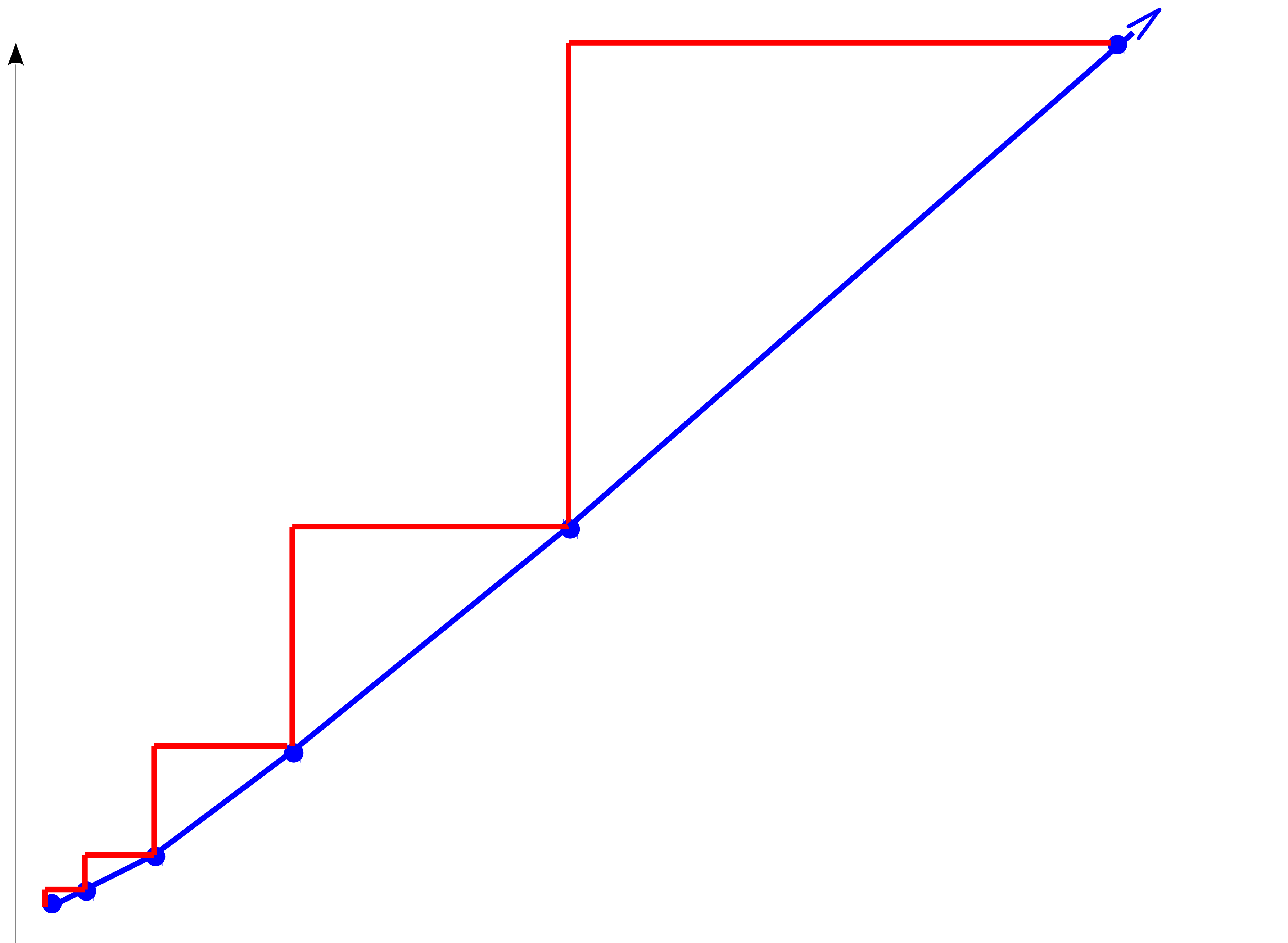
$$f(n) = \begin{cases} n \log_2 n & \text{if } n = 2^k \\ n! & \text{otherwise} \end{cases}$$

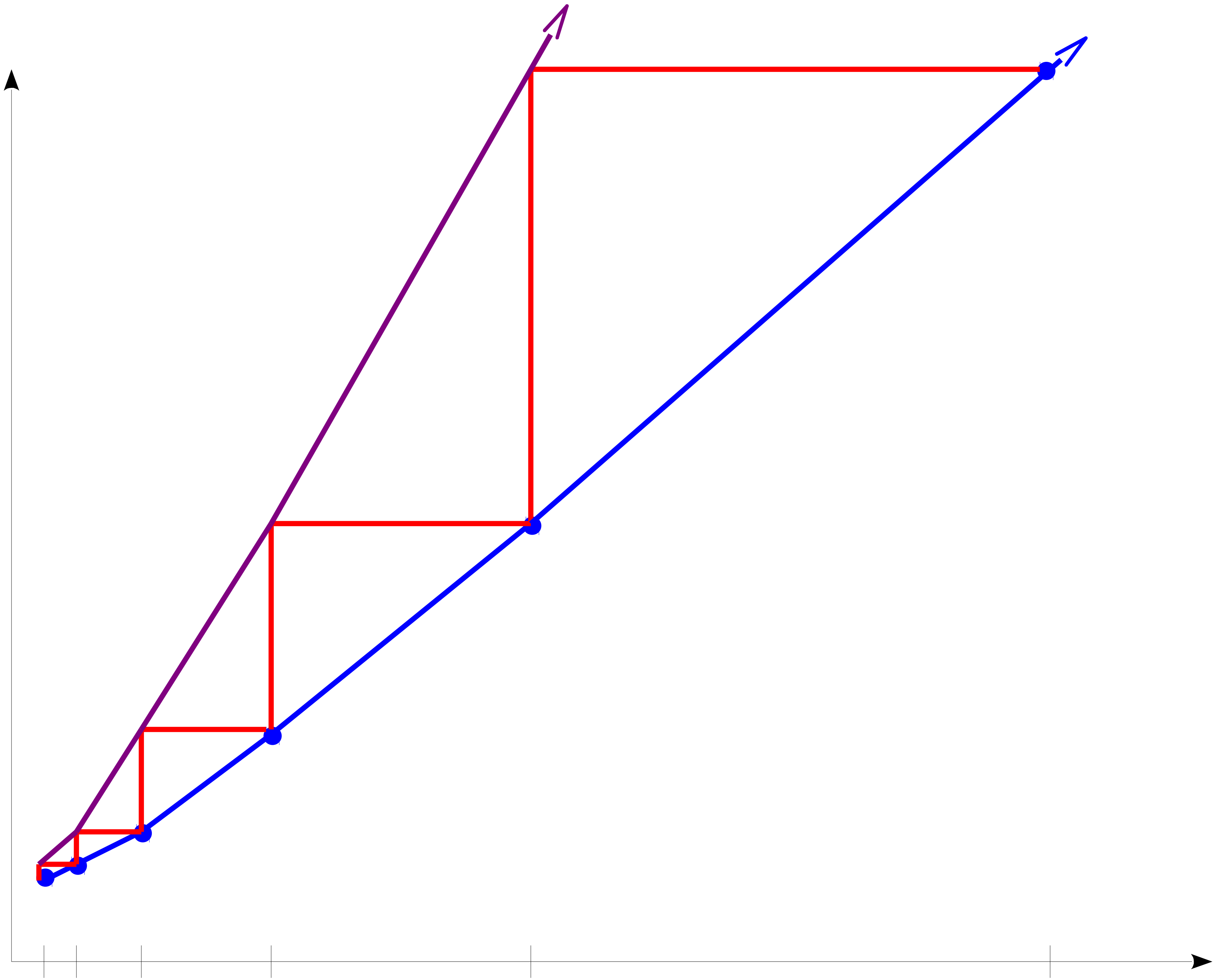
- Only looking at inputs that are powers of two, we might claim that $f(n) = \Theta(n \log n)$, even though this isn't the case!
- We need to do extra work to show that $T(n)$ is "well-behaved" enough to extrapolate.

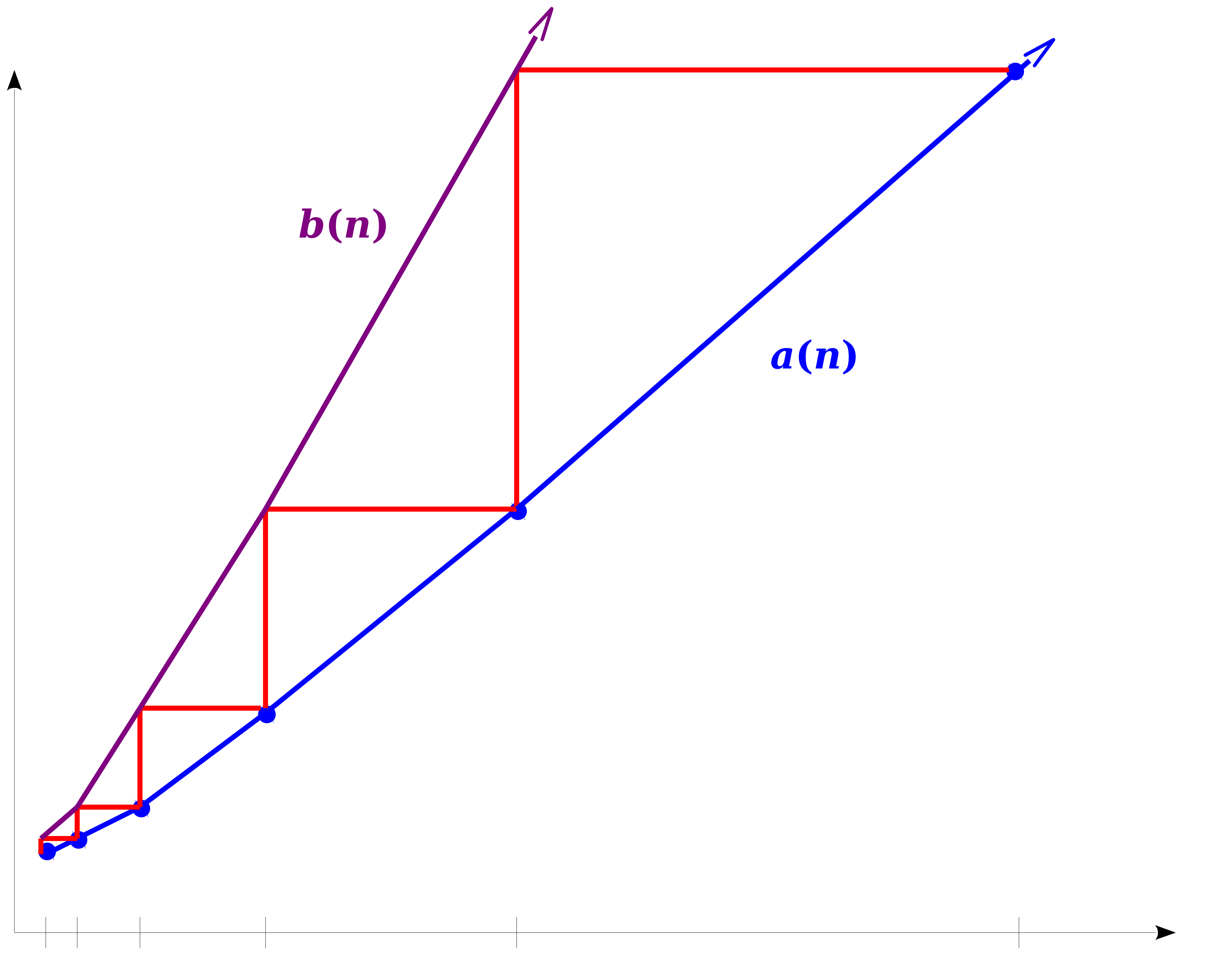




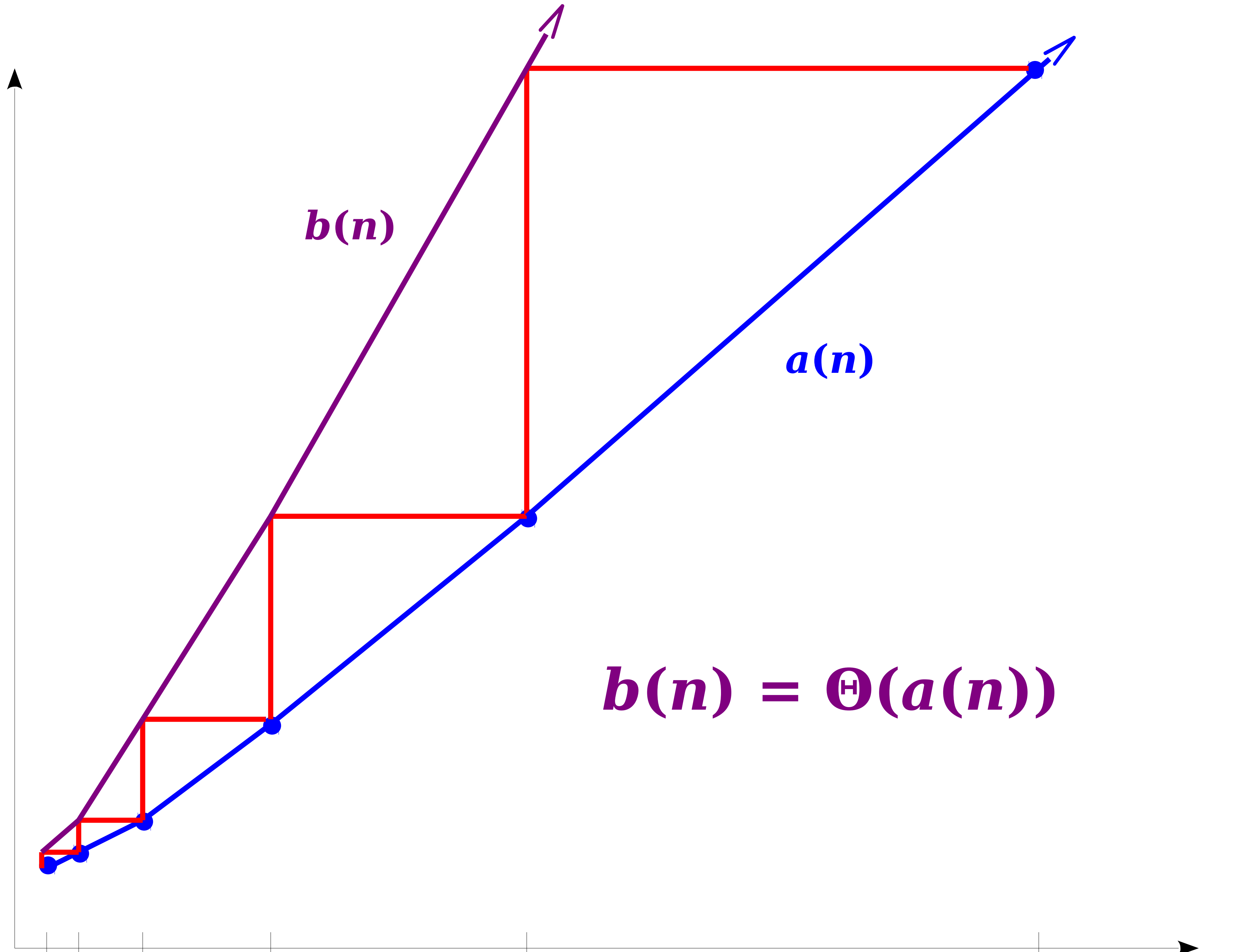








b(n)
a(n)



b(n)
a(n)
b(n) = Θ(a(n))

Our Proof Strategy

- We will proceed as follows:
  - Show that the values generated by the recurrence are nondecreasing.
  - For each non power-of-two n, provide an upper bound $T(n)$ using our upper bound on the next power of two greater than n.
  - Show that the upper bound we find this way is asymptotically equivalent (in terms of Θ) to our original bound.

Making Things Easier

- We are given this recurrence:

$$\boxed{\begin{aligned}&T(0) \le c\\ &T(1) \le c\\ &T(n) \le T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + cn\end{aligned}}$$

- This only gives an upper bound on $T(n)$; we don't know the exact values.

Making Things Easier

- We are given this recurrence:

$$\boxed{\begin{aligned}&T(0) \leq c\\&T(1) \leq c\\&T(n) \leq T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + cn\end{aligned}}$$

- This only gives an upper bound on $T(n)$; we don't know the exact values.
- Let's define a new function $f(n)$ as follows:

$$\boxed{\begin{aligned}&f(0) = c\\&f(1) = c\\&f(n) = f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn\end{aligned}}$$

- Note that $T(n) \leq f(n)$ for all $n \in \mathbb{N}$.

$$f(0) = c$$
$$f(1) = c$$
$$f(n) = f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn$$

***Lemma:* $f(n + 1) \geq f(n)$ for all $n \in \mathbb{N}$.**

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$$f(1) = c$$
$$f(n) = f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn$$

***Lemma:* $f(n + 1) \geq f(n)$ for all $n \in \mathbb{N}$.**

Proof: By induction on n.

$$\boxed{\begin{aligned} &f(0) = c \\ &f(1) = c \\ &f(n) = f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn \end{aligned}}$$

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Proof: By induction on n. As a base case, note that

$$f(1) = c \geq c = f(0)$$

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$$f(n) = f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn$$

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Proof: By induction on n. As a base case, note that

$$f(1) = c \geq c = f(0)$$

For the inductive step, assume that for some n that the lemma holds for all $n' < n$.

$$\boxed{\begin{aligned} &f(0) = c \\ &f(1) = c \\ &f(n) = f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn \end{aligned}}$$

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Proof: By induction on n. As a base case, note that

$$f(1) = c \geq c = f(0)$$

For the inductive step, assume that for some n that the lemma holds for all $n' < n$. Then

$$f(n + 1) = f(\lceil (n+1) / 2 \rceil) + f(\lfloor (n+1) / 2 \rfloor) + c(n+1)$$

$$f(0) = c$$
$$f(1) = c$$
$$f(n) = f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn$$

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$$\begin{aligned} f(n + 1) &= f(\lceil (n+1) / 2 \rceil) + f(\lfloor (n+1) / 2 \rfloor) + c(n+1) \\ &\geq f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn \end{aligned}$$

$$
\boxed{\begin{aligned} f(0) &= c \\ f(1) &= c \\ f(n) &= f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn \end{aligned}}
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$$
\begin{aligned}
f(n + 1) &= f(\lceil (n+1) / 2 \rceil) + f(\lfloor (n+1) / 2 \rfloor) + c(n+1) \\
&\geq f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn \\
&= f(n)
\end{aligned}
$$

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$$\begin{aligned} f(n + 1) &= f(\lceil (n+1) / 2 \rceil) + f(\lfloor (n+1) / 2 \rfloor) + c(n+1) \\ &\geq f(\lceil n / 2 \rceil) + f(\lfloor n / 2 \rfloor) + cn \\ &= f(n) \blacksquare \end{aligned}$$

Theorem: **T(n) = O(n log n)**

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Proof: Consider any $n \in \mathbb{N}$ with $n \geq 1$.

Theorem: **T(*n*) = O(*n* log *n*)**

Proof: Consider any $n \in \mathbb{N}$ with $n \geq 1$. Let k be such that $2^k \leq n < 2^{k+1}$.

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Proof: Consider any $n \in \mathbb{N}$ with $n \geq 1$. Let k be such that $2^k \leq n < 2^{k+1}$. Thus $2^{k+1} \leq 2n < 2^{k+2}$.

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$$T(n) \leq f(n) \leq f(2^{k+1})$$

Using our upper bound for powers of two:

$$f(2^{k+1}) \leq c(2^{k+1}) \log_2 (2^{k+1}) + c(2^{k+1})$$

Theorem: $\mathbf{T(n) = O(n \log n)}$

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Therefore

$$T(n) \leq c(2^{k+1}) \log_2 (2^{k+1}) + c(2^{k+1})$$

Theorem: **T(*n*) = O(*n* log *n*)**

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Therefore

$$\begin{aligned} T(n) &\leq c(2^{k+1}) \log_2 (2^{k+1}) + c(2^{k+1}) \\ &\leq c(2n) \log_2 (2n) + 2cn \end{aligned}$$

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***Theorem:* $\mathbf{T}(\boldsymbol{n}) = \mathbf{O}(\boldsymbol{n} \log \boldsymbol{n})$**

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From our lemma, we know that

$$\mathrm{T}(n) \leq f(n) \leq f(2^{k+1})$$

Using our upper bound for powers of two:

$$f(2^{k+1}) \leq \mathrm{c}(2^{k+1}) \log_2 (2^{k+1}) + \mathrm{c}(2^{k+1})$$

Therefore

$$\begin{aligned}
\mathrm{T}(n) &\leq c(2^{k+1}) \log_2 (2^{k+1}) + c(2^{k+1}) \\
&\leq c(2n) \log_2 (2n) + 2cn \\
&= 2cn (\log_2 n + 1) + 2cn \\
&= 2cn \log_2 n + 4cn
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So for any $n \geq 1$, $\mathrm{T}(n) \leq 2cn \log_2 n + 4cn$.

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So for any $n \geq 1$, $T(n) \leq 2cn \log_2 n + 4cn$. Thus $T(n) = O(n \log n)$. ■

Summary

- We can safely extrapolate from the runtime bounds at powers of two for the following reasons:
  - The runtime is nondecreasing, so we can use powers of two to provide upper bounds on other points.
  - The runtime grows only polynomially, so this upper bounding strategy does not produce values that are “too much” bigger than the actual values.
- **In the future, we will assume that this line of proof works and will not repeat it.**

Perfectly Safe Assumptions

- For the purposes of this class, you can safely simplify recurrences by
  - Only evaluating the recurrences at powers of some number to avoid ceilings and floors.
  - Replace $\Theta(f(n))$ or $O(f(n))$ terms in a recurrence with a constant multiple of $f(n)$.
  - Replace all constants with a single constant equal to the max of all of the constants.

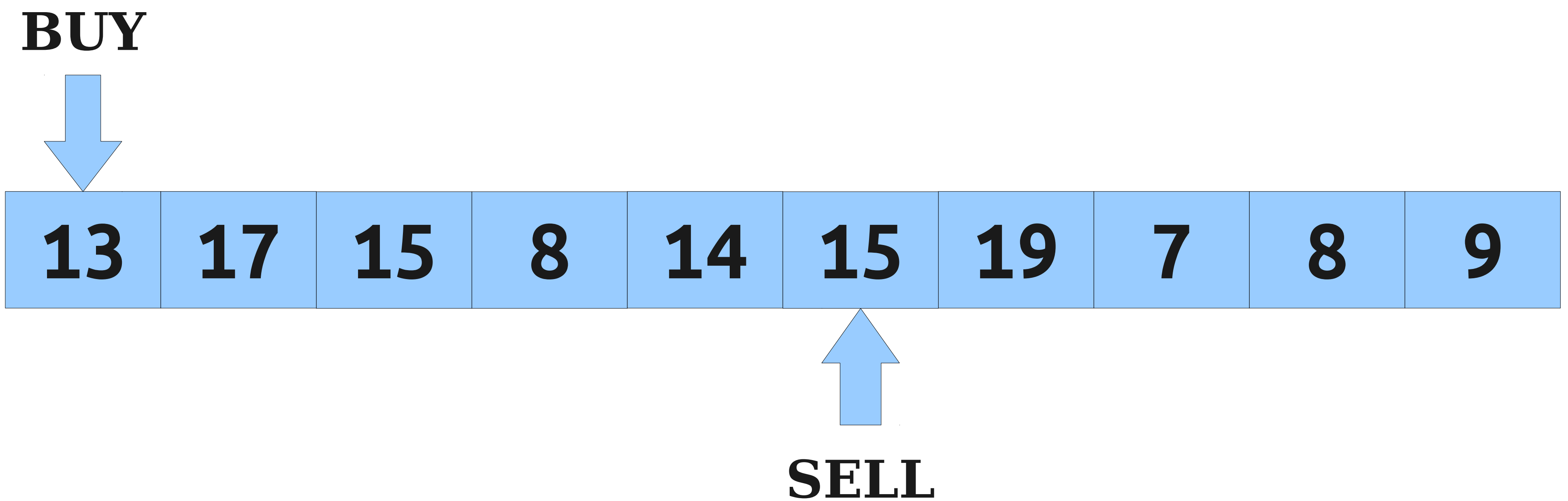
A Different Problem:

Maximum Single-Sell Profit

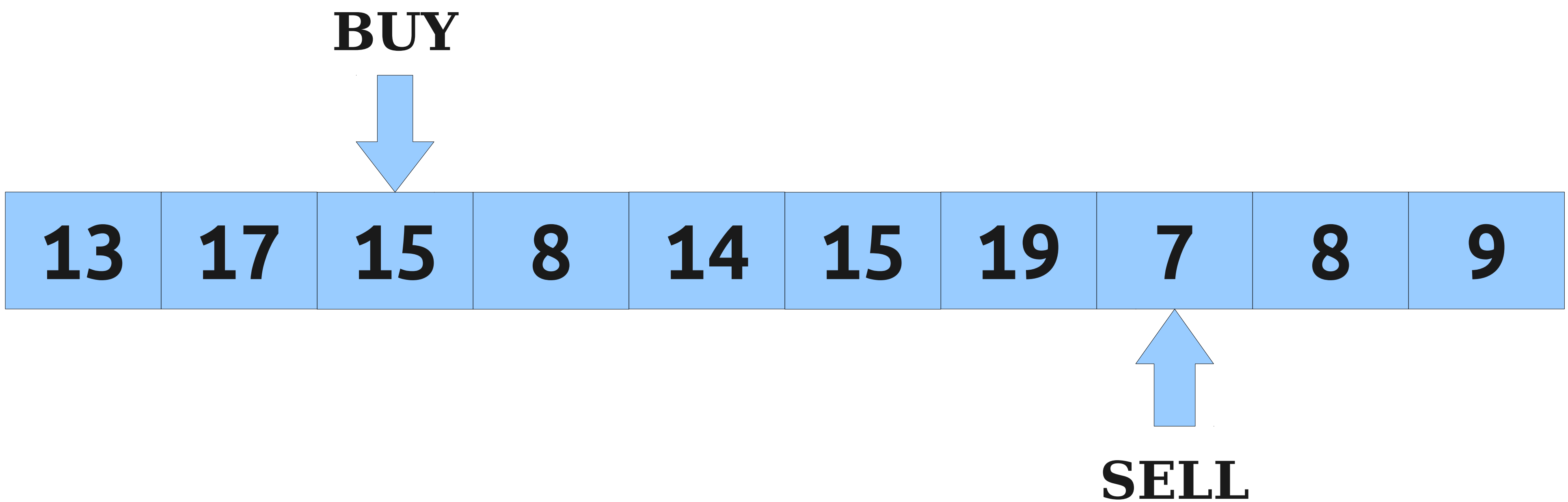
Maximum Single-Sell Profit

13	17	15	8	14	15	19	7	8	9

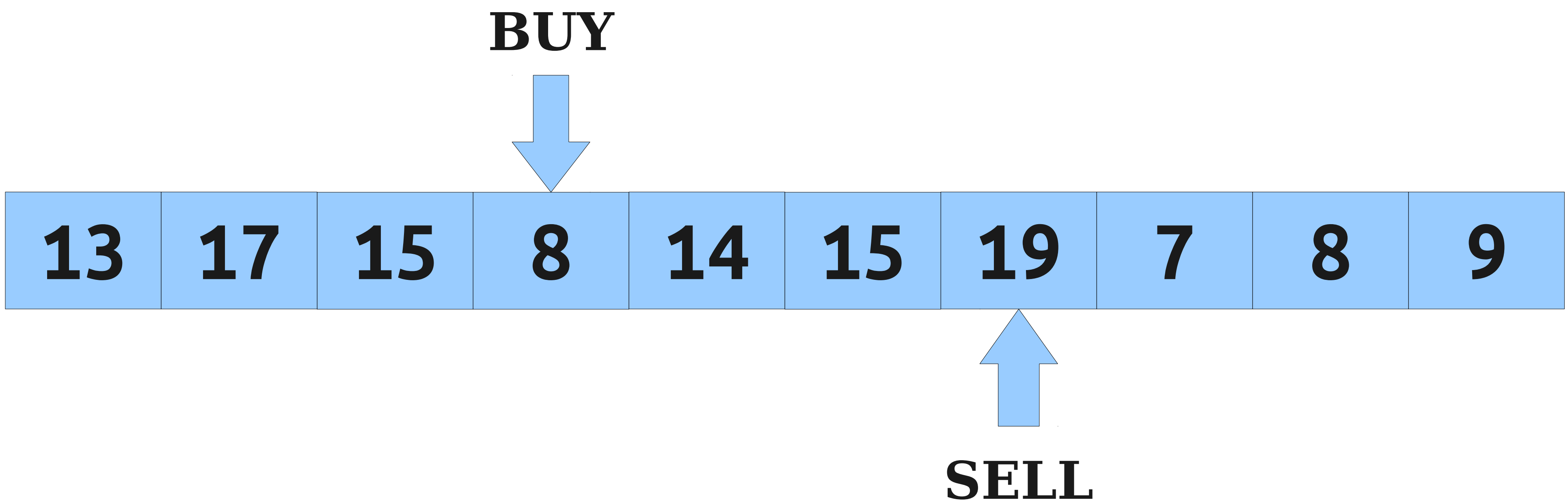
Maximum Single-Sell Profit



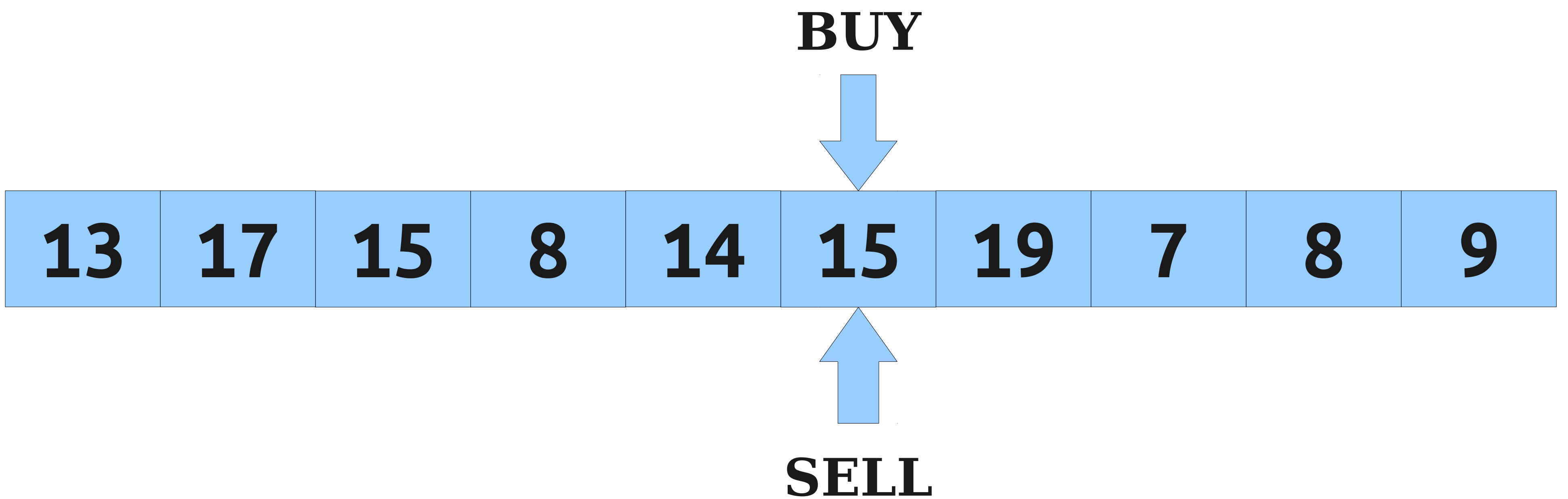
Maximum Single-Sell Profit



Maximum Single-Sell Profit



Maximum Single-Sell Profit



Maximum Single-Sell Profit

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Maximum Single-Sell Profit

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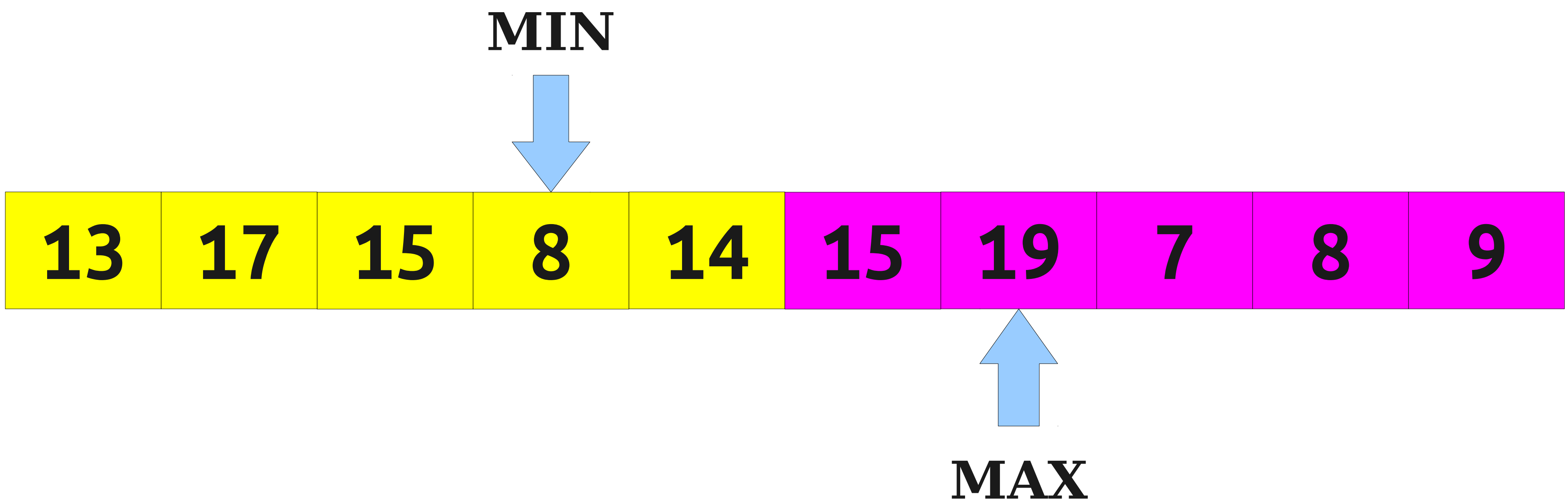
```
procedure maxProfit(list prices):
  let best = 0
  for i = 0 to length(prices) - 1:
     for j = i + 1 to length(prices) - 1:
        if prices[j] - prices[i] > best:
           best = prices[j] - prices[i]
  return best
```

Maximum Single-Sell Profit

13	17	15	8	14	15	19	7	8	9

Maximum Single-Sell Profit



Maximum Single-Sell Profit

13	17	15	8	14	15	19	7	8	9

```
procedure maxProfit(list prices):
  if length(prices) ≤ 1:
     return 0

  let left  be the first  half of prices
  let right be the second half of prices

  return max(maxProfit(left), maxProfit(right),
             max(right) - min(left))
```

Analyzing the Algorithm

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$T(1) = \Theta(1)$

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$$T(0) = \Theta(1)$$
$$T(1) = \Theta(1)$$
$$T(n) \leq T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n)$$

Analyzing the Algorithm

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$T(0) = \Theta(1)$
$T(1) = \Theta(1)$
$T(n) \leq T(\lceil n / 2 \rceil) + T(\lfloor n / 2 \rfloor) + \Theta(n)$

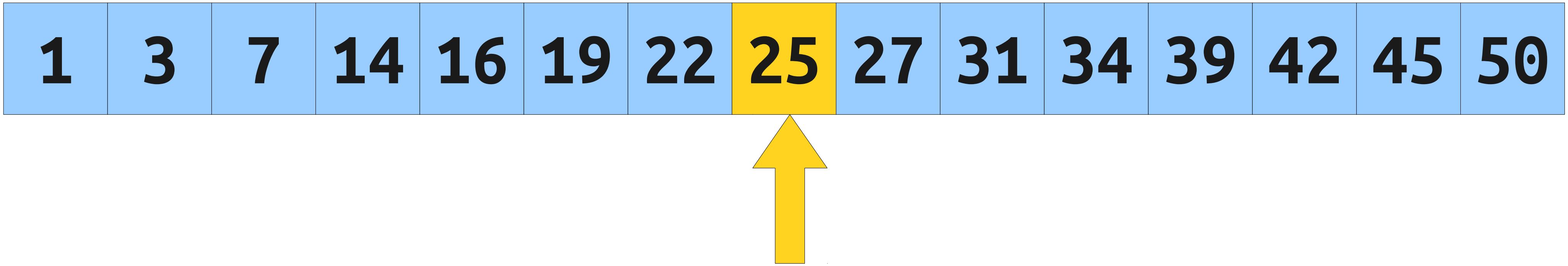
$T(n) = O(n \log n)$

The Divide-and-Conquer Framework

- The two algorithms we have just seen are examples of **divide-and-conquer** algorithms.
- These algorithms usually have two steps:
  - **(Divide)** Split the input apart into multiple smaller pieces, recursively solving each piece.
  - **(Conquer)** Combine the solutions to each smaller piece together into the overall solution.
- Typically, correctness is proven inductively and runtime is proven by solving a recurrence relation.
- In many cases, the runtime is determined without actually solving the recurrence; more on that later.

Another Algorithm: **Binary Search**

1	3	7	14	16	19	22	25	27	31	34	39	42	45	50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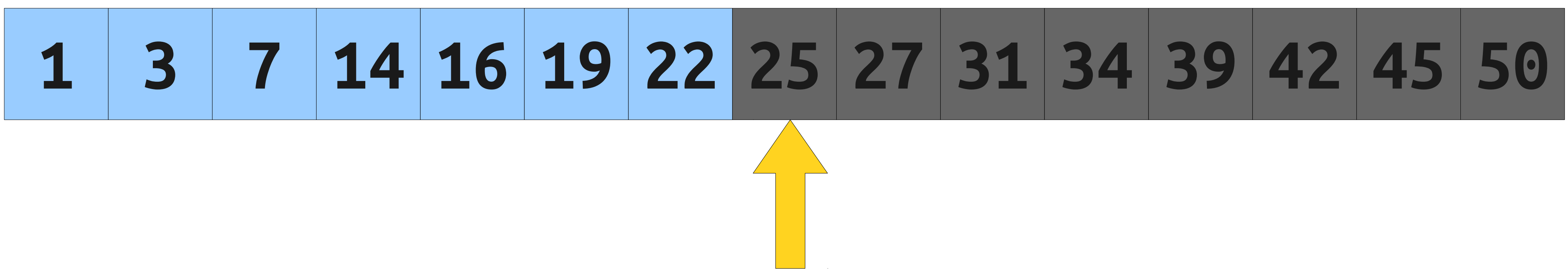
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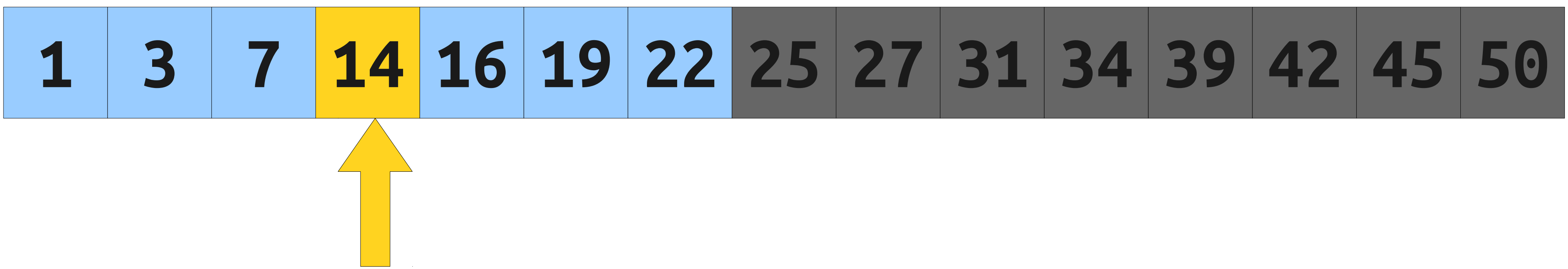
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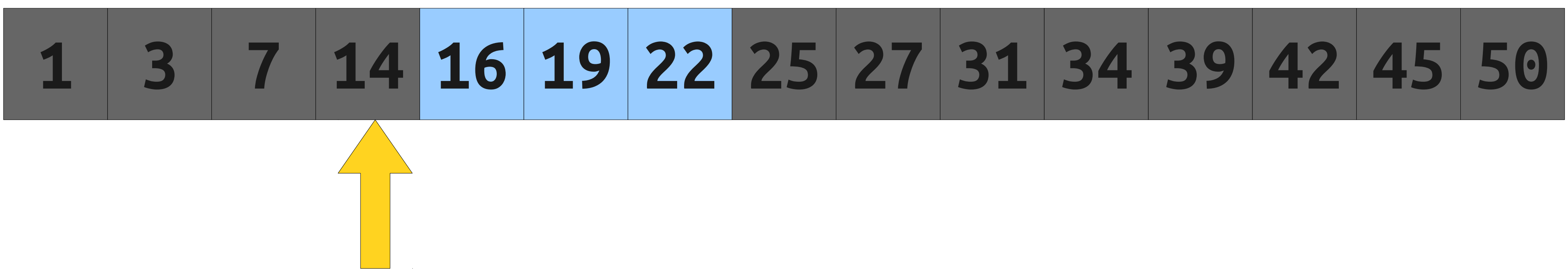
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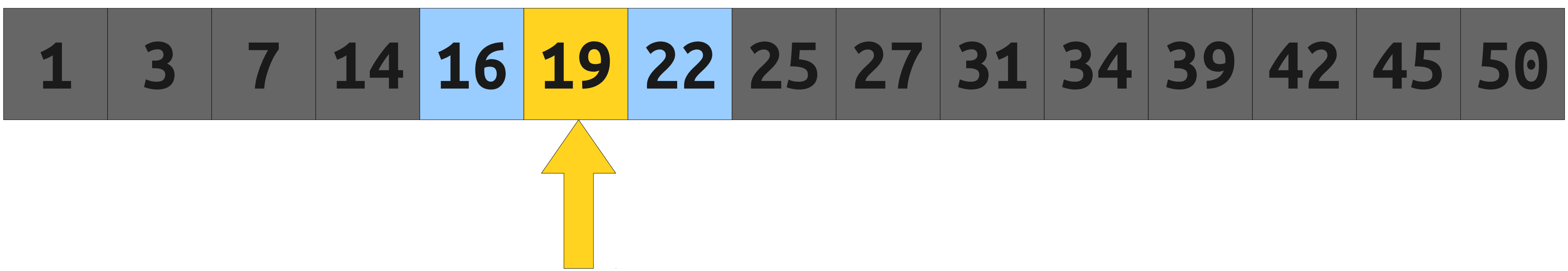
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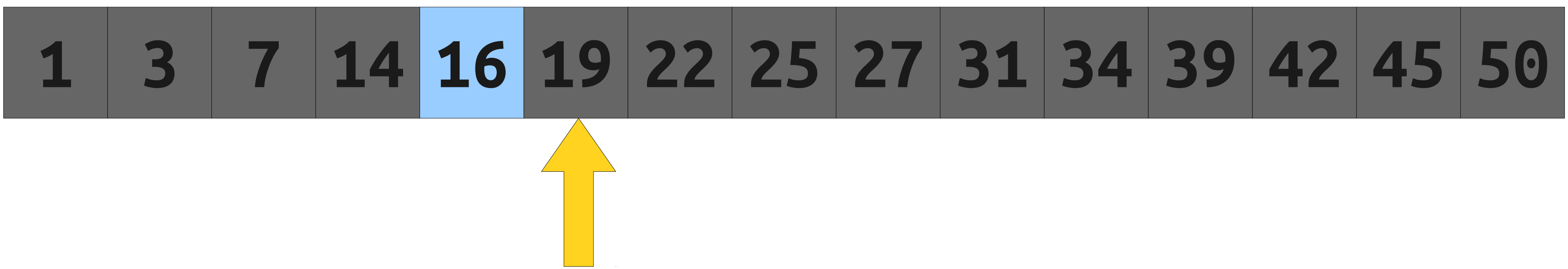
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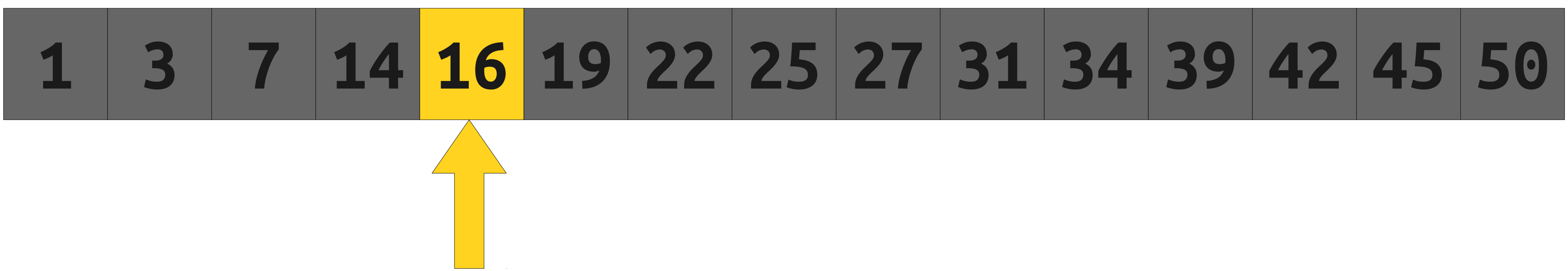
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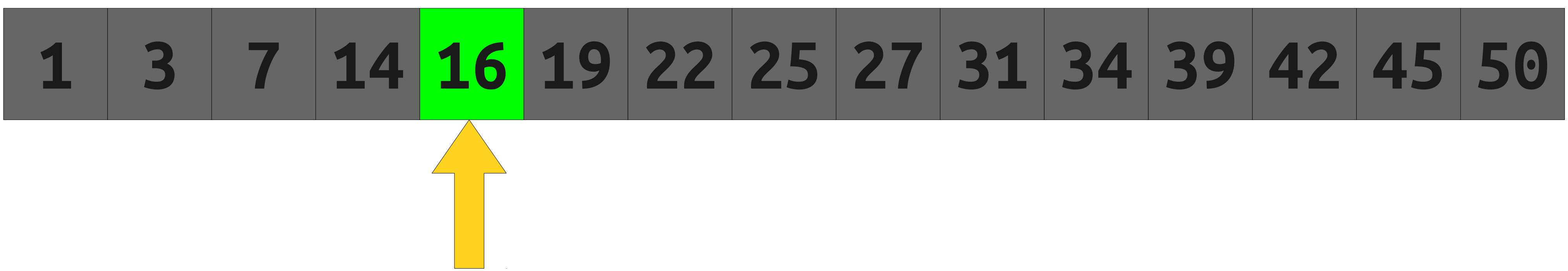

1 3 7 14 16 19 22 25 27 31 34 39 42 45 50

```
procedure binarySearch(list A, int low, int high,
                                  value key):
  if low ≥ high:
     return false

  let mid = ⌊(high + low) / 2⌋
  if A[mid] = key:
     return true
  else if A[mid] > key:
     return binarySearch(a, low, mid)
  else (A[mid] < key):
     return binarySearch(a, mid + 1, high)
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$$T(n) \le T(\lfloor n / 2 \rfloor) + \Theta(1)$$

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     return binarySearch(a, mid + 1, high)
```

$$T(1) \le c$$
$$T(n) \le T(n / 2) + c$$

The Iteration Method

$$\begin{aligned} T(1) &\le c \\ T(n) &\le T(n / 2) + c \end{aligned}$$

The Iteration Method

$T(1) \le c$
$T(n) \le T(n / 2) + c$

$$T(n) \le T\left(\frac{n}{2}\right) + c$$

The Iteration Method

$$\boxed{\begin{array}{l} T(1) \le c \\ T(n) \le T(n/2) + c \end{array}}$$

$$\begin{aligned} \mathrm{T}(n) &\le T\left(\frac{n}{2}\right) + c \\ &\le \left(T\left(\frac{n}{4}\right) + c\right) + c \end{aligned}$$

The Iteration Method

$T(1) \le c$
$T(n) \le T(n / 2) + c$

$$
\begin{aligned}
\mathrm{T}(n) &\le T\left(\frac{n}{2}\right) + c \\
&\le \left(T\left(\frac{n}{4}\right) + c\right) + c \\
&= T\left(\frac{n}{4}\right) + 2c
\end{aligned}
$$

The Iteration Method

$T(1) \leq c$
$T(n) \leq T(n / 2) + c$

$$
\begin{aligned}
\mathrm{T}(n) &\leq T\left(\frac{n}{2}\right)+c \\
&\leq \left(T\left(\frac{n}{4}\right)+c\right)+c \\
&= T\left(\frac{n}{4}\right)+2c \\
&\leq \left(T\left(\frac{n}{8}\right)+c\right)+2c
\end{aligned}
$$

The Iteration Method

T(1) ≤ c
T(n) ≤ T(n / 2) + c

$$
\begin{aligned}
\mathrm{T}(n) &\le T\left(\frac{n}{2}\right) + c \\
&\le \left(T\left(\frac{n}{4}\right) + c\right) + c \\
&= T\left(\frac{n}{4}\right) + 2c \\
&\le \left(T\left(\frac{n}{8}\right) + c\right) + 2c \\
&= T\left(\frac{n}{8}\right) + 3c
\end{aligned}
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The Iteration Method

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$\mathrm{T}(n) \leq \mathrm{T}(n / 2) + c$

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&= T\left(\frac{n}{4}\right) + 2c \\
&\leq \left(T\left(\frac{n}{8}\right) + c\right) + 2c \\
&= T\left(\frac{n}{8}\right) + 3c \\
&\dots \\
&\leq T\left(\frac{n}{2^k}\right) + kc
\end{aligned}
$$

The Iteration Method

$T(1) \leq c$
$T(n) \leq T(n / 2) + c$

$$T(n) \leq T\left(\frac{n}{2^k}\right) + k\,c$$

The Iteration Method

T(1) ≤ c
T(n) ≤ T(n / 2) + c

$$\begin{aligned} \mathrm{T}(n) &\leq T\left(\frac{n}{2^k}\right) + k\,c \\ &= T(1) + c\log_2 n \end{aligned}$$

The Iteration Method

$\mathrm{T}(1) \leq c$
$\mathrm{T}(n) \leq \mathrm{T}(n / 2) + c$

$$\begin{aligned} \mathrm{T}(n) &\leq T\left(\frac{n}{2^k}\right) + k\,c \\ &= T(1) + c\log_2 n \\ &\leq c + c\log_2 n \end{aligned}$$

The Iteration Method

$T(1) \le c$
$T(n) \le T(n / 2) + c$

$$\begin{aligned} \mathrm{T}(n) &\le T\left(\frac{n}{2^k}\right) + kc \\ &= T(1) + c\log_2 n \\ &\le c + c\log_2 n \\ &= O(\log n) \end{aligned}$$

The Recursion Tree Method

The Recursion Tree Method

$$T(1) \leq c$$
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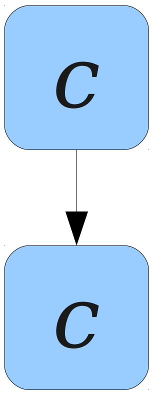
$$T(1) \leq c$$
$$T(n) \leq T(n / 2) + c$$

c

The Recursion Tree Method

$T(1) \leq c$
$T(n) \leq T(n / 2) + c$



The Recursion Tree Method

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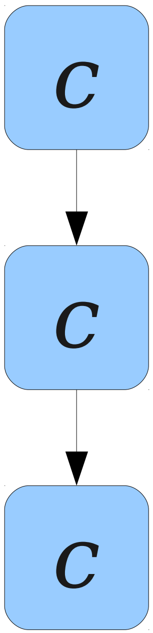

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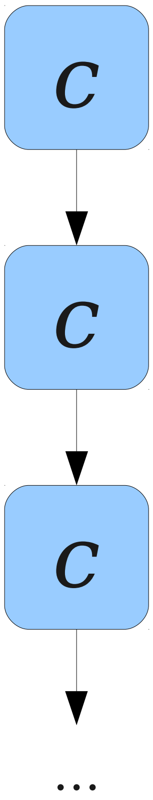

The Recursion Tree Method

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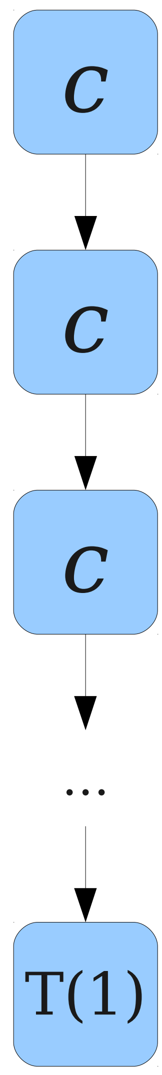

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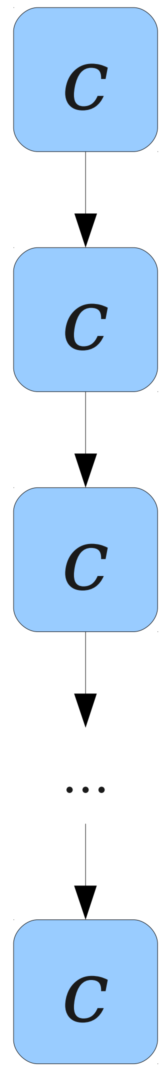

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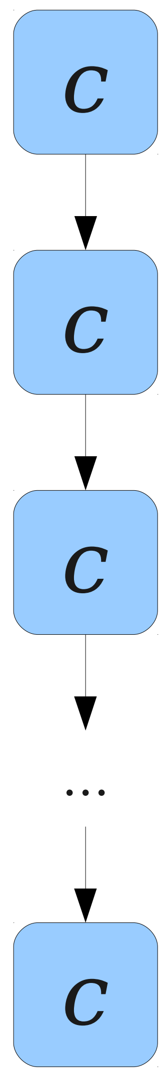

$c \log_2 n + c$

Formalizing Our Argument

- To formalize correctness, it's useful to use this invariant:

If $key = A[i]$ for some i, then $low \leq i < high$

- You can prove this is true by induction on the number of calls made.
- We can also formalize the runtime bound by induction to prove the O(log n) upper bound, but it's not super exciting to do so.