Optimal Control Problems II

April 7, 2006

1

1.1

Find and classify the critical points and the critical value of $L(u) = \frac{1}{2}u^\top Qu + S^\top u$ if
a. $Q = \begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}, \quad S = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
b. $Q = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}, \quad S = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

1.2

A meteor is in a hyperbolic orbit with respect to the earth, described by $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Find the minimum distance to a satellite at a fixed position (x_1, y_1).

1.3

a. Find the rectangle of maximum perimeter that can be inscribed inside an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.
b. Find the rectangle of maximum area that can be inscribed inside an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

2

2.1

For the bilinear system $\dot{x} = Ax + Bu + Dxu$ with a scalar input $u \in \mathbf{R}$, minimize the cost

$$J = \frac{1}{2}x^\top Sx|_T + \frac{1}{2}\int_0^T x^\top Qx + ru^2\,dt$$

Show that the optimal control involves a state-costate inner product. The optimal state-costate equations contain cubic terms and are very difficult to solve.

2.2

Find the optimal control for the scalar plant $\dot{x} = u$, $x(t_0) = x_0$, with performance index

$$J(t_0) = \frac{1}{2}x^\top Sx|_T + \frac{1}{2}\int_{t_0}^T ru^2\,dt$$

a. Solve the Riccati using separation of variables.
b. Suppose x(T) is fixed. Find the optimal control as a function of $x(t_0), x(T)$.
c. Use the results of Part b to develop a state-feedback control law. Solve for $x(t_0)$ and substitute to get an optimal input of the form $u(t) = g(t)x(t) + h(t)$. Compare with the optimal control minimizing the cost to go $J(t)$ in $[t, t+T]$.

2.3

Let V,W be the $n \times n$ solutions to the Hamiltonian system

$$\begin{pmatrix} \dot{V} \\ \dot{W} \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^\top \\ -Q & -A^\top \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}$$

with the boundary condition $W(T) = S(T)V(T)$. Show that the solution to associated Riccati differential equation $-\dot{S} = A^\top S + SA - SBR^{-1}B^\top S + Q$ is given by $S(t) = W(t)V(t)^{-1}$.

2.4

For the cart system $\dot{x}_1 = x_2$, $\dot{x}_2 = u$, minimize the cost

$$J = \frac{1}{2}\int_0^\infty x_1^2 + 2vx_1x_2 + qx_2^2 + u^2\,dt$$

where $q - v^2 > 0$. Find the solution to the ARE, the optimal control and the optimal closed-loop system. Also, plot the loci of the closed-loop poles as q varies from 0 to ∞.

3

3.1

Consider the harmonic oscillator

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -\omega_n^2 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u$$

Find the optimal control to drive any initial state to zero in minimum time, subject to $|u(t)| \leq 1,\ \forall t$.
a. Find and solve the costate equations.
b. Sketch the phase-plane trajectories for $u = 1$ and $u = -1$.
c. Find the switching curve and derive a minimum-time feedback control law.

3.2

Develop a minimum-fuel control law for Problem 3.1.

Ref: F. Lewis and V. Syrmos, Optimal Control. Wiley, New York, 1995.

Solutions to Optimal Control Problems II

1.1 $\quad L = \frac{1}{2} u^T Q u + S^T u$

1) Critical pt: $\quad L_u = 0 = Qu + S \Rightarrow u_* = -Q^{-1}S$

$$= \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Optimal Cost: $L(u^*) = 1/2$. Hessian $L_{uu} = Q$ is negative definite $\Rightarrow$ u_* is a maximum

2) Critical point: $\quad L_u = 0 \Rightarrow u_* = -Q^{-1}S = \begin{bmatrix} -1/3 \\ -1/3 \end{bmatrix}$

Optimal cost is $L(u^*) = -1/6$. Hessian L_{uu} is indefinite $\Rightarrow$ u_* is a saddle point.

1.2 $\quad$ The cost function is the distance

$$L = \sqrt{(x-x_1)^2 + (y-y_1)^2}$$

Equivalently, and for convenience, we can choose to minimize

$$L = (x-x_1)^2 + (y-y_1)^2$$

For this, the Hamiltonian is

$$H = (x-x_1)^2 + (y-y_1)^2 + \lambda\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right)$$

And the necessary conditions for a minimum become

$$H_\lambda = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1 = 0$$

$$H_x = 2(x-x_1) + \frac{2\lambda x}{a^2} = 0 \Rightarrow x = \frac{a^2 x_1}{\lambda + a^2}$$

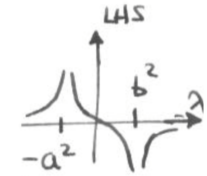
$$H_y = 2(y-y_1) - \frac{2\lambda y}{b^2} = 0 \Rightarrow y = \frac{b^2 y_1}{b^2 - \lambda}$$

Substituting the last two in the equation of the hyperbola,

$$\frac{a^2 x_1^2}{(\lambda + a^2)^2} - \frac{b^2 y_1^2}{(\lambda - b^2)^2} = 1.$$

This equation has two solutions for λ.

The left-hand side as a function of λ looks like:



The roots can be found by means of numerical methods, or as roots of polynomials (note that the conversion of this eqn to a polynomial will introduce new roots that must be discarded).

Finally, at the minimum, the curvature matrix

$$L_{uu} = \left[-f_u^T f_x^{-T} \, , \, I \right] \begin{bmatrix} H_{xx} & H_{xu} \\ H_{ux} & H_{uu} \end{bmatrix} \begin{bmatrix} -f_u^T f_x^{-T} \\ I \end{bmatrix}$$

can be used to specify the type of the critical point. (min)

1.3

(i) The optimization problem is

$$\min L = -4(x+y)$$

$$\text{s.t. } f(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$$

From this, $H = -4(x+y) + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$

and $H_\lambda = 0 = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$ $\Rightarrow \lambda^* = 2\sqrt{a^2+b^2}$

$$H_x = 0 = -4 + \frac{2\lambda x}{a^2} \Rightarrow x = \frac{2a^2}{\lambda}$$

$$H_y = 0 = -4 + \frac{2\lambda y}{b^2} \Rightarrow y = \frac{2b^2}{\lambda}$$

$$\Rightarrow x^* = \frac{a^2}{\sqrt{a^2+b^2}}, \quad y^* = \frac{b^2}{\sqrt{a^2+b^2}}$$

$\Rightarrow$ Max Perimeter of a rectangle is $4(x^*+y^*) = \boxed{4\sqrt{a^2+b^2}}$

(ii) The optimization problem now is

$$\min -4xy \qquad \text{s.t. } f(x,y) = 0$$

$$H(x,y,\lambda) = -4xy + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right)$$

$$\Rightarrow H_\lambda = 0 = f(x,y)$$

$$H_x = 0 \Rightarrow -4y + \frac{2\lambda x}{a^2} \Rightarrow \lambda = \frac{2a^2 y}{x}$$

$$H_y = 0 = -4x + \frac{2\lambda y}{b^2} \Rightarrow \lambda = \frac{2b^2 x}{y}$$

$$\Rightarrow \frac{x^2}{a^2} = \frac{y^2}{b^2}$$

$$\Rightarrow x^* = \frac{a}{\sqrt{2}}, \quad y^* = \frac{b}{\sqrt{2}} \quad \text{and} \quad \lambda^* = 2ab$$

Then the max. area of a rectangle is $4x^*y^* = \boxed{2ab}$

2.1

$$\dot{x} = Ax + Dxu + bu$$

$$J = \frac{1}{2} x^T S x \Big|_T + \frac{1}{2}\int_0^T x^T Q x + ru^2$$

$$H = \frac{1}{2} x^T Q x + \frac{1}{2} r u^2 + \lambda^T (Ax + Dxu + bu)$$

Stationarity cond. $\frac{\partial H}{\partial u} = 0 \Rightarrow ru + (Dx+b)^T \lambda = 0$

$$\Rightarrow u_* = -\frac{1}{r}(Dx+b)^T\lambda$$

$$= -\frac{b^T\lambda}{r} - \frac{x^T(D+D^T)\lambda}{2r}$$

∴ The optimal input contains a state-costate "inner product" (It is formally an inner product if $D+D^T > 0$).

Substituting the optimal input into the state & costate equs.:

$$\dot{x} = Ax + (Dx+b)\left(\frac{-1}{r}\right)(Dx+b)^T\lambda$$

$$= Ax - (Dx+b)(Dx+b)^T\frac{\lambda}{r}. \longrightarrow \text{last term is cubic in } x, \lambda$$

$$\dot{\lambda} = -Qx - (A^T + D^T u)\lambda$$

$$= -Qx - A^T\lambda + \underbrace{\frac{1}{r}D^T\lambda\lambda^T b}_{\text{quadratic}} + \underbrace{\frac{1}{2r}D^T\lambda\lambda^T(D+D^T)x}_{\text{cubic}}$$

2.2

$$\dot{x} = u$$

$$J(x_0, t_0) = \frac{1}{2} s(T)\, x^2(T) + \frac{1}{2}\int_{t_0}^{T} r u^2\, dt$$

a) i Riccati

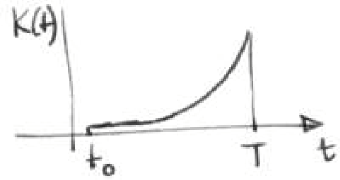
$$-\dot{S} = A^T S + SA - SBR^{-1}B^T S + Q$$

$$= -\frac{S^2}{r} \qquad ; \quad BC \quad S(T)$$

$$\Rightarrow \frac{ds}{S^2} = \frac{dt}{r} \Rightarrow -\int_{1/S(t)}^{1/S(T)} d(1/S) = \int_t^T dt/r$$

$$\Rightarrow \frac{1}{S(T)} - \frac{1}{S(t)} = \frac{-(T-t)}{r} \Rightarrow \boxed{S(t) = \frac{S(T)\, r}{r + S(T)(T-t)}}$$

ii) $K = R^{-1}B^T S = \dfrac{S(T)}{r + S(T)(T-t)}$

iii) 

$u_* = -Kx$

b) i) $G = \displaystyle\int_t^T e^{A(T-\tau)} BB^T e^{A^T(T-\tau)}\, d\tau$; Weighted: $\displaystyle\int e^{A(T-\tau)} BR^{-1}B^{-1} e^{A^T(T-\tau)}$

$= T - t$ $\qquad\qquad = \dfrac{T-t}{r}$

ii) $\boxed{u_*(t) = \dfrac{x(T) - x(t_0)}{T - t_0}}$ (constant)

iii) $x(t) = x(t_0) + \dfrac{t - t_0}{T - t_0}\left(x(T) - x(t_0)\right)$ (linear)

c) $x(t_0) = \dfrac{T-t_0}{T-t}\, x(t) - \dfrac{t-t_0}{T-t}\, x(T)$

$$\Rightarrow u_*(t) = \frac{1}{T-t_0} x(T) - \frac{1}{T-t_0}\left[\frac{T-t_0}{T-t} x(t) - \frac{t-t_0}{T-t} x(T)\right]$$

$$= \frac{1}{T-t}\left[x(T) - x(t)\right] \Rightarrow \left|\begin{array}{l} h(t) = \dfrac{x(T)}{T-t} \\ g(t) = \dfrac{1}{T-t} \end{array}\right.$$

Comparing with (a).:

We want $x(T) \to 0$ with a high penalty to emulate the fixed final state, so $S(T) \to \infty$.

Then $K \to \dfrac{1}{T-t}$, $u_*(t) \to -\dfrac{x(t)}{T-t}$

iii) $u_*(t) = \dfrac{1}{T-t}\left[x(T) - x(t)\right]$

min Cost-to-go $J(t_0)$ at time t_0 : $u_*(t_0) = -K(t_0)\, x(t_0)$

$$= -\frac{S(T)}{r + S(T)(T-t_0)}\, x(t_0)$$

They approach each other for $S(T) \to \infty$
$x(T) \to 0$

(otherwise, J must be reformulated in terms of a target state)

2.3

$$\begin{pmatrix} \dot{V} \\ W \end{pmatrix} = \begin{pmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{pmatrix} \begin{pmatrix} V \\ W \end{pmatrix}$$

$$W(T) = S(T)V(T)$$

Show $\Rightarrow$ $S(t) = W(t)V^{-1}(t)$.

We verify the Riccati: $-\dot{S} = A^TS + SA - SBR^{-1}B^TS + Q$

$$\dot{S} = \dot{W}V^{-1} + W\dot{V^{-1}} = \dot{W}V^{-1} - WV^{-1}\dot{V}V^{-1}$$

$$= (-QV - A^TW)V^{-1} - WV^{-1}(AV - BR^{-1}B^TW)V^{-1}$$

$$= -Q - A^TS - SA + SBR^{-1}B^TS \quad \text{(verified)}$$

Then, at T, $S(T) = W(T)V^{-1}(T)$ $\Rightarrow$ S satisfies ODE + BC

$\Rightarrow$ sol'n.

2.4

$$\dot{x}_1 = x_2 \qquad J = \int_0^\infty x^T \underbrace{\begin{pmatrix} 1 & v \\ v & q \end{pmatrix}}_{Q} x + u^2$$
$$\dot{x}_2 = u$$

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad BB^T = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

RICCATI: $A^TS + SA - S^TBB^TS + Q = 0$ (Algebraic because of infinite horizon)

Let $S = \begin{pmatrix} s_1 & s_2 \\ s_2 & s_3 \end{pmatrix}$. Then,

$$\begin{pmatrix} 0 & s_1 \\ s_1 & 2s_2 \end{pmatrix} - \begin{pmatrix} s_2^2 & s_2s_3 \\ s_2s_3 & s_3^2 \end{pmatrix} + \begin{pmatrix} 1 & v \\ v & q \end{pmatrix} = 0$$

$\Rightarrow$ $s_2^2 = 1$, $s_3^2 = q + 2s_2$, $s_1 = s_2s_3 - v$

S PD $\quad s_1s_3 > s_2^2$, $s_1 > 0$, $s_3 > 0$

$\Rightarrow$ $s_2 = +\sqrt{1} = 1$, $s_3 = \sqrt{q+2}$, $s_1 = \sqrt{q+2} - v$

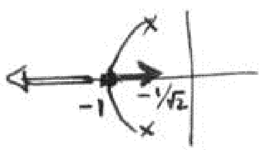
$$\Rightarrow \quad S = \begin{pmatrix} \sqrt{q+2} - v & 1 \\ 1 & \sqrt{q+2} \end{pmatrix} \quad \text{which is PD as long as } q > v^2$$

$$K = R^{-1} B^T S = \left[1, \ \sqrt{q+2} \right]$$

$$u_*(t) = -K x(t)$$

Optimal closed loop: $\dot{x} = (A - BK)\, x$; $A - BK = \begin{pmatrix} 0 & 1 \\ -1 & -\sqrt{q+2} \end{pmatrix}$

Roots of Char. Eqn: $\dfrac{-\sqrt{q+2} \pm \sqrt{q-2}}{2}$ $\begin{cases} -\frac{1}{\sqrt{2}} \pm j \frac{1}{\sqrt{2}} \ ; \ q \to 0 \\ -1, -1 \ ; \ q = 2 \\ -\sqrt{q}, -1/\sqrt{q} \ ; \ q \to \infty \end{cases}$



Stable for $q > 0$. (guaranteed from LQR theory)

Also for $q > -2$

3.1

$$\dot{x} = Ax + Bu \qquad A = \begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\min \int_0^T 1 \; ; \quad x(T) = 0$$

$$H = 1 + \lambda^T A x + \lambda^T B u \;\Rightarrow\; \left| \begin{array}{l} u_* = \arg\min_u H = -\operatorname{sign} \lambda_2 \\ \dot{\lambda}_2 = -A^T \lambda \end{array} \right.$$

Finding the matrix exponential (Laplace, Cayley, etc)

$$e^{At} = \begin{bmatrix} \cos \omega t & \frac{1}{\omega} \sin \omega t \\ -\omega \sin \omega t & \cos \omega t \end{bmatrix}, \quad e^{-A^T t} = \left(e^{At}\right)^{-T} = \begin{bmatrix} \cos \omega t & \omega \sin \omega t \\ -\frac{1}{\omega} \sin \omega t & \cos \omega t \end{bmatrix}$$

and $\lambda_2(t) = \lambda_{02} \cos \omega t - \frac{\lambda_{01}}{\omega} \sin \omega t \quad \left(= e^{-A^T t} \lambda_0 \right)$

$\lambda_{01}, \lambda_{02}$ will be chosen to satisfy the BC, $x(0) = x_0$, $x(T) = 0$.

Thus the optimal input has the form

$$u_*(t) = -\operatorname{sign}\left(\alpha \cos \omega t + \beta \sin \omega t \right)$$

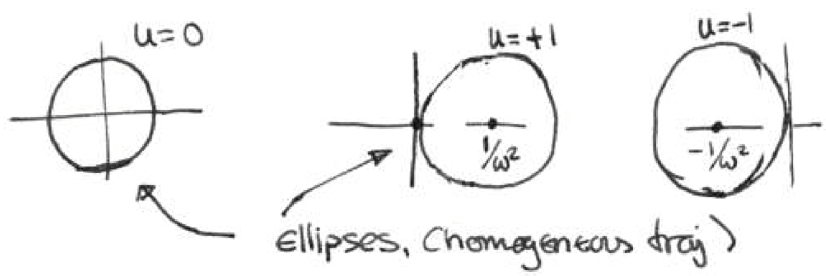
Rewrite in magnitude-phase

$$= \operatorname{sign}\left(\rho \cos(\omega t + \phi) \right)$$

$$= \operatorname{sign}\left[\cos(\omega t + \phi) \right]$$

$\Rightarrow$ it is a square wave with period $T = \frac{2\pi}{\omega}$, same as the oscillator, and its only free parameter is the starting phase. Effectively it will look like a number of complete periods with beginning and ending segments of arbitrary duration ($< T/2$)

Trajectories:



Notice that when $u = +1$, $\quad \mathring{x}_1 = x_2$

$$\mathring{x}_2 = -\omega^2 x_1 + u = -\omega^2 (x_1 - 1/\omega^2)$$

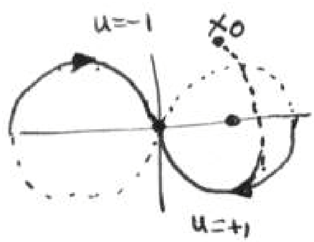
Letting $\bar{x}_1 = x_1 - 1/\omega^2$, $\bar{x}_2 = x_2 \Rightarrow \mathring{\bar{x}}_1 = \bar{x}_2$

$$\mathring{\bar{x}}_2 = -\omega^2 \bar{x}_1$$

$\Rightarrow$ in the shifted coordinates the system is an unforced oscillato

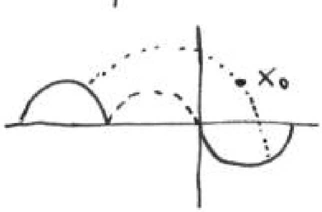
$\Rightarrow$ The trajectories will be ellipses centered at $0, -1/\omega^2, +1/\omega^2$ for $u = 0, -1, +1$ respectively. (With a transformation $\bar{x}_1 = \omega x_1$, $\bar{x}_2 = x_2 \Rightarrow \mathring{\bar{x}}_1 = \omega \bar{x}_2$, $\mathring{\bar{x}}_2 = -\omega \bar{x}_1$, the ellipses become circles).

The last part of the switching curve is quite obvious. It is the ellipse (circle in normalized coordinates) that passes thru the origin (notice the orientation)



Starting with an IC that does not belong to these two arcs, there will be at least one switching before the state becomes zero. (Since $\lambda(A) \neq$ real, the number of switchings is not constrained)

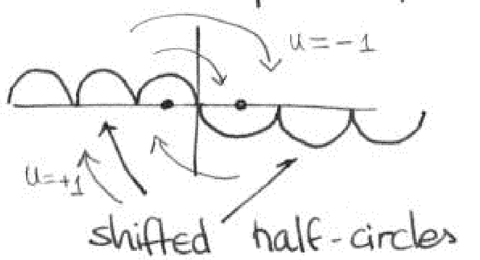
So, starting from an IC x_0 (see graph) the input would be initially -1, switching to $+1$ when $x(t)$ hits the switching curve. The -1 part of the trajectory is a circle centered at $-1/\omega^2$. To find the next switching point we observe that it must come from a square wave



input that should switch in a half-period. In the normalized coordinates, the entire circle is covered in one period and equal time segments correspond to equal arcs.

Hence the other switching point would be the arc that is
- symmetric to the $\{u=+1\}$-switching
- about $-1/\omega^2$

Hence, because of input symmetry, the symmetry about a point can be viewed as a shifting, that would produce a much simpler expression for the switching curve.



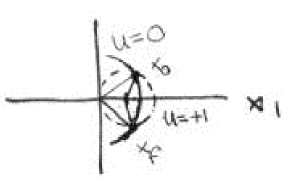
Notes:

1) It is instructive to look at the evolution of the system backwards in time. Starting with $x_0 = 0$, solve (in Simulink)
$\mathring{x} = -(Ax + Bu)$, $\mathring{\lambda} = A^T\lambda$, $u = -\text{sign}\,\lambda_2$.
Different λ_0's would correspond to different switching points, covering all trajectories that pass thru $\left(-\frac{3}{\omega^2}, +\frac{3}{\omega^2}\right)$

Then, to arrive to any point in the state space, we simply need to follow the appropriate trajectory thru the switching curve

2). Elapsed time is proportional to the arc angle.* To illustrate this point consider the following two trajectories that have the same initial & final points (symmetric about x_1-axis)

* in normalized coordinates



The angle for $u=+1$ is clearly larger than the angle for $u=0$ (similarly for $u=-1$)
So, between x_0 and x_f the elapsed time will be: $t_{u=+1} > t_{u=0} > t_{u=-1}$

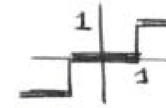
3.2

The last comment from the previous problem is important here.

For the min fuel problem (fixed final time)

$$H = |u| + \lambda^T A x + \lambda^T B u \quad , \quad |u| \le 1$$

$$u_* = \arg\min H = \text{bang-off-bang}\left(-B^T\lambda\right)$$



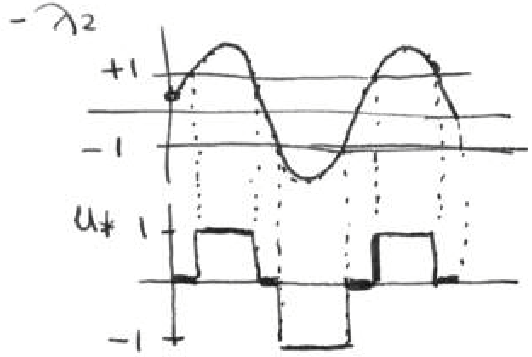
$$\dot{\lambda} = -A^T\lambda$$

This is similar to the min-time problem:

$$-B^T\lambda = -\lambda_2 = \alpha\cos\omega t + \beta\sin(\omega t)$$
$$= \rho\cos(\omega t + \phi)$$

where (ρ, ϕ) have 1-1 correspondence with $\lambda_0 = \begin{bmatrix}\lambda_{01} \\ \lambda_{02}\end{bmatrix}$

But here, ρ is important:



Small ρ $(\rho < 1)$ implies that $u_* = 0 \Rightarrow x(0) = 0$

Large ρ $(\rho \gg 1)$ implies that $u_* = \pm 1$ except for very short time intervals $\Rightarrow$ u_* approaches the time-optimal input. This situation occurs when $T \to T_{min}$

Clearly, if $T < T_{min}$ there is no solution.

The min fuel trajectory is now a function of two parameters ρ and ϕ (or $\lambda_{01}, \lambda_{02}$). In the min-time problem we found that each value of ϕ would be associated with a "spiral" of initial conditions, and ρ was irrelevant.

In the min-fuel problem the extra parameter ρ is associated with $T - T_{min}$. A rough harmonic analysis shows that the control is applied ($u = \pm 1$) when x_1 is small and the system "coasts" ($u=0$) when x_1 is large. But the thresholds are not simple functions. Again, solving the optimal equations backwards in time provides an interesting illustration of the optimal trajectories.