

GAME THEORY

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Part II. Two-Person Zero-Sum Games

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Part II. Two-Person Zero-Sum Games

1. The Strategic Form of a Game.

The individual most closely associated with the creation of the theory of games is John von Neumann, one of the greatest mathematicians of the 20th century. Although others preceded him in formulating a theory of games - notably Émile Borel - it was von Neumann who published in 1928 the paper that laid the foundation for the theory of two-person zero-sum games. Von Neumann's work culminated in a fundamental book on game theory written in collaboration with Oskar Morgenstern entitled *Theory of Games and Economic Behavior*, 1944. Other discussions of the theory of games relevant for our present purposes may be found in the text book, *Game Theory* by Guillermo Owen, 2nd edition, Academic Press, 1982, and the expository book, *Game Theory and Strategy* by Philip D. Straffin, published by the Mathematical Association of America, 1993.

The theory of von Neumann and Morgenstern is most complete for the class of games called two-person zero-sum games, i.e. **games with only two players in which one player wins what the other player loses**. In Part II, we restrict attention to such games. We will refer to the players as Player I and Player II.

1.1 Strategic Form. The simplest mathematical description of a game is the strategic form, mentioned in the introduction. For a two-person zero-sum game, the payoff function of Player II is the negative of the payoff of Player I, so we may restrict attention to the single payoff function of Player I, which we call here L .

Definition 1. The **strategic form**, or **normal form**, of a two-person zero-sum game is given by a triplet (X, Y, A) , where

- (1) X is a nonempty set, the set of strategies of Player I
- (2) Y is a nonempty set, the set of strategies of Player II
- (3) A is a real-valued function defined on $X \times Y$. (Thus, $A(x, y)$ is a real number for every $x \in X$ and every $y \in Y$.)

The interpretation is as follows. Simultaneously, Player I chooses $x \in X$ and Player II chooses $y \in Y$, each unaware of the choice of the other. Then their choices are made known and I wins the amount $A(x, y)$ from II. Depending on the monetary unit involved, $A(x, y)$ will be cents, dollars, pesos, beads, etc. If A is negative, I pays the absolute value of this amount to II. Thus, **$A(x, y)$ represents the winnings of I and the losses of II.**

This is a very simple definition of a game; yet it is broad enough to encompass the finite combinatorial games and games such as tic-tac-toe and chess. This is done by being sufficiently broadminded about the definition of a strategy. A strategy for a game of chess,

for example, is a complete description of how to play the game, of what move to make in every possible situation that could occur. It is rather time-consuming to write down even one strategy, good or bad, for the game of chess. However, several different programs for instructing a machine to play chess well have been written. Each program constitutes one strategy. The program Deep Blue, that beat then world chess champion Gary Kasparov in a match in 1997, represents one strategy. The set of all such strategies for Player I is denoted by X . Naturally, in the game of chess it is physically impossible to describe all possible strategies since there are too many; in fact, there are more strategies than there are atoms in the known universe. On the other hand, the number of games of tic-tac-toe is rather small, so that it is possible to study all strategies and find an optimal strategy for each player. Later, when we study the extensive form of a game, we will see that many other types of games may be modeled and described in strategic form.

To illustrate the notions involved in games, let us consider the simplest non-trivial case when both X and Y consist of two elements. As an example, take the game called Odd-or-Even.

1.2 Example: Odd or Even. Players I and II simultaneously call out one of the numbers one or two. Player I's name is Odd; he wins if the sum of the numbers is odd. Player II's name is Even; she wins if the sum of the numbers is even. The amount paid to the winner by the loser is always the sum of the numbers in dollars. To put this game in strategic form we must specify X , Y and A . Here we may choose $X = \{1, 2\}$, $Y = \{1, 2\}$, and A as given in the following table.

		II (even)		y	
				1	2
I (odd)	x	1	$\begin{pmatrix} -2 & +3 \\ +3 & -4 \end{pmatrix}$		
		2			

$$A(x, y) = \text{I's winnings} = \text{II's losses.}$$

It turns out that one of the players has a distinct advantage in this game. Can you tell which one it is?

Let us analyze this game from Player I's point of view. Suppose he calls 'one' $3/5$ ths of the time and 'two' $2/5$ ths of the time at random. In this case,

1. If II calls 'one', I loses 2 dollars $3/5$ ths of the time and wins 3 dollars $2/5$ ths of the time; on the average, he wins $-2(3/5) + 3(2/5) = 0$ (he breaks even in the long run).
2. If II call 'two', I wins 3 dollars $3/5$ ths of the time and loses 4 dollars $2/5$ ths of the time; on the average he wins $3(3/5) - 4(2/5) = 1/5$.

That is, if I mixes his choices in the given way, the game is even every time II calls 'one', but I wins 20¢ on the average every time II calls 'two'. By employing this simple strategy, I is assured of at least breaking even on the average no matter what II does. Can Player I fix it so that he wins a positive amount no matter what II calls?

Let p denote the proportion of times that Player I calls ‘one’. Let us try to choose p so that Player I wins the same amount on the average whether II calls ‘one’ or ‘two’. Then since I’s average winnings when II calls ‘one’ is $-2p + 3(1 - p)$, and his average winnings when II calls ‘two’ is $3p - 4(1 - p)$ Player I should choose p so that

$$\begin{aligned} -2p + 3(1 - p) &= 3p - 4(1 - p) \\ 3 - 5p &= 7p - 4 \\ 12p &= 7 \\ p &= 7/12. \end{aligned}$$

Hence, I should call ‘one’ with probability $7/12$, and ‘two’ with probability $5/12$. On the average, I wins $-2(7/12) + 3(5/12) = 1/12$, or $8\frac{1}{3}$ cents every time he plays the game, no matter what II does. Such a strategy that produces the same average winnings no matter what the opponent does is called an **equalizing strategy**.

Therefore, the game is clearly in I’s favor. Can he do better than $8\frac{1}{3}$ cents per game on the average? The answer is: Not if II plays properly. In fact, II could use the same procedure:

call ‘one’ with probability $7/12$

call ‘two’ with probability $5/12$.

If I calls ‘one’, II’s average loss is $-2(7/12) + 3(5/12) = 1/12$. If I calls ‘two’, II’s average loss is $3(7/12) - 4(5/12) = 1/12$.

Hence, I has a procedure that guarantees him at least $1/12$ on the average, and II has a procedure that keeps her average loss to at most $1/12$. $1/12$ is called the **value** of the game, and the procedure each uses to insure this return is called an **optimal strategy** or a **minimax strategy**.

If instead of playing the game, the players agree to call in an arbitrator to settle this conflict, it seems reasonable that the arbitrator should require II to pay $8\frac{1}{3}$ cents to I. For I could argue that he should receive at least $8\frac{1}{3}$ cents since his optimal strategy guarantees him that much on the average no matter what II does. On the other hand II could argue that she should not have to pay more than $8\frac{1}{3}$ cents since she has a strategy that keeps her average loss to at most that amount no matter what I does.

1.3 Pure Strategies and Mixed Strategies. It is useful to make a distinction between a pure strategy and a mixed strategy. We refer to elements of X or Y as pure strategies. The more complex entity that chooses among the pure strategies at random in various proportions is called a mixed strategy. Thus, I’s optimal strategy in the game of Odd-or-Even is a mixed strategy; it mixes the pure strategies one and two with probabilities $7/12$ and $5/12$ respectively. Of course every pure strategy, $x \in X$, can be considered as the mixed strategy that chooses the pure strategy x with probability 1.

In our analysis, we made a rather subtle assumption. We assumed that when a player uses a mixed strategy, he is only interested in his average return. He does not care about his

maximum possible winnings or losses — only the average. This is actually a rather drastic assumption. We are evidently assuming that a player is indifferent between receiving 5 million dollars outright, and receiving 10 million dollars with probability $1/2$ and nothing with probability $1/2$. I think nearly everyone would prefer the \$5,000,000 outright. This is because the utility of having 10 megabucks is not twice the utility of having 5 megabucks.

The main justification for this assumption comes from **utility theory** and is treated in Appendix 1. The basic premise of utility theory is that one should evaluate a payoff by its utility to the player rather than on its numerical monetary value. Generally a player's utility of money will not be linear in the amount. The main theorem of utility theory states that under certain reasonable assumptions, a player's preferences among outcomes are consistent with the existence of a utility function and the player judges an outcome only on the basis of the *average* utility of the outcome.

However, utilizing utility theory to justify the above assumption raises a new difficulty. Namely, the two players may have different utility functions. The same outcome may be perceived in quite different ways. This means that the game is no longer zero-sum. We need an assumption that says the utility functions of two players are the same (up to change of location and scale). This is a rather strong assumption, but for moderate to small monetary amounts, we believe it is a reasonable one.

A mixed strategy may be implemented with the aid of a suitable outside random mechanism, such as tossing a coin, rolling dice, drawing a number out of a hat and so on. The seconds indicator of a watch provides a simple personal method of randomization provided it is not used too frequently. For example, Player I of Odd-or-Even wants an outside random event with probability $7/12$ to implement his optimal strategy. Since $7/12 = 35/60$, he could take a quick glance at his watch; if the seconds indicator showed a number between 0 and 35, he would call 'one', while if it were between 35 and 60, he would call 'two'.

1.4 The Minimax Theorem. A two-person zero-sum game (X, Y, A) is said to be a **finite game** if both strategy sets X and Y are finite sets. The fundamental theorem of game theory due to von Neumann states that the situation encountered in the game of Odd-or-Even holds for all finite two-person zero-sum games. Specifically,

The Minimax Theorem. *For every finite two-person zero-sum game,*

- (1) *there is a number V , called the value of the game,*
- (2) *there is a mixed strategy for Player I such that I's average gain is at least V no matter what II does, and*
- (3) *there is a mixed strategy for Player II such that II's average loss is at most V no matter what I does.*

This is one form of the minimax theorem to be stated more precisely and discussed in greater depth later. If V is zero we say the game is fair. If V is positive, we say the game favors Player I, while if V is negative, we say the game favors Player II.

1.5 Exercises.

1. Consider the game of Odd-or-Even with the sole change that the loser pays the winner the product, rather than the sum, of the numbers chosen (who wins still depends on the sum). Find the table for the payoff function A , and analyze the game to find the value and optimal strategies of the players. Is the game fair?

2. Player I holds a black Ace and a red 8. Player II holds a red 2 and a black 7. The players simultaneously choose a card to play. If the chosen cards are of the same color, Player I wins. Player II wins if the cards are of different colors. The amount won is a number of dollars equal to the number on the winner's card (Ace counts as 1.) Set up the payoff function, find the value of the game and the optimal mixed strategies of the players.

3. Sherlock Holmes boards the train from London to Dover in an effort to reach the continent and so escape from Professor Moriarty. Moriarty can take an express train and catch Holmes at Dover. However, there is an intermediate station at Canterbury at which Holmes may detrain to avoid such a disaster. But of course, Moriarty is aware of this too and may himself stop instead at Canterbury. Von Neumann and Morgenstern (loc. cit.) estimate the value to Moriarty of these four possibilities to be given in the following matrix (in some unspecified units).

		Holmes	
		Canterbury	Dover
Moriarty	Canterbury	100	-50
	Dover	0	100

What are the optimal strategies for Holmes and Moriarty, and what is the value? (Historically, as related by Dr. Watson in "The Final Problem" in Arthur Conan Doyle's *The Memoires of Sherlock Holmes*, Holmes detrained at Canterbury and Moriarty went on to Dover.)

4. The entertaining book *The Compleat Strategyst* by John Williams contains many simple examples and informative discussion of strategic form games. Here is one of his problems.

"I know a good game," says Alex. "We point fingers at each other; either one finger or two fingers. If we match with one finger, you buy me one Daiquiri, If we match with two fingers, you buy me two Daiquiris. If we don't match I let you off with a payment of a dime. It'll help pass the time."

Olaf appears quite unmoved. "That sounds like a very dull game — at least in its early stages." His eyes glaze on the ceiling for a moment and his lips flutter briefly; he returns to the conversation with: "Now if you'd care to pay me 42 cents before each game, as a partial compensation for all those 55-cent drinks I'll have to buy you, then I'd be happy to pass the time with you."

Olaf could see that the game was inherently unfair to him so he insisted on a side payment as compensation. Does this side payment make the game fair? What are the optimal strategies and the value of the game?

2. Matrix Games — Domination

A finite two-person zero-sum game in strategic form, (X, Y, A) , is sometimes called a matrix game because the payoff function A can be represented by a matrix. If $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$, then by the *game matrix* or *payoff matrix* we mean the matrix

$$\mathbf{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \text{where } a_{ij} = A(x_i, y_j),$$

In this form, Player I chooses a row, Player II chooses a column, and II pays I the entry in the chosen row and column. Note that the entries of the matrix are the winnings of the row chooser and losses of the column chooser.

A mixed strategy for Player I may be represented by an m -tuple, $\mathbf{p} = (p_1, p_2, \dots, p_m)^T$ of probabilities that add to 1. If I uses the mixed strategy $\mathbf{p} = (p_1, p_2, \dots, p_m)^T$ and II chooses column j , then the (average) payoff to I is $\sum_{i=1}^m p_i a_{ij}$. Similarly, a mixed strategy for Player II is an n -tuple $\mathbf{q} = (q_1, q_2, \dots, q_n)^T$. If II uses \mathbf{q} and I uses row i the payoff to I is $\sum_{j=1}^n a_{ij} q_j$. More generally, if I uses the mixed strategy \mathbf{p} and II uses the mixed strategy \mathbf{q} , the (average) payoff to I is $\mathbf{p}^T \mathbf{A} \mathbf{q} = \sum_{i=1}^m \sum_{j=1}^n p_i a_{ij} q_j$.

Note that the pure strategy for Player I of choosing row i may be represented as the mixed strategy \mathbf{e}_i , the unit vector with a 1 in the i th position and 0's elsewhere. Similarly, the pure strategy for II of choosing the j th column may be represented by \mathbf{e}_j . In the following, we shall be attempting to 'solve' games. This means finding the value, and at least one optimal strategy for each player. Occasionally, we shall be interested in finding all optimal strategies for a player.

2.1 Saddle points. Occasionally it is easy to solve the game. If some entry a_{ij} of the matrix \mathbf{A} has the property that

- (1) a_{ij} is the minimum of the i th row, and
- (2) a_{ij} is the maximum of the j th column,

then we say a_{ij} is a saddle point. If a_{ij} is a saddle point, then Player I can then win at least a_{ij} by choosing row i , and Player II can keep her loss to at most a_{ij} by choosing column j . Hence a_{ij} is the value of the game.

Example 1.

$$\mathbf{A} = \begin{pmatrix} 4 & 1 & -3 \\ 3 & 2 & 5 \\ 0 & 1 & 6 \end{pmatrix}$$

The central entry, 2, is a saddle point, since it is a minimum of its row and maximum of its column. Thus it is optimal for I to choose the second row, and for II to choose the second column. The value of the game is 2, and $(0, 1, 0)$ is an optimal mixed strategy for both players.

For large $m \times n$ matrices it is tedious to check each entry of the matrix to see if it has the saddle point property. It is easier to compute the minimum of each row and the maximum of each column to see if there is a match. Here is an example of the method.

$$\begin{array}{ccccc}
 & & & & \text{row min} \\
 & & & & \\
 \mathbf{A} = \begin{pmatrix} 3 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 2 \end{pmatrix} & & 0 & & \\
 & & 0 & & \\
 & & 0 & & \\
 & & 1 & & \\
 \text{col max} & 3 & 2 & 2 & 2
 \end{array}
 \qquad
 \begin{array}{ccccc}
 & & & & \text{row min} \\
 & & & & \\
 \mathbf{B} = \begin{pmatrix} 3 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 1 \\ 3 & 1 & 2 & 2 \end{pmatrix} & & 0 & & \\
 & & 0 & & \\
 & & 0 & & \\
 & & 1 & & \\
 \text{col max} & 3 & 1 & 2 & 2
 \end{array}$$

In matrix \mathbf{A} , no row minimum is equal to any column maximum, so there is no saddle point. However, if the 2 in position a_{12} were changed to a 1, then we have matrix \mathbf{B} . Here, the minimum of the fourth row is equal to the maximum of the second column; so b_{42} is a saddle point.

2.2 Solution of All 2 by 2 Matrix Games. Consider the general 2×2 game matrix

$$\mathbf{A} = \begin{pmatrix} a & b \\ d & c \end{pmatrix}.$$

To solve this game (i.e. to find the value and at least one optimal strategy for each player) we proceed as follows.

1. *Test for a saddle point.*
2. *If there is no saddle point, solve by finding equalizing strategies.*

We now prove the method of finding equalizing strategies of Section 1.2 works whenever there is no saddle point by deriving the value and the optimal strategies.

Assume there is no saddle point. If $a \geq b$, then $b < c$, as otherwise b is a saddle point. Since $b < c$, we must have $c > d$, as otherwise c is a saddle point. Continuing thus, we see that $d < a$ and $a > b$. In other words, if $a \geq b$, then $a > b < c > d < a$. By symmetry, if $a \leq b$, then $a < b > c < d > a$. This shows that

If there is no saddle point, then either $a > b$, $b < c$, $c > d$ and $d < a$, or $a < b$, $b > c$, $c < d$ and $d > a$.

In equations (1), (2) and (3) below, we develop formulas for the optimal strategies and value of the general 2×2 game. If I chooses the first row with probability p (i.e. uses the mixed strategy $(p, 1 - p)$), we equate his average return when II uses columns 1 and 2.

$$ap + d(1 - p) = bp + c(1 - p).$$

Solving for p , we find

$$p = \frac{c - d}{(a - b) + (c - d)}. \quad (1)$$

Since there is no saddle point, $(a - b)$ and $(c - d)$ are either both positive or both negative; hence, $0 < p < 1$. Player I's average return using this strategy is

$$v = ap + d(1 - p) = \frac{ac - bd}{a - b + c - d}.$$

If II chooses the first column with probability q (i.e. uses the strategy $(q, 1 - q)$), we equate his average losses when I uses rows 1 and 2.

$$aq + b(1 - q) = dq + c(1 - q)$$

Hence,

$$q = \frac{c - b}{a - b + c - d}. \quad (2)$$

Again, since there is no saddle point, $0 < q < 1$. Player II's average loss using this strategy is

$$aq + b(1 - q) = \frac{ac - bd}{a - b + c - d} = v, \quad (3)$$

the same value achievable by I. This shows that the game has a value, and that the players have optimal strategies. (something the minimax theorem says holds for all finite games).

Example 2.

$$\mathbf{A} = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix} \quad \begin{aligned} p &= \frac{-4 - 3}{-2 - 3 - 4 - 3} = 7/12 \\ q &= \text{same} \\ v &= \frac{8 - 9}{-2 - 3 - 4 - 3} = 1/12 \end{aligned}$$

Example 3.

$$\mathbf{A} = \begin{pmatrix} 0 & -10 \\ 1 & 2 \end{pmatrix} \quad \begin{aligned} p &= \frac{2 - 1}{0 + 10 + 2 - 1} = 1/11 \\ q &= \frac{2 + 10}{0 + 10 + 2 - 1} = 12/11. \end{aligned}$$

But q must be between zero and one. What happened? The trouble is we “forgot to test this matrix for a saddle point, so of course it has one”. (J. D. Williams *The Compleat Strategyst* Revised Edition, 1966, McGraw-Hill, page 56.) The lower left corner is a saddle point. So $p = 0$ and $q = 1$ are optimal strategies, and the value is $v = 1$.

2.3 Removing Dominated Strategies. Sometimes, large matrix games may be reduced in size (hopefully to the 2×2 case) by deleting rows and columns that are obviously bad for the player who uses them.

Definition. We say the i th row of a matrix $\mathbf{A} = (a_{ij})$ *dominates* the k th row if $a_{ij} \geq a_{kj}$ for all j . We say the i th row of \mathbf{A} *strictly dominates* the k th row if $a_{ij} > a_{kj}$ for all j . Similarly, the j th column of \mathbf{A} *dominates* (strictly *dominates*) the k th column if $a_{ij} \leq a_{ik}$ (resp. $a_{ij} < a_{ik}$) for all i .

Anything Player I can achieve using a dominated row can be achieved at least as well using the row that dominates it. Hence dominated rows may be deleted from the matrix. A similar argument shows that dominated columns may be removed. To be more precise, *removal of a dominated row or column does not change the value of a game*. However, there may exist an optimal strategy that uses a dominated row or column (see Exercise 9). If so, removal of that row or column will also remove the use of that optimal strategy (although there will still be at least one optimal strategy left). However, in the case of removal of a *strictly dominated* row or column, the set of optimal strategies does not change.

We may iterate this procedure and successively remove several rows and columns. As an example, consider the matrix, \mathbf{A} .

The last column is dominated by the middle column. Deleting the last column we obtain:

$$\mathbf{A} = \begin{pmatrix} 2 & 0 & 4 \\ 1 & 2 & 3 \\ 4 & 1 & 2 \end{pmatrix}$$

Now the top row is dominated by the bottom row. (Note this is not the case in the original matrix). Deleting the top row we obtain:

$$\begin{pmatrix} 2 & 0 \\ 1 & 2 \\ 4 & 1 \end{pmatrix}$$

This 2×2 matrix does not have a saddle point, so $p = 3/4$, $q = 1/4$ and $v = 7/4$. I's optimal strategy in the original game is $(0, 3/4, 1/4)$; II's is $(1/4, 3/4, 0)$.

$$\begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$$

A row (column) may also be removed if it is dominated by a probability combination of other rows (columns).

If for some $0 < p < 1$, $pa_{i_1j} + (1-p)a_{i_2j} \geq a_{kj}$ for all j , then the k th row is dominated by the mixed strategy that chooses row i_1 with probability p and row i_2 with probability $1 - p$. Player I can do at least as well using this mixed strategy instead of choosing row k . (In addition, any mixed strategy choosing row k with probability p_k may be replaced by the one in which k 's probability is split between i_1 and i_2 . That is, i_1 's probability is increased by pp_k and i_2 's probability is increased by $(1-p)p_k$.) A similar argument may be used for columns.

Consider the matrix $\mathbf{A} = \begin{pmatrix} 0 & 4 & 6 \\ 5 & 7 & 4 \\ 9 & 6 & 3 \end{pmatrix}$.

The middle column is dominated by the outside columns taken with probability $1/2$ each. With the central column deleted, the middle row is dominated by the combination of the top row with probability $1/3$ and the bottom row with probability $2/3$. The reduced matrix, $\begin{pmatrix} 0 & 6 \\ 9 & 3 \end{pmatrix}$, is easily solved. The value is $V = 54/12 = 9/2$.

Of course, mixtures of more than two rows (columns) may be used to dominate and remove other rows (columns). For example, the mixture of columns one two and three with probabilities $1/3$ each in matrix $\mathbf{B} = \begin{pmatrix} 1 & 3 & 5 & 3 \\ 4 & 0 & 2 & 2 \\ 3 & 7 & 3 & 5 \end{pmatrix}$ dominates the last column,

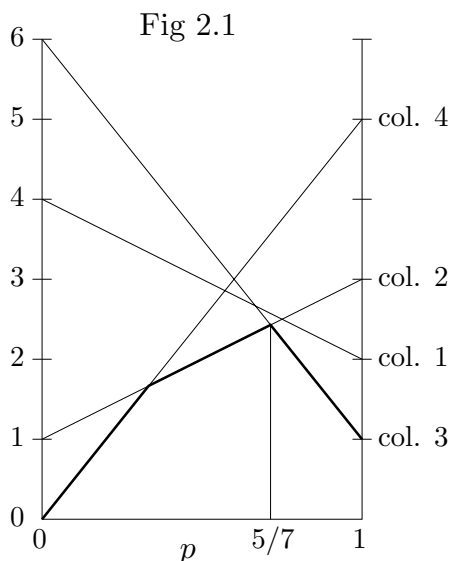
and so the last column may be removed.

Not all games may be reduced by dominance. In fact, even if the matrix has a saddle point, there may not be any dominated rows or columns. The 3×3 game with a saddle point found in Example 1 demonstrates this.

2.4 Solving $2 \times n$ and $m \times 2$ games. Games with matrices of size $2 \times n$ or $m \times 2$ may be solved with the aid of a graphical interpretation. Take the following example.

$$\begin{array}{c} p \\ 1-p \end{array} \quad \begin{pmatrix} 2 & 3 & 1 & 5 \\ 4 & 1 & 6 & 0 \end{pmatrix}$$

Suppose Player I chooses the first row with probability p and the second row with probability $1-p$. If II chooses Column 1, I's average payoff is $2p + 4(1-p)$. Similarly, choices of Columns 2, 3 and 4 result in average payoffs of $3p + (1-p)$, $p + 6(1-p)$, and $5p$ respectively. We graph these four linear functions of p for $0 \leq p \leq 1$. For a fixed value of p , Player I can be sure that his average winnings is at least the minimum of these four functions evaluated at p . This is known as the lower envelope of these functions. Since I wants to maximize his guaranteed average winnings, he wants to find p that achieves the maximum of this lower envelope. According to the drawing, this should occur at the intersection of the lines for Columns 2 and 3. This essentially, involves solving the game in which II is restricted to Columns 2 and 3. The value of the game $\begin{pmatrix} 3 & 1 \\ 1 & 6 \end{pmatrix}$ is $v = 17/7$, I's optimal strategy is $(5/7, 2/7)$, and II's optimal strategy is $(5/7, 2/7)$. Subject to the accuracy of the drawing, we conclude therefore that in the original game I's optimal strategy is $(5/7, 2/7)$, II's is $(0, 5/7, 2/7, 0)$ and the value is $17/7$.

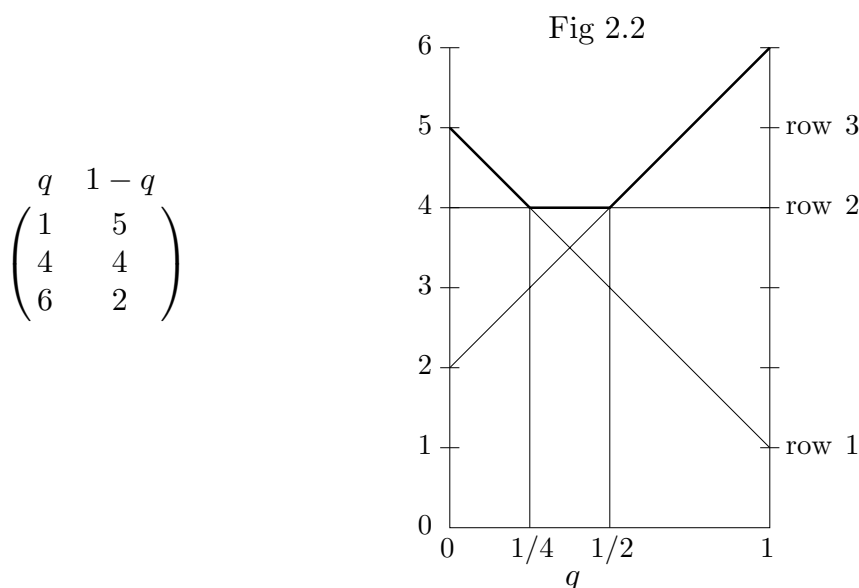


The accuracy of the drawing may be checked: *Given any guess at a solution to a game, there is a sure-fire test to see if the guess is correct*, as follows. If I uses the strategy $(5/7, 2/7)$, his average payoff if II uses Columns 1, 2, 3 and 4, is $18/7$, $17/7$, $17/7$, and $25/7$

respectively. Thus his average payoff is at least $17/7$ no matter what II does. Similarly, if II uses $(0, 5/7, 2/7, 0)$, her average loss is (at most) $17/7$. Thus, $17/7$ is the value, and these strategies are optimal.

We note that the line for Column 1 plays no role in the lower envelope (that is, the lower envelope would be unchanged if the line for Column 1 were removed from the graph). *This is a test for domination.* Column 1 is, in fact, dominated by Columns 2 and 3 taken with probability $1/2$ each. The line for Column 4 does appear in the lower envelope, and hence Column 4 cannot be dominated.

As an example of a $m \times 2$ game, consider the matrix associated with Figure 2.2. If q is the probability that II chooses Column 1, then II's average loss for I's three possible choices of rows is given in the accompanying graph. Here, Player II looks at the largest of her average losses for a given q . This is the upper envelope of the function. II wants to find q that minimizes this upper envelope. From the graph, we see that any value of q between $1/4$ and $1/3$ inclusive achieves this minimum. The value of the game is 4, and I has an optimal pure strategy: row 2.



These techniques work just as well for $2 \times \infty$ and $\infty \times 2$ games.

2.5 Latin Square Games. A Latin square is an $n \times n$ array of n different letters such that each letter occurs once and only once in each row and each column. The 5×5 array at the right is an example. If in a Latin square each letter is assigned a numerical value, the resulting matrix is the matrix of a Latin square game. Such games have simple solutions. The value is the average of the numbers in a row, and the strategy that chooses each pure strategy with equal probability $1/n$ is optimal for both players. The reason is not very deep. The conditions for optimality are satisfied.

$$a = 1, b = 2, c = d = 3, e = 6$$

$$\begin{pmatrix} a & b & c & d & e \\ b & e & a & c & d \\ c & a & d & e & b \\ d & c & e & b & a \\ e & d & b & a & c \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 3 & 6 \\ 2 & 6 & 1 & 3 & 3 \\ 3 & 1 & 3 & 6 & 2 \\ 3 & 3 & 6 & 2 & 1 \\ 6 & 3 & 2 & 1 & 3 \end{pmatrix}$$

In the example above, the value is $V = (1+2+3+3+6)/5 = 3$, and the mixed strategy $\mathbf{p} = \mathbf{q} = (1/5, 1/5, 1/5, 1/5, 1/5)$ is optimal for both players. The game of matching pennies is a Latin square game. Its value is zero and $(1/2, 1/2)$ is optimal for both players.

2.6 Exercises.

1. Solve the game with matrix $\begin{pmatrix} -1 & -3 \\ -2 & 2 \end{pmatrix}$, that is find the value and an optimal (mixed) strategy for both players.

2. Solve the game with matrix $\begin{pmatrix} 0 & 2 \\ t & 1 \end{pmatrix}$ for an arbitrary real number t . (Don't forget to check for a saddle point!) Draw the graph of $v(t)$, the value of the game, as a function of t , for $-\infty < t < \infty$.

3. Show that if a game with $m \times n$ matrix has two saddle points, then they have equal values.

4. Reduce by dominance to 2×2 games and solve.

$$(a) \quad \begin{pmatrix} 5 & 4 & 1 & 0 \\ 4 & 3 & 2 & -1 \\ 0 & -1 & 4 & 3 \\ 1 & -2 & 1 & 2 \end{pmatrix} \quad (b) \quad \begin{pmatrix} 10 & 0 & 7 & 1 \\ 2 & 6 & 4 & 7 \\ 6 & 3 & 3 & 5 \end{pmatrix}.$$

5. (a) Solve the game with matrix $\begin{pmatrix} 3 & 2 & 4 & 0 \\ -2 & 1 & -4 & 5 \end{pmatrix}$.

(b) Reduce by dominance to a 3×2 matrix game and solve: $\begin{pmatrix} 0 & 8 & 5 \\ 8 & 4 & 6 \\ 12 & -4 & 3 \end{pmatrix}$.

6. Players I and II choose integers i and j respectively from the set $\{1, 2, \dots, n\}$ for some $n \geq 2$. Player I wins 1 if $|i - j| = 1$. Otherwise there is no payoff. If $n = 7$, for example, the game matrix is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

(a) Using dominance to reduce the size of the matrix, solve the game for $n = 7$ (i.e. find the value and one optimal strategy for each player).

(b) See if you can solve the game for arbitrary n .

7. In general, the sure-fire test may be stated thus: For a given game, conjectured optimal strategies (p_1, \dots, p_m) and (q_1, \dots, q_n) are indeed optimal if the minimum of I's average payoffs using (p_1, \dots, p_m) is equal to the maximum of II's average payoffs using (q_1, \dots, q_n) . Show that for the game with the following matrix the mixed strategies $\mathbf{p} = (6/37, 20/37, 0, 11/37)$ and $\mathbf{q} = (14/37, 4/37, 0, 19/37, 0)$ are optimal for I and II respectively. What is the value?

$$\begin{pmatrix} 5 & 8 & 3 & 1 & 6 \\ 4 & 2 & 6 & 3 & 5 \\ 2 & 4 & 6 & 4 & 1 \\ 1 & 3 & 2 & 5 & 3 \end{pmatrix}$$

8. Given that $\mathbf{p} = (52/143, 50/143, 41/143)$ is optimal for I in the game with the following matrix, what is the value?

$$\begin{pmatrix} 0 & 5 & -2 \\ -3 & 0 & 4 \\ 6 & -4 & 0 \end{pmatrix}$$

9. Player I secretly chooses one of the numbers, 1, 2 and 3, and Player II tries to guess which. If II guesses correctly, she loses nothing; otherwise, she loses the absolute value of the difference of I's choice and her guess. Set up the matrix and reduce it by dominance to a 2 by 2 game and solve. Note that II has an optimal pure strategy that was eliminated by dominance. Moreover, this strategy dominates the optimal mixed strategy in the 2 by 2 game.

10. **Magic Square Games.** A magic square is an $n \times n$ array of the first n integers with the property that all row and column sums are equal. Show how to solve all games with magic square game matrices. Solve the example,

$$\begin{pmatrix} 16 & 3 & 2 & 13 \\ 5 & 10 & 11 & 8 \\ 9 & 6 & 7 & 12 \\ 4 & 15 & 14 & 1 \end{pmatrix}.$$

(This is the magic square that appears in Albrecht Dürer's engraving, *Melencolia*. See <http://freemasonry.bcy.ca/art/melencolia.html>)

11. In an article, "Normandy: Game and Reality" by W. Drakert in *Moves*, No. 6 (1972), an analysis is given of the invasion of Europe at Normandy in World War II. Six possible attacking configurations (1 to 6) by the Allies and six possible defensive strategies (A to F) by the Germans were simulated and evaluated, 36 simulations in all. The following

table gives the estimated value to the Allies of each hypothetical battle in some numerical units.

	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>E</i>	<i>F</i>
1	13	29	8	12	16	23
2	18	22	21	22	29	31
3	18	22	31	31	27	37
4	11	22	12	21	21	26
5	18	16	19	14	19	28
6	23	22	19	23	30	34

- (a) Assuming this is a matrix of a six by six game, reduce by dominance and solve.
 (b) The historical defense by the Germans was *B*, and the historical attack by the Allies was 1. Criticize these choices.

3. The Principle of Indifference.

For a matrix game with $m \times n$ matrix \mathbf{A} , if Player I uses the mixed strategy $\mathbf{p} = (p_1, \dots, p_m)^T$ and Player II uses column j , Player I's average payoff is $\sum_{i=1}^m p_i a_{ij}$. If V is the value of the game, an optimal strategy, \mathbf{p} , for I is characterized by the property that Player I's average payoff is at least V no matter what column j Player II uses, i.e.

$$\sum_{i=1}^m p_i a_{ij} \geq V \quad \text{for all } j = 1, \dots, n. \quad (1)$$

Similarly, a strategy $\mathbf{q} = (q_1, \dots, q_n)^T$ is optimal for II if and only if

$$\sum_{j=1}^n a_{ij} q_j \leq V \quad \text{for all } i = 1, \dots, m. \quad (2)$$

When both players use their optimal strategies the average payoff, $\sum_i \sum_j p_i a_{ij} q_j$, is exactly V . This may be seen from the inequalities

$$\begin{aligned} V &= \sum_{j=1}^n V q_j \leq \sum_{j=1}^n \left(\sum_{i=1}^m p_i a_{ij} \right) q_j = \sum_{i=1}^m \sum_{j=1}^n p_i a_{ij} q_j \\ &= \sum_{i=1}^m p_i \left(\sum_{j=1}^n a_{ij} q_j \right) \leq \sum_{i=1}^m p_i V = V. \end{aligned} \quad (3)$$

Since this begins and ends with V we must have equality throughout.

3.1 The Equilibrium Theorem. The following simple theorem – the Equilibrium Theorem – gives conditions for equality to be achieved in (1) for certain values of j , and in (2) for certain values of i .

Theorem 3.1. Consider a game with $m \times n$ matrix \mathbf{A} and value V . Let $\mathbf{p} = (p_1, \dots, p_m)^T$ be any optimal strategy for I and $\mathbf{q} = (q_1, \dots, q_n)^T$ be any optimal strategy for II. Then

$$\sum_{j=1}^n a_{ij} q_j = V \quad \text{for all } i \text{ for which } p_i > 0 \quad (4)$$

and

$$\sum_{i=1}^m p_i a_{ij} = V \quad \text{for all } j \text{ for which } q_j > 0. \quad (5)$$

Proof. Suppose there is a k such that $p_k > 0$ and $\sum_{j=1}^n a_{kj} q_j \neq V$. Then from (2), $\sum_{j=1}^n a_{kj} q_j < V$. But then from (3) with equality throughout

$$V = \sum_{i=1}^m p_i \left(\sum_{j=1}^n a_{ij} q_j \right) < \sum_{i=1}^m p_i V = V.$$

The inequality is strict since it is strict for the k th term of the sum. This contradiction proves the first conclusion. The second conclusion follows analogously. ■

Another way of stating the first conclusion of this theorem is: If there exists an optimal strategy for I giving positive probability to row i , then every optimal strategy of II gives I the value of the game if he uses row i .

This theorem is useful in certain classes of games for helping direct us toward the solution. The procedure this theorem suggests for Player 1 is to try to find a solution to the set of equations (5) formed by those j for which you think it likely that $q_j > 0$. One way of saying this is that Player 1 searches for a strategy that makes Player 2 *indifferent* as to which of the (good) pure strategies to use. Similarly, Player 2 should play in such a way to make Player 1 indifferent among his (good) strategies. This is called the **Principle of Indifference**.

Example. As an example of this consider the game of Odd-or-Even in which both players simultaneously call out one of the numbers zero, one, or two. The matrix is

$$\begin{array}{cc} & \text{Even} \\ \text{Odd} & \begin{pmatrix} 0 & 1 & -2 \\ 1 & -2 & 3 \\ -2 & 3 & -4 \end{pmatrix} \end{array}$$

Again it is difficult to guess who has the advantage. If we play the game a few times we might become convinced that Even's optimal strategy gives positive weight (probability) to each of the columns. If this assumption is true, Odd should play to make Player 2 indifferent; that is, Odd's optimal strategy \mathbf{p} must satisfy

$$\begin{aligned} p_2 - 2p_3 &= V \\ p_1 - 2p_2 + 3p_3 &= V \\ -2p_1 + 3p_2 - 4p_3 &= V, \end{aligned} \tag{6}$$

for some number, V — three equations in four unknowns. A fourth equation that must be satisfied is

$$p_1 + p_2 + p_3 = 1. \tag{7}$$

This gives four equations in four unknowns. This system of equations is solved as follows. First we work with (6); add the first equation to the second.

$$p_1 - p_2 + p_3 = 2V \tag{8}$$

Then add the second equation to the third.

$$-p_1 + p_2 - p_3 = 2V \tag{9}$$

Taken together (8) and (9) imply that $V = 0$. Adding (7) to (9), we find $2p_2 = 1$, so that $p_2 = 1/2$. The first equation of (6) implies $p_3 = 1/4$ and (7) implies $p_1 = 1/4$. Therefore

$$\mathbf{p} = (1/4, 1/2, 1/4)^T \tag{10}$$

is a strategy for I that keeps his average gain to zero no matter what II does. Hence the value of the game is at least zero, and $V = 0$ if our assumption that II's optimal strategy gives positive weight to all columns is correct. To complete the solution, we note that if the optimal \mathbf{p} for I gives positive weight to all rows, then II's optimal strategy \mathbf{q} must satisfy the same set of equations (6) and (7) with \mathbf{p} replaced by \mathbf{q} (because the game matrix here is symmetric). Therefore,

$$\mathbf{q} = (1/4, 1/2, 1/4)^T \quad (11)$$

is a strategy for II that keeps his average loss to zero no matter what I does. Thus the value of the game is zero and (10) and (11) are optimal for I and II respectively. The game is fair.

3.2 Nonsingular Game Matrices. Let us extend the method used to solve this example to arbitrary nonsingular square matrices. Let the game matrix \mathbf{A} be $m \times m$, and suppose that \mathbf{A} is nonsingular. Assume that I has an optimal strategy giving positive weight to each of the rows. (This is called the *all-strategies-active* case.) Then by the principle of indifference, every optimal strategy \mathbf{q} for II satisfies (4), or

$$\sum_{j=1}^m a_{ij} q_j = V \quad \text{for } i = 1, \dots, m. \quad (12)$$

This is a set of m equations in m unknowns, and since \mathbf{A} is nonsingular, we may solve for the q_i . Let us write this set of equations in vector notation using \mathbf{q} to represent the column vector of II's strategy, and $\mathbf{1} = (1, 1, \dots, 1)^T$ to represent the column vector of all 1's:

$$\mathbf{A}\mathbf{q} = V\mathbf{1} \quad (13)$$

We note that V cannot be zero since (13) would imply that \mathbf{A} was singular. Since \mathbf{A} is non-singular, \mathbf{A}^{-1} exists. Multiplying both sides of (13) on the left by \mathbf{A}^{-1} yields

$$\mathbf{q} = V\mathbf{A}^{-1}\mathbf{1}. \quad (14)$$

If the value of V were known, this would give the unique optimal strategy for II. To find V , we may use the equation $\sum_{j=1}^m q_j = 1$, or in vector notation $\mathbf{1}^T \mathbf{q} = 1$. Multiplying both sides of (14) on the left by $\mathbf{1}^T$ yields $1 = \mathbf{1}^T \mathbf{q} = V\mathbf{1}^T \mathbf{A}^{-1} \mathbf{1}$. This shows that $\mathbf{1}^T \mathbf{A}^{-1} \mathbf{1}$ cannot be zero so we can solve for V :

$$V = 1/\mathbf{1}^T \mathbf{A}^{-1} \mathbf{1}. \quad (15)$$

The unique optimal strategy for II is therefore

$$\mathbf{q} = \mathbf{A}^{-1} \mathbf{1} / \mathbf{1}^T \mathbf{A}^{-1} \mathbf{1}. \quad (16)$$

However, if some component, q_j , turns out to be negative, then our assumption that I has an optimal strategy giving positive weight to each row is false.

However, if $q_j \geq 0$ for all j , we may seek an optimal strategy for I by the same method. The result would be

$$\mathbf{p}^T = \mathbf{1}^T \mathbf{A}^{-1} / \mathbf{1}^T \mathbf{A}^{-1} \mathbf{1}. \quad (17)$$

Now, if in addition $p_i \geq 0$ for all i , then both \mathbf{p} and \mathbf{q} are optimal since both guarantee an average payoff of V no matter what the other player does. Note that we do not require the p_i to be strictly positive as was required by our original “all-strategies-active” assumption.

We summarize this discussion as a theorem.

Theorem 3.2. *Assume the square matrix \mathbf{A} is nonsingular and $\mathbf{1}^T \mathbf{A}^{-1} \mathbf{1} \neq 0$. Then the game with matrix \mathbf{A} has value $V = 1/\mathbf{1}^T \mathbf{A}^{-1} \mathbf{1}$ and optimal strategies $\mathbf{p}^T = V \mathbf{1}^T \mathbf{A}^{-1}$ and $\mathbf{q} = V \mathbf{A}^{-1} \mathbf{1}$, provided both $\mathbf{p} \geq \mathbf{0}$ and $\mathbf{q} \geq \mathbf{0}$.*

If the value of a game is zero, this method cannot work directly since (13) implies that \mathbf{A} is singular. However, the addition of a positive constant to all entries of the matrix to make the value positive, *may* change the game matrix into being nonsingular. The previous example of Odd-or-Even is a case in point. The matrix is singular so it would seem that the above method would not work. Yet if 1, say, were added to each entry of the matrix to obtain the matrix \mathbf{A} below, then \mathbf{A} is nonsingular and we may apply the above method. Let us carry through the computations. By some method or another \mathbf{A}^{-1} is obtained.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & -1 & 4 \\ -1 & 4 & -3 \end{pmatrix} \quad \mathbf{A}^{-1} = \frac{1}{16} \begin{pmatrix} 13 & -2 & -7 \\ -2 & 4 & 6 \\ -7 & 6 & 5 \end{pmatrix}$$

Then $\mathbf{1}^T \mathbf{A}^{-1} \mathbf{1}$, the sum of the elements of \mathbf{A}^{-1} , is found to be 1, so from (15), $V = 1$. Therefore, we compute $\mathbf{p}^T = \mathbf{1}^T \mathbf{A}^{-1} = (1/4, 1/2, 1/4)^T$, and $\mathbf{q} = \mathbf{A}^{-1} \mathbf{1} = (1/4, 1/2, 1/4)^T$. Since both are nonnegative, both are optimal and 1 is the value of the game with matrix \mathbf{A} .

What do we do if either \mathbf{p} or \mathbf{q} has negative components? A complete answer to questions of this sort is given in the comprehensive theorem of Shapley and Snow (1950). This theorem shows that an arbitrary $m \times n$ matrix game whose value is not zero may be solved by choosing some suitable square submatrix \mathbf{A} , and applying the above methods and checking that the resulting optimal strategies are optimal for the whole matrix, \mathbf{A} . Optimal strategies obtained in this way are called **basic**, and it is noted that every optimal strategy is a probability mixture of basic optimal strategies. Such a submatrix, \mathbf{A} , is called an *active submatrix* of the game. See Karlin (1959, Vol. I, Section 2.4) for a discussion and proof. The problem is to determine which square submatrix to use. The simplex method of linear programming is simply an efficient method not only for solving equations of the form (13), but also for finding which square submatrix to use. This is described in Section 4.4.

3.3 Diagonal Games. We apply these ideas to the class of diagonal games - games whose game matrix \mathbf{A} is square and diagonal,

$$\mathbf{A} = \begin{pmatrix} d_1 & 0 & \dots & 0 \\ 0 & d_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & d_m \end{pmatrix} \quad (18)$$

Suppose all diagonal terms are positive, $d_i > 0$ for all i . (The other cases are treated in Exercise 2.) One may apply Theorem 3.2 to find the solution, but it is as easy to proceed directly. The set of equations (12) becomes

$$p_i d_i = V \quad \text{for } i = 1, \dots, m \quad (19)$$

whose solution is simply

$$p_i = V/d_i \quad \text{for } i = 1, \dots, m. \quad (20)$$

To find V , we sum both sides over i to find

$$1 = V \sum_{i=1}^m 1/d_i \quad \text{or} \quad V = \left(\sum_{i=1}^m 1/d_i \right)^{-1}. \quad (21)$$

Similarly, the equations for Player II yield

$$q_i = V/d_i \quad \text{for } i = 1, \dots, m. \quad (22)$$

Since V is positive from (21), we have $p_i > 0$ and $q_i > 0$ for all i , so that (20) and (22) give optimal strategies for I and II respectively, and (21) gives the value of the game.

As an example, consider the game with matrix \mathbf{C} .

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

From (20) and (22) the optimal strategy is proportional to the reciprocals of the diagonal elements. The sum of these reciprocals is $1 + 1/2 + 1/3 + 1/4 = 25/12$. Therefore, the value is $V = 12/25$, and the optimal strategies are $\mathbf{p} = \mathbf{q} = (12/25, 6/25, 4/25, 3/25)^T$.

3.4 Triangular Games. Another class of games for which the equations (12) are easy to solve are the games with triangular matrices - matrices with zeros above or below the main diagonal. Unlike for diagonal games, the method does not always work to solve triangular games because the resulting \mathbf{p} or \mathbf{q} may have negative components. Nevertheless, it works often enough to merit special mention. Consider the game with triangular matrix \mathbf{T} .

$$\mathbf{T} = \begin{pmatrix} 1 & -2 & 3 & -4 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The equations (12) become

$$\begin{aligned} p_1 &= V \\ -2p_1 + p_2 &= V \\ 3p_1 - 2p_2 + p_3 &= V \\ -4p_1 + 3p_2 - 2p_3 + p_4 &= V. \end{aligned}$$

These equations may be solved one at a time from the top down to give

$$p_1 = V \quad p_2 = 3V \quad p_3 = 4V \quad p_4 = 4V.$$

Since $\sum p_i = 1$, we find $V = 1/12$ and $\mathbf{p} = (1/12, 1/4, 1/3, 1/3)$. The equations for the q 's are

$$\begin{aligned} q_1 - 2q_2 + 3q_3 - 4q_4 &= V \\ q_2 - 2q_3 + 3q_4 &= V \\ q_3 - 2q_4 &= V \\ q_4 &= V. \end{aligned}$$

The solution is

$$q_1 = 4V \quad q_2 = 4V \quad q_3 = 3V \quad q_4 = V.$$

Since the p 's and q 's are non-negative, $V = 1/12$ is the value, $\mathbf{p} = (1/12, 1/4, 1/3, 1/3)$ is optimal for I, and $\mathbf{q} = (1/3, 1/3, 1/4, 1/12)$ is optimal for II.

3.5 Symmetric Games. A game is symmetric if the rules do not distinguish between the players. For symmetric games, both players have the same options (the game matrix is square), and the payoff if I uses i and II uses j is the negative of the payoff if I uses j and II uses i . This means that the game matrix should be **skew-symmetric**: $\mathbf{A} = -\mathbf{A}^T$, or $a_{ij} = -a_{ji}$ for all i and j .

Definition 3.1. A finite game is said to be **symmetric** if its game matrix is square and skew-symmetric.

Speaking more generally, we may say that a game is symmetric if after some rearrangement of the rows or columns the game matrix is skew-symmetric.

The game of paper-scissors-rock is an example. In this game, Players I and II simultaneously display one of the three objects: paper, scissors, or rock. If they both choose the same object to display, there is no payoff. If they choose different objects, then scissors win over paper (scissors cut paper), rock wins over scissors (rock breaks scissors), and paper wins over rock (paper covers rock). If the payoff upon winning or losing is one unit, then the matrix of the game is as follows.

$$\begin{array}{cc} & \begin{array}{ccc} & \text{II} \\ & \text{paper} & \text{scissors} & \text{rock} \end{array} \\ \begin{array}{c} \text{I} \\ \text{paper} \\ \text{scissors} \\ \text{rock} \end{array} & \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix} \end{array}$$

This matrix is skew-symmetric so the game is symmetric. The diagonal elements of the matrix are zero. This is true of any skew-symmetric matrix, since $a_{ii} = -a_{ii}$ implies $a_{ii} = 0$ for all i .

A contrasting example is the game of matching pennies. The two players simultaneously choose to show a penny with either the heads or the tails side facing up. One of the

players, say Player I, wins if the choices match. The other player, Player II, wins if the choices differ. Although there is a great deal of symmetry in this game, we do not call it a symmetric game. Its matrix is

$$\begin{array}{cc} & \begin{array}{cc} \text{II} \\ \text{heads} & \text{tails} \end{array} \\ \begin{array}{c} \text{I} \\ \text{heads} \\ \text{tails} \end{array} & \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \end{array}$$

This matrix is not skew-symmetric.

We expect a symmetric game to be fair, that is to have value zero, $V = 0$. This is indeed the case.

Theorem 3.3. *A finite symmetric game has value zero. Any strategy optimal for one player is also optimal for the other.*

Proof. Let \mathbf{p} be an optimal strategy for I. If II uses the same strategy the average payoff is zero, because

$$\mathbf{p}^T \mathbf{A} \mathbf{p} = \sum \sum p_i a_{ij} p_j = \sum \sum p_i (-a_{ji}) p_j = - \sum \sum p_j a_{ji} p_i = -\mathbf{p}^T \mathbf{A} \mathbf{p} \quad (23)$$

implies that $\mathbf{p}^T \mathbf{A} \mathbf{p} = 0$. This shows that the value $V \leq 0$. A symmetric argument shows that $V \geq 0$. Hence $V = 0$. Now suppose \mathbf{p} is optimal for I. Then $\sum_{i=1}^m p_i a_{ij} \geq 0$ for all j . Hence $\sum_{j=1}^m a_{ij} p_j = -\sum_{j=1}^m p_j a_{ji} \leq 0$ for all i , so that \mathbf{p} is also optimal for II. By symmetry, if \mathbf{q} is optimal for II, it is optimal for I also. ■

Mendelsohn Games. (N. S. Mendelsohn (1946)) In Mendelsohn games, two players simultaneously choose a positive integer. Both players want to choose an integer larger but not too much larger than the opponent. Here is a simple example. The players choose an integer between 1 and 100. If the numbers are equal there is no payoff. The player that chooses a number one larger than that chosen by his opponent wins 1. The player that chooses a number two or more larger than his opponent loses 2. Find the game matrix and solve the game.

Solution. The payoff matrix is

$$\begin{array}{c} \begin{matrix} & 1 & 2 & 3 & 4 & 5 & \dots \end{matrix} \\ \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ \vdots \end{pmatrix} \begin{pmatrix} 0 & -1 & 2 & 2 & 2 & \dots \\ 1 & 0 & -1 & 2 & 2 & \dots \\ -2 & 1 & 0 & -1 & 2 & \dots \\ -2 & -2 & 1 & 0 & -1 & \dots \\ -2 & -2 & -2 & 1 & 0 & \dots \\ \vdots & \vdots & & & & \ddots \end{pmatrix} \end{array} \quad (24)$$

The game is symmetric so the value is zero and the players have identical optimal strategies. We see that row 1 dominates rows 4, 5, 6, ... so we may restrict attention to the upper left

3×3 submatrix. We suspect that there is an optimal strategy for I with $p_1 > 0$, $p_2 > 0$ and $p_3 > 0$. If so, it would follow from the principle of indifference (since $q_1 = p_1 > 0$, $q_2 = p_2 > 0$, $q_3 = p_3 > 0$ is optimal for II) that

$$\begin{aligned} p_2 - 2p_3 &= 0 \\ -p_1 + p_3 &= 0 \\ 2p_1 - p_2 &= 0. \end{aligned} \tag{25}$$

We find $p_2 = 2p_3$ and $p_1 = p_3$ from the first two equations, and the third equation is redundant. Since $p_1 + p_2 + p_3 = 1$, we have $4p_3 = 1$; so $p_1 = 1/4$, $p_2 = 1/2$, and $p_3 = 1/4$. Since p_1 , p_2 and p_3 are positive, this gives the solution: $\mathbf{p} = \mathbf{q} = (1/4, 1/2, 1/4, 0, 0, \dots)^T$ is optimal for both players.

3.6 Invariance. Consider the game of matching pennies: Two players simultaneously choose heads or tails. Player I wins if the choices match and Player II wins otherwise.

There doesn't seem to be much of a reason for either player to choose heads instead of tails. In fact, the problem is the same if the names of heads and tails are interchanged. In other words, the problem is *invariant* under interchanging the names of the pure strategies. In this section, we make the notion of invariance precise. We then define the notion of an invariant strategy and show that in the search for a minimax strategy, a player may restrict attention to invariant strategies. Use of this result greatly simplifies the search for minimax strategies in many games. In the game of matching pennies for example, there is only one invariant strategy for either player, namely, choose heads or tails with probability $1/2$ each. Therefore this strategy is minimax without any further computation.

We look at the problem from Player II's viewpoint. Let Y denote the pure strategy space of Player II, assumed finite. A transformation, g of Y into Y is said to be *onto* Y if the range of g is the whole of Y , that is, if for every $y_1 \in Y$ there is $y_2 \in Y$ such that $g(y_2) = y_1$. A transformation, g , of Y into itself is said to be *one-to-one* if $g(y_1) = g(y_2)$ implies $y_1 = y_2$.

Definition 3.2. Let $G = (X, Y, A)$ be a finite game, and let g be a one-to-one transformation of Y onto itself. The game G is said to be **invariant** under g if for every $x \in X$ there is a unique $x' \in X$ such that

$$A(x, y) = A(x', g(y)) \quad \text{for all } y \in Y. \tag{26}$$

The requirement that x' be unique is not restrictive, for if there were another point $x'' \in X$ such that

$$A(x, y) = A(x'', g(y)) \quad \text{for all } y \in Y, \tag{27}$$

then, we would have $A(x', g(y)) = A(x'', g(y))$ for all $y \in Y$, and since g is onto,

$$A(x', y) = A(x'', y) \quad \text{for all } y \in Y. \tag{28}$$

Thus the strategies x' and x'' have identical payoffs and we could remove one of them from X without changing the problem at all.

To keep things simple, we assume without loss of generality that all duplicate pure strategies have been eliminated. That is, we assume

$$\begin{aligned} A(x', y) &= A(x'', y) && \text{for all } y \in Y \text{ implies that } x' = x'', \text{ and} \\ A(x, y') &= A(x, y'') && \text{for all } x \in X \text{ implies that } y' = y''. \end{aligned} \quad (29)$$

Unicity of x' in Definition 3.2 follows from this assumption.

The given x' in Definition 3.2 depends on g and x only. We denote it by $x' = \bar{g}(x)$. We may write equation (26) defining invariance as

$$A(x, y) = A(\bar{g}(x), g(y)) \quad \text{for all } x \in X \text{ and } y \in Y. \quad (26')$$

The mapping \bar{g} is a one-to-one transformation of X since if $\bar{g}(x_1) = \bar{g}(x_2)$, then

$$A(x_1, y) = A(\bar{g}(x_1), g(y)) = A(\bar{g}(x_2), g(y)) = A(x_2, y) \quad (30)$$

for all $y \in Y$, which implies $x_1 = x_2$ from assumption (29). Therefore the inverse, g^{-1} , of g , defined by $g^{-1}(g(x)) = g(g^{-1}(x)) = x$, exists. Moreover, any one-to-one transformation of a finite set is automatically onto, so \bar{g} is a one-to-one transformation of X onto itself.

Lemma 1. *If a finite game, $G = (X, Y, A)$, is invariant under a one-to-one transformation, g , then G is also invariant under g^{-1} .*

Proof. We are given $A(x, y) = A(\bar{g}(x), g(y))$ for all $x \in X$ and all $y \in Y$. Since true for all x and y , it is true if y is replaced by $g^{-1}(y)$ and x is replaced by $\bar{g}^{-1}(x)$. This gives $A(\bar{g}^{-1}(x), g^{-1}(y)) = A(x, y)$ for all $x \in X$ and all $y \in Y$. This shows that G is invariant under g^{-1} . ■

Lemma 2. *If a finite game, $G = (X, Y, A)$, is invariant under two one-to-one transformations, g_1 and g_2 , then G is also invariant under the composition transformation, g_2g_1 , defined by $g_2g_1(y) = g_2(g_1(y))$.*

Proof. We are given $A(x, y) = A(\bar{g}_1(x), g_1(y))$ for all $x \in X$ and all $y \in Y$, and $A(x, y) = A(\bar{g}_2(x), g_2(y))$ for all $x \in X$ and all $y \in Y$. Therefore,

$$A(x, y) = A(\bar{g}_2(\bar{g}_1(x)), g_2(g_1(y))) = A(\bar{g}_2(\bar{g}_1(x)), g_2g_1(y)) \quad \text{for all } y \in Y \text{ and } x \in X. \quad (31)$$

which shows that G is invariant under g_2g_1 . ■

Furthermore, these proofs show that

$$\overline{g_2g_1} = \bar{g}_2 \bar{g}_1, \quad \text{and} \quad \overline{g^{-1}} = \bar{g}^{-1}. \quad (32)$$

Thus the class of transformations, g on Y , under which the problem is invariant forms a group, \mathcal{G} , with composition as the multiplication operator. The identity element, e of the group is the identity transformation, $e(y) = y$ for all $y \in Y$. The set, $\bar{\mathcal{G}}$ of corresponding transformations \bar{g} on X is also a group, with identity $\bar{e}(x) = x$ for all $x \in X$. Equations (32) say that $\bar{\mathcal{G}}$ is isomorphic to \mathcal{G} ; as groups, they are indistinguishable.

This shows that we could have analyzed the problem from Player I's viewpoint and arrived at the same groups $\bar{\mathcal{G}}$ and \mathcal{G} .

Definition 3.3. A finite game $G = (X, Y, A)$ is said to be invariant under a group, \mathcal{G} of transformations, if (26') holds for all $g \in \mathcal{G}$.

We now define what it means for a mixed strategy, \mathbf{q} , for Player II to be invariant under a group \mathcal{G} . Let m denote the number of elements in X and n denote the number of elements in Y .

Definition 3.4. Given that a finite game $G = (X, Y, A)$ is invariant under a group, \mathcal{G} , of one-to-one transformations of Y , a mixed strategy, $\mathbf{q} = (q(1), \dots, q(n))$, for Player II is said to be invariant under \mathcal{G} if

$$q(g(y)) = q(y) \quad \text{for all } y \in Y \text{ and all } g \in \mathcal{G}. \quad (33)$$

Similarly a mixed strategy $\mathbf{p} = (p(1), \dots, p(m))$, for Player I is said to be invariant under \mathcal{G} (or $\overline{\mathcal{G}}$) if

$$p(\overline{g}(x)) = p(x) \quad \text{for all } x \in X \text{ and all } \overline{g} \in \overline{\mathcal{G}}. \quad (34)$$

Two points y_1 and y_2 in Y are said to be equivalent if there exists a g in \mathcal{G} such that $g(y_2) = y_1$. It is an easy exercise to show that this is an equivalence relation. The set of points, $E_y = \{y' : g(y') = y \text{ for some } g \in \mathcal{G}\}$, is called an equivalence class, or an orbit. Thus, y_1 and y_2 are equivalent if they lie in the same orbit. Definition 3.4 says that a mixed strategy \mathbf{q} for Player II is invariant if it is constant on orbits, that is, if it assigns the same probability to all pure strategies in the orbit. The power of this notion is contained in the following theorem.

Theorem 3.4. If a finite game $G = (X, Y, A)$ is invariant under a group \mathcal{G} , then there exist invariant optimal strategies for the players.

Proof. It is sufficient to show that Player II has an invariant optimal strategy. Since the game is finite, there exists a value, V , and an optimal mixed strategy for player II, \mathbf{q}^* . This is to say that

$$\sum_{y \in Y} A(x, y) q^*(y) \leq V \quad \text{for all } x \in X. \quad (35)$$

We must show that there is an invariant strategy $\tilde{\mathbf{q}}$ that satisfies this same condition. Let $N = |\mathcal{G}|$ be the number of elements in the group \mathcal{G} . Define

$$\tilde{q}(y) = \frac{1}{N} \sum_{g \in \mathcal{G}} q^*(g(y)) \quad (36)$$

(This takes each orbit and replaces each probability by the average of the probabilities in the orbit.) Then $\tilde{\mathbf{q}}$ is invariant since for any $g' \in \mathcal{G}$,

$$\begin{aligned} \tilde{q}(g'(y)) &= \frac{1}{N} \sum_{g \in \mathcal{G}} q^*(g(g'(y))) \\ &= \frac{1}{N} \sum_{g \in \mathcal{G}} q^*(g(y)) = \tilde{q}(y) \end{aligned} \quad (37)$$

since applying g' to $Y = \{1, 2, \dots, n\}$ is just a reordering of the points of Y . Moreover, \tilde{q} satisfies (35) since

$$\begin{aligned}
\sum_{y \in Y} A(x, y) \tilde{q}(y) &= \sum_{y \in Y} A(x, y) \frac{1}{N} \sum_{g \in \mathcal{G}} q^*(g(y)) \\
&= \frac{1}{N} \sum_{g \in \mathcal{G}} \sum_{y \in Y} A(x, y) q^*(g(y)) \\
&= \frac{1}{N} \sum_{g \in \mathcal{G}} \sum_{y \in Y} A(\bar{g}(x), g(y)) q^*(g(y)) \\
&= \frac{1}{N} \sum_{g \in \mathcal{G}} \sum_{y \in Y} A(\bar{g}(x), y) q^*(y) \\
&\leq \frac{1}{N} \sum_{g \in \mathcal{G}} V = V.
\end{aligned} \tag{38}$$

■

In matching pennies, $X = Y = \{1, 2\}$, and $A(1, 1) = A(2, 2) = 1$ and $A(1, 2) = A(2, 1) = -1$. The Game $G = (X, Y, A)$ is invariant under the group $\mathcal{G} = \{e, g\}$, where e is the identity transformation, and g is the transformation, $g(1) = 2, g(2) = 1$. The (mixed) strategy $(q(1), q(2))$ is invariant under \mathcal{G} if $q(1) = q(2)$. Since $q(1) + q(2) = 1$, this implies that $q(1) = q(2) = 1/2$ is the only invariant strategy for Player II. It is therefore minimax. Similarly, $p(1) = p(2) = 1/2$ is the only invariant, and hence minimax, strategy for Player I.

Similarly, the game of paper-scissors-rock is invariant under the group $\mathcal{G} = \{e, g, g^2\}$, where $g(\text{paper}) = \text{scissors}$, $g(\text{scissors}) = \text{rock}$ and $g(\text{rock}) = \text{paper}$. The unique invariant, and hence minimax, strategy gives probability $1/3$ to each of paper, scissors and rock.

Colonel Blotto Games. For more interesting games reduced by invariance, we consider a class of tactical military games called Blotto Games, introduced by Tukey (1949). There are many variations of these games; see for example the web page, <http://www.amsta.leeds.ac.uk/~pmt6jrp/personal/blotto.html>.

We describe the discrete version treated in Williams (1954), Karlin (1959) and Dresher (1961).

Colonel Blotto has 4 regiments with which to occupy two posts. The famous Lieutenant Kije has 3 regiments with which to occupy the same posts. The payoff is defined as follows. The army sending the most units to either post captures it and all the regiments sent by the other side, scoring one point for the captured post and one for each captured regiment. If the players send the same number of regiments to a post, both forces withdraw and there is no payoff.

Colonel Blotto must decide how to split his forces between the two posts. There are 5 pure strategies he may employ, namely, $X = \{(4, 0), (3, 1), (2, 2), (1, 3), (0, 4)\}$, where

(n_1, n_2) represents the strategy of sending n_1 units to post number 1, and n_2 units to post number two. Lieutenant Kije has 4 pure strategies, $Y = \{(3, 0), (2, 1), (1, 2), (0, 3)\}$. The payoff matrix is

$$\begin{array}{c} (3, 0) \quad (2, 1) \quad (1, 2) \quad (0, 3) \\ \begin{array}{c} (4, 0) \\ (3, 1) \\ (2, 2) \\ (1, 3) \\ (0, 4) \end{array} \begin{pmatrix} 4 & 2 & 1 & 0 \\ 1 & 3 & 0 & -1 \\ -2 & 2 & 2 & -2 \\ -1 & 0 & 3 & 1 \\ 0 & 1 & 2 & 4 \end{pmatrix} \end{array} \quad (39)$$

Unfortunately, the 5 by 4 matrix game cannot be reduced by removing dominated strategies. So it seems that to solve it, we must use the simplex method. However, there is an invariance in this problem that simplifies it considerably. This involves the symmetry between the posts. This leads to the group, $\mathcal{G} = \{e, g\}$, where

$$g((3, 0)) = (0, 3) \quad g((0, 3)) = (3, 0) \quad g((2, 1)) = (1, 2) \quad g((1, 2)) = (2, 1)$$

and the corresponding group, $\bar{\mathcal{G}} = \{\bar{e}, \bar{g}\}$, where

$$\begin{aligned} \bar{g}((4, 0)) &= (0, 4) \quad \bar{g}((0, 4)) = (4, 0) \quad \bar{g}((3, 1)) = (1, 3) \quad \bar{g}((1, 3)) = (3, 1) \\ \text{and } \bar{g}((2, 2)) &= (2, 2) \end{aligned}$$

The orbits for Kije are $\{(3, 0), (0, 3)\}$ and $\{(2, 1), (1, 2)\}$. Therefore a strategy, \mathbf{q} , is invariant if $q((3, 0)) = q((0, 3))$ and $q((2, 1)) = q((1, 2))$. Similarly, the orbits for Blotto are $\{(4, 0), (0, 4)\}$, $\{(3, 1), (1, 3)\}$ and $\{(2, 2)\}$. So a strategy, \mathbf{p} , for Blotto is invariant if $p((4, 0)) = p((0, 4))$ and $p((3, 1)) = p((1, 3))$.

We may reduce Kije's strategy space to two elements, defined as follows:

$(3, 0)^*$: use $(3, 0)$ and $(0, 3)$ with probability 1/2 each.

$(2, 1)^*$: use $(2, 1)$ and $(1, 2)$ with probability 1/2 each.

Similarly, Blotto's strategy space reduces to three elements:

$(4, 0)^*$: use $(4, 0)$ and $(0, 4)$ with probability 1/2 each.

$(3, 1)^*$: use $(3, 1)$ and $(1, 3)$ with probability 1/2 each.

$(2, 2)$: use $(2, 2)$.

With these strategy spaces, the payoff matrix becomes

$$\begin{array}{c} (3, 0)^* \quad (2, 1)^* \\ \begin{array}{c} (4, 0)^* \\ (3, 1)^* \\ (2, 2) \end{array} \begin{pmatrix} 2 & 1.5 \\ 0 & 1.5 \\ -2 & 2 \end{pmatrix} \end{array} \quad (40)$$

As an example of the computations used to arrive at these payoffs, consider the upper left entry. If Blotto uses (4,0) and (0,4) with probability 1/2 each, and if Kije uses (3,0) and (0,3) with probability 1/2 each, then the four corners of the matrix (39) occur with probability 1/4 each, so the expected payoff is the average of the four numbers, 4, 0, 0, 4, namely 2.

To complete the analysis, we solve the game with matrix (40). We first note that the middle row is dominated by the top row (even though there was no domination in the original matrix). Removal of the middle row reduces the game to a 2 by 2 matrix game whose solution is easily found. The mixed strategy (8/9,0,1/9) is optimal for Blotto, the mixed strategy (1/9,8/9) is optimal for Kije, and the value is $V = 14/9$.

Returning now to the original matrix (39), we find that (4/9,0,1/9,0,4/9) is optimal for Blotto, (1/18,4/9,4/9,1/18) is optimal for Kije, and $V = 14/9$ is the value.

3.7 Exercises.

1. Consider the game with matrix
$$\begin{pmatrix} -2 & 2 & -1 \\ 1 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix}.$$

(a) Note that this game has a saddle point.

(b) Show that the inverse of the matrix exists.

(c) Show that II has an optimal strategy giving positive weight to each of his columns.

(d) Why then, don't equations (16) give an optimal strategy for II?

2. Consider the diagonal matrix game with matrix (18).

(a) Suppose one of the diagonal terms is zero. What is the value of the game?

(b) Suppose one of the diagonal terms is positive and another is negative. What is the value of the game?

(c) Suppose all diagonal terms are negative. What is the value of the game?

3. Player II chooses a number $j \in \{1, 2, 3, 4\}$, and Player I tries to guess what number II has chosen. If he guesses correctly and the number was j , he wins 2^j dollars from II. Otherwise there is no payoff. Set up the matrix of this game and solve.

4. Player II chooses a number $j \in \{1, 2, 3, 4\}$ and I tries to guess what it is. If he guesses correctly, he wins 1 from II. If he overestimates he wins 1/2 from II. If he underestimates, there is no payoff. Set up the matrix of this game and solve.

5. Player II chooses a number $j \in \{1, 2, \dots, n\}$ and I tries to guess what it is. If he guesses correctly, he wins 1. If he guesses too high, he loses 1. If he guesses too low, there is no payoff. Set up the matrix and solve.

6. Player II chooses a number $j \in \{1, 2, \dots, n\}$, $n \geq 2$, and Player I tries to guess what it is by guessing some $i \in \{1, 2, \dots, n\}$. If he guesses correctly, i.e. $i = j$, he wins 1. If $i > j$, he wins b^{i-j} for some number $b < 1$. Otherwise, if $i < j$, he wins nothing. Set up

the matrix and solve. Hint: If $\mathbf{A}_n = (a_{ij})$ denotes the game matrix, then show the inverse matrix is $\mathbf{A}_n^{-1} = (a^{ij})$, where $a^{ij} = \begin{cases} 1 & \text{if } i = j \\ -b & \text{if } i = j + 1, \\ 0 & \text{otherwise} \end{cases}$, and use Theorem 3.2.

7. The Pascal Matrix Game. The Pascal matrix of order n is the $n \times n$ matrix \mathbf{B}_n of elements b_{ij} , where

$$b_{ij} = \binom{i-1}{j-1} \quad \text{if } i \geq j, \quad \text{and} \quad b_{ij} = 0 \quad \text{if } i < j.$$

The i th row of \mathbf{B}_n consists of the binomial coefficients in the expansion of $(x + y)^i$. Call and Velleman (1993) show that the inverse of \mathbf{B}_n is the matrix \mathbf{A}_n with entries a_{ij} , where $a_{ij} = (-1)^{i+j} b_{ij}$. Using this, find the value and optimal strategies for the matrix game with matrix \mathbf{A}_n .

8. Solve the games with the following matrices.

$$(a) \quad \begin{pmatrix} 1 & -1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix} \quad (b) \quad \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 3/2 & 1 & 1 \\ 1 & 1 & 4/3 & 1 \\ 1 & 1 & 1 & 5/4 \end{pmatrix}$$

$$(c) \quad \begin{pmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 4 & 3 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

9. Another Mendelsohn game. Two players simultaneously choose an integer between 1 and n inclusive, where $n \geq 5$. If the numbers are equal there is no payoff. The player that chooses a number one larger than that chosen by his opponent wins 2. The player that chooses a number two or more larger than that chosen by his opponent loses 1.

(a) Set up the game matrix.

(b) It turns out that the optimal strategy satisfies $p_i > 0$ for $i = 1, \dots, 5$, and $p_i = 0$ for all other i . Solve for the optimal \mathbf{p} . (It is not too difficult since you can argue that $p_1 = p_5$ and $p_2 = p_4$ by symmetry of the equations.) Check that in fact the strategy you find is optimal.

10. Silverman Games. (See R. T. Evans (1979) and Heuer and Leopold-Wildburger (1991).) Two players simultaneously choose positive integers. As in Mendelsohn games, a player wants to choose an integer larger but not too much larger than the opponent, but in Silverman games “too much larger” is determined multiplicatively rather than additively. Solve the following example: The player whose number is larger but less than three times as large as the opponent’s wins 1. But the player whose number is three times as large or larger loses 2. If the numbers are the same, there is no payoff.

(a) Note this is a symmetric game, and show that dominance reduces the game to a 3 by

3 matrix.

(b) Solve.

11. Solve the following games.

$$(a) \quad \begin{pmatrix} 0 & 1 & -2 \\ -1 & 0 & 3 \\ 2 & -3 & 0 \end{pmatrix} \qquad (b) \quad \begin{pmatrix} 0 & 1 & -2 \\ -2 & 0 & 1 \\ 1 & -2 & 0 \end{pmatrix}$$
$$(c) \quad \begin{pmatrix} 1 & 4 & -1 & 5 \\ 4 & -1 & 5 & 1 \\ -1 & 5 & 1 & 4 \\ 5 & 1 & 4 & -1 \\ 2 & 2 & 2 & 2 \end{pmatrix}$$

12. Run the original Blotto matrix (39) through the Matrix Game Solver, on the web at: <http://www.math.ucla.edu/~tom/gamesolve.html>, and note that it gives different optimal strategies than those found in the text. What does this mean? Show that $(3, 1)^*$ is strictly dominated in (40). This means that no optimal strategy can give weight to $(3, 1)^*$. Is this true for the solution found?

13. (a) Suppose Blotto has 2 units and Kije just 1 unit, with 2 posts to capture. Solve.

(b) Suppose Blotto has 3 units and Kije 2 units, with 2 posts to capture. Solve.

14. (a) Suppose there are 3 posts to capture. Blotto has 4 units and Kije has 3. Solve. (Reduction by invariance leads to a 4 by 3 matrix, reducible further by domination to 2 by 2.)

(b) Suppose there are 4 posts to capture. Blotto has 4 units and Kije has 3. Solve. (A 5 by 3 reduced matrix, reducible by domination to 4 by 3. But you may as well use the Matrix Game Solver to solve it.)

15. **Battleship.** The game of Battleship, sometimes called Salvo, is played on a two square boards, usually 10 by 10. Each player hides a fleet of ships on his own board and tries to sink the opponent's ships before the opponent sinks his. (For one set of rules, see <http://www.kielack.de/games/destroya.htm>, and while you are there, have a game.)

For simplicity, consider a 3 by 3 board and suppose that Player I hides a destroyer (length 2 squares) horizontally or vertically on this board. Then Player II shoots by calling out squares of the board, one at a time. After each shot, Player I says whether the shot was a hit or a miss. Player II continues until both squares of the destroyer have been hit. The payoff to Player I is the number of shots that Player II has made. Let us label the squares from 1 to 9 as follows.

1	2	3
4	5	6
7	8	9

The problem is invariant under rotations and reflections of the board. In fact, of the 12 possible positions for the destroyer, there are only two distinct invariant choices available to Player I: the strategy, $[1, 2]^*$, that chooses one of $[1, 2]$, $[2, 3]$, $[3, 6]$, $[6, 9]$, $[8, 9]$, $[7, 8]$, $[4, 7]$, and $[1, 4]$, at random with probability $1/8$ each, and the strategy, $[2, 5]^*$, that chooses one of $[2, 5]$, $[5, 6]$, $[5, 8]$, and $[4, 5]$, at random with probability $1/4$ each. This means that invariance reduces the game to a 2 by n game where n is the number of invariant strategies of Player II. Domination may reduce it somewhat further. Solve the game.

16. Dresher's Guessing Game. Player I secretly writes down one of the numbers $1, 2, \dots, n$. Player II must repeatedly guess what I's number is until she guesses correctly, losing 1 for each guess. After each guess, Player I must say whether the guess is correct, too high, or too low. Solve this game for $n = 3$. (This game was investigated by Dresher (1961) and solved for $n \leq 11$ by Johnson (1964). A related problem is treated in Gal (1974).)

17. Thievery. Player I wants to steal one of $m \geq 2$ items from Player II. Player II can only guard one item at a time. Item i is worth $u_i > 0$, for $i = 1, \dots, m$. This leads to a matrix game with $m \times m$ matrix,

$$\mathbf{A} = \begin{pmatrix} 0 & u_1 & u_1 & \dots & u_1 \\ u_2 & 0 & u_2 & \dots & u_2 \\ u_3 & u_3 & 0 & \dots & u_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_m & u_m & u_m & \dots & 0 \end{pmatrix}$$

Solve!

Hint: It might be expected that for some $k \leq m$ Player I will give all his probability to stealing one of the k most expensive items. Order the items from most expensive to least expensive, $u_1 \geq u_2 \geq \dots \geq u_m > 0$, and use the principle of indifference on the upper left $k \times k$ submatrix of \mathbf{A} for some k .

18. Player II chooses a number $j \in \{1, 2, \dots, n\}$, $n \geq 2$, and Player I tries to guess what it is by guessing some $i \in \{1, 2, \dots, n\}$. If he guesses correctly, i.e. $i = j$, he wins 2. If he misses by exactly 1, i.e. $|i - j| = 1$, then he loses 1. Otherwise there is no payoff. Solve. Hint: Let \mathbf{A}_n denote the n by n payoff matrix, and show that $\mathbf{A}_n^{-1} = \mathbf{B}_n = (b_{ij})$, where $b_{ij} = i(n + 1 - j)/(n + 1)$ for $i \leq j$, and $b_{ij} = b_{ji}$ for $i > j$.

19. The Number Hides Game. The Number Hides Game, introduced by Ruckle (1983) and solved by Baston, Bostock and Ferguson (1989), may be described as follows. From the set $S = \{1, 2, \dots, k\}$, Player I chooses an interval of m_1 consecutive integers and Player II chooses an interval of m_2 consecutive integers. The payoff to Player I is the number of integers in the intersection of the two intervals. When $k = n + 1$ and $m_1 = m_2 = 2$, this game is equivalent to the game with $n \times n$ matrix $\mathbf{A}_n = (a_{ij})$, where

$$a_{ij} = \begin{cases} 2 & \text{if } i = j \\ 1 & \text{if } |i - j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

[In this form, the game is also a special case of the **Helicopter versus Submarine Game**, solved in the book of Garnaev (2000), in which the payoff for $|i - j| = 1$ is allowed to be an arbitrary number a , $0 \leq a \leq 1$.] Since \mathbf{A}_n^{-1} is just \mathbf{B}_n of the previous exercise with b_{ij} replaced by $(-1)^{i+j}b_{ij}$, the solution can be derived as in that exercise. Instead, just show the following.

(a) For n odd, the value is $V_n = 4/(n + 1)$. There is an optimal equalizing strategy (the same for both players) that is proportional to $(1, 0, 1, 0, \dots, 0, 1)$.

(b) For n even, the value is $4(n + 1)/(n(n + 2))$. There is an optimal equalizing strategy (the same for both players) that is proportional to $(k, 1, k - 1, 2, k - 2, 3, \dots, 2, k - 1, 1, k)$, where $k = n/2$.

4. Solving Finite Games.

Consider an arbitrary finite two-person zero-sum game, (X, Y, A) , with $m \times n$ matrix, A . Let us take the strategy space X to be the first m integers, $X = \{1, 2, \dots, m\}$, and similarly, $Y = \{1, 2, \dots, n\}$. A mixed strategy for Player I may be represented by a column vector, $(p_1, p_2, \dots, p_m)^T$ of probabilities that add to 1. Similarly, a mixed strategy for Player II is an n -tuple $\mathbf{q} = (q_1, q_2, \dots, q_n)^T$. The sets of mixed strategies of players I and II will be denoted respectively by X^* and Y^* ,

$$X^* = \{\mathbf{p} = (p_1, \dots, p_m)^T : p_i \geq 0, \text{ for } i = 1, \dots, m \text{ and } \sum_1^m p_i = 1\}$$

$$Y^* = \{\mathbf{q} = (q_1, \dots, q_n)^T : q_j \geq 0, \text{ for } j = 1, \dots, n \text{ and } \sum_1^n q_j = 1\}$$

The m -dimensional unit vector $\mathbf{e}_k \in X^*$ with a one for the k th component and zeros elsewhere may be identified with the pure strategy of choosing row k . Thus, we may consider the set of Player I's pure strategies, X , to be a subset of X^* . Similarly, Y may be considered to be a subset of Y^* . We could if we like consider the game (X, Y, A) in which the players are allowed to use mixed strategies as a new game (X^*, Y^*, A) , where $A(\mathbf{p}, \mathbf{q}) = \mathbf{p}^T \mathbf{A} \mathbf{q}$, though we would no longer call this game a finite game.

In this section, we give an algorithm for solving finite games; that is, we show how to find the value and at least one optimal strategy for each player. Occasionally, we shall be interested in finding all optimal strategies for a player.

4.1 Best Responses. Suppose that Player II chooses a column at random using $\mathbf{q} \in Y^*$. If Player I chooses row i , the average payoff to I is

$$\sum_{j=1}^n a_{ij} q_j = (\mathbf{A} \mathbf{q})_i, \quad (1)$$

the i th component of the vector $\mathbf{A} \mathbf{q}$. Similarly, if Player I uses $\mathbf{p} \in X^*$ and Player II chooses column j , Then I's average payoff is

$$\sum_{i=1}^m p_i a_{ij} = (\mathbf{p}^T \mathbf{A})_j, \quad (2)$$

the j th component of the vector $\mathbf{p}^T \mathbf{A}$. More generally, if Player I uses $\mathbf{p} \in X^*$ and Player II uses $\mathbf{q} \in Y^*$, the average payoff to I becomes

$$\sum_{i=1}^m \left(\sum_{j=1}^n a_{ij} q_j \right) p_i = \sum_{i=1}^m \sum_{j=1}^n p_i a_{ij} q_j = \mathbf{p}^T \mathbf{A} \mathbf{q}. \quad (3)$$

Suppose it is known that Player II is going to use a particular strategy $\mathbf{q} \in Y^*$. Then Player I would choose that row i that maximizes (1); or, equivalently, he would choose that $\mathbf{p} \in X^*$ that maximizes (3). His average payoff would be

$$\max_{1 \leq i \leq m} \sum_{j=1}^n a_{ij} q_j = \max_{\mathbf{p} \in X^*} \mathbf{p}^T \mathbf{A} \mathbf{q} \quad (4)$$

To see that these quantities are equal, note that the left side is the maximum of $\mathbf{p}^T \mathbf{A} \mathbf{q}$ over $\mathbf{p} \in X^*$, and so, since $X \subset X^*$, must be less than or equal to the right side. The reverse inequality follows since (3) is an average of the quantities in (1) and so must be less than or equal to the largest of the values in (1).

Any $\mathbf{p} \in X^*$ that achieves the maximum of (3) is called a *best response* or a *Bayes strategy* against \mathbf{q} . In particular, any row i that achieves the maximum of (1) is a (pure) Bayes strategy against \mathbf{q} . There always exist pure Bayes strategies against \mathbf{q} for every $\mathbf{q} \in Y^*$ in finite games.

Similarly, if it is known that Player I is going to use a particular strategy $\mathbf{p} \in X^*$, then Player II would choose that column j that minimizes (2), or, equivalently, that $\mathbf{q} \in Y^*$ that minimizes (3). Her average payoff would be

$$\min_{1 \leq j \leq n} \sum_{i=1}^m p_i a_{ij} = \min_{\mathbf{q} \in Y^*} \mathbf{p}^T \mathbf{A} \mathbf{q}. \quad (5)$$

Any $\mathbf{q} \in Y^*$ that achieves the minimum in (5) is called a *best response* or a *Bayes strategy* for Player II against \mathbf{p} .

The notion of a best response presents a practical way of playing a game: Make a guess at the probabilities that you think your opponent will play his/her various pure strategies, and choose a best response against this. This method is available in quite complex situations. In addition, it allows a player to take advantage of an opponent's perceived weaknesses. Of course this may be a dangerous procedure. Your opponent may be better at this type of guessing than you. (See Exercise 1.)

4.2 Upper and Lower Values of a Game. Suppose now that II is required to announce her choice of a mixed strategy $\mathbf{q} \in Y^*$ before I makes his choice. This changes the game to make it apparently more favorable to I. If II announces \mathbf{q} , then certainly I would use a Bayes strategy against \mathbf{q} and II would lose the quantity (4) on the average. Therefore, II would choose to announce that \mathbf{q} that minimizes (4). The minimum of (4) over all $\mathbf{q} \in Y^*$ is denoted by \bar{V} and called the *upper value of the game* (X, Y, A) .

$$\bar{V} = \min_{\mathbf{q} \in Y^*} \max_{1 \leq i \leq m} \sum_{j=1}^n a_{ij} q_j = \min_{\mathbf{q} \in Y^*} \max_{\mathbf{p} \in X^*} \mathbf{p}^T \mathbf{A} \mathbf{q}. \quad (6)$$

Any $\mathbf{q} \in Y^*$ that achieves the minimum in (6) is called a *minimax strategy for II*. It minimizes her maximum loss. There always exists a minimax strategy in finite games: the quantity (4), being the maximum of m linear functions of \mathbf{q} , is a continuous function of \mathbf{q} and since Y^* is a closed bounded set, this function assumes its minimum over Y^* at some point of Y^* .

In words, \bar{V} is the smallest average loss that Player II can assure for herself no matter what I does.

A similar analysis may be carried out assuming that I must announce his choice of a mixed strategy $\mathbf{p} \in X^*$ before II makes her choice. If I announces \mathbf{p} , then II would choose

that column with the smallest average payoff, or equivalently that $\mathbf{q} \in Y^*$ that minimizes the average payoff (5). Given that (5) is the average payoff to I if he announces \mathbf{p} , he would therefore choose \mathbf{p} to maximize (5) and obtain on the average

$$\underline{V} = \max_{\mathbf{p} \in X^*} \min_{1 \leq j \leq n} \sum_{i=1}^m p_i a_{ij} = \max_{\mathbf{p} \in X^*} \min_{\mathbf{q} \in Y^*} \mathbf{p}^T \mathbf{A} \mathbf{q}. \quad (7)$$

The quantity \underline{V} is called the *lower value of the game*. It is the maximum amount that I can guarantee himself no matter what II does. Any $\mathbf{p} \in X^*$ that achieves the maximum in (7) is called a *minimax strategy for I*. Perhaps *maximin strategy* would be more appropriate terminology in view of (7), but from symmetry (either player may consider himself Player II for purposes of analysis) the same word to describe the same idea may be preferable and it is certainly the customary terminology. As in the analysis for Player II, we see that Player I always has a minimax strategy. The existence of minimax strategies in matrix games is worth stating as a lemma.

Lemma 1. *In a finite game, both players have minimax strategies.*

It is easy to argue that the lower value is less than or equal to the upper value. For if $\bar{V} < \underline{V}$ and if I can assure himself of winning at least \underline{V} , Player II cannot assure herself of not losing more than \bar{V} , an obvious contradiction. It is worth stating this fact as a lemma too.

Lemma 2. *The lower value is less than or equal to the upper value,*

$$\underline{V} \leq \bar{V}.$$

This lemma also follows from the general mathematical principle that for *any* real-valued function, $f(x, y)$, and *any* sets, X^* and Y^* ,

$$\max_{x \in X^*} \min_{y \in Y^*} f(x, y) \leq \min_{y \in Y^*} \max_{x \in X^*} f(x, y).$$

To see this general principle, note that $\min_{y'} f(x, y') \leq f(x, y) \leq \max_{x'} f(x', y)$ for every fixed x and y . Then, taking \max_x on the left does not change the inequality, nor does taking \min_y on the right, which gives the result.

If $\underline{V} < \bar{V}$, the average payoff should fall between \underline{V} and \bar{V} . Player II can keep it from getting larger than \bar{V} and Player I can keep it from getting smaller than \bar{V} . When $\underline{V} = \bar{V}$, a very nice stable situation exists.

Definition. *If $\underline{V} = \bar{V}$, we say the value of the game exists and is equal to the common value of \underline{V} and \bar{V} , denoted simply by V . If the value of the game exists, we refer to minimax strategies as optimal strategies.*

The Minimax Theorem, stated in Chapter 1, may be expressed simply by saying that for finite games, $\underline{V} = \bar{V}$.

The Minimax Theorem. *Every finite game has a value, and both players have minimax strategies.*

We note one remarkable corollary of this theorem. If the rules of the game are changed so that Player II is required to announce her choice of a mixed strategy before Player I makes his choice, then the apparent advantage given to Player I by this is illusory. Player II can simply announce her minimax strategy.

4.3 Invariance under Change of Location and Scale. Another simple observation is useful in this regard. This concerns the *invariance* of the minimax strategies under the operations of adding a constant to each entry of the game matrix, and of multiplying each entry of the game matrix by a positive constant. The game having matrix $\mathbf{A} = (a_{ij})$ and the game having matrix $\mathbf{A}' = (a'_{ij})$ with $a'_{ij} = a_{ij} + b$, where b is an arbitrary real number, are very closely related. In fact, the game with matrix \mathbf{A}' is equivalent to the game in which II pays I the amount b , and then I and II play the game with matrix \mathbf{A} . Clearly any strategies used in the game with matrix \mathbf{A}' give Player I b plus the payoff using the same strategies in the game with matrix \mathbf{A} . Thus, any minimax strategy for either player in one game is also minimax in the other, and the upper (lower) value of the game with matrix \mathbf{A}' is b plus the upper (lower) value of the game with matrix \mathbf{A} .

Similarly, the game having matrix $\mathbf{A}'' = (a''_{ij})$ with $a''_{ij} = ca_{ij}$, where c is a positive constant, may be considered as the game with matrix \mathbf{A} with a change of scale (a change of monetary unit if you prefer). Again, minimax strategies do not change, and the upper (lower) value of \mathbf{A}'' is c times the upper (lower) value of \mathbf{A} . We combine these observations as follows. (See Exercise 2.)

Lemma 3. *If $\mathbf{A} = (a_{ij})$ and $\mathbf{A}' = (a'_{ij})$ are matrices with $a'_{ij} = ca_{ij} + b$, where $c > 0$, then the game with matrix \mathbf{A} has the same minimax strategies for I and II as the game with matrix \mathbf{A}' . Also, if V denotes the value of the game with matrix \mathbf{A} , then the value V' of the game with matrix \mathbf{A}' satisfies $V' = cV + b$.*

4.4 Reduction to a Linear Programming Problem. There are several nice proofs of the Minimax Theorem. The simplest proof from scratch seems to be that of G. Owen (1982). However, that proof is by contradiction using induction on the size of the matrix. It gives no insight into properties of the solution or on how to find the value and optimal strategies. Other proofs, based on the Separating Hyperplane Theorem or the Brouwer Fixed Point Theorem, give some insight but are based on nontrivial theorems not known to all students.

Here we use a proof based on linear programming. Although based on material not known to all students, it has the advantage of leading to a simple algorithm for solving finite games. For a background in linear programming, the book by Chvátal (1983) can be recommended. A short course on Linear Programming more in tune with the material as it is presented here may be found on the web at <http://www.math.ucla.edu/~tom/LP.pdf>.

A Linear Program is defined as the problem of choosing real variables to maximize or minimize a linear function of the variables, called the objective function, subject to linear

constraints on the variables. The constraints may be equalities or inequalities. A standard form of this problem is to choose y_1, \dots, y_n , to

$$\text{maximize} \quad b_1 y_1 + \dots + b_n y_n, \quad (8)$$

subject to the constraints

$$\begin{aligned} a_{11} y_1 + \dots + a_{1n} y_n &\leq c_1 \\ &\vdots \\ a_{m1} y_1 + \dots + a_{mn} y_n &\leq c_m \end{aligned} \quad (9)$$

and

$$y_j \geq 0 \quad \text{for } j = 1, \dots, n.$$

Let us consider the game problem from Player I's point of view. He wants to choose p_1, \dots, p_m to maximize (5) subject to the constraint $\mathbf{p} \in X^*$. This becomes the mathematical program: choose p_1, \dots, p_m to

$$\text{maximize} \quad \min_{1 \leq j \leq n} \sum_{i=1}^m p_i a_{ij} \quad (10)$$

subject to the constraints

$$p_1 + \dots + p_m = 1 \quad (11)$$

and

$$p_i \geq 0 \quad \text{for } i = 1, \dots, m.$$

Although the constraints are linear, the objective function is not a linear function of the p 's because of the min operator, so this is not a linear program. However, it can be changed into a linear program through a trick. Add one new variable v to Player I's list of variables, restrict it to be less than the objective function, $v \leq \min_{1 \leq j \leq n} \sum_{i=1}^m p_i a_{ij}$, and try to make v as large as possible subject to this new constraint. The problem becomes:

Choose v and p_1, \dots, p_m to

$$\text{maximize} \quad v \quad (12)$$

subject to the constraints

$$\begin{aligned} v &\leq \sum_{i=1}^m p_i a_{i1} \\ &\vdots \\ v &\leq \sum_{i=1}^m p_i a_{in} \\ p_1 + \dots + p_m &= 1 \end{aligned} \quad (13)$$

and

$$p_i \geq 0 \quad \text{for } i = 1, \dots, m.$$

This is indeed a linear program. For solving such problems, there exists a simple algorithm known as the simplex method.

In a similar way, one may view the problem from Player II's point of view and arrive at a similar linear program. II's problem is: choose w and q_1, \dots, q_n to

$$\text{minimize } w \tag{14}$$

subject to the constraints

$$\begin{aligned} w &\geq \sum_{j=1}^n a_{1j} q_j \\ &\vdots \\ w &\geq \sum_{j=1}^n a_{mj} q_j \\ q_1 + \dots + q_n &= 1 \end{aligned} \tag{15}$$

and

$$q_j \geq 0 \quad \text{for } j = 1, \dots, n.$$

In Linear Programming, there is a theory of duality that says these two programs, (12)-(13), and (14)-(15), are dual programs. And there is a remarkable theorem, called the Duality Theorem, that says dual programs have the same value. The maximum Player I can achieve in (14) is equal to the minimum that Player II can achieve in (12). But this is exactly the claim of the Minimax Theorem. In other words, the Duality Theorem implies the Minimax Theorem.

There is another way to transform the linear program, (12)-(13), into a linear program that is somewhat simpler for computations when it is known that the value of the game is positive. So suppose $v > 0$ and let $x_i = p_i/v$. Then the constraint $p_1 + \dots + p_m = 1$ becomes $x_1 + \dots + x_m = 1/v$, which looks nonlinear. But maximizing v is equivalent to minimizing $1/v$, so we can remove v from the problem by minimizing $x_1 + \dots + x_m$ instead. The problem, (12)-(13), becomes: choose x_1, \dots, x_m to

$$\text{minimize } x_1 + \dots + x_m \tag{16}$$

subject to the constraints

$$\begin{aligned} 1 &\leq \sum_{i=1}^m x_i a_{i1} \\ &\vdots \\ 1 &\leq \sum_{i=1}^m x_i a_{in} \end{aligned} \tag{17}$$

and

$$x_i \geq 0 \quad \text{for } i = 1, \dots, m.$$

When we have solved this problem, the solution of the original game may be easily found. The value will be $v = 1/(x_1 + \dots + x_m)$ and the optimal strategy for Player I will be $p_i = vx_i$ for $i = 1, \dots, m$.

4.5 Description of the Pivot Method for Solving Games The following algorithm for solving finite games is essentially the simplex method for solving (16)-(17) as described in Williams (1966).

Step 1. Add a constant to all elements of the game matrix if necessary to insure that the value is positive. (If you do, you must remember at the end to subtract this constant from the value of the new matrix game to get the value of the original matrix game.)

Step 2. Create a *tableau* by augmenting the game matrix with a border of -1 's along the lower edge, $+1$'s along the right edge, and zero in the lower right corner. Label I's strategies on the left from x_1 to x_m and II's strategies on the top from y_1 to y_n .

	y_1	y_2	\cdots	y_n	
x_1	a_{11}	a_{12}	\cdots	a_{1n}	1
x_2	a_{21}	a_{22}	\cdots	a_{2n}	1
\vdots	\vdots	\vdots		\vdots	\vdots
x_m	a_{m1}	a_{m2}	\cdots	a_{mn}	1
	-1	-1	\cdots	-1	0

Step 3. Select any entry in the interior of the tableau to be the *pivot*, say row p column q , subject to the properties:

- The border number in the pivot column, $a(m+1, q)$, must be negative.
- The pivot, $a(p, q)$, itself must be positive.
- The pivot row, p , must be chosen to give the smallest of the ratios the border number in the pivot row to the pivot, $a(p, n+1)/a(p, q)$, among all positive pivots for that column.

Step 4. Pivot as follows:

- Replace each entry, $a(i, j)$, not in the row or column of the pivot by $a(i, j) - a(p, j) \cdot a(i, q)/a(p, q)$.
- Replace each entry in the pivot row, except for the pivot, by its value divided by the pivot value.
- Replace each entry in the pivot column, except for the pivot, by the negative of its value divided by the pivot value.
- Replace the pivot value by its reciprocal.

This may be represented symbolically by

$$\begin{array}{|c|c|} \hline \textcircled{p} & r \\ \hline c & q \\ \hline \end{array} \longrightarrow \begin{array}{|cc|} \hline 1/p & r/p \\ -c/p & q - (rc/p) \\ \hline \end{array}$$

where p stands for the pivot, r represents any number in the same row as the pivot, c represents any number in the same column as the pivot, and q is an arbitrary entry not in the same row or column as the pivot.

Step 5. Exchange the label on the left of the pivot row with the label on the top of the pivot column.

Step 6. If there are any negative numbers remaining in the lower border row, go back to step 3.

Step 7. Otherwise, a solution may now be read out:

a. The value, v , is the reciprocal of the number in the lower right corner. (If you subtracted a number from each entry of the matrix in step 1, it must be added to v here.)

b. I's optimal strategy is constructed as follows. Those variables of Player I that end up on the left side receive probability zero. Those that end up on the top receive the value of the bottom edge in the same column divided by the lower right corner.

c. II's optimal strategy is constructed as follows. Those variables of Player II that end up on the top receive probability zero. Those that end up on the left receive the value of the right edge in the same row divided by the lower right corner.

4.6 A Numerical Example. Let us illustrate these steps using an example. Let's take a three-by-three matrix since that is the simplest example one we cannot solve using previous methods. Consider the matrix game with the following matrix,

$$B = \begin{pmatrix} 2 & -1 & 6 \\ 0 & 1 & -1 \\ -2 & 2 & 1 \end{pmatrix}$$

We might check for a saddle point (there is none) and we might check for domination (there is none). Is the value positive? We might be able to guess by staring at the matrix long enough, but why don't we simply make the first row positive by adding 2 to each entry of the matrix:

$$B' = \begin{pmatrix} 4 & 1 & 8 \\ 2 & 3 & 1 \\ 0 & 4 & 3 \end{pmatrix}$$

The value of this game is at least one since Player I can guarantee at least 1 by using the first (or second) row. We will have to remember to subtract 2 from the value of B' to get the value of B . This completes Step 1 of the algorithm.

In Step 2, we set up the tableau for the matrix B' as follows.

	y_1	y_2	y_3	
x_1	4	1	8	1
x_2	2	3	1	1
x_3	0	4	3	1
	-1	-1	-1	0

In Step 3, we must choose the pivot. Since all three columns have a negative number in the lower edge, we may choose any of these columns as the pivot column. Suppose we choose column 1. The pivot row must have a positive number in this column, so it must be one of the top two rows. To decide which row, we compute the ratios of border numbers to pivot. For the first row it is $1/4$; for the second row it is $1/2$. the former is smaller, so the pivot is in row 1. We pivot about the 4 in the upper left corner.

Step 4 tells us how to pivot. The pivot itself gets replaced by its reciprocal, namely $1/4$. The rest of the numbers in the pivot row are simply divided by the pivot, giving $1/4$, 2, and $1/4$. Then the rest of the numbers in the pivot column are divided by the pivot and changed in sign. The remaining nine numbers are modified by subtracting $r \cdot c/p$ for the corresponding r and c . For example, from the 1 in second row third column we subtract $8 \times 2/4 = 4$, leaving -3 . The complete pivoting operation is

$$\begin{array}{c|ccc|c} & y_1 & y_2 & y_3 & \\ \hline x_1 & \textcircled{4} & 1 & 8 & 1 \\ x_2 & 2 & 3 & 1 & 1 \\ x_3 & 0 & 4 & 3 & 1 \\ \hline & -1 & -1 & -1 & 0 \end{array} \longrightarrow \begin{array}{c|ccc|c} & x_1 & y_2 & y_3 & \\ \hline y_1 & 1/4 & 1/4 & 2 & 1/4 \\ x_2 & -1/2 & 5/2 & -3 & 1/2 \\ x_3 & 0 & 4 & 3 & 1 \\ \hline & 1/4 & -3/4 & 1 & 1/4 \end{array}$$

In the fifth step, we interchange the labels of the pivot row and column. Here we interchange x_1 and y_1 . This has been done in the display.

For Step 6, we check for negative entries in the lower edge. Since there is one, we return to Step 3.

This time, we must pivot in column 2 since it has the unique negative number in the lower edge. All three numbers in this column are positive. We find the ratios of border numbers to pivot for rows 1, 2 and 3 to be 1, $1/5$, and $1/4$. The smallest occurs in the second row, so we pivot about the $5/2$ in the second row, second column. Completing Steps 4 and 5, we obtain

$$\begin{array}{c|ccc|c} & x_1 & y_2 & y_3 & \\ \hline y_1 & 1/4 & 1/4 & 2 & 1/4 \\ x_2 & -1/2 & \textcircled{5/2} & -3 & 1/2 \\ x_3 & 0 & 4 & 3 & 1 \\ \hline & 1/4 & -3/4 & 1 & 1/4 \end{array} \longrightarrow \begin{array}{c|ccc|c} & x_1 & x_2 & y_3 & \\ \hline y_1 & .3 & -.1 & 2.3 & .2 \\ y_2 & -.2 & .4 & -1.2 & .2 \\ x_3 & .8 & -1.6 & 7.8 & .2 \\ \hline & .1 & .3 & .1 & .4 \end{array}$$

At Step 6 this time, all values on the lower edge are non-negative so we pass to Step 7. We may now read the solution to the game with matrix B' .

The value is the reciprocal of .4, namely $5/2$.

Since x_3 is on the left in the final tableau, the optimal p_3 is zero. The optimal p_1 and p_2 are the ratios, $.1/.4$ and $.3/.4$, namely $1/4$ and $3/4$. Therefore, I's optimal mixed strategy is $(p_1, p_2, p_3) = (.25, .75, 0)$.

Since y_3 is on the top in the final tableau, the optimal q_3 is zero. The optimal q_1 and q_2 are the ratios, $.2/.4$ and $.2/.4$, namely $1/2$ and $1/2$. Therefore, II's optimal mixed strategy is $(q_1, q_2, q_3) = (.5, .5, 0)$.

The game with matrix B has the same optimal mixed strategies but the value is $5/2 - 2 = 1/2$.

Remarks. 1. The reason that the pivot row is chosen according to the rule in Step 3(c) is so that the numbers in the resulting right edge of the tableau stay non-negative. If after pivoting you find a negative number in the last column, you have made a mistake, either in the choice of the pivot row, or in your numerical calculations for pivoting.

2. There may be ties in comparing the ratios to choose the pivot row. The rule given allows you to choose among those rows with the smallest ratios. The smallest ratio may be zero.

3. The value of the number in the lower right corner never decreases. (Can you see why this is?) In fact, the lower right corner is always equal to the sum of the values in the lower edge corresponding to Player I's labels along the top. Similarly, it is also the sum of the values on the right edge corresponding to Player II's labels on the left. This gives another small check on your arithmetic.

4. One only pivots around numbers in the main body of the tableau, never in the lower or right edges.

5. This method gives one optimal strategy for each player. If other optimal strategies exist, there will be one or more zeros in the bottom edge or right edge in the final tableau. Other optimal basic strategies can be found by pivoting further, in a column with a zero in the bottom edge or a row with a zero in the right edge, in such a way that the bottom row and right edge stay nonnegative.

4.7 Exercises.

1. Consider the game with matrix A . Past experience in playing the game with Player II enables Player I to arrive at a set of probabilities reflecting his belief of the column that II will choose. I thinks that with probabilities $1/5$, $1/5$, $1/5$, and $2/5$, II will choose columns 1, 2, 3, and 4 respectively.

$$A = \begin{pmatrix} 0 & 7 & 2 & 4 \\ 1 & 4 & 8 & 2 \\ 9 & 3 & -1 & 6 \end{pmatrix}.$$

(a) Find for I a Bayes strategy (best response) against $(1/5, 1/5, 1/5, 2/5)$.

(b) Suppose II guesses correctly that I is going to use a Bayes strategy against $(1/5, 1/5, 1/5, 2/5)$. Instruct II on the strategy she should use - that is, find II's Bayes strategy against I's Bayes strategy against $(1/5, 1/5, 1/5, 2/5)$.

2. The game with matrix \mathbf{A} has value zero, and $(6/11, 3/11, 2/11)$ is optimal for I.

$$\mathbf{A} = \begin{pmatrix} 0 & -1 & 1 \\ 2 & 0 & -2 \\ -3 & 3 & 0 \end{pmatrix} \quad \mathbf{B} = \begin{pmatrix} 5 & 3 & 7 \\ 9 & 5 & 1 \\ -1 & 11 & 5 \end{pmatrix}$$

- (a) Find the value of the game with matrix \mathbf{B} and an optimal strategy for I.
 (b) Find an optimal strategy for II in both games.

3. **A game without a value.** Let $X = \{1, 2, 3, \dots\}$, let $Y = \{1, 2, 3, \dots\}$ and

$$A(i, j) = \begin{cases} +1 & \text{if } i > j \\ 0 & \text{if } i = j \\ -1 & \text{if } i < j \end{cases}$$

This is the game “The player that chooses the larger integer wins”. Here we may take for the space X^* of mixed strategies of Player I

$$X^* = \{(p_1, p_2, \dots) : p_i \geq 0 \text{ for all } i, \text{ and } \sum_{i=1}^{\infty} p_i = 1\}.$$

Similarly,

$$Y^* = \{(q_1, q_2, \dots) : q_j \geq 0 \text{ for all } j, \text{ and } \sum_{j=1}^{\infty} q_j = 1\}.$$

The payoff for given $\mathbf{p} \in X^*$ and $\mathbf{q} \in Y^*$ is

$$A(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} p_i A(i, j) q_j.$$

- (a) Show that for all $\mathbf{q} \in Y^*$, $\sup_{1 \leq i < \infty} \sum_{j=1}^{\infty} A(i, j) q_j = +1$.
 (b) Conclude that $\bar{V} = +1$.
 (c) Using symmetry, argue that $\underline{V} = -1$.
 (d) What are I’s minimax strategies in this game?

4. Use the method presented in Section 4.5 to solve the game with matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 2 \\ 2 & -1 & -2 \\ 3 & -3 & 0 \end{pmatrix}.$$

Either argue that the value is positive, or add +1 to the elements of the matrix. To go easy on the homework grader, make the first pivot in the second column.

5. An Example In Which the Lower Value is Greater than the Upper Value? Consider the infinite game with strategy spaces $X = Y = \{0, 1, 2, \dots\}$, and payoff function,

$$A(i, j) = \begin{cases} 0 & \text{if } i = j \\ 4^j & \text{if } i > j \\ -4^i & \text{if } i < j. \end{cases}$$

Note that the game is symmetric. Let $\mathbf{p} = (p_0, p_1, p_2, \dots) = (1/2, 1/4, 1/8, \dots)$ be a mixed strategy for Player I, $p_i = 2^{-(i+1)}$.

(a) Show that if Player I uses this strategy, his average return, $\sum_{i=0}^{\infty} p_i A(i, j)$, is equal to $1/2$ for all pure strategies j for Player II.

(b) So \mathbf{p} is an equalizer strategy that guarantees Player I at least $1/2$. So the lower value is at least $1/2$. Perhaps he can do better. In fact he can, but ... Wait a minute! The game is symmetric. Shouldn't the value be zero? Worse, suppose Player II uses the same strategy. By symmetry, she can keep Player I's winnings down to $-1/2$ no matter what pure strategy he chooses. So the upper value is at most $-1/2$. What is wrong? What if both players use the mixed strategy, \mathbf{p} ? We haven't talked much about infinite games, but what restrictions would you place on infinite games to avoid such absurd examples? Should the restrictions be placed on the payoff function, A , or on the notion of a mixed strategy?

5. The Extensive Form of a Game

The strategic form of a game is a compact way of describing the mathematical aspects of a game. In addition, it allows a straightforward method of analysis, at least in principle. However, the flavor of many games is lost in such a simple model. Another mathematical model of a game, called the extensive form, is built on the basic notions of position and move, concepts not apparent in the strategic form of a game. In the extensive form, we may speak of other characteristic notions of games such as bluffing, signaling, sandbagging, and so on. Three new concepts make their appearance in the extensive form of a game: the game tree, chance moves, and information sets.

5.1 The Game Tree. The extensive form of a game is modelled using a directed graph. A **directed graph** is a pair (T, F) where T is a nonempty set of vertices and F is a function that gives for each $x \in T$ a subset, $F(x)$ of T called the followers of x . When a directed graph is used to represent a game, the vertices represent positions of the game. The followers, $F(x)$, of a position, x , are those positions that can be reached from x in one move.

A **path from a vertex t_0 to a vertex t_1** is a sequence, x_0, x_1, \dots, x_n , of vertices such that $x_0 = t_0$, $x_n = t_1$ and x_i is a follower of x_{i-1} for $i = 1, \dots, n$. For the extensive form of a game, we deal with a particular type of directed graph called a tree.

Definition. A **tree** is a directed graph, (T, F) in which there is a special vertex, t_0 , called the root or the initial vertex, such that for every other vertex $t \in T$, there is a unique path beginning at t_0 and ending at t .

The existence and uniqueness of the path implies that a tree is connected, has a unique initial vertex, and has no circuits or loops.

In the extensive form of a game, play starts at the initial vertex and continues along one of the paths eventually ending in one of the terminal vertices. At terminal vertices, the rules of the game specify the payoff. For n -person games, this would be an n -tuple of payoffs. Since we are dealing with two-person zero-sum games, we may take this payoff to be the amount Player I wins from Player II. For the nonterminal vertices there are three possibilities. Some nonterminal vertices are assigned to Player I who is to choose the move at that position. Others are assigned to Player II. However, some vertices may be singled out as positions from which a chance move is made.

Chance Moves. Many games involve chance moves. Examples include the rolling of dice in board games like monopoly or backgammon or gambling games such as craps, the dealing of cards as in bridge or poker, the spinning of the wheel of fortune, or the drawing of balls out of a cage in lotto. In these games, chance moves play an important role. Even in chess, there is generally a chance move to determine which player gets the white pieces (and therefore the first move which is presumed to be an advantage). It is assumed that the players are aware of the probabilities of the various outcomes resulting from a chance move.

Information. Another important aspect we must consider in studying the extensive form of games is the amount of information available to the players about past moves of the game. In poker for example, the first move is the chance move of shuffling and dealing the cards, each player is aware of certain aspects of the outcome of this move (the cards he received) but he is not informed of the complete outcome (the cards received by the other players). This leads to the possibility of “bluffing.”

5.2 Basic Endgame in Poker. One of the simplest and most useful mathematical models of a situation that occurs in poker is called the “classical betting situation” by Friedman (1971) and “basic endgame” by Cutler (1976). These papers provide explicit situations in the game of stud poker and of lowball stud for which the model gives a very accurate description. This model is also found in the exercises of the book of Ferguson (1967). Since this is a model of a situation that occasionally arises in the last round of betting when there are two players left, we adopt the terminology of Cutler and call it Basic Endgame in poker. This will also emphasize what we feel is an important feature of the game of poker, that like chess, go, backgammon and other games, there is a distinctive phase of the game that occurs at the close, where special strategies and tactics that are analytically tractable become important.

Basic Endgame is played as follows. Both players put 1 dollar, called the ante, in the center of the table. The money in the center of the table, so far two dollars, is called the pot. Then Player I is dealt a card from a deck. It is a winning card with probability $1/4$ and a losing card with probability $3/4$. Player I sees this card but keeps it hidden from Player II. (Player II does not get a card.) Player I then checks or bets. If he checks, then his card is inspected; if he has a winning card he wins the pot and hence wins the 1 dollar ante from II, and otherwise he loses the 1 dollar ante to II. If I bets, he puts 2 dollars more into the pot. Then Player II – not knowing what card Player I has – must fold or call. If she folds, she loses the 1 dollar ante to I no matter what card I has. If II calls, she adds 2 dollars to the pot. Then Player I’s card is exposed and I wins 3 dollars (the ante plus the bet) from II if he has a winning card, and loses 3 dollars to II otherwise.

Let us draw the tree for this game. There are at most three moves in this game: (1) the chance move that chooses a card for I, (2) I’s move in which he checks or bets, and (3) II’s move in which she folds or calls. To each vertex of the game tree, we attach a label indicating which player is to move from that position. Chance moves we generally refer to as moves by nature and use the label N. See Figure 1.

Each edge is labelled to identify the move. (The arrows are omitted for the sake of clarity. Moves are assumed to proceed down the page.) Also, the moves leading from a vertex at which nature moves are labelled with the probabilities with which they occur. At each terminal vertex, we write the numerical value of I’s winnings (II’s losses).

There is only one feature lacking from the above figure. From the tree we should be able to reconstruct all the essential rules of the game. That is not the case with the tree of Figure 1 since we have not indicated that at the time II makes her decision she does not know which card I has received. That is, when it is II’s turn to move, she does not know at which of her two possible positions she is. We indicate this on the diagram by encircling the

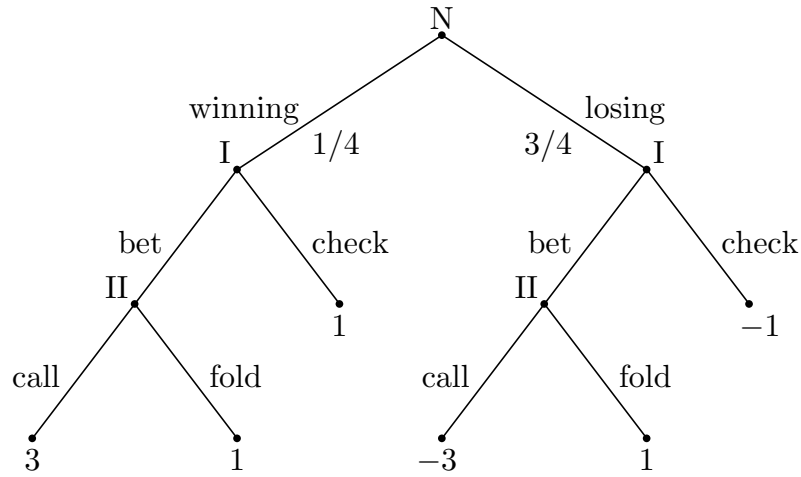


Figure 1.

two positions in a closed curve, and we say that these two vertices constitute an information set. The two vertices at which I is to move constitute two separate information sets since he is told the outcome of the chance move. To be complete, this must also be indicated on the diagram by drawing small circles about these vertices. We may delete one of the labels indicating II's vertices since they belong to the same information set. It is really the information set that must be labeled. The completed game tree becomes

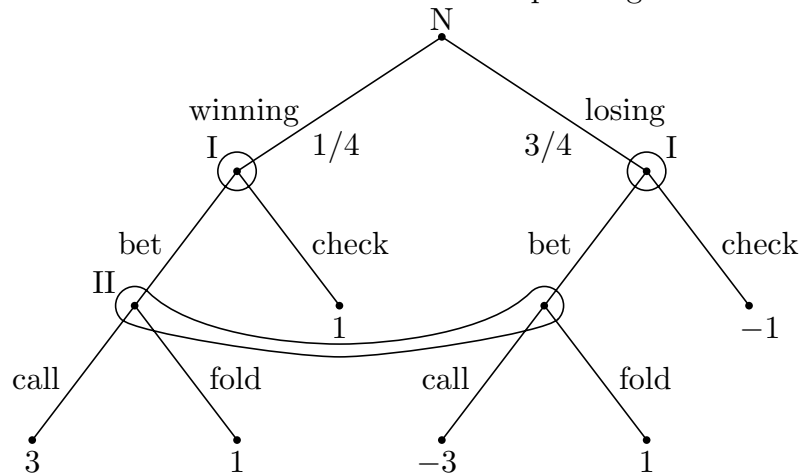


Figure 2.

The diagram now contains all the essential rules of the game.

5.3 The Kuhn Tree. The game tree with all the payoffs, information sets, and labels for the edges and vertices included is known as the Kuhn Tree. We now give the formal definition of a Kuhn tree.

Not every set of vertices can form an information set. In order for a player not to be aware of which vertex of a given information set the game has come to, each vertex in that information set must have the same number of edges leaving it. Furthermore, it is important that the edges from each vertex of an information set have the same set of labels. The player moving from such an information set really chooses a label. It is presumed that a player makes just one choice from each information set.

Definition. A finite two-person zero-sum game in extensive form is given by

- 1) a finite tree with vertices T ,
- 2) a payoff function that assigns a real number to each terminal vertex,
- 3) a set T_0 of non-terminal vertices (representing positions at which chance moves occur) and for each $t \in T_0$, a probability distribution on the edges leading from t ,
- 4) a partition of the rest of the vertices (not terminal and not in T_0) into two groups of information sets $T_{11}, T_{12}, \dots, T_{1k_1}$ (for Player I) and $T_{21}, T_{22}, \dots, T_{2k_2}$ (for Player II), and
- 5) for each information set T_{jk} for player j , a set of labels L_{jk} , and for each $t \in T_{jk}$, a one-to-one mapping of L_{jk} onto the set of edges leading from t .

The information structure in a game in extensive form can be quite complex. It may involve lack of knowledge of the other player's moves or of some of the chance moves. It may indicate a lack of knowledge of how many moves have already been made in the game (as is the case With Player II in Figure 3).

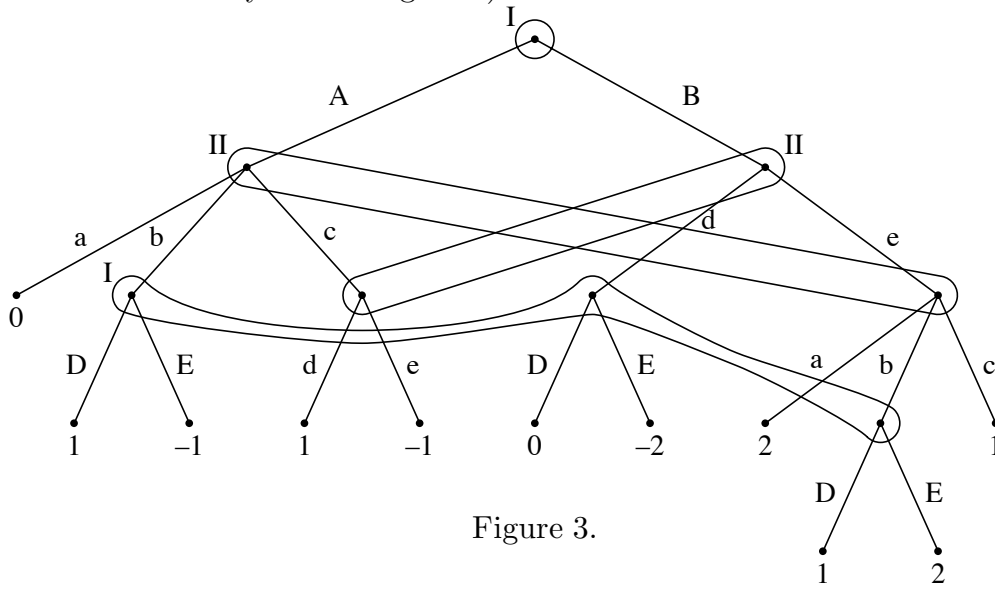


Figure 3.

It may describe situations in which one player has forgotten a move he has made earlier (as is the case With Player I in Figures 3 or 4). In fact, one way to try to model the game of bridge as a two-person zero-sum game involves the use of this idea. In bridge, there are four individuals forming two teams or partnerships of two players each. The interests of the members of a partnership are identical, so it makes sense to describe this as a two-person game. But the members of one partnership make bids alternately based on cards that one member knows and the other does not. This may be described as a single player who alternately remembers and forgets the outcomes of some of the previous random moves. Games in which players remember all past information they once knew and all past moves they made are called games of **perfect recall**.

A kind of degenerate situation exists when an information set contains two vertices which are joined by a path, as is the case with I's information set in Figure 5.

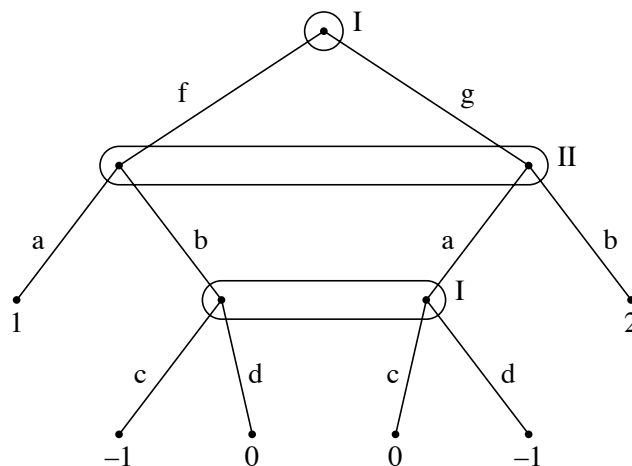


Figure 4.

We take it as a convention that a player makes one choice from each information set during a game. That choice is used no matter how many times the information set is reached. In Figure 5, if I chooses option a there is no problem. If I chooses option b , then in the lower of I's two vertices the a is superfluous, and the tree is really equivalent to Figure 6. Instead of using the above convention, we may if we like assume in the definition of a game in extensive form that no information set contains two vertices joined by a path.

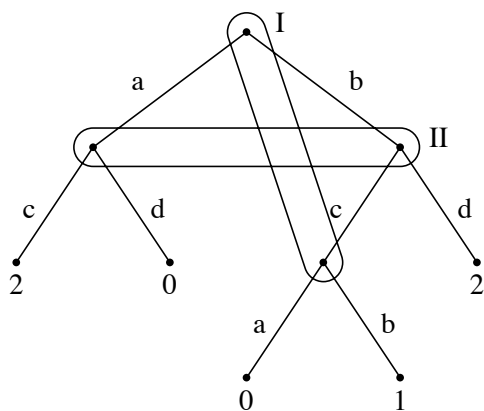


Figure 5.

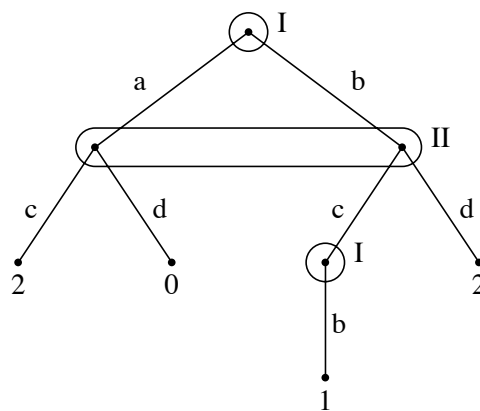


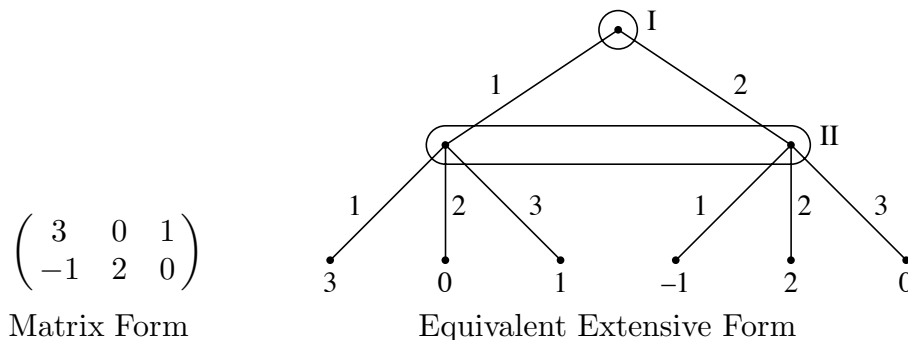
Figure 6.

Games in which both players know the rules of the game, that is, in which both players know the Kuhn tree, are called **games of complete information**. Games in which one or both of the players do not know some of the payoffs, or some of the probabilities of chance moves, or some of the information sets, or even whole branches of the tree, are called **games with incomplete information**, or **pseudogames**. We assume in the following that we are dealing with games of complete information.

5.4 The Representation of a Strategic Form Game in Extensive Form. The notion of a game in strategic form is quite simple. It is described by a triplet (X, Y, A) as in Section 1. The extensive form of a game on the other hand is quite complex. It is described

by the game tree with each non-terminal vertex labeled as a chance move or as a move of one of the players, with all information sets specified, with probability distributions given for all chance moves, and with a payoff attached to each terminal vertex. It would seem that the theory of games in extensive is much more comprehensive than the theory of games in strategic form. However, by taking a game in extensive form and considering only the strategies and average payoffs, we may reduce the game to strategic form.

First, let us check that a game in strategic form can be put into extensive form. In the strategic form of a game, the players are considered to make their choices simultaneously, while in the extensive form of a game simultaneous moves are not allowed. However, simultaneous moves may be made sequentially as follows. We let one of the players, say Player I, move first, and then let player II move without knowing the outcome of I's move. This lack of knowledge may be described by the use of an appropriate information set. The example below illustrates this.



Player I has 2 pure strategies and Player II has 3. We pretend that Player I moves first by choosing row 1 or row 2. Then Player II moves, not knowing the choice of Player I. This is indicated by the information set for Player II. Then Player II moves by choosing column 1, 2 or 3, and the appropriate payoff is made.

5.5 Reduction of a Game in Extensive Form to Strategic Form. To go in the reverse direction, from the extensive form of a game to the strategic form, requires the consideration of pure strategies and the usual convention regarding random payoffs.

Pure strategies. Given a game in extensive form, we first find X and Y , the sets of pure strategies of the players to be used in the strategic form. A pure strategy for Player I is a rule that tells him exactly what move to make in each of his information sets. Let T_{11}, \dots, T_{1k_1} be the information sets for Player I and let L_{11}, \dots, L_{1k_1} be the corresponding sets of labels. A pure strategy for I is a k_1 -tuple $\mathbf{x} = (x_1, \dots, x_{k_1})$ where for each i , x_i is one of the elements of L_{1i} . If there are m_i elements in L_{1i} , the number of such k_1 -tuples and hence the number of I's pure strategies is the product $m_1 m_2 \cdots m_{k_1}$. The set of all such strategies is X . Similarly, if T_{21}, \dots, T_{2k_2} represent II's information sets and L_{21}, \dots, L_{2k_2} the corresponding sets of labels, a pure strategy for II is a k_2 -tuple, $\mathbf{y} = (y_1, \dots, y_{k_2})$ where $y_j \in L_{2j}$ for each j . Player II has $n_1 n_2 \cdots n_{k_2}$ pure strategies if there are n_j elements in L_{2j} . Y denotes the set of these strategies.

Random payoffs. A referee, given $\mathbf{x} \in X$ and $\mathbf{y} \in Y$, could play the game, playing the appropriate move from \mathbf{x} whenever the game enters one of I's information sets, playing the

appropriate move from \mathbf{y} whenever the game enters one of II's information sets, and playing the moves at random with the indicated probabilities at each chance move. The actual outcome of the game for given $\mathbf{x} \in X$ and $\mathbf{y} \in Y$ depends on the chance moves selected, and is therefore a random quantity. Strictly speaking, random payoffs were not provided for in our definition of games in normal form. However, we are quite used to replacing random payoffs by their average values (expected values) when the randomness is due to the use of mixed strategies by the players. We adopt the same convention in dealing with random payoffs when the randomness is due to the chance moves. The justification of this comes from utility theory.

Convention. *If for fixed pure strategies of the players, $\mathbf{x} \in X$ and $\mathbf{y} \in Y$, the payoff is a random quantity, we replace the payoff by the average value, and denote this average value by $A(\mathbf{x}, \mathbf{y})$.*

For example, if for given strategies $\mathbf{x} \in X$ and $\mathbf{y} \in Y$, Player I wins 3 with probability $1/4$, wins 1 with probability $1/4$, and loses 1 with probability $1/2$, then his average payoff is $\frac{1}{4}(3) + \frac{1}{4}(1) + \frac{1}{2}(-1) = 1/2$ so we let $A(\mathbf{x}, \mathbf{y}) = 1/2$.

Therefore, given a game in extensive form, we say (X, Y, A) is the equivalent strategic form of the game if X and Y are the pure strategy spaces of players I and II respectively, and if $A(\mathbf{x}, \mathbf{y})$ is the average payoff for $\mathbf{x} \in X$ and $\mathbf{y} \in Y$.

5.6 Example. Let us find the equivalent strategic form to Basic Endgame in Poker described in the Section 5.2, whose tree is given in Figure 2. Player I has two information sets. In each set he must make a choice from among two options. He therefore has $2 \cdot 2 = 4$ pure strategies. We may denote them by

(b, b) : bet with a winning card or a losing card.

(b, c) : bet with a winning card, check with a losing card.

(c, b) : check with a winning card, bet with a losing card.

(c, c) : check with a winning card or a losing card.

Therefore, $X = \{(b, b), (b, c), (c, b), (c, c)\}$. We include in X all pure strategies whether good or bad (in particular, (c, b) seems a rather perverse sort of strategy.)

Player II has only one information set. Therefore, $Y = \{c, f\}$, where

c : if I bets, call.

f : if I bets, fold.

Now we find the payoff matrix. Suppose I uses (b, b) and II uses c . Then if I gets a winning card (which happens with probability $1/4$), he bets, II calls, and I wins 3 dollars. But if I gets a losing card (which happens with probability $3/4$), he bets, II calls, and I loses 3 dollars. I's average or expected winnings is

$$A((b, b), c) = \frac{1}{4}(3) + \frac{3}{4}(-3) = -\frac{3}{2}.$$

This gives the upper left entry in the following matrix. The other entries may be computed similarly and are left as exercises.

$$\begin{array}{c} \begin{array}{cc} & c & f \\ (b, b) & -3/2 & 1 \\ (b, c) & 0 & -1/2 \\ (c, b) & -2 & 1 \\ (c, c) & -1/2 & -1/2 \end{array} \end{array}$$

Let us solve this 4 by 2 game. The third row is dominated by the first row, and the fourth row is dominated by the second row. In terms of the original form of the game, this says something you may already have suspected: that if I gets a winning card, it cannot be good for him to check. By betting he will win at least as much, and maybe more. With the bottom two rows eliminated the matrix becomes $\begin{pmatrix} -3/2 & 1 \\ 0 & -1/2 \end{pmatrix}$, whose solution is easily found. The value is $V = -1/4$. I's optimal strategy is to mix (b, b) and (b, c) with probabilities $1/6$ and $5/6$ respectively, while II's optimal strategy is to mix c and f with equal probabilities $1/2$ each. The strategy (b, b) is Player I's bluffing strategy. Its use entails betting with a losing hand. The strategy (b, c) is Player I's "honest" strategy, bet with a winning hand and check with a losing hand. I's optimal strategy requires some bluffing and some honesty.

In Exercise 4, there are six information sets for I each with two choices. The number of I's pure strategies is therefore $2^6 = 64$. II has 2 information sets each with two choices. Therefore, II has $2^2 = 4$ pure strategies. The game matrix for the equivalent strategic form has dimension 64 by 4. Dominance can help reduce the dimension to a 2 by 3 game! (See Exercise 10(d).)

5.7 Games of Perfect Information. Now that a game in extensive form has been defined, we may make precise the notion of a game of perfect information.

Definition. *A game of perfect information is a game in extensive form in which each information set of every player contains a single vertex.*

In a game of perfect information, each player when called upon to make a move knows the exact position in the tree. In particular, each player knows all the past moves of the game including the chance ones. Examples include tic-tac-toe, chess, backgammon, craps, etc.

Games of perfect information have a particularly simple mathematical structure. The main result is that *every game of perfect information when reduced to strategic form has a saddle point*; both players have optimal pure strategies. Moreover, *the saddle point can be found by removing dominated rows and columns*. This has an interesting implication for the game of chess for example. Since there are no chance moves, every entry of the game matrix for chess must be either $+1$ (a win for Player I), or -1 (a win for Player

II), or 0 (a draw). A saddle point must be one of these numbers. Thus, either Player I can guarantee himself a win, or Player II can guarantee himself a win, or both players can assure themselves at least a draw. From the game-theoretic viewpoint, chess is a very simple game. One needs only to write down the matrix of the game. If there is a row of all +1's, Player I can win. If there is a column of all -1's, then Player II can win. Otherwise, there is a row with all +1's and 0's and a column with all -1's and 0's, and so the game is drawn with best play. Of course, the real game of chess is so complicated, there is virtually no hope of ever finding an optimal strategy. In fact, it is not yet understood how humans can play the game so well.

5.8 Behavioral Strategies. For games in extensive form, it is useful to consider a different method of randomization for choosing among pure strategies. All a player really needs to do is to make one choice of an edge for each of his information sets in the game. A **behavioral strategy** is a strategy that assigns to each information set a probability distributions over the choices of that set.

For example, suppose the first move of a game is the deal of one card from a deck of 52 cards to Player I. After seeing his card, Player I either bets or passes, and then Player II takes some action. Player I has 52 information sets each with 2 choices of action, and so he has 2^{52} pure strategies. Thus, a mixed strategy for I is a vector of 2^{52} components adding to 1. On the other hand, a behavioral strategy for I simply given by the probability of betting for each card he may receive, and so is specified by only 52 numbers.

In general, the dimension of the space of behavioral strategies is much smaller than the dimension of the space of mixed strategies. The question arises – Can we do as well with behavioral strategies as we can with mixed strategies? The answer is we can if both players in the game have *perfect recall*. The basic theorem, due to Kuhn in 1953 says that in finite games with perfect recall, any distribution over the payoffs achievable by mixed strategies is achievable by behavioral strategies as well.

To see that behavioral strategies are not always sufficient, consider the game of imperfect recall of Figure 4. Upon reducing the game to strategic form, we find the matrix

$$\begin{array}{c} \begin{array}{cc} & a & b \\ (f, c) & \left(\begin{array}{cc} 1 & -1 \\ 1 & 0 \\ 0 & 2 \\ -1 & 2 \end{array} \right) \\ (f, d) & \\ (g, c) & \\ (g, d) & \end{array} \end{array}$$

The top and bottom rows may be removed by domination, so it is easy to see that the unique optimal mixed strategies for I and II are $(0, 2/3, 1/3, 0)$ and $(2/3, 1/3)$ respectively. The value is $2/3$. However, Player I's optimal strategy is not achievable by behavioral strategies. A behavioral strategy for I is given by two numbers, p_f , the probability of choice f in the first information set, and p_c , the probability of choice c in the second information set. This leads to the mixed strategy, $(p_f p_c, p_f(1 - p_c), (1 - p_f)p_c, (1 - p_f)(1 - p_c))$. The strategy $(0, 2/3, 1/3, 0)$ is not of this form since if the first component is zero, that is if

$p_f p_c = 0$, then either $p_f = 0$ or $p_c = 0$, so that either the second or third component must be zero also.

If the rules of the game require players to use behavioral strategies, as is the case for certain models of bridge, then the game may not have a value. This means that if Player I is required to announce his behavioral strategy first, then he is at a distinct disadvantage. The game of Figure 4 is an example of this. (see Exercise 11.)

5.9 Exercises.

1. **The Silver Dollar.** Player II chooses one of two rooms in which to hide a silver dollar. Then, Player I, not knowing which room contains the dollar, selects one of the rooms to search. However, the search is not always successful. In fact, if the dollar is in room #1 and I searches there, then (by a chance move) he has only probability $1/2$ of finding it, and if the dollar is in room #2 and I searches there, then he has only probability $1/3$ of finding it. Of course, if he searches the wrong room, he certainly won't find it. If he does find the coin, he keeps it; otherwise the dollar is returned to Player II. Draw the game tree.

2. **Two Guesses for the Silver Dollar.** Draw the game tree for problem 1, if when I is unsuccessful in his first attempt to find the dollar, he is given a second chance to choose a room and search for it with the same probabilities of success, independent of his previous search. (Player II does not get to hide the dollar again.)

3. **A Statistical Game.** Player I has two coins. One is fair (probability $1/2$ of heads and $1/2$ of tails) and the other is biased with probability $1/3$ of heads and $2/3$ of tails. Player I knows which coin is fair and which is biased. He selects one of the coins and tosses it. The outcome of the toss is announced to II. Then II must guess whether I chose the fair or biased coin. If II is correct there is no payoff. If II is incorrect, she loses 1. Draw the game tree.

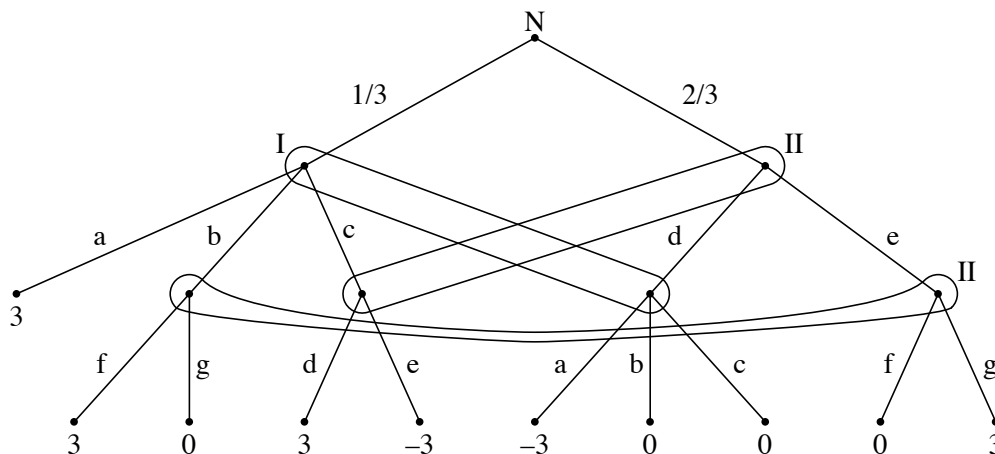
4. **A Forgetful Player.** A fair coin (probability $1/2$ of heads and $1/2$ of tails) is tossed and the outcome is shown to Player I. On the basis of the outcome of this toss, I decides whether to bet 1 or 2. Then Player II hearing the amount bet but not knowing the outcome of the toss, must guess whether the coin was heads or tails. Finally, Player I (or, more realistically, his partner), remembering the amount bet and II's guess, but not remembering the outcome of the toss, may double or pass. II wins if her guess is correct and loses if her guess is incorrect. The absolute value of the amount won is [the amount bet (+1 if the coin comes up heads)] ($\times 2$ if I doubled). Draw the game tree.

5. **A One-Shot Game of Incomplete Information.** Consider the two games $G_1 = \begin{pmatrix} 6 & 0 \\ 0 & 0 \end{pmatrix}$ and $G_2 = \begin{pmatrix} 3 & 0 \\ 0 & 6 \end{pmatrix}$. One of these games is chosen to be played at random with probability $1/3$ for G_1 and probability $2/3$ for G_2 . The game chosen is revealed to Player I but not to Player II. Then Player I selects a row, 1 or 2, and simultaneously Player II chooses a column, 1 or 2, with payoff determined by the selected game. Draw the game tree. (If the game chosen by nature is played repeatedly with Player II learning only the pure strategy choices of Player I and not the payoffs, this is called a *repeated*

game of incomplete information. There is a large literature concerning such games; see for example, the books of Aumann and Maschler (1995) and Sorin (2002).)

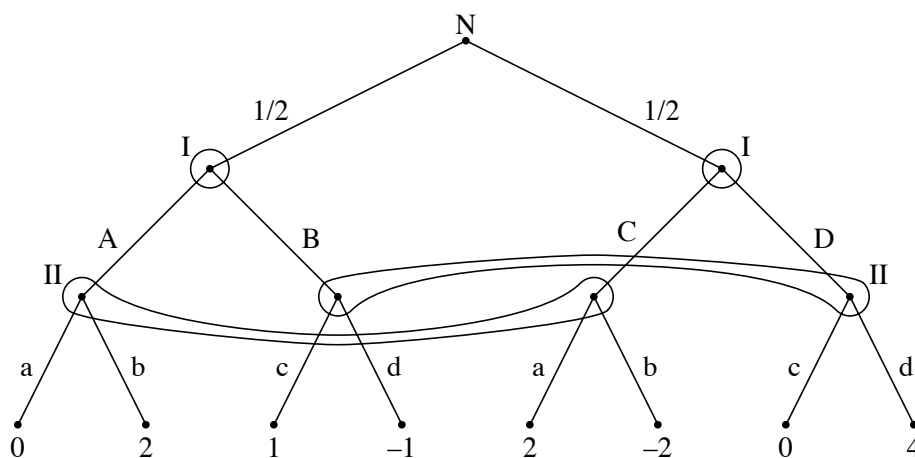
6. Basic Endgame in Poker. Generalize Basic Endgame in poker by letting the probability of receiving a winning card be an arbitrary number p , $0 \leq p \leq 1$, and by letting the bet size be an arbitrary number $b > 0$. (In Figure 2, $1/4$ is replaced by p and $3/4$ is replaced by $1 - p$. Also 3 is replaced by $1 + b$ and -3 is replaced by $-(1 + b)$.) Find the value and optimal strategies. (Be careful. For $p \geq (2 + b)/(2 + 2b)$ there is a saddle point. When you are finished, note that for $p < (2 + b)/(2 + 2b)$, Player II's optimal strategy does not depend on p !) For other generalizations, see Ferguson and Ferguson (2007).

7. (a) Find the equivalent strategic form of the game with the game tree:



- (b) Solve the game.

8. (a). Find the equivalent strategic form of the game with the game tree:



- (b). Solve the game.

9. Coin A has probability $1/2$ of heads and $1/2$ of tails. Coin B has probability $1/3$ of heads and $2/3$ of tails. Player I must predict “heads” or “tails”. If he predicts heads, coin A is tossed. If he predicts tails, coin B is tossed. Player II is informed as to whether I's prediction was right or wrong (but she is not informed of the prediction or the coin that was used), and then must guess whether coin A or coin B was used. If II guesses correctly

she wins 1 dollar from I. If II guesses incorrectly and I's prediction was right, I wins 2 dollars from II. If both are wrong there is no payoff.

- (a) Draw the game tree.
- (b) Find the equivalent strategic form of the game.
- (c) Solve.

10. Find the equivalent strategic form and solve the game of

- (a) Exercise 1.
- (b) Exercise 2.
- (c) Exercise 3.
- (d) Exercise 4.
- (e) Exercise 5.

11. Suppose, in the game of Figure 4, that Player I is required to use behavioral strategies. Show that if Player I is required to announce his behavioral strategy first, he can only achieve a lower value of $1/2$. Whereas, if Player II is required to announce her strategy first, Player I has a behavioral strategy reply that achieves the upper value of $2/3$ at least.

12. (Beasley (1990), Chap. 6.) Player I draws a card at random from a full deck of 52 cards. After looking at the card, he bets either 1 or 5 that the card he drew is a face card (king, queen or jack, probability $3/13$). Then Player II either concedes or doubles. If she concedes, she pays I the amount bet (no matter what the card was). If she doubles, the card is shown to her, and Player I wins twice his bet if the card is a face card, and loses twice his bet otherwise.

- (a) Draw the game tree. (You may argue first that Player I always bets 5 with a face card and Player II always doubles if Player I bets 1.)
- (b) Find the equivalent normal form.
- (c) Solve.

6. Recursive and Stochastic Games

6.1 Matrix Games with Games as Components. We consider now matrix games in which the outcome of a particular choice of pure strategies of the players may be that the players have to play another game. Let us take a simple example.

Let G_1 and G_2 denote 2×2 games with matrices

$$G_1 = \begin{pmatrix} 0 & 3 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad G_2 = \begin{pmatrix} 0 & 1 \\ 4 & 3 \end{pmatrix}$$

and let G denote the 2×2 game whose matrix is represented by

$$G = \begin{pmatrix} G_1 & 4 \\ 5 & G_2 \end{pmatrix}.$$

The game G is played in the usual manner with Player I choosing a row and Player II choosing a column. If the entry in the chosen row and column is a number, II pays I that amount and the game is over. If I chooses row 1 and II chooses column 1, then the game G_1 is played. If I chooses row 2 and II chooses column 2, then G_2 is played.

We may analyze the game G by first analyzing G_1 and G_2 .

G_1 : Optimal for I is (1/2,1/2)
 Optimal for II is (2/3,1/3)
 Val(G_1) = 1.

G_2 : Optimal for I is (0,1)
 Optimal for II is (0,1)
 Val(G_2) = 3.

If after playing the game G the players end up playing G_1 , then they can expect a payoff of the value of G_1 , namely 1, on the average. If the players end up playing G_2 , they can expect an average payoff of the value of G_2 , namely 3. Therefore, the game G can be considered equivalent to the game with matrix

$$\begin{pmatrix} 1 & 4 \\ 5 & 3 \end{pmatrix}$$

G : Optimal for I is (2/5,3/5)
 Optimal for II is (1/5,4/5)
 Val(G)=17/5.

This method of solving the game G may be summarized as follows. *If the matrix of a game G has other games as components, the solution of G is the solution of the game whose matrix is obtained by replacing each game in the matrix of G by its value.*

Decomposition. This example may be written as a 4×4 matrix game. The four pure strategies of Player I may be denoted $\{(1, 1), (1, 2), (2, 1), (2, 2)\}$, where (i, j) represents: use row i in G , and if this results in G_i being played use row j . A similar notation may be used for Player II. The 4×4 game matrix becomes

$$G = \left(\begin{array}{cc|cc} 0 & 3 & 4 & 4 \\ 2 & -1 & 4 & 4 \\ \hline 5 & 5 & 0 & 1 \\ 5 & 5 & 4 & 3 \end{array} \right)$$

We can solve this game by the methods of Chapter 4.

Conversely, suppose we are given a game G and suppose after some rearrangement of the rows and of the columns the matrix may be decomposed into the form

$$G = \left(\begin{array}{c|c} G_{11} & G_{12} \\ \hline G_{21} & G_{22} \end{array} \right)$$

where G_{11} and G_{22} are arbitrary matrices and G_{12} and G_{21} are constant matrices. (A constant matrix has the same numerical value for all of its entries.) Then we can solve G by the above method, pretending that as the first move the players choose a row and column from the 2×2 decomposed matrix. See Exercise 1(b).

6.2 Multistage Games. Of course, a game that is the component of some matrix game may itself have other games as components, in which case one has to iterate the above method to obtain the solution. This works if there are a finite number of stages.

Example 1. The Inspection Game. (M. Dresher (1962)) Player II must try to perform some forbidden action in one of the next n time periods. Player I is allowed to inspect II secretly just once in the next n time periods. If II acts while I is inspecting, II loses 1 unit to I. If I is not inspecting when II acts, the payoff is zero.

Let G_n denote this game. We obtain the iterative equations

$$G_n = \begin{array}{cc} & \begin{array}{cc} \text{act} & \text{wait} \end{array} \\ \begin{array}{c} \text{inspect} \\ \text{wait} \end{array} & \left(\begin{array}{cc} 1 & 0 \\ 0 & G_{n-1} \end{array} \right) \end{array} \quad \text{for } n = 2, 3, \dots$$

with boundary condition $G_1 = (1)$. We may solve iteratively.

$$\text{Val}(G_1) = 1$$

$$\text{Val}(G_2) = \text{Val} \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) = 1/2$$

$$\text{Val}(G_3) = \text{Val} \left(\begin{array}{cc} 1 & 0 \\ 0 & 1/2 \end{array} \right) = 1/3$$

$$\vdots$$

$$\text{Val}(G_n) = \text{Val} \left(\begin{array}{cc} 1 & 0 \\ 0 & 1/(n-1) \end{array} \right) = 1/n$$

since inductively, $\text{Val}(G_n) = \frac{1}{n-1} / (1 + \frac{1}{n-1}) = 1/n$. The optimal strategy in the game G_n for both players is $(1/n, (n-1)/n)$. For other games of this sort, see the book by Garnaev (2000).

Example 2. Guess it! (Rufus Isaacs (1955); see also Martin Gardner (1978), p. 40.) As a more complex example of a multistage game, consider the following game loosely related to the game Cluedo. From a deck with $m + n + 1$ distinct cards, m cards are dealt to Player I, n cards are dealt to Player II, and the remaining card, called the “target card”, is placed face down on the table. Players know their own cards but not those of their opponent. The objective is to guess correctly the target card. Players alternate moves, with Player I starting. At each move, a player may either

- (1) guess at the target card, in which case the game ends, with the winner being the player who guessed if the guess is correct, and his opponent if the guess is incorrect, or
- (2) ask if the other player holds a certain card. If the other player has the card, that card must be shown and is removed from play.

With a deck of say 11 cards and each player receiving 5 cards, this is a nice playable game that illustrates need for bluffing in a clear way. If a player asks about a card that is in his own hand, he knows what the answer will be. We call such a play a *bluff*. If a player asks about a card not in his hand, we say he is *honest*. If a player is always honest and the card he asks about is the target card, the other player will know that the requested card is the target card and so will win. Thus a player must bluff occasionally. Bluffing may also lure the opponent into a wrong guess at the target card.

Let us denote this game with Player I to move by $G_{m,n}$. The game $G_{m,0}$ is easy to play. Player I can win immediately. Since his opponent has no cards, he can tell what the target card is. Similarly, the game $G_{0,n}$ is easy to solve. If Player I does not make a guess immediately, his opponent will win on the next move. However, his probability of guessing correctly is only $1/(n+1)$. Valuing 1 for a win and 0 for a loss from Player I’s viewpoint, the value of the game is just the probability I wins under optimal play. We have

$$\text{Val}(G_{m,0}) = 1 \quad \text{for all } m \geq 0, \quad \text{and} \quad \text{Val}(G_{0,n}) = \frac{1}{n+1} \quad \text{for all } n \geq 0. \quad (1)$$

If Player I asks for a card that Player II has, that card is removed from play and it is Player II’s turn to move, holding $n-1$ cards to her opponent’s m cards. This is exactly the game $G_{n-1,m}$ but with Player II to move. We denote this game by $\bar{G}_{n-1,m}$. Since the probability that Player I wins is one minus the probability that Player II wins, we have

$$\text{Val}(\bar{G}_{n,m}) = 1 - \text{Val}(G_{n,m}) \quad \text{for all } m \text{ and } n. \quad (2)$$

Suppose Player I asks for a card that Player II does not have. Player II must immediately decide whether or not Player I was bluffing. If she decides Player I was honest, she will announce the card Player I asked for as her guess at the target card, and win if she was right and lose if she was wrong. If she decides Player I was bluffing and she is wrong, Player I will win on his turn. If she is correct, the card Player I asked for is removed from his hand, and the game played next is $\bar{G}_{n,m-1}$.

Using such considerations, we may write the game as a multistage game in which a stage consists of three pure strategies for Player I (honest, bluff, guess) and two pure strategies for Player II (*ignore* the asked card, *call* the bluff by guessing the asked card). The game matrix becomes, for $m \geq 1$ and $n \geq 1$,

$$G_{m,n} = \begin{array}{c} \text{honest} \\ \text{bluff} \\ \text{guess} \end{array} \begin{array}{cc} \begin{array}{c} \text{ignore} \\ \text{call} \end{array} \\ \left(\begin{array}{cc} \frac{n}{n+1}\bar{G}_{n-1,m} + \frac{1}{n+1} & \frac{n}{n+1}\bar{G}_{n-1,m} \\ \bar{G}_{n,m-1} & 1 \\ \frac{1}{n+1} & \frac{1}{n+1} \end{array} \right) \end{array} \quad (3)$$

This assumes that if Player I asks honestly, he chooses among the $n + 1$ unknown cards with probability $1/(n + 1)$ each; also if he bluffs, he chooses among his m cards with probability $1/m$ each. That this may be done follows from the invariance considerations of Section 3.6.

As an example, the upper left entry of the matrix is found as follows. With probability $n/(n + 1)$, Player I asks a card that is in Player II's hand and the game becomes $\bar{G}_{n-1,m}$; with probability $1/(n + 1)$, Player I asks the target card, Player II ignores it and Player I wins on his next turn, i.e. gets 1. The upper right entry is similar, except this time if the asked card is the target card, Player II guesses it and Player I gets 0.

It is reasonable to assume that if $m \geq 1$ and $n \geq 1$, Player I should not guess, because the probability of winning is too small. In fact if $m \geq 1$ and $n \geq 1$, there is a strategy for Player I that dominates guessing, so that the last row of the matrix may be deleted. This strategy is: On the first move, ask any of the $m + n + 1$ cards with equal probability $1/(m + n + 1)$ (i.e. use row 1 with probability $(n + 1)/(m + n + 1)$ and row 2 with probability $m/(m + n + 1)$), and if Player II doesn't guess at her turn, then guess at the next turn. We must show that Player I wins with probability at least $1/(n + 1)$ whether or not Player II guesses at her next turn. If Player II guesses, her probability of win is exactly $1/(m + 1)$ whether or not the asked card is one of hers. So Player I's win probability is $m/(m + 1) \geq 1/2 \geq 1/(n + 1)$. If Player II does not guess, then at Player I's next turn, Player II has at most n cards (she may have $n - 1$) so again Player I's win probability is at least $1/(n + 1)$.

So the third row may be removed in (3) and the games reduce to

$$G_{m,n} = \begin{array}{c} \text{honest} \\ \text{bluff} \end{array} \begin{array}{cc} \begin{array}{c} \text{ignore} \\ \text{call} \end{array} \\ \left(\begin{array}{cc} \frac{n}{n+1}\bar{G}_{n-1,m} + \frac{1}{n+1} & \frac{n}{n+1}\bar{G}_{n-1,m} \\ \bar{G}_{n,m-1} & 1 \end{array} \right) \end{array} \quad (4)$$

for $m \geq 1$ and $n \geq 1$. These 2 by 2 games are easily solved recursively, using the boundary conditions (1). One can find the value and optimal strategies of $G_{m,n}$ after one finds the values of $G_{n,m-1}$ and $G_{n-1,m}$ and uses (2). For example, the game $G_{1,1}$ reduces to the game with matrix $\begin{pmatrix} 3/4 & 1/4 \\ 0 & 1 \end{pmatrix}$. The value of this game is $1/2$, an optimal strategy for

Player I is (2/3,1/3) (i.e. bluff with probability 1/3), and the optimal strategy of player II is (1/2,1/2).

One can also show that for all $m \geq 1$ and $n \geq 1$ these games do not have saddle points. In fact, one can show more: that $\text{Val}(G_{m,n})$ is nondecreasing in m and nonincreasing in n . (The more cards you have in your hand, the better.) Let $V_{m,n} = \text{Val}(G_{m,n})$. Then using $\text{Val} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc)/(a + d - b - c)$, we have after some algebra

$$\begin{aligned} V_{m,n} &= \text{Val} \begin{pmatrix} \frac{n}{n+1}(1 - V_{n-1,m}) + \frac{1}{n+1} & \frac{n}{n+1}(1 - V_{n-1,m}) \\ (1 - V_{n,m-1}) & 1 \end{pmatrix} \\ &= \frac{1 + n(1 - V_{n-1,m})V_{n,m-1}}{1 + (n+1)V_{n,m-1}}. \end{aligned}$$

for $m \geq 1$ and $n \geq 1$. This provides a simple direct way to compute the values recursively.

The following table gives the computed values as well as the optimal strategies for the players for small values of m and n .

$m \backslash n$	1	2	3	4	5	6
1	0.5000	0.5000	0.4000	0.3750	0.3333	0.3125
	0.3333	0.2500	0.2000	0.1667	0.1429	0.1250
	0.5000	0.5000	0.4000	0.3750	0.3333	0.3125
2	0.6667	0.5556	0.5111	0.4500	0.4225	0.3871
	0.5000	0.3333	0.2667	0.2143	0.1818	0.1563
	0.3333	0.3333	0.2889	0.2500	0.2301	0.2055
3	0.6875	0.6250	0.5476	0.5126	0.4667	0.4411
	0.5000	0.3750	0.2857	0.2361	0.1967	0.1701
	0.3750	0.3250	0.2762	0.2466	0.2167	0.1984
4	0.7333	0.6469	0.5966	0.5431	0.5121	0.4749
	0.5556	0.3947	0.3134	0.2511	0.2122	0.1806
	0.3333	0.3092	0.2634	0.2342	0.2118	0.1899
5	0.7500	0.6809	0.6189	0.5810	0.5389	0.5112
	0.5714	0.4255	0.3278	0.2691	0.2229	0.1917
	0.3333	0.2908	0.2566	0.2284	0.2062	0.1885
6	0.7714	0.6972	0.6482	0.6024	0.5704	0.5353
	0.6000	0.4410	0.3488	0.2808	0.2362	0.2003
	0.3143	0.2834	0.2461	0.2236	0.2028	0.1854

Table of values and optimal strategies of $G_{m,n}$ for $1 \leq m, n \leq 6$. The top number in each box is the value, the middle number is the probability with which Player I should bluff, and the bottom number is the probability with which Player II should call the asked card.

6.3 Recursive Games. ϵ -Optimal Strategies. In some games with games as components, it may happen that the original game comes up again. Such games are called **recursive**. A simple example is

$$G = \begin{pmatrix} G & 1 \\ 1 & 0 \end{pmatrix}$$

This is an infinite game. If the players always play the first row and first column, the game will be played forever. No matter how unlikely such a possibility is, the mathematical definition is not complete until we say what the payoff is if G is played forever. Let us say that II pays I Q units if they both choose their first pure strategy forever, and write

$$G = \begin{pmatrix} G & 1 \\ 1 & 0 \end{pmatrix}, Q.$$

We are not automatically assured the existence of a value or the existence of optimal strategies in infinite games. However, it is easy to see that the value of G exists and is equal to 1 no matter what the value of the number Q is. The analysis can be made as follows.

II can restrict her losses to at most 1 by choosing the second column. If $Q \geq 1$, I can guarantee winning at least 1 by playing his first row forever. But if $Q < 1$, this won't work. It turns out that an optimal strategy for I, guaranteeing him at least 1, does not exist in this case. However, for any $\epsilon > 0$ there is a strategy for I that guarantees him an average gain of at least $1 - \epsilon$. Such a strategy, that guarantees a player an average payoff within ϵ of the value, is called **ϵ -optimal**. In this case, the strategy that continually uses the mixed strategy $(1 - \epsilon, \epsilon)$ (top row with probability $1 - \epsilon$ and bottom row with probability ϵ) is ϵ -optimal for I. The use of such a strategy by I insures that he will eventually choose row 2, so that the payoff is bound to be 0 or 1 and never Q . The best that Player II can do against this strategy is to choose column 2 immediately, hoping that I chooses row 2. The expected payoff would then be $1 \cdot (1 - \epsilon) + 0 \cdot \epsilon = 1 - \epsilon$.

In summary, for the game G above, the value is 1; Player II has an optimal strategy, namely column 2; If $Q \geq 1$, the first row forever is optimal for I; if $Q < 1$, there is no optimal strategy for I, but the strategy $(1 - \epsilon, \epsilon)$ forever is ϵ -optimal for I.

Consider now the game

$$G_0 = \begin{pmatrix} G_0 & 5 \\ 1 & 0 \end{pmatrix}, Q.$$

For this game, the value depends on Q . If $Q \geq 1$, the first row forever is optimal for I, and the value is Q if $1 \leq Q \leq 5$, and the value is 5 if $Q \geq 5$. For $Q < 1$, the value is 1; however, in contrast to the game G above, I has an optimal strategy for the game G_0 , for example $(1/2, 1/2)$ forever. II's optimal strategy is the first column forever if $Q < 5$, the second column if $Q > 5$ and anything if $Q = 5$.

In analogy to what we did for games with games as components, we might attempt to find the value v of such a game by replacing G_0 by v in the matrix and solving the

equation

$$v = \text{Val} \begin{pmatrix} v & 5 \\ 1 & 0 \end{pmatrix}$$

for v . Here there are many solutions to this equation. The set of all solutions to this equation is the set of numbers v in the interval $1 \leq v \leq 5$. (Check this!)

This illustrates a general result that the equation, given by equating v to the value of the game obtained by replacing the game in the matrix by v , always has a solution equal to the value of the game. It may have more solutions but the value of the game is that solution that is closest to Q . For more information on these points, consult the papers of Everett (1957) and of Milnor and Shapley (1957).

Example 3. Let

$$G = \begin{pmatrix} G & 1 & 0 \\ 1 & 0 & G \\ 0 & G & 1 \end{pmatrix}, Q.$$

Then, if the value of G is v ,

$$v = \text{Val} \begin{pmatrix} v & 1 & 0 \\ 1 & 0 & v \\ 0 & v & 1 \end{pmatrix} = \frac{1+v}{3}.$$

This equation has a unique solution, $v = 1/2$. This must be the value for all Q . The strategy $(1/3, 1/3, 1/3)$ forever is optimal for both players.

Example 4. The basic game of Dare is played as follows. Player I, the leader, and Player II, the challenger, simultaneously “pass” or “dare”. If both pass, the payoff is zero (and the game is over). If I passes and II dares, I wins 1. If I dares and II passes, I wins 3. If both dare, the basic game is played over with the roles of the players reversed (the leader becomes the challenger and vice versa). If the players keep daring forever, let the payoff be zero. We might write

$$G = \begin{array}{cc} & \begin{matrix} \text{pass} & \text{dare} \end{matrix} \\ \begin{matrix} \text{pass} \\ \text{dare} \end{matrix} & \begin{pmatrix} 0 & 1 \\ 3 & -G^T \end{pmatrix} \end{array}$$

where $-G^T$ represents the game with the roles of the players reversed. (Its matrix is the negative of the transpose of the matrix G .) The value of $-G^T$ is the negative of the value of G .

If v represents the value of G , then $v \geq 0$ because of the top row. Therefore the matrix for G with $-G^T$ replaced by $-v$ does not have a saddle point, and we have

$$v = \text{Val} \begin{pmatrix} 0 & 1 \\ 3 & -v \end{pmatrix} = \frac{3}{4+v}.$$

This gives the quadratic equation, $v^2 + 4v - 3 = 0$. The only nonnegative solution is $v = \sqrt{7} - 2 = .64575 \dots$. The optimal strategy for I is $((5 - \sqrt{7})/3, (\sqrt{7} - 2)/3)$ and the optimal strategy for II is $(3 - \sqrt{7}, \sqrt{7} - 2)$.

Example 5. Consider the following three related games.

$$G_1 = \begin{pmatrix} G_2 & 0 \\ 0 & G_3 \end{pmatrix} \quad G_2 = \begin{pmatrix} G_1 & 1 \\ 1 & 0 \end{pmatrix} \quad G_3 = \begin{pmatrix} G_1 & 2 \\ 2 & 0 \end{pmatrix}$$

and suppose the payoff if the games are played forever is Q . Let us attempt to solve these games. Let $v_1 = \text{Val}(G_1)$, $v_2 = \text{Val}(G_2)$, and $v_3 = \text{Val}(G_3)$. Player I can guarantee that $v_1 > 0$, $v_2 > 0$ and $v_3 > 0$ by playing $(1/2, 1/2)$ forever. In addition, $v_2 \leq 1$ and $v_3 \leq 2$, which implies $v_1 < 1$. Therefore none of the games has a saddle point and we may write

$$v_1 = \frac{v_2 v_3}{v_2 + v_3}, \quad v_2 = \frac{1}{2 - v_1}, \quad v_3 = \frac{4}{4 - v_1}.$$

Substituting the latter two equations into the former, we obtain

$$\begin{aligned} \frac{v_1}{2 - v_1} + \frac{4v_1}{4 - v_1} &= \frac{4}{(2 - v_1)(4 - v_1)} \\ 5v_1^2 - 12v_1 + 4 &= 0 \\ (5v_1 - 2)(v_1 - 2) &= 0 \end{aligned}$$

Since $0 < v_1 < 1$, this implies that $v_1 = 2/5$. Hence

Game	value	opt. for I= opt for II
G_1	$2/5$	$(16/25, 9/25)$
G_2	$5/8$	$(5/8, 3/8)$
G_3	$10/9$	$(5/9, 4/9)$

independent of the value of Q .

6.4 Stochastic Movement Among Games. We may generalize the notion of a recursive game by allowing the choice of the next game played to depend not only upon the pure strategy choices of the players, but also upon chance. Let G_1, \dots, G_n be games and let p_1, \dots, p_n be probabilities that sum to one. We use the notation, $p_1 G_1 + \dots + p_n G_n$, to denote the situation where the game to be played next is chosen at random, with game G_i being chosen with probability p_i , $i = 1, \dots, n$. Since, for a given number z , the 1×1 matrix (z) denotes the trivial game in which II pays I z , we may, for example, use $\frac{1}{2}G_1 + \frac{1}{2}(3)$ to represent the situation where G_1 is played if a fair coin comes up heads, and II pays I 3 otherwise.

Example 6. Let G_1 and G_2 be related as follows.

$$G_1 = \begin{pmatrix} \frac{1}{2}G_2 + \frac{1}{2}(0) & 1 \\ 2 & 0 \end{pmatrix} \quad G_2 = \begin{pmatrix} \frac{2}{3}G_1 + \frac{1}{3}(-2) & 0 \\ 0 & -1 \end{pmatrix}$$

The game must eventually end (with probability 1). In fact, the players could not play forever even if they wanted to. Even if they choose the first row and first column forever,

eventually the game would end with a payoff of 0 or -2 . Thus we do not need to specify any payoff if play continues forever. To solve, let $v_i = \text{Val}(G_i)$ for $i = 1, 2$. Then $0 \leq v_1 \leq 1$ and $-1 \leq v_2 \leq 0$, so neither game has a saddle point. Hence,

$$\begin{aligned} v_1 &= \text{Val} \begin{pmatrix} \frac{1}{2}v_2 & 1 \\ 2 & 0 \end{pmatrix} = \frac{4}{6 - v_2} \quad \text{and} \\ v_2 &= \text{Val} \begin{pmatrix} \frac{2}{3}v_1 - \frac{2}{3} & 0 \\ 0 & -1 \end{pmatrix} = -\frac{2(1 - v_1)}{5 - 2v_1} \end{aligned}$$

Thus

$$v_1 = \frac{4}{6 + \frac{2(1-v_1)}{5-2v_1}} = \frac{2(5 - 2v_1)}{16 - 7v_1}.$$

This leads to the quadratic equation, $7v_1^2 - 20v_1 + 10 = 0$, with solution, $v_1 = (10 - \sqrt{30})/7 = .646 \dots$. We may substitute back into the equation for v_2 to find $v_2 = -(2\sqrt{30} - 10)/5 = -.191 \dots$. From these values one can easily find the optimal strategies for the two games.

Example 7. A coin with probability $2/3$ of heads is tossed. Both players must guess whether the coin will land heads or tails. If I is right and II is wrong, I wins 1 if the coin is heads and 4 if the coin is tails and the game is over. If I is wrong and II is right, there is no payoff and the game is over. If both players are right, the game is played over. But if both players are wrong, the game is played over with the roles of the players reversed. If the game never ends, the payoff is Q .

If we denote this game by G , then

$$G = \begin{pmatrix} \frac{2}{3}G + \frac{1}{3}(-G^T) & \frac{2}{3}(1) + \frac{1}{3}(0) \\ \frac{2}{3}(0) + \frac{1}{3}(4) & \frac{2}{3}(-G^T) + \frac{1}{3}G \end{pmatrix}$$

If we let its value be denoted by v , then

$$v = \text{Val} \begin{pmatrix} \frac{1}{3}v & \frac{2}{3} \\ \frac{4}{3} & -\frac{1}{3}v \end{pmatrix}$$

If $v \geq 2$, then there is a saddle at the upper right corner with $v = 2/3$. This contradiction shows that $v < 2$ and there is no saddle. Therefore,

$$v = \frac{8 + v^2}{18} \quad \text{or} \quad v^2 - 18v + 8 = 0.$$

This has a unique solution less than two,

$$v = 9 - \sqrt{73} = .456 \dots$$

from which we may calculate the optimal strategy for I:

$$\left(\frac{13 - \sqrt{73}}{6}, \frac{\sqrt{73} - 7}{6} \right) = (.743 \dots, .256 \dots)$$

and the optimal strategy for II:

$$\left(\frac{11 - \sqrt{73}}{6}, \frac{\sqrt{73} - 5}{6}\right) = (.409 \dots, .591 \dots).$$

The value and optimal strategies are independent of Q .

6.5 Stochastic Games. If to the features of the games of the previous section is added the possibility of a payoff at each stage until the game ends, the game is called a **Stochastic Game**. This seems to be the proper level of generality for theoretical treatment of multistage games. It is an area of intense contemporary research. See for example the books of Filar and Vrieze (1997) and Maitra and Sudderth (1996). Stochastic games were introduced by Shapley in (1953) in a beautiful paper that has been reprinted in Raghavan et al. (1991), and more recently in Kuhn (1997). In this section, we present Shapley's main result.

A Stochastic Game, \mathbf{G} , consists of a finite set of positions or states, $\{1, 2, \dots, N\}$, one of which is specified as the starting position. We denote by $G^{(k)}$ the game in which k is the starting position. Associated with each state, k , is a matrix game, $\mathbf{A}^{(k)} = (a_{ij}^{(k)})$. If the stochastic game is in state k , the players simultaneously choose a row and column of $\mathbf{A}^{(k)}$, say i and j . As a result, two things happen. First, Player I wins the amount $a_{ij}^{(k)}$ from Player II. Second, with probabilities that depend on i, j and k , the game either stops, or it moves to another state (possibly the same one). The probability that the game stops is denoted by $s_{ij}^{(k)}$, and the probability that the next state is ℓ is denoted by $P_{ij}^{(k)}(\ell)$, where

$$s_{ij}^{(k)} + \sum_{\ell=1}^N P_{ij}^{(k)}(\ell) = 1 \quad (5)$$

for all i, j and k .

The payoffs accumulate throughout the game until it stops. To make sure the game eventually stops, we make the assumption that all the stopping probabilities are positive. Let s denote the smallest of these probabilities.

$$s = \min_{i,j,k} s_{ij}^{(k)} > 0 \quad (6)$$

Under this assumption, the probability is one that the game ends in a finite number of moves. This assumption also makes the expected accumulated payoff finite no matter how the game is played, since if M denotes the largest of the absolute values of the payoffs, $M = \max_{i,j,k} |a_{ij}^{(k)}|$, then the total expected payoff to either player is bounded by

$$M + (1 - s)M + (1 - s)^2 M + \dots = M/s. \quad (7)$$

Player I wishes to maximize the total accumulated payoff and Player II to minimize it. We use a modification of the notation of the previous section to describe this game.

$$G^{(k)} = \left(a_{ij}^{(k)} + \sum_{\ell=1}^N P_{ij}^{(k)}(\ell) G^{(\ell)} \right). \quad (8)$$

Note that the probabilities in each component of this matrix sum to less than one. It is understood that with the remaining probability, $s_{ij}^{(k)}$, the game ends. It should be noted that in contrast to the previous section, a payoff does not end the game. After a payoff is made, it is then decided at random whether the game ends and, if not, which state should be played next.

Since no upper bound can be placed on the length of the game, this is an infinite game. A strategy for a player must specify for every n how to choose an action if the game reaches stage n . In general, theory does not guarantee a value. Moreover, the choice of what to do at stage n may depend on what happened at all previous stages, so the space of possible strategies is extremely complex.

Nevertheless, in stochastic games, the value and optimal strategies for the players exist for every starting position. Moreover, optimal strategies exist that have a very simple form. Strategies that prescribe for a player a probability distribution over his choices that depends only on the game, G_k , being played and not on the stage n or past history are called **stationary strategies**. The following theorem states that there exist stationary optimal strategies.

Theorem 1. (Shapley (1952)) *Each game $G^{(k)}$ has a value, $v(k)$. These values are the unique solution of the set of equations,*

$$v(k) = \text{Val} \left(a_{ij}^{(k)} + \sum_{\ell=1}^N P_{ij}^{(k)}(\ell) v(\ell) \right) \quad \text{for } k = 1, \dots, N. \quad (9)$$

Each player has a stationary optimal strategy that in state k uses the optimal mixed strategy for the game with matrix

$$\mathbf{A}^{(k)}(\mathbf{v}) = \left(a_{ij}^{(k)} + \sum_{\ell=1}^N P_{ij}^{(k)}(\ell) v(\ell) \right) \quad (10)$$

where \mathbf{v} represents the vector of values, $\mathbf{v} = (v(1), \dots, v(N))$.

In equations (9), we see the same principle as in the earlier sections: the value of a game is the value of the matrix game (8) with the games replaced by their values. A proof of this theorem may be found in Appendix 2.

Example 8. As a very simple example, consider the following stochastic game with one state, call it G .

$$G = \begin{pmatrix} 1 + (3/5)G & 3 + (1/5)G \\ 1 + (4/5)G & 2 + (2/5)G \end{pmatrix}$$

From Player II's viewpoint, column 1 is better than column 2 in terms of immediate payoff, but column 2 is more likely to end the game sooner than column 1, so that it should entail smaller future payoffs. Which column should she choose?

Assume that all strategies are active, i.e. that the game does not have a saddle point. We must check when we are finished to see if the assumption was correct. Then

$$\begin{aligned} v &= \text{Val} \begin{pmatrix} 1 + (3/5)v & 3 + (1/5)v \\ 1 + (4/5)v & 2 + (2/5)v \end{pmatrix} \\ &= \frac{(1 + (4/5)v)(3 + (1/5)v) - (1 + (3/5)v)(2 + (2/5)v)}{1 + (4/5)v + 3 + (1/5)v - 1 - (3/5)v - 2 - (2/5)v} = 1 + v - (2/25)v^2 \end{aligned}$$

This leads to

$$(2/25)v^2 = 1.$$

Solving this quadratic equation gives two possible solutions $v = \pm\sqrt{25/2} = \pm(5/2)\sqrt{2}$. Since the value is obviously positive, we must use the plus sign. This is $v = (5/2)\sqrt{2} = 3.535$. If we put this value into the matrix above, it becomes

$$\begin{pmatrix} 1 + (3/2)\sqrt{2} & 3 + (1/2)\sqrt{2} \\ 1 + 2\sqrt{2} & 2 + \sqrt{2} \end{pmatrix}$$

The optimal strategy for Player I in this matrix is $\mathbf{p} = (\sqrt{2} - 1, 2 - \sqrt{2}) = (.414, .586)$, and the optimal strategy for Player II is $\mathbf{q} = (1 - \sqrt{2}/2, \sqrt{2}/2) = (.293, .707)$. Since these are probability vectors, our assumption is correct and these are the optimal strategies, and $v = (5/2)\sqrt{2}$ is the value of the stochastic game.

6.6 Approximating the solution. For a general stochastic game with many states, equations (9) become a rather complex system of simultaneous nonlinear equations. We cannot hope to solve such systems in general. However, there is a simple iterative method of approximating the solution. This is based on Shapley's proof of Theorem 1, and is called **Shapley iteration**.

First we make a guess at the solution, call it $\mathbf{v}_0 = (v_0(1), \dots, v_0(N))$. Any guess will do. We may use all zero's as the initial guess, $\mathbf{v}_0 = \mathbf{0} = (0, \dots, 0)$. Then given \mathbf{v}_n , we define inductively, \mathbf{v}_{n+1} , by the equations,

$$v_{n+1}(k) = \text{Val} \left(a_{ij}^{(k)} + \sum_{\ell=1}^N P_{ij}^{(k)}(\ell) v_n(\ell) \right) \quad \text{for } k = 1, \dots, N. \quad (11)$$

With $\mathbf{v}_0 = \mathbf{0}$, the $v_n(k)$ have an easily understood interpretation. $v_n(k)$ is the value of the stochastic game starting in state k if there is forced stopping if the game reaches stage n . In particular, $v_1(k) = \text{Val}(\mathbf{A}_k)$ for all k .

The proof of Theorem 1 shows that $v_n(k)$ converges to the true value, $v(k)$, of the stochastic game starting at k . Two useful facts should be noted. First, the convergence

is at an exponential rate: the maximum error goes down at least as fast as $(1 - s)^n$. (See Corollary 1 of Appendix 2.) Second, the maximum error at stage $n + 1$ is at most the maximum change from stage n to $n + 1$ multiplied by $(1 - s)/s$. (See Corollary 2 of Appendix 2.)

Let us take an example of a stochastic game with two positions. The corresponding games $G^{(1)}$ and $G^{(2)}$, are related as follows.

$$G^{(1)} = \begin{pmatrix} 4 + .3G^{(1)} & 0 + .4G^{(2)} \\ 1 + .4G^{(2)} & 3 + .5G^{(1)} \end{pmatrix} \quad G^{(2)} = \begin{pmatrix} 0 + .5G^{(1)} & -5 \\ -4 & 1 + .5G^{(2)} \end{pmatrix}$$

Using $\mathbf{v}_0 = (0, 0)$ as the initial guess, we find $\mathbf{v}_1 = (2, -2)$, since

$$v_1(1) = \text{Val} \begin{pmatrix} 4 & 0 \\ 1 & 3 \end{pmatrix} = 2 \quad v_1(2) = \text{Val} \begin{pmatrix} 0 & -5 \\ -4 & 1 \end{pmatrix} = -2.$$

The next iteration gives

$$v_2(1) = \text{Val} \begin{pmatrix} 4.6 & -.8 \\ .2 & 4 \end{pmatrix} = 2.0174 \quad v_2(2) = \text{Val} \begin{pmatrix} 1 & -5 \\ -4 & 0 \end{pmatrix} = -2.$$

Continuing, we find

$$\begin{array}{ll} v_3(1) = 2.0210 & v_3(2) = -1.9983 \\ v_4(1) = 2.0220 & v_4(2) = -1.9977 \\ v_5(1) = 2.0224 & v_5(2) = -1.9974 \\ v_6(1) = 2.0225 & v_6(2) = -1.9974 \end{array}$$

The smallest stopping probability is .5, so the rate of convergence is at least $(.5)^n$ and the maximum error of \mathbf{v}_6 is at most .0002.

The optimal strategies using \mathbf{v}_6 are easily found. For game $G^{(1)}$, the optimal strategies are $\mathbf{p}^{(1)} = (.4134, .5866)$ for Player I and $\mathbf{q}^{(1)} = (.5219, .4718)$ for Player II. For game $G^{(2)}$, the optimal strategies are $\mathbf{p}^{(2)} = (.3996, .6004)$ for Player I and $\mathbf{q}^{(2)} = (.4995, .5005)$ for Player II.

6.7 Exercises

1.(a) Solve the system of games

$$G = \begin{pmatrix} 0 & G_1 \\ G_2 & G_3 \end{pmatrix} \quad G_1 = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \quad G_2 = \begin{pmatrix} 0 & 6 \\ 5 & 1 \end{pmatrix} \quad G_3 = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}.$$

(b). Solve the games with matrices

(b1)

$$\begin{pmatrix} 0 & 6 & 0 & 2 \\ 0 & 3 & 0 & 5 \\ 5 & 0 & 2 & 0 \\ 1 & 0 & 4 & 0 \end{pmatrix}$$

(b2)

$$\begin{pmatrix} 3 & 1 & 5 & 2 & 2 \\ 1 & 3 & 5 & 2 & 2 \\ 4 & 4 & 1 & 2 & 2 \\ 1 & 1 & 1 & 6 & 3 \\ 1 & 1 & 1 & 4 & 7 \end{pmatrix}$$

2. **The Inspection Game.** Let $G_{m,n}$ denote the inspection game in which I is allowed m inspections in the n time periods. (Thus, for $1 \leq n \leq m$, $\text{Val}(G_{m,n}) = 1$, while for $n \geq 1$, $\text{Val}(G_{0,n}) = 0$.) Find the iterative structure of the games and solve.

3. **A Game of Endurance.** II must count from n down to zero by subtracting either one or two at each stage. I must guess at each stage whether II is going to subtract one or two. If I ever guesses incorrectly at any stage, the game is over and there is no payoff. Otherwise, if I guesses correctly at each stage, he wins 1 from II. Let G_n denote this game, and use the initial conditions $G_0 = (1)$ and $G_1 = (1)$. Find the recursive structure of the games and solve. (In the solution, you may use the notation F_n to denote the Fibonacci sequence, 1, 1, 2, 3, 5, 8, 13, ..., with definition $F_0 = 1$, $F_1 = 1$, and for $n \geq 2$, $F_n = F_{n-1} + F_{n-2}$.)

4. Solve the sequence of games, G_0, G_1, \dots , where

$$G_0 = \begin{pmatrix} 3 & 2 \\ 1 & G_1 \end{pmatrix}, \dots, G_n = \begin{pmatrix} n+3 & n+2 \\ n+1 & G_{n+1} \end{pmatrix}, \dots$$

Assume that if play continues forever, the payoff is zero.

5. (a) In the game "Guess it!", $G_{1,n}$, with $m = 1$ and arbitrary n , show that Player I's optimal strategy is to bluff with probability $1/(n+2)$.

(b) Show that Player II's optimal strategy in $G_{1,n}$ is to call the asked card with probability $V_{1,n}$, the value of $G_{1,n}$.

6. **Recursive Games.** (a) Solve the game $G = \begin{pmatrix} G & 2 \\ 0 & 1 \end{pmatrix}$, Q .

(b) Solve the game $G = \begin{pmatrix} G & 1 & 1 \\ 1 & 0 & G \\ 1 & G & 0 \end{pmatrix}$, Q .

7. Consider the following three related games.

$$G_1 = \begin{pmatrix} G_2 & 1 \\ 1 & 0 \end{pmatrix} \quad G_2 = \begin{pmatrix} G_3 & 0 \\ 0 & 2 \end{pmatrix} \quad G_3 = \begin{pmatrix} G_1 & 1 \\ 1 & 0 \end{pmatrix}$$

and suppose the payoff is Q if the games are played forever. Solve.

8. Consider the following three related games.

$$G_1 = \begin{pmatrix} G_1 & G_2 & G_3 \\ G_2 & G_3 & G_1 \\ G_3 & G_1 & G_2 \end{pmatrix} \quad G_2 = \begin{pmatrix} G_1 & 0 \\ 0 & 2 \end{pmatrix} \quad G_3 = \begin{pmatrix} G_2 & 1 \\ 1 & 0 \end{pmatrix}$$

and suppose the payoff is Q if the games are played forever. Solve.

9. There is one point to go in the match. The player that wins the last point while serving wins the match. The server has two strategies, high and low. The receiver has two strategies, near and far. The probability the server wins the point is given in the accompanying table.

	near	far
high	.8	.5
low	.6	.7

If the server misses the point, the roles of the players are interchanged and the win probabilities for given pure strategies are the same for the new server. Find optimal strategies for server and receiver, and find the probability the server wins the match.

10. Player I tosses a coin with probability p of heads. For each $k = 1, 2, \dots$, if I tosses k heads in a row he may stop and challenge II to toss the same number of heads; then II tosses the coin and wins if and only if he tosses k heads in a row. If I tosses tails before challenging II, then the game is repeated with the roles of the players reversed. If neither player ever challenges, the game is a draw.

(a) Solve when $p = 1/2$.

(b) For arbitrary p , what are the optimal strategies of the players? Find the limit as $p \rightarrow 1$ of the probability that I wins.

11. Solve the following stochastic game.

$$G = \begin{pmatrix} 4 & 1 + (1/3)G \\ 0 & 1 + (2/3)G \end{pmatrix}.$$

12. Consider the following stochastic game with two positions.

$$G^{(1)} = \begin{pmatrix} 2 & 2 + (1/2)G^{(2)} \\ 0 & 4 + (1/2)G^{(2)} \end{pmatrix} \quad G^{(2)} = \begin{pmatrix} -4 & 0 \\ -2 + (1/2)G^{(1)} & -4 + (1/2)G^{(1)} \end{pmatrix}$$

(a) Solve the equations (9) exactly for the values $v(1)$ and $v(2)$.

(b) Carry out Shapley iteration to find \mathbf{v}_2 starting with the initial guess $\mathbf{v}_0 = (0, 0)$, and compare with the exact values found in (a).

7. Continuous Poker Models.

As an example of a game with more than a countable number of pure strategies, we look at some poker models in which the set of possible “hands” of the players is the interval, $[0, 1]$. This presents a clear example of the use of the principle of indifference in solving games.

The study of two-person zero-sum poker models with independent uniform hands goes back to Borel and von Neumann. Borel discusses a form of poker in Chapter 5, “Le jeu de poker” of his 1938 book, *Applications aux Jeux des Hazard*. Von Neumann presents his analysis of a similar form of poker in the seminal book on game theory — *Theory of Games and Economic Behavior* by von Neumann and Morgenstern (1944). Chapter 19 of the book is devoted to certain mathematical models of poker, with both discrete and continuous hands, and with both simultaneous bets and alternating bets.

Player I is dealt a hand $x \in [0, 1]$ completely at random, by which we mean that x has a uniform distribution over the interval $[0, 1]$; the prior probability that x is in any subinterval of $[0, 1]$ is the length of the subinterval. Similarly, Player II receives a hand y completely at random in $[0, 1]$. Throughout the play, both players know the value of their own hand, but not that of the opponent.

We assume that x and y are *independent* random variables; that is, learning the value of his own hand gives a player no information about the hand of his opponent. (This assumption would not be satisfied if the players were dealt distinct hands from a finite deck, so it would be nice to weaken this assumption. Some work has been done by Sakaguchi and Sakai (1981) in the case of a negative dependence of the hands (i.e. a high hand for one player tends to go with a low hand of the opponent), but the positive dependent case (when higher hands tend to occur together) is completely open.)

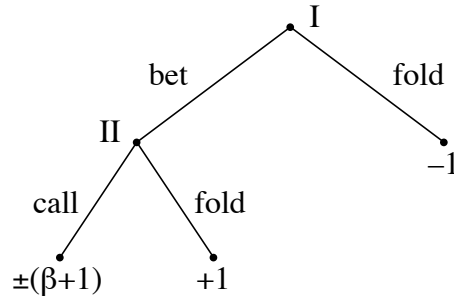
There follows some rounds of betting in which the players take turns acting. After the dealing of the hands, all actions that the players take are announced. Thus, except for the dealing of the hands at the start of the game, this would be a game of perfect information. Games of this sort, where, after an initial random move giving secret information to the players, the game is played with no further introduction of hidden information, are called games of *almost perfect information*. Techniques for solving such games have been studied by Ponssard (1975) and applied to a poker model by Sorin and Ponssard (1980).

It is convenient to study the action part of games of almost complete information by what we call the betting tree. This is distinct from the Kuhn tree in that it neglects the information sets that may arise from the initial distribution of hands. The examples below illustrate this concept.

7.1 La Relance. In his book, Borel discusses a model of poker he calls “la relance”. The players receive independent uniform hands on the interval $[0, 1]$, and each contributes an ante of 1 unit into the pot. Player I acts first either by folding and thus conceding the pot to Player II, or by betting a prescribed amount $\beta > 0$ which he adds to the pot. If Player I bets, then Player II acts either by folding and thus conceding the pot to Player I,

or by calling and adding β to the pot. If Player II calls the bet of Player I, the hands are compared and the player with the higher hand wins the entire pot. That is, if $x > y$ then Player I wins the pot; if $x < y$ then Player II wins the pot. We do not have to consider the case $x = y$ since this occurs with probability 0.

The betting tree is



In this diagram, the plus-or-minus sign indicates that the hands are compared, and the higher hand wins the amount $\beta + 1$.

In the analysis, Borel finds that there is a unique optimal strategy for Player II. The optimal strategy for Player II is of the form for some number b in the interval $[0,1]$, to fold if $y < b$ and to call if $y > b$. The optimal value of b may be found using the principle of indifference. Player II chooses b to make I indifferent between betting and folding when I has some hand $x < b$. If I bets with such an x , he, wins 2 (the pot) if II has $y < b$ and loses β if II has $y > b$. His expected winnings are in this case, $2b - \beta(1 - b)$. On the other hand, if I folds he wins nothing. (This views the game as a constant-sum game. It views the money already put into the pot as a sunk cost, and so the sum of the player's winnings is 2 whatever the outcome. This is a minor point but it is the way most poker players view the pot.) He will be indifferent between betting and folding if

$$2b - \beta(1 - b) = 0$$

from which we conclude

$$b = \beta/(2 + \beta). \quad (1)$$

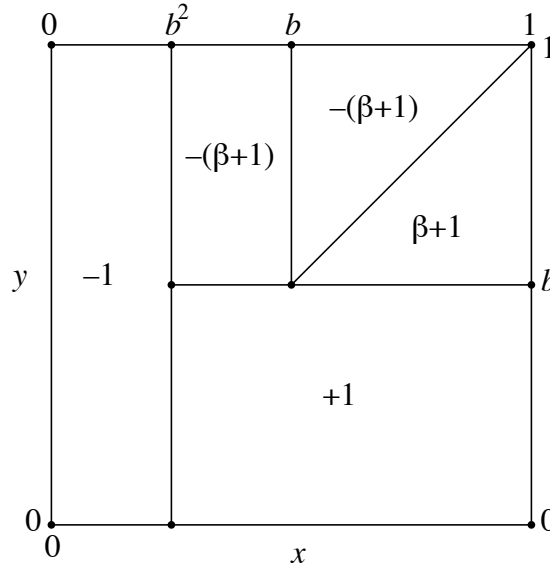
Player I's optimal strategy is not unique, but Borel find all of them. These strategies are of the form: if $x > b$, bet; and if $x < b$, do anything provided the total probability that you fold is b^2 . For example, I may fold with his worst hands, i.e. with $x < b^2$, or he may fold with the best of his hands less than b , i.e. with $b - b^2 < x < b$, or he may, for all $0 < x < b$, simply toss a coin with probability b of heads and fold if the coin comes up heads.

The value of the game may easily be computed. Suppose Player I folds with any $x < b^2$ and bets otherwise and suppose Player II folds with $y < b$. Then the payoff in the unit square has the values given in the following diagram. The values in the upper right

corner cancel and the rest is easy to evaluate. The value is $v(\beta) = -(\beta + 1)(1 - b)(b - b^2) + (1 - b^2)b - b^2$, or, recalling $b = \beta/(2 + \beta)$,

$$v(\beta) = -b^2 = -\frac{\beta^2}{(2 + \beta)^2}. \quad (2)$$

Thus, the game is in favor of Player II.



We summarize in

Theorem 7.1. *The value of la relance is given by (2). An optimal strategy for Player I is to bet if $x > b - b^2$ and to fold otherwise, where b is given in (1). An optimal strategy for Player II is to call if $y > b$ and to fold otherwise.*

As an example, suppose $\beta = 2$. (When the size of the bet is restricted to be no larger than the size of the pot, it is called *pot-limit poker*). Then $b = 1/2$; an optimal strategy for Player I is to bet if $x > 1/4$ and fold otherwise; the optimal strategy of Player II is to call if $y > 1/2$. The game favors Player II, whose expected return is $1/4$ unit each time the game is played.

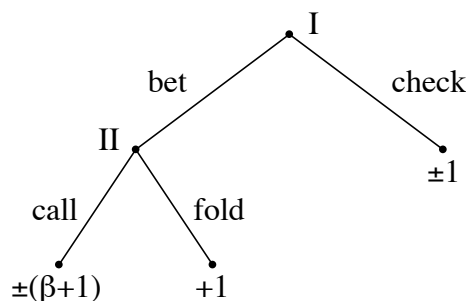
If I bets when $x < b$, he knows he will lose if called, assuming II is using an optimal strategy. Such a bet is called a *bluff*. In la relance, it is necessary for I to bluff with probability b^2 . Which of the hands below b he chooses to bluff with is immaterial as far as the value of the game is concerned. However, there is a secondary advantage to bluffing (betting) with the hands just below b , that is, with the hands from b^2 to b . Such a strategy takes maximum advantage of a mistake the other player may make.

A particular strategy σ for a player is called a *mistake* if there exists an optimal strategy for the opponent when used against σ gives the opponent an expected payoff better than the value of the game. In la relance, it is a mistake for Player II to call with

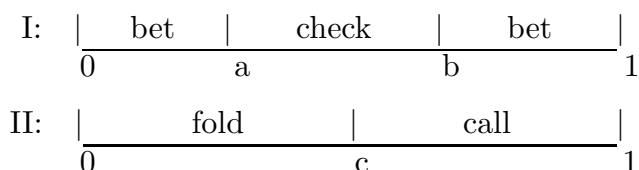
some $y < b$ or to fold with some $y > b$. If II calls with some $y < b$, then I can gain from the mistake most profitably if he bluffs only with his best hands below b .

A strategy is said to be *admissible* for a player if no other strategy for that player does better against one strategy of the opponent without doing worse against some other strategy of the opponent. The rule of betting if and only if $x > b^2$ is the unique admissible optimal strategy for Player I.

7.2 The von Neumann Model. The model of von Neumann differs from the model of Borel in one small but significant respect. If Player I does not bet, he does not necessarily lose the pot. Instead the hands are immediately compared and the higher hand wins the pot. We say Player I checks rather than folds. This provides a better approximation to real poker and a clearer example of the concept of “bluffing” in poker. The betting tree of von Neumann’s poker is the same as Borel’s except that the -1 payoff on the right branch is changed to ± 1 .



This time it is Player I that has a unique optimal strategy. It is of the form for some numbers a and b with $a < b$: bet if $x < a$ or if $x > b$, and check otherwise. Although there are many optimal strategies for Player II (and von Neumann finds all of them), one can show that there is a unique admissible one and it has the simple form: call if $y > c$ for some number c . It turns out that $0 < a < c < b < 1$.



The region $x < a$ is the region in which Player I bluffs. It is noteworthy that Player I must bluff with his worst hands, and not with his moderate hands. It is a mistake for Player I to do otherwise. Here is a rough explanation of this somewhat counterintuitive feature. Hands below c may be used for bluffing or checking. For bluffing it doesn’t matter much which hands are used; one expects to lose them if called. For checking though it certainly matters; one is better off checking with the better hands.

Let us apply the principle of indifference to find the optimal values of a , b and c . This will lead to three equations in three unknowns, known as the *indifference equations* (not

to be confused with difference equations). First, Player II should be indifferent between folding and calling with a hand $y = c$. Again we use the gambler's point of view of the game as a constant sum game, where winning what is already in the pot is considered as a bonus. If II folds, she wins zero. If she calls with $y = c$, she wins $(\beta + 2)$ if $x < a$ and loses β if $x > b$. Equating her expected winnings gives the first indifference equation,

$$(\beta + 2)a - \beta(1 - b) = 0. \quad (3)$$

Second, Player I should be indifferent between checking and betting with $x = a$. If he checks with $x = a$, he wins 2 if $y < a$, and wins nothing otherwise, for an expected return of $2a$. If he bets, he wins 2 if $y < c$ and loses β if $y > c$, for an expected return of $2c - \beta(1 - c)$. Equating these gives the second indifference equation,

$$2c - \beta(1 - c) = 2a. \quad (4)$$

Third, Player I should be indifferent between checking and betting with $x = b$. If he checks, he wins 2 if $y < b$. If he bets, he wins 2 if $y < c$ and wins $\beta + 2$ if $c < y < b$, and loses β if $y > b$, for an expected return of $2c + (\beta + 2)(b - c) - \beta(1 - b)$. This gives the third indifference equation,

$$2c + (\beta + 2)(b - c) - \beta(1 - b) = 2b,$$

which reduces to

$$2b - c = 1. \quad (5)$$

The optimal values of a , b and c can be found by solving equations (4) (5) and (6) in terms of β . The solution is

$$a = \frac{\beta}{(\beta + 1)(\beta + 4)} \quad b = \frac{\beta^2 + 4\beta + 2}{(\beta + 1)(\beta + 4)} \quad c = \frac{\beta(\beta + 3)}{(\beta + 1)(\beta + 4)}. \quad (6)$$

The value is

$$v(\beta) = a = \beta / ((\beta + 1)(\beta + 4)). \quad (7)$$

This game favors Player I. We summarize this in

Theorem 7.2. *The value of von Neumann's poker is given by (7). An optimal strategy for Player I is to check if $a < x < b$ and to bet otherwise, where a and b are given in (6). An optimal strategy for Player II is to call if $y > c$ and to fold otherwise, where c is given in (6).*

For pot-limit poker where $\beta = 2$, we have $a = 1/9$, $b = 7/9$, and $c = 5/9$, and the value is $v(2) = 1/9$.

It is interesting to note that there is an optimal bet size for Player I. It may be found by setting the derivative of $v(\beta)$ to zero and solving the resulting equation for β . It is $\beta = 2$. In other words, the optimal bet size is the size of the pot, exactly pot-limit poker!

7.3 Other Models. Most of the subsequent uniform poker models treated in the mathematical literature have been extensions of *la relance*. Such extensions might not be as interesting as extensions of the von Neumann model, but they are somewhat simpler. Bellman and Blackwell (1949) extend *la relance* to allow Player I to choose from two different sizes of bets. In Bellman and Blackwell (1950), *la relance* is extended to allow Player II to raise a bet by Player I, the size of the raise being equal to the size of the bet (see Bellman (1952).) These two papers are extended in Karlin and Restrepo (1957), the 1949 model to allow a finite number of allowable bet sizes for Player I, and the (1950) model to allow any finite number of raises. In Goldman and Stone (1960a), the (1950) model of Blackwell and Bellman is extended to allow the raise size to be different from the bet size. All of these models and a few others are summarized in Chapter 9 of Volume 2 of Karlin (1959). An extension of *la relance* to allow dependent hands has been made by Sakaguchi and Sakai (1981), and to more complex information structure by Sakaguchi (1993). In Karlin (1959) (Exercise 9.3) and Sakaguchi (1984), the hands x and y are allowed to have arbitrary distinct continuous distributions, not necessarily uniform.

There are two important extensions of the von Neumann model. One, by Newman (1959), allows Player I to choose the bet size, $\beta(x)$, as an arbitrary nonnegative quantity depending on the hand, x , he received. Betting 0 is equivalent to checking. Player I must be careful letting the bet size depend on the hand because this gives information to Player II. However with a judicious use of bluffing, Player I's optimal strategy bets arbitrarily large amounts. Newman's result may be summarized as follows. An optimal strategy for Player I is to check if $1/7 < x < 4/7$, to bet β if $x = 1 - (7/12)(\beta + 2)^{-2}$ for $x > 4/7$, and to bet β if $x = (7/36)(2 + 3\beta)(\beta + 2)^{-3}$ for $x < 1/7$. An optimal strategy for Player II is to call a bet of β if $y > 1 - (12/7)(\beta + 2)^{-1}$. The value is $1/7$. Cutler (1976) solves the problem when the bets are restricted to an interval $[a, b]$ on the positive real line. Sakaguchi (1994) treats the case where Player I is told Player II's card.

The other important extension, by Cutler (1975), is to allow an unlimited number of raises under pot-limit rules. Cutler treats two cases. The first is when Player I is forced to bet on the first round. Thereafter the players in turn may fold, call or raise indefinitely until someone folds or calls. This is solved by recursion. In the other case, Player I is allowed to check on the first round, but he is forbidden to raise if he checks and Player II bets. This game is also solved completely. He says, "However, solving the problem with check raises appears to be quite difficult ...".

Some other uniform poker models that have been considered are the simultaneous move models of Gillies, Mayberry and von Neumann (1953) and Goldman and Stone (1960b), the high-low models with negatively dependent hands of Sakaguchi and Sakai (1982), and the three-person poker models of Nash and Shapley (1950), Sakai (1984) and Sakaguchi and Sakai (1992b), a stud poker model of Pruitt (1961), and the multistage models of Sakaguchi and Sakai (1992a).

7.4 Exercises.

1. Suppose that when Player I checks, Player II is given a choice between checking in which case there is no payoff, and calling in which case the hands are compared and

the higher hand wins the antes. Otherwise, the game is the same as la relance or von Neumann's poker.

(a) Draw the betting tree.

(b) Find the indifference equations. (Assume that I checks if and only if $a < x < b$. Also assume that if I bets, then II calls iff $y > c$, and if Player I checks, Player II calls iff $y > d$, where $a \leq c \leq b$ and $a \leq d \leq b$.)

(c) Solve the equations when $\beta = 2$.

(d)* Does the game favor Player I or Player II, or does it depend on the size of β ?

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