Chapter 1

Introduction

1.1 Objectives

⋆ Financial processes are known to exhibit a combination of properties including semi-heavy-tailed marginal distributions, short- or long-range dependence, volatility and scaling. Current models of their non-Gaussian behaviour relies on a form of Lévy-type distributions such as an α-stable or log normal distribution. These distributions have heavy tails, hence would not be suitable for financial data, which have high volatility but usually not to the extent of turbulence processes. Some variants, such as a truncated stable process, have been suggested, but their implementation is still involved. The first part of the thesis aims to investigate a new way to describe the semi-heavy-tailed behaviour of financial processes. We treat these processes as continuous-time random walks characterised by a transition probability density governed by a fractional diffusion equation. This equation extends the Feller fractional heat equation, which generates α-stable processes, in the sense that the Riesz operator in the Feller equation is composed with the Bessel operator to yield a faster decay in the diffusion. The processes generated by the resulting fractional Riesz-Bessel equation will have semi-heavy tails, which are more suitable for financial data.

$\star$ Many financial time series are now confirmed to possess correlations. On the other hand, most current market models allow no form of correlations. Another major aim of the thesis is to investigate a mechanism to incorporate memory into a class of market models. In particular, the traditional Black-Scholes model will be expanded to incorporate memory in such a way that the model still preserves the completeness and arbitrage-free conditions needed for replication of contingent claims. This approach will be used to estimate the historical and implied volatility of the resulting model.

$\star$ Financial processes are also known to display scaling, in other words, a portion of the data has the same behaviour as the entire data set when zoomed in and rescaled. We aim to model this scaling behaviour explicitly via a well-known technique in fractal geometry, namely iterated function systems. The model will be demonstrated in the problem of market classification.

$\star$ The final aim of the thesis is to apply the new methodology developed to analyse some stock prices, stock indices, foreign exchange rates and other financial time series of some major markets. The tools will be shown to work well for these time series.

1.2 Motivation and literature review

1.2.1 Long-memory in financial processes

Long memory or long-range dependence (LRD) has been investigated extensively in a variety of applied fields, especially in finance (Willinger et al. 1999, Baillie 1996, Granger and Ding 1996, Comte and Renault 1996, 1998, Heyde and Liu 2001). Since the concept of long-range dependence is incompatible with the efficient market hypothesis, a key assumption in mathematical finance, it is still a controversial issue whether market models should include long memory (Lo 1991, Baillie 1996, Willinger et al. 1999).

A second-order stationary process $\xi(t)$ with discrete time is said to possess long-range dependence if its covariance function $R(s) = cov(\xi(t), \xi(t+s))$, $s \in \mathbb{Z}_+$ decays at a hyperbolic rate as $s \to \infty$. In particular, the covariance function $R(s)$ of a process with LRD can be approximated as

$$R(s) \sim K s^{2d-1}, \ \ |d| < \frac{1}{2}, \ \ s \to \infty \tag{1.1}$$

for some constant $K > 0$, or

$$R(s) \sim K s^{2d-1} \cos(\varkappa s), \ \ s \to \infty, \tag{1.2}$$

where $K > 0$, $|d| < 1/2$, $\varkappa = \cos^{-1}\phi \in [0, \pi]$, $|\phi| \leq 1$. Here, $\sim$ means the limit of the ratio of the left-hand side to the right-hand side is equal to 1 as $s \to \infty$. The covariance function (1.1) decays slowly at a hyperbolic rate, while the covariance function (1.2) resembles a hyperbolically damped cosine wave.

The simplest model of a stationary process with LRD and covariance function (1.1) was first proposed by Granger and Joyeux [1980] and Hosking [1981]. This process can be defined by the difference equation

$$(1-B)^d \xi(t) = \varepsilon(t), \ \ t \in \mathbb{Z}, \ \ |d| < \frac{1}{2}, \tag{1.3}$$

where $\varepsilon(t)$ is white noise with $E\varepsilon(t) = 0$, $E\varepsilon^2(t) = \sigma^2 > 0$. The backshift operator B is defined by $B^k\xi(t) = \xi(t-k)$, $k = 0, 1, 2, ...,$ and

$$(1-B)^d = \sum_{j=0}^{\infty} a_j B^j, \ a_0 = 1, \ \ a_j = \frac{\Gamma(j-d)}{\Gamma(j+1)\Gamma(-d)}, \ \ j = 1, 2, ...$$

For its application to empirical data, see, for example, Beran [1992, 1994]. A unique stationary solution of the difference equation (1.3) has the moving-average representation

$$\xi(t) = (1-B)^{-d}\varepsilon(t) = \sum_{j=0}^{\infty} \psi_j \varepsilon(t-j), \ \ \psi_j = \frac{\Gamma(j+d)}{\Gamma(d)\Gamma(1+j)},$$

with coefficients ψ_j, $j = 0, 1, ...,$ satisfying $\sum_{j=0}^{\infty} \psi_j^2 < \infty$, but $\sum_{j=0}^{\infty} \psi_j = \infty$. The covariance function of this process can be approximated by (1.1) and the spectral density has the form

$$f_d(\lambda) = \frac{\sigma^2}{2\pi} |1 - \exp(-i\lambda)|^{-2d} = \frac{\sigma^2}{2\pi}\left(2\sin\frac{\lambda}{2}\right)^{-2d}, \ \ \lambda \in [-\pi, \pi). \tag{1.4}$$

Note that for $-1/2 < d < 0$, we have

$$f_d(0) = \frac{1}{2\pi} \sum_{s \in \mathbb{Z}} R(s) = 0,$$

but for $0 < d < 1/2$,

$$f_d(0) = \frac{1}{2\pi} \sum_{s \in \mathbb{Z}} R(s) = \infty. \tag{1.5}$$

More general models can be proposed via spectral densities of the form $g(\lambda) = f_0(\lambda) f(\lambda), \lambda \in [-\pi, \pi)$, where $f(\lambda)$ is as defined in (1.4) or (1.5) and $f_0(\lambda)$ is a rational spectral density of the ARMA type (see Granger and Joyeux 1980, Hosking 1981, Samarov and Taqqu 1988, Gray et al. 1989, Viano et al. 1995, Giraitis and Leipus 1995, Chung 1996a,b Leipus and Viano 2000).

In continuous time, a fundamental process which may exhibit long memory is fractional Brownian motion. For any H in $(0, 1)$, fractional Brownian motion (FBM) with Hurst index H is a centered Gaussian process $B^H = \left\{B_t^H, t \geq 0\right\}$ with covariance

$$E\left(B_s^H B_t^H\right) = \frac{V_H}{2}(s^{2H} + t^{2H} - |t - s|^{2H}),$$

where V_H is a normalizing constant given by

$$V_H = \frac{\Gamma(2 - 2H)\cos(\pi H)}{\pi H(1 - 2H)}$$

(Mandelbrot and Ness 1968). It is a process starting from zero with stationary increments, $E(B_t^H - B_s^H)^2 = V_H |t - s|^{2H}$, and is self-similar, that is , $B_{\alpha t}^H$ has the same distribution as $\alpha^H B_t^H$ (Decreusefond and Üstünel 1998, Alòs et al. 2000). The constant H determines the sign of the covariance of the future and past increments. This covariance is positive when $H > \frac{1}{2}$ and negative when $H < \frac{1}{2}$. The case $H = \frac{1}{2}$ corresponds to the ordinary Brownian motion. Furthermore, as the covariance between increments at a distance u decreases to zero as u^{2H-2}, FBM exhibits long-range dependence when $H > \frac{1}{2}$.

An approach to model financial processes with long memory is via the theory of stochastic differential equations driven by fractional Brownian motion

(Comte and Renault 1996, 1998, Dai and Heyde 1996, Norros et al. 1999, Iglói and Terdik 1999a,b, Alòs et al. 2000). In this approach, the effect of LRD can be obtained from the noise term. However, such models have inherent difficulties because FBM is not a semimartingale and the resulting Black-Scholes market contains arbitrage opportunities (Rogers 1997). Recently, Hu and Øksendal [1999] have shown that fractional Black-Scholes markets offer no arbitrage if stochastic integrals with respect to FBM are defined via the Wick product. This opens new opportunities for application of FBM to financial modelling. A generalization of FBM is fractional Riesz-Bessel motion (FRBM) proposed in Anh et al. [1999]. A stochastic calculus for FRBM including the corresponding Itô formula was developed in Anh and Nguyen [2000]. Recently, Heyde [1999] proposed a risky asset model with LRD through fractal activity time. The idea is to replace Brownian time in geometric Brownian motion by some process with stationary LRD increments and heavy tails.

Another approach is to incorporate LRD by replacing ordinary differential operators by fractional differential operators in differential or partial differential equations driven by white noise (Gay and Heyde 1990, Inoue 1993, Viano et al. 1994, Chambers 1996, Anh et al. 1999, Anh and Leonenko 2000). The main advantage of this approach is that LRD can be effected via the Green function of the fractional operator involved, hence freeing up the noise term to represent the effects of non-Gaussianity. This approach will be further explored in Subsection 1.2.3.

1.2.2 Heavy tails in financial processes

We first recall some properties of Lévy processes and particularly α-stable processes. These processes have played an important role in modelling the tail behaviour of financial processes.

Let $L = \{L(t) = L(t, \omega), \, t \geq 0\}$ be a stochastic process defined on a complete probability space $(\Omega, \mathcal{F}, P)$. We say that the process has independent increments if, for each $n \in \mathbb{N}$ and $0 \leq t_1 < t_2 < ... < t_{n+1} < \infty$, the random variables

$$L\left(t_{j+1}\right) - L\left(t_j\right), \quad 1 \leq j \leq n,$$

are independent, and it has stationary increments if the random variables $L\left(t\right)$ satisfy

$$L\left(t_{j+1}\right) - L\left(t_j\right) \stackrel{d}{=} L\left(t_{j+1} - t_j\right) - L\left(0\right).$$

Here $\stackrel{d}{=}$ denotes equality in finite-dimensional distribution. We say that L is a Lévy process if

(L1) $L\left(0\right) = 0$ almost surely;

(L2) L has independent and stationary increments;

(L3) L is stochastic continuous, i.e. for all $a > 0$ and for all $s \geq 0$

$$\lim_{t \to s} P\left\{\left|L\left(t\right) - L\left(s\right)\right| > a\right\} = 0.$$

Recall that the sample paths of a process are the maps $t \to L\left(t\right)\left(\omega\right)$ from $\mathbb{R}_+$ into $\mathbb{R}$, for each $\omega \in \Omega$. A Lévy process has a càdlàg modification, that is, its sample paths are right-continuous and possess limits to the left, and we will always assume that the càdlàg version is used (see Protter 1992, p. 21).

A random variable Y is said to be stable, hence have a stable distribution, if there exist $\sigma \geq 0$, $-1 \leq \beta \leq 1$ and $\mu \in \mathbb{R}$ such that, for all $\zeta \in \mathbb{R}$, its characteristic function ϕ_Y satisfies

1) $\phi_Y\left(\zeta\right) = \exp\left\{i\mu\zeta - \frac{1}{2}\sigma^2\zeta^2\right\}$ when $\alpha = 2$;

2) $\phi_Y\left(\zeta\right) = \exp\left\{i\mu\zeta - \sigma^\alpha\left|\zeta\right|^\alpha\left(1 - i\beta\, sgn\ \left(\zeta\right)\tan\frac{\pi\alpha}{2}\right)\right\}$ when $\alpha \neq 1, 2$;

3) $\phi_Y\left(\zeta\right) = \exp\left\{i\mu\zeta - \sigma\left|\zeta\right|\left(1 - i\beta\frac{2}{\pi}\, sgn\ \left(\zeta\right)\log\left|\zeta\right|\right)\right\}$ when $\alpha = 1$.

The parameter α is the index of stability of the stable law, while β is the skewness of the stable law, $\gamma = \sigma^\alpha$ is the scale parameter and μ is the location parameter. All stable random variables have densities f_Y, which in general can be expressed in the form of series expansion (Feller 1971, Chapter 17, Section 6). In three important cases, these densities have a closed form:

The normal distribution

$$\alpha = 2, Y \sim \mathcal{N}\left(\mu, \sigma^2\right), \; f_Y(u) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(u-\mu)^2/\left(2\sigma^2\right)}, u \in \mathbb{R}^1;$$

The Cauchy distribution

$$\alpha = 1, Y \sim \mathcal{C}\left(\mu, \sigma^2\right), \; f_Y(u) = \frac{\sigma}{\pi\left[(u-\mu)^2 + \sigma^2\right]}, u \in \mathbb{R}^1;$$

The Lévy distribution

$$\alpha = \frac{1}{2}, Y \sim \mathcal{L}\left(\mu, \sigma^2\right), \; f_Y(u) = \left(\frac{\sigma}{2\pi}\right)^{1/2} \frac{1}{(u-\mu)^{3/2}} e^{-\sigma/(2(u-\mu))}, u > \mu.$$

One of the reasons why these processes are important in applications is that they display self-similarity:

$$L(at) \stackrel{d}{=} a^H L(t)$$

for all t, $a \geq 0$ and $H = 1/\alpha$, $0 < \alpha < 2$. The symmetric stable Lévy process is also called the α-stable Lévy motion (see Samrodnitsky and Taqqu 1994, Section 7.5). In the case $\alpha = 2,$ its tail behaviour is given by

$$P(Y > u) \sim \frac{e^{-u^2/2}}{\sqrt{2\pi}u}, \quad u \to \infty$$

(see Feller 1971, Chapter 7, Section 1). When $\alpha \neq 2$, there is a weaker polynomial decay as expressed in the following results:

$$\begin{aligned} \lim_{u\to\infty} u^\alpha P(Y > u) &= c_\alpha \frac{1+\beta}{2}\sigma^\alpha; \\ \lim_{u\to\infty} u^\alpha P(Y < -u) &= c_\alpha \frac{1-\beta}{2}\sigma^\alpha, \end{aligned}$$

where $c_\alpha > 1$ (see Samrodnitsky and Taqqu 1994, pp. 16-18 for the proof and an explicit expression for the constant c_α).

Many large data sets exhibit heavy tails and skewness. The strong empirical evidence for these features combined with the generalized central limit theorem is used by many researchers to justify the use of stable models in economics and finance (see Mandelbrot 1967, Fama 1965a,b, Roll 1970, Embrechts et al.

1997, Cheng and Rachev 1993, McCulloch 1996, Adler et al. 1998). Generalised hyperbolic forms with finite moments of all orders have been used to fit these stable laws (see Eberlein and Keller 1995, Bibby and Sørensen 1997, Barndorff-Nielsen 1998, 2001, Eberlein and Raible 1999, Rydberg 1999, Barndorff-Nielsen and Shepard 2001). However, sample paths of Lévy processes, particularly those of α-stable processes, may be too irregular to be able to represent the tail nature of financial processes. Their marginal distributions have semi-heavy tails rather than Lévy-type heavy tails. This analysis is detailed in Voit [2001], where rescaling has been used to demonstrate the faster decay of financial time series. In this thesis, we will develop a new class of distributions, which we call the Riesz-Bessel distributions. These distributions, which are defined by a fractional diffusion equation, have semi-heavy tails as required.

1.2.3 Volatility in financial processes

Hobson [1998, 2004] described the volatility of a financial asset as the variance per unit time of the logarithm of the price of the asset. Volatility is an important tool to investigate a risk in the valuation of options and other derivative securities. In the well-known Black-Scholes model, volatility is a crucial parameter of the underlying price process. Empirical analyses of the volatility of many stock prices indicated that volatility is not constant (Blattberg and Gonedes 1974, Scott 1987). In fact, Scott [1997] found that the volatility of stock returns changes randomly over time, and on occasions there are large, rapid price movements resembling jumps.

In the current literature, volatility has been learned by (i) fitting parametric econometric models such as ARCH (Engle 1982) and GARCH (Bollerslev 1986, Duan 1995); (ii) modelling stochastic volatility; or (iii) studying volatility implied by options prices in conjunction with specific option pricing models such as the Black-Scholes model.

Figure 1.1 shows the volatility of the SET index from 4 August 1997 to 17

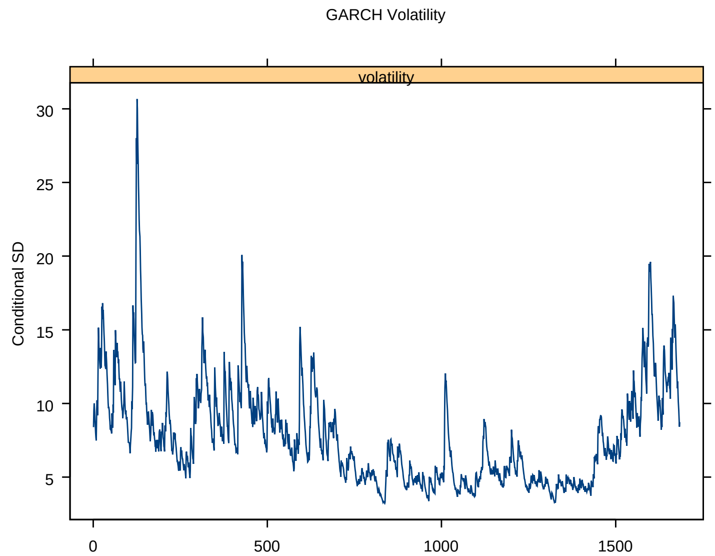


Figure 1.1: the volatility of the SET index from 4 August 1997 to 17 June 2004 by a GARCH(1,1)

June 2004 estimated by a GARCH (1,1) model. In this thesis, we will not pay attention to this approach, which is extensively developed.

Stochastic volatility modelling was first introduced by Hull and White [1987], Scott [1987], and Wiggins [1987] to price options where the volatility of the underlying asset price $S\left(t\right)$ is believed to be stochastic:

$$dS = rdt + S\sqrt{V}dW_s$$

$$dV = a(b - V)dt + \xi V^{\alpha}dW_v$$

where a, b, ξ and α are constants, V is variance rate of the stock, which is the square of its volatility, and W_s and W_v are Wiener processes. The variance rate is assumed to revert to a level b at a rate a (Hull 1997). Hull and White [1987] showed that the volatility is stochastic but uncorrelated with the stock price, the price of a European option is the Black-Scholes price integrated over the probability distribution of average variance rate during the life of the option. This means that the price of European call option is given by $\int c(\overline{v})g(\overline{v})d\overline{v}$, where $\overline{v}$ is the average value of the variance rate σ^2, c is the Black-Scholes price expressed as a function of $\overline{v}$ and g is the probability density function of $\overline{v}$ in a risk neutral world (Hull 1997). When Hull and White [1987] compared the price given by their stochastic volatility model with the price given by the Black-Scholes model with the variance rate equal to the average value $\overline{v}$, they found that the Black-Scholes model overprices in or deep out of the money. The pricing bias caused by a stochastic volatility depends on the correlation between the volatility and the asset price. When the correlation is significantly positive, the Black-scholes model tends to underestimate the price for out-of-the-money call options and overestimate the price for out-of-the-money put options. The reason for this (Hull 1997) is that when the stock price increases, volatility tends to increase. This means that very high stock prices are more likely than those under geometric Brownian motion. When the stock price decreases, volatility tends to decrease. This means that very low stock prices are less likely than those under geometric Brownian motion.

In this thesis, we will pay attention to approach (iii), namely, studying volatility implied by options prices in conjunction with specific option pricing models.

We first recapture some elements of the Black-Scholes model in the theory of option pricing. Options are financial instruments designed to protect investors from the stock market randomness. A European option is a financial instrument giving to its owner the right but not an obligation to buy (European call) or to sell (European put) a share at the maturity time T. Therefore, the purchaser of a European call option on an asset with strike price K and expiry T has the right, but not an obligation, to buy one unit of the asset at time T for a price K. On the other hand, the seller of a European put option on an asset with strike price K and expiry T has the right, but not an obligation, to sell one unit of the asset at time T for a price K. However, this right will only be exercised if the price P_t of the asset at time T is above K; otherwise at expiry the option is worthless (Hobson 1998, Perelló et al. 2000)

The Black-Scholes pricing formula for stock options assumes that the price $(P_t)_{t\leq T}$ of a stock is the solution to a stochastic differential equation

$$dP_t = P_t(\sigma dB_t + \tau dt),$$

where σ is the volatility parameter and B_t is a Brownian motion. The Black-Scholes price C of a call is given explicitly by

$$C(P_t, t; K, T; \sigma, \tau) \equiv C = Ke^{-\tau(T-t)}(M_t\phi(d_1) - \phi(d_2)),$$

where $M_t \equiv (P_t/Ke^{-\tau(T-t)})$ is the moneyness of the option and d_1 and d_2 are given by

$$d_1 = \frac{\ln(M_t) + \frac{1}{2}\sigma^2(T-t)}{\sigma\sqrt{T-t}},$$
$$d_2 = d_1 - \sigma\sqrt{T-t}$$

respectively. The term moneyness refers to the fact that if $M_t > 1$ the option is said to be in-the-money and $M_t < 1$ is said to be out-of-the money. Moreover, the option price depends on the volatility only through the quantity $\sigma^2(T-t)$, which is the integrated squared volatility over the remaining lifetime of the option (Hull and White 1987, Hobson 1998).

A well-known drawback of the Black–Scholes model is that it does not explain the difference between historical volatility HV and implied volatility IV. Anh and Inoue [2005] introduced a dynamic model of complete financial markets, in which the prices of European calls and puts are given by the Black–Scholes formula but HV and IV may be different. The price process $(S(t) : t \in \mathbf{R})$ of this model is defined via an AR(∞)-type equation for the log-price process $Z(t) := \log S(t)$. In the simplest case, this equation takes the form

$$\frac{dZ}{dt}(t) - m = -\int_{-\infty}^{t} pe^{-q(t-s)} \left\{\frac{dZ}{dt}(s) - m\right\} ds + \sigma\frac{dW}{dt}(t)$$

where $m \in \mathbf{R}$, $\sigma, q \in (0, \infty)$, $p \in (-q, \infty)$ and $(W(t) : t \in \mathbf{R})$ is a one-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. The above equation can be explicitly solved to obtain, for $t \in \mathbf{R}$,

$$S(t) = S(0)\exp\left\{mt - \sigma \int_0^t \left(\int_{-\infty}^s pe^{-(p+q)(s-u)} dW(u) \right) ds + \sigma W(t) \right\}.$$

Compared with the Black–Scholes model, the above equation defined by has two additional parameters p and q which describe the *memory of the market*. When $p = 0$, the following equation produces the Black–Scholes price process given by

$$S(t) = S(0)\exp\left(mt + \sigma W(t)\right)$$

This solution is known as geometric Brownian motion. We will develop a new method to estimate the implied volatility based on this class of models and apply the theory to estimate the historical volatility of the S&P500 index.

Although there are many stock indices in the American stock market such as the Dow Jones (DJ) index, the Standard & Poor 500 (S&P500) index, the Nasdaq index, we will consider the daily data of the S&P500 index in particular to estimate its historical volatility for 90 days in April and May 2002. The S&P500 index is constructed based on 500 companies chosen for the market size, liquidity, and industry group representation in the United States of America. So the S&P500 index is one of the most widely used benchmarks of the U.S. equity performance.

1.2.4 Scaling in financial processes

As noted above, many macroeconomic and financial time series or their transforms apparently display the characteristics of anomalous diffusion, namely long-range dependence and heavy-tailed marginal distributions. Barndorff-Nielsen [1998, 1999] used discrete or continuous-type superposition of Ornstein-Uhlenbeck processes with Lévy motion input to obtain a class of random processes with LRD and infinitely divisible marginal distributions, while Iglói and Terdik [1999b] and Oppenheim and Viano [1999] obtained long memory by aggregating continuous-time short-memory Gaussian processes with random coefficients.

We next recall some basic definitions on the local scaling properties of the paths of a process $X(t)$ on some interval $[0,T]$; for further details see, for example, Jaffard [1997a,b] or Riedi [1999]. A typical feature of a scaling process $X(t)$ is that it has a non-integer degree of differentiability, characterised by its local Hölder exponent $h(t)$ defined by

$$h(t) := \sup_{l} \left\{ l : |X(t') - P_t(t')| < C|t'-t|^l \right\}$$

for t' sufficiently close to t, $P_t(.)$ being the Taylor polynomial of X at t, and C is a positive constant. The sets

$$E_h := \{t : h(t) = h\}, \tag{1.6}$$

which form a decomposition of the support of X according to its singularity exponents, can be highly interwoven and dense on $[0,T]$. The *singularity spectrum* of X is then defined as $d(h) = \dim(E_h)$ where dim is the Hausdorff dimension. A process X is said to be *multifractal/multiscaling* if the support of its singularity spectrum has a non-empty interior. A classical example of a multifractal process is the multiplicative cascade on the interval $[0,1]$ (Mandelbrot 1974). However, such multiplicative cascades are not suitable models for financial time series as they do not possess stationary increments and are only defined on some finite interval. An example of a stochastic process with stationary increments and defined on $[0,\infty)$ which is also a multifractal is Lévy motion. Jaffard [1999a,b] showed that all Lévy motions are multifractal with the exception of Brownian motion, compound Poisson processes, deterministic motion and their convolutions. The singularity spectrum of a Lévy motion without Brownian component was shown to be

$$d(h) = \begin{cases} \gamma h, & h \in [0, 1/\gamma], \\ -\infty, & \text{elsewhere}, \end{cases} \tag{1.7}$$

where γ is given by

$$\gamma = \inf \left\{ \eta : \int_{|x|<1} |x|^{\eta} \, \nu \, (dx) \right\} \tag{1.8}$$

and is called the upper index of the Lévy measure ν.

The increasing availability of intraday (high-frequency) data opens new frontiers for financial market research (Lee and Ready 1991). For example, Holthausen et al. [1987] used a buy-sell classification to examine the differential effect of buyer-initiated and seller-initiated block trades. Hasbrouck [1988] used the classification of trades as buy or sell to test asymmetric-information and inventory-control theories of specialist behaviour. Harries [1989] used an increase in the ratio of buys and sells to explain the anomalous behaviour of closing prices. Lee and Ready [1991] used intraday trade and quote data to classify individual trades as market buy or market sell. Ait-Sahalia [1998] used intraday data of the New York Stock Exchange to study the behaviour of the market.

In this thesis, we are interested in the scaling behaviour of high-frequency data via their tick-test form. A tick test is a technique which infers the direction of a trade by comparing its price to the price of the preceding trades (Lee and Ready 1991). The test classifies each trade into four categories: an uptick, a downtick, a zero-uptick, and a zero-downtick. A trade is an uptick (downtick) if the price is higher (lower) than the price of the previous trade. When the price is the same as the previous trade (a zero tick), if the last price change was an uptick, then the trade is a zero-uptick. Similarly, if the last price change was a downtick, then the trade is a zero-downtick. A trade is classified as a buy if it occurs on an uptick or a zero-uptick; otherwise it is classified as a sell. The tick test has been used by many academic researchers and practitioners (see Holthausen et al. 1987, Lee and Ready 1991, Ait-Sahalia 1998).

An intraday time series is first transformed into a tick-test series, which is then converted into a measure representation. The scaling behaviour, possibly multifractal, of this representation is then modelled and analysed via an iterated function system.

1.3 Approach and contributions of the thesis

Chapter 2 of the thesis will investigate the semi-heavy-tailed behaviour of financial processes. We will treat these processes as continuous-time random walks characterised by a transition probability density governed by the fractional Riesz-Bessel equation

$$\frac{\partial p}{\partial t} = -\left(-\Delta\right)^{\alpha}\left(I-\Delta\right)^{\gamma} p\left(t,x\right), \quad p\left(0,x\right) = \delta\left(x\right), \tag{1.9}$$

where Δ is the Laplace operator, the operators $-\left(I-\Delta\right)^{\frac{\gamma}{2}}$, $\gamma \geq 0$, and $\left(-\Delta\right)^{\frac{\alpha}{2}}$, $\alpha > 0$, are inverses of the Bessel and Riesz potentials respectively, and $\delta\left(x\right)$ is the Dirac delta function. The solution $p\left(t,x\right)$ of the above equation is given in terms of its spatial Fourier transform

$$\hat{p}\left(t,\lambda\right) = \exp\left[-t\left|\lambda\right|^{2\alpha}\left(1+\left|\lambda\right|^{2}\right)^{\gamma}\right], \quad \lambda \in \mathbb{R}^{d}.$$

The function $\hat{p}\left(t,z\right)$ is the characteristic function of a distribution for all $t \geq 0$ if and only if $\alpha \in \left(0,1\right], \alpha+\gamma \in \left[0,1\right]$, in which case, the resulting process is called the Riesz-Bessel Lévy motion (RBLm). Since there is no closed form for the density function of the Riesz-Bessel distribution, it must be computed via numerical inversion of $\hat{p}\left(t,z\right)$ using the Fast Fourier Transform.

We establish that, for $\alpha+\gamma < 1/2$, the density of the Lévy measure of RBLm is completely monotone on $(0,\infty)$. This implies that RBLm can be written as the difference of two subordinators whose distribution belongs to the class of generalized convolutions of mixtures of exponentials. As a result, simulation of a general Riesz-Bessel Lévy motion can be carried out by simulating an appropriate stable subordinator. The algorithms for this simulation are provided for two special but important cases.

The next component develops a method for statistical estimation of the

RBLm. We will do this via the empirical characteristic function

$$\hat{\phi}_n(\lambda) = \frac{1}{n}\sum_{j=1}^{n}\cos(\lambda X_j),$$

since a closed form for the density function of the Riesz-Bessel distribution is not available, hence the maximum likelihood approach is not feasible. The minimal distance estimate can be written as the solution to an estimating equation of the form

$$\sum_i a(\lambda_i;\theta)\left(\hat{\phi}_n(\lambda_i) - \phi(\lambda_i;\theta)\right) = 0$$

for particular choices of $a(\lambda;\theta)$. The theory of quasi-likelihood provides a framework in which an optimal choice for $a(\lambda;\theta)$ can be made within a given class of estimating functions. The optimal estimating equation is given by

$$Z(\theta)^T V^{-1}(\theta)\left(\hat{\phi}_n - \phi(\theta)\right) = 0,$$

where

$$Z(\theta)_{ij} = \frac{\partial\phi(\lambda_i;\theta)}{\partial\theta_j}, \quad V(\theta)_{ij} = \operatorname{cov}\left(\hat{\phi}_n(\lambda_i), \hat{\phi}_n(\lambda_j)\right).$$

The above estimating equation can be solved iteratively given a good initial estimate θ_0 as

$$Z(\theta_m)^T V(\theta_m)^{-1} Z(\theta_m)\delta_m = Z(\theta_m)^T V(\theta_m)^{-1}\left(\hat{\phi}_n - \phi(\theta_m)\right),$$

$$\theta_{m+1} = \theta_m + \delta_m.$$

The resulting estimator is consistent, asymptotically normal (Heyde 1997, p. 15) with covariance matrix given by

$$E\ (\hat{\theta}-\theta)(\hat{\theta}-\theta)^T = \left[Z(\theta)^T V^{-1}(\theta) Z(\theta)\right]^{-1}.$$

The solution is just generalised least squares and so the method can be easily implemented in most statistical packages.

This estimation method is performed on a number of financial time series such as stock prices and exchange rates to verify and model their non-Gaussian behaviour. The method works very well and give insights into the scaling behaviour of these financial processes.

Chapter 3 provides a method to estimate the historical volatility of a financial time series based on the Anh-Inoue dynamic model of complete financial markets in which the prices of European calls and puts are given by the Black-Scholes formula. This model incorporates a crucial aspect, namely, memory, into the Black–Scholes model, without losing its usefulness and simplicity, particularly, the Black-Scholes formula. Furthermore, the model can distinguish between historical volatility (HV) and implied volatility (IV). The price process $(S(t) : t \in \mathbf{R})$ of this model is defined via an AR(∞)-type equation for the log-price process $Z(t) := \log S(t)$. In the simplest case, this equation takes the form

$$\frac{dZ}{dt}(t) - m = -\int_{-\infty}^{t} pe^{-q(t-s)} \left\{ \frac{dZ}{dt}(s) - m \right\} ds + \sigma \frac{dW}{dt}(t), \tag{1.10}$$

where $m \in \mathbf{R}$, $\sigma, q \in (0,\infty)$, $p \in (-q,\infty)$ and $(W(t) : t \in \mathbf{R})$ is a one-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Equation (1.10) can be solved explicitly to obtain, for $t \in \mathbf{R}$,

$$S(t) = S(0) \exp\left\{ mt - \sigma \int_0^t \left(\int_{-\infty}^{s} pe^{-(p+q)(s-u)} dW(u) \right) ds + \sigma W(t) \right\}. \tag{1.11}$$

Compared with the Black–Scholes model, the model defined by (1.10) has two additional parameters p and q which describe the memory of the market. When $p = 0$, Eq. (1.11) produces the Black–Scholes price process given by

$$S(t) = S(0) \exp\left(mt + \sigma W(t) \right). \tag{1.12}$$

The log-price process $(Z(t) : t \in \mathbf{R})$ of (1.10) is a Gaussian process with stationary increments which has memory. In this model, the constant σ of (1.10) is equal to the implied volatility of the model, defined via the Black–Scholes formula. In this setting, we show that the historical volatility is given explicitly as

$$\mathrm{HV}(t) = \sigma \sqrt{\frac{q^2}{(p+q)^2} + \frac{p(2q+p)}{(p+q)^3} \cdot \frac{(1 - e^{-(p+q)t})}{t}} \qquad (t > 0).$$

We estimated HV(t) ($t = 1, 2, 3, \dots$) from real market data such as closing values of S&P 500 index via nonlinear least squares. Our finding is that HV(t) is not constant, and very often reveals features in agreement with market conditions. This work provides clear evidence that financial markets have memory and that the model defined above can capture some movement of stock indices reasonably well.

Chapter 4 presents a new method to study the scaling behaviour of stock markets based on their high-frequency data and the theory of iterated function systems. High-frequency data are first converted to the tick-test form, then their measure representation is next generated. We call any string made of K letters from the set $\{0, 1\}$ a K-string. Letting $s = s_1 \cdots s_K$, $s_i \in \{0, 1\}, i = 1, \cdots, K$, be a substring with length K, we define

$$x_{left}(s) = \sum_{i=1}^{K} \frac{s_i}{2^i},$$

and

$$x_{right}(s) = x_{left}(s) + \frac{1}{2^K}.$$

We then use the subinterval $[x_{left}(s), x_{right}(s))$ to represent substring s. Let $N_K(s)$ be the number of times that substring s with length K appears in the tick test time series. If the number of bases in the time series is L, we define

$$F_K(s) = N_K(s)/(L - K + 1)$$

to be the frequency of substring s. It follows that $\sum_{\{s\}} F_K(s) = 1$. Now we can define a measure μ_K on $[0, 1[$ by $d\mu_K(x) = Y_K(x)dx$, where

$$Y_K(x) = 2^K F_K(s), \qquad x \in [x_{left}(s), x_{right}(s)).$$

It is easy to see $\int_0^1 d\mu_K(x) = 1$ and $\mu_K([x_{left}(s), x_{right}(s))) = F_K(s)$. We call μ_K the *measure representation* of the intraday stock data corresponding to the given K. We demonstrate that these probability measures can be modelled as recurrent iterated function systems. Each RIFS consists of N contractive maps $S = \{S_1, S_2, \cdots, S_N\}$ with an associated matrix of probabilities $\mathbf{P} = (p_{ij})$ which satisfy $\sum_j p_{ij} = 1,\ i = 1, 2, \cdots, N$. A sample can be generated from an RIFS via the scheme

$$x_{n+1} = S_{\sigma_n}(x_n), \quad n = 0, 1, 2, 3, \cdots,$$

where x_0 is any starting point. The indices σ_n are not chosen independently, but rather with a probability that depends on the previous index σ_{n-1}:

$$P(\sigma_{n+1} = i) = p_{\sigma_n, i}.$$

For the data at hand, we find that a suitable RIFS can be constructed from two contractive similarities whose parameters are estimated from tick test data by the method of moments. Each market is then represented by a two-dimensional vector constructed from the estimated RIFS. We then classify the markets using the Euclidean distance of these vectors. In this chapter, we provide an application of the RIFS technique on tick test data of the Stock Exchanges of Singapore, Shanghai, Shenzhen and New York. We find that the vectors of the time series from the same market are very close to each other based on the Euclidean distance. This application demonstrates the power of the RIFS technique in modeling the multiple scaling of high-frequency data and in market classification via this scaling.

Chapter 5 pays attention to some key indices of the Stock Exchange of Thailand. In particular, we model the non-Gaussian behaviour of its SET index, SET50 index and MAI index. We demonstrate that the Riesz-Bessel density provides good fit for 1-day returns. We then look at the behaviour of these time series at small lags via rescaling. We find that there appears to be departure from the assumption of a Lévy motion for the SET index. In fact, the rescaled results, where the probability density p is rescaled and plotted as $\tau^{2(\alpha+\gamma)} p\left(X_{rescaled}\right)$ against $X_{rescaled} = \frac{X(t,\tau)}{\tau^{1/(2(\alpha+\gamma))}}$, with $X(t, \tau) = \log S(t) - \log S(t - \tau)$, $S(t)$ being any of the given time series, indicate that convergence to the symmetric $2(\alpha + \gamma)$-stable distribution holds for the series

SET50 and MAI studied, while SET presents a clear departure from it. This provides evidence of second- and/or higher-order correlations in the SET series and gives a clear example in which a model which exhibits both Lévy-type behaviour and short- or long-range dependence of the process is needed.

Chapter 2

Estimation and simulation of the Riesz-Bessel distribution

2.1 Introduction

Bochner [1949] and Feller [1952] demonstrated the connection between the stable distribution and fractional calculus by proposing a Cauchy problem whose solution is in the class of stable distributions. Specifically, the Cauchy problem studied by Bochner [1949] was

$$\frac{\partial p}{\partial t} = -\left(-\Delta\right)^{\alpha} p\left(t, x\right), \quad p\left(0, x\right) = \delta\left(x\right), \tag{2.1}$$

where $\alpha \in (0, 1]$, $\delta(x)$ is the Dirac delta function and the operator $(-\Delta)^{\alpha}$ is understood as the inverse of the Riesz potential defined by the kernel

$$J_{\alpha}\left(x\right) = \frac{\Gamma\left(n/2 - \alpha\right)}{\pi^{n/2} 4^{\alpha} \Gamma\left(\alpha\right)} \left|x\right|^{2\alpha - n}. \tag{2.2}$$

The solution is the symmetric 2α-stable distribution. The operator $-(-\Delta)^{\alpha}$ and its generalization by Feller [1952] are part of a general theory concerning infinitesimal generators of Lévy semigroups, that is, the transition probability density functions of Lévy motions. Despite a large number of fractional operators (see Samko et al. 1993) and the connection established by Bochner [1949], there remain few specific examples of Cauchy problems generating Lévy semigroups. Most of the works in this direction concentrated on the stable distribution (see Gorenflo and Mainardi 1998, 1999). In Anh and McVinish [2004], the Riesz-Bessel distribution is proposed as the solution to the Cauchy problem

$$\frac{\partial p}{\partial t} = -\left(-\Delta\right)^{\alpha}\left(I-\Delta\right)^{\gamma} p\left(t,x\right), \quad p\left(0,x\right) = \delta\left(x\right), \tag{2.3}$$

where the operator $(I-\Delta)^{\gamma}$ is understood as the inverse of the Bessel potential defined by the kernel

$$I_{\gamma}\left(x\right) = \frac{\left(4\pi\right)^{\gamma}}{\Gamma\left(\gamma\right)} \int_0^{\infty} e^{-\pi|x|^2/s - s/4\pi} s^{\gamma - n/2} \frac{ds}{s}. \tag{2.4}$$

The solution of (2.3) is given in terms of its spatial Fourier transform

$$\hat{p}\left(t,\lambda\right) = \exp\left[-t\left|\lambda\right|^{2\alpha}\left(1+\left|\lambda\right|^2\right)^{\gamma}\right], \quad \lambda \in R^d, \tag{2.5}$$

and $\hat{p}\left(t,\lambda\right)$ is a characteristic function under certain conditions on α and γ.

As with the stable and Linnik distributions, despite the simple form of the characteristic function, there is no closed form expression for the probability density function of the Riesz Bessel distribution. When there is no closed form expression for the density, the problem of simulating random variables is sometimes addressed via special representations. An example of this is the simulation algorithm proposed by Kozubowski [2000] which makes use of the mixture representation of the Linnik distribution derived by Kotz and Ostrovskii [1996]. Also, the method of simulating stable random variables proposed by Chambers et al. [1976] (see also Weron 1996) is based on an integral representation due to Zolotarev [1966].

In the estimation problem, the lack of a closed form for the density means direct maximum likelihood estimation is usually abandoned. Numerous methods have been proposed for the stable and Linnik distributions, though they can be applied more generally. An incomplete list of these methods include the fractional moment estimation (Nikias and Shao 1995, Kozubowski 2001), method of moment type (Press 1972, Anderson 1992), minimal distance method (Paulson et al. 1975, Anderson and Arnold 1993), log-log regression of characteristic function (Koutrouvelis 1980) and the k–L procedure of Feuerverger and McDunnough [1981a,b]. The use of these methods is usually supported by some asymptotic results together with a simulation study to suggest their accuracy on small samples.

In this chapter, further properties of the Riesz-Bessel distribution are provided. These properties allow for the simulation of random variables from the Riesz-Bessel distribution. Estimation is addressed by nonlinear generalized least squares on the empirical characteristic function. The estimator is shown to approximate the maximum likelihood estimator. The Riesz-Bessel distribution is then illustrated with financial data.

The chapter is organized as follows: In section 2.2, properties of the Riesz-Bessel distribution are reviewed and two new properties are presented. In Section 2.3, based on one of the new properties, a method for simulating a random variable from the Riesz-Bessel distribution is proposed. In Section 2.4, the estimation problem for the Riesz-Bessel distribution is studied within the quasi-likelihood framework (see Heyde 1997). This enables us to see the k–L procedure as an approximate maximum likelihood approach. The chapter concludes with an illustration of the fitting method by application to some financial data.

2.2 The Riesz-Bessel distribution

A Lévy motion such that the characteristic function of its distribution at time t is given by (2.5) is called a Riesz-Bessel-Lévy motion (RBLm) and will be denoted by $RB\,(t)$. As stated in the Introduction, (2.5) is a characteristic function, but only for a specific range of values of α and γ. The conditions for $p\,(t,x)$ to be a probability distribution are given in the following theorem.

Theorem 2.2.1. *The function $\hat{p}\,(t,z)$ is the characteristic function of a distribution for all $t \geq 0$ if and only if $\alpha \in (0,1]\,,\ \alpha+\gamma \in [0,1]$.*

This class of distributions can be made strictly type equivalent by setting

$$\hat{p}(t,\lambda) = \exp\left[-t\left|\lambda\right|^{2\alpha}(c^2+|\lambda|^2)^{\gamma}\right], \quad \lambda \in R^d, \tag{2.6}$$

with $c > 0$. However, it will be assumed throughout that $c = 1$, unless stated otherwise. Theorem 2.1 was proved in Anh and McVinish [2004] by first showing that

$$\phi(\lambda) = \exp\left[-t\lambda^{\alpha}(1+\lambda)^{\gamma}\right], \quad \lambda > 0. \tag{2.7}$$

is the Laplace-Stieltjes transformation of a probability distribution for all $t > 0,\ \alpha \in (0,1],\ \alpha+\gamma \in [0,1]$. The Lévy motion whose distribution at time t has Laplace-Stieltjes transform (2.7) is called the Riesz-Bessel-Lévy subordinator (RBLs) and will be denoted by $RBS(t)$. Simple conditioning arguments then show that

$$RB(t) \stackrel{d}{=} W(RBS(t)), \tag{2.8}$$

where $W(t)$ is a Brownian motion with variance $2t$ and equality is in the sense of finite dimensional distributions. A distribution whose Lévy motion can be written in the form (2.8) is said to be of Type-G. Type-G distributions were introduced in Marcus [1987] and defined on $\mathbb{R}^1$ as being the distribution of a random variable that is equal in law to σZ, where Z is a standard normal and σ^2 is a non-negative infinitely divisible random variable. An extension to $\mathbb{R}^d$ is given in Barndorff-Nielsen and Pérez-Abreu [2000]. It should be noted that representation (2.8) can also be interpreted in terms of a transformation of the heat (Gaussian) semigroup to a new semigroup.

The role of the parameters α, γ in RBLm is clear from (2.5): The parameter α determines which moments are finite and so, as $t \to \infty$, the distribution can be re-scaled to converge to a symmetric 2α-stable distribution. The parameter γ acts together with α to determine the small time behaviour, that is, as $t \to 0$ the distribution can be re-scaled to converge to a symmetric $2(\alpha+\gamma)$-stable distribution. A similar interpretation of the parameters can be applied to RBLs. This type of behaviour is consistent with experience in applying stable distributions to financial returns data. Taylor [1986] notes that the index of stability estimated from returns data tend to increase with time horizon from ~ 1.6 to near 2. By taking $\alpha = 1$ and $\gamma < 0$, the RBLm is able to incorporate this observation in a parsimonious manner. Simulated sample paths demonstrating this property are given in Section 2.3.

The Laplace-Stieltjes transform of the RBLs has the Lévy representation

$$E\left(e^{-\lambda RBS(t)}\right) = \exp\left[-at\lambda - t\int_0^\infty \left(1 - e^{-x\lambda}\right)\nu_S\left(dx\right)\right], \tag{2.9}$$

where $\nu_S\left(dx\right)$ is called the Lévy measure. The Lévy measure can be expressed in terms of Kummer's confluent hypergeometric function

$$ {}_1F_1\left(a;b;x\right) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\frac{\left(a\right)_k}{\left(b\right)_k} \tag{2.10}$$

and

$$\left(a\right)_k = \begin{cases} 1 & k = 0 \\ a\left(a+1\right)\ldots\left(a+k\right) & k \geq 1 \end{cases} \tag{2.11}$$

(see Andrews et al. 1999 for details). For $\alpha + \gamma \in [0,1)\,, a = 0$ and

$$\nu_S\left(dx\right) = \left[\frac{\alpha\ {}_1F_1\left(1-\gamma;2-\alpha-\gamma;-x\right)}{\Gamma\left(2-\alpha-\gamma\right)x^{\alpha+\gamma}} + \right.$$

$$\left.\frac{\left(\alpha+\gamma\right)\ {}_1F_1\left(1-\gamma;1-\alpha-\gamma;-x\right)}{\Gamma\left(1-\alpha-\gamma\right)x^{1+\alpha+\gamma}}\right]dx, \tag{2.12}$$

and for $\alpha + \gamma = 1$, $a = 1$ and

$$\nu_S\left(dx\right) = \alpha x^{-1}\left[{}_1F_1\left(\alpha;1;-x\right) - {}_1F_1\left(\alpha+1;2;-x\right)\right]dx. \tag{2.13}$$

The qualitative behaviour of the paths of RBLs changes with the value of $\alpha+\gamma$: when $\alpha+\gamma = 0$, the process is a compound Poisson process; when $\alpha+\gamma \in (0,1)$, the process is a pure jump process with jumping times dense in $(0,\infty)$; when $\alpha+\gamma = 1$, the process is a compound Poisson process with drift.

The characteristic function of RBLm has Lévy representation

$$E\left(e^{i\lambda RB(t)}\right) = \exp\left[-at\lambda^2 - t\int_R \left(\cos\left(\lambda x\right) - 1\right)\nu\left(dx\right)\right], \tag{2.14}$$

where $\nu(d\lambda)$ is the Lévy measure. As RBLm is a subordinated Brownian motion, its Lévy measure is of the form

$$\nu(dx) = \int_0^\infty (4\pi s)^{-1/2} \exp\left(\frac{-x^2}{4s}\right) \nu_S(ds)\, dx, \tag{2.15}$$

with $\nu_S(dx)$ given by either (2.12) or (2.13). As with RBLs, the qualitative behaviour of the paths of RBLm changes with the value of $\alpha+\gamma$. For $\alpha+\gamma < 1$, RBLs and RBLm display similar behaviour. For $\alpha+\gamma = 1$, RBLm is the sum of a compound Poisson process and an independent Brownian motion.

Despite there being no closed form for the density function of the Riesz-Bessel distribution, it is still possible to visualize the density by numerical inversion of the characteristic function. In Figures 2.1 and 2.2, the density of the Riesz-Bessel distribution is plotted for $t = 1$, $\alpha = 1$ and γ varying. The graphs were generated using the method described in Mittnik et al. [1999] for the stable distributions. Note that in these plots the variance of the distribution is held constant.

2.2.1 Generalized convolutions of mixtures of exponentials

We now consider the problem of determining if RBLs is a member of the class of generalized convolutions of mixtures of exponentials (GCMED), that is, can RBLs be obtained as a weak limit of sums of random variables with completely monotone densities. These results rely on chapter nine of Bondesson [1992]. A distribution of the class of GCMED is a distribution on $[0,\infty)$ with Laplace-Stieltjes transform

$$\phi(\lambda) = \exp\left[-a\lambda + \int_{(0,\infty)} \left(\frac{1}{x+\lambda} - \frac{1}{x}\right) Q(dx)\right], \quad \lambda \geq 0, \tag{2.16}$$

where $a \geq 0$ and the non-negative measure Q on $(0,\infty)$ satisfies

$$\int_{(0,\infty)} \frac{1}{x(1+x)} Q(dx) < \infty. \tag{2.17}$$

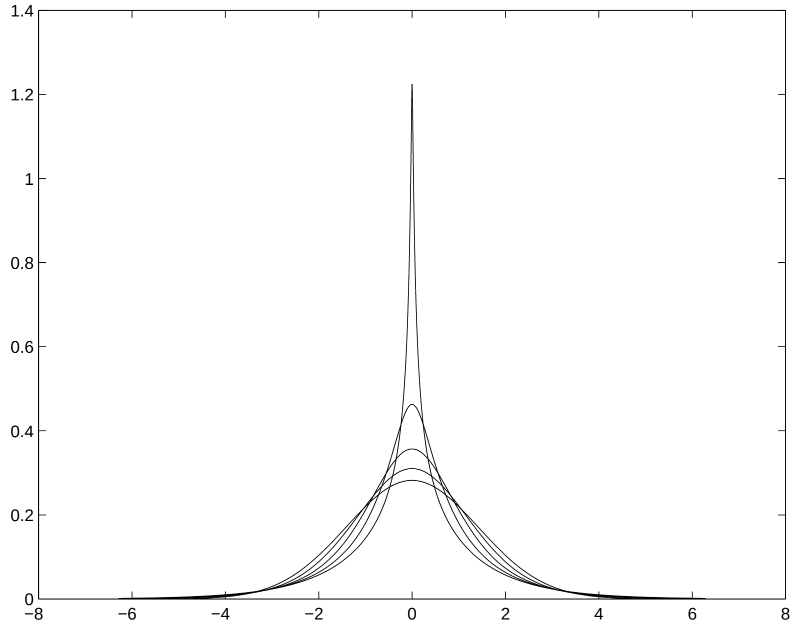


Figure 2.1: The Riesz-Bessel density calculated by numerical inversion of the Fourier transform, $t = 1, \alpha = 1; \gamma = -0.8, -0.6, \ldots, 0$. As $\gamma \downarrow -1$ the density becomes more peaked at $x = 0$. Note we have not included that case of $\alpha + \gamma = 0$ as in this case the distribution possess an atom at $x = 0$.

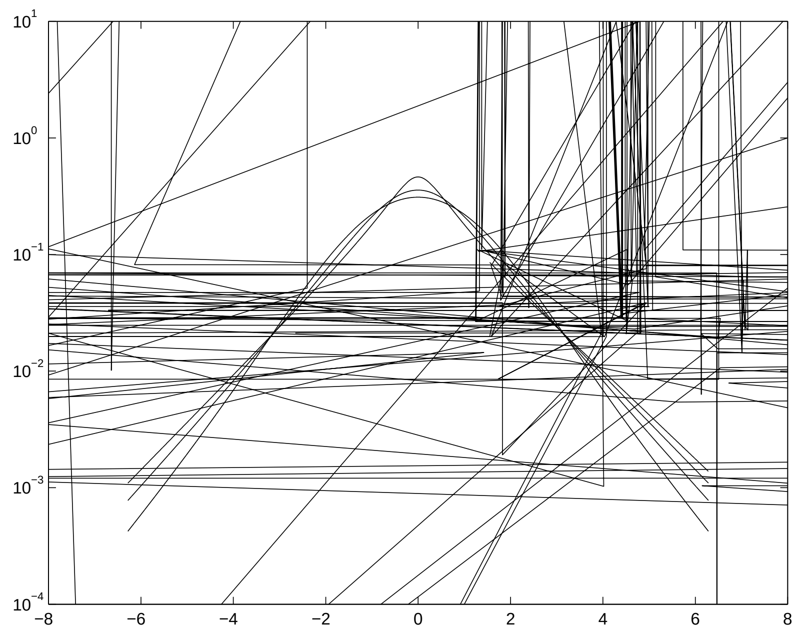


Figure 2.2: Densities from Figure 2.1 plotted on the semi-log scale. As $\gamma \downarrow -1$ the tails of the distribution become heavier which can also be seen from the Lévy density.

This class of distributions can be characterized as those infinitely divisible distributions whose Lévy measure has a completely monotone derivative, in which case the Lévy measure is given by $\nu_S(x)\,dx = \int e^{-xy} Q(dy)\,dx$, (Theorem 9.1.2 of Bondesson [1992]). It is noted that this class is closed under weak limits (Theorem 9.1.1 of Bondesson [1992]). A special subclass is obtained by restricting $Q(dx)$ to have an increasing density. The resulting class is the generalized Gamma convolutions (GGC), that is the class of distributions obtained as weak limits of sums of Gamma random variables. As GGC contain a large number of interesting distributions, such as positive stable, Mittag-Leffler, log-normal and generalized inverse Gaussian to name a few, it is also of interest to establish if RBLs is a member of GGC.

Proposition 2.2.1. *The Riesz-Bessel-Lévy subordinator is a member of the generalized convolutions of mixtures of exponentials.*

Proof. Assume $\alpha + \gamma \in (0, 1)$ and $\gamma > 0$. The inverse Laplace transform of the Lévy density can be obtained from Equation 3.33.1.3 of Prudnikov et al. [1992] and basic properties of the Laplace transform as

$$\frac{1}{\pi}\int_0^x \Big[\sin(\alpha\pi)\, u^\alpha (1-u)_+^{\gamma-1}\left(\frac{\alpha}{u} - \alpha - \gamma\right) + \sin(\pi(\alpha+\gamma))\, u^\alpha (u-1)_+^{\gamma-1}\left(\alpha+\gamma-\frac{\alpha}{u}\right)\Big]\, du. \tag{2.18}$$

The first term of the integrand is positive for $u \in [0, \alpha/(\alpha+\gamma))$ and negative for $u \in (\alpha/(\alpha+\gamma), 1]$ and zero for $u > 1$. The second term of the integrand is zero for $u \in [0, 1]$ and positive for all $u > 1$. It follows that if the integral is non-negative at $x = 1$ then the integral is non-negative for all $x > 0$. From elementary properties of the Gamma function, the integral at $x = 1$ is zero and hence the Lévy density is completely monotone on $(0, \infty)$. Now we assume that $\alpha + \gamma \in (0, 1)$ and $\gamma < 0$. The inverse Laplace transform of the Lévy density is given in Equation 3.33.1.2 of Prudnikov et al. [1992] as

$$\frac{x^\alpha}{\Gamma(\alpha)\Gamma(1-\alpha)}\Big[{}_2F_1(1-\gamma; \alpha; 1+\alpha; x)$$

$$- \frac{\alpha+\gamma}{1+\alpha} \; {}_2F_1\left(1-\gamma; 1+\alpha; 2+\alpha; x\right)\Big] \, 1_{(0<x\leq 1)} \tag{2.19}$$

$$+ \frac{x^{\alpha+\gamma-1}}{\Gamma(\alpha+\gamma)\,\Gamma(2-\alpha-\gamma)} \left[\alpha \; {}_2F_1\left(1-\gamma; 1-\alpha-\gamma; 2-\alpha-\gamma; 1/x\right)\right. \tag{2.20}$$

$$\left. + \left(1-\alpha-\gamma\right) x \; {}_2F_1\left(1-\gamma; -\alpha-\gamma; 1-\alpha-\gamma; 1/x\right)\right] 1_{(x>1)}. \tag{2.21}$$

For $x \in [0,1]$, taking the series expansion of the Gaussian hypergeometric function yields

$$\begin{aligned}
& {}_2F_1\left(1-\gamma, \alpha; 1+\alpha; x\right) - \frac{\alpha+\gamma}{1+\alpha} \; {}_2F_1\left(1-\gamma, 1+\alpha; 2+\alpha; x\right) \\
= \; & \sum_{k=0}^{\infty} \frac{(1-\gamma)_k\,(\alpha)_k}{(1+\alpha)_k} \frac{x^k}{k!} - \frac{\alpha+\gamma}{1+\alpha} \sum_{k=0}^{\infty} \frac{(1-\gamma)_k\,(1+\alpha)_k}{(2+\alpha)_k} \frac{x^k}{k!} \\
= \; & 1 - \frac{\alpha+\gamma}{1+\alpha} + \sum_{k=1}^{\infty} \frac{\alpha\,(1-\gamma)_k}{k+\alpha} \frac{x^k}{k!} - \sum_{k=1}^{\infty} \frac{(\alpha+\gamma)\,(1-\gamma)_k}{k+1+\alpha} \frac{x^k}{k!} \\
= \; & \frac{1-\gamma}{1+\alpha} + \sum_{k=1}^{\infty} (1-\gamma)_k \left(\frac{\alpha}{\alpha+k} - \frac{\alpha+\gamma}{k+1+\alpha}\right) \frac{x^k}{k!}.
\end{aligned}$$

As $\gamma < 0$ it follows that (2.19) is non-negative for $x \in [0,1]$. For $x > 1$, the series expansion of the Gaussian hypergeometric function yields,

$$\begin{aligned}
& \alpha {}_2F_1\left(1-\gamma, 1-\alpha-\gamma; 2-\alpha-\gamma; x\right) \\
- \; & \left(1-\alpha-\gamma\right) x \; {}_2F_1\left(1-\gamma, -\alpha-\gamma; 1-\alpha-\gamma; x\right) \\
= \; & \alpha + \left(1-\alpha-\gamma\right) x + \alpha \sum_{k=1}^{\infty} \frac{(1-\alpha-\gamma)\,(1-\gamma)_k}{k+1-\alpha-\gamma} \frac{x^{-k}}{k!} \\
& + \; (1-\alpha-\gamma) \sum_{k=1}^{\infty} \frac{(-\alpha-\gamma)\,(1-\gamma)_k}{k-\alpha-\gamma} \frac{x^{1-k}}{k!} \\
= \; & \alpha + (1-\gamma)(-\alpha-\gamma) + (1-\alpha-\gamma)\,x \\
& + \; \sum_{k=1}^{\infty} \left\{\alpha + \frac{(-\alpha-\gamma)(k+1-\gamma)}{(k+1)}\right\} \frac{(1-\alpha-\gamma)\,(1-\gamma)_k\,x^{-k}}{k!\,(k+1-\alpha-\gamma)} \\
= \; & (-\gamma)(1-\alpha-\gamma) + (1-\alpha-\gamma)\,x \\
& + \; (-\gamma)(1-\alpha-\gamma) \sum_{k=1}^{\infty} \frac{(1-\gamma)_k\,x^{-k}}{k!\,(k+1-\alpha-\gamma)} \left(1 - \frac{\alpha+\gamma}{k+1}\right).
\end{aligned}$$

As $\gamma < 0$ and $\alpha + \gamma < 1$ it follows that (2.20)-(2.21) is positive for $x > 1$. Hence, the Lévy density is completely monotone for this range of parameters. The remaining cases, $\alpha + \gamma = 1$ and $\alpha + \gamma = 0$, are members of GCMED as this class is closed under weak limits. This completes the proof. □

The functions (2.18) and (2.19)-(2.21) give the density q of the measure Q in the respective parameter ranges. For $\gamma > 0,\ \alpha + \gamma < 1$, it is clear that q is not increasing and from Theorem 9.1.4 of Bondesson [1992] it follows that RBLs is not a member of the class GGC. Furthermore, q is not bounded as $x \to \infty$ and hence from Theorem 9.1.5 of Bondesson [1992] it follows that the distribution of RBLs does not have a completely monotone derivative for any $t > 0$. These statements also hold for $\gamma < 0$. Recall the following property due to Gauss of the hypergeometric function. If $\Re(c - a - b) < 0$, then

$$\lim_{x \to 1^-} \frac{{}_2F_1(a;b;c;x)}{(1-x)^{c-a-b}} = \frac{\Gamma(c)\,\Gamma(a+b-c)}{\Gamma(a)\,\Gamma(b)}, \tag{2.22}$$

(see Andrews et al. 1999 for details). For $\gamma < 0,\ q$ satisfies

$$\lim_{x \to 1^-} \frac{q(x)}{(1-x)^{\gamma}} = \frac{1}{\Gamma(\alpha)\,\Gamma(1-\alpha)}, \tag{2.23}$$

thus, q cannot be increasing and so RBLs is not a member of GGC. Furthermore, as q is unbounded, the distribution of RBLs does not have a completely monotone derivative for any $t > 0$. The remaining case of $\alpha + \gamma = 1$ cannot be a member of GGC as RBLs is a compound Poisson process with drift in this case and hence is not self-decomposable. However, the distribution of RBLs does have a completely monotone derivative for some $t > 0$ in this case.

Proposition 2.2.2. *Let $Y_t = RBS(t) - t$ and assume $\alpha + \gamma = 1$. The distribution function of Y_t has an atom at zero with mass $e^{-t(1-\alpha)}$. The absolutely continuous component of the distribution has a density given by*

$$\frac{1}{\pi}\int_0^1 \left\{\exp\left[-ux - t\left(u^{\alpha}(1-u)^{1-\alpha}\cos(\alpha\pi) + u\right)\right] \cdot \right.$$
$$\left. \sin\left(t\sin(\alpha\pi)\,u^{\alpha}(1-u)^{1-\alpha}\right)\right\} du, \quad x \geq 0, \tag{2.24}$$

provided $t \leq \alpha^{-\alpha}(1-\alpha)^{\alpha-1}\pi/\sin(\alpha\pi)$.

Proof. The Laplace-Stieltjes transform of the distribution of Y_t is given by

$$\phi(\lambda) = \exp\left\{-t\left[\lambda^{\alpha}(1+\lambda)^{1-\alpha} - \lambda\right]\right\}. \tag{2.25}$$

From Proposition 2.2.1 it is known that Y_t is a member of the class of GCMED. By application of some elementary properties of the Laplace transform to (2.13), the density of the Q measure in this case is given by

$$q(x) = \frac{t}{\Gamma(\alpha)\Gamma(1-\alpha)} \int_0^x (1-u)_+^{-\alpha} u^{\alpha-1} (\alpha-u)\, du.$$

The above integral is bounded by $t\alpha^\alpha (1-\alpha)^{1-\alpha} \pi/\sin(\alpha\pi)$ for all $\alpha \in (0,1)$ and hence, the density $q(x)$ is finite for all $x > 0$. Application of Theorem 9.1.5 of Bondesson [1992] gives that the density of RBLs is completely monotone on $x > t, t \in \left(0, \alpha^{-\alpha}(1-\alpha)^{\alpha-1} \pi/\sin(\alpha\pi)\right]$ for $\alpha + \gamma = 1$. It follows that (2.25) is the Stieltjes transform of a Borel measure $\mu(du)$ which may be obtained by application of the Stieltjes complex inversion formula (Widder 1941, Chapter VIII, Theorem 7a). Now,

$$\lim_{\eta \to 0^+} \frac{1}{2\pi i} \int_0^u [\phi(-\sigma - i\eta) - \phi(-\sigma + i\eta)]\, d\sigma = \frac{\mu(u+) + \mu(u-)}{2} - \frac{\mu(0+) + \mu(0)}{2} \tag{2.26}$$

for $\lambda > 0$. The integrand in (2.26) remains bounded as $\eta \to 0$. Applying the Lebesgue dominated convergence theorem we see that μ is absolutely continuous and hence has representation (2.24). This completes the proof. □

From (2.15) and Proposition 2.2.1 it follows that the density of the Lévy measure of RBLm is completely monotone on $(0, \infty)$. This implies that RBLm for $\alpha + \gamma < 1/2$ (this condition ensures RBLm has paths of bounded variation) can be written as the difference of two subordinators whose distribution belongs to the class of GCMED. Geman et al. [2001] provide the following interpretation of these processes in a financial setting: Let S_t be the price of some traded asset. The log price process is given by

$$\log(S_t/S_0) = U(t) - V(t) \tag{2.27}$$

where $U(t), V(t)$ are the prevailing buy/sell orders which are modeled as independent subordinators. The representation of a Lévy measure with a completely monotone derivative as the Laplace transform of some measure is interpreted as an economy populated by individuals who submit prevailing price buy or sell orders with an exponential distribution. The measure from the representation of the Lévy measure relates to the number of orders per unit time at a particular mean level, with exponential size distribution. A further property of subordinators with GCMED distributions is that by solely observing small jumps information can be obtained on the larger jumps of the process.

2.3 Simulation

It is noted that subordination of a Riesz-Bessel motion by a stable subordinator is again a Riesz-Bessel motion with a change of its parameters. Precisely, if $RB(t)$ is a Riesz-Bessel motion with characteristic function (2.3) and S_t is a stable subordinator with Laplace transform $\exp\left(-tz^{\beta}\right)$ then $RB(S_t)$ is a Riesz-Bessel motion with $\alpha := \alpha\beta$ and $\gamma := \gamma\beta$. Thus, all Riesz-Bessel motions can be reduced to the subordination of one of the two following cases: If $\gamma < 0$ then it can be obtained by subordination of a Riesz-Bessel motion with $\beta := \alpha, \gamma := \gamma/\beta$ and $\alpha := 1$. If $\gamma > 0$ then it can be obtained by subordination of a Riesz-Bessel motion with $\beta := \alpha + \gamma, \alpha := \alpha/\beta$ and $\gamma := \gamma/\beta$.

Simulation of a general Riesz-Bessel motion can be carried out by simulating an appropriate stable subordinator and one of the special cases. Simulation of stable random variables is detailed in Weron [1996]. Details on the simulation of stable processes can be found in Janicki and Weron [1994]. Simulation of the two special cases of Riesz-Bessel motion is discussed below.

First we consider the case of $\alpha + \gamma = 1$. From Proposition 2.2.2 it follows that for $t \leq \alpha^{-\alpha}(1-\alpha)^{\alpha-1}\pi/\sin(\alpha\pi)$ a Riesz-Bessel random variable with $\alpha + \gamma = 1$ can be represented by

$$RB_t \stackrel{d}{=} \sigma_t Z, \quad \sigma_t^2 \stackrel{d}{=} t + \delta W/A, \tag{2.28}$$

where $Z \sim N(0,2)$, δ is a Bernoulli random variable with $\Pr(\delta = 0) = \exp(-t\gamma)$, W is a exponential random variable with unit mean and A is a random variable with density

$$\exp\left[-t\left(u^{\alpha}\left(1-u\right)^{1-\alpha}\cos\left(\alpha\pi\right)+u\right)\right]\sin\left(t\sin\left(\alpha\pi\right)u^{\alpha}\left(1-u\right)^{1-\alpha}\right)u^{-1}. \tag{2.29}$$

To use representation (2.28) it is necessary that $t \le \alpha^{-\alpha}\left(1-\alpha\right)^{\alpha-1}\pi/\sin\left(\alpha\pi\right)$. If $t > \alpha^{-\alpha}\left(1-\alpha\right)^{\alpha-1}\pi/\sin\left(\alpha\pi\right)$, then we can use the property that the Riesz-Bessel distribution is closed under convolution, that is,

$$RB\left(t\right)=\sum_{k=1}^{M}RB_{k}\left(t_{k}\right),\quad t=\sum_{k=1}^{M}t_{k} \tag{2.30}$$

with $t_k \le \alpha^{-\alpha}\left(1-\alpha\right)^{\alpha-1}\pi/\sin\left(\alpha\pi\right)$ for all k and $RB_k\left(t\right)$ are independent. Simulation from a density proportional to (2.29) can be achieved by a rejection sampling algorithm. The algorithm for simulating $RB\left(t\right)$ with $\alpha+\gamma=1$ is given below:

Algorithm 1: Case of $\alpha+\gamma=1$.

1. Repeat

   - Generate two independent random variates $U_1,\ U_2$ from the uniform distribution on $[0,1]$.
   - Set $V \leftarrow U_1^{1/\alpha}$.
   - Set $G \leftarrow t\sin\left(\alpha\pi\right)V^{\alpha-1}\times \max_{u\in[0,1]}\exp\left[-t\left(u^{\alpha}\left(1-u\right)^{1-\alpha}\cos\left(\alpha\pi\right)+u\right)\right]$.
   - Set $g \leftarrow \sin\left(t\sin\left(\alpha\pi\right)V^{\alpha}\left(1-V\right)^{1-\alpha}\right)\times$ $\exp\left[-t\left(V^{\alpha}\left(1-V\right)^{1-\alpha}\cos\left(\alpha\pi\right)+V\right)\right]/V$.
   - Until $U_2 < g/G$.

2. Generate an exponential random variable W with unit mean and a Bernoulli random variable δ such that $\Pr\left(\delta=0\right)=\exp\left(-t\gamma\right)$.

3. Set $V=t+\delta W/V$.

4. Generate a Gaussian random variable Z with mean zero and variance 2.

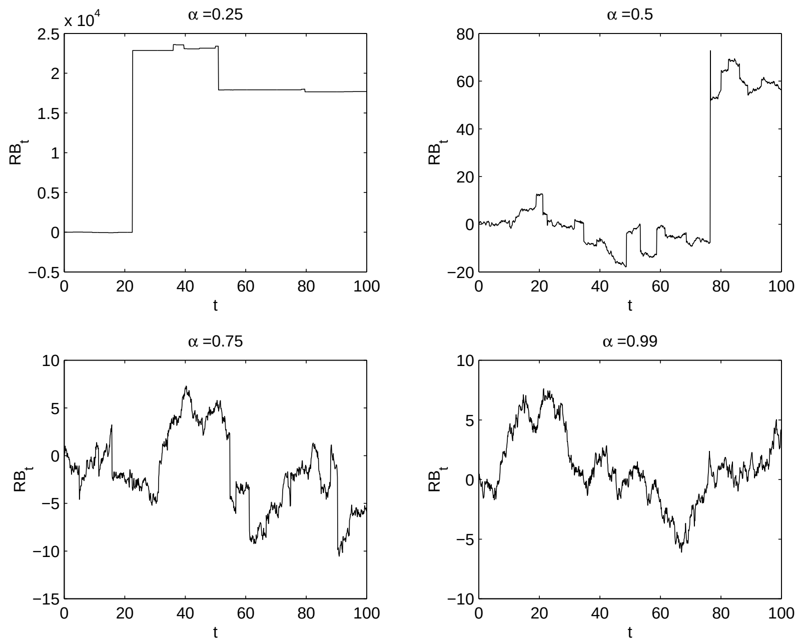


Figure 2.3: Sample paths of Riesz-Bessel motion with $\alpha + \gamma = 1$. Note that as $\alpha \to 1$ the size of the jumps in the process becomes smaller.

5. Return $Z\sqrt{V}$.

The expected number of iterations required to generate a single random variable is given by

$$\frac{t \sin(\alpha\pi)}{\alpha\pi\left(1 - \exp\left(-t\left(1-\alpha\right)\right)\right)} \max_{u \in [0,1]} \exp\left[-t\left(u^{\alpha}\left(1-u\right)^{1-\alpha}\cos\left(\alpha\pi\right) + u\right)\right].$$

It is seen that for moderate values of t the efficiency of the algorithm increases with α while for t small the efficiency is symmetric about $\alpha = 1/2$ and increases as α approaches 1 and 0. The sample paths of Figure 2.3 were generated using Algorithm 1 and time increments of 0.1.

Now we consider the case of $\alpha = 1$. This was briefly considered in Anh and McVinish [2004] where it was noted that the Lévy measure is given by

$$\nu\left(dx\right) = \frac{1}{\Gamma\left(-\gamma\right)}\left(x^{(1+\gamma)}e^{-x} + \left(1+\gamma\right)x^{-(2+\gamma)}e^{-x}\right)dx$$

from which it can be seen that RBLs is the sum of a tempered stable (TS) subordinator, also called a CGMY process (see Carr et al. 2002 for details) and an independent compound Poisson process with Gamma distributed jumps. In this chapter, the TS component will be simulated using a rejection sampling algorithm. The algorithm for simulating $RB(t)$ with $\alpha = 1$ is given below.

Algorithm 2: Case of $\alpha = 1$.

1. Generate a Poisson random variable N with mean t.
2. Generate a Gamma random variable G with shape parameter $-\gamma N$ and scale parameter 1.
3. Repeat
   - Generate a positive stable random variable V, with index $1+\gamma$ and scale parameter $t\cos(\pi(1+\gamma)/2)$.
   - Generate a random variable U from the uniform distribution of $[0,1]$.
   - Until $U < \exp(-V)$.
4. Generate a Gaussian random variable Z with mean zero and variance 2.
5. Return $Z\sqrt{V+G}$.

In this algorithm most computation is required for the rejection step which generates the TS component. The efficiency of the rejection algorithm will decrease quickly as t increase, through for t small the efficiency is near one. The sample paths of Figure 2.4 were generated using this algorithm with time increments of 0.1.

How to simulate random variables from the Riesz-Bessel distribution for the case of $\{\alpha \neq 1\} \cap \{\alpha + \gamma \neq 1\}$ from these special cases was described at the beginning of this section. The efficiency of these algorithms will only be reasonable when the parameter values are near these special cases, that is, only if α or $\alpha + \gamma$ are not too far from 1. Further research will hopefully provide more efficient algorithms.

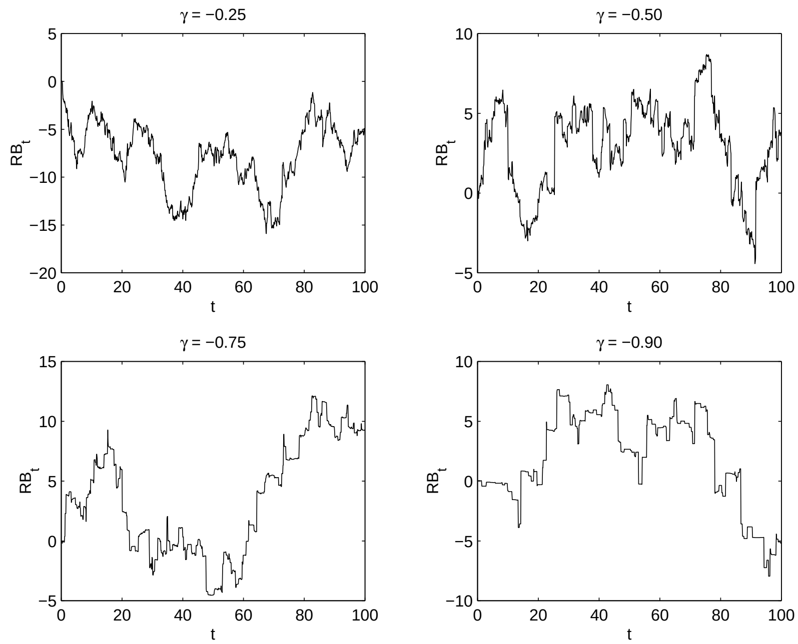


Figure 2.4: Sample paths of Riesz-Bessel motion with $\alpha = 1$. Note that as $\gamma \to -1$ RBLm approaches a compound Poisson process.

2.4 Parameter estimation

For distributions which lack a closed form for the density function, maximum likelihood estimation of parameters is generally not feasible and so one needs an alternative approach. For the stable and Linnik distributions, estimates based on the empirical characteristic function have proven useful. For a symmetric distribution, the empirical characteristic function is defined as

$$\hat{\phi}_n(\lambda) = \frac{1}{n} \sum_{j=1}^{n} \cos(\lambda X_j), \tag{2.31}$$

from which it is clear that

$$\begin{aligned} E\ \hat{\phi}_n(\lambda) &= \phi(\lambda), \\ \operatorname{cov}\left(\hat{\phi}_n(\lambda_1), \hat{\phi}_n(\lambda_2)\right) &= \frac{1}{2n}\left[\phi(\lambda_1 - \lambda_2) + \phi(\lambda_1 + \lambda_2) - 2\phi(\lambda_1)\phi(\lambda_2)\right]. \end{aligned} \tag{2.32}$$

The strong law of large number implies that $\hat{\phi}_n(\lambda) \to \phi(\lambda)$, almost surely, and hence estimators based on the empirical characteristic function are usually strongly consistent.

A number of estimators based on the empirical characteristic function are described in the literature. In the context of the stable distribution, Koutrouvelis [1980] proposed the least squares regression of $\log(-\log|\hat{\phi}_n(\lambda)|^2)$ on $\log|t|$, Press [1972] suggested a method of moments, Paulson et al. [1975] gave a minimal distance method of estimation, where the estimate is given by

$$\min_{\theta} \int_{-\infty}^{\infty} \left| \hat{\phi}_n(\lambda) - \phi(\lambda) \right|^2 e^{-\lambda^2} d\lambda \tag{2.33}$$

and the integral is approximated by Gauss-Hermite quadrature, and Feuerverger and McDunnough [1981a,b] proposed the k–L procedure. The method of moments type estimate, minimal distance estimate and k–L procedure can be written as the solution to an estimating equation

$$\sum_i a(\lambda_i;\theta)\left(\hat{\phi}_n(\lambda_i) - \phi(\lambda_i;\theta)\right) = 0 \tag{2.34}$$

for particular choices of $a(\lambda;\theta)$. The theory of quasi-likelihood provides a framework in which an optimal choice for $a(\lambda;\theta)$ can be made within a given class of estimating functions such as (2.34) (see Heyde 1997). Taking the sequence $\{\lambda_i\}$ as fixed, the optimal estimating equation within the class (2.34) is

$$Z(\theta)^T V^{-1}(\theta)\left(\hat{\phi}_n - \phi(\theta)\right) = 0, \tag{2.35}$$

where

$$Z(\theta)_{ij} = \frac{\partial \phi(\lambda_i;\theta)}{\partial \theta_j}, \quad V(\theta)_{ij} = \operatorname{cov}\left(\hat{\phi}_n(\lambda_i), \hat{\phi}_n(\lambda_j)\right) \tag{2.36}$$

(Heyde 1997, p. 15), which is precisely the estimator obtained from the k–L procedure. It was proved by Feuerverger and McDunnough [1981a] that this estimator can be made to have arbitrarily high asymptotically efficiency. Using the quasi-likelihood framework it is possible to see that for any finite sample size the estimator is an approximate maximum likelihood estimator. Note that Equation (2.35) can be written in the form

$$\frac{1}{n}\sum_{j=1}^{n}\sum_i a(\lambda_i;\theta)\left(\cos(\lambda_i X_j) - \phi(\lambda_i;\theta)\right) = 0. \tag{2.37}$$

The quasi-score function minimises the distance to the true score function, that is, it minimises

$$E\left[\left(\frac{\partial \log f\left(X;\theta\right)}{\partial\theta}-\sum_{i} a\left(\lambda_i;\theta\right)\left(\cos\left(\lambda_i X_j\right)-\phi\left(\lambda_i;\theta\right)\right)\right)^2\right], \tag{2.38}$$

(see Heyde 1997, p. 13 for details). Now consider the space $L_2\left(f\right)$ of functions which are square integrable with respect to $f\left(x;\theta\right)$. This Hilbert space can be decomposed into the subspace of functions that are constant almost everywhere, denoted by H, and the subspace of functions orthogonal to it, that is

$$L_2\left(f\right)=H\oplus H^{\perp}. \tag{2.39}$$

Clearly, the score function belongs to the subspace $H^{\perp}$. It is known that the space of trigonometric functions is dense in $L_2\left(f\right)$. The function $\cos\left(\lambda x\right)-\phi\left(\lambda;\theta\right)$ is in $H^{\perp}$ as it is the result of $\cos\left(\lambda x\right)$ being made orthogonal to H. It follows that the functions $\cos\left(\lambda x\right)-\phi\left(\lambda;\theta\right)$ are dense in $H^{\perp}$ and hence (2.38) can be made arbitrarily small. In summary, the estimating equation (2.35) can be made arbitrarily close to the true score function by taking a sufficiently fine sequence of $\{\lambda_i\}$.

The estimating equation (2.35) can be solved iteratively given a good initial estimate θ_0 as follows:

$$Z\left(\theta_m\right)^T V\left(\theta_m\right)^{-1} Z\left(\theta_m\right)\delta_m=Z\left(\theta_m\right)^T V\left(\theta_m\right)^{-1}\left(\hat{\phi}_n-\phi\left(\theta_m\right)\right), \tag{2.40}$$

$$\theta_{m+1}=\theta_m+\delta_m. \tag{2.41}$$

The resulting estimator is consistent, asymptotically normal with covariance matrix given by

$$E\ (\hat{\theta}-\theta)(\hat{\theta}-\theta)^T=\left[Z\left(\theta\right)^T V^{-1}\left(\theta\right)Z\left(\theta\right)\right]^{-1}. \tag{2.42}$$

Equation (2.40) is just generalised least squares and so the method can be easily implemented in most statistical packages. As the parameters of the

Riesz-Bessel distribution need to satisfy certain constraints, it is advisable to transform the parameters to remove these constraints. For example, set

$$\alpha^* = \log\left(\frac{\alpha}{1-\alpha}\right), \quad \gamma^* = \log\left(\frac{\alpha+\gamma}{1-\alpha-\gamma}\right), \quad t^* = \log(t), \tag{2.43}$$

and the appropriate changes made to the matrix Z.

The only choice to be made is the sequence $\{\lambda_i\}$, which is a common problem to the other methods previously mentioned. A large number of ordinates at which the empirical characteristic function is computed will lead to more efficient estimates; however there is also an increase in computational cost, and stability problems may arise with a near singular matrix. A simulation study was performed to assess the method in small samples for the Riesz-Bessel distribution. For each value of the parameter, the estimation scheme was applied 50 times to a sample of size 250. The $\{\lambda_i\}$ was taken to be a sequence from 0.1 to 5 with spacing of 0.1 The results of the simulation study are reported in Table I.

Table I: Performance of estimator in the simulation study.

(α, γ, t)	Average	Standard Error
(0.9,-0.8,1)	(0.9050, -0.7936, 0.9930)	(0.0766, 0.1210, 0.1764)
(0.9,-0.6,1)	(0.9229, -0.6045, 0.9933)	(0.0673, 0.1406, 0.1712)
(0.9,-0.4,1)	(0.9524, -0.4247, 1.0001)	(0.0653, 0.1855, 0.1587)
(0.9,-0.2,1)	(0.9637, -0.2497, 1.0008)	(0.0526, 0.1879, 0.1431)
(0.7,-0.6,1)	(0.7276, -0.6253, 1.0083)	(0.0884, 0.1369, 0.1709)
(0.7,-0.4,1)	(0.7579, -0.4476, 1.0176)	(0.0797, 0.1513, 0.1553)
(0.7,-0.2,1)	(0.8269, -0.3072, 1.0080)	(0.0611, 0.1698, 0.1717)
(0.7,0.2,1)	(0.7098, 0.1376, 1.0482)	(0.0661, 0.1958, 0.1617)
(0.5,0.2,1)	(0.4826, 0.2672, 0.9589)	(0.0664, 0.2142, 0.1808)
(0.5,0.4,1)	(0.5181, 0.3335, 1.0633)	(0.0739, 0.2074, 0.1889)

We note that the bias appears to be considerable for small negative values of γ. The bias does not appear to be as great for small positive values of γ.

2.5 Application to financial data

In this application, the Riesz-Bessel distribution is fitted to six typical financial time series, namely the time series of the Japanese yen (JY), the Deutsche mark (DM), the British pound (BP) and the French franc (FF), all against the US dollar, the Dow Jones index (DJ) and the IBM stock price (IBM), using the method described in the previous section. We will also look at the scaling behaviour of these time series based on the results of the estimation.

The above daily time series cover the following periods, yielding the sample sizes cited in brackets:

JY, 12 December 1983 - 8 October 2001 ($n = 4511$);
DM, 4 January 1971 - 8 April 1996 ($n = 6334$);
BP, 12 December 1983 - 8 October 2001 ($n = 4511$);
FF, 4 January 1971 - 31 December 1998 ($n = 6429$);
DJ, 2 January 1990 - 27 August 2002 ($n = 3193$);
IBM, 2 January 1990 - 3 September 2002 ($n = 3197$).

The FF time series together with its returns $\log FF(t) - \log FF(t-\tau)$ at different lag lengths τ are plotted in Figures (2.5)-(2.8) as illustrative examples. It is observed that 1-day returns have dense fluctuations with many large spikes. When τ increases, the paths thin out (i.e., the time series display more correlation due to overlapping information), and have larger spikes; these properties result in longer tails and a higher peak at 0 in the respective sample density functions. This observation confirms that a non-Gaussian model is more appropriate to fit the probability density functions of returns data as commonly reported in the literature.

We assume that the financial processes are exponential transformations of a Riesz-Bessel-Lévy motion whose distribution is parameterised as

$$\hat{p}(t, \lambda) = \exp\left[-\kappa t |\lambda|^{2\alpha} (c^2 + \lambda^2)^{\gamma}\right], \quad \lambda \in \mathbb{R},$$

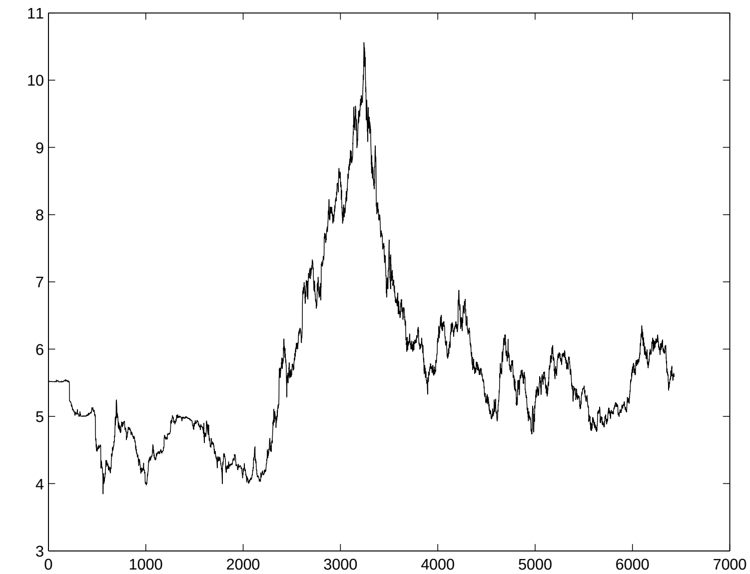


Figure 2.5: The daily time series FF(t) of French franc over the period 4 January 1971 to 31 December 1998

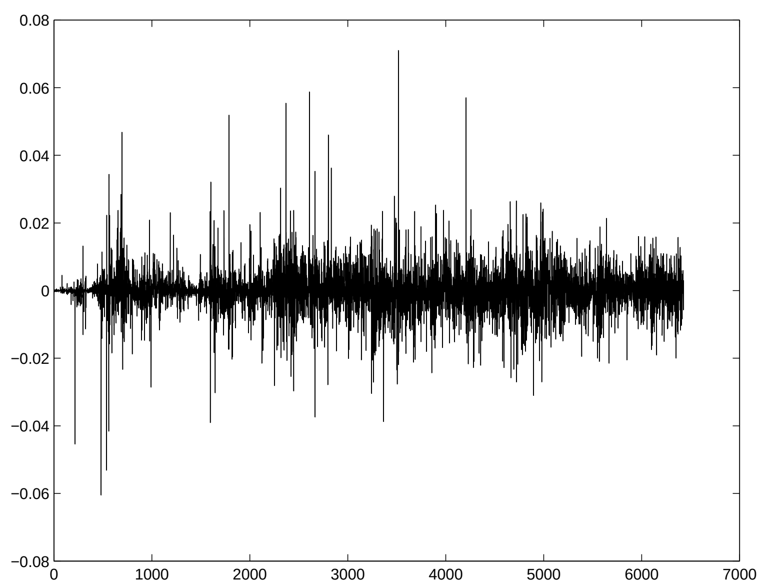


Figure 2.6: The 1-day returns of log FF(t)- log FF(t-1) of the time series of FF

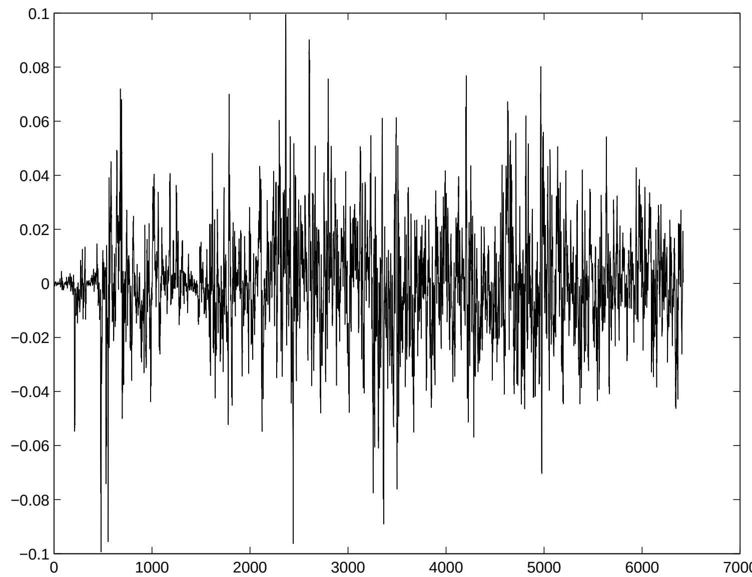


Figure 2.7: The 8-day returns of log FF(t)- log FF(t-8) of the time series of FF

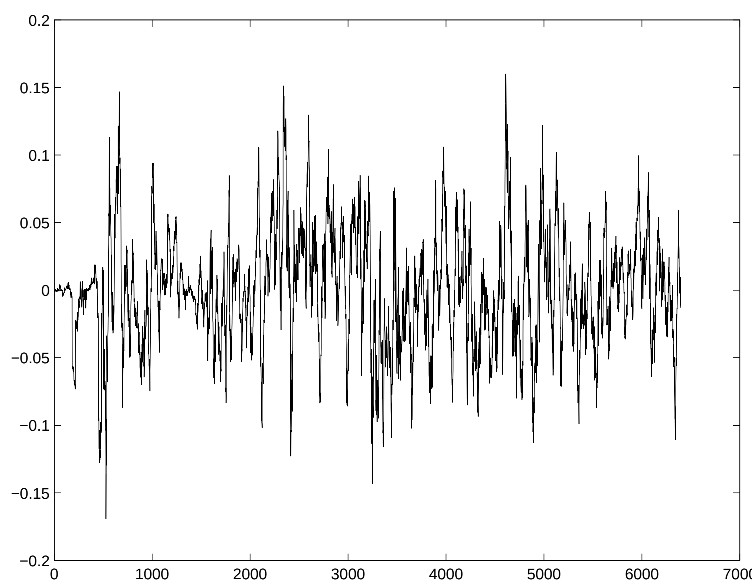


Figure 2.8: The 16-day returns of log FF(t)- log FF(t-32) of the time series of FF

where the additional parameter κ is introduced to allow t to be a time parameter. The method of Section 2.4 is then performed to estimate the parameters κ, α, γ and c for the daily returns with $t = 1$. Admittedly, this ignores certain dependencies that financial data are known to display, however, the aim of this application is to demonstrate the fit of the distribution to some real data. Applying the estimation method described above the following parameter estimates were obtained with $t = 1$:

Financial time series	κ	α	γ	c
DJ	0.0115	0.9590	-0.4878	198.0287
DM	0.0908	0.9809	-0.7270	282.4812
FF	0.1300	0.9380	-0.7158	331.0499
BP	0.0004	0.9999	-0.2587	130.9644
IBM	0.0011	0.9999	-0.2295	32.4711
JY	0.0057	0.9999	-0.4869	181.1382

The standard errors are also given in the following table which corresponds to the table above:

Financial time series	κ	α	γ	c
DJ	0.0040	0.0183	0.1397	57.1182
DM	0.0183	0.0101	0.0551	26.9374
FF	0.0288	0.0148	0.0669	39.9615
BP	0.0001	0.0018	0.0461	23.2543
IBM	0.0001	0.0032	0.0322	6.6078
JY	0.0015	0.0015	0.0703	24.1096

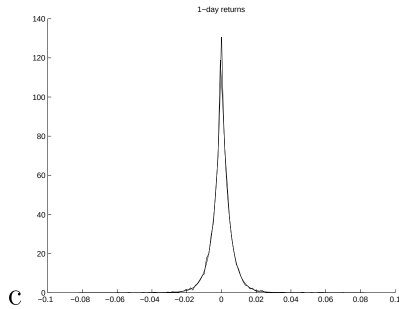
It is noted that the values of α are close to 1 in all cases, those of $\alpha + \gamma$ are less than 1, and the values of γ for DM and FF are closer to -1 than the other time series. The value of α being very close to 1 indicates that the distribution appears to have finite variance but not Gaussian as γ is significantly different from 0. These scenarios agree quite well with those depicted in Figure 2.4. Especially, the DM and FF time series seem to approach a compound Poisson process. A plot of the fitted Riesz-Bessel density with a non-parametric density estimate of the data is given below.

As noted in Section 2.2, the parameters α and γ act together to indicate the behaviour of the time series at small lags τ. Under the hypothesis of a Riesz-Bessel-Lévy motion, the distribution can be rescaled to converge to a symmetric $2(\alpha+\gamma)$-stable distribution. To evaluate this hypothesis, we compute the histograms, of the FF and JY series in particular, and the corresponding Riesz-Bessel densities, which are obtained by inverting the characteristic function with the parameter estimates from daily returns and letting t change. The results are plotted in Figures 2.9 and 2.10. It is seen that the hypothesis seems consistent with the JY series. On the other hand, for the FF series, while the Riesz-Bessel density provides good fit for 1-day returns, there appears to be departure from the assumption of a Lévy motion. Due possibly to second- and/or higher-order correlations in the data, 4-, 8- and 16-day returns of the FF series are, on average, more concentrated at 0 than predicted by Lévy motion with Riesz-Bessel density.

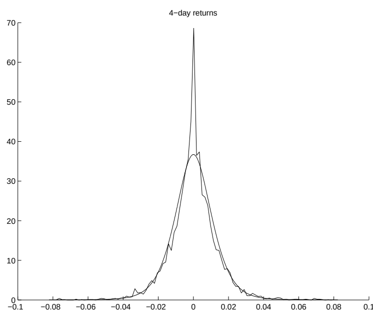
The rescaled results, where the probability density p is rescaled and plotted as $\tau^{2(\alpha+\gamma)}p(X_{rescaled})$ against $X_{rescaled}$, are displayed in Figures 2.11 and 2.12 for the FF series and the JY series respectively. Here,

$$X_{rescaled} = \frac{X(t,\tau)}{\tau^{1/(2(\alpha+\gamma))}},$$

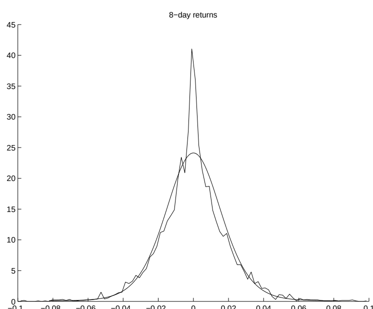
with $X(t,\tau) = \log S(t) - \log S(t-\tau)$, $S(t)$ being any of the given time series. If the time series follows a Riesz-Bessel-Lévy motion, the probability densities of their returns at different lags will collapse to a limiting density, namely, that of the symmetric $2(\alpha+\gamma)$-stable distribution. This pattern seems to hold for the series JY, BP, DJ and IBM studied, while FF and DM present a clear departure from it. As noted above for the FF series, which is also apparent in the DM series, there is evidence of second- and/or higher-order correlations. In this case, a model which exhibits both Lévy-type behaviour and short- or long-range dependence of the process is warranted.



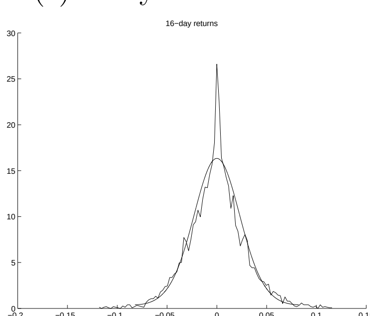
(a)1-day returns of FF



(b)4-day returns of FF

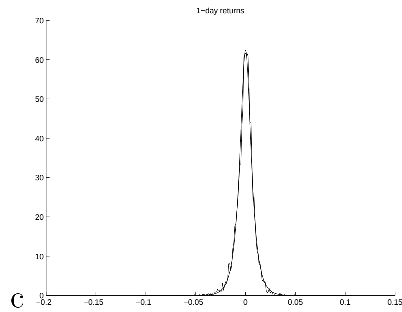


(c) 8-day returns of FF

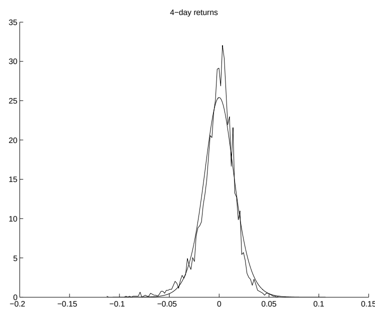


(d)16-day returns of FF

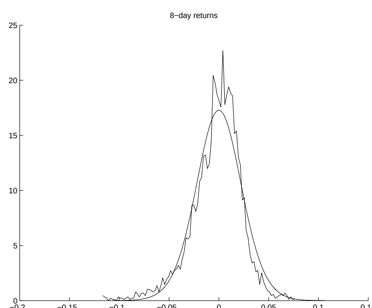
Figure 2.9: The sample densities(rugged curves) of the returns of French franc series at the different lags and their estimation (smooth curves)



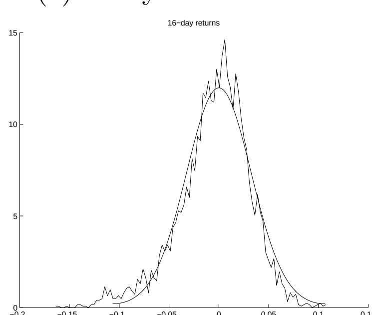
(a)1-day returns of JY



(b)4-day returns of JY



(c) 8-day returns of JY



(d)16-day returns of JY

Figure 2.10: The sample densities(rugged curves) of the returns of Japanese Yen series at the different lags and their estimation (smooth curves)

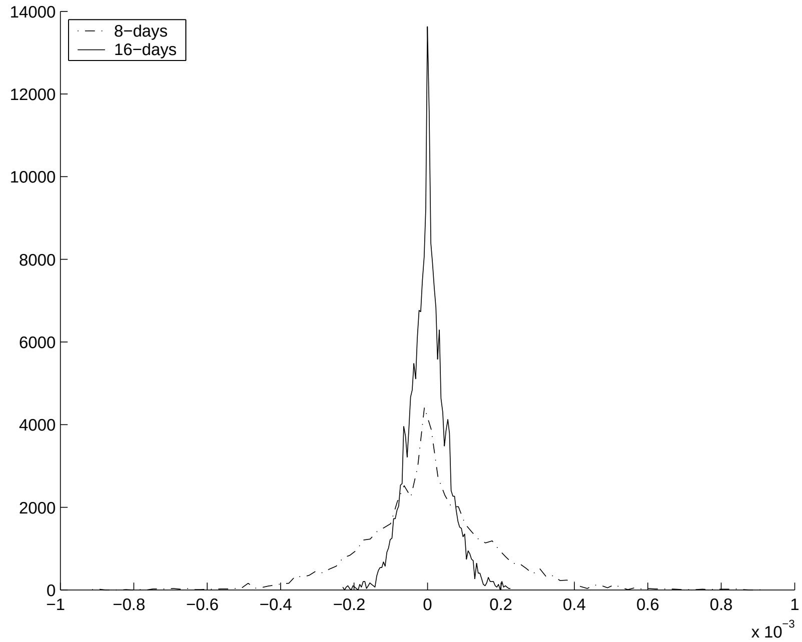


Figure 2.11: The rescaled sample densities of the returns of FF at 8, 16 days based on the estimates of the Riesz-Bessel distribution

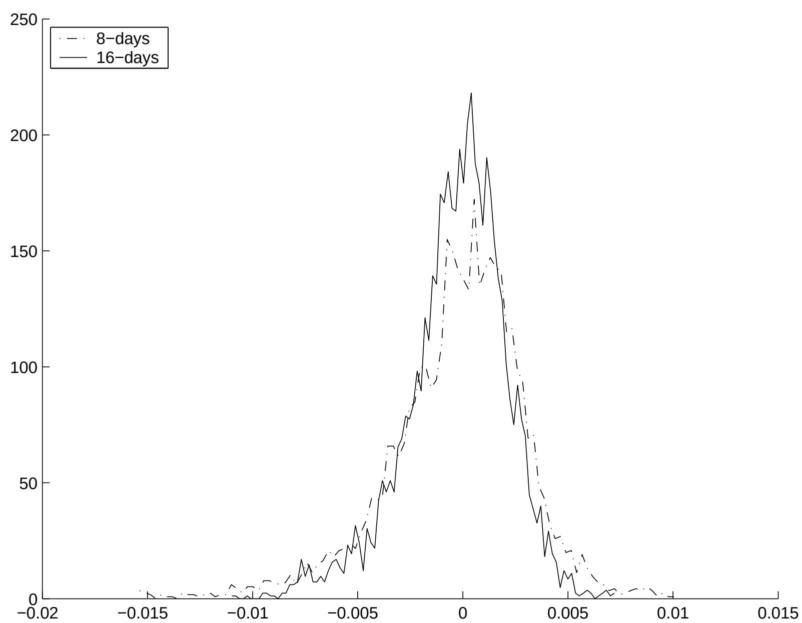


Figure 2.12: The rescaled sample densities of the returns of JY at 8, 16 days based on the estimates of the Riesz-Bessel distribution

Chapter 3

Incorporation of memory into the Black-Scholes-Merton theory and estimation of volatility

3.1 Introduction

This chapter considers a dynamic model of complete financial markets in which the prices of European calls and puts are given by the Black-Scholes formula. The model has memory and can distinguish between historical volatility HV and implied volatility IV. A new method is then provided to estimate the implied volatility from the model. Our IV estimate for real data such as S&P 500 is almost always larger than the value obtained from traditional methods, namely HV(1). Since IV>HV(1) more often than IV<HV(1) in real markets, our result has the effect of narrowing the gap between HV and IV, and making the Black-Scholes-Merton theory more consistent with empirical data.

A well-known drawback of the Black–Scholes model is that it does not explain the difference between historical volatility HV and implied volatility

IV. In Anh and Inoue [2005] (see also Anh, Inoue, and Kasahara [2005]), a dynamic model of complete financial markets was introduced, in which the prices of European calls and puts are given by the Black–Scholes formula but HV and IV may be different. The price process $(S(t) : t \in \mathbf{R})$ of this model is defined via an AR(∞)-type equation for the log-price process $Z(t) := \log S(t)$. In the simplest case, this equation takes the form

$$\frac{dZ}{dt}(t) - m = -\int_{-\infty}^{t} pe^{-q(t-s)} \left\{ \frac{dZ}{dt}(s) - m \right\} ds + \sigma \frac{dW}{dt}(t), \qquad (3.1)$$

where $m \in \mathbf{R}$, $\sigma, q \in (0, \infty)$, $p \in (-q, \infty)$ and $(W(t) : t \in \mathbf{R})$ is a one-dimensional standard Brownian motion on a probability space $(\Omega, \mathcal{F}, P)$. Equation (3.1) can be solved explicitly to obtain, for $t \in \mathbf{R}$,

$$S(t) = S(0) \exp \left\{ mt - \sigma \int_{0}^{t} \left(\int_{-\infty}^{s} pe^{-(p+q)(s-u)} dW(u) \right) ds + \sigma W(t) \right\}. \quad (3.2)$$

Compared with the Black–Scholes model, the model defined by (3.1) has two additional parameters p and q which describe the *memory of the market*. When $p = 0$, Eq. (3.2) produces the Black–Scholes price process given by

$$S(t) = S(0) \exp \left(mt + \sigma W(t) \right). \qquad (3.3)$$

The log-price process $(Z(t) : t \in \mathbf{R})$ of (3.1) is a Gaussian process with stationary increments which has memory. In view of some recent studies, it is unlikely that observed stock prices and stock indices are log-normal (Eberlein and Keller 1995, Bibby and Sørensen 1997, Barndorff-Nielsen 1998, Rydberg 1999, Barndorff-Nielsen and Shepard 2001). However, Anh and Inoue [2005] neglected such a consideration, recalling the fact that, after all, the Black–Scholes model, which is log-normal, is still dominant among a large number of market models used by practitioners. Their intention was to incorporate a crucial aspect, namely, memory, into the Black–Scholes model, without losing its usefulness and simplicity, particularly, the *Black–Scholes formula*. This chapter continues this work and will provide clear evidence that financial markets have memory and that the model defined by (3.1) can capture some movement of stock indices reasonably well.

By Theorem 3.3 in Anh and Inoue [2005] (see also Theorem 3.3.1 below), the constant σ of (3.1) is equal to the implied volatility of the model, defined via the Black–Scholes formula. Notice that, in the model defined by (3.1), the prices of European calls and puts are given by the Black-Scholes formula as in the Black–Scholes model. We now define

$$\mathrm{HV}(t-s) := \sqrt{\frac{\mathrm{Var}\{\log(S(t)/S(s))\}}{t-s}} \qquad (t > s \geq 0). \tag{3.4}$$

If $(S(t))$ is Black–Scholes, then we have $\mathrm{HV}(t) = \sigma$ for every $t > 0$. However, in the present model, we have $\mathrm{HV}(t) = f(t)$, where the function $f(t)$ is given by

$$f(t) = \sigma\sqrt{\frac{q^2}{(p+q)^2} + \frac{p(2q+p)}{(p+q)^3} \cdot \frac{(1-e^{-(p+q)t})}{t}} \qquad (t > 0) \tag{3.5}$$

(see Examples 4.3 and 4.5 in Anh and Inoue 2005). We see that if $p > 0$, then $f(t)$ is decreasing, while if $p < 0$, then $f(t)$ is increasing. Moreover, we have

$$\lim_{t\to 0+} f(t) = \sigma, \qquad \lim_{t\to\infty} f(t) = \frac{\sigma q}{p+q}.$$

We estimated $\mathrm{HV}(t)$ $(t = 1, 2, 3, \ldots)$ from real market data such as closing values of S&P 500 index. Our finding is that $\mathrm{HV}(t)$ is not constant, and very often reveals features in agreement with those described above (Section 3.2). We fitted the function $f(t)$ by using nonlinear least squares, and found that $f(t)$ approximates $\mathrm{HV}(t)$ rather well when the market is stable (Section 3.4).

This model suggests a new method for historical estimation of implied volatility (Section 3.4). In fact, in the traditional method, it is $\mathrm{HV}(1)$ that is regarded as the historical estimate of volatility (see, e.g., Hull 1997). The choice of time lag 1 (day) has been adopted because it can be conveniently computed from closing data; after all, in the Black–Scholes model, $\mathrm{HV}(t)$ is constant, whence the choice of t is not so relevant. Thus, in this traditional method, only one value of $\mathrm{HV}(t)$ is used to estimate volatility. However, as stated above, since observed market data show that $\mathrm{HV}(t)$ is not constant, such a method would give only a partial description of the market. We propose a new method for the estimation of implied volatility based on the model defined by (3.1), in which we use several values of $\mathrm{HV}(t)$. As stated above, we can fit $f(t)$ to $\mathrm{HV}(t)$ rather well by using nonlinear least squares, and in so doing we obtain estimated values of σ, p and q. Since σ is equal to the implied volatility in the model defined by (3.1), this estimated value of σ is *our historically estimated value of implied volatility*.

3.2 Estimating $\mathrm{HV}(t)$ from historical data

The definition of $\mathrm{HV}(t)$ is given by (3.4). It should be noted that the variance in (3.4) is not defined with respect to the equivalent martingale measure but with respect to the physical probability measure P. To estimate $\mathrm{HV}(t)$, we use stock prices observed at a fixed time interval. Let N be the number of trading days in the interval. For $i = 1, 2, \ldots, N$, let S_i be the closing price on the i-th day. We fix $t \in \{1, 2, \ldots, \}$, which is reasonably smaller than N. We define $u_i = u_{i,t}$ by

$$u_i := \log\left(S_{i+t}/S_i\right) \qquad (i = 1, 2, \ldots, N-t).$$

Then the estimate of $\mathrm{HV}(t)$, for which we write as $\mathrm{hv}(t)$, is given by

$$\mathrm{hv}(t) = 100\sqrt{\frac{252}{t(N-t-1)}\sum_{i=1}^{N-t}(u_i - \overline{u})^2}, \tag{3.6}$$

where $\overline{u}$ is the sample mean of u_i defined by

$$\overline{u} := \frac{1}{N-t}\sum_{i=1}^{N-t} u_i.$$

The number 252 in (3.6) is the number of trading days in one year, and has the effect of converting the return into that per annum. On the other hand, the number 100 in (3.6) gives the return in terms of percentage.

Figure 1 shows the plotting of $(t, \mathrm{hv}(t))$ $(t = 1, 2, \ldots, 25)$ for S&P 500 closing indices from 14 January through 22 May 2002, for which we have $N = 90$ trading days. We remark that the market was relatively stable during this period. If the market followed the Black–Scholes model (3.3), then $\mathrm{hv}(t)$

would be approximately a constant, that is, the implied volatility. However, Figure 1 clearly shows that this is not the case; we cannot regard $\mathrm{hv}(t)$ as constant here. Thus considering the market as Black–Scholes and $\mathrm{hv}(1)$ as the historical estimate of implied volatility, as still in common practice, are both problematic.

3.3 A financial market model with memory

We briefly recall from Anh and Inoue [2005] some basic results on the financial market model defined by (3.1). We also refer to Anh, Inoue, and Kasahara [2005], where the expected log-utility maximization problem for the model is solved using a new method in prediction theory.

Let $m \in \mathbf{R}$, $\sigma, q \in (0, \infty)$, $p \in (-q, \infty)$ and $(W(t) : t \in \mathbf{R})$ be a Brownian motion defined on $(\Omega, \mathcal{F}, P)$ such that $W(0) = 0$. We consider a price process $S(t)$ of the form

$$S(t) = S(0) \exp Z(t) \qquad (t \in \mathbf{R}), \tag{3.7}$$

where $S(0)$ is a positive constant and $(Z(t) : t \in \mathbf{R})$ is a mean-square continuous process with stationary increments defined on $(\Omega, \mathcal{F}, P)$ such that $Z(0) = 0$. We also assume that the process $(Z(t))$ is the solution to Eq. (3.1) in the random distribution sense. Then $(S(t) : t \in \mathbf{R})$ is given by (3.2).

The integral on the right-hand side of (3.1) describes the memory of the market. For a rough interpretation of this concept, let us put $a(t) = pe^{-qt}$ for $t > 0$. Then for ϵ sufficiently small, we would have approximately

$$\int_u^{u+\epsilon} a(t-s) \left\{ \frac{dZ}{dt}(s) - m \right\} ds \approx a(t-u)\{Z(u+\epsilon) - Z(u) - m\epsilon\}.$$

Thus, in this model, if $p > 0$, then a good performance of past prices, especially the most recent one, makes the market cautious of the next price fall.

Let $T \in (0, \infty)$ be the maturity date. To define the financial market, we use the following filtration:

$$\mathcal{G}_t := \sigma(\sigma(S(u) : 0 \le u \le t) \cup \mathcal{N}^T) \qquad (0 \le t \le T),$$

where $\mathcal{N}^T$ is the class of all P-negligible sets from $\sigma(S(u) : 0 \leq u \leq T)$. Thus, in this model, we consider only $\mathcal{G}_T$-measurable contingent claims and $(\mathcal{G}_t)$-adapted strategies. This setting is the same as that of the Black–Scholes model. Also, as in the Black–Scholes model, we suppose that we are in a market in which the riskless asset price $S_0(t)$ follows $S_0(t) = \exp(rt)$ for $t \geq 0$, where r is a nonnegative constant instantaneous rate. We put

$$\mathcal{F}_T := \bigcap_{\epsilon>0} \sigma\left[\sigma(W(s) : -\infty < s \leq T + \epsilon) \cup \mathcal{N}\right],$$

where $\mathcal{N}$ is the class of all P-negligible sets from $\mathcal{F}$.

We have the following theorem.

Theorem 3.3.1 (Anh and Inoue 2005, Theorem 3.3). *The market*

$$\{(\Omega, \mathcal{F}_T, P), (\mathcal{G}_t)_{0\leq t\leq T}, (S(t))_{0\leq t\leq T}, (S_0(t))_{0\leq t\leq T}\}$$

as defined above is complete. In this market, the prices of European calls and puts with maturity T are given by the Black–Scholes formula, and the constant σ serves as the implied volatility.

By Theorem 3.3.1, we can define the implied volatility of the financial market model above by the Black–Scholes formula, as in the Black–Scholes model. Also we see that the constant σ in (3.1) is equal to the implied volatility of the market model.

3.4 Model fitting and estimation of implied volatility

If a financial market could be regarded as following the model defined by (3.1), then, by the results stated in Section 3.3, we may use the estimate of σ as that of the implied volatility. To estimate σ, we can use $\mathrm{HV}(t)$ defined by (3.4). By Examples 4.3 and 4.5 in Anh and Inoue [2005], we have $\mathrm{HV}(t) = f(t)$, where the function $f(t) = f(t; \sigma, p, q)$ is given by (3.5). We fit $f(t)$ to the estimate

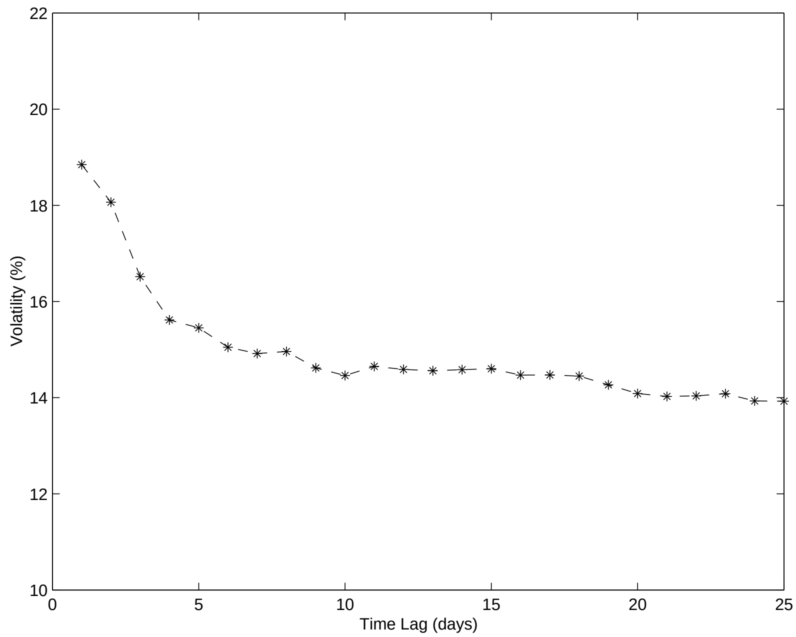


Figure 3.1: Plotting of hv(t) for S&P500, 14 Jan 02- 22 May 02, 90 trading days. Here $\mathrm{D}_{\max} = 25$

hv(t) of HV(t) by using nonlinear least squares, and, in this way, we obtain the estimated values of σ, p and q.

For example, as in Section 3.2, we consider the S&P 500 closing indices of 90 trading days from 14 January through 22 May 2002, and use the values of hv(t) plotted in Figure 3.1 By numerically calculating the triple (σ, p, q) that minimizes

$$\sum_{t=1}^{25} \{\mathrm{hv}(t) - f(t; \sigma, p, q)\}^2 ,$$

we obtain the following result:

$$\sigma = 21.7\,(\%)\,, \quad p = 0.384, \quad q = 0.660.$$

Figure 3.2 plots the fitted function $f(t) = f(t; 21.7, 0.384, 0.660)$ and hv(t). It is seen that $f(t)$ approximates hv(t) very well. In this case, we have ARN $=$ 0.195, where ARN is the *average residue norm* given by

$$\mathrm{ARN} = \sqrt{\frac{1}{D_{\max}} \sum_{t=1}^{D_{\max}} \{\mathrm{hv}(t) - f(t; \sigma_0, p_0, q_0)\}^2}, \tag{3.8}$$

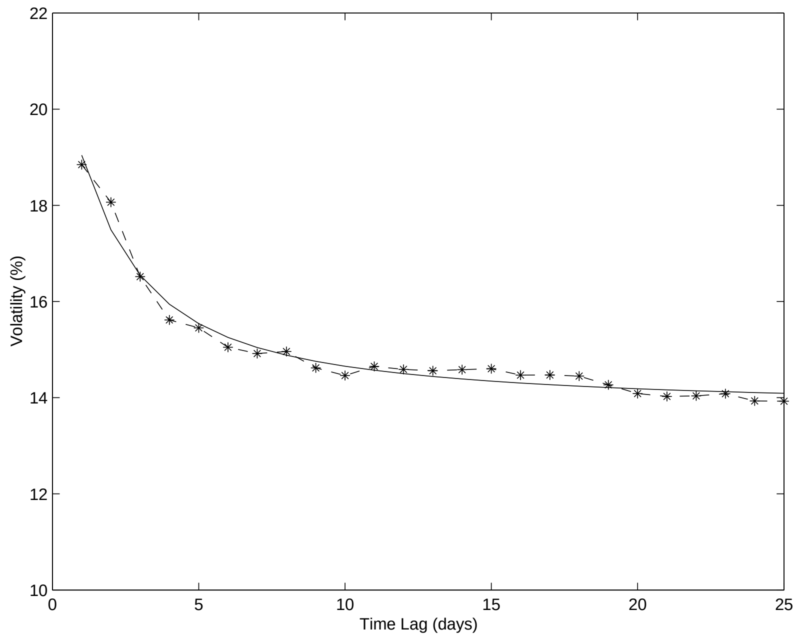


Figure 3.2: Fitting of f(t)(solid line) to hv(t)(dashed line)for S&P500,14 Jan 02 - 22 May 02, 90 trading days. Here $D_{\max} = 25$

with $(\sigma_0, p_0, q_0) = (21.7, 0.384, 0.660)$ and $D_{\max} = 25$. The value of ARN describes how close $f(\cdot)$ is to $\mathrm{hv}(\cdot)$ in average. We remark that, in the Black–Scholes framework, $f(t) = \sigma$, whence we cannot reasonably fit it to such $\mathrm{hv}(t)$. Our historical estimate of implied volatility is 21.7%, while the traditional one based on the Black–Scholes model is $\mathrm{hv}(1) = 18.8\%$. Thus the former is larger than the latter by $21.7 - 18.8 = 2.9\%$.

In general, the procedure to estimate σ, p and q can be described as follows:

1. For the closing prices S_i $(i = 1, 2, \ldots, N)$, we calculate $\mathrm{hv}(t)$ by (3.6) up to sufficiently large t.

2. We choose a suitable positive integer $D_{\max}$.

3. We numerically calculate the triple (σ, p, q) that minimizes

$$\sum_{t=1}^{D_{\max}} \left\{\mathrm{hv}(t) - f(t; \sigma, p, q)\right\}^2 . \tag{3.9}$$

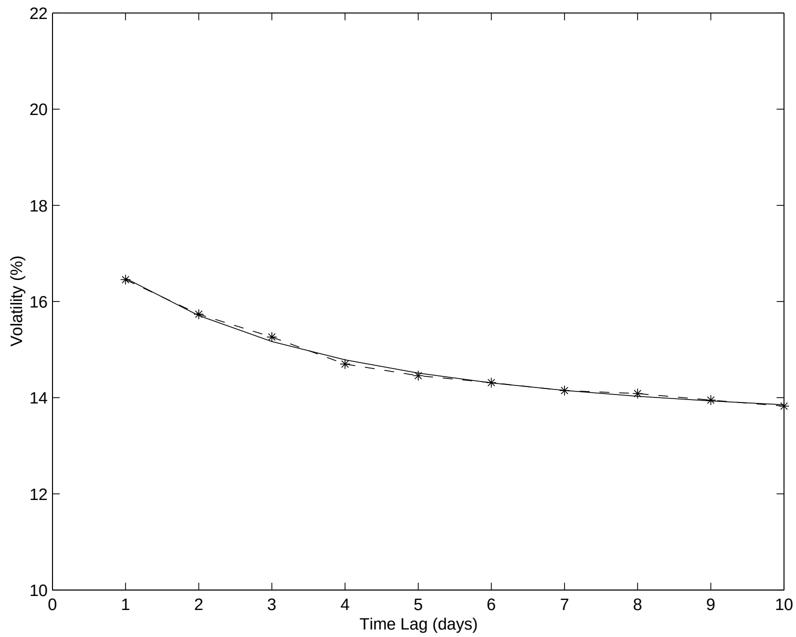


Figure 3.3: Fitting of f(t)(solid line) to hv(t)(dashed line)for S&P500,24 Dec 01 - 03 May 02, 90 trading days. Here $\mathrm{D_{max}} = 10$

Figure 3.3 shows the plotting of the fitted function $f(t) = f(t; 17.6, 0.176, 0.514)$ to hv(t) for the S&P 500 closing index data, 24 December 2001–3 May 2002, 90 trading days, with $D_{\max} = 10$.

Table 1 provides the resulting values of σ, p, q and ARN for the 90-day data sets of S&P 500 closing indices ending in May 2002. Thus $N = 90$, while we choose $D_{\max} = 25$. For example, the period 14Jan02–22May02 in Table 1 means that we use the closing indices of the 90 trading days from 14 January 2002 through 22 May 2002. Each value of ARN in the table is given by (3.8) with $D_{\max} = 25$, where σ_0, p_0 and q_0 are the corresponding estimates of σ, p and q, respectively.

The estimates of σ, p and q depend on the choice of $D_{\max}$. In Table 2, we provide the estimates obtained by using the same S&P 500 index data as those in Table 1 but using $D_{\max} = 10$ instead of $D_{\max} = 25$. As Tables 1 and 2 suggest, the estimate of σ tends to increase as we increase $D_{\max}$. Choosing a suitable value for $D_{\max}$ may be regarded as a model selection problem, which is not easy. If we choose a small value of $D_{\max}$, the fitting tends to be good. On the other hand, by choosing a large value of $D_{\max}$, we can incorporate the global shape of the graph of $\mathrm{hv}(t)$, which is likely to be closely related to the option pricing, into consideration. So here we have two contradicting goals, and this situation is similar to that of usual model selection.

Tables 3 and 4 provide the results for 90-day data sets of S&P 500 indices ending in April 2002, with $D_{\max} = 25$ and $D_{\max} = 8$, respectively. As Tables 1–4 suggest, we almost always have $p > 0$ for the market data under investigation. We remark that $f(t)$ is decreasing if $p > 0$. Since $f(0+) = \sigma$ and $f(1) \approx \mathrm{hv}(1)$, our historical estimate of implied volatility is almost always larger than the traditional one, that is, $\mathrm{hv}(1)$. Since IV$>$ $\mathrm{hv}(1)$ much more often than IV$<$ $\mathrm{hv}(1)$ in real markets, this has the effect of narrowing the gap between HV and IV.

Period	hv(1)(%)	σ(%)	p	q	ARN
20Dec01–01May02	16.4	19.1	0.487	1.303	0.309
21Dec01–02May02	16.4	18.8	0.447	1.265	0.327
24Dec01–03May02	16.5	19.6	0.641	1.590	0.325
26Dec01–06May02	16.7	21.2	0.981	1.971	0.337
27Dec01–07May02	16.7	20.3	0.764	1.834	0.365
28Dec01–08May02	17.9	22.9	1.021	1.686	0.393
31Dec01–09May02	17.9	22.9	0.984	1.612	0.415
02Jan02–10May02	18.1	23.1	0.996	1.620	0.462
03Jan02–13May02	18.3	23.3	1.000	1.583	0.448
04Jan02–14May02	18.6	23.6	0.866	1.299	0.430
07Jan02–15May02	18.6	23.1	0.718	1.110	0.406
08Jan02–16May02	18.7	22.3	0.539	0.901	0.353
09Jan02–17May02	18.7	21.6	0.408	0.732	0.293
10Jan02–20May02	18.8	21.7	0.405	0.711	0.235
11Jan02–21May02	18.8	21.8	0.399	0.686	0.212
14Jan02–22May02	18.8	21.7	0.384	0.660	0.195
15Jan02–23May02	18.9	21.7	0.379	0.649	0.220
16Jan02–24May02	18.8	21.5	0.359	0.623	0.229
17Jan02–28May02	18.8	21.3	0.334	0.585	0.241
18Jan02–29May02	18.7	21.1	0.297	0.528	0.220
22Jan02–30May02	18.7	20.8	0.262	0.472	0.212
23Jan02–31May02	18.7	20.7	0.251	0.459	0.210
Average	18.1	21.6	0.587	1.085	0.311

Table 1. The results for S&P 500 closing index data sets of 90 trading days ending in May 2002. Here $D_{max} = 25$.

Period	hv(1)(%)	σ(%)	p	q	ARN
20Dec01–01May02	16.4	17.3	0.142	0.434	0.082
21Dec01–02May02	16.4	17.3	0.141	0.422	0.080
24Dec01–03May02	16.5	17.6	0.176	0.514	0.050
26Dec01–06May02	16.7	18.2	0.218	0.571	0.102
27Dec01–07May02	16.7	18.0	0.179	0.480	0.189
28Dec01–08May02	17.9	20.5	0.394	0.705	0.187
31Dec01–09May02	17.9	20.5	0.373	0.652	0.232
02Jan02–10May02	18.1	21.0	0.427	0.696	0.263
03Jan02–13May02	18.3	21.5	0.476	0.741	0.193
04Jan02–14May02	18.6	21.6	0.406	0.607	0.256
07Jan02–15May02	18.6	21.1	0.335	0.508	0.253
08Jan02–16May02	18.7	20.8	0.286	0.446	0.208
09Jan02–17May02	18.7	20.7	0.264	0.442	0.204
10Jan02–20May02	18.8	21.1	0.308	0.522	0.168
11Jan02–21May02	18.8	21.2	0.320	0.531	0.207
14Jan02–22May02	18.8	21.3	0.326	0.543	0.218
15Jan02–23May02	18.9	21.5	0.346	0.576	0.254
16Jan02–24May02	18.8	21.4	0.340	0.573	0.267
17Jan02–28May02	18.8	21.2	0.318	0.541	0.298
18Jan02–29May02	18.7	21.0	0.293	0.509	0.284
22Jan02–30May02	18.7	20.9	0.265	0.473	0.281
23Jan02–31May02	18.7	20.8	0.268	0.491	0.275
Average	18.1	20.3	0.300	0.544	0.207

Table 2. The results for S&P 500 closing index data sets of 90 trading days ending in May 2002. Here $D_{max} = 10$.

Period	hv(1)(%)	σ(%)	p	q	ARN
19Nov01–01Apr02	16.3	17.6	0.154	0.326	0.246
20Nov01–02Apr02	16.4	17.6	0.153	0.327	0.232
21Nov01–03Apr02	16.4	17.7	0.153	0.325	0.250
23Nov01–04Apr02	16.3	17.5	0.136	0.305	0.277
26Nov01–05Apr02	16.3	17.4	0.130	0.301	0.303
27Nov01–08Apr02	16.2	17.2	0.124	0.305	0.303
28Nov01–09Apr02	16.0	16.9	0.106	0.287	0.321
29Nov01–10Apr02	16.0	17.0	0.123	0.336	0.348
30Nov01–11Apr02	16.5	17.8	0.185	0.446	0.356
03Dec01–12Apr02	16.5	18.0	0.215	0.549	0.370
04Dec01–15Apr02	16.4	17.8	0.232	0.659	0.361
05Dec01–16Apr02	16.4	17.9	0.244	0.682	0.359
06Dec01–17Apr02	16.4	18.4	0.330	0.842	0.374
07Dec01–18Apr02	16.4	18.9	0.459	1.086	0.391
10Dec01–19Apr02	16.2	18.7	0.491	1.217	0.387
11Dec01–22Apr02	16.4	19.3	0.597	1.333	0.363
12Dec01–23Apr02	16.4	19.2	0.534	1.240	0.338
13Dec01–24Apr02	16.2	18.5	0.415	1.103	0.317
14Dec01–25Apr02	16.2	18.5	0.427	1.131	0.299
17Dec01–26Apr02	16.3	18.7	0.461	1.206	0.290
18Dec01–29Apr02	16.3	18.3	0.352	1.056	0.292
19Dec01–30Apr02	16.4	18.4	0.374	1.121	0.299
Average	16.3	18.1	0.291	0.736	0.322

Table 3. The results for S&P 500 closing index data sets of 90 trading days ending in April 2002. Here $D_{max} = 25$.

Period	hv(1)(%)	σ(%)	p	q	ARN
19Nov01–01Apr02	16.3	17.2	0.109	0.086	0.158
20Nov01–02Apr02	16.4	17.2	0.110	0.097	0.145
21Nov01–03Apr02	16.4	17.3	0.114	0.115	0.168
23Nov01–04Apr02	16.3	17.1	0.105	0.065	0.147
26Nov01–05Apr02	16.3	17.0	0.116	0.020	0.148
27Nov01–08Apr02	16.2	16.8	0.101	0.035	0.079
28Nov01–09Apr02	16.0	16.6	0.108	0.005	0.115
29Nov01–10Apr02	16.0	16.7	0.092	0.075	0.139
30Nov01–11Apr02	16.5	17.5	0.143	0.284	0.115
03Dec01–12Apr02	16.5	17.7	0.172	0.378	0.149
04Dec01–15Apr02	16.4	17.8	0.220	0.586	0.035
05Dec01–16Apr02	16.4	17.7	0.205	0.534	0.068
06Dec01–17Apr02	16.4	17.8	0.206	0.465	0.087
07Dec01–18Apr02	16.4	17.9	0.244	0.537	0.137
10Dec01–19Apr02	16.2	17.7	0.254	0.599	0.150
11Dec01–22Apr02	16.4	18.2	0.306	0.665	0.153
12Dec01–23Apr02	16.4	18.0	0.257	0.559	0.153
13Dec01–24Apr02	16.2	17.3	0.165	0.368	0.157
14Dec01–25Apr02	16.2	17.2	0.157	0.356	0.099
17Dec01–26Apr02	16.3	17.2	0.160	0.415	0.078
18Dec01–29Apr02	16.3	17.1	0.134	0.406	0.058
19Dec01–30Apr02	16.4	17.4	0.178	0.590	0.056
Average	16.3	17.4	0.166	0.329	0.118

Table 4. The results for S&P 500 closing index data sets of 90 trading days ending in April 2002. Here $\mathrm{D}_{\mathrm{max}} = 8$.

Chapter 4

Classification of financial markets via recurrent iterated function systems

4.1 Introduction

This chapter presents the basic theory of iterated function systems (IFS) and an application of this theory to study the scaling behaviour of stock markets. A measure representation is proposed for prices of the stock exchanges of Singapore, Shanghai, Shenzhen and New York. We first demonstrate that these probability measures can be modelled as recurrent iterated function systems (RIFS) consisting of two contractive similarities whose parameters are estimated from tick test data by the method of moments. Each market is then represented by a two-dimensional vector constructed from the estimated RIFS. We then classify the markets using the Euclidean distance of these vectors. It will be seen that stock prices of the same market tend to have the same shape, hence will be closer to each other in the above Euclidean distance.

In current practice, daily or weekly financial time series have been modelled in the ARIMA framework popularised by Box and Jenkins [1976], and their volatility has been captured via an autoregressive conditionally heteroskedastic (ARCH) model introduced by Engle [1982]. Nowadays, the increasing availability of intraday trade and quote data has had an important impact on research in finance, economics and mathematics including financial market microstructure theory. These intraday data are now available for most exchanges such as the Singapore stock exchange (SGSE), Shanghai stock exchange (SHSE), Shenzen stock exchange (SZSE), New York stock exchange (NYSE), and so on. An important form of intraday data is that of the so-called tick-test data. The improved ability to discern whether a trade is a buy order or a sell order is of particular importance. Most studies have classified trades as buys or sells by comparing the trade prices to the quote prices in effect at the time of the trade (Lee and Ready 1991). The intraday trade and quote data do not identify whether a trade was triggered by a market buy or sell order, so this information must be inferred from the data. One of the general approach to infer the direction of a trade is to compare the trade price to adjacent trades, a technique commonly known as a "tick test" (Lee and Ready 1991). A trade is an uptick (downtick) if the price is higher (lower) than the price of the previous trade. A trade is classified as a buy if it occurs on an uptick; otherwise it is classified as a sell. The tick test has been used by many reasearchers (e.g. Holthausen et al. 1987; Lee and Ready 1991; Ait-Sahalia 1998) and by market regulators. In this chapter, we simplify the tick test to the following: A financial time series $t_i,\ i = 1, 2, \cdots, L+1$, is converted to another time series $X_i, i = 1, 2, \cdots, L$, where $X_i = 1$ if $t_{i+1} \geq t_i$ and $X_i = 0$ if $t_{i+1} < t_i$. The converted time series X_i is called the *tick-test time series* of the original financial time series t_i.

To pursue analogies between stock market dynamics and stochastic models commonly used in statistical physics of complex systems have attracted considerable interest for many years (Anderson et al. 1988). In 1900, the first stochastic and scaling model in finance, Brownian motion, was proposed by Bachelier. Almost seven decades later, some generalisations were made by Madelbrot and his followers involving either fractional Brownian motion (Mandelbrot and Ness 1968) or Lévy motion (e.g. Mandelbrot 1967). Multiple scaling/multifractality is an important property related to the correlations and heavy-tailed marginal distributions of financial processes. Fractal geometry provides a mathematical formalism for describing complex spatial and dynamical structures with multiple scaling (Mandelbrot 1983; Feder 1988; Barnsley and Hurd 1993; Peruggia 1993; Lu 1997). Multifractal analysis has been used to study stock prices (Canessa 2000; Bouchaud et al. 2000) and foreign exchange rates (Anh et al. 2000; Schmitt et al. 1999). In fractal geometry, the iterated function system is a useful model to generate fractal and multifractal structures. The recurrent iterated function system is an extension of an IFS in which memory is allowed.

In this chapter, we will demonstrate the technique of RIFS on tick test data of four different stock exchanges (SGSE, SFSE, SZSE and NYSE). We first give the measure representation of the intraday data in Section 4.2. We then demonstrate that these probability measures of the K-strings can be modelled by RIFS consisting of two contractive similarities in Section 4.3. Each of these RIFS is specified by a matrix of incidence probabilities $P = (p_{ij})$,$i, j = 1, 2$ with $p_{i1} + p_{i2} = 1$ for $i = 1, 2$. It is our hypothesis that the measure representation of stock data can be captured by the matrix P. If we denote $p_{11} = p_1$ and $p_{21} = p_2$ in the matrix P, then each stock series can be represented by a vector (p_1, p_2) in $\mathbb{R}^2$. We will see in Section 4.4 that the vectors of the time series from the same market are very close to each other based on the Euclidean distance. Some conclusions are drawn in Section 4.5.

4.2 Measure representation of intraday stock data

Following Yu et al. [2001, 2003], we first derive the measure representation of intraday stock data. We call any string made of K letters from the set

$\{0,1\}$ a K-string. For a given K, there are in total 2^K different K-strings. In order to count the number of each kind of K-strings in a given tick test time series, 2^K counters are needed. We divide the interval $[0,1)$ into 2^K disjoint subintervals, and use each subinterval to represent a counter. Letting $s = s_1 \cdots s_K,\ s_i \in \{0,1\}, i = 1, \cdots, K$, be a substring with length K, we define

$$x_{left}(s) = \sum_{i=1}^{K} \frac{s_i}{2^i}, \tag{4.1}$$

and

$$x_{right}(s) = x_{left}(s) + \frac{1}{2^K}. \tag{4.2}$$

We then use the subinterval $[x_{left}(s), x_{right}(s))$ to represent substring s. Let $N_K(s)$ be the number of times that substring s with length K appears in the tick test time series. If the number of bases in the time series is L, we define

$$F_K(s) = N_K(s)/(L - K + 1) \tag{4.3}$$

to be the frequency of substring s. It follows that $\sum_{\{s\}} F_K(s) = 1$. Now we can define a measure μ_K on $[0, 1[$ by $d\mu_K(x) = Y_K(x)dx$, where

$$Y_K(x) = 2^K F_K(s), \quad \text{when} \quad x \in [x_{left}(s), x_{right}(s)). \tag{4.4}$$

It is easy to see $\int_0^1 d\mu_K(x) = 1$ and $\mu_K([x_{left}(s), x_{right}(s))) = F_K(s)$. We call μ_K the *measure representation* of the intraday stock data corresponding to the given K. As an example, the histogram of substrings in the tick test data of the intraday stock prices of the NYSE for $K = 10$ is given in the upper-right panel of Figure 4.1. Self-similarity is apparent in the measure.

For simplicity of notation, the index K is dropped in $F_K(s)$, etc. from now on, where its meaning is clear.

4.3 RIFS model and the moment method for parameter estimation

We propose to model the measure representation of intraday stock data by an RIFS. We consider a system of N contractive maps $S = \{S_1, S_2, ..., S_N\}$. This

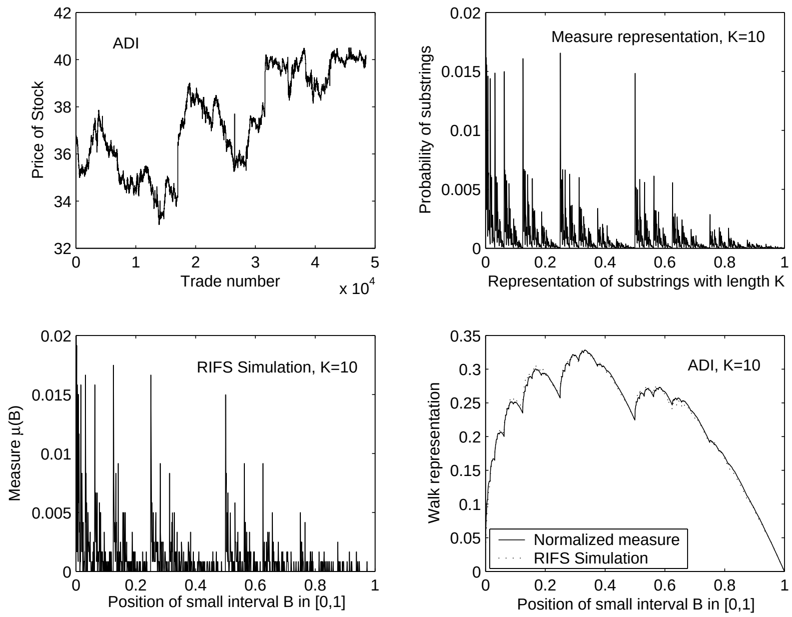


Figure 4.1: ADI of New York stock exchange.

system is called an iterated functions system (Peruggia 1993). Letting E_0 be a compact set in a compact metric space, we define $E_{\sigma_1\sigma_2\cdots\sigma_n} = S_{\sigma_1} \circ S_{\sigma_2} \circ \cdots \circ S_{\sigma_n}(E_0)$ and

$$E_n = \cup_{\sigma_1,\cdots,\sigma_n \in \{1,2,\cdots,N\}} E_{\sigma_1\sigma_2\cdots\sigma_n}.$$

Then $E = \cap_{n=1}^{\infty} E_n$ is called the *attractor* of the IFS. Given a set of probabilities $P_i > 0$, $\sum_{i=1}^{N} P_i = 1$, pick an $x_0 \in E$ and define the iteration sequence

$$x_{n+1} = S_{\sigma_n}(x_n), \qquad n = 0, 1, 2, 3, \cdots, \tag{4.5}$$

where the indices σ_n are chosen randomly and independently from the set $\{1, 2, \cdots, N\}$ with probabilities $P(\sigma_n = i) = P_i$. Then every orbit $\{x_n\}$ is dense in the attractor E (Vrscay 1991; Anh et al. 2002). For n large enough, we can view the orbit $\{x_0, x_1, \cdots, x_n\}$ as an approximation of E. This process is called a *chaos game*.

Given a system of contractive maps $S = \{S_1, S_2, \cdots, S_N\}$ on a compact metric space E^*, we associate with these maps a matrix of probabilities $\mathbf{P} = (p_{ij})$ which is row stochastic, i.e. $\sum_j p_{ij} = 1$, $i = 1, 2, \cdots, N$. Consider a random chaos game sequence generated by

$$x_{n+1} = S_{\sigma_n}(x_n), \quad n = 0, 1, 2, 3, \cdots,$$

where x_0 is any starting point. The fundamental difference between this process and the usual chaos game Eq. (4.5) is that the indices σ_n are not chosen independently, but rather with a probability that depends on the previous index σ_{n-1}:

$$P(\sigma_{n+1} = i) = p_{\sigma_n, i}$$

Then $(E^*, w, \mathbf{P})$ is called a *recurrent IFS*. The flexibility of RIFS permits the construction of more general sets and measures which do not have to exibit the strict self-similarity of IFS. This would offer a more suitable framework to model fractal-like objects and measures in nature.

Let μ be the invariant measure on the attractor E of an IFS or RIFS, χ_B the characteristic function for the Borel subset $B \subset E$; then from the ergodic theorem for IFS or RIFS (Barnsley and Hurd 1993),

$$\mu(B) = \lim_{n\to\infty} [\frac{1}{n+1} \sum_{k=0}^{n} \chi_B(x_k)].$$

In other words, $\mu(B)$ is the relative visitation frequency of B during the chaos game. A histogram approximation of the invariant measure may then be obtained by counting the number of visits made to each pixel on the computer screen.

The coefficients in the contractive maps and the probabilities in the IFS or RIFS model are the parameters to be estimated for a real measure which we want to simulate. Vrscay [1991] introduced a moment method to perform this task. If μ is the invariant measure and E the attractor of IFS or RIFS in $\mathbb{R}$, the moments of μ are

$$g_i = \int_E x^i d\mu, \qquad g_0 = \int_E d\mu = 1. \tag{4.6}$$

If $S_i(x) = c_i x + d_i$, $i = 1, \cdots, N$, then the following well-known recursion relations hold for the IFS model:

$$[1-\sum_{i=1}^{N} p_i c_i^n] g_n = \sum_{j=1}^{n} \left(\begin{array}{c} n \\ j \end{array} \right) g_{n-j} (\sum_{i=1}^{N} p_i c_i^{n-j} d_i^j). \tag{4.7}$$

Thus, setting $g_0 = 1$, the moments $g_n,\ n \geq 1$, may be computed recursively from a knowledge of $g_0, \cdots, g_{n-1}$ (Vrscay 1991).

For the RIFS model, we have

$$g_n = \sum_{j=1}^{N} g_n^{(j)}, \tag{4.8}$$

where $g_n^{(j)},\ j = 1, \cdots, N$, are given by the solution of the following system of linear equations:

$$\sum_{j=1}^{N} (p_{ji} c_i^n - \delta_{ij}) g_n^{(j)} = -\sum_{k=0}^{n-1} \left(\begin{array}{c} n \\ k \end{array} \right) [\sum_{j=1}^{N} c_i^k d_i^{n-k} p_{ji} g_k^{(j)}],\ i = 1, \cdots, N,\ n \geq 1. \tag{4.9}$$

For $n = 0$, we set $g_0^{(i)} = m_i$, where m_i are given by the solution of the linear equations

$$\sum_{j=1}^{N} p_{ji} m_j = m_i, \quad i = 1, 2, \cdots, N, \quad \text{and } g_0 = \sum_{i=1}^{N} m_i = 1. \tag{4.10}$$

If we denote by G_k the moments obtained directly from the real measure using (4.6), and g_k the formal expression of moments obtained from (4.7) for IFS model and from (4.8-4.10) for RIFS model, then through solving the optimal problem

$$\min_{c_i, d_i, p_i \text{ or } p_{ij}} \sum_{k=1}^{n} (g_k - G_k)^2, \qquad \text{for some chosen } n, \tag{4.11}$$

we will obtain the estimates of the parameters in the IFS or RIFS model.

4.4 Results and discussion

From the measure representation of stock data, it is natural to choose $N = 2$ and $S_1(x) = x/2, S_2(x) = x/2 + 1/2$ in the IFS or RIFS model. Based on the estimated values of the probabilities, we can use the chaos game to generate a histogram approximation of the invariant measure of the IFS or RIFS , which then can be compared with the given measure of the intraday stock data.

We analyse several intraday stock prices of the SGSE from 17 May 2002 to 24 June 2002, the SHSE from 1 April 2002 to 12 April 2002, the SZSE from 1 April 2002 to 12 April 2002 and the NYSE from 1 May 2002 to 15 May 2002. As an example, we plot in Figure 4.1 (upper-left) the ADI price in the NYSE whose sample size is 48,487.

A measure representation of each tick test time series for a given $K = 10$ is then obtained. There are in total $2^{10} = 1024$ different 10-strings; hence we devide the interval $[0, 1)$ into 1024 disjoint subintervals, and use each subinterval to represent a counter as can be seen in Figure 4.1 (upper right) for ADI of the NYSE.

We then estimate an RIFS model for each measure using the moment method. The resulting RIFS models are used to simulate the measures of the selected stocks. The RIFS simulation for the measure representation of ADI in the NYSE is shown in Figure 4.1 (lower-left). The results shown in Figure 4.1 indicate that the RIFS simulation fits the original measure of intraday stock data very well. In order to clarify how close the simulated measure is to the original measure, we convert a measure to its walk representation: We denote by $\{t_j,\ j = 1, 2, \cdots, 2^K\}$ the density of a measure and t_{ave} its average, then define the walk $T_j = \sum_{k=1}^{j}(t_k - t_{ave}),\ j = 1, 2, \cdots, 2^K$. The two walks of the given measure and the measure generated by the chaos game of RIFS are then plotted in the same figure for comparison. The walk representation for the original measure representation and its RIFS simulation of ADI in the NYSE are shown in Figure 4.1 (lower-right). It is seen that the two curves in the walk representations are very close to each other. This provides evidence that the RIFS model is a good model to fit the measure representation of intraday data.

The walk representations for different stocks in the same market are com-

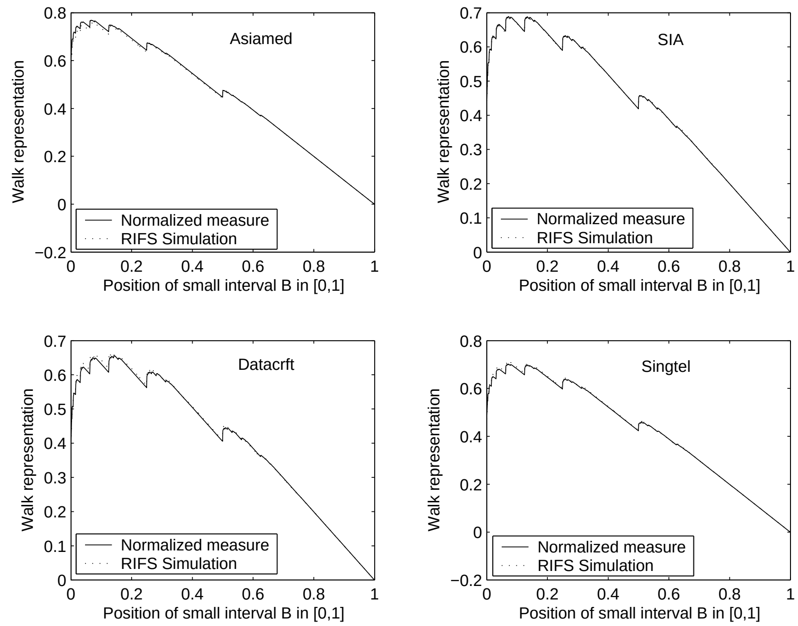


Figure 4.2: Normalized and simulated measure for the same market.

pared. We just show the results for several stocks in the SGSE in Figure 4.2.

We found that the walk representations have similar shapes for stocks in the same market. Then we can compare the walks of normalized measure representations and the corresponding simulated measures by the RIFS models between the different stock exchanges in Figure 4.3.

We found that the walk representations have different shapes for different markets. Moreover, it will be useful to know the scaling behaviour of a market via the RIFS model. The parameters in the RIFS model estimated for the selected stocks are shown in Table 1.

The classification of the markets by the Euclidean distance is shown in Figure 4.4, which shows that stock prices of the same market tend to be closer to each other in the Euclidean distance.

From our analysis, once we know the parameters p_1 and p_2 for each stock, we can generate the RIFS measure by the chaos game algorithm, which then yields the probability of all K-strings. Then we can use this information to predict whether the price of the next trade for this stock would go up or down

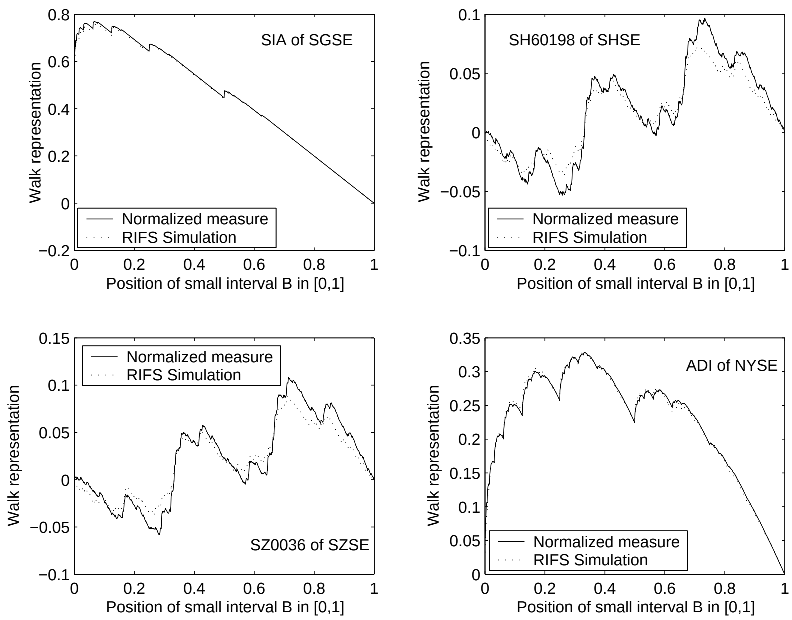


Figure 4.3: Normalized and simulated measure for the different market.

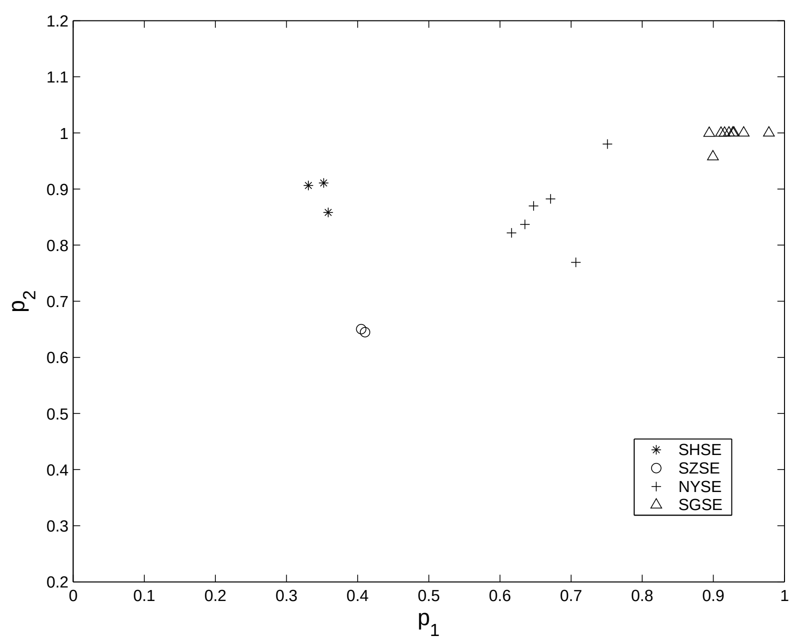


Figure 4.4: Classification of the markets by the Euclidean distance.

Table 4.1: Parameters from RIFS model for different stocks.

Market	Company	Number of ticks	p_1	p_2
New York Stock Exchange	ADI	48487	0.70680	0.76936
New York Stock Exchange	AOL	61299	0.67117	0.88243
New York Stock Exchange	GE	52892	0.64715	0.86990
New York Stock Exchange	IBM	62744	0.61632	0.82195
New York Stock Exchange	LU	65365	0.75113	0.98014
New York Stock Exchange	PFE	63512	0.63507	0.83687
Shanghai Stock Exchange	SH60198	2511	0.35846	0.85823
Shanghai Stock Exchange	SH60813	2436	0.35225	0.91080
Shanghai Stock Exchange	SH60839	2376	0.33077	0.90622
Shenzen Stock Exchange	SZ0036	2748	0.41032	0.64488
Shenzen Stock Exchange	SZ0592	1884	0.40504	0.65036
Singapore Stock Exchange	BIL	5086	0.92193	0.99989
Singapore Stock Exchange	SIA	2938	0.91070	0.99977
Singapore Stock Exchange	OCBC	3605	0.92876	0.99999
Singapore Stock Exchange	SINGTEL	4791	0.91575	0.99982
Singapore Stock Exchange	ASIAMED	1721	0.94269	0.99998
Singapore Stock Exchange	DATACRFT	8906	0.89936	0.95756
Singapore Stock Exchange	LEONGHIN	1677	0.97809	0.99999
Singapore Stock Exchange	CHARTERED	6990	0.89420	0.99953

depending on the price pattern of the previous few trades. For example, if the probabilities of the strings "011011" and "011010" are 0.012 and 0.035 respectively, and the pattern of the previous five trades is "01101", meaning the "down, up, up, down, up" pattern, then the price of next trade tends to go down because this resulting pattern has probability 0.035, which is larger than 0.012.

4.5 Conclusions

The results of the above sections provide support to the following assertions:

1. The tick test time series is a useful characteristic of the price of a stock.

2. The measure representation can be used to characterise the information from a tick test time series.

3. An RIFS model fits the measure representation of intraday data very well.

4. The parameters estimated from the RIFS can be used to classify the stocks from different stock exchange markets.

5. The RIFS simulation can be used to predict the movement of the next trade based on the pattern of the previous few trades.

Chapter 5

Analysis of the Stock Exchange of Thailand

5.1 Introduction

The inception of the Thai stock market began in July 1962 with the creation of the Bangkok Stock Exchange (BSE) as a limited company by private investors. However, the BSE was rather inactive due to non-professional management. Its annual turnover was only 160 million baht in 1968, and 114 million baht in 1969. Moreover, the trading volumes were only 46 million baht in 1970, and then 28 million baht in 1971. Although the turnover reached 87 million baht in 1972, the stock continued to perform poorly with the turnover dropped to 26 million baht as shown in Figure 5.1.

The BSE finally stopped operations in the early 1970s when it was disbanded as a result of inactivity (see Banks 1996,Centre 2004). It is generally accepted that the BSE failed to succeed because of a lack of official government support and a limited investor understanding of the equity market. In 1974 the Thai government passed legislation to create a new trading centre, the Securities Exchange of Thailand, and formal trading of equities began on the Securities Exchange in 1975. Then the Exchange was renamed in the lat-

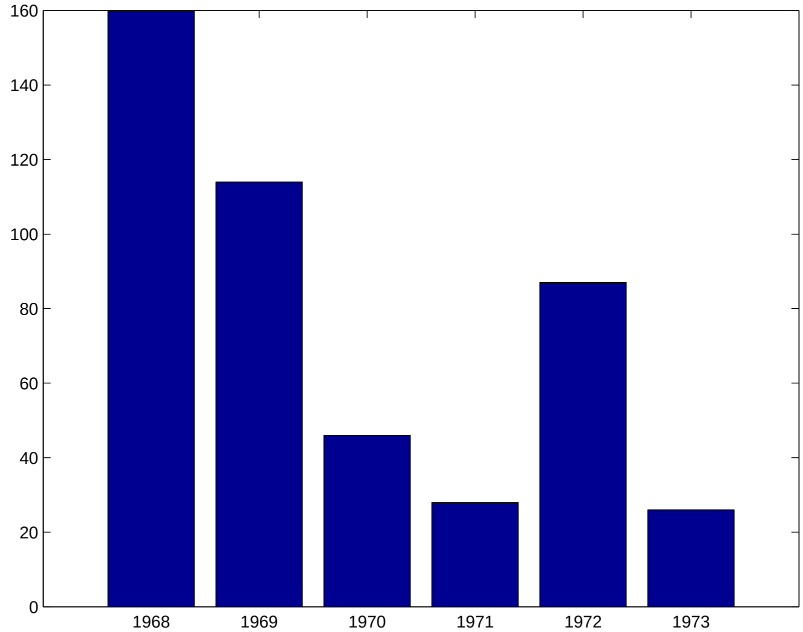


Figure 5.1: BSE performance from 1968 to 1973.

ter part of the 1980s and is now known as the Stock Exchange of Thailand (SET). However, Banks [1996] described the Thai financial market as in the early stages of diversification and expansion because of limitations in cash markets. Moreover the Thai fixed income market also remains relatively small in comparison with other Asia Pacific debt markets.

As an example, Figure 5.2 shows a data set, $S(t)$, of the SET from 4 August 1997 to 17 June 2004. Figure 5.3 shows the log returns, $R(t,\tau)$, at time t of the SET time series in the form

$$R(t,\tau) = \log S(t) - \log S(t-\tau)$$

where τ=1-day interval, derived from the SET data set in this period. As can be seen in Figure 5.3, the first half of the time series has higher volatility with high (positive and negative) returns than the second half of the SET time series.

In this chapter, we will use the Riesz-Bessel distribution to fit the SET

Figure 5.2: SET index values from 4 August 1997 to 17 June 2004.

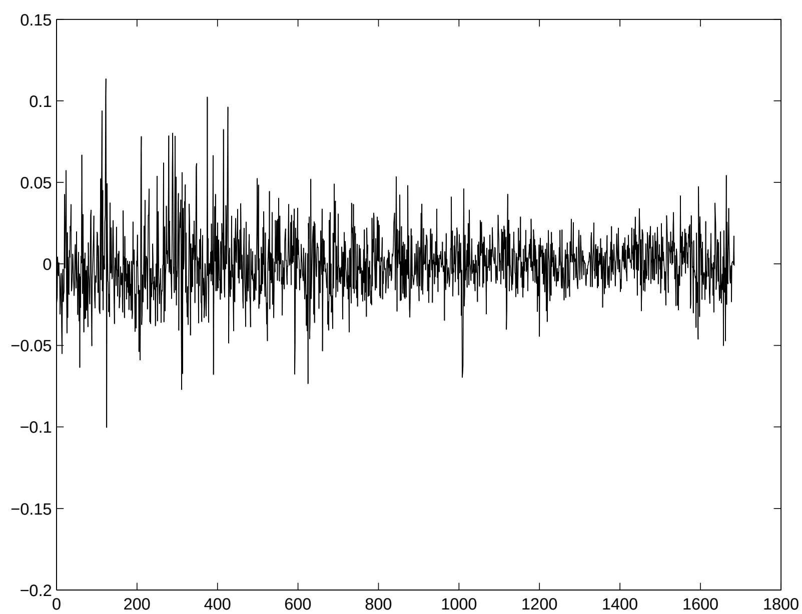


Figure 5.3: The 1-day Log Returns of SET index from 4 August 1997

index, the SET50 index and the MAI index of the Thai market, to be defined below. This model fits the time series DJ, DM, FF, BP, IBM and JY very well in Chapter two.

5.2 Definitions of some indices

The SET Index is a composite index calculated from prices of common stocks (including Property Fund) on the main board. It is a market capitalization weighted price index of the current market values of all listed common shares with the value on the base date 1975, which was when the SET Index was established, set at 100 points. The SET calculation is adjusted in line with new listings, delistings, and capitalization changes with other effects beyond price movement eliminated from the index (see Centre 2004).

The SET50 Index is a market capitalization weighted index, calculated from share prices of the top 50 listed companies on the SET. These 50 companies have capitalization and high levels of liquidity. The base date used is 16 August 1995. The component stocks in the SET50 Index are revised every 6 months. Since the Thai stock market had an objective to accommodate the issuing of index futures and index options in the future and to benchmark for measuring the performance of mutual funds invested in the SET, the SET50 Index was launched on 17 June 1996 to reach the goal (see Centre 2004).

Calculation of the SET50 Index. Normally, the SET50 Index is a market capitalization weighted average index calculated by using the same method for the existing SET Index. While the SET Index is calculated from all of the companies listed on the SET, the SET50 Index is calculated from the 50 common stocks. The basic formula is

$$\text{SET50 Index} = (CMV/BMV) * 100 \tag{5.1}$$

where CMV=Current Market Value of 50 Component Stocks and BMV= Base Market Value of 50 Component Stocks. The SET50 Index has been available since 16 August 1995 (see Centre 2004).

The MAI Index. The Market for Alternative Investment (MAI) is a business unit of the Stock Exchange of Thailand, which was established on 11 November 1998. It officially commenced operation on 21 June 1999. The objective is to create new fund-raising opportunities for Small and Medium-sized Enterprises (SMEs) as well as provide a greater range of investment alternatives for investors. Since the recent economic crisis in 1997 has conclusively demonstrated the advantage of selling equity over debt as a means of raising funds, MAI has therefore been established to answer the need. Moreover, MAI has been set to suit the character of the SME businesses as well. Therefore, the MAI index was launched on 3 September 2002 to serve as a center for the trading of listed securities. It was calculated from the trading value of listed securities based on 2 September 2002 (see Centre 2004).

The Stock Exchange of Thailand fully recognizes the importance of Small and Medium-sized Enterprises along with large firms in supporting the full development of economic growth. One of the major problems faced by Thailand currently is how to establish businesses efficiently at the lowest cost possible in order to maximize returns.

5.3 Modelling the SET indices

In this application, the Riesz-Bessel distribution as described in Chapter 2 is fitted to the SET index, the SET50 index and the MAI index. We will also look at the scaling behaviour of these time series based on the results of the estimation.

The above daily time series cover the following periods, yielding the sample sizes cited in brackets.

SET, 4 August 1997 - 17 June 2004 ($n = 1685$);

SET50, 19 December 2001 - 17 June 2004 $(n = 608)$;

MAI, 3 September 2002 - 22 October 2004 $(n = 528)$;

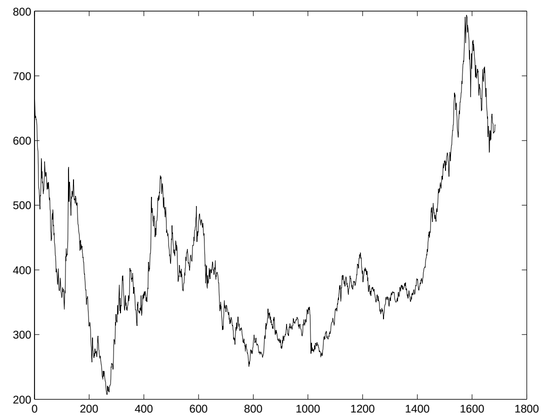
The SET time series together with its returns $\log SET\,(t) - \log SET\,(t-\tau)$ at different lag lengths τ are plotted in Figure 5.4(b) and Figure 5.4(c) . As can be seen in Figure 5.4, distinctive periods with high returns appear in the first half of the SET time series, i.e., high volatility, with moderate returns in the second half, i.e. low volatility.

Similarly, we plot the log returns of the SET50 time series as $\log SET50\,(t) - \log SET50\,(t-\tau)$ at different lag lengths τ and the log returns of the MAI time series as $\log MAI\,(t) - \log MAI\,(t-\tau)$ at different lag lengths τ. It is seen in Figures 5.4 and 5.5, when τ increases from 1-day to 32-day, that the paths thin out but have larger spikes since the time series display more correlation due to overlapping information. These properties result in longer tails and a higher peak at 0 in the respective sample density functions. This observation confirms that a non-Gaussian model is more appropriate to fit the probability density functions of returns data as commonly reported in the literature.

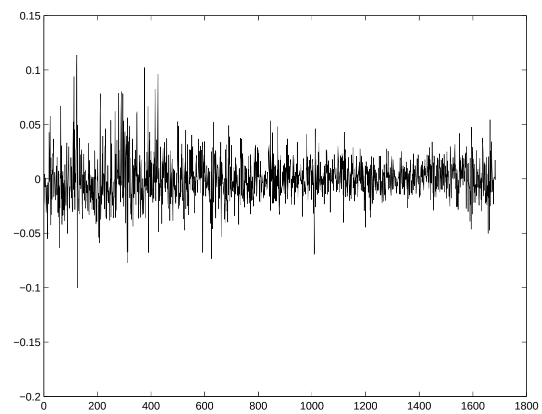
We assume that the above time series are exponential transformations of a Riesz-Bessel-Lévy motion whose distribution is characterised by the Fourier transform of its density function:

$$\hat{p}\,(t,\lambda) = \exp\left[-\kappa t\,|\lambda|^{2\alpha}\,(c^2+\lambda^2)^{\gamma}\right], \quad \lambda \in \mathbb{R},$$

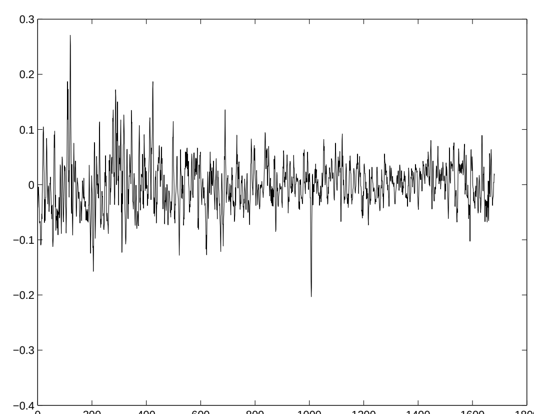
where the additional parameter κ is introduced to allow t to be a time parameter. The method of Section 2.3 is then performed to estimate the parameters κ, α, γ and c for the daily returns with $t = 1$. They are reported in the following table:



(a) SET Index from 4 August 1997 to 17 June 2004

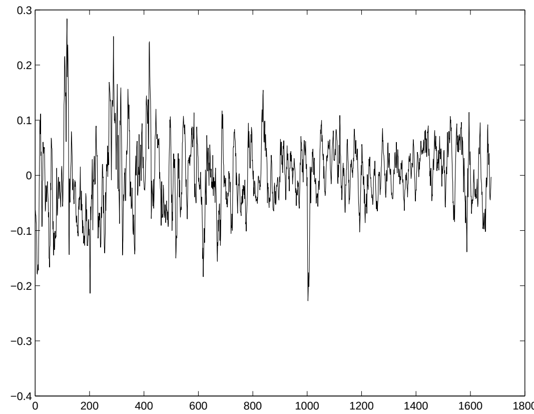


(b) The 1-day returns log SET(t)- log SET(t-1)

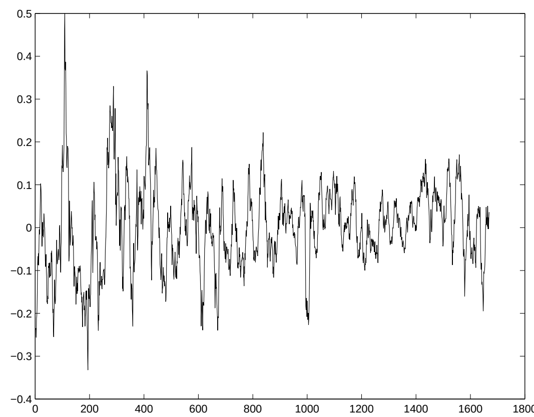


(c) The 4-day returns log SET(t)- log SET(t-4)

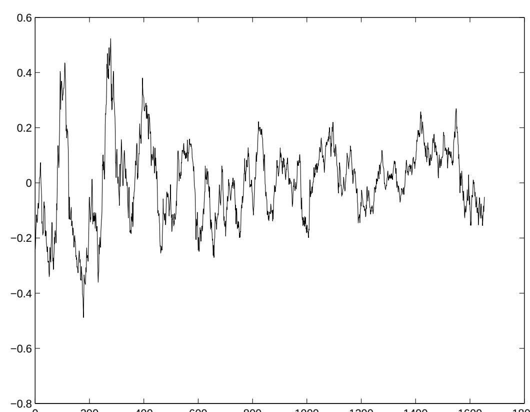
Figure 5.4: The time series and the Log Returns of SET Index



(a) The 8-day returns log SET(t)- log SET(t-8)



(b) The 16-day returns log SET(t)- log SET(t-16)



(c) The 32-day returns log SET(t)- log SET(t-32)

Figure 5.5: The Log Returns of SET Index

Financial time series	κ	α	γ	c
SET	5.5743	0.9307	-0.9307	3.9802
SET50	0.8984	0.9999	-0.3070	2.5728
MAI	0.7676	0.9999	-0.5629	1.4529

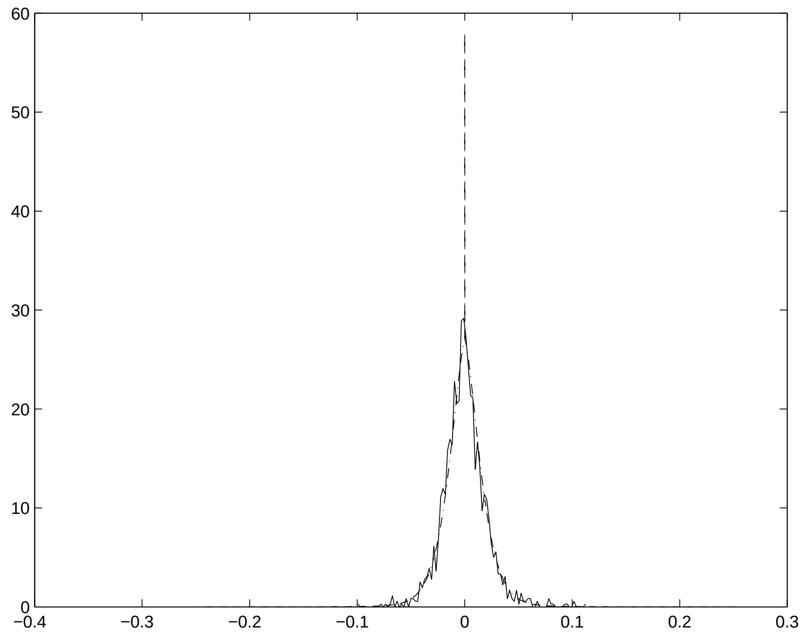
The standard errors are given in the following table which corresponds to the above table:

Financial time series	κ	α	γ	c
SET	19.1266	0.0241	0.8230	2.5670
SET50	2.1914	0.0022	0.8047	3.9191
MAI	0.3167	0.0041	0.1545	0.4310

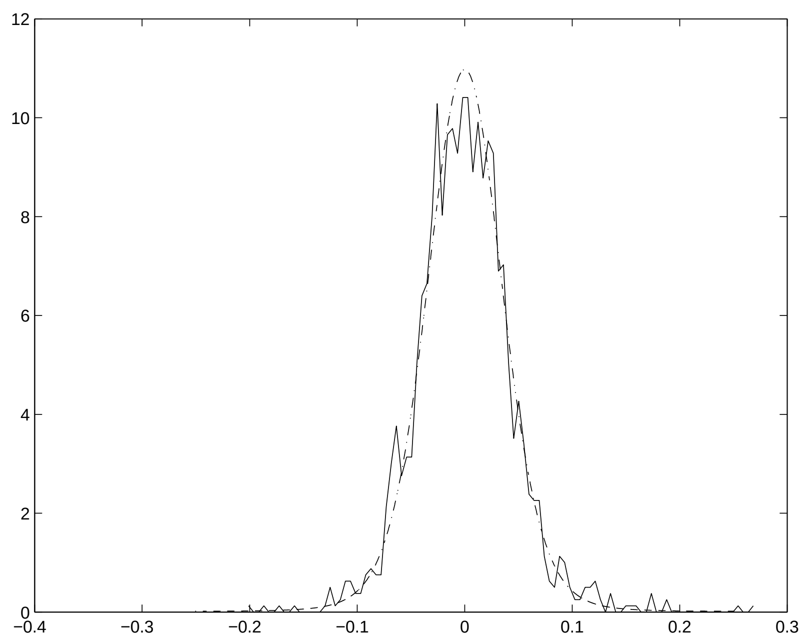
The estimation of α is precise while that of the other parameters has too high standard errors, which may be due to the relatively short length of these time series. The values of α are close to 1 in all cases and those of $\alpha + \gamma$ are less than 1; also the value of γ for the SET time series is closer to -1 than the other time series. Therefore, the SET time series seems to approach a compound Poisson process, while the SET50 and the MAI seem to be pure jump processes.

For the SET time series, the Riesz-Bessel density provides good fit for 1-day returns. There appears to be departure from the assumption of a Lévy motion. Due possibly to second- and/or higher-order correlations in the data, 4-day, 8-day, and 16-day returns of the SET time series are, on average, more concentrated at 0 than predicted by Lévy motion with Riesz-Bessel density. The hypothesis seems consistent with the SET time series in Figures 5.6 and 5.7. But this does not seem to hold for the SET50 in Figures 5.8 , 5.9 and the MAI time series in Figures 5.10 and 5.11.

As noted in Chapter two, the parameters α and γ act together to indicate the behaviour of the time series at small lags τ. Under the hypothesis of a Riesz-Bessel-Lévy motion, the distribution can be rescaled to converge to a symmetric $2(\alpha + \gamma)$-stable distribution. To evaluate this hypothesis, we

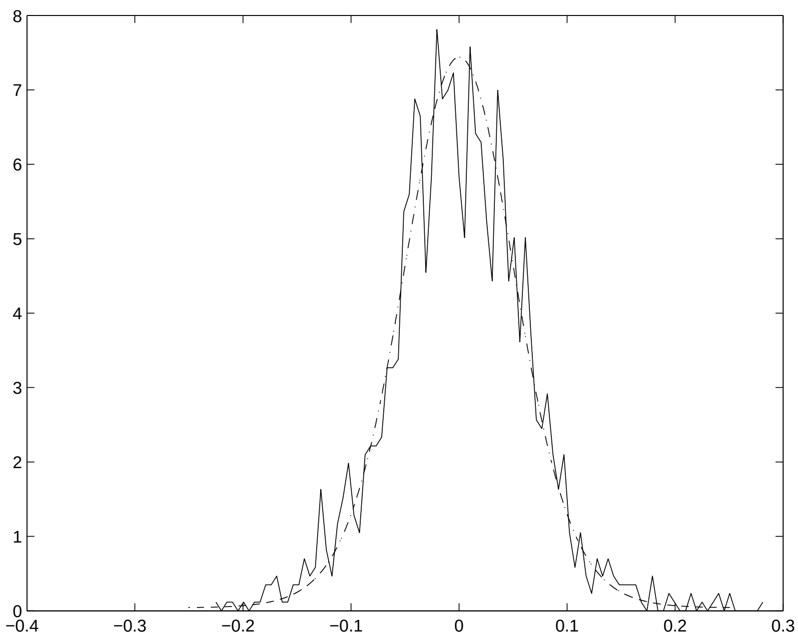


(a) Fitting the sample density of 1-day returns of SET Index

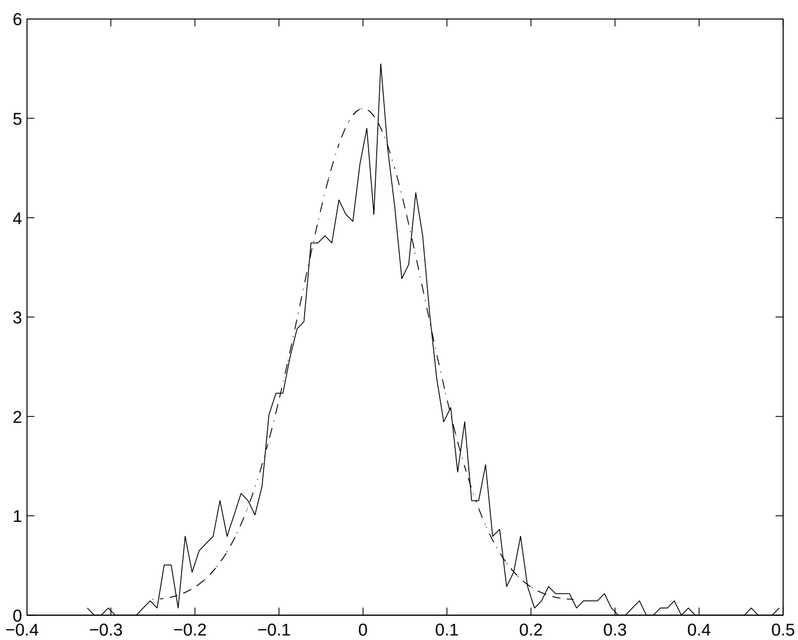


(b) Fitting the sample density of 4-day returns of SET Index

Figure 5.6: The sample densities(solid curves) of the returns of the SET Index series at different lags and their estimation(dashed curves)by the method of Section 3

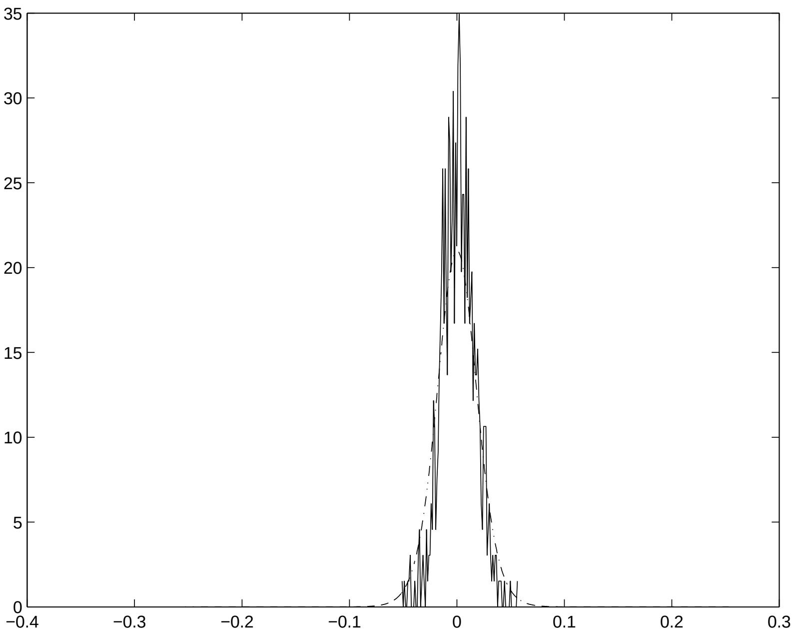


(a) Fitting the sample density of 8-day returns of SET Index

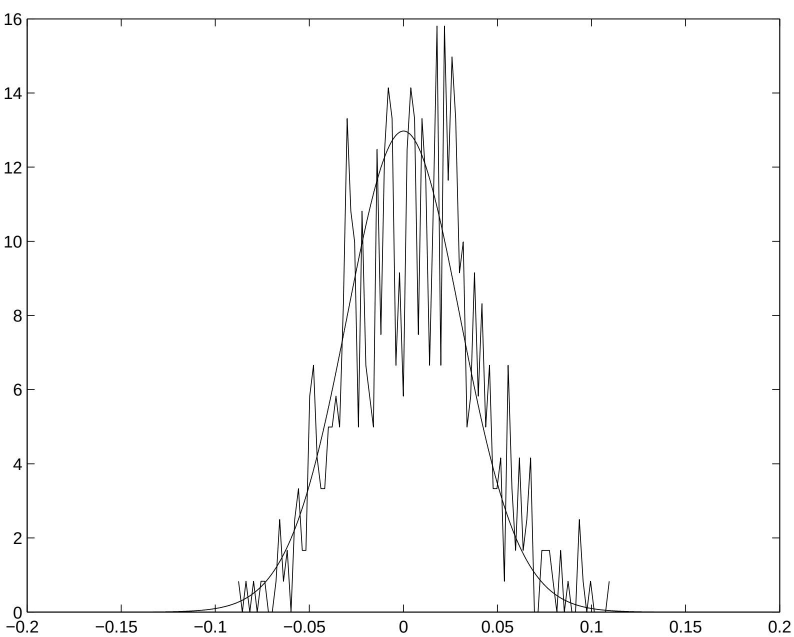


(b) Fitting the sample density of 16-day returns of SET Index

Figure 5.7: The sample densities(solid curves) of the returns of the SET Index series at different lags and their estimation(dashed curves)by the method of Section 3

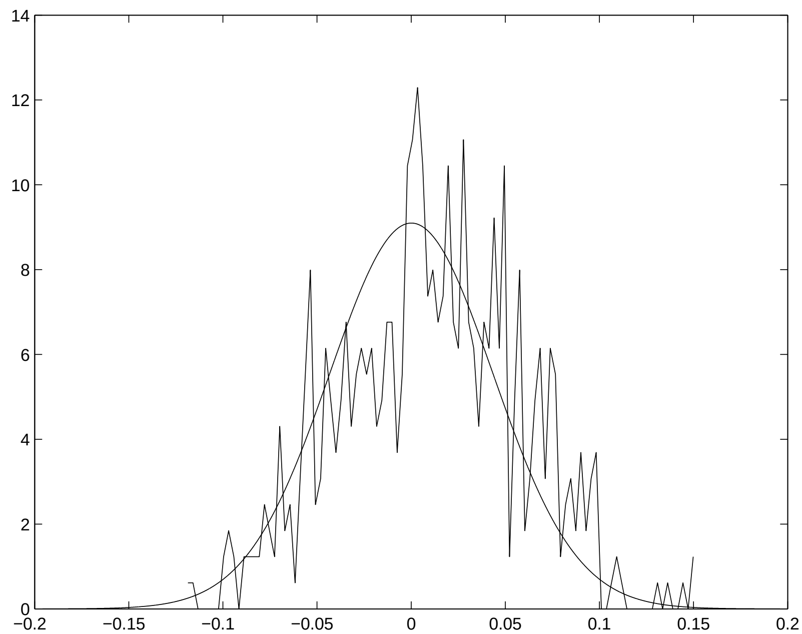


(a) Fitting the sample density of 1-day returns of SET50 Index

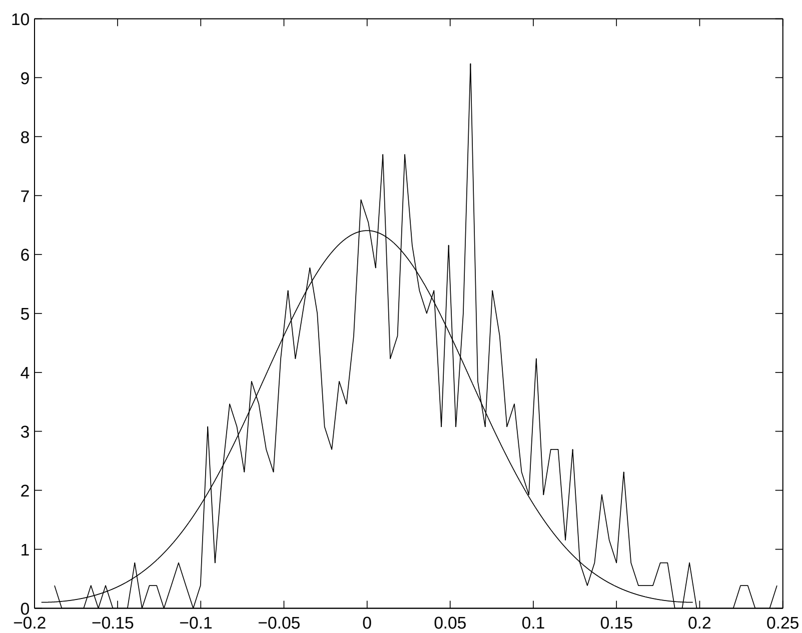


(b) Fitting the sample density of 4-day returns of SET50 Index

Figure 5.8: The sample densities(rugged curves) of the returns of the SET50 Index series at different lags and their estimation(smooth curves)by the method of Section 3

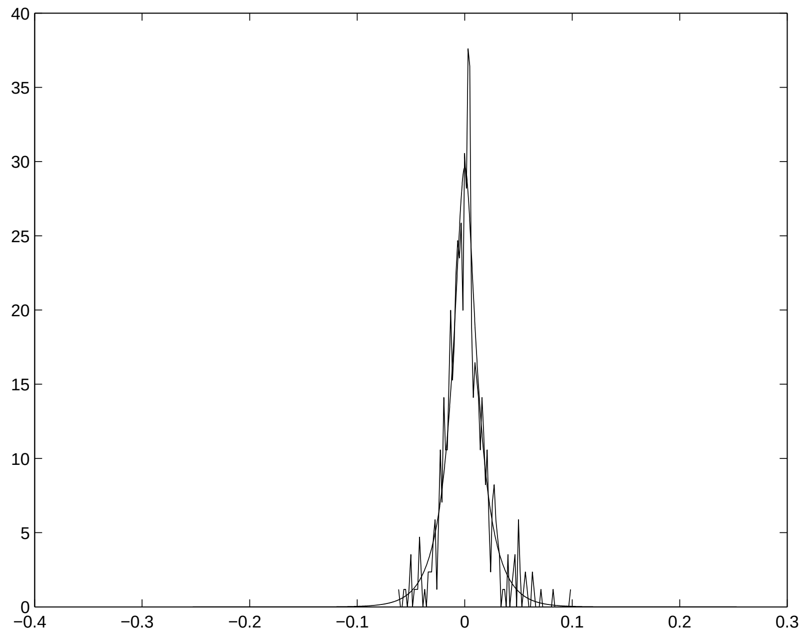


(a) Fitting the sample density of 8-day returns of SET50 Index

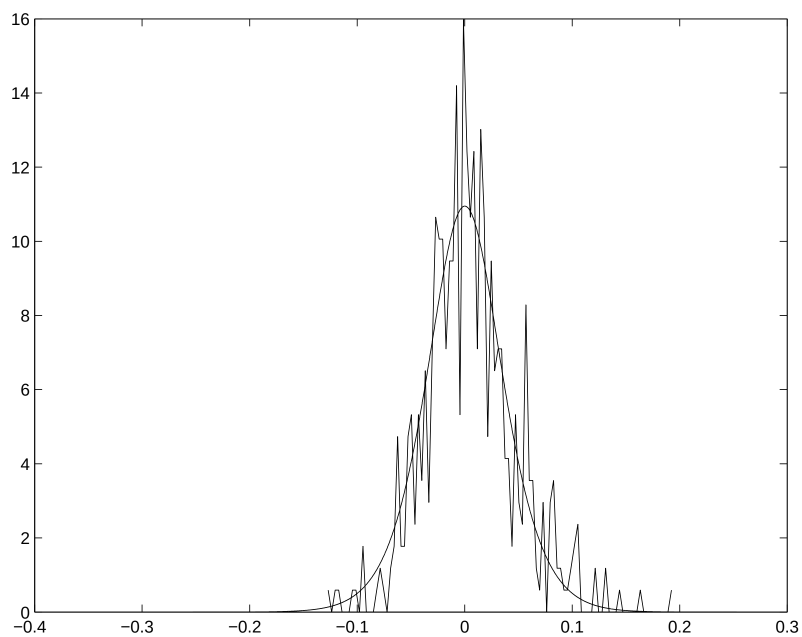


(b) Fitting the sample density of 16-day returns of SET50 Index

Figure 5.9: The sample densities(rugged curves) of the returns of the SET50 Index series at different lags and their estimation(smooth curves)by the method of Section 3

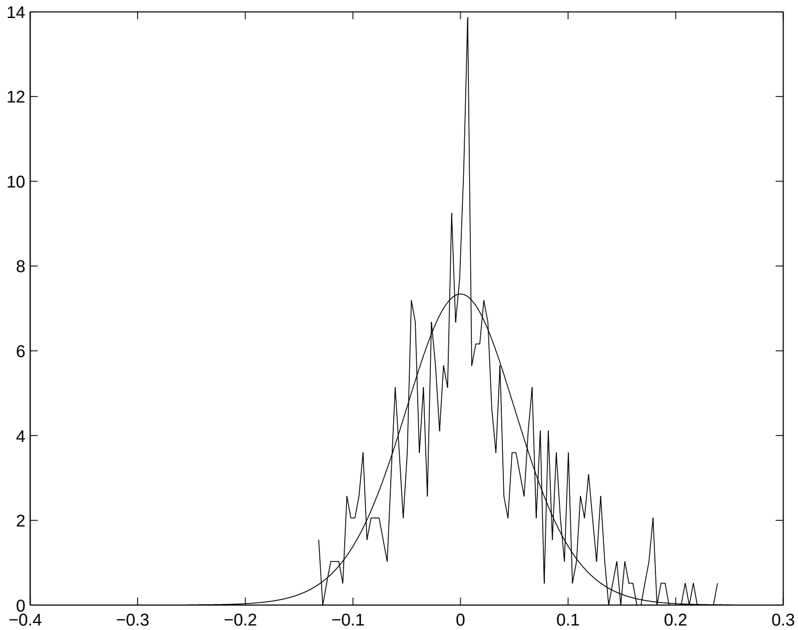


(a) Fitting the sample density of 1-day returns of MAI Index

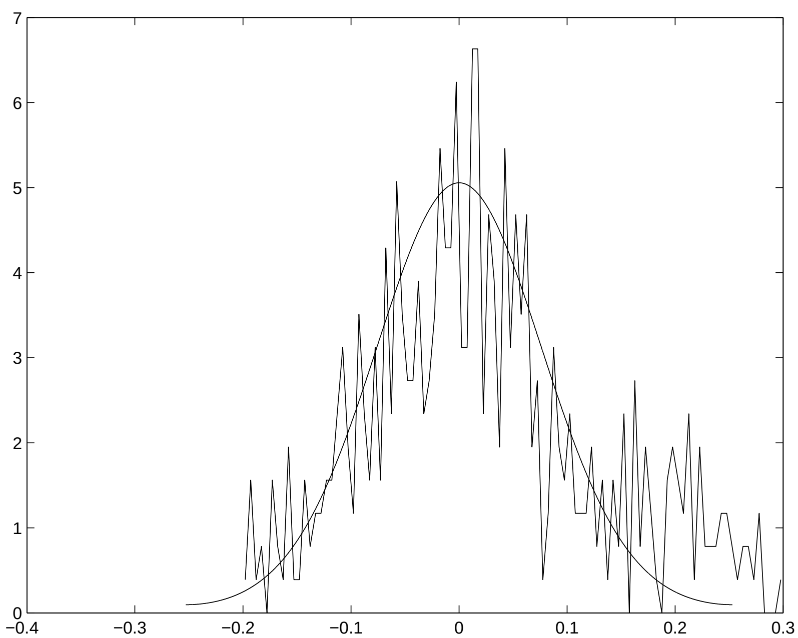


(b) Fitting the sample density of 4-day returns of MAI Index

Figure 5.10: The sample densities(rugged curves) of the returns of the MAI Index series at different lags and their estimation(smooth curves)by the method of Section 3



(a) Fitting the sample density of 8-day returns of MAI Index



(b) Fitting the sample density of 16-day returns of MAI Index

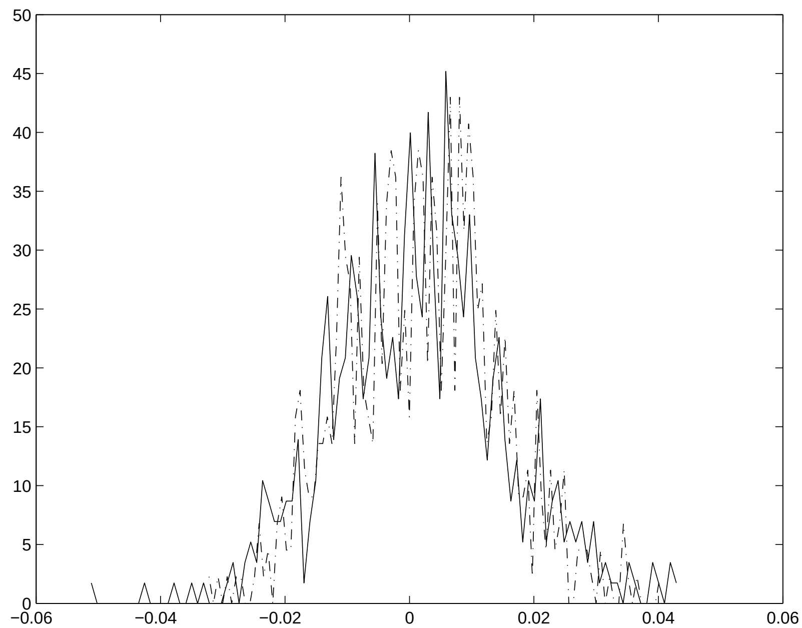
Figure 5.11: The sample densities(rugged curves) of the returns of the MAI Index series at different lags and their estimation(smooth curves)by the method of Section 3

compute the histograms of the SET, SET50 and MAI time series, and the corresponding Riesz-Bessel densities, which are obtained by inverting the characteristic function with the parameter estimated from daily returns.

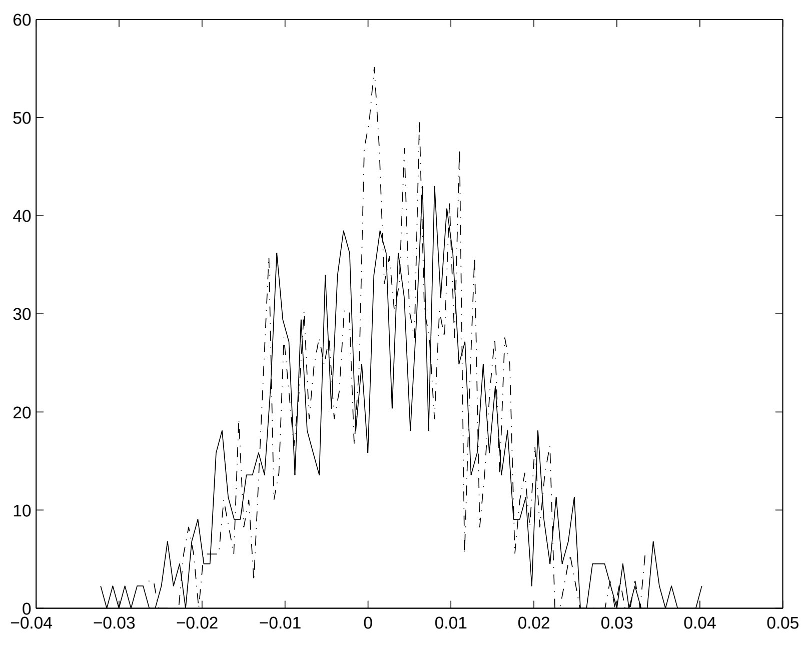
The rescaled results, where the probability density p is rescaled and plotted as $\tau^{2(\alpha+\gamma)}p\left(X_{rescaled}\right)$ against $X_{rescaled}$, are displayed in Figures 5.12 and 5.13 for the SET50 series and in Figures 5.14 and 5.15 for the MAI series respectively. Here,

$$X_{rescaled} = \frac{X\left(t,\tau\right)}{\tau^{1/(2(\alpha+\gamma))}},$$

with $X\left(t,\tau\right) = \log S\left(t\right) - \log S\left(t-\tau\right)$, $S\left(t\right)$ being any of the given time series. If the time series follows a Riesz-Bessel-Lévy motion, the probability densities of their returns at different lags will collapse to a limiting density, namely, that of the symmetric $2\left(\alpha+\gamma\right)$-stable distribution. This pattern seems to hold for the series SET50 and MAI studied, while SET presents a clear departure from it. This indicates that there is evidence of second- and/or higher-order correlations in the SET series. In this case, a model which exhibits both Lévy-type behaviour and short- or long-range dependence of the process is warranted.

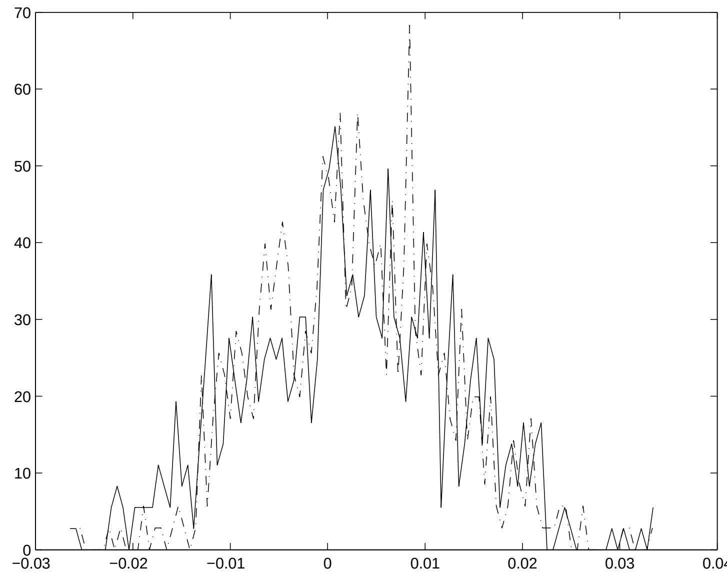


(a) The rescaled sample density,2-day(solid curve)&4-day(dashed curve)

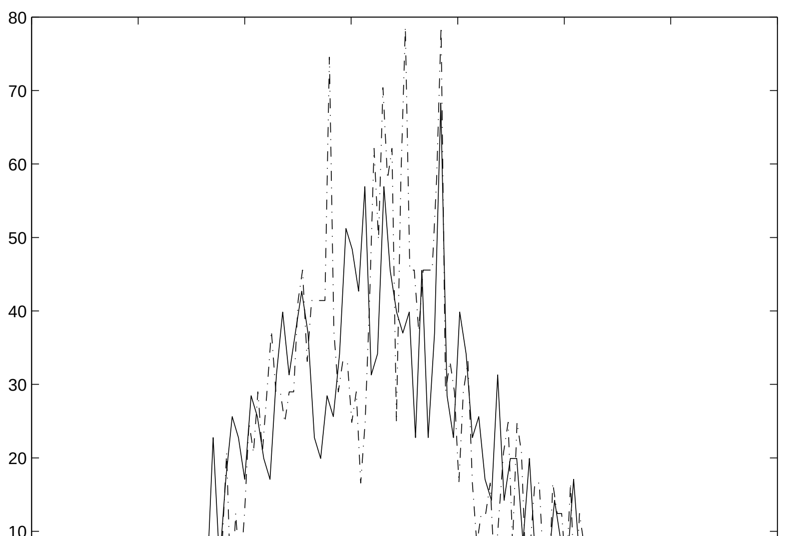


(b) The rescaled sample density,4-day(solid curve)&8-day(dashed curve)

Figure 5.12: The rescaled sample density of the SET50 Index based on the estimates of the Riesz-Bessel distribution.

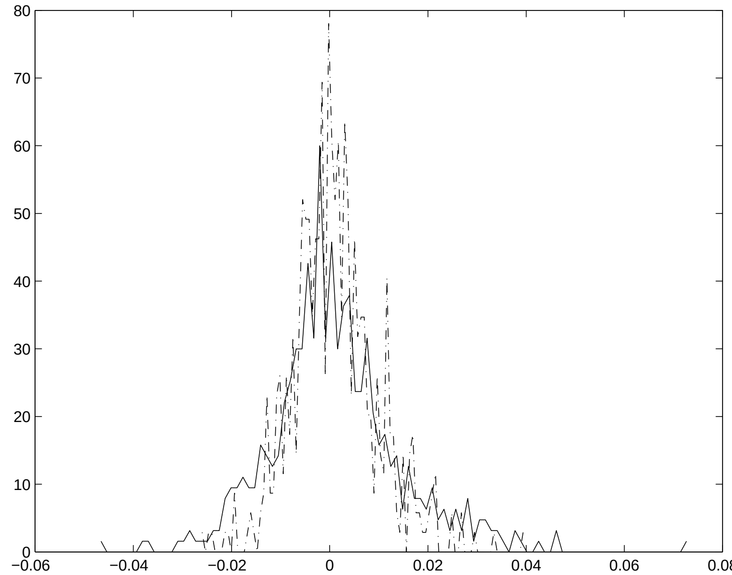


(a) The rescaled sample density,8-day(solid curve)&16-day(dashed curve)

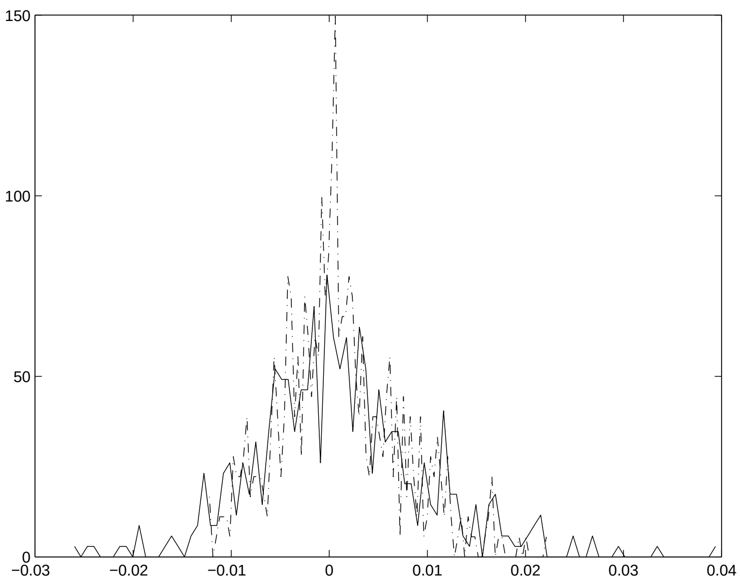


(b) The rescaled sample density,16-day(solid curve)&32-day(dashed curve)

Figure 5.13: The rescaled sample density of the SET50 Index based on the estimates of the Riesz-Bessel distribution.

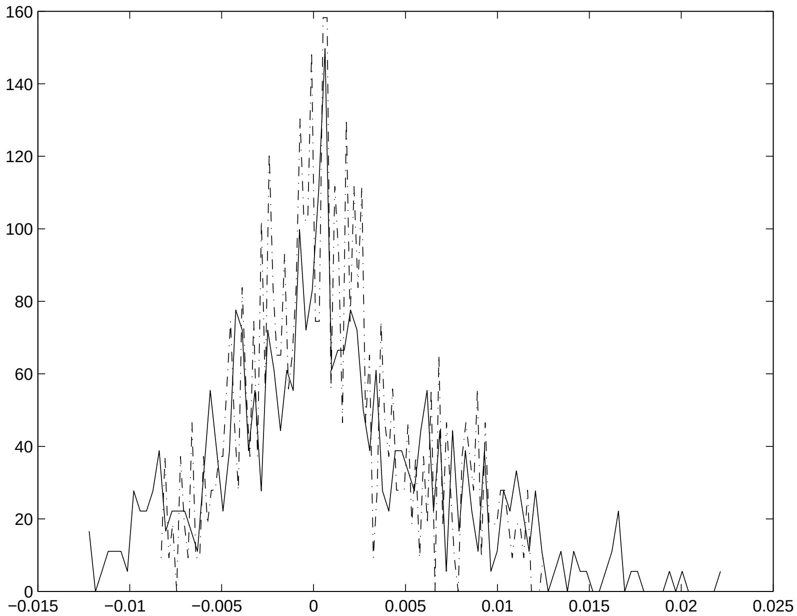


(a) The rescaled sample density,2-day(solid curve)&4-day(dashed curve)

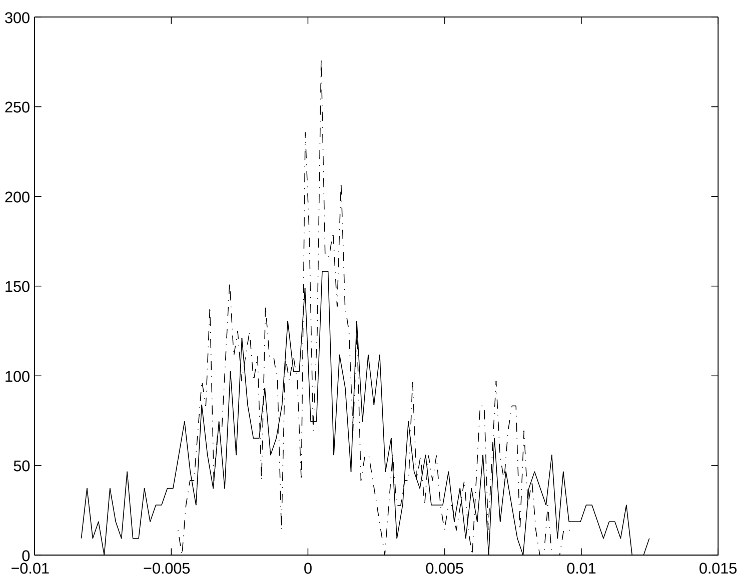


(b) The rescaled sample density,4-day(solid curve)&8-day(dashed curve)

Figure 5.14: The rescaled sample density of the MAI Index based on the estimates of the Riesz-Bessel distribution.



(a) The rescaled sample density,8-day(solid curve)&16-day(dashed curve)



(b) The rescaled sample density,16-day(solid curve)&32-day(dashed curve)

Figure 5.15: The rescaled sample density of the MAI Index based on the estimates of the Riesz-Bessel distribution.

Chapter 6

Conclusions and some open problems

This thesis has detailed a theory of the Riesz-Bessel distribution. This provides a parsimonious yet versatile setting to model the semi-heavy tailed behaviour of financial processes. This distribution can be selected to model, for example, a compound Poisson process, a pure jump process, or a compound Poisson process with drift. The Riesz-Bessel distribution can be rescaled to converge to a stable distribution. For some range of its parameters, there exists a Lévy process corresponding to a Riesz-Bessel distribution, which we call a Riesz-Bessel Lévy motion. The density of the Lévy measure of RBLm is completely monotone on $(0, \infty)$. This implies that RBLm for $\alpha + \gamma < 1/2$ can be written as the difference of two subordinators whose distribution belongs to the class of generalized convolutions of mixtures of exponentials. These results lead to efficient algorithms to simulate paths of RBLm in some special but important cases. Furthermore, a quasilikelihood-type method has been developed to estimate the two key parameters of the Riesz-Bessel distribution. This method uses the empirical characteristic function and can be conveniently implemented via generalised least squares.

The methods and techniques of this development have been applied to a number of financial time series including some important exchange rates such as the Japanese yen, the Deutche mark, the British pound and the French franc, and stock prices such as the IBM, the Dow Jones, and several stock indices such as those from the Stock Exchange of Thailand. The Riesz-Bessel distribution has been found to model very well the seme-heavy tailed behaviour of these processes. However, when a process possesses significant correlation, such as in the case of the French franc or the Deutche mark, it seems that the Riesz-Bessel model is not adequate in capturing the behaviour of the distribution at the origin. A further extension to the following setting is promising:

$$\frac{\partial^{\beta} p}{\partial t^{\beta}} == - \left(-\Delta\right)^{\alpha} \left(I - \Delta\right)^{\gamma} p\left(t, x\right), \quad p\left(0, x\right) = \delta\left(x\right).$$

Here, the fractional-in-time derivative is the regularized fractional derivative or fractional derivative in the Caputo-Djrbashian sense :

$$\frac{\partial^{\beta} u}{\partial t^{\beta}} = \begin{cases} \frac{\partial^{m} u}{\partial t^{m}}\left(t, x\right), & \text{if } \beta = m \in \mathbb{N}, \\ \frac{1}{\Gamma(m-\beta)} \int_0^t \left(t-\tau\right)^{m-\beta-1} \frac{\partial^{m} u(\tau, x)}{\partial \tau^{m}} d\tau, & \text{if } m-1 < \beta < m \end{cases}$$

Some preliminary simulation of this extended model indicates that it captures reasonably well the behaviour of the distribution at the origin. Statistical estimation of the model is therefore warranted.

An analysis of the time series of the French franc and the Deutche mark provided evidence of long memory in these data sets. This has led to a consideration of market models with memory. In financial applications, such models must preserve the completeness and arbitrage-free conditions needed for replication of contingent claims. The Anh-Inoue model provides such a framework. It has a mechanism to incorporate memory (short or long) and yields a means to compute the implied volatility. Via this setting, the parameters of the model can be estimated using nonlinear least squares. It is still observed that the volatility is characteristic of the period in which it is estimated; in other words, volatility changes its character over time. This change may be the result of an underlying scaling mechanism. It seems possible to model volatility as a stochastic process with multiple scaling. Incorporating this volatility process into a market model will be challenging.

The classification of financial markets using high-frequency data was another objective that we emphasized in the thesis. We transformed the intraday or minute data into tick data in the form of measure representation, then modelled them as recurrent iterated function systems. A measure representation is an extension of the concept of the histogram: We look at the probability for a k-string instead of the probability for a single value, hence uncover more patterns in the data, particularly their multiple scaling. This new methodology provides a convenient framework for market classification. It also yields a mechanism for short-term prediction, in fact for the next few ticks. Some preliminary results indicate its potential, hence warrant further investigations.

Appendix

Programme source code listing

```
//This program simulates Fractional Brownian process
//This algorithm is written by Saupe
//
#include <iostream.h>
#include <math.h>
#include <stdlib.h>
#include <fstream.h>
#include <iomanip.h>
ofstream output4;

// Functions to be used in the program

int fbm(int maxlevel,double H,double sigma,double delta[],double X[]);
double boxmuller(double mu,double sigma);
int main()
{
        double X[513];
    double delta[9];
    int maxlevel=9;
        double sigma=1.0;
        double fracdim=0.5;
        int i;
        int j=0;
        for  (i=0; i<513; i++)
                X[i]=0.0;
        for (i=0; i<8; i++)
                delta[i]=0.0;
        double t=fbm(maxlevel,fracdim,sigma,delta,X);
        return 0;
}

//The following procedure will make the fractional brownian motion
int fbm(int maxlevel,double H,double sigma, double delta[],double X[])
{
        int i;
        double D;
        int d;
        int level;
        int N;
```

```
output4.open("fbm5.txt");
    {
        for (i=1; i <= maxlevel; i++)
            delta [i-1] =sigma*pow(0.5,1.0*i*H)*sqrt(0.5)*sqrt(1.0-
            pow(2.0,2.0*H-2.0));
        N = pow(2.0,1.0*maxlevel);
        X[0]=0.0;
        X[N]=sigma*boxmuller(0.0,1.0);
        D=N;
        d=D/2;
        level=1;
        while (level <= maxlevel)
        {
            for (i=d; D<0.0 && N-d-i <=0.0 || -D<=0.0 && D<=0.0 || -D
                < 0.0 && i-N+d <= 0.0 ;i += D)
            X[i] =0.5*X[i-d]+0.5*X[i+d];
            for (i=0;D <0.0 && N-i<=0.0 || -D <=0.0 && D<=0.0 || -D<
                0.0 && i-N <= 0.0 ; i +=D)
            X[i] +=delta[level-1]*boxmuller(0.0,1.0);
            D=D/2.0;
            d=d/2;
            level ++;
        }
        double retx[513];
        for (i=0; i<512; i++)
        {
            retx[i]=X[i];
            cout<< X[i] << "\n";
            output4<< X[i] << "\n";
        }
    output4.close();
        return retx[1];
    }
}
```

```
// This procedure will simulate random Normal R.V.
// ,Gaussian R.V. using the box-muller method
double boxmuller (double mu,double sigma)
{
        double rmax=RAND_MAX;
        double u1=rand()/rmax;
        double u2=rand()/rmax;
        double choice=rand()/rmax;
        if (u1 == 0)
                return mu;
        else
        {
                double z1=(sqrt(-2*log(u1))*cos(2*3.141592654*u2));
        double z2=(sqrt(-2*log(u1))*sin(2*3.141592654*u2));
        if (choice >=0.5)
            return sigma*z1+mu;
                 else
                  return sigma*z2+mu;
        }
}
```

```
        program main
          parameter(N=1721)
          real x1(N),x2(N-1)
        integer x3(N-1)

                open(1,file='Asiamedtotal.txt',status='old')

          do 1 I=1,N
          read(1,*) x1(I)
1         continue
        close(1)


       open(1,file='Asiamed_Price_lisan.dat',status='new')
          do 3 I=1,N-1
          if (x1(I).LE. x1(I+1)) then
          x3(I)=0
          else
          x3(I)=1
          end if
          write(1,*) x3(I)
3              continue
          close(1)
          end
```

```
#include <stdio.h>
int frequency[2][2][2][2][2][2][2][2][2][2];

void main () {

      FILE *fp;
      int i,j,k,l,m,n,o,p,q,r;
      char ch;
      long time=0;
      int flag[10],temp;

     if ((fp=fopen("Asiamed_Price_lisan.dat", "r"))==NULL) {
           printf("can not open file lisan.dat\n");
           exit (0);
           }
```

```
for(i=0;i<9;i++) {
        do {
                ch=fgetc(fp);
                }while(ch!='0'&ch!='1');
        if(ch=='0') flag[i]=0;
        else flag[i]=1;
        }

do {
        do      {
                ch=fgetc(fp);
                }while(ch!='0'&ch!='1'&ch!=EOF);

     if(ch=='0') flag[9]=0;
     else flag[9]=1;



frequency[flag[0]][flag[1]][flag[2]][flag[3]][flag[4]][flag[5]][flag[6]][flag[7]][flag
[8]][flag[9]]++;
              temp=flag[0];
              for(i=0;i<9;i++) flag[i]=flag[i+1];
              time++;
              }while(ch!=EOF);

    frequency[temp][flag[0]][flag[1]][flag[2]][flag[3]][flag[4]][flag[5]][flag[6]][
flag[7]][flag[8]]--;

    fclose(fp);

    if ((fp=fopen("Asiamed_Price_str10.dat", "a+"))==NULL) {
         printf("can not open file confmer22.dat\n");
         exit (0);
         }
    for(i=0;i<2;i++) {
    for(j=0;j<2;j++) {
    for(k=0;k<2;k++) {
    for(l=0;l<2;l++) {
    for(m=0;m<2;m++) {
    for(n=0;n<2;n++) {
    for(o=0;o<2;o++) {
    for(p=0;p<2;p++) {
```

```
for(q=0;q<2;q++) {
for(r=0;r<2;r++) {
	fprintf(fp,"%d\n ",frequency[i][j][k][l][m][n][o][p][q][r]);
	}
	}}}}}}}}}
fclose(fp);
}
```

```
      program main
      parameter (N=2**10)

      real f1(N)
      real B_1(N),G(20),B_11(N)

      Integer f_1(N)

      open(2,file='Asiamed_Price_str10.dat',status='old')
        sumcd=0.0
      do 20 I=1,N
        read(2,*) B_11(I)
          sumcd=sumcd+B_11(I)
20        continue
        close(2)
      open(1,file='Asiamed_Price_str10cd.dat',status='new')
          do 25 I=1,N
        f1(I)=1.0*(I-1)/N+1.0/(2*N)
        B_1(I)=1.0*B_11(I)/sumcd
        write(1,*) f1(I),'   ',B_1(I)
25      continue
        close(1)


      open(2,file='Asiamed_Price_str10_ju.dat',status='new')
      Do 35 I=1,20
      sum_g=0
      do 45 J=1,N
      sum_g=sum_g+(f1(J)**I) *B_1(J)
45      continue
      G(I)=sum_g
      write(2,*) G(I)
35      continue
      close(2)
      end
```

```
  program main
  use Numerical_libraries

  integer IBTYPE,IPRINT,M_1,MAXITN,ME,N_1,J
  parameter (IBTYPE=0,IPRINT=0,M_1=2,MAXITN=10000,ME=2,N_1=4)
    double precision X_1(N_1),FVALUE,XGuess(N_1),
&   XLB(N_1),XSCALE(N_1),XUB(N_1)
    external moment

    DATA Xguess/0.5,0.5,0.5,0.5/
    data XSCALE/0.001,0.001,0.001,0.001/
    DATA XLB/0.0,0.0,0.0,0.0/
    DATA XUB/1.0,1.0,1.0,1.0/

    call DNCONF (moment,M_1,ME,N_1,XGuess,IBTYPE,XLB,XUB,
&            XSCALE,IPRINT,MAXITN,X_1,FVALUE)

  write(*,*) FVALUE
    write(*,*) X_1(1),X_1(2)
    write(*,*) X_1(3),X_1(4)
    end


  Subroutine moment(M_1,ME,N_1,X_1,ACTIVE,FC,Grad)
  use Numerical_libraries

  Parameter (IPATH=1, LDA=1,N=2,M=15)
  integer M_1,ME,N_1
    double precision X_1(N_1),P(N,N),ju(M+1),S
    double precision A0(LDA,LDA),g(M+1,N),B0(N-1),X0(N-1),sum_N_1
    double precision A(N,N),B(N),X(N),a1(N),judata(M),FC,Grad(*)
  integer delta
    LOGICAL ACTIVE(*)
    EXTERNAL  delta


  Double precision cnj,sum_b,sumgm

  S=0.5
    a1(1)=0
    a1(2)=1-S
```

```
      do 1 I=1,N
        do 2 J=1,N
       P(I,J)=X_1(2*(I-1)+J)
2      continue
1      continue
C       above is to obtain the probability matrix

      do 10 I=1,N-1
        do 20 J=1,N-1
        A0(I,J)=P(J,I)-P(N,I)-delta(I,J)
20    continue
10      continue
      do 25 I=1,N-1
        B0(I)=-P(N,I)
25      continue
        call DLSARG (N-1,A0,LDA,B0,IPATH,X0)
        sum_N_1=0.0
        ju(1)=0.0
        do 30 I=1,N-1
        g(1,I)=X0(I)
        sum_N_1=sum_N_1+g(1,I)
        ju(1)=ju(1)+g(1,I)
30    continue
      g(1,N)=1.0-sum_N_1
        ju(1)=ju(1)+g(1,N)
C        above ju(1) means ju(0), g(1,I) means g(0,I)

      sumgm=0.0
     open(1,file='Asiamed_Price_str10_ju.dat',status='old')
      do 40 n1=1,M
        read(1,*) judata(n1)
      do 45 I=1,N
        do 46 J=1,N
        A(I,J)=P(J,I)*(S**(n1))-delta(I,J)


46      continue
      B(I)=0.0
      do 47 k=0,n1-1
       sum_b=0.0D0
        do 48 J=1,N
        if (k.EQ. 0) then
```

```
          sum_b=sum_b+(a1(J)**(n1-k))*P(J,I)*g(k+1,J)
          else
          sum_b=sum_b+(S**k)*(a1(J)**(n1-k))*P(J,I)*g(k+1,J)
        end if
48      continue
        call combinator(n1,k,cnj)
C         write(*,*) 'cnj=',cnj
C         write(*,*) n1,k
        B(I)=B(I)-cnj*sum_b
47      continue
45      continue
          call DLSARG (N,A,LDA+1,B,IPATH,X)
        ju(n1+1)=0.0
          do 49 I=1,N
          g(n1+1,I)=X(I)
          ju(n1+1)=ju(n1+1)+g(n1+1,I)
49      continue
       sumgm=sumgm+(ju(n1+1)-judata(n1))*(ju(n1+1)-judata(n1))
40      continue
        FC=sumgm
       write(*,*) FC,X_1(1),X_1(2)
         write(*,*) X_1(3),X_1(4)
         If (ACTIVE(1)) Grad(1)=X_1(1)+X_1(2)-1.0D0
         If (ACTIVE(2)) Grad(2)=X_1(3)+X_1(4)-1.0D0
         close(1)
       return
   end
```

```
      Function delta(x,y)
      integer x, y
      integer delta
      If (x.EQ.y) then
      delta=1
      else
      delta=0
      end if
      end

      Subroutine combinator(l1,l2,lc)
      integer l1,l2,I
      Double precision lc, gra1,gra2,gra3

        if (l2.EQ. 0) then
        lc=1.0D0
        else
      gra1=1
      do 200 I=1,L1
      gra1=gra1*I
200       continue
      gra2=1
      do 210 I=1,l2
      gra2=gra2*I
210       continue
      gra3=1
      do 220 I=1,l1-l2
      gra3=gra3*I
220       continue
      lc=1.0D0*gra1/(gra2*gra3)
        end if
      return
      end
```

```
          program main
          integer n_1,n_2
          parameter(n_1=1200,n_2=2**10,Num=2)
          integer B(n_2),I,J,sigma(n_1+1)
          real p(Num,Num),x(n_1+1),y(n_2),t(n_2),kai(n_1)
          real s1,s2,a1,a2

      Data P/0.942686,0.9999840,
     &         0.057314,0.0000160/
          s1=0.5
          s2=s1
          a1=0.0
          a2=1-s1
          x(1)=0
            sigma(1)=1
          open(2,file='random.dat',status='old')
          do 20  I=2,n_1+1
          read(2,*) kai(I)
C           write(*,*) 'random number is ',kai(I)
        if ((kai(I).GE.0) .AND. (kai(I).LT.p(sigma(I-1),1))) then
          x(i)=s1*x(i-1)+a1
            sigma(I)=1
         else
          x(i)=s2*x(i-1)+a2
            sigma(I)=2
          end if
C           write(*,*) 'x(i)=',x(i)
          do 10 J=1,n_2
          fuzu=1.0*(j-1)/n_2
          fuzu1=1.0*j/n_2
          if ((x(i).GE.fuzu) .AND. (x(i).LT.fuzu1)) then
          B(j)=B(j)+1
          else
          B(j)=B(j)
          end if
10          continue
20          continue
           close(2)
```

```
      open(1,file='Asiamed_Price_rifscd.dat',status='new')
        do 30 j=1,n_2
         t(j)=1.0*j/n_2-1.0*1/(2*n_2)
         y(j)=1.0*B(j)/(n_1+1)
         write(1,*) t(j),'    ',y(j)
30          continue
                        close(1)
           end
```

```
      program main
        parameter(N=2**10)
        real x1(N),x2(N),y1(N),y2(N),sum,ave

     open(1,file='Asiamed_Price_str10cd.dat',status='old')
   open(2,file='Asiamed_Price_str10walk.dat',status='new')
         sum=0.0
        do 1 I=1,N
        read(1,*) x1(I),y1(I)
        sum=sum+y1(I)
1       continue
      ave=sum/N
        x2(1)=x1(1)
        y2(1)=y1(1)-ave
        write(2,*) x2(1),y2(1)
C       write(2,*) y2(1)
        do 5 I=2,N
        x2(I)=x1(I)
        y2(I)=y2(I-1)+(y1(I)-ave)
        write(2,*) x2(I),y2(I)
C       write(2,*) y2(I)
5      continue
      close(2)
        close(1)
        end
```

```
         program main
               parameter(N=2**10)
               real x1(N),x2(N),y1(N),y2(N),sum,ave

          open(1,file='Asiamed_Price_rifscd.dat',status='old')
       open(2,file='Asiamed_Price_rifswalk.dat',status='new')
                sum=0.0
               do 1 I=1,N
               read(1,*) x1(I),y1(I)
               sum=sum+y1(I)
1              continue
            ave=sum/N
               x2(1)=x1(1)
               y2(1)=y1(1)-ave
               write(2,*) x2(1),y2(1)
C              write(2,*) y2(1)
               do 5 I=2,N
               x2(I)=x1(I)
               y2(I)=y2(I-1)+(y1(I)-ave)
               write(2,*) x2(I),y2(I)
C              write(2,*) y2(I)
5            continue
            close(2)
               close(1)
               end
```

```
% The lines 5, 6 and 40 should be changed according to the period.
% The value of Ds should be changed as you like.
% (1)Calculation of the function HV
Ds = 8;
load SpApril1.txt -ascii;
clear S; S =SpApril1;
N = length(S);
clear u;
for i = 1:Ds
    for k = 1:(N-i)
        u(i,k) = log(S(N-k+1-i)/S(N-k+1));
    end
end
delete hv;
clear hv;
for i = 1:Ds
 K1 = 0; K2 =0;
  for k = 1:(N-i)
   K1 = K1 + u(i,k); K2 = K2 + u(i,k)^2;
  end
 hv(i) = 100*sqrt(252)*sqrt((K2/(N-1-i) - (K1)^2/((N-i)*(N-1-i)))/i);
end
save hv hv -ascii;
plot(hv, '--*')
axis([0 Ds 10 22])
hold on
% (2) Non Linear Least square estimation by toolbox of matlab
x0 = [hv(1) 0.34 0.56]
LB = [hv(1) 0 0]


UB  = [(hv(1) + 3) 2.0 5.0]
[x,resnorm] = lsqnonlin(@myfun1,x0,LB,UB)
%(3)Error estimation
error = sqrt(resnorm/Ds)


%(4)Estimated plotting
sigma = x(1);
p = x(2);
q = x(3);
for i = 1:Ds
```

```
f(i) = sigma*sqrt((q^2)/((p+q)^2)+p*(2*q+p)*(1-exp(-(p+q)*i))/(i*(p+q)^3));
end
plot(f, 'r')
%(5)Title and Label
title(['S&P500 19Nov2001-1 April2002 (',num2str(N),' business days)'])
text(0.8,21.5,['\sigma = ',num2str(sigma),',   p = ',num2str(p),',   q = ',num2str(q),',
error = ',num2str(error)])
text(0.8,20.5,['HV(1) = ',num2str(hv(1))])
text(0.8,12, ['Initial Parameter for (sigma  p  q) =  ',num2str(x0)])
text(0.8,11.2, ['Lower Bound for (sigma p q) =  ',num2str(LB)])
text(0.8,10.4, ['Upper Bound for (sigma p q) =  ',num2str(UB)])
%text(0.8,10, ['Initial Parameter for (sigma  p  q) =  ',num2str(x0)])
%text(0.8,9.2, ['Lower Bound for (sigma p q) =  ',num2str(LB)])
%text(0.8,8.4, ['Upper Bound for (sigma p q) =  ',num2str(UB)])
xlabel('Time Lag (days)')
ylabel('Volatility (%)')
delete hv;
hold off
```

```
function F = myfun(x)
load hv -ascii;
t0 = length(hv);
% x(1)=sigma, x(2)=p, x(3)=q
for i = 1:t0
G(i) = x(1)*sqrt((x(3)^2)/((x(2)+x(3))^2)+x(2)*(2*x(3)+x(2))*(1-exp(-
(x(2)+x(3))*i))/(i*(x(2)+x(3))^3));
end
F = G – hv
```

```
function [theta0,I]=RBMLE(theta0,X,lambda);

delta=ones(4,1);
theta=zeros(4,1);
theta(1)=log(theta0(1));
theta(2)=log(theta0(2)/(1-theta0(2)));
theta(3)=log((theta0(2)+theta0(3))/(1-theta0(2)-theta0(3)));
theta(4)=log(theta0(4));
theta
while(max(abs(delta))>0.00001)
  [Z,V,Y]=RBMLE2(theta0,X,lambda);
  delta=lscov(Z,Y,V);
  theta=theta+0.2*delta;
  theta0=[exp(theta(1)); exp(theta(2))/(1+exp(theta(2)));
exp(theta(3))/(1+exp(theta(3))) - exp(theta(2))/(1+exp(theta(2))); exp(theta(4))]
      if (theta0(2)>0.999)
      delta=0;
      theta0=[NaN;NaN;NaN];
      end
  end

  I=inv(Z'*inv(V)*Z);
```

```
function [Z,V,Y]=RBMLE2(theta,X,lambda);

N=length(lambda);
M=length(X);

% Write parameters in useful form.
k=theta(1);
a=theta(2);
g=theta(3);
scale=theta(4);

% Transform parameters onto the whole real line.
k1=log(k);
a1=log(a/(1-a));
g1=log((g+a)/(1-g-a));
s1=log(scale);
% Computes the empirical characteristic function.
PhiHat=zeros(N,1);
for i=1:N
  PhiHat(i)=mean(cos(lambda(i)*X));
end

% Setting up the IGLS scheme, i.e. computing the variance matrix V and
% matrix of derivatives Z.
Z=zeros(N,4);
V=zeros(N);
lPhi1=-k*lambda.^(2*a).*(scale^2+lambda.^2).^g;
Phi1=exp(lPhi1);
Y=PhiHat-Phi1;
Z(:,1)=lPhi1.*Phi1;
Z(:,2)=(2*log(lambda)-log(scale^2+lambda.^2)).*lPhi1.*Phi1 *
exp(a1)*(1+exp(a1))^(-2);
Z(:,3)=log(scale^2+lambda.^2) .*lPhi1.*Phi1 * exp(g1)*(1+exp(g1))^(-2);
Z(:,4)=2*scale*g*lPhi1.*Phi1 ./(scale^2+lambda.^2);
```

```
for i=1:N
  for j=1:i
    lPhi2=-k*((lambda(j)+lambda(i))^2)^a
*(scale^2+(lambda(j)+lambda(i))^2).^g;
    lPhi3=-k*((lambda(j)-lambda(i))^2)^a*(scale^2+(lambda(j)-lambda(i))^2).^g;
    Phi2=exp(lPhi2);
    Phi3=exp(lPhi3);
    V(i,j)=(0.5*(Phi2+Phi3)-Phi1(i)*Phi1(j))/M;
    V(j,i)=V(i,j);
  end
end
```

```
function y= RieszBesselChFun(z,alpha,gamma,kappa,A)
y=exp(-kappa*(z.^2).^alpha.*(A^2+z.^2).^gamma);
```

```
function [p,zeta]= RieszBesselDensity(alpha,gamma,kappa,A,N,R)
Delta=2*R/N;
z=0:Delta:2*R-Delta;
z=z-R;
fHat=RieszBesselChFun(z,alpha,gamma,kappa,A);


fTrans=fft(fHat)/N;
fTrans=fftshift(fTrans);
zeta=((1-N/2):N/2)*pi/R;
fTrans=fTrans.*exp(-i*R*zeta);

p=-real(fTrans)/(zeta(2) - zeta(1));
```

```
function [V,I]=RieszBesselInfo(theta0,x,h,N,R);

I=zeros(4);
lik=RieszBesselLikelihood(theta0,x,N,R);

theta(1)=exp(theta0(1))/(1+exp(theta0(1)));
theta(2)=exp(theta0(2))/(1+exp(theta0(2))) - theta(1);
theta(3)=exp(theta0(3));
theta(4)=exp(theta0(4));

for j=1:4;
  for k=j:4;
    thetaJ1=theta; thetaJ2=theta; thetaK1=theta; thetaK2=theta;
    thetaJ1(j)=thetaJ1(j)+h;
    thetaJ1(k)=thetaJ1(k)+h;
    [pdf, z]=RieszBesselDensity(thetaJ1(1),thetaJ1(2),thetaJ1(3),thetaJ1(4),N,R);
            likJ1=spline(z,pdf,x);
            likJ1=-sum(log(likJ1.^2)/2);
    thetaJ2(j)=thetaJ2(j)-h;
    thetaJ2(k)=thetaJ2(k)+h;
    [pdf, z]=RieszBesselDensity(thetaJ2(1),thetaJ2(2),thetaJ2(3),thetaJ2(4),N,R);
            likJ2=spline(z,pdf,x);
            likJ2=-sum(log(likJ2.^2)/2);
    thetaK1(k)=thetaK1(k)+h;
    thetaK1(j)=thetaK1(j)-h;
            [pdf,
z]=RieszBesselDensity(thetaK1(1),thetaK1(2),thetaK1(3),thetaK1(4),N,R);
            likK1=spline(z,pdf,x);
            likK1=-sum(log(likK1.^2)/2);
    thetaK2(k)=thetaK2(k)-h;
    thetaK2(j)=thetaK2(j)-h;
    [pdf,
z]=RieszBesselDensity(thetaK2(1),thetaK2(2),thetaK2(3),thetaK2(4),N,R);
            likK2=spline(z,pdf,x);
            likK2=-sum(log(likK2.^2)/2);
    I(j,k)=(likJ1-likJ2-likK1+likK2)/((2*h)^2);
  end
end

I=I+I'-diag(diag(I));
V=I^(-1);
```

```
function lik=RieszBesselLikelihood(theta0,x,N,R)

alpha=exp(theta0(1))/(1+exp(theta0(1)));
gamma=exp(theta0(2))/(1+exp(theta0(2))) - alpha;
kappa=exp(theta0(3));
A=exp(theta0(4));

[pdf, z]=RieszBesselDensity(alpha,gamma,kappa,A,N,R);
lik=spline(z,pdf,x);
lik=-sum(log(lik.^2)/2);
```

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