Eighth Edition

VECTOR MECHANICS FOR ENGINEERS:

DYNAMICS

CHAPTER

12

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Kinetics of Particles: Newton's Second Law

Contents

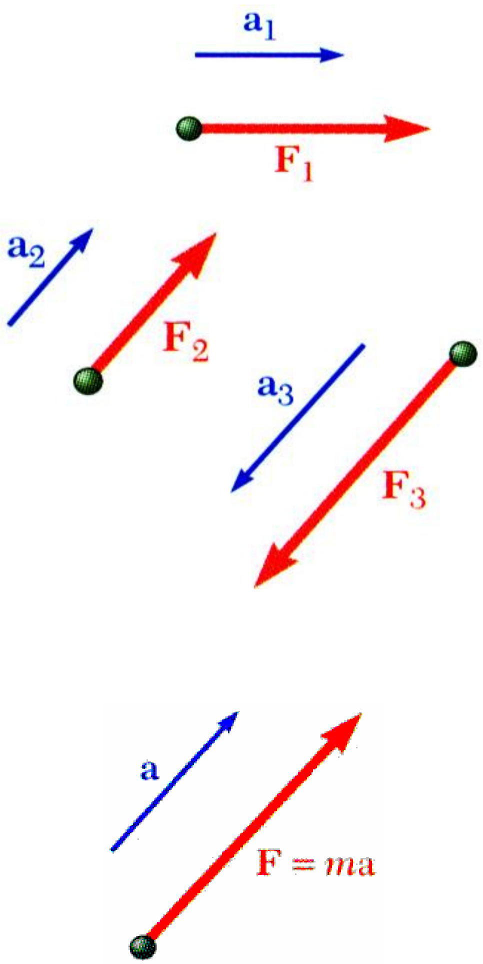
Introduction

- Newton’s first and third laws are sufficient for the study of bodies at rest (statics) or bodies in motion with no acceleration.

- When a body accelerates (changes in velocity magnitude or direction), Newton’s second law is required to relate the motion of the body to the forces acting on it.

- Newton’s second law:
  - A particle will have an acceleration proportional to the magnitude of the resultant force acting on it and in the direction of the resultant force.
  - The resultant of the forces acting on a particle is equal to the rate of change of linear momentum of the particle.
  - The sum of the moments about *O* of the forces acting on a particle is equal to the rate of change of angular momentum of the particle about *O*.

Newton's Second Law of Motion



- *Newton's Second Law*: If the *resultant force* acting on a particle is not zero, the particle will have an acceleration *proportional to the magnitude of resultant* and in the *direction of the resultant*.

- Consider a particle subjected to constant forces,

$$\frac{F_1}{a_1} = \frac{F_2}{a_2} = \frac{F_3}{a_3} = \cdots = \text{constant} = \text{mass},\ m$$

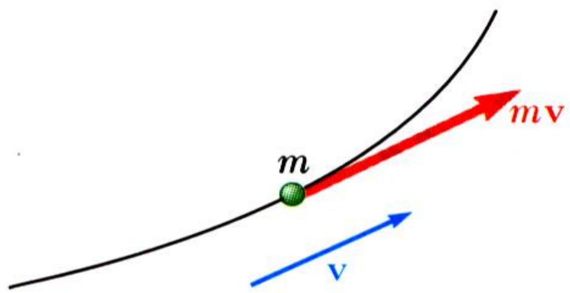
- When a particle of mass m is acted upon by a force $\vec{F}$, the acceleration of the particle must satisfy

$$\vec{F} = m\,\vec{a}$$

- Acceleration must be evaluated with respect to a *Newtonian frame of reference*, i.e., one that is not accelerating or rotating.

- If force acting on particle is zero, particle will not accelerate, i.e., it will remain stationary or continue on a straight line at constant velocity.

Linear Momentum of a Particle



- Replacing the acceleration by the derivative of the velocity yields

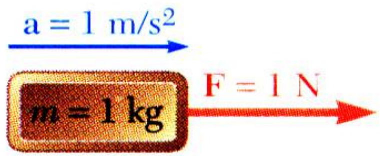
$$\sum \vec{F} = m\frac{d\vec{v}}{dt}$$

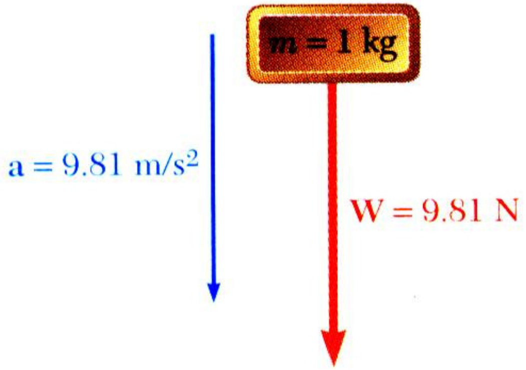
$$= \frac{d}{dt}(m\vec{v}) = \frac{d\vec{L}}{dt}$$

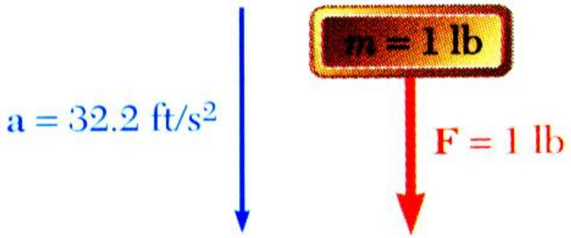
$$\vec{L} = \text{linear momentum of the particle}$$

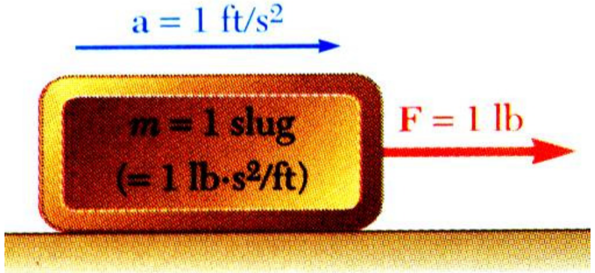
- *Linear Momentum Conservation Principle*:
If the resultant force on a particle is zero, the linear momentum of the particle remains constant in both magnitude and direction.

Systems of Units









- Of the units for the four primary dimensions (force, mass, length, and time), three may be chosen arbitrarily. The fourth must be compatible with Newton's 2nd Law.

- *International System of Units* (SI Units): base units are the units of length (m), mass (kg), and time (second). The unit of force is derived,

$$1\,\mathrm{N} = (1\,\mathrm{kg})\left(1\,\frac{\mathrm{m}}{\mathrm{s}^2}\right) = 1\,\frac{\mathrm{kg}\cdot\mathrm{m}}{\mathrm{s}^2}$$

- *US customary units* – these units are, respectively the foot (ft), the pound (lb) and second (s)

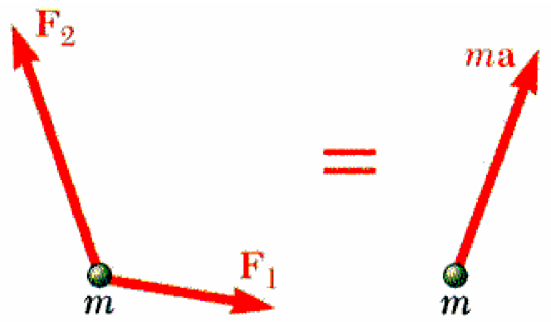
1 foot = 0.3048 m　　　　1 lb = 0.4535 kg

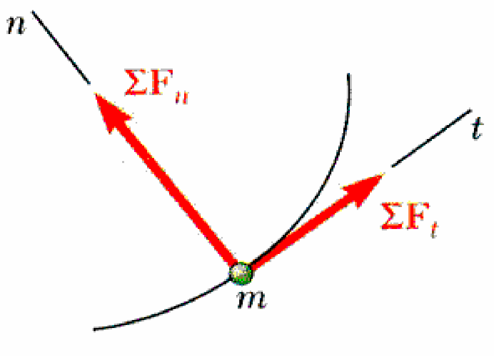
g = 32.2 ft/s^2= 9.81 m/s^2

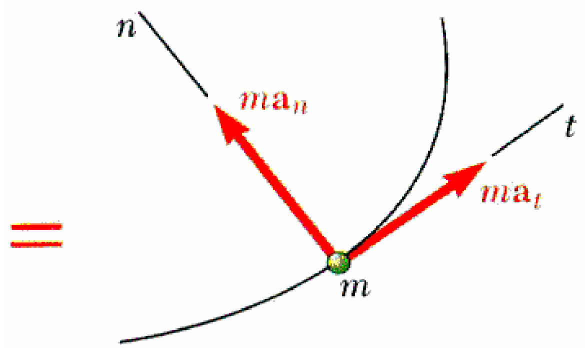
The unit of mass is derived,

$$1\,\mathrm{slug} = \frac{1\,\mathrm{lb}}{1\,\mathrm{ft}/\mathrm{s}^2} = 1\,\frac{\mathrm{lb}\cdot\mathrm{s}^2}{\mathrm{ft}}$$

Equations of Motion







- Newton's second law provides

$$\sum \vec{F} = m\vec{a}$$

- Solution for particle motion is facilitated by resolving vector equation into scalar component equations, e.g., for rectangular components,

$$\sum\left(F_x\vec{i} + F_y\vec{j} + F_z\vec{k}\right) = m\left(a_x\vec{i} + a_y\vec{j} + a_z\vec{k}\right)$$

$$\sum F_x = ma_x \qquad \sum F_y = ma_y \qquad \sum F_z = ma_z$$

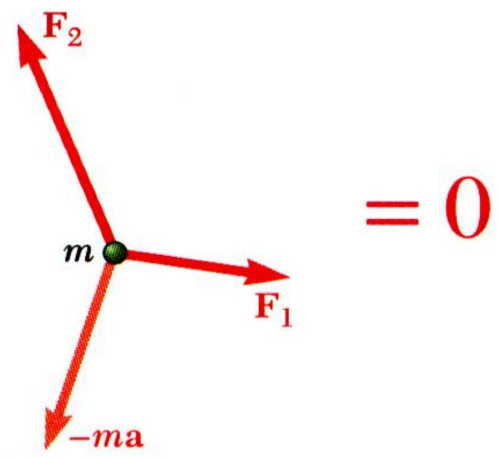
$$\sum F_x = m\ddot{x} \qquad \sum F_y = m\ddot{y} \qquad \sum F_z = m\ddot{z}$$

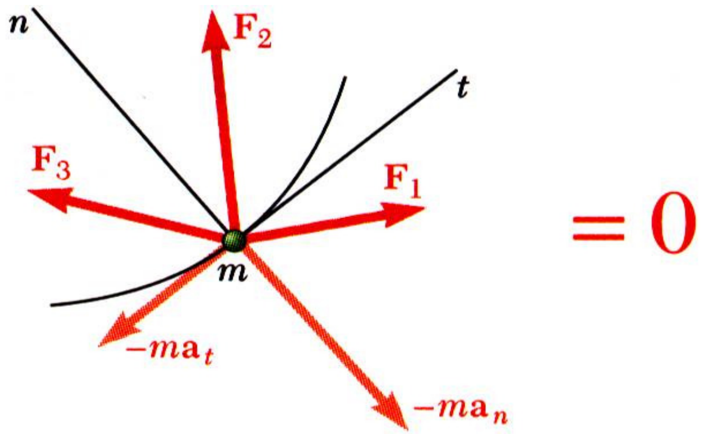
- For tangential and normal components,

$$\sum F_t = ma_t \qquad \sum F_n = ma_n$$

$$\sum F_t = m\frac{dv}{dt} \qquad \sum F_n = m\frac{v^2}{\rho}$$

Dynamic Equilibrium





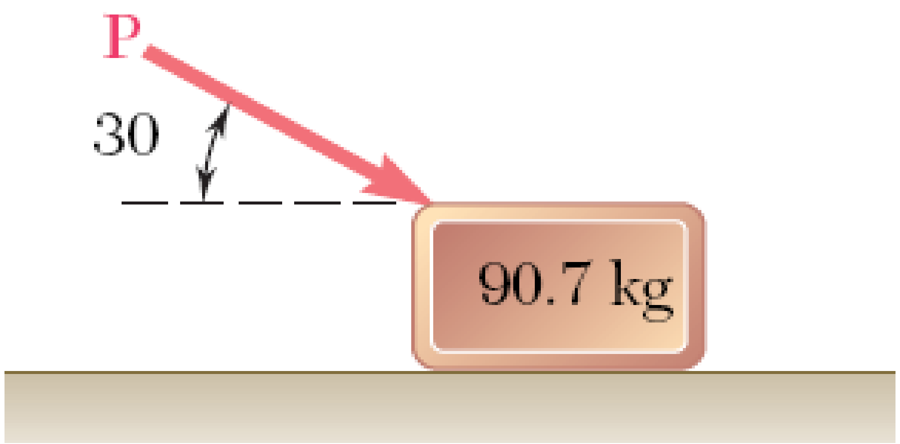
- Alternate expression of Newton's second law,

$$\sum \vec{F} - m\vec{a} = 0$$

$$-m\vec{a} \equiv \textit{inertial vector}$$

- With the inclusion of the inertial vector, the system of forces acting on the particle is equivalent to zero. The particle is in *dynamic equilibrium*.
- Methods developed for particles in static equilibrium may be applied, e.g., coplanar forces may be represented with a closed vector polygon.
- Inertia vectors are often called *inertial forces* as they measure the resistance that particles offer to changes in motion, i.e., changes in speed or direction.
- Inertial forces may be conceptually useful but are not like the contact and gravitational forces found in statics.

Sample Problem 12.1

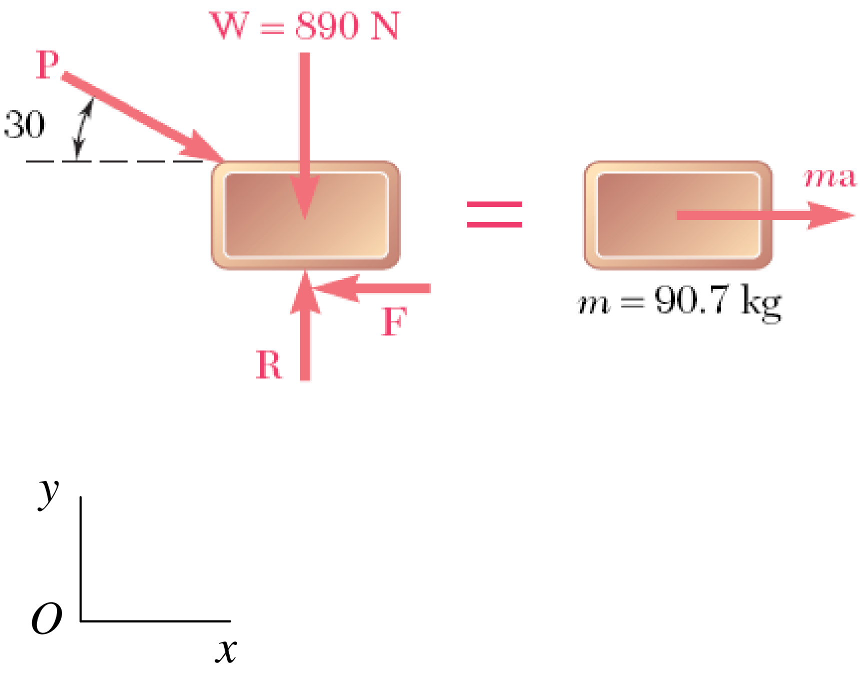


A 90.7 kg block rests on a horizontal plane. Find the magnitude of the force ***P*** required to give the block an accelera-tion or 3 m/s^2 to the right. The coef-ficient of kinetic friction between the block and plane is μ_k = 0.25.

SOLUTION:

- Resolve the equation of motion for the block into two rectangular component equations.
- Unknowns consist of the applied force ***P*** and the normal reaction ***N*** from the plane. The two equations may be solved for these unknowns.

Sample Problem 12.1



$$W = mg = 890\ \mathrm{N}$$

$$F = \mu_k N$$
$$= 0.25N$$

SOLUTION:

- Resolve the equation of motion for the block into two rectangular component equations.

$$\sum F_x = ma:$$

$$P\cos 30° - 0.25R = (90.7\ \mathrm{kg})(3\ \mathrm{m/s^2})$$
$$= 272\ \mathrm{N}$$

$$\sum F_y = 0:$$

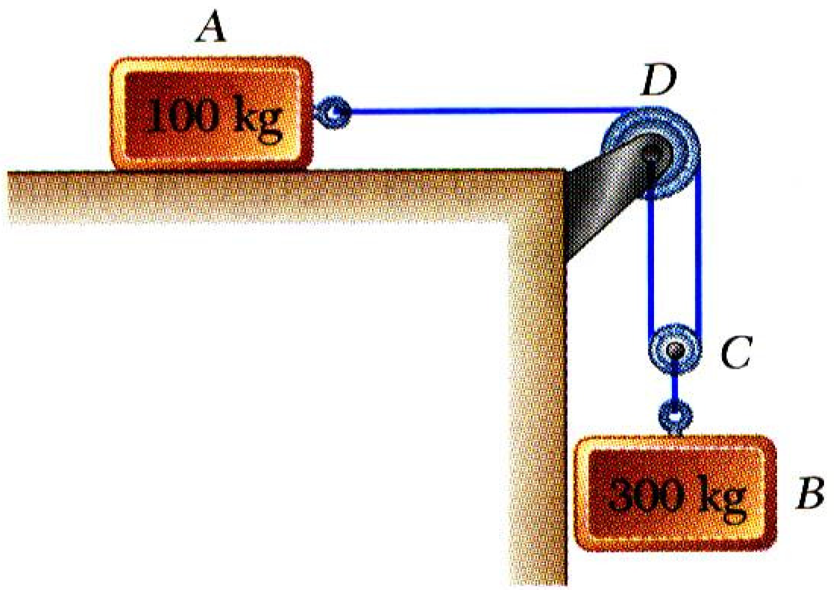
$$R - P\sin 30° - 890\ \mathrm{N} = 0$$

- Unknowns consist of the applied force ***P*** and the normal reaction ***N*** from the plane. The two equations may be solved for these unknowns.

$$N = P\sin 30° + 890\ \mathrm{N}$$
$$P\cos 30° - 0.25(P\sin 30° + 890\ \mathrm{N}) = 272\ \mathrm{N}$$

$$\boxed{P = 667.3\ \mathrm{N}}$$

Sample Problem 12.3

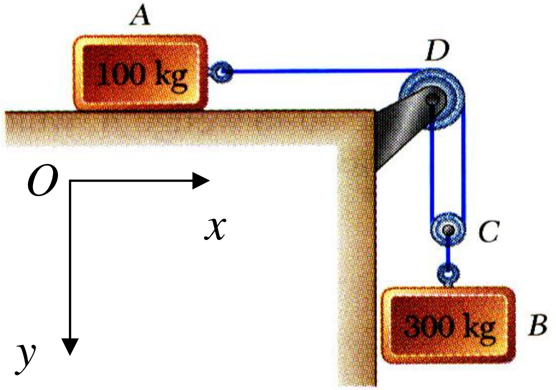


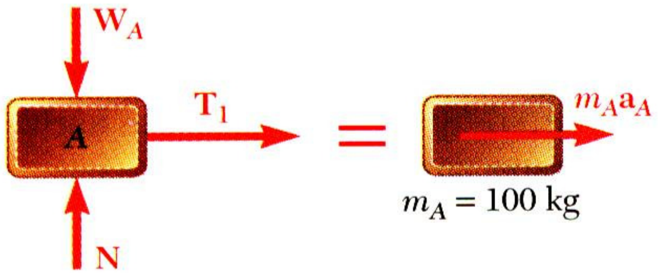
The two blocks shown start from rest. The horizontal plane and the pulley are frictionless, and the pulley is assumed to be of negligible mass. Determine the acceleration of each block and the tension in the cord.

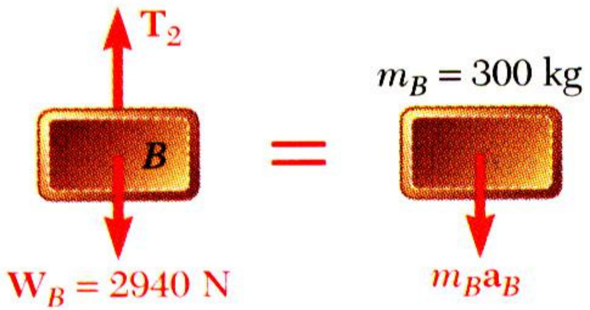
SOLUTION:

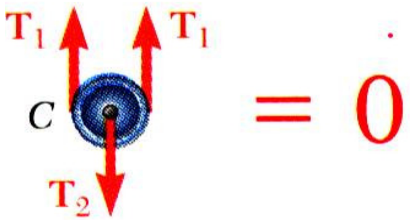
- Write the kinematic relationships for the dependent motions and accelerations of the blocks.
- Write the equations of motion for the blocks and pulley.
- Combine the kinematic relationships with the equations of motion to solve for the accelerations and cord tension.

Sample Problem 12.3









SOLUTION:

- Write the kinematic relationships for the dependent motions and accelerations of the blocks.

$$y_B = \frac{1}{2} x_A \qquad a_B = \frac{1}{2} a_A$$

- Write equations of motion for blocks and pulley.

$$\sum F_x = m_A a_A :$$

$$T_1 = (100\ \text{kg}) a_A$$

$$\sum F_y = m_B a_B :$$

$$m_B g - T_2 = m_B a_B$$

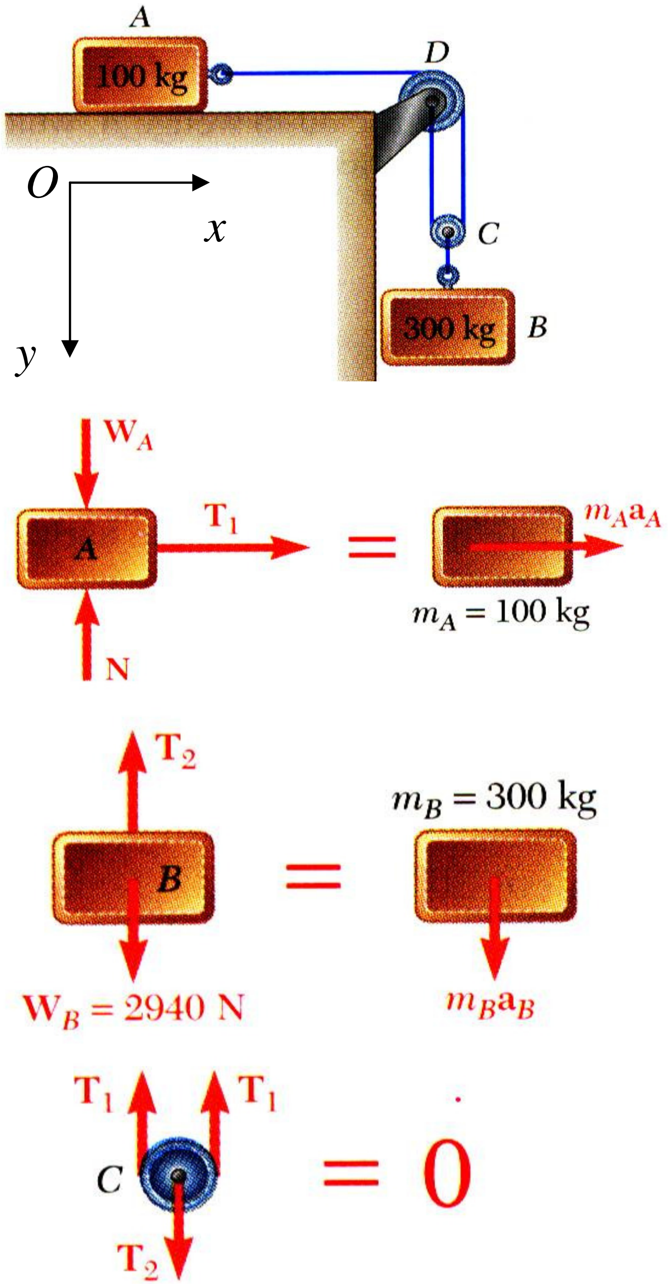
$$(300\ \text{kg})\left(9.81\ \text{m/s}^2\right) - T_2 = (300\ \text{kg}) a_B$$

$$T_2 = 2940\ \text{N} - (300\ \text{kg}) a_B$$

$$\sum F_y = m_C a_C = 0 :$$

$$T_2 - 2T_1 = 0$$

Sample Problem 12.3



- Combine kinematic relationships with equations of motion to solve for accelerations and cord tension.

$$y_B = \tfrac{1}{2} x_A \qquad a_B = \tfrac{1}{2} a_A$$

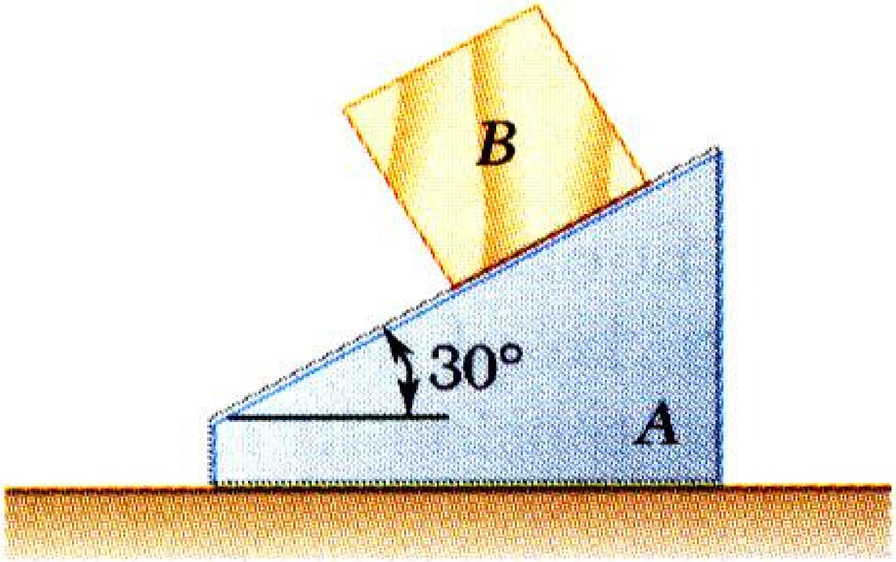
$$T_1 = (100\text{ kg})a_A$$

$$T_2 = 2940\text{ N} - (300\text{ kg})a_B$$
$$= 2940\text{ N} - (300\text{ kg})\left(\tfrac{1}{2} a_A\right)$$

$$T_2 - 2T_1 = 0$$
$$2940\text{ N} - (150\text{ kg})a_A - 2(100\text{ kg})a_A = 0$$

$$\boxed{\begin{aligned} a_A &= 8.40\text{ m/s}^2 \\ a_B &= \tfrac{1}{2} a_A = 4.20\text{ m/s}^2 \\ T_1 &= (100\text{ kg})a_A = 840\text{ N} \\ T_2 &= 2T_1 = 1680\text{ N} \end{aligned}}$$

Sample Problem 12.4



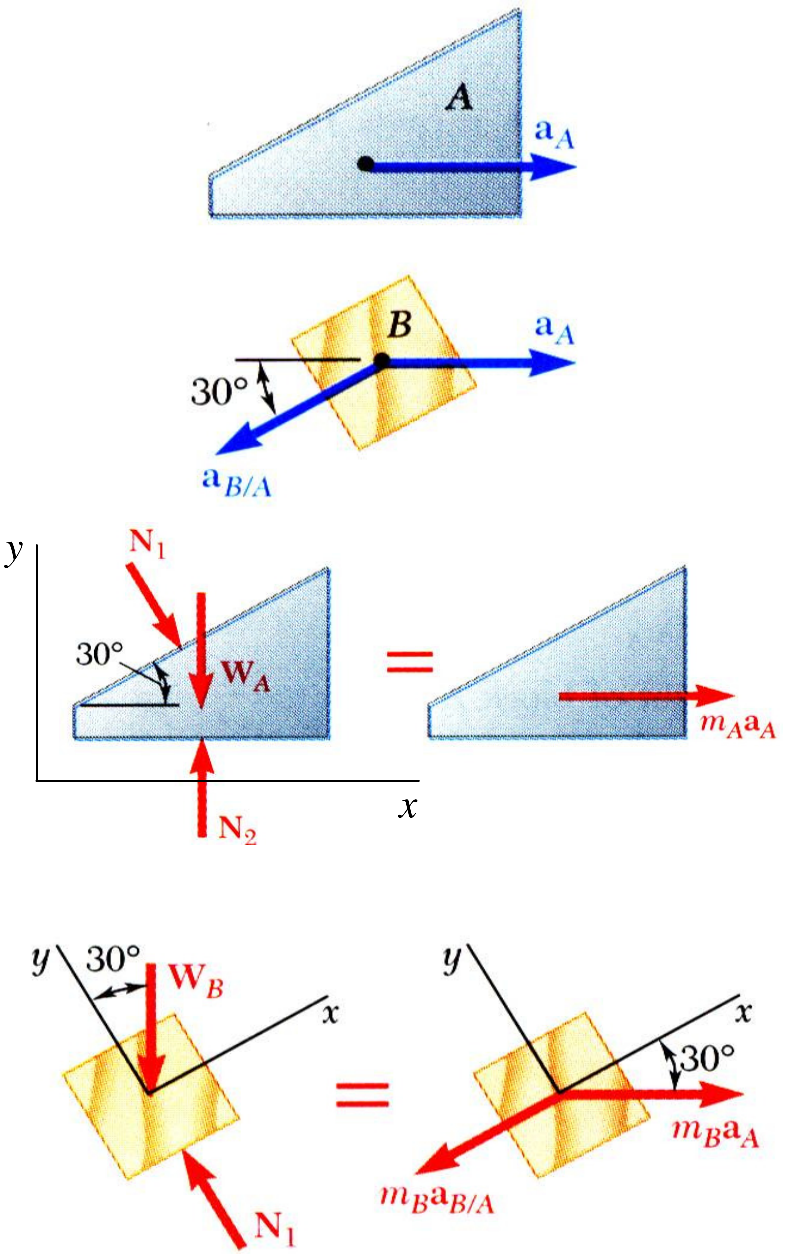
The 5.4 kg block *B* starts from rest and slides on the 13.6 kg wedge *A*, which is supported by a horizontal surface.

Neglecting friction, determine *(a)* the acceleration of the wedge, and *(b)* the acceleration of the block relative to the wedge.

SOLUTION:

- The block is constrained to slide down the wedge. Therefore, their motions are dependent. Express the acceleration of block as the acceleration of wedge plus the acceleration of the block relative to the wedge.
- Write the equations of motion for the wedge and block.
- Solve for the accelerations.

Sample Problem 12.4



SOLUTION:

- The block is constrained to slide down the wedge. Therefore, their motions are dependent.

$$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$$

- Write equations of motion for wedge and block.

$$\sum F_x = m_A a_A :$$

$$N_1 \sin 30° = m_A a_A$$

$$0.5 N_1 = (W_A / g) a_A$$

$$\sum F_x = m_B a_x = m_B \left(a_A \cos 30° - a_{B/A}\right):$$

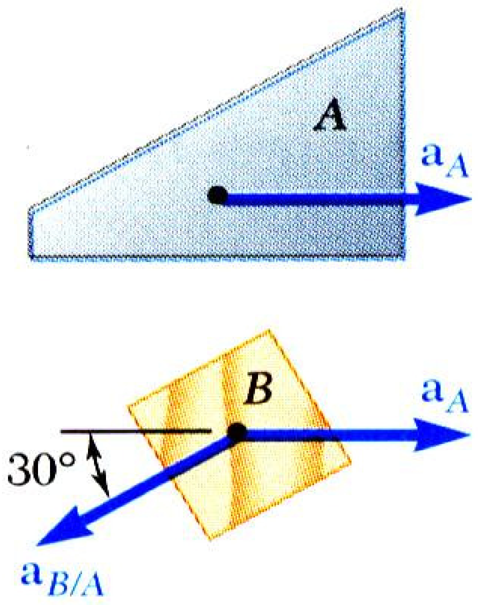
$$-W_B \sin 30° = (W_B / g)\left(a_A \cos 30° - a_{B/A}\right)$$

$$a_{B/A} = a_A \cos 30° + g \sin 30°$$

$$\sum F_y = m_B a_y = m_B \left(-a_A \sin 30°\right):$$

$$N_1 - W_B \cos 30° = -(W_B / g) a_A \sin 30°$$

Sample Problem 12.4



mass of block B m_B = 5.4 kg
Weight of block B $W_B = m_B g$ = 53 N
mass of Wedge A m_A = 13.6 kg
Weight of Wedge A $W_A = m_A g$ = 133.4 N

- Solve for the accelerations.

$$0.5N_1 = (W_A/g)a_A$$

$$N_1 - W_B \cos 30° = -(W_B/g)a_A \sin 30°$$

$$2(W_A/g)a_A - W_B \cos 30° = -(W_B/g)a_A \sin 30°$$

$$a_A = \frac{gW_B \cos 30°}{2W_A + W_B \sin 30°}$$

$$a_A = \frac{(53\,\text{N})\cos 30°}{2(133.4\,\text{N}) + (53\,\text{N})\sin 30°}(9.81\ \text{m/s}^2)$$

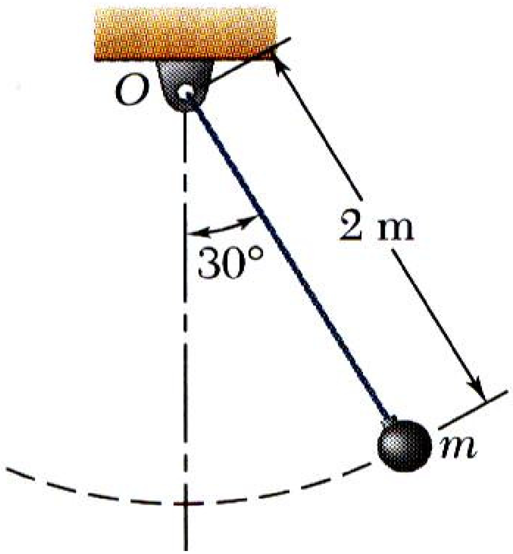
$$\boxed{a_A = 1.53\,\text{m/s}^2}$$

$$a_{B/A} = a_A \cos 30° + g \sin 30°$$

$$a_{B/A} = (1.54\,\text{m/s}^2)\cos 30° + (9.81\,\text{m/s}^2)\sin 30°$$

$$\boxed{a_{B/A} = 6.24\,\text{m/s}^2}$$

Sample Problem 12.5

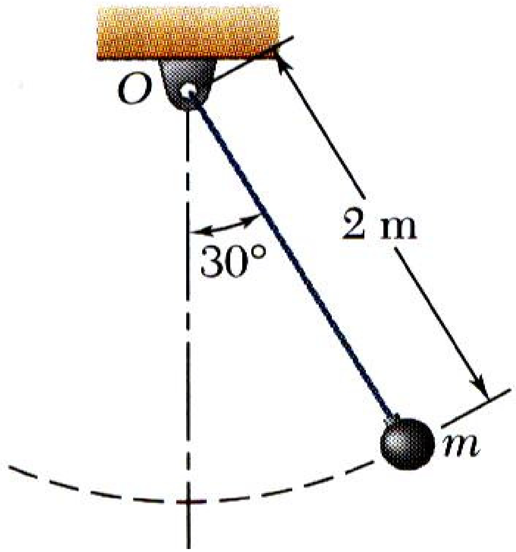


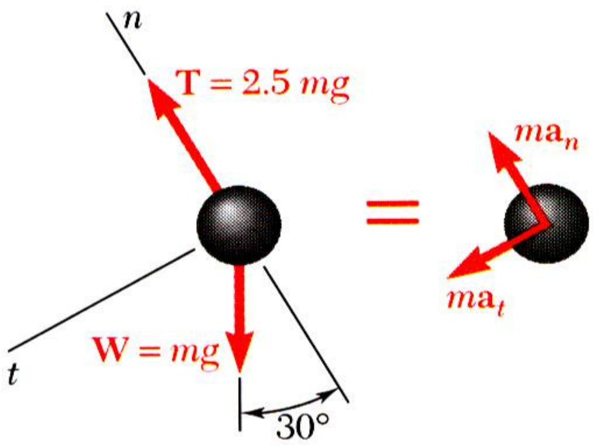
The bob of a 2-m pendulum describes an arc of a circle in a vertical plane. If the tension in the cord is 2.5 times the weight of the bob for the position shown, find the velocity and accel-eration of the bob in that position.

SOLUTION:

- Resolve the equation of motion for the bob into tangential and normal components.
- Solve the component equations for the normal and tangential accelerations.
- Solve for the velocity in terms of the normal acceleration.

Sample Problem 12.5





SOLUTION:

- Resolve the equation of motion for the bob into tangential and normal components.
- Solve the component equations for the normal and tangential accelerations.

$$\sum F_t = ma_t : \qquad mg \sin 30° = ma_t$$

$$a_t = g \sin 30°$$

$$\boxed{a_t = 4.9\,\mathrm{m/s^2}}$$

$$\sum F_n = ma_n : \qquad 2.5mg - mg\cos 30° = ma_n$$

$$a_n = g(2.5 - \cos 30°)$$

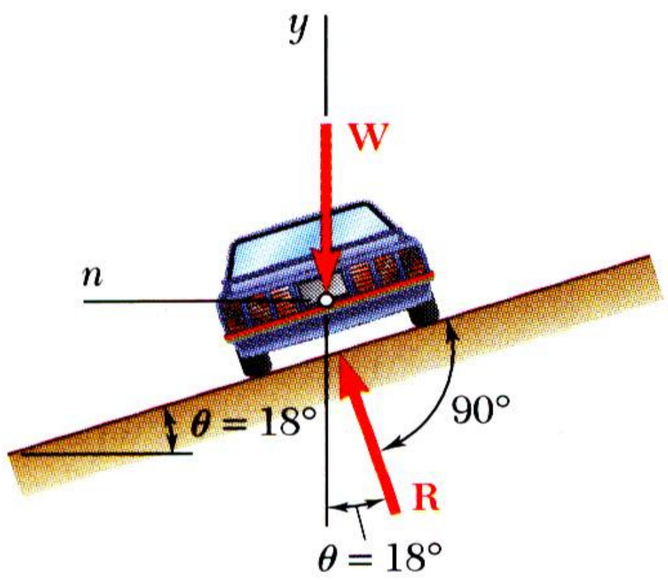
$$\boxed{a_n = 16.01\,\mathrm{m/s^2}}$$

- Solve for velocity in terms of normal acceleration.

$$a_n = \frac{v^2}{\rho} \qquad v = \sqrt{\rho a_n} = \sqrt{(2\,\mathrm{m})\left(16.03\,\mathrm{m/s^2}\right)}$$

$$\boxed{v = \pm 5.66\,\mathrm{m/s}}$$

Sample Problem 12.6

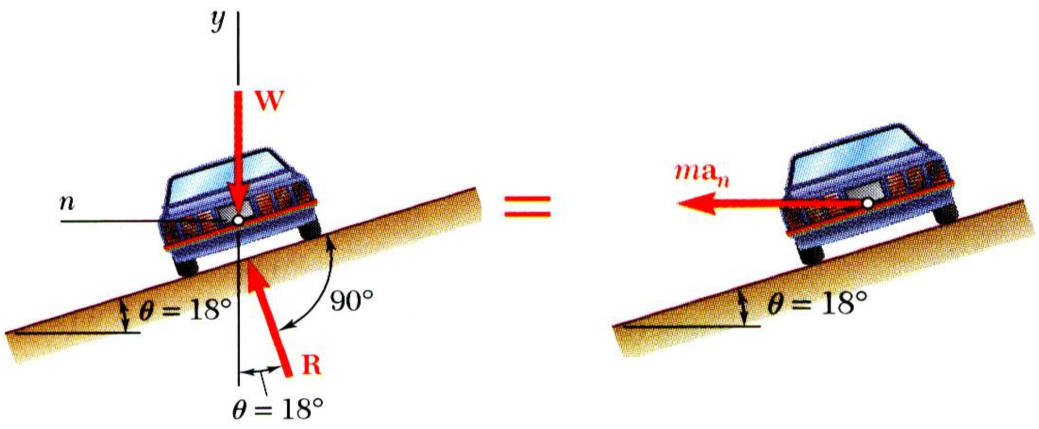


Determine the rated speed of a highway curve of radius $\rho = 122$ m banked through an angle $\theta = 18^{o}$. The rated speed of a banked highway curve is the speed at which a car should travel if no lateral friction force is to be exerted at its wheels.

SOLUTION:

- The car travels in a horizontal circular path with a normal component of acceleration directed toward the center of the path.The forces acting on the car are its weight and a normal reaction from the road surface.
- Resolve the equation of motion for the car into vertical and normal components.
- Solve for the vehicle speed.

Sample Problem 12.6



SOLUTION:

- The car travels in a horizontal circular path with a normal component of acceleration directed toward the center of the path.The forces acting on the car are its weight and a normal reaction from the road surface.

- Resolve the equation of motion for the car into vertical and normal components.

$$\sum F_y = 0: \qquad R\cos\theta - W = 0$$

$$R = \frac{W}{\cos\theta}$$

$$\sum F_n = ma_n: \qquad R\sin\theta = \frac{W}{g}a_n$$

$$\frac{W}{\cos\theta}\sin\theta = \frac{W}{g}\frac{v^2}{\rho}$$

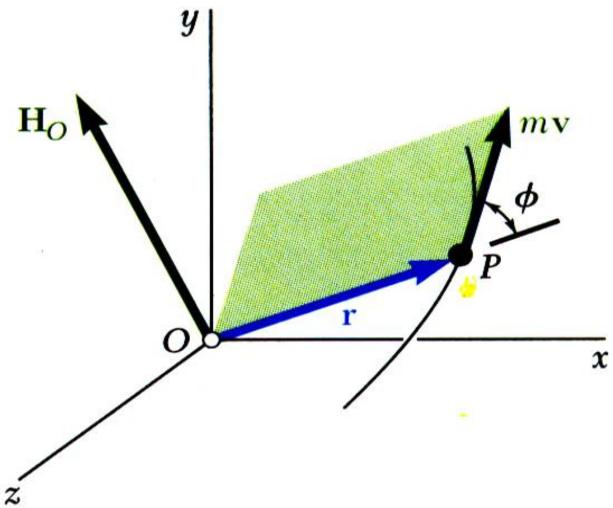
- Solve for the vehicle speed.

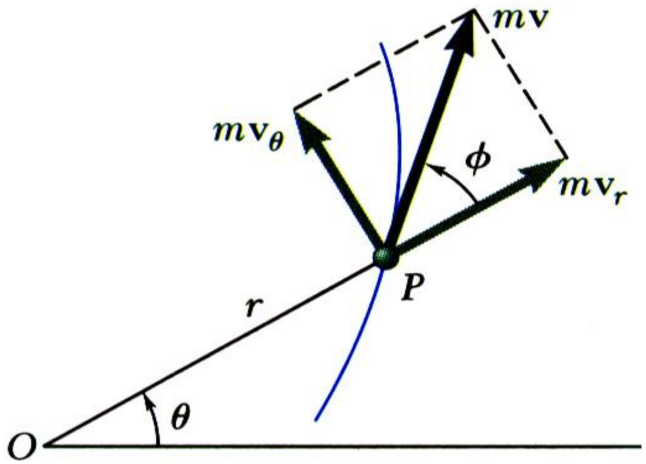
$$v^2 = g\rho\tan\theta$$

$$= \left(9.81\,\mathrm{m/s^2}\right)\left(122\,\mathrm{m}\right)\tan 18°$$

$$\boxed{v = 19.7\ \mathrm{m/s}}$$

Angular Momentum of a Particle





- $\vec{H}_O = \vec{r} \times m\vec{V} =$ *moment of momentum* or the *angular momentum* of the particle about O.

- $\vec{H}_O$ is perpendicular to plane containing $\vec{r}$ and $m\vec{V}$

$$\begin{aligned} H_O &= rmV\sin\phi \\ &= rmv_\theta \\ &= mr^2\dot{\theta} \end{aligned}$$

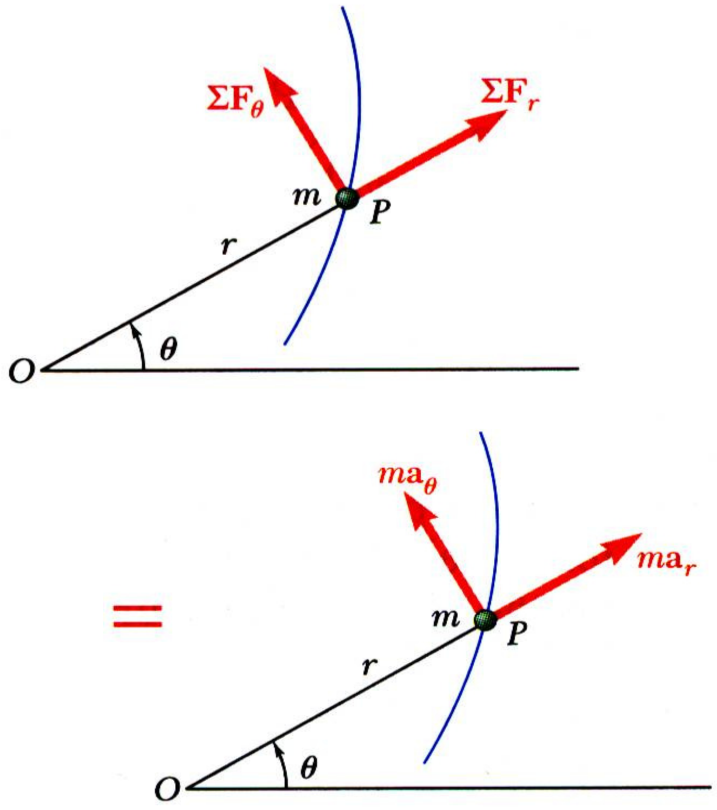
$$\vec{H}_O = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x & y & z \\ mv_x & mv_y & mv_z \end{vmatrix}$$

- Derivative of angular momentum with respect to time,

$$\begin{aligned} \dot{\vec{H}}_O &= \dot{\vec{r}} \times m\vec{V} + \vec{r} \times m\dot{\vec{V}} = \vec{V} \times m\vec{V} + \vec{r} \times m\vec{a} \\ &= r \times \sum \vec{F} \\ &= \sum \vec{M}_O \end{aligned}$$

- It follows from Newton's second law that the sum of the moments about O of the forces acting on the particle is equal to the rate of change of the angular momentum of the particle about O.

Eqs of Motion in Radial & Transverse Components



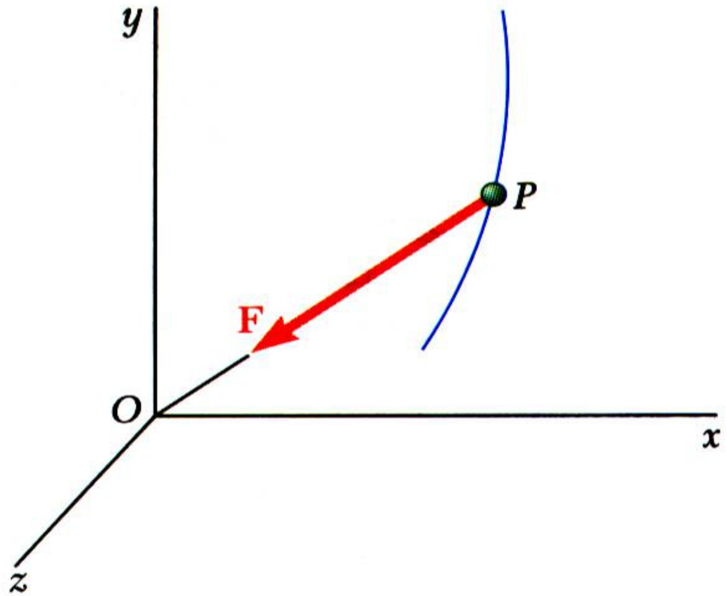
- Consider particle at r and θ, in polar coordinates,

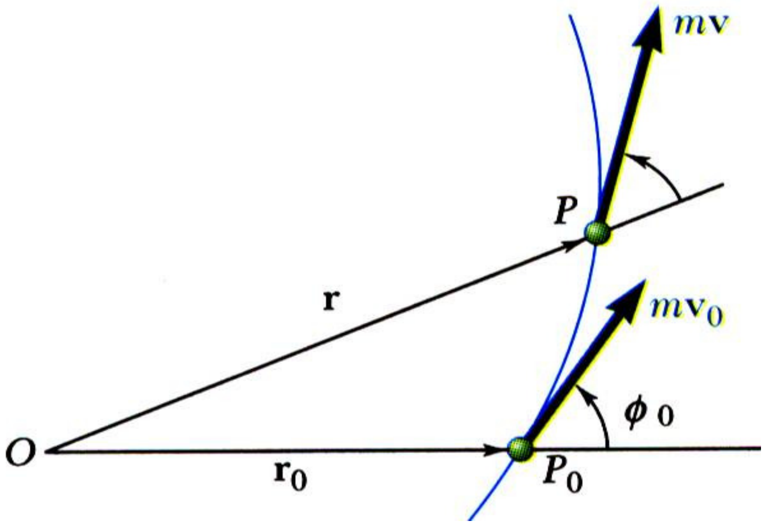
$$\sum F_r = ma_r = m\left(\ddot{r} - r\dot{\theta}^2\right)$$
$$\sum F_\theta = ma_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

- This result may also be derived from conservation of angular momentum,

$$H_O = mr^2\dot{\theta}$$
$$r\sum F_\theta = \frac{d}{dt}\left(mr^2\dot{\theta}\right)$$
$$= m\left(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}\right)$$
$$\sum F_\theta = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

Conservation of Angular Momentum





- When only force acting on particle is directed toward or away from a fixed point O, the particle is said to be *moving under a central force*.
- Since the line of action of the central force passes through O, $\sum \vec{M}_O = \dot{\vec{H}}_O = 0$ and

$$\vec{r} \times m\vec{V} = \vec{H}_O = \text{constant}$$

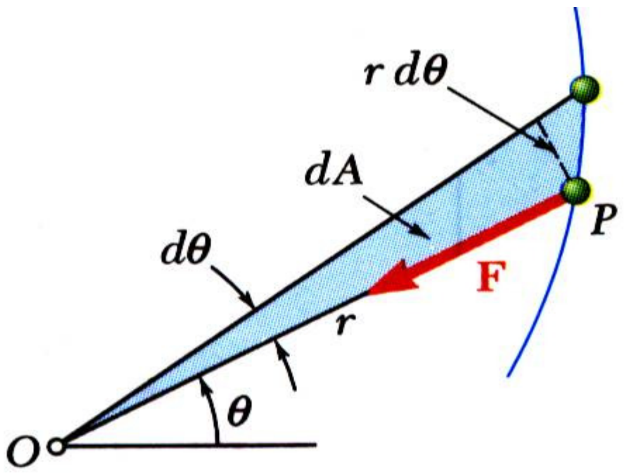
- Position vector and motion of particle are in a plane perpendicular to $\vec{H}_O$.
- Magnitude of angular momentum,

$$H_O = rmV \sin\phi = \text{constant}$$
$$= r_0 m V_0 \sin\phi_0$$

or $$H_O = mr^2\dot{\theta} = \text{constant}$$

$$\frac{H_O}{m} = r^2\dot{\theta} = h = \frac{\text{angular momentum}}{\text{unit mass}}$$

Conservation of Angular Momentum



- Radius vector OP sweeps infinitesimal area

$$dA = \frac{1}{2} r^2 d\theta$$

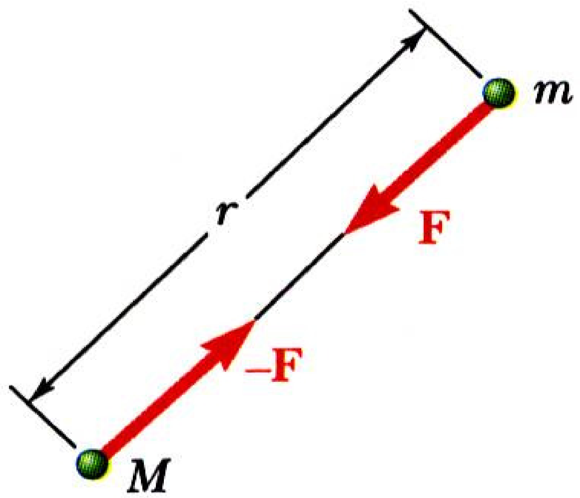
- Define $$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = \textit{areal velocity}$$

- Recall, for a body moving under a central force,

$$h = r^2 \dot{\theta} = \text{constant}$$

- When a particle moves under a central force, its areal velocity is constant.

Newton's Law of Gravitation



- Gravitational force exerted by the sun on a planet or by the earth on a satellite is an important example of gravitational force.
- *Newton's law of universal gravitation* - two particles of mass M and m attract each other with equal and opposite force directed along the line connecting the particles,

$$F = G\frac{Mm}{r^2}$$

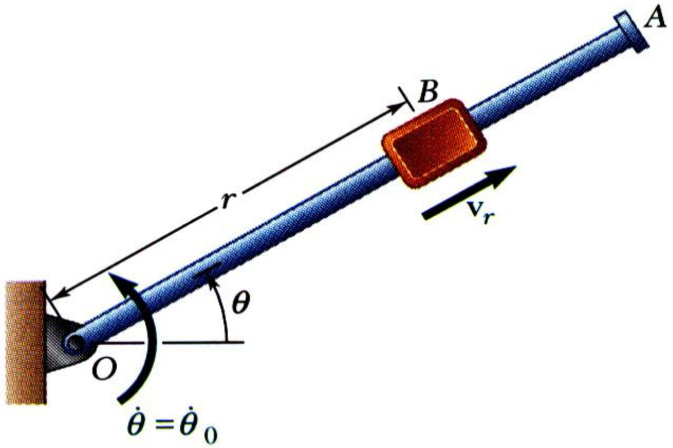
$$G = \text{constant of gravitation}$$

$$= 66.73\times10^{-12}\,\frac{\mathrm{m}^3}{\mathrm{kg}\cdot\mathrm{s}^2} = 34.4\times10^{-9}\,\frac{\mathrm{ft}^4}{\mathrm{lb}\cdot\mathrm{s}^4}$$

- For particle of mass m on the earth's surface,

$$W = m\frac{MG}{R^2} = mg \qquad g = 9.81\,\frac{\mathrm{m}}{\mathrm{s}^2} = 32.2\,\frac{\mathrm{ft}}{\mathrm{s}^2}$$

Sample Problem 12.7



A block *B* of mass *m* can slide freely on a frictionless arm *OA* which rotates in a horizontal plane at a constant rate $\dot{\theta}_0$.

Knowing that *B* is released at a distance r_0 from *O*, express as a function of *r*

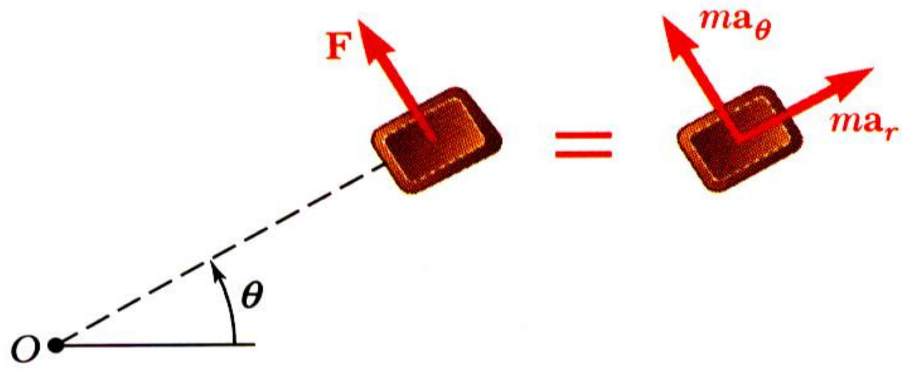
a) the component v_r of the velocity of *B* along *OA*, and

b) the magnitude of the horizontal force exerted on *B* by the arm *OA*.

SOLUTION:

- Write the radial and transverse equations of motion for the block.
- Integrate the radial equation to find an expression for the radial velocity.
- Substitute known information into the transverse equation to find an expression for the force on the block.

Sample Problem 12.7



SOLUTION:

- Write the radial and transverse equations of motion for the block.

$$\sum F_r = m\,a_r: \quad 0 = m\left(\ddot{r} - r\dot{\theta}^2\right)$$

$$\sum F_\theta = m\,a_\theta: \quad F = m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)$$

- Integrate the radial equation to find an expression for the radial velocity.

$$\ddot{r} = \dot{v}_r = \frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r\frac{dv_r}{dr}$$

$$\ddot{r} = \dot{v}_r = \frac{dv_r}{dt} = \frac{dv_r}{dr}\frac{dr}{dt} = v_r\frac{dv_r}{dr}$$

$$v_r dv_r = r\dot{\theta}^2 dr = r\dot{\theta}_0^2 dr$$

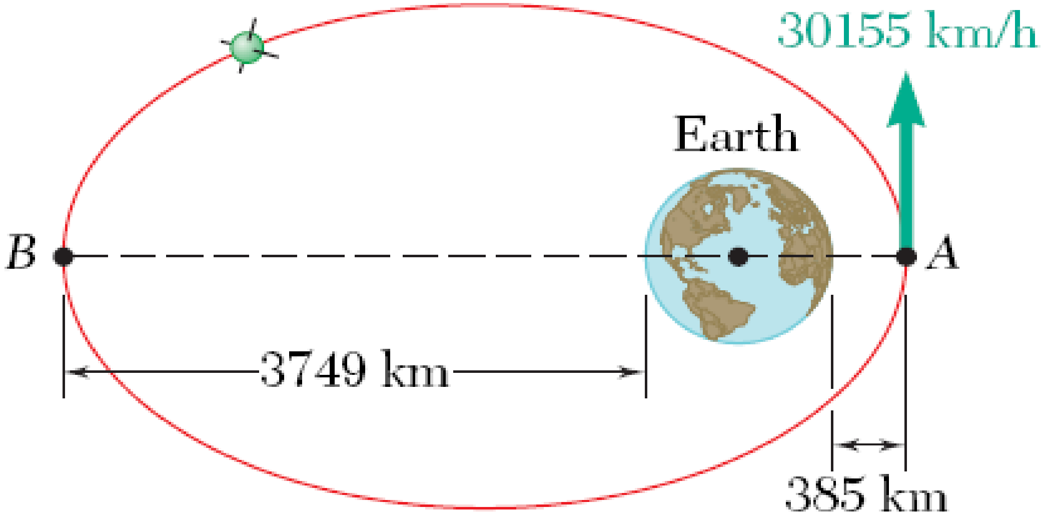
$$\int_0^{v_r} v_r dv_r = \dot{\theta}_0^2 \int_{r_0}^{r} r\,dr$$

$$\boxed{v_r^2 = \theta_0^2\left(r^2 - r_0^2\right)}$$

- Substitute known information into the transverse equation to find an expression for the force on the block.

$$\boxed{F = 2m\,\theta_0^2\left(r^2 - r_0^2\right)^{1/2}}$$

Sample Problem 12.8

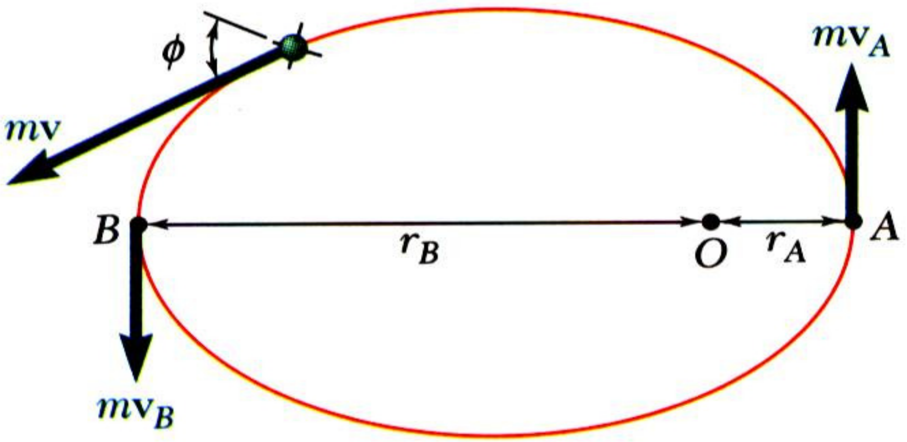


A satellite is launched in a direction parallel to the surface of the earth with a velocity of 30155 km/h from an altitude of 385 km. Determine the velocity of the satellite as it reaches it maximum altitude of 3749 km. The radius of the earth is 6345 km.

SOLUTION:

- Since the satellite is moving under a central force, its angular momentum is constant. Equate the angular momentum at A and B and solve for the velocity at B.

Sample Problem 12.8



SOLUTION:

- Since the satellite is moving under a central force, its angular momentum is constant. Equate the angular momentum at *A* and *B* and solve for the velocity at *B*.

$$rmv\sin\phi = H_O = \text{constant}$$

$$r_A m v_A = r_B m v_B$$

$$v_B = v_A \frac{r_A}{r_B}$$

$$= (30155\ \text{km/h})\frac{(6345\text{km} + 385\ \text{km})}{(6345\text{km} + 3749\text{km})}$$

$$\boxed{v_B = 20105\ \text{km/h}}$$

Trajectory of a Particle Under a Central Force

- For particle moving under central force directed towards force center,

$$m\left(\ddot{r} - r\dot{\theta}^2\right) = \sum F_r = -F \qquad m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right) = \sum F_\theta = 0$$

- Second expression is equivalent to $r^2\dot{\theta} = h = \text{constant}$, from which,

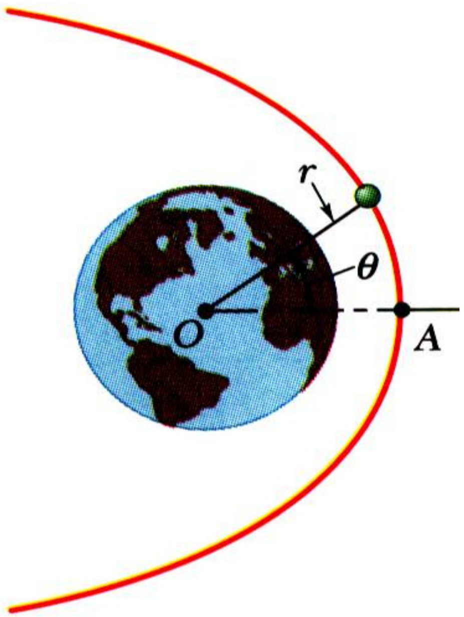
$$\dot{\theta} = \frac{h}{r^2} \quad \text{and} \quad \ddot{r} = -\frac{h^2}{r^2}\frac{d^2}{d\theta^2}\left(\frac{1}{r}\right)$$

- After substituting into the radial equation of motion and simplifying,

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{mh^2u^2} \quad \text{where} \quad u = \frac{1}{r}$$

- If F is a known function of r or u, then particle trajectory may be found by integrating for $u = f(\theta)$, with constants of integration determined from initial conditions.

Application to Space Mechanics



- Consider earth satellites subjected to only gravitational pull of the earth,

$$\frac{d^2u}{d\theta^2}+u=\frac{F}{mh^2u^2} \quad \text{where} \quad u=\frac{1}{r} \qquad F=\frac{GMm}{r^2}=GMmu^2$$

$$\frac{d^2u}{d\theta^2}+u=\frac{GM}{h^2}=\text{constant}$$

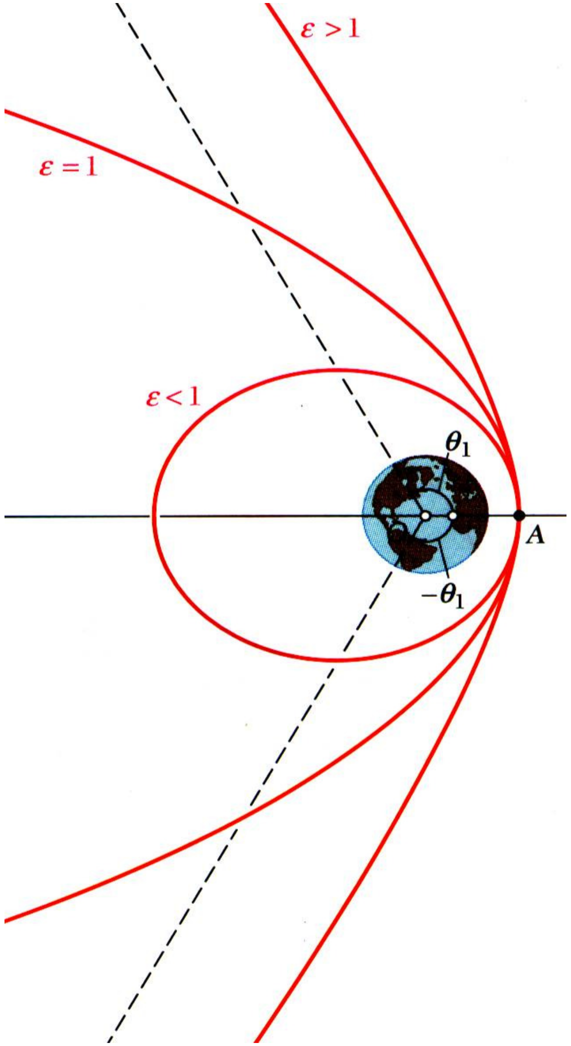
- Solution is equation of conic section,

$$u=\frac{1}{r}=\frac{GM}{h^2}\left(1+\varepsilon\cos\theta\right) \qquad \varepsilon=\frac{Ch^2}{GM}=\text{eccentricity}$$

- Origin, located at earth's center, is a focus of the conic section.

- Trajectory may be ellipse, parabola, or hyperbola depending on value of eccentricity.

Application to Space Mechanics



- Trajectory of earth satellite is defined by

$$\frac{1}{r} = \frac{GM}{h^2}\left(1 + \varepsilon\cos\theta\right) \qquad \varepsilon = \frac{Ch^2}{GM} = \text{eccentricity}$$

- *hyperbola*, $\varepsilon > 1$ or $C > GM/h^2$. The radius vector becomes infinite for

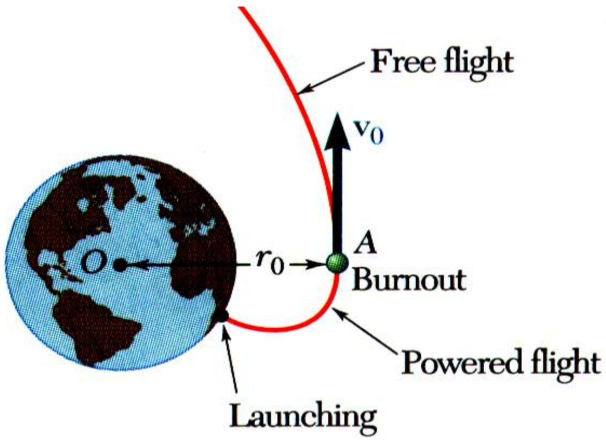
$$1 + \varepsilon\cos\theta_1 = 0 \quad \theta_1 = \pm\cos^{-1}\left(-\frac{1}{\varepsilon}\right) = \pm\cos^{-1}\left(-\frac{GM}{Ch^2}\right)$$

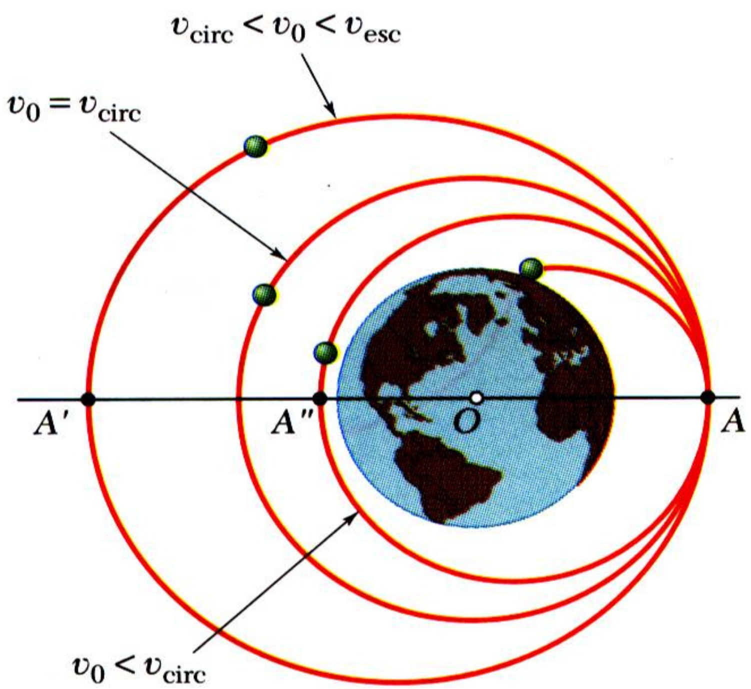
- *parabola*, $\varepsilon = 1$ or $C = GM/h^2$. The radius vector becomes infinite for

$$1 + \cos\theta_2 = 0 \quad \theta_2 = 180°$$

- *ellipse*, $\varepsilon < 1$ or $C < GM/h^2$. The radius vector is finite for θ and is constant, i.e., a circle, for $\varepsilon < 0$.

Application to Space Mechanics





- Integration constant C is determined by conditions at beginning of free flight, $\theta = 0$, $r = r_0$,

$$\frac{1}{r_0} = \frac{GM}{h^2}\left(1 + \frac{Ch^2}{GM}\cos 0°\right)$$

$$C = \frac{1}{r_0} - \frac{GM}{h^2} = \frac{1}{r_0} - \frac{GM}{(r_0 v_0)^2}$$

- Satellite escapes earth orbit for

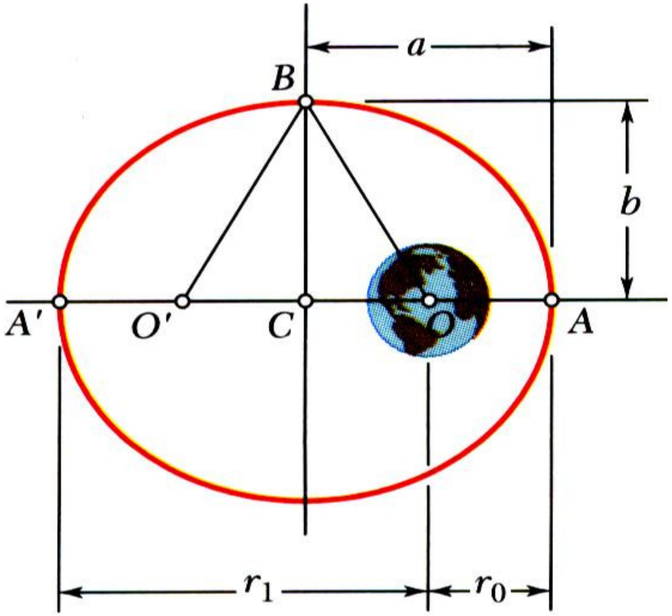
$$\varepsilon \geq 1 \text{ or } C \geq GM/h^2 = GM/(r_0 v_0)^2$$

$$v_{esc} = v_0 = \sqrt{\frac{2GM}{r_0}}$$

- Trajectory is elliptic for $v_0 < v_{esc}$ and becomes circular for $\varepsilon = 0$ or $C = 0$,

$$v_{circ} = \sqrt{\frac{GM}{r_0}}$$

Application to Space Mechanics



- Recall that for a particle moving under a central force, the *areal velocity* is constant, i.e.,

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{1}{2} h = \text{constant}$$

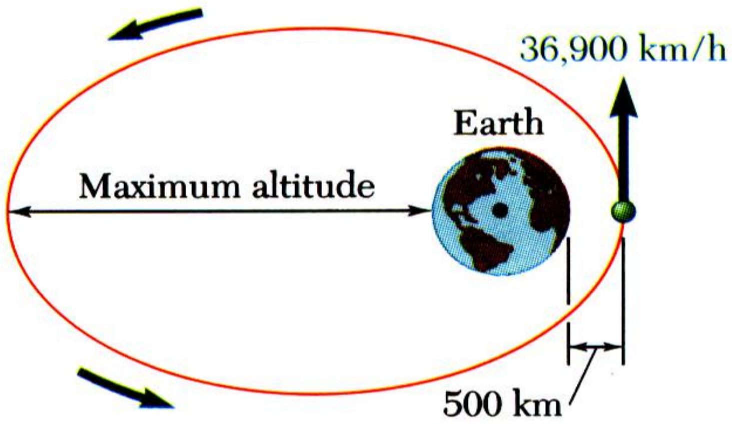
- *Periodic time* or time required for a satellite to complete an orbit is equal to area within the orbit divided by *areal velocity*,

$$\tau = \frac{\pi ab}{h/2} = \frac{2\pi ab}{h}$$

where $\quad a = \frac{1}{2}(r_0 + r_1)$

$$b = \sqrt{r_0 r_1}$$

Sample Problem 12.9



A satellite is launched in a direction parallel to the surface of the earth with a velocity of 36,900 km/h at an altitude of 500 km.

Determine:

a) the maximum altitude reached by the satellite, and

b) the periodic time of the satellite.

SOLUTION:

- Trajectory of the satellite is described by

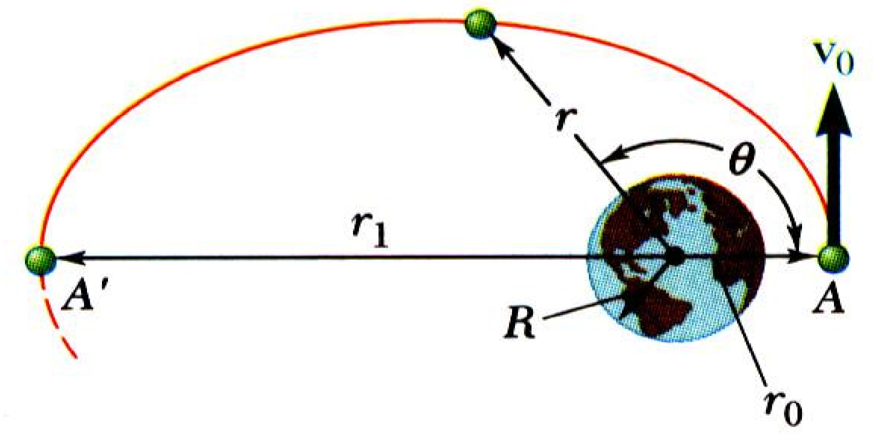
$$\frac{1}{r} = \frac{GM}{h^2} + C\cos\theta$$

Evaluate C using the initial conditions at $\theta = 0$.

- Determine the maximum altitude by finding r at $\theta = 180^o$.

- With the altitudes at the perigee and apogee known, the periodic time can be evaluated.

Sample Problem 12.9



$$r_0 = (6370+500)\,\text{km} = 6.87\times10^6\,\text{m}$$

$$v_0 = 36900\frac{\text{km}}{\text{h}}\times\frac{1000\,\text{m/km}}{3600\,\text{s/h}} = 10.25\times10^3\,\text{m/s}$$

$$h = r_0 v_0 = \left(6.87\times10^6\,\text{m}\right)\left(10.25\times10^3\,\text{m/s}\right) = 70.4\times10^9\,\text{m}^2/\text{s}$$

$$GM = gR^2 = \left(9.81\,\text{m/s}^2\right)\left(6.37\times10^6\,\text{m}\right)^2 = 397.6\times10^{12}\,\text{m}^3/\text{s}^2$$

SOLUTION:

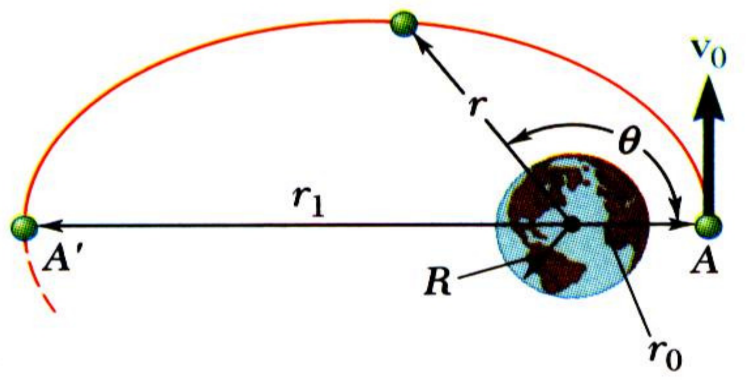
- Trajectory of the satellite is described by

$$\frac{1}{r} = \frac{GM}{h^2} + C\cos\theta$$

Evaluate C using the initial conditions at $\theta = 0$.

$$C = \frac{1}{r_0} - \frac{GM}{h^2} = \frac{1}{6.87\times10^6\,\text{m}} - \frac{397.6\times10^{12}\,\text{m}^3/\text{s}^2}{\left(70.4\,\text{m}^2/\text{s}\right)^2} = 65.3\times10^{-9}\,\text{m}^{-1}$$

Sample Problem 12.9

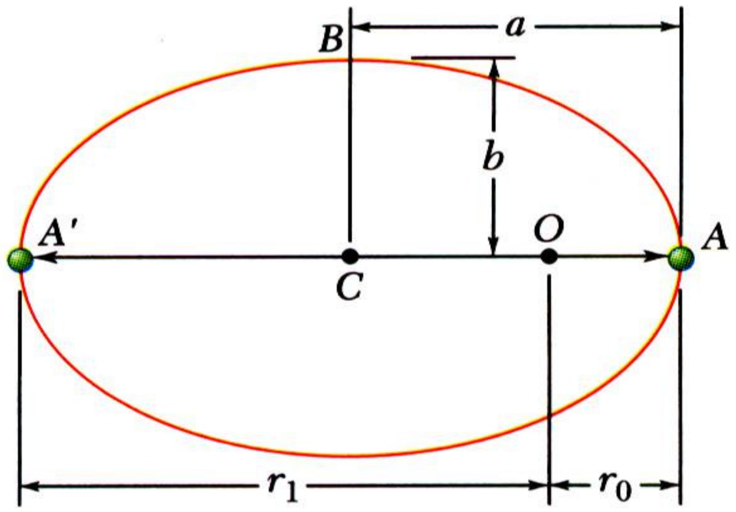


- Determine the maximum altitude by finding r_1 at $\theta = 180°$.

$$\frac{1}{r_1} = \frac{GM}{h^2} - C = \frac{397.6\times10^{12}\ \mathrm{m^3/s^2}}{\left(70.4\ \mathrm{m^2/s}\right)^2} - 65.3\times10^{-9}\ \frac{1}{\mathrm{m}}$$

$$r_1 = 67.1\times10^6\ \mathrm{m} = 67100\ \mathrm{km}$$

$$\boxed{\text{max altitude} = (67100 - 6370)\,\mathrm{km} = 60730\ \mathrm{km}}$$



- With the altitudes at the perigee and apogee known, the periodic time can be evaluated.

$$a = \tfrac{1}{2}\left(r_0 + r_1\right) = \tfrac{1}{2}(6.87 + 66.7)\times10^6\ \mathrm{m} = 36.8\times10^6\ \mathrm{m}$$

$$b = \sqrt{r_0 r_1} = \sqrt{6.87\times66.7}\times10^6\ \mathrm{m} = 21.4\times10^6\ \mathrm{m}$$

$$\tau = \frac{2\pi ab}{\mathrm{h}} = \frac{2\pi\left(36.8\times10^6\ \mathrm{m}\right)\left(21.4\times10^6\ \mathrm{m}\right)}{70.4\times10^9\ \mathrm{m^2/s}}$$

$$\boxed{\tau = 70.3\times10^3\ \mathrm{s} = 19\ \mathrm{h}\ 43\ \mathrm{min}}$$

Kepler's Laws of Planetary Motion

- Results obtained for trajectories of satellites around earth may also be applied to trajectories of planets around the sun.
- Properties of planetary orbits around the sun were determined astronomical observations by Johann Kepler (1571-1630) before Newton had developed his fundamental theory.

1) Each planet describes an ellipse, with the sun located at one of its foci.
2) The radius vector drawn from the sun to a planet sweeps equal areas in equal times.
3) The squares of the periodic times of the planets are proportional to the cubes of the semimajor axes of their orbits.