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subject to

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} 400\mu_{13} \\ 600\mu_{12} + 200\mu_{13} \end{Bmatrix} + \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} 1000\mu_{22} + 1000\mu_{23} \\ 2000\mu_{22} \end{Bmatrix} \le \begin{Bmatrix} 1000 \\ 500 \end{Bmatrix}$$

that is,

$$\begin{aligned} 600\mu_{12} + 600\mu_{13} + 3000\mu_{22} + 1000\mu_{23} &\le 1000 \\ 400\mu_{13} + 1000\mu_{22} + 1000\mu_{23} &\le 500 \\ \mu_{11} + \mu_{12} + \mu_{13} &= 1 \\ \mu_{21} + \mu_{22} + \mu_{23} &= 1 \end{aligned}$$

with

$$\mu_{11} \ge 0, \quad \mu_{12} \ge 0, \quad \mu_{13} \ge 0, \quad \mu_{21} \ge 0, \quad \mu_{22} \ge 0, \quad \mu_{23} \ge 0$$

The optimization problem can be stated in standard form (after adding the slack variables α and β) as:

$$\text{Minimize } f = -1200\mu_{12} - 800\mu_{13} - 8000\mu_{22} - 2000\mu_{23}$$

subject to

$$\begin{aligned} 600\mu_{12} + 600\mu_{13} + 3000\mu_{22} + 1000\mu_{23} + \alpha &= 1000 \\ 400\mu_{13} + 1000\mu_{22} + 1000\mu_{23} + \beta &= 500 \\ \mu_{11} + \mu_{12} + \mu_{13} &= 1 \\ \mu_{21} + \mu_{22} + \mu_{23} &= 1 \\ \mu_{ij} \ge 0\ (i = 1,2; j = 1,2,3), \quad \alpha \ge 0, \quad \beta &\ge 0 \end{aligned} \tag{E$_{10}$}$$

Step 3: The problem (E_{10}) can now be solved by using the simplex method.

4.5 SENSITIVITY OR POSTOPTIMALITY ANALYSIS

In most practical problems, we are interested not only in optimal solution of the LP problem, but also in how the solution changes when the parameters of the problem change. The change in the parameters may be discrete or continuous. The study of the effect of discrete parameter changes on the optimal solution is called *sensitivity analysis* and that of the continuous changes is termed *parametric programming*. One way to determine the effects of changes in the parameters is to solve a series of new problems once for each of the changes made. This is, however, very inefficient from a computational point of view. Some techniques that take advantage of the properties of the simplex solution are developed to make a sensitivity analysis. We study some of these techniques in this section. There are five basic types of parameter changes that affect the optimal solution. They are:

1. Changes in the right-hand-side constants b_i
2. Changes in the cost coefficients c_j
3. Changes in the coefficients of the constraints a_{ij}
4. Addition of new variables
5. Addition of new constraints

In general, when a parameter is changed, it results in one of the three cases:

1. The optimal solution remains unchanged; that is, the basic variables and their values remain unchanged.
2. The basic variables remain the same but their values are changed.
3. The basic variables as well as their values are changed.

4.5.1 Changes in the Right-Hand-Side Constants b_i

Suppose that we have found the optimal solution to a LP problem. Let us now change the b_i to $b_i + \Delta b_i$ so that the new problem differs from the original only on the right-hand side. Our interest is to investigate the effect of changing b_i to $b_i + \Delta b_i$ on the original optimum. We know that a basis is optimal if the relative cost coefficients corresponding to the nonbasic variables $\bar{c}_j$ are nonnegative. By considering the procedure according to which $\bar{c}_j$ are obtained, we can see that the values of $\bar{c}_j$ are not related to the b_i. The values of $\bar{c}_j$ depend only on the basis, on the coefficients of the constraint matrix, and the original coefficients of the objective function. The relation is given in Eq. (4.10):

$$\bar{c}_j = c_j - \boldsymbol{\pi}^T\mathbf{A}_j = c_j - \mathbf{c}_B^T\mathbf{B}^{-1}\mathbf{A}_j \tag{4.33}$$

Thus changes in b_i will affect the values of basic variables in the optimal solution and the optimality of the basis will not be affected provided that the changes made in b_i do not make the basic solution infeasible. Thus if the new basic solution remains feasible for the new right-hand side, that is, if

$$\mathbf{X}_B' = \mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b}) \geq \mathbf{0} \tag{4.34}$$

then the original optimal basis, **B**, also remains optimal for the new problem. Since the original solution, say†

$$\mathbf{X}_B = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{Bmatrix}$$

is given by

$$\mathbf{X}_B = \mathbf{B}^{-1}\mathbf{b} \tag{4.35}$$

Eq. (4.34) can also be expressed as

$$x_i' = x_i + \sum_{j=1}^{m} \beta_{ij}\,\Delta b_j \geq 0, \qquad i = 1,2,\ldots,m \tag{4.36}$$

where

$$\mathbf{B}^{-1} = [\beta_{ij}] \tag{4.37}$$

Hence the original optimal basis **B** remains optimal provided that the changes made in b_i, Δb_i, satisfy the inequalities (4.36). The change in the value of the ith optimal basic variable, Δx_i, due to the change in b_i is given by

$$\mathbf{X}_B' - \mathbf{X}_B = \Delta\mathbf{X}_B = \mathbf{B}^{-1}\Delta\mathbf{b}$$

that is,

$$\Delta x_i = \sum_{j=1}^{m} \beta_{ij}\Delta b_j, \qquad i = 1,2,\ldots,m \tag{4.38}$$

Finally, the change in the optimal value of the objective function (Δf) due to the change Δb_i can be obtained as

$$\Delta f = \mathbf{c}_B^T\Delta\mathbf{X}_B = \mathbf{c}_B^T\mathbf{B}^{-1}\Delta\mathbf{b} = \boldsymbol{\pi}^T\,\Delta\mathbf{b} = \sum_{j=1}^{m} \pi_j\,\Delta\mathbf{b}_j \tag{4.39}$$

Suppose that the changes made in $b_i(\Delta b_i)$ are such that the inequality (4.34) is violated for some variables so that these variables become infeasible for the

†It is assumed that the variables are renumbered such that the first m variables represent the basic variables and the remaining $n - m$ the nonbasic variables.

new right-hand-side vector. Our interest in this case will be to determine the new optimal solution. This can be done without reworking the problem from the beginning by proceeding according to the following steps.

1. Replace the $\bar{b}_i$ of the original optimal tableau by the new values, $\bar{\mathbf{b}}' = \mathbf{B}^{-1}(\mathbf{b} + \Delta\mathbf{b})$ and change the signs of all the numbers that are lying in the rows in which the infeasible variables appear, that is, in rows for which $\bar{b}_i' < 0$.
2. Add artificial variables to these rows, thereby replacing the infeasible variables in the basis by the artificial variables.
3. Go through the phase I calculations to find a basic feasible solution for the problem with the new right-hand side.
4. If the solution found at the end of phase I is not optimal, we go through the phase II calculations to find the new optimal solution.

The procedure outlined above saves considerable time and effort compared to the reworking of the problem from the beginning if only a few variables become infeasible with the new right-hand side. However, if the number of variables that become infeasible are not few, the procedure above might also require as much effort as the one involved in reworking of the problem from the beginning.

Example 4.5 A manufacturer produces four products, A, B, C, and D, by using two types of machines (lathes and milling machines). The times required on the two machines to manufacture 1 unit of each of the four products, the profit per unit of the product, and the total time available on the two types of machines per day are given below.

	Time Required per Unit (min) for Product:				Total Time Available per Day (min)
Machine	A	B	C	D	
Lathe machine	7	10	4	9	1200
Milling machine	3	40	1	1	800
Profit per unit ($)	45	100	30	50	

Find the number of units to be manufactured of each product per day for maximizing the profit.

Note: This is an ordinary LP problem and is given to serve as a reference problem for illustrating the sensitivity analysis.

SOLUTION Let x_1, x_2, x_3, and x_4 denote the number of units of products A, B, C, and D produced per day. Then the problem can be stated in standard form as follows:

$$\text{Minimize } f = -45x_1 - 100x_2 - 30x_3 - 50x_4$$

subject to

$$7x_1 + 10x_2 + 4x_3 + 9x_4 \leq 1200$$
$$3x_1 + 40x_2 + x_3 + x_4 \leq 800$$
$$x_i \geq 0, \quad i = 1 \text{ to } 4$$

By introducing the slack variables $x_5 \geq 0$ and $x_6 \geq 0$, the problem can be stated in canonical form and the simplex method can be applied. The computations are shown in tableau form below.

Basic Variables	Variables x_1	x_2	x_3	x_4	x_5	x_6	$-f$	$\bar{b}_i$	Ratio $\bar{b}_i/\bar{a}_{is}$ for $\bar{a}_{is} > 0$
x_5	7	10	4	9	1	0	0	1200	120
x_6	3	$\boxed{40}$ Pivot element	1	1	0	1	0	800	20←Smaller one, x_6 leaves the basis
$-f$	−45	−100	−30	−50	0	0	1	0	
		↑ Minimum $\bar{c}_j < 0$; x_2 enters the next basis							

Result of pivot operation:

x_5	$\frac{25}{4}$	0	$\frac{15}{4}$	$\boxed{\frac{35}{4}}$ Pivot element	1	$-\frac{1}{4}$	0	1000	$\frac{4000}{35}$←Smaller one, x_5 leaves the basis
x_2	$\frac{3}{40}$	1	$\frac{1}{40}$	$\frac{1}{40}$	0	$\frac{1}{40}$	0	20	800
$-f$	$-\frac{75}{2}$	0	$-\frac{55}{2}$	$-\frac{95}{2}$	0	$\frac{5}{2}$	1	2000	
				↑ Minimum $\bar{c}_j < 0$, x_4 enters the basis					

Result of pivot operation:

x_4	$\frac{5}{7}$	0	$\boxed{\frac{3}{7}}$ Pivot element	1	$\frac{4}{35}$	$-\frac{1}{35}$	0	$\frac{4,000}{35}$	$\frac{800}{3}$	←Smaller one, x_4 leaves the basis
x_2	$\frac{2}{35}$	1	$\frac{1}{70}$	0	$-\frac{1}{350}$	$\frac{9}{350}$	0	$\frac{120}{7}$	1200	
$-f$	$-\frac{25}{7}$	0	$-\frac{50}{7}$	0	$\frac{38}{7}$	$\frac{8}{7}$	1	$\frac{52,000}{7}$		
			↑ Minimum $c_j < 0$, x_3 enters the basis							

Result of pivot operation:

x_3	$\frac{5}{3}$	0	1	$\frac{7}{3}$	$\frac{4}{15}$	$-\frac{1}{15}$	0	$\frac{800}{3}$
x_2	$\frac{1}{30}$	1	0	$-\frac{1}{30}$	$-\frac{1}{150}$	$\frac{2}{75}$	0	$\frac{40}{3}$
$-f$	$\frac{25}{3}$	0	0	$\frac{50}{3}$	$\frac{22}{3}$	$\frac{2}{3}$	1	$\frac{28,000}{3}$

The optimum solution is given by

$$x_2 = \frac{40}{3}, \quad x_3 = \frac{800}{3} \quad \text{(basic variables)}$$

$$x_1 = x_4 = x_5 = x_6 = 0 \quad \text{(nonbasic variables)}$$

$$f_{\min} = \frac{-28{,}000}{3} \quad \text{or} \quad \text{maximum profit} = \frac{\$28{,}000}{3}$$

From the final tableau, one can find that

$$\mathbf{X}_B = \begin{Bmatrix} x_3 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} \frac{800}{3} \\ \frac{40}{3} \end{Bmatrix} = \text{vector of basic variables in the optimum solution} \tag{E$_1$}$$

$$\mathbf{c}_B = \begin{Bmatrix} c_3 \\ c_2 \end{Bmatrix} = \begin{Bmatrix} -30 \\ -100 \end{Bmatrix} = \text{vector of original cost coefficients corresponding to the basic variables} \tag{E$_2$}$$

$$\mathbf{B} = \begin{bmatrix} 4 & 10 \\ 1 & 40 \end{bmatrix} = \text{matrix of original coefficients corresponding to the basic variables} \tag{E$_3$}$$

$$\mathbf{B}^{-1} = \begin{bmatrix} \beta_{33} & \beta_{32} \\ \beta_{23} & \beta_{22} \end{bmatrix} = \begin{bmatrix} \frac{4}{15} & -\frac{1}{15} \\ -\frac{1}{150} & \frac{2}{75} \end{bmatrix} = \text{inverse of the coefficient matrix } \mathbf{B} \text{ which appears in the final tableau also} \tag{E$_4$}$$

$$\boldsymbol{\pi} = \mathbf{c}_B^T \mathbf{B}^{-1} = (-30 \quad -100) \begin{bmatrix} \frac{4}{15} & -\frac{1}{15} \\ -\frac{1}{150} & \frac{2}{75} \end{bmatrix}$$

$$= \begin{Bmatrix} -\frac{22}{3} \\ -\frac{2}{3} \end{Bmatrix} = \begin{array}{l} \text{simplex multipliers, the} \\ \text{negatives of which appear} \\ \text{in the final tableau also} \end{array} \qquad (E_5)$$

Example 4.6 Find the effect of changing the total time available per day on the two machines from 1200 and 800 min to 1500 and 1000 min in Example 4.5.

SOLUTION Equation (4.36) gives

$$x_i + \sum_{j=1}^{m} \beta_{ij}\, \Delta b_j \geq 0, \qquad i = 1,2,\ldots,m \qquad (4.36)$$

where x_i is the optimum value of the ith basic variable. (This equation assumes that the variables are renumbered such that x_1 to x_m represent the basic variables.)

If the variables are not renumbered, Eq. (4.36) will be applicable for $i = 3$ and 2 in the present problem with $\Delta b_3 = 300$ and $\Delta b_2 = 200$. From Eqs. (E_1) to (E_5) of Example 4.5, the left-hand sides of Eq. (4.36) become

$$x_3 + \beta_{33}\, \Delta b_3 + \beta_{32}\, \Delta b_2 = \tfrac{800}{3} + \tfrac{4}{15}(300) - \tfrac{1}{15}(200) = \tfrac{5000}{15}$$

$$x_2 + \beta_{23}\, \Delta b_3 + \beta_{22}\, \Delta b_2 = \tfrac{40}{3} - \tfrac{1}{150}(300) + \tfrac{2}{75}(200) = \tfrac{2500}{150}$$

Since both these values are ≥ 0, the original optimal basis $\mathbf{B}$ remains optimal even with the new values of b_i. The new values of the (optimal) basic variables are given by Eq. (4.38) as

$$\mathbf{X}'_B = \begin{Bmatrix} x'_3 \\ x'_2 \end{Bmatrix} = \mathbf{X}_B + \Delta\mathbf{X}_B = \mathbf{X}_B + \mathbf{B}^{-1}\, \Delta\mathbf{b}$$

$$= \begin{Bmatrix} \frac{800}{3} \\ \frac{40}{3} \end{Bmatrix} + \begin{bmatrix} \frac{4}{15} & -\frac{1}{15} \\ -\frac{1}{150} & \frac{2}{75} \end{bmatrix} \begin{Bmatrix} 300 \\ 200 \end{Bmatrix} = \begin{Bmatrix} \frac{1000}{3} \\ \frac{50}{3} \end{Bmatrix}$$

and the optimum value of the objective function by Eq. (4.39) as

$$f'_{\min} = f_{\min} + \Delta f = f_{\min} + \mathbf{c}_B^T\, \Delta\mathbf{X}_B = -\frac{28{,}000}{3} + (-30 \quad -100) \begin{Bmatrix} \frac{200}{3} \\ \frac{10}{3} \end{Bmatrix}$$

$$= -\frac{35{,}000}{3}$$

Thus the new profit will be \$35,000/3.

4.5.2 Changes in the Cost Coefficients c_j

The problem here is to find the effect of changing the cost coefficients from c_j to $c_j + \Delta c_j$ on the optimal solution obtained with c_j. The relative cost coefficients corresponding to the nonbasic variables, $x_{m+1}, x_{m+2}, \ldots, x_n$ are given by Eq. (4.10):

$$\overline{c}_j = c_j - \boldsymbol{\pi}^T \mathbf{A}_j = c_j - \sum_{i=1}^{m} \pi_i a_{ij}, \qquad j = m+1, m+2, \ldots, n \tag{4.40}$$

where the simplex multipliers π_i are related to the cost coefficients of the basic variables by the relation

$$\boldsymbol{\pi}^T = \mathbf{c}_B^T \mathbf{B}^{-1}$$

that is,

$$\pi_i = \sum_{k=1}^{m} c_k \beta_{ki}, \qquad i = 1, 2, \ldots, m \tag{4.41}$$

From Eqs. (4.40) and (4.41), we obtain

$$\overline{c}_j = c_j - \sum_{i=1}^{m} a_{ij} \left(\sum_{k=1}^{m} c_k \beta_{ki} \right) = c_j - \sum_{k=1}^{m} c_k \left(\sum_{i=1}^{m} a_{ij} \beta_{ki} \right),$$
$$i = m+1, m+2, \ldots, n \tag{4.42}$$

If the c_j are changed to $c_j + \Delta c_j$, the original optimal solution remains optimal, provided that the new values of $\overline{c}_j$, $\overline{c}_j'$, satisfy the relation

$$\overline{c}_j' = c_j + \Delta c_j - \sum_{k=1}^{m} (c_k + \Delta c_k) \left(\sum_{i=1}^{m} a_{ij} \beta_{ki} \right) \geq 0$$
$$= \overline{c}_j + \Delta c_j - \sum_{k=1}^{m} \Delta c_k \left(\sum_{i=1}^{m} a_{ij} \beta_{ki} \right) \geq 0,$$
$$j = m+1, m+2, \ldots, n \tag{4.43}$$

where $\overline{c}_j$ indicate the values of the relative cost coefficients corresponding to the original optimal solution.

In particular, if changes are made only in the cost coefficients of the nonbasic variables, Eq. (4.43) reduces to

$$\overline{c}_j + \Delta c_j \geq 0, \qquad j = m+1, m+2, \ldots, n \tag{4.44}$$

If Eq. (4.43) is satisfied, the changes made in c_j, Δc_j, will not affect the optimal basis and the values of the basic variables. The only change that occurs is in the optimal value of the objective function according to

$$\Delta f = \sum_{j=1}^{m} x_j \, \Delta c_j \tag{4.45}$$

and this change will be zero if only the c_j of nonbasic variables are changed.

Suppose that Eq. (4.43) is violated for some of the nonbasic variables. Then it is possible to improve the value of the objective function by bringing any nonbasic variable that violates Eq. (4.43) into the basis provided that it can be assigned a nonzero value. This can be done easily with the help of the previous optimal tableau. Since some of the $\bar{c}_j'$ are negative, we start the optimization procedure again, by using the old optimum as an initial feasible solution. We continue the iterative process until the new optimum is found. As in the case of changing the right-hand-side b_i, the effectiveness of this procedure depends on the number of violations made in Eq. (4.43) by the new values $c_j + \Delta c_j$.

In some of the practical problems, it may become necessary to solve the optimization problem with a series of objective functions. This can be accomplished without reworking the entire problem for each new objective function. Assume that the optimum solution for the first objective function is found by the regular procedure. Then consider the second objective function as obtained by changing the first one and evaluate Eq. (4.43). If the resulting $\bar{c}_j' \geq 0$, the old optimum still remains as optimum and one can proceed to the next objective function in the same manner. On the other hand, if one or more of the resulting $\bar{c}_j' < 0$, we can adopt the procedure outlined above and continue the iterative process using the old optimum as the starting feasible solution. After the optimum is found, we switch to the next objective function.

Example 4.7 Find the effect of changing c_3 from -30 to -24 in Example 4.5.

SOLUTION Here $\Delta c_3 = 6$ and Eq. (4.43) gives that

$$\bar{c}_1' = \bar{c}_1 + \Delta c_1 - \Delta c_3[a_{21}\beta_{32} + a_{31}\beta_{33}] = \tfrac{25}{3} + 0 - 6[3(-\tfrac{1}{15}) + 7(\tfrac{4}{15})] = -\tfrac{5}{3}$$

$$\bar{c}_4' = \bar{c}_4 + \Delta c_4 - \Delta c_3[a_{24}\beta_{32} + a_{34}\beta_{33}] = \tfrac{50}{3} + 0 - 6[1(-\tfrac{1}{15}) + 9(\tfrac{4}{15}] = \tfrac{8}{3}$$

$$\bar{c}_5' = \bar{c}_5 + \Delta c_5 - \Delta c_3[a_{25}\beta_{32} + a_{35}\beta_{33}] = \tfrac{22}{3} + 0 - 6[0(-\tfrac{1}{15}) + 1(\tfrac{4}{15})] = \tfrac{86}{15}$$

$$\bar{c}_6' = \bar{c}_6 + \Delta c_6 - \Delta c_3[a_{26}\beta_{32} + a_{36}\beta_{33}] = \tfrac{2}{3} + 0 - 6[1(-\tfrac{1}{15}) + 0(\tfrac{4}{15})] = \tfrac{16}{15}$$

The change in the value of the objective function is given by Eq. (4.45) as

$$\Delta f = \Delta c_3 \, x_3 = \frac{4800}{3} \quad \text{so that} \quad f = -\frac{28{,}000}{3} + \frac{4800}{3} = -\frac{23{,}200}{3}$$

Since $\bar{c}_1'$ is negative, we can bring x_1 into the basis. Thus we start with the optimal tableau of the original problem with the new values of relative cost coefficients and improve the solution according to the regular procedure.

Basic Variables	Variables								Ratio $\bar{b}_i/\bar{a}_{ij}$ for $\bar{a}_{ij} > 0$
	x_1	x_2	x_3	x_4	x_5	x_6	$-f$	$\bar{b}_i$	
x_3	$\boxed{\frac{5}{3}}$ Pivot element	0	1	$\frac{7}{3}$	$\frac{4}{15}$	$-\frac{1}{15}$	0	$\frac{800}{3}$	160 ←
x_2	$\frac{1}{30}$	1	0	$-\frac{1}{30}$	$-\frac{1}{150}$	$\frac{2}{75}$	0	$\frac{40}{3}$	400
$-f$	$-\frac{5}{3}$ ↑	0	0	$\frac{8}{3}$	$\frac{86}{15}$	$\frac{16}{15}$	1	$\frac{23{,}200}{3}$	
x_1	1	0	$\frac{3}{5}$	$\frac{7}{5}$	$\frac{4}{25}$	$-\frac{1}{25}$	0	160	
x_2	0	1	$-\frac{1}{50}$	$-\frac{2}{25}$	$-\frac{3}{250}$	$\frac{7}{250}$	0	8	
$-f$	0	0	1	5	6	1	1	8000	

Since all the relative cost coefficients are nonnegative, the present solution is optimum with

$$x_1 = 160, \quad x_2 = 8 \quad \text{(basic variables)}$$

$$x_3 = x_4 = x_5 = x_6 = 0 \quad \text{(nonbasic variables)}$$

$$f_{\min} = -8000 \quad \text{and} \quad \text{maximum profit} = \$8000$$

4.5.3 Addition of New Variables

Suppose that the optimum solution of a LP problem with n variables $x_1, x_2, \ldots, x_n$ has been found and we want to examine the effect of adding some more variables x_{n+k}, $k = 1, 2, \ldots$, on the optimum solution. Let the constraint coefficients and the cost coefficients corresponding to the new variables x_{n+k} be denoted by $a_{i,n+k}$, $i = 1$ to m and c_{n+k}, respectively. If the new variables are treated as additional nonbasic variables in the old optimum solution, the corresponding relative cost coefficients are given by

$$\bar{c}_{n+k} = c_{n+k} - \sum_{i=1}^{m} \pi_i a_{i,n+k} \tag{4.46}$$

where $\pi_1, \pi_2, \ldots, \pi_m$ are the simplex multipliers corresponding to the original optimum solution. The original optimum remains optimum for the new problem also provided that $\bar{c}_{n+k} \geq 0$ for all k. However, if one or more $\bar{c}_{n+k} < 0$, it pays to bring some of the new variables into the basis provided that they can be assigned a nonzero value. For bringing a new variable into the basis, we first have to transform the coefficients $a_{i,n+k}$ into $\overline{a}_{i,n+k}$ so that the columns of the new variables correspond to the canonical form of the old optimal basis. This can be done by using Eq. (4.9) as

$$\underset{m \times 1}{\overline{\mathbf{A}}_{n+k}} = \underset{m \times m}{\mathbf{B}^{-1}} \underset{m \times 1}{\mathbf{A}_{n+k}}$$

that is,

$$\overline{a}_{i,n+k} = \sum_{j=1}^{m} \beta_{ij} a_{j,n+k}, \qquad i = 1 \text{ to } m \tag{4.47}$$

where $\mathbf{B}^{-1} = [\beta_{ij}]$ is the inverse of the old optimal basis. The rules for bringing a new variable into the basis, finding a new basic feasible solution, testing this solution for optimality, and the subsequent procedure is same as the one outlined in the regular simplex method.

Example 4.8 In Example 4.5, if a new product, E, which requires 15 min of work on the lathe and 10 min on the milling machine per unit, is available, will it be worthwhile to manufacture it if the profit per unit is \$40?

SOLUTION Let x_k be the number of units of product E manufactured per day. Then $c_k = -40$, $a_{1k} = 15$, and $a_{2k} = 10$; therefore,

$$\overline{c}_k = c_k - \pi_1 a_{1k} - \pi_2 a_{2k} = -40 + (\tfrac{22}{3})(15) + (\tfrac{2}{3})(10) = \tfrac{230}{3} \geq 0$$

Since the relative cost coefficient $\overline{c}_k$ is nonnegative, the original optimum solution remains optimum for the new problem also and the variable x_k will remain as a nonbasic variable. This means that it is not worth manufacturing product E.

4.5.4 Changes in the Constraint Coefficients a_{ij}

Here the problem is to investigate the effect of changing the coefficient a_{ij} to $a_{ij} + \Delta a_{ij}$ after finding the optimum solution with a_{ij}. There are two possibilities in this case. The first possibility occurs when all the coefficients a_{ij}, in which changes are made, belong to the columns of those variables which are nonbasic in the old optimal solution. In this case, the effect of changing a_{ij} on the optimal solution can be investigated by adopting the procedure outlined in the preceding section. The second possibility occurs when the coefficients changed a_{ij} correspond to a basic variable, say, x_{j0} of the old optimal solution. The following procedure can be adopted to examine the effect of changing $a_{i,j0}$ to $a_{i,j0} + \Delta a_{i,j0}$.

1. Introduce a new variable x_{n+1} to the original system with constraint coefficients

$$a_{i,n+1} = a_{i,j0} + \Delta a_{i,j0} \tag{4.48}$$

and cost coefficient

$$c_{n+1} = c_{j0} \quad \text{(original value itself)} \tag{4.49}$$

2. Transform the coefficients $a_{i,n+1}$ to $\bar{a}_{i,n+1}$ by using the inverse of the old optimal basis, $\mathbf{B}^{-1} = [\beta_{ij}]$, as

$$\bar{a}_{i,n+1} = \sum_{j=1}^{m} \beta_{ij} a_{j,n+1}, \qquad i = 1 \text{ to } m \tag{4.50}$$

3. Replace the original cost coefficient (c_{j0}) of x_{j0} by a large positive number N, but keep c_{n+1} equal to the old value c_{j0}.
4. Compute the modified cost coefficients using Eq. (4.43):

$$\bar{c}_j' = \bar{c}_j + \Delta c_j - \sum_{k=1}^{m} \Delta c_k \left(\sum_{i=1}^{m} a_{ij} \beta_{ki} \right),$$

$$j = m+1, m+2, \ldots, n, n+1 \tag{4.51}$$

where $\Delta c_k = 0$ for $k = 1, 2, \ldots, j_0 - 1, j_0 + 1, \ldots, m$ and $\Delta c_{j0} = N - c_{j0}$.

5. Carry the regular iterative procedure of simplex method with the new objective function and the augmented matrix found in Eqs. (4.50) and (4.51) until the new optimum is found.

Remarks:

1. The number N has to be taken sufficiently large to ensure that x_{j0} cannot be contained in the new optimal basis that is ultimately going to be found.
2. The procedure above can easily be extended to cases where changes in coefficients a_{ij} of more than one column are made.
3. The present procedure will be computationally efficient (compared to reworking of the problem from the beginning) only for cases where there are not too many number of basic columns in which the a_{ij} are changed.

Example 4.9 Find the effect of changing $\mathbf{A}_1$ from $\begin{Bmatrix} 7 \\ 3 \end{Bmatrix}$ to $\begin{Bmatrix} 6 \\ 10 \end{Bmatrix}$ in Example 4.5 (i.e., changes are made in the coefficients a_{ij} of nonbasic variables only).

SOLUTION The relative cost coefficients of the nonbasic variables (of the original optimum solution) corresponding to the new a_{ij} are given by

$$\bar{c}_j = c_j - \boldsymbol{\pi}^T \mathbf{A}_j, \qquad j = \text{nonbasic } (1, 4, 5, 6)$$

Since $\mathbf{A}_1$ is changed, we have

$$\bar{c}_1 = c_1 - \boldsymbol{\pi}^T\mathbf{A}_1 = -45 - (-\tfrac{22}{3} \ -\tfrac{2}{3})\begin{Bmatrix} 6 \\ 10 \end{Bmatrix} = \tfrac{17}{3}$$

As $\bar{c}_1$ is positive, the original optimum solution remains optimum for the new problem also.

Example 4.10 Find the effect of changing $\mathbf{A}_1$ from $\begin{Bmatrix} 7 \\ 3 \end{Bmatrix}$ to $\begin{Bmatrix} 5 \\ 6 \end{Bmatrix}$ in Example 4.5.

SOLUTION The relative cost coefficient of the nonbasic variable x_1 for the new $\mathbf{A}_1$ is given by

$$\bar{c}_1 = c_1 - \boldsymbol{\pi}^T\mathbf{A}_1 = -45 - (-\tfrac{22}{3} \ -\tfrac{2}{3})\begin{Bmatrix} 5 \\ 6 \end{Bmatrix} = -\tfrac{13}{3}$$

Since $\bar{c}_1$ is negative, x_1 can be brought into the basis to reduce the objective function further. For this we start with the original optimum tableau with the new values of $\overline{\mathbf{A}}_1$ given by

$$\overline{\mathbf{A}}_1 = \mathbf{B}^{-1}\mathbf{A}_1 = \begin{bmatrix} \frac{4}{15} & -\frac{1}{15} \\ -\frac{1}{150} & \frac{2}{75} \end{bmatrix}\begin{Bmatrix} 5 \\ 6 \end{Bmatrix} = \begin{bmatrix} \frac{20}{15} & -\frac{6}{15} \\ -\frac{1}{30} & +\frac{4}{25} \end{bmatrix} = \begin{Bmatrix} \frac{14}{15} \\ \frac{19}{150} \end{Bmatrix}$$

Basic Variables	Variables x_1	x_2	x_3	x_4	x_5	x_6	$-f$	$\bar{b}_i$	$(\bar{b}_i/\bar{a}_{is})$
x_3	$\frac{14}{15}$	0	1	$\frac{7}{3}$	$\frac{4}{15}$	$-\frac{1}{15}$	0	$\frac{800}{3}$	$\frac{4000}{14}$
x_2	$\boxed{\frac{19}{150}}$ Pivot element	1	0	$-\frac{1}{30}$	$-\frac{1}{150}$	$\frac{2}{75}$	0	$\frac{40}{3}$	$\frac{2000}{19} \leftarrow$
$-f$	$-\frac{13}{3}$	0	0	$\frac{50}{3}$	$\frac{22}{3}$	$\frac{2}{3}$	1	$\frac{28{,}000}{3}$	
	$\uparrow$								
x_3	0	$-\frac{140}{19}$	1	$\frac{49}{19}$	$\frac{6}{19}$	$-\frac{5}{19}$	0	$\frac{3{,}200}{19}$	
x_1	1	$\frac{150}{19}$	0	$-\frac{5}{19}$	$-\frac{1}{19}$	$\frac{4}{19}$	0	$\frac{2{,}000}{19}$	
$-f$	0	$\frac{650}{19}$	0	$\frac{295}{19}$	$\frac{135}{19}$	$\frac{30}{19}$	1	$\frac{186{,}000}{19}$	

Since all $\bar{c}_j$ are nonnegative, the present tableau gives the new optimum solution as

$$x_1 = 2000/19, \quad x_3 = 3200/19 \quad \text{(basic variables)}$$

$$x_2 = x_4 = x_5 = x_6 = 0 \quad \text{(nonbasic variables)}$$

$$f_{\min} = -\frac{186{,}000}{19} \quad \text{and} \quad \text{maximum profit} = \frac{\$186{,}000}{19}$$

4.5.5 Addition of Constraints

Suppose that we have solved a LP problem with m constraints and obtained the optimal solution. We want to examine the effect of adding some more inequality constraints on the original optimum solution. For this we evaluate the new constraints by substituting the old optimal solution and see whether they are satisfied. If they are satisfied, it means that the inclusion of the new constraints in the old problem would not have affected the old optimum solution, and hence the old optimal solution remains optimal for the new problem also. On the other hand, if one or more of the new constraints are not satisfied by the old optimal solution, we can solve the problem without reworking the entire problem by proceeding as follows.

1. The simplex tableau corresponding to the old optimum solution expresses all the basic variables in terms of the nonbasic ones. With this information, eliminate the basic variables from the new constraints.
2. Transform the constraints thus obtained by multiplying throughout by -1.
3. Add the resulting constraints to the old optimal tableau and introduce one artificial variable for each new constraint added. Thus the enlarged system of equations will be in canonical form since the old basic variables were eliminated from the new constraints in step 1. Hence a new basis, consisting of the old optimal basis plus the artificial variables in the new constraint equations, will be readily available from this canonical form.
4. Go through phase I computations to eliminate the artificial variables.
5. Go through phase II computations to find the new optimal solution.

Example 4.11 If each of the products A, B, C, and D require respectively 2, 5, 3, and 4 min of time per unit on grinding machine in addition to the operations specified in Example 4.5, find the new optimum solution. Assume that the total time available on grinding machine per day is 600 min and all this time has to be utilized fully.

SOLUTION The present data corresponds to the addition of a constraint which can be stated as

$$2x_1 + 5x_2 + 3x_3 + 4x_4 = 600 \tag{E$_1$}$$

By substituting the original optimum solution,

$$x_2 = \tfrac{40}{3}, \quad x_3 = \tfrac{800}{3}, \quad x_1 = x_4 = x_5 = x_6 = 0$$

the left-hand side of Eq. (E_1) gives

$$2(0) + 5\left(\tfrac{40}{3}\right) + 3\left(\tfrac{800}{3}\right) + 4(0) = \tfrac{2600}{3} \neq 600$$

Thus the new constraint is not satisfied by the original optimum solution. Hence we proceed as follows:

Step 1: From the original optimum tableau, we can express the basic variables as

$$x_3 = \tfrac{800}{3} - \tfrac{5}{3}x_1 - \tfrac{7}{3}x_4 - \tfrac{4}{15}x_5 + \tfrac{1}{15}x_6$$

$$x_2 = \tfrac{40}{3} - \tfrac{1}{30}x_1 + \tfrac{1}{30}x_4 + \tfrac{1}{150}x_5 - \tfrac{1}{75}x_6$$

Thus Eq. (E_1) can be expressed as

$$2x_1 + 5\left(\tfrac{40}{3} - \tfrac{1}{30}x_1 + \tfrac{1}{30}x_4 + \tfrac{1}{150}x_5 - \tfrac{2}{75}x_6\right)$$

$$+ 3\left(\tfrac{800}{3} - \tfrac{5}{3}x_1 - \tfrac{7}{3}x_4 - \tfrac{4}{15}x_5 + \tfrac{1}{15}x_6\right) + 4x_4 = 600$$

that is,

$$-\tfrac{19}{6}x_1 - \tfrac{17}{6}x_4 - \tfrac{23}{30}x_5 + \tfrac{1}{15}x_6 = -\tfrac{800}{3} \tag{E$_2$}$$

Step 2: Transform this constraint such that the right-hand side becomes positive, that is,

$$\tfrac{19}{6}x_1 + \tfrac{17}{6}x_4 + \tfrac{23}{30}x_5 - \tfrac{1}{15}x_6 = \tfrac{800}{3} \tag{E$_3$}$$

Step 3: Add an artifical variable, say, x_k, the new constraint given by Eq. (E_3) and the infeasibility form $w = x_k$ into the original optimum tableau to obtain the new canonical system as follows:

Basic Variables	Variables										
	x_1	x_2	x_3	x_4	x_5	x_6	x_k	$-f$	$-w$	$\bar{b}_i$	$(\bar{b}_i/\bar{a}_{is})$
x_3	$\frac{5}{3}$	0	1	$\frac{7}{3}$	$\frac{4}{5}$	$-\frac{1}{15}$	0	0	0	$\frac{800}{3}$	160
x_2	$\frac{1}{30}$	1	0	$-\frac{1}{30}$	$-\frac{1}{150}$	$\frac{2}{75}$	0	0	0	$\frac{40}{3}$	400
x_k	$\boxed{\frac{19}{6}}$ Pivot element	0	0	$\frac{17}{6}$	$\frac{23}{30}$	$-\frac{1}{15}$	1	0	0	$\frac{800}{3}$	$\frac{1600}{19}$
$-f$	$\frac{25}{3}$	0	0	$\frac{50}{3}$	$\frac{22}{3}$	$\frac{2}{3}$	0	1	0	$\frac{28,000}{3}$	
$-w$	$-\frac{19}{6}$	0	0	$-\frac{17}{6}$	$-\frac{23}{30}$	$\frac{1}{15}$	0	0	1	$-\frac{800}{3}$	
	↑										

Step 4: Eliminate the artificial variable by applying the phase I procedure:

Basic Variables	Variables									
	x_1	x_2	x_3	x_4	x_5	x_6	x_k	$-f$	$-w$	$\bar{b}_i$
x_3	0	0	1	$\frac{16}{19}$	$\frac{113}{285}$	$-\frac{3}{95}$	$-\frac{10}{19}$	0	0	$\frac{2,400}{19}$
x_2	0	1	0	$-\frac{6}{95}$	$-\frac{7}{475}$	$\frac{13}{475}$	$-\frac{1}{95}$	0	0	$\frac{200}{19}$
x_1	1	0	0	$\frac{17}{19}$	$\frac{23}{95}$	$-\frac{2}{95}$	$\frac{6}{19}$	0	0	$\frac{1,600}{19}$
$-f$	0	0	0	$\frac{175}{19}$	$\frac{101}{19}$	$\frac{16}{19}$	$-\frac{50}{19}$	1	0	$\frac{164,000}{19}$
$-w$	0	0	0	0	0	0	0	0	1	0

Thus the new optimum solution is given by

$$x_1 = \frac{1600}{19}, \quad x_2 = \frac{200}{19}, \quad x_3 = \frac{2400}{19} \quad \text{(basic variables)}$$

$$x_4 = x_5 = x_6 = 0 \quad \text{(nonbasic variables)}$$

$$f_{\min} = -\frac{164,000}{19} \quad \text{and} \quad \text{maximum profit} = \frac{\$164,000}{19}$$

4.6 TRANSPORTATION PROBLEM

This section deals with an important class of LP problems called the transportation problem. As the name indicates, a *transportation problem* is one in which the objective for minimization is the cost of transporting a certain commodity from a number of origins to a number of destinations. Although the transportation problem can be solved using the regular simplex method, its special structure offers a more convenient procedure for solving this type of problems.

This procedure is based on the same theory of the simplex method, but it makes use of some shortcuts that yield a simpler computational scheme.

Suppose that there are m origins $R_1, R_2, \ldots, R_m$ (e.g., warehouses) and n destinations, $D_1, D_2, \ldots, D_n$ (e.g., factories). Let a_i be the amount of a commodity available at origin i ($i = 1, 2, \ldots, m$) and b_j be the amount required at destination j ($j = 1, 2, \ldots, n$). Let c_{ij} be the cost per unit of transporting the commodity from origin i to destination j. The objective is to determine the amount of commodity (x_{ij}) transported from origin i to destination j such that the total transportation costs are minimized. This problem can be formulated mathematically as:

$$\text{Minimize } f = \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} \tag{4.52}$$

subject to

$$\sum_{j=1}^{n} x_{ij} = a_i, \qquad i = 1,2,\ldots,m \tag{4.53}$$

$$\sum_{i=1}^{m} x_{ij} = b_j, \qquad j = 1,2,\ldots,n \tag{4.54}$$

$$x_{ij} \geq 0, \qquad i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n \tag{4.55}$$

Clearly, this is a LP problem in mn variables and $m + n$ equality constraints.

Equations (4.53) state that the total amount of the commodity transported from the origin i to the various destinations must be equal to the amount available at origin i ($i = 1,2,\ldots,m$), while Eqs. (4.54) state that the total amount of the commodity received by destination j from all the sources must be equal to the amount required at the destination j ($j = 1,2,\ldots,n$). The nonnegativity conditions Eqs. (4.55) are added since negative values for any x_{ij} have no physical meaning. It is assumed that the total demand equals the total supply, that is,

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j \tag{4.56}$$

Equation (4.56), called the *consistency condition*, must be satisfied if a solution is to exist. This can be seen easily since

$$\sum_{i=1}^{m} a_i = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} x_{ij} \right) = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} x_{ij} \right) = \sum_{j=1}^{n} b_j \tag{4.57}$$

The problem stated in Eqs. (4.52) to (4.56) was originally formulated and solved by Hitchcock in 1941 [4.6]. This was also considered independently by

Koopmans in 1947 [4.7]. Because of these early investigations the problem is sometimes called the *Hitchcock–Koopmans transportation problem*. The special structure of the transportation matrix can be seen by writing the equations in standard form:

$$\begin{array}{lllr}
x_{11} + x_{12} + \cdots + x_{1n} & & & = a_1 \\
& x_{21} + x_{22} + \cdots + x_{2n} & & = a_2 \\
& & \vdots & \vdots \\
& & x_{m1} + x_{m2} + \cdots + x_{mn} & = a_m
\end{array} \tag{4.58a}$$

$$\begin{array}{llll}
x_{11} & + x_{21} & + x_{m1} & = b_1 \\
x_{12} & + x_{22} & + x_{m2} & = b_2 \\
\vdots & \vdots & \vdots & \\
x_{1n} & + x_{2n} & + x_{mn} & = b_n
\end{array} \tag{4.58b}$$

$$\begin{aligned} c_{11}x_{11} + c_{12}x_{12} + \cdots + c_{1n}x_{1n} + c_{21}x_{21} + \cdots + c_{2n}x_{2n} + \cdots \\ + c_{m1}x_{m1} + \cdots + c_{mn}x_{mn} = f \end{aligned} \tag{4.58c}$$

We notice the following properties from Eqs. (4.58).

1. All the nonzero coefficients of the constraints are equal to 1.
2. The constraint coefficients appear in a triangular form.
3. Any variable appears only once in the first m equations and once in the next n equations.

These are the special properties of the transportation problem that allow development of the *transportation technique*. To facilitate the identification of a starting solution, the system of equations (4.58) is represented in the form of an array, called the *transportation array*, as shown in Fig. 4.2. In all the techniques developed for solving the transportation problem, the calculations are made directly on the transportation array.

Computational Procedure. The solution of a LP problem, in general, requires a calculator or, if the problem is large, a high-speed digital computer. On the other hand, the solution of a transportation problem can often be obtained with the use of a pencil and paper since additions and subtractions are the only

To / From		Destination j 1	2	3	$\cdots$	n	Amount available a_i
Origin i	1	x_{11} c_{11}	x_{12} c_{12}	x_{13} c_{13}	$\cdots$	x_{1n} c_{1n}	a_1
	2	x_{21} c_{21}	x_{22} c_{22}	x_{23} c_{23}	$\cdots$	x_{2n} c_{2n}	a_2
	3	x_{31} c_{31}	x_{32} c_{32}	x_{33} c_{33}	$\cdots$	x_{3n} c_{3n}	a_3
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	m	x_{m1} c_{m1}	x_{m2} c_{m2}	x_{m3} c_{m3}	$\cdots$	x_{mn} c_{mn}	a_m
Amount required b_j		b_1	b_2	b_3	$\cdots$	b_n	

Figure 4.2 Transportation array.

calculations required. The basic steps involved in the solution of a transportation problem are:

1. Determine a starting basic feasible solution.
2. Test the current basic feasible solution for optimality. If the current solution is optimal, stop the iterative process; otherwise, go to step 3.
3. Select a variable to enter the basis from among the current nonbasic variables.
4. Select a variable to leave from the basis from among the current basic variables (using the feasibility condition).
5. Find a new basic feasible solution and return to step 2.

The details of these steps are given in Ref. [4.10].

4.7 KARMARKAR'S METHOD

Karmarkar proposed a new method in 1984 for solving large-scale linear programming problems very efficiently. The method is known as an *interior* *method* since it finds improved search directions strictly in the interior of the feasible space. This is in contrast with the simplex method, which searches along the boundary of the feasible space by moving from one feasible vertex to an adjacent one until the optimum point is found. For large LP problems, the number of vertices will be quite large and hence the simplex method would become very expensive in terms of computer time. Along with many other applications, Karmarkar's method has been applied to aircraft route scheduling problems. It was reported [4.19] that Karmarkar's method solved problems involving 150,000 design variables and 12,000 constraints in 1 hour while the simplex method required 4 hours for solving a smaller problem involving only 36,000 design variables and 10,000 constraints. In fact, it was found that Karmarkar's method is as much as 50 times faster than the simplex method for large problems.

Karmarkar's method is based on the following two observations:

1. If the current solution is near the center of the polytope, we can move along the steepest descent direction to reduce the value of f by a maximum amount. From Fig. 4.3, we can see that the current solution can be improved substantially by moving along the steepest descent direction if it is near the center (point 2) but not near the boundary point (points 1 and 3).
2. The solution space can always be transformed without changing the nature of the problem so that the current solution lies near the center of the polytope.

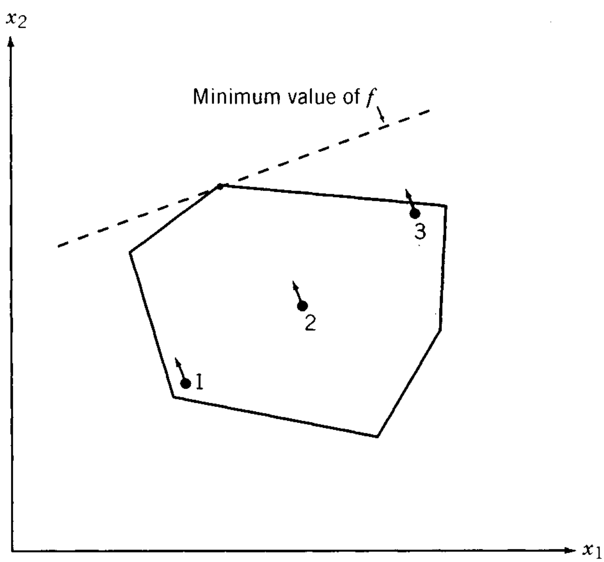

Figure 4.3 Improvement of objective function from different points of a polytope.

It is well known that in many numerical problems, by changing the units of data or rescaling (e.g., using feet instead of inches), we may be able to reduce the numerical instability. In a similar manner, Karmarkar observed that the variables can be transformed (in a more general manner than ordinary rescaling) so that straight lines remain straight lines while angles and distances change for the feasible space.

4.7.1 Statement of the Problem

Karmarkar's method requires the LP problem in the following form:

$$\text{Minimize } f = \mathbf{c}^T\mathbf{X}$$

subject to

$$\begin{aligned} [a]\mathbf{X} &= \mathbf{0} \\ x_1 + x_2 + \cdots + x_n &= 1 \\ \mathbf{X} &\geq \mathbf{0} \end{aligned} \tag{4.59}$$

where $\mathbf{X} = \{x_1 \ \ x_2 \ \cdots \ x_n\}^T$, $\mathbf{c} = \{c_1 \ \ c_2 \ \cdots \ c_n\}^T$, and $[a]$ is an $m \times n$ matrix. In addition, an interior feasible starting solution to Eqs. (4.59) must be known. Usually, $\mathbf{X} = \left\{\frac{1}{n} \ \ \frac{1}{n} \ \cdots \ \frac{1}{n}\right\}^T$ is chosen as the starting point. In addition, the optimum value of f must be zero for the problem. Thus

$$\begin{aligned} \mathbf{X}^{(1)} &= \left\{\frac{1}{n} \ \ \frac{1}{n} \ \cdots \ \frac{1}{n}\right\}^T = \text{interior feasible} \\ f_{\min} &= 0 \end{aligned} \tag{4.60}$$

Although most LP problems may not be available in the form of Eq. (4.59) while satisfying the conditions of Eq. (4.60), it is possible to put any LP problem in a form that satisfies Eqs. (4.59) and (4.60) as indicated below.

4.7.2 Conversion of an LP Problem into the Required Form

Let the given LP problem be of the form:

$$\text{Minimize } \mathbf{d}^T\mathbf{X}$$

subject to

$$\begin{aligned} [\alpha]\mathbf{X} &= \mathbf{b} \\ \mathbf{X} &\geq \mathbf{0} \end{aligned} \tag{4.61}$$

To convert this problem into the form of Eq. (4.59), we use the procedure suggested in Ref. [4.20] and define integers m and n such that $\mathbf{X}$ will be an $(n-3)$-component vector and $[\alpha]$ will be a matrix of order $m - 1 \times n - 3$. We now define the vector $\overline{\mathbf{z}} = \{z_1 \quad z_2 \quad \cdots \quad z_{n-3}\}^T$ as

$$\overline{\mathbf{z}} = \frac{\mathbf{X}}{\beta} \tag{4.62}$$

where β is a constant chosen to have a sufficiently large value such that

$$\beta > \sum_{i=1}^{n-3} x_i \tag{4.63}$$

for any feasible solution $\mathbf{X}$ (assuming that the solution is bounded). By using Eq. (4.62), the problem of Eq. (4.61) can be stated as follows:

$$\text{Minimize } \beta \mathbf{d}^T \overline{\mathbf{z}}$$

subject to

$$\begin{aligned} [\alpha]\, \overline{\mathbf{z}} &= \frac{1}{\beta} \mathbf{b} \\ \overline{\mathbf{z}} &\geq \mathbf{0} \end{aligned} \tag{4.64}$$

We now define a new vector $\mathbf{z}$ as

$$\mathbf{z} = \begin{Bmatrix} \overline{\mathbf{z}} \\ z_{n-2} \\ z_{n-1} \\ z_n \end{Bmatrix}$$

and solve the following related problem instead of the problem in Eqs. (4.64):

$$\text{Minimize } \{\beta \mathbf{d}^T \quad 0 \quad 0 \quad M\}\, \mathbf{z}$$

subject to

$$\begin{aligned} &\begin{bmatrix} [\alpha] & \mathbf{0} & -\dfrac{n}{\beta}\mathbf{b} & \left(\dfrac{n}{\beta}\mathbf{b} - [\alpha]\,\mathbf{e}\right) \\ 0 & 0 & n & 0 \end{bmatrix} \mathbf{z} = \begin{Bmatrix} \mathbf{0} \\ 1 \end{Bmatrix} \\ &\mathbf{e}^T \overline{\mathbf{z}} + z_{n-2} + z_{n-1} + z_n = 1 \\ &\mathbf{z} \geq \mathbf{0} \end{aligned} \tag{4.65}$$

where $\mathbf{e}$ is an $(m - 1)$-component vector whose elements are all equal to 1, z_{n-2} is a slack variable that absorbs the difference between 1 and the sum of other variables, z_{n-1} is constrained to have a value of $1/n$, and M is given a large value (corresponding to the artificial variable z_n) to force z_n to zero when the problem stated in Eqs. (4.61) has a feasible solution. Equations (4.65) are developed such that if $\mathbf{z}$ is a solution to these equations, $\mathbf{X} = \beta\bar{\mathbf{z}}$ will be a solution to Eqs. (4.61) if Eqs. (4.61) have a feasible solution. Also, it can be verified that the interior point $\mathbf{z} = (1/n)\mathbf{e}$ is a feasible solution to Eqs. (4.65). Equations (4.65) can be seen to be the desired form of Eqs. (4.61) except for a 1 on the right-hand side. This can be eliminated by subtracting the last constraint from the next-to-last constraint, to obtain the required form:

$$\text{Minimize } \{\beta\mathbf{d}^T \quad 0 \quad 0 \quad M\}\ \mathbf{z}$$

subject to

$$\begin{bmatrix} [\alpha] & \mathbf{0} & -\dfrac{n}{\beta}\mathbf{b} & \left(\dfrac{n}{\beta}\mathbf{b} - [\alpha]\,\mathbf{e}\right) \\ -\mathbf{e}^T & -1 & (n-1) & -1 \end{bmatrix} \mathbf{z} = \begin{Bmatrix} \mathbf{0} \\ 0 \end{Bmatrix}$$

$$\mathbf{e}^T\bar{\mathbf{z}} + z_{n-2} + z_{n-1} + z_n = 1 \tag{4.66}$$

$$\mathbf{z} \geq \mathbf{0}$$

Note: When Eqs. (4.66) are solved, if the value of the artificial variable $z_n > 0$, the original problem in Eqs. (4.61) is infeasible. On the other hand, if the value of the slack variable $z_{n-2} = 0$, the solution of the problem given by Eqs. (4.61) is unbounded.

Example 4.12 Transform the following LP problem into a form required by Karmarkar's method:

$$\text{Minimize } 2x_1 + 3x_2$$

subject to

$$3x_1 + x_2 - 2x_3 = 3$$

$$5x_1 - 2x_2 = 2$$

$$x_i \geq 0, \qquad i = 1,2,3$$

SOLUTION It can be seen that $\mathbf{d} = \{2 \quad 3 \quad 0\}^T$, $[\alpha] = \begin{bmatrix} 3 & 1 & -2 \\ 5 & -2 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{Bmatrix} 3 \\ 2 \end{Bmatrix}$, and $\mathbf{X} = \{x_1 \quad x_2 \quad x_3\}^T$. We define the integers m and n as $n = 6$ and

$m = 3$ and choose $\beta = 10$ so that

$$\bar{\mathbf{z}} = \frac{1}{10}\begin{Bmatrix} z_1 \\ z_2 \\ z_3 \end{Bmatrix}$$

Noting that $\mathbf{e} = \{1 \quad 1 \quad 1\}^T$, Eqs. (4.66) can be expressed as

$$\text{Minimize } \{20 \quad 30 \quad 0 \quad 0 \quad 0 \quad M\}\ \mathbf{z}$$

subject to

$$\left[\begin{bmatrix} 3 & 1 & -2 \\ 5 & -2 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} - \frac{6}{10}\begin{Bmatrix} 3 \\ 2 \end{Bmatrix} \right.$$

$$\left. \left(\frac{6}{10}\begin{Bmatrix} 3 \\ 2 \end{Bmatrix} - \begin{bmatrix} 3 & 1 & -2 \\ 5 & -2 & 0 \end{bmatrix}\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}\right)\right] \mathbf{z} = \mathbf{0}$$

$$\{-\{1 \quad 1 \quad 1\} \quad -1 \quad 5 \quad -1\}\ \mathbf{z} = 0$$

$$z_1 + z_2 + z_3 + z_4 + z_5 + z_6 = 1$$

$$\mathbf{z} = \{z_1 \quad z_2 \quad z_3 \quad z_4 \quad z_5 \quad z_6\}^T \geq \mathbf{0}$$

where M is a very large number. These equations can be seen to be in the desired form.

4.7.3 Algorithm

Starting from an interior feasible point $\mathbf{X}^{(1)}$, Karmarkar's method finds a sequence of points $\mathbf{X}^{(2)}, \mathbf{X}^{(3)}, \ldots$ using the following iterative procedure:

1. Initialize the process. Being at the center of the simplex as the initial feasible point $\mathbf{X}^{(1)} = \begin{Bmatrix} \dfrac{1}{n} & \dfrac{1}{n} & \cdots & \dfrac{1}{n} \end{Bmatrix}^T$ Set the iteration number as $k = 1$.
2. Test for optimality. Since $f = 0$ at the optimum point, we stop the procedure if the following convergence criterion is satisfied:

$$\|\mathbf{c}^T\mathbf{X}^{(k)}\| \leq \epsilon \tag{4.67}$$

where ϵ is a small number. If Eq. (4.67) is not satisfied, go to step 3.

3. Compute the next point, $\mathbf{X}^{(k+1)}$. For this, we first find a point $\mathbf{Y}^{(k+1)}$ in the transformed unit simplex as

$$\mathbf{Y}^{(k+1)} = \left\{\frac{1}{n}\ \frac{1}{n} \cdots \frac{1}{n}\right\}^T - \frac{\alpha\,([I] - [P]^T([P]\,[P]^T)^{-1}[P])\,[D(\mathbf{X}^{(k)})]\mathbf{c}}{\|\mathbf{c}\|\sqrt{n\,(n-1)}} \tag{4.68}$$

where $\|\mathbf{c}\|$ is the length of the vector $\mathbf{c}$, $[I]$ the identity matrix of order n, $[D(\mathbf{X}^{(k)})]$ an $n \times n$ matrix with all off-diagonal entries equal to 0, and diagonal entries are equal to the components of the vector $\mathbf{X}^{(k)}$ as

$$[D(\mathbf{X}^{(k)})]_{ii} = x_i^{(k)}, \qquad i = 1,2,\ldots,n \tag{4.69}$$

$[P]$ is an $(m + 1) \times n$ matrix whose first m rows are given by $[a]\,[D(\mathbf{X}^{(k)})]$ and the last row is composed of 1's:

$$[P] = \begin{bmatrix} & [a][D(\mathbf{X}^{(k)})] & & \\ 1 & 1 & \cdots & 1 \end{bmatrix} \tag{4.70}$$

and the value of the parameter α is usually chosen as $\alpha = \frac{1}{4}$ to ensure convergence. Once $\mathbf{Y}^{(k+1)}$ is found, the components of the new point $\mathbf{X}^{(k+1)}$ are determined as

$$x_i^{(k+1)} = \frac{x_i^{(k)} y_i^{(k+1)}}{\sum_{r=1}^{n} x_r^{(k)} y_r^{(k+1)}}, \qquad i = 1,2,\ldots,n \tag{4.71}$$

Set the new iteration number as $k = k + 1$ and go to step 2.

Example 4.13 Find the solution of the following problem using Karmarkar's method:

$$\text{Minimize } f = 2x_1 + x_2 - x_3$$

subject to

$$\begin{aligned} x_2 - x_3 &= 0 \\ x_1 + x_2 + x_3 &= 1 \\ x_i \geq 0, \quad i &= 1,2,3 \end{aligned} \tag{E.1}$$

Use the value of $\epsilon = 0.05$ for testing the convergence of the procedure.

SOLUTION The problem is already in the required form of Eq. (4.59), and hence the following iterative procedure can be used to find the solution of the problem.

Step 1: We choose the initial feasible point as

$$\mathbf{X}^{(1)} = \begin{Bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{Bmatrix}$$

and set $k = 1$.

Step 2: Since $|f(\mathbf{X}^{(1)})| = |\frac{2}{3}| > 0.05$, we go to step 3.

Step 3: Since $[a] = \{0 \quad 1 \quad -1\}$, $\mathbf{c} = \{2 \quad 1 \quad -1\}^T$, $\|\mathbf{c}\| = \sqrt{(2)^2 + (1)^2 + (-1)^2} = \sqrt{6}$, we find that

$$[D(\mathbf{X}^{(1)})] = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

$$[a]\,[D(\mathbf{X}^{(1)})] = \{0 \quad \tfrac{1}{3} \quad -\tfrac{1}{3}\}$$

$$[P] = \begin{bmatrix} [a][D(\mathbf{X}^{(1)})] \\ 1 \quad 1 \quad 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix}$$

$$([P]\,[P]^T)^{-1} = \begin{bmatrix} \frac{2}{9} & 0 \\ 0 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{9}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$$

$$[D(\mathbf{X}^{(1)})]\,\mathbf{c} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{Bmatrix} 2 \\ 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{Bmatrix}$$

$$([I] - [P]^T([P]\,[P]^T)^{-1}[P])\,[D(\mathbf{X}^{(1)})]\,\mathbf{c}$$

$$= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & 1 \\ -\frac{1}{3} & 1 \end{bmatrix} \begin{bmatrix} \frac{9}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{3} & -\frac{1}{3} \\ 1 & 1 & 1 \end{bmatrix} \right) \begin{Bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{Bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{6} \end{bmatrix} \begin{Bmatrix} \frac{2}{3} \\ \frac{1}{3} \\ -\frac{1}{3} \end{Bmatrix} = \begin{Bmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{Bmatrix}$$

Using $\alpha = \frac{1}{4}$, Eq. (4.68) gives

$$\mathbf{Y}^{(2)} = \begin{Bmatrix} \frac{1}{3} \\ \frac{1}{3} \\ \frac{1}{3} \end{Bmatrix} - \frac{1}{4} \begin{Bmatrix} \frac{4}{9} \\ -\frac{2}{9} \\ -\frac{2}{9} \end{Bmatrix} \frac{1}{\sqrt{3}\,(2)\,\sqrt{6}} = \begin{Bmatrix} \frac{34}{108} \\ \frac{37}{108} \\ \frac{37}{108} \end{Bmatrix}$$

Noting that

$$\sum_{r=1}^{n} x_r^{(1)} y_r^{(2)} = \frac{1}{3}\left(\frac{34}{108}\right) + \frac{1}{3}\left(\frac{37}{108}\right) + \frac{1}{3}\left(\frac{37}{108}\right) = \frac{1}{3}$$

Eq. (4.71) can be used to find

$$\{x_i^{(2)}\} = \left\{ \frac{x_i^{(1)} y_i^{(2)}}{\sum_{r=1}^{3} x_r^{(1)} y_r^{(2)}} \right\} = 3 \begin{Bmatrix} \frac{34}{324} \\ \frac{37}{324} \\ \frac{37}{324} \end{Bmatrix} = \begin{Bmatrix} \frac{34}{108} \\ \frac{37}{108} \\ \frac{37}{108} \end{Bmatrix}$$

Set the new iteration number as $k = k + 1 = 2$ and go to step 2. The procedure is to be continued until convergence is achieved.

Notes:

1. Although $\mathbf{X}^{(2)} = \mathbf{Y}^{(2)}$ in this example, they need not be, in general, equal to one another.
2. The value of f at $\mathbf{X}^{(2)}$ is

$$f(\mathbf{X}^{(2)}) = 2\left(\frac{34}{108}\right) + \frac{37}{108} - \frac{37}{108} = \frac{17}{27} < f(\mathbf{X}^{(1)}) = \frac{18}{27}$$

4.8 QUADRATIC PROGRAMMING

A quadratic programming problem can be stated as:

$$\text{Minimize } f(\mathbf{X}) = \mathbf{C}^T\mathbf{X} + \tfrac{1}{2}\mathbf{X}^T\mathbf{D}\mathbf{X} \tag{4.72}$$

subject to

$$\mathbf{A}\,\mathbf{X} \leq \mathbf{B} \tag{4.73}$$

$$\mathbf{X} \geq \mathbf{0} \tag{4.74}$$

where

$$\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}, \quad \mathbf{C} = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{Bmatrix}, \quad \mathbf{B} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{Bmatrix},$$

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & d_{2n} \\ \vdots & & & \\ d_{n1} & d_{n2} & \cdots & d_{nn} \end{bmatrix}, \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

In Eq. (4.72) the term $\mathbf{X}^T\mathbf{D}\mathbf{X}/2$ represents the quadratic part of the objective function with $\mathbf{D}$ being a symmetric positive-definite matrix. If $\mathbf{D} = \mathbf{O}$, the problem reduces to a LP problem. The solution of the quadratic programming problem stated in Eqs. (4.72) to (4.74) can be obtained by using the Lagrange multiplier technique. By introducing the slack variables s_i^2, $i = 1, 2, \ldots, m$, in Eqs. (4.73) and the surplus variables t_j^2, $j = 1, 2, \ldots, n$, in Eqs. (4.74), the quadratic programming problem can be written as:

$$\text{Minimize } f(\mathbf{X}) = \mathbf{C}^T\mathbf{X} + \tfrac{1}{2}\mathbf{X}^T\mathbf{D}\mathbf{X} \tag{4.72}$$

subject to the equality constraints

$$\mathbf{A}_i^T\mathbf{X} + s_i^2 = b_i, \quad i = 1,2,\ldots,m \tag{4.75}$$

$$-x_j + t_j^2 = 0, \quad j = 1,2,\ldots,n \tag{4.76}$$

where

$$\mathbf{A}_i = \begin{Bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{Bmatrix}$$

The Lagrange function can be written as

$$L(\mathbf{X},\mathbf{S},\mathbf{T},\boldsymbol{\lambda},\boldsymbol{\theta}) = \mathbf{C}^T\mathbf{X} + \tfrac{1}{2}\mathbf{X}^T\mathbf{D}\mathbf{X} + \sum_{i=1}^{m} \lambda_i\,(\mathbf{A}_i^T\mathbf{X} + s_i^2 - b_i) + \sum_{j=1}^{n} \theta_j(-x_j + t_j^2) \tag{4.77}$$

The necessary conditions for the stationariness of L give

$$\frac{\partial L}{\partial x_j} = c_j + \sum_{i=1}^{n} d_{ij}x_i + \sum_{i=1}^{m} \lambda_i a_{ij} - \theta_j = 0, \qquad j = 1,2,\ldots,n \tag{4.78}$$

$$\frac{\partial L}{\partial s_i} = 2\lambda_i s_i = 0, \qquad i = 1,2,\ldots,m \tag{4.79}$$

$$\frac{\partial L}{\partial t_j} = 2\theta_j t_j = 0, \qquad j = 1,2,\ldots,n \tag{4.80}$$

$$\frac{\partial L}{\partial \lambda_i} = \mathbf{A}_i^T\mathbf{X} + s_i^2 - b_i = 0, \qquad i = 1,2,\ldots,m \tag{4.81}$$

$$\frac{\partial L}{\partial \theta_j} = -x_j + t_j^2 = 0, \qquad j = 1,2,\ldots,n \tag{4.82}$$

By defining a set of new variables Y_i as

$$Y_i = s_i^2 \geq 0, \qquad i = 1,2,\ldots,m \tag{4.83}$$

Equations (4.81) can be written as

$$\mathbf{A}_i^T\mathbf{X} - b_i = -s_i^2 = -Y_i, \qquad i = 1,2,\ldots,m \tag{4.84}$$

Multiplying Eq. (4.79) by s_i and Eq. (4.80) by t_j, we obtain

$$\lambda_i s_i^2 = \lambda_i Y_i = 0, \qquad i = 1,2,\ldots,m \tag{4.85}$$

$$\theta_j t_j^2 = 0, \qquad j = 1,2,\ldots,n \tag{4.86}$$

Combining Eqs. (4.84) and (4.85), and Eqs. (4.82) and (4.86), we obtain

$$\lambda_i(\mathbf{A}_i^T\mathbf{X} - b_i) = 0, \qquad i = 1,2,\ldots,m \tag{4.87}$$

$$\theta_j x_j = 0, \qquad j = 1,2,\ldots,n \tag{4.88}$$

Thus the necessary conditions can be summarized as follows:

$$c_j - \theta_j + \sum_{i=1}^{n} x_i d_{ij} + \sum_{i=1}^{m} \lambda_i a_{ij} = 0, \qquad j = 1,2,\ldots,n \tag{4.89}$$

$$\mathbf{A}_i^T\mathbf{X} - b_i = -Y_i, \qquad i = 1,2,\ldots,m \tag{4.90}$$

$$x_j \geq 0, \qquad j = 1,2,\ldots,n \tag{4.91}$$

$$Y_i \geq 0, \qquad i = 1,2,\ldots,m \tag{4.92}$$

$$\lambda_i \geq 0, \qquad i = 1,2,\ldots,m \tag{4.93}$$

$$\theta_j \geq 0, \qquad j = 1,2,\ldots,n \tag{4.94}$$

$$\lambda_i Y_i = 0, \qquad i = 1,2,\ldots,m \tag{4.95}$$

$$\theta_j x_j = 0, \qquad j = 1,2,\ldots,n \tag{4.96}$$

We can notice one important thing in Eqs. (4.89) to (4.96). With the exception of Eqs. (4.95) and (4.96), the necessary conditions are linear functions of the variables x_j, Y_i, λ_i, and θ_j. Thus the solution of the original quadratic programming problem can be obtained by finding a nonnegative solution to the set of $m + n$ linear equations given by Eqs. (4.89) and (4.90), which also satisfies the $m + n$ equations stated in Eqs. (4.95) and (4.96).

Since $\mathbf{D}$ is a positive-definite matrix, $f(\mathbf{X})$ will be a strictly convex function,† and the feasible space is convex (because of linear equations), any local minimum of the problem will be the global minimum. Further, it can be seen that there are $2\,(n + m)$ variables and $2\,(n + m)$ equations in the necessary conditions stated in Eqs. (4.89) to (4.96). Hence the solution of the Eqs. (4.89), (4.90), (4.95), and (4.96) must be unique. Thus the feasible solution satisfying all the Eqs. (4.89) to (4.96), if it exists, must give the optimum solution of the quadratic programming problem directly. The solution of the system of equations above can be obtained by using phase I of the simplex method. The only restriction here is that the satisfaction of the nonlinear relations, Eqs. (4.95) and (4.96), has to be maintained all the time. Since our objective is just to find a feasible solution to the set of Eqs. (4.89) to (4.96), there is no necessity of phase II computations. We shall follow the procedure developed by Wolfe [4.21] to apply phase I. This procedure involves the introduction of n nonnegative artificial variables z_i into the Eqs. (4.89) so that

$$c_j - \theta_j + \sum_{i=1}^{n} x_i d_{ij} + \sum_{i=1}^{m} \lambda_i a_{ij} + z_j = 0, \qquad j = 1,2,\ldots,n \tag{4.97}$$

Then we minimize

$$F = \sum_{j=1}^{n} z_j \tag{4.98}$$

subject to the constraints

$$c_j - \theta_j + \sum_{i=1}^{n} x_i d_{ij} + \sum_{i=1}^{m} \lambda_i a_{ij} + z_j = 0, \qquad j = 1,2,\ldots,n$$

$$\mathbf{A}_i^T\mathbf{X} + Y_i = b_i, \qquad i = 1,2,\ldots,m$$

$$\mathbf{X} \geq \mathbf{0}, \quad \mathbf{Y} \geq \mathbf{0}, \quad \boldsymbol{\lambda} \geq \mathbf{0}, \quad \boldsymbol{\theta} \geq \mathbf{0}$$

†See Appendix A for the definition and properties of a convex function.

While solving this problem, we have to take care of the additional conditions

$$\begin{aligned} \lambda_i Y_i &= 0, \qquad i = 1,2,\ldots,m \\ \theta_j x_j &= 0, \qquad j = 1,2,\ldots,n \end{aligned} \tag{4.99}$$

Thus when deciding whether to introduce Y_i into the basic solution, we first have to ensure that either λ_i is not in the solution or λ_i will be removed when Y_i enters the basis. Similar care has to be taken regarding the variables θ_j and x_j. These additional checks are not very difficult to make during the solution procedure.

Example 4.14

$$\text{Minimize } f = -4x_1 + x_1^2 - 2x_1x_2 + 2x_2^2$$

subject to

$$2x_1 + x_x \leq 6$$

$$x_1 - 4x_2 \leq 0$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

SOLUTION By introducing the slack variables $Y_1 = s_1^2$ and $Y_2 = s_2^2$ and the surplus variables $\theta_1 = t_1^2$ and $\theta_2 = t_2^2$, the problem can be stated as follows:

$$\text{Minimize } f = (-4 \;\; 0)\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \frac{1}{2}(x_1 \;\; x_2)\begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}$$

subject to

$$\begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} + \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} = \begin{Bmatrix} 6 \\ 0 \end{Bmatrix}$$

$$\begin{aligned} -x_1 + \theta_1 &= 0 \\ -x_2 + \theta_2 &= 0 \end{aligned} \tag{E$_1$}$$

By comparing this problem with the one stated in Eqs. (4.72) to (4.74), we find that

$$c_1 = -4, \quad c_2 = 0, \quad \mathbf{D} = \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 1 & -4 \end{bmatrix},$$

$$\mathbf{A}_1 = \begin{Bmatrix} 2 \\ 1 \end{Bmatrix}, \quad \mathbf{A}_2 = \begin{Bmatrix} 1 \\ -4 \end{Bmatrix}, \quad \text{and} \quad \mathbf{B} = \begin{Bmatrix} 6 \\ 0 \end{Bmatrix}$$

The necessary conditions for the solution of the problem stated in Eqs. (E_1) can be obtained, using Eqs. (4.89) to (4.96), as

$$\begin{aligned} -4 - \theta_1 + 2x_1 - 2x_2 + 2\lambda_1 + \lambda_2 &= 0 \\ 0 - \theta_2 - 2x_1 + 4x_2 + \lambda_1 - 4\lambda_2 &= 0 \end{aligned} \tag{E_2}$$

$$2x_1 + x_2 - 6 = -Y_1$$

$$x_1 - 4x_2 - 0 = -Y_2$$

$$\begin{aligned} &x_1 \geq 0, \quad x_2 \geq 0, \quad Y_1 \geq 0, \quad Y_2 \geq 0, \quad \lambda_1 \geq 0, \\ &\lambda_2 \geq 0, \quad \theta_1 \geq 0, \quad \theta_2 \geq 0 \end{aligned} \tag{E_3}$$

$$\begin{aligned} \lambda_1 Y_1 = 0, \quad \theta_1 x_1 = 0 \\ \lambda_2 Y_2 = 0, \quad \theta_2 x_2 = 0 \end{aligned} \tag{E_4}$$

(If Y_i is in the basis, λ_i cannot be in the basis, and if x_j is in the basis, θ_j cannot be in the basis to satisfy these equations.) Equations (E_2) can be rewritten as

$$\begin{aligned} 2x_1 - 2x_2 + 2\lambda_1 + \lambda_2 - \theta_1 + z_1 &= 4 \\ -2x_1 + 4x_2 + \lambda_1 - 4\lambda_2 - \theta_2 + z_2 &= 0 \\ 2x_1 + x_2 \qquad\qquad + Y_1 \qquad &= 6 \\ x_1 - 4x_2 \qquad\qquad\qquad + Y_2 &= 0 \end{aligned} \tag{E_5}$$

where z_1 and z_2 are artificial variables. To find a feasible solution to Eqs. (E_2) to (E_4) by using phase I of simplex method, we minimize $w = z_1 + z_2$ with constraints stated in Eqs. (E_5), (E_3), and (E_4).

The initial simplex tableau is shown below.

Basic Variables	Variables x_1	x_2	λ_1	λ_2	θ_1	θ_2	Y_1	Y_2	z_1	z_2	w	$\bar{b}_i$	$\bar{b}_i/\bar{a}_{is}$ for $\bar{a}_{is} > 0$
Y_1	2	1	0	0	0	0	1	0	0	0	0	6	6
Y_2	1	−4	0	0	0	0	0	1	0	0	0	0	
z_1	2	−2	2	1	−1	0	0	0	1	0	0	4	
z_2	−2	$\boxed{4}$	1	−4	0	−1	0	0	0	1	0	0	0←Smaller one
$-w$	0	−2	−3	3	1	1	0	0	0	0	1	−4	
		↑ x_2 selected for entering next basis	↑ Most negative										

According to the regular procedure of simplex method, λ_1 enters the next basis since the cost coefficient of λ_1 is most negative and z_2 leaves the basis since the ratio $\bar{b}_i/\bar{a}_{is}$ is smaller for z_2. However, λ_1 cannot enter the basis, as Y_1 is already in the basis [to satisfy Eqs. (E_4)]. Hence we select x_2 for entering the next basis. According to this choice, z_2 leaves the basis. By carrying out the required pivot operation, we obtain the following tableau.

Basic Variables	Variables												$\bar{b}_i/\bar{a}_{is}$ for
	x_1	x_2	λ_1	λ_2	θ_1	θ_2	Y_1	Y_2	z_1	z_2	w	$\bar{b}_i$	$\bar{a}_{is} > 0$
Y_1	$\boxed{\frac{5}{2}}$	0	$-\frac{1}{4}$	1	0	$\frac{1}{4}$	1	0	0	$-\frac{1}{4}$	0	6	$\frac{12}{5}$ ← Smaller one
Y_2	-1	0	1	-4	0	-1	0	1	0	1	0	0	
z_1	1	0	$\frac{5}{2}$	-1	-1	$-\frac{1}{2}$	0	0	1	$\frac{1}{2}$	0	4	4
x_2	$-\frac{1}{2}$	1	$\frac{1}{4}$	-1	0	$-\frac{1}{4}$	0	0	0	$\frac{1}{4}$	0	0	
$-w$	-1	0	$-\frac{5}{2}$	1	1	$\frac{1}{2}$	0	0	0	$\frac{1}{2}$	1	-4	
	↑ x_1 selected to enter the basis		↑ Most negative										

This tableau shows that λ_1 has to enter the basis and Y_2 or x_2 has to leave the basis. However, λ_1 cannot enter the basis since Y_1 is already in the basis [to satisfy the requirement of Eqs. (E_4)]. Hence x_1 is selected to enter the basis and this gives Y_1 as the variable that leaves the basis. The pivot operation on the element $\frac{5}{2}$ results in the following tableau:

Basic Variables	Variables												$\bar{b}_i/\bar{a}_{is}$ for
	x_1	x_2	λ_1	λ_2	θ_1	θ_2	Y_1	Y_2	z_1	z_2	w	$\bar{b}_i$	$\bar{a}_{is} > 0$
x_1	1	0	$-\frac{1}{10}$	$\frac{2}{5}$	0	$\frac{1}{10}$	$\frac{2}{5}$	0	0	$-\frac{1}{10}$	0	$\frac{12}{5}$	
Y_2	0	0	$\frac{9}{10}$	$-\frac{18}{5}$	0	$-\frac{9}{10}$	$\frac{2}{5}$	1	0	$\frac{9}{10}$	0	$\frac{12}{5}$	$\frac{8}{3}$
z_1	0	0	$\boxed{\frac{13}{5}}$	$-\frac{7}{5}$	-1	$-\frac{3}{5}$	$-\frac{2}{5}$	0	1	$\frac{3}{5}$	0	$\frac{8}{5}$	$\frac{8}{13}$ ← Smaller one
x_2	0	1	$\frac{1}{5}$	$-\frac{4}{5}$	0	$-\frac{1}{5}$	$\frac{1}{5}$	0	0	$\frac{1}{5}$	0	$\frac{6}{5}$	6
$-w$	0	0	$-\frac{13}{5}$	$\frac{7}{5}$	1	$\frac{3}{5}$	$\frac{2}{5}$	0	0	$\frac{2}{5}$	1	$-\frac{8}{5}$	
			↑ Most negative										

From this tableau we find that λ_1 enters the basis (this can be permitted this time since Y_1 is not in the basis) and z_1 leaves the basis. The necessary pivot operation gives the following tableau:

Basic Variables	Variables												$\bar{b}_i/\bar{a}_{is}$ for
	x_1	x_2	λ_1	λ_2	θ_1	θ_2	Y_1	Y_2	z_1	z_2	w	$\bar{b}_i$	$\bar{a}_{is} > 0$
x_1	1	0	0	$\frac{9}{26}$	$-\frac{1}{26}$	$\frac{1}{13}$	$\frac{5}{13}$	0	$\frac{1}{26}$	$-\frac{1}{13}$	0	$\frac{32}{13}$	
Y_2	0	0	0	$-\frac{81}{26}$	$\frac{9}{26}$	$-\frac{9}{13}$	$\frac{7}{13}$	1	$-\frac{9}{26}$	$\frac{9}{13}$	0	$\frac{24}{13}$	
λ_1	0	0	1	$-\frac{7}{13}$	$-\frac{5}{13}$	$-\frac{3}{13}$	$-\frac{2}{13}$	0	$\frac{5}{13}$	$\frac{3}{13}$	0	$\frac{8}{13}$	
x_2	0	1	0	$-\frac{9}{13}$	$\frac{1}{13}$	$-\frac{2}{13}$	$\frac{3}{13}$	0	$-\frac{1}{13}$	$\frac{2}{13}$	0	$\frac{14}{13}$	
$-w$	0	0	0	0	0	0	0	0	1	1	1	0	

Since both the artificial variables z_1 and z_2 are driven out of the basis, the present tableau gives the desired solution as $x_1 = \frac{32}{13}$, $x_2 = \frac{14}{13}$, $Y_2 = \frac{24}{13}$, $\lambda_1 = \frac{8}{13}$ (basic variables), $\lambda_2 = 0$, $Y_1 = 0$, $\theta_1 = 0$, $\theta_2 = 0$ (nonbasic variables). Thus the solution of the original quadratic programming problem is given by

$$x_1^* = \frac{32}{13}, \quad x_2^* = \frac{14}{13}, \quad \text{and} \quad f_{\min} = f(x_1^*, x_2^*) = -\frac{88}{13}$$

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REVIEW QUESTIONS

4.1 Is the decomposition method efficient for all LP problems?

4.2 What is the scope of postoptimality analysis?

4.3 Why is Karmarkar's method called an interior method?

4.4 What is the major difference between the simplex and Karmarkar methods?

4.5 State the form of LP problem required by Karmarkar's method.

4.6 What are the advantages of the revised simplex method?

4.7 Match the following terms and descriptions.

(a) Karmarkar's method	Moves from one vertex to another
(b) Simplex method	Interior point algorithm
(c) Quadratic programming	Phase I computations not required
(d) Dual simplex method	Dantzig and Wolfe method
(e) Decomposition method	Wolfe's method

4.8 Answer true or false.

(a) The quadratic programming problem is a convex programming problem.

(b) It is immaterial whether a given LP problem is designated the primal or dual.

(c) If the primal problem involves minimization of f subject to greater-than constraints, its dual deals with the minimization of f subject to less-than constraints.

(d) If the primal problem has an unbounded solution, its dual will also have an unbounded solution.

(e) The transportation problem can be solved by simplex method.

4.9 Match the following in the context of duality theory.

(a) x_i is nonnegative	ith constraint is of less-than or equal-to type
(b) x_i is unrestricted	Maximization type
(c) ith constraint is of equality type	ith variable is unrestricted
(d) ith constraint is of greater-than or equal-to type	ith variable is nonnegative
(e) Minimization type	ith constraint is of equality type

PROBLEMS

Solve the following LP problems by the revised simplex method.

4.1
$$\text{Minimize } f = -5x_1 + 2x_2 + 5x_3 - 3x_4$$

subject to

$$2x_1 + x_2 - x_3 = 6$$
$$3x_1 + 8x_3 + x_4 = 7$$
$$x_i \geq 0, \qquad i = 1 \text{ to } 4$$

4.2
$$\text{Maximize } f = 15x_1 + 6x_2 + 9x_3 + 2x_4$$

subject to

$$10x_1 + 5x_2 + 25x_3 + 3x_4 \leq 50$$
$$12x_1 + 4x_2 + 12x_3 + x_4 \leq 48$$
$$7x_1 + x_4 \leq 35$$
$$x_i \geq 0, \qquad i = 1 \text{ to } 4$$

4.3 $$\text{Minimize } f = 2x_1 + 3x_2 + 2x_3 - x_4 + x_5$$

subject to

$$3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 = 0$$

$$x_1 + x_2 + x_3 + 3x_4 + x_5 = 2$$

$$x_i \geq 0, \qquad i = 1,2,\ldots,5$$

4.4 Discuss the relationships between the regular simplex method and the revised simplex method.

4.5 Solve the following LP problem graphically and by the revised simplex method:

$$\text{Maximize } f = x_2$$

subject to

$$-x_1 + x_2 \leq 0$$

$$-2x_1 - 3x_2 \leq 6$$

$$x_1, x_2 \text{ unrestricted in sign}$$

4.6 Consider the LP problem:

$$\text{Minimize } f = 3x_1 + x_3 + 2x_5$$

subject to

$$x_1 + x_3 - x_4 + x_5 = -1$$

$$x_2 - 2x_3 + 3x_4 + 2x_5 = -2$$

$$x_i \geq 0, \quad i = 1 \text{ to } 5$$

Solve this problem using the dual simplex method.

4.7 $$\text{Maximize } f = 4x_1 + 2x_2$$

subject to

$$x_1 - 2x_2 \geq 2$$

$$x_1 + 2x_2 = 8$$

$$x_1 - x_2 \leq 11$$

$$x_1 \geq 0, \quad x_2 \text{ unrestricted in sign}$$

(a) Write the dual of this problem.

(b) Find the optimum solution of the dual.

(c) Verify the solution obtained in part (b) by solving the primal problem graphically.

4.8 A water resource system consisting of two reservoirs is shown in Fig. 4.4. The flows and storages are expressed in a consistent set of units. The following data are available:

Quantity	Stream 1 ($i = 1$)	Stream 2 ($i = 2$)
Capacity of reservoir i	9	7
Available release from reservoir i	9	6
Capacity of channel below reservoir i	4	4
Actual release from reservoir i	x_1	x_2

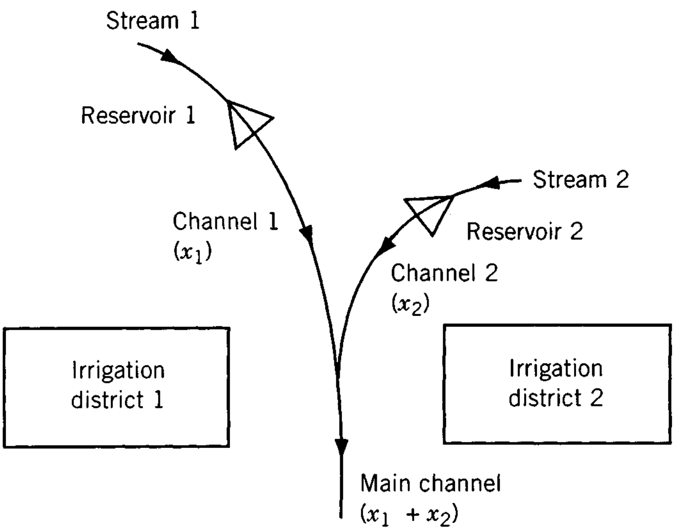

Figure 4.4 Water-resource system.

The capacity of the main channel below the confluence of the two streams is 5 units. If the benefit is equivalent to \$2 × 10^6 and \$3 × 10^6 per unit of water released from reservoirs 1 and 2, respectively, determine the releases x_1 and x_2 from the reserovirs to maximize the benefit. Solve this problem using duality theory.

4.9 Solve the following LP problem by the dual simplex method:

$$\text{Minimize } f = 2x_1 + 9x_2 + 24x_3 + 8x_4 + 5x_5$$

subject to

$$x_1 + x_2 + 2x_3 - x_5 - x_6 = 1$$

$$-2x_1 + x_3 + x_4 + x_5 - x_7 = 2$$

$$x_i \geq 0, \qquad i = 1 \text{ to } 7$$

4.10 Solve Problem 3.1 by solving its dual.

4.11 Show that neither the primal nor the dual of the problem,

$$\text{Maximize } f = -x_1 + 2x_2$$

subject to

$$-x_1 + x_2 \leq -2$$

$$x_1 - x_2 \leq 1$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

has a feasible solution. Verify your result graphically.

4.12 Solve the following LP problem by decomposition principle, and verify your result by solving it by the revised simplex method:

$$\text{Maximize } f = 8x_1 + 3x_2 + 8x_3 + 6x_4$$

subject to

$$4x_1 + 3x_2 + x_3 + 3x_4 \leq 16$$

$$4x_1 - x_2 + x_3 \leq 12$$

$$x_1 + 2x_2 \leq 8$$

$$3x_1 + x_2 \leq 10$$

$$2x_3 + 3x_4 \leq 9$$

$$4x_3 + x_4 \leq 12$$

$$x_i \geq 0, \qquad i = 1 \text{ to } 4$$

4.13 Apply the decomposition principle to the dual of the following problem and solve it.

$$\text{Minimize } f = 10x_1 + 2x_2 + 4x_3 + 8x_4 + x_5$$

subject to

$$x_1 + 4x_2 - x_3 \geq 16$$

$$2x_1 + x_2 + x_3 \geq 4$$

$$3x_1 + x_4 + x_5 \geq 8$$

$$x_1 + 2x_4 - x_5 \geq 20$$

$$x_i \geq 0, \quad i = 1 \text{ to } 5$$

4.14 Express the dual of the following LP problem:

$$\text{Maximize } f = 2x_1 + x_2$$

subject to

$$x_1 - 2x_2 \geq 2$$

$$x_1 + 2x_2 = 8$$

$$x_1 - x_2 \leq 11$$

$$x_1 \geq 0, \quad x_2 \text{ is unrestricted in sign}$$

4.15 Find the effect of changing $\mathbf{b} = \begin{Bmatrix} 1200 \\ 800 \end{Bmatrix}$ to $\begin{Bmatrix} 1180 \\ 120 \end{Bmatrix}$ in Example 4.5 using sensitivity analysis.

4.16 Find the effect of changing the cost coefficients c_1 and c_4 from -45 and -50 to -40 and -60, respectively, in Example 4.5 using sensitivity analysis.

4.17 Find the effect of changing c_1 from -45 to -40 and c_2 from -100 to -90 in Example 4.5 using sensitivity analysis.

4.18 If a new product, E, which requires 10 min of work on lathe and 10 min of work on milling machine per unit, with a profit of \$120 per unit is available in Example 4.5, determine whether it is worth manufacturing E.

4.19 A metallurgical company produces four products, A, B, C, and D, by using copper and zinc as basic materials. The material requirements and the profit per unit of each of the four products, and the maximum quantities of copper and zinc available are given below.

	Product				Maximum Quantity Available
	A	B	C	D	
Copper (lb)	4	9	7	10	6000
Zinc (lb)	2	1	3	20	4000
Profit per unit (\$)	15	25	20	60	

Find the number of units of the various products to be produced for maximizing the profit.

Solve problems 4.20–4.28 using the data of problem 4.19.

4.20 Find the effect of changing the profit per unit of product D to \$30.

4.21 Find the effect of changing the profit per unit of product A to \$10, and of product B to \$20.

4.22 Find the effect of changing the profit per unit of product B to \$30 and of product C to \$25.

4.23 Find the effect of changing the available quantities of copper and zinc to 4000 and 6000 lb, respectively.

4.24 What is the effect of introducing a new product, E, which requires 6 lb of copper and 3 lb of zinc per unit if it brings a profit of \$30 per unit?

4.25 Assume that products A, B, C, and D require, in addition to the stated amounts of copper and zinc, 4, 3, 2 and 5 lb of nickel per unit, respectively. If the total quantity of nickel available is 2000 lb, in what way the original optimum solution is affected?

4.26 If product A requires 5 lb of copper and 3 lb of zinc (instead of 4 lb of copper and 2 lb of zinc) per unit, find the change in the optimum solution.

4.27 If product C requires 5 lb of copper and 4 lb of zinc (instead of 7 lb of copper and 3 lb of zinc) per unit, find the change in the optimum solution.

4.28 If the available quantities of copper and zinc are changed to 8000 lb and 5000 lb, respectively, find the change in the optimum solution.

4.29 Solve the following LP problem:

$$\text{Minimize } f = 8x_1 - 2x_2$$

subject to

$$-4x_1 + 2x_2 \le 1$$

$$5x_1 - 4x_2 \le 3$$

$$x_1 \ge 0, \quad x_2 \ge 0$$

Investigate the change in the optimum solution of Problem 4.29 when the following changes are made **(a)** by using sensitivity analysis and **(b)** by solving the new problem graphically.

4.30 $b_1 = 2$

4.31 $b_2 = 4$

4.32 $c_1 = 10$

4.33 $c_2 = -4$

4.34 $a_{11} = -5$

4.35 $a_{22} = -2$

4.36 Perform one iteration of Karmarkar's method for the LP problem:

$$\text{Minimize } f = 2x_1 - 2x_2 + 5x_3$$

subject to

$$x_1 - x_2 = 0$$

$$x_1 + x_2 + x_3 = 1$$

$$x_i \geq 0, \qquad i = 1,2,3$$

4.37 Perform one iteration of Karmarkar's method for the following LP problem:

$$\text{Minimize } f = 3x_1 + 5x_2 - 3x_3$$

subject to

$$x_1 - x_3 = 0$$

$$x_1 + x_2 + x_3 = 1$$

$$x_i \geq 0, \qquad i = 1,2,3$$

4.38 Transform the following LP problem into the form required by Karmarkar's method:

$$\text{Minimize } f = x_1 + x_2 + x_3$$

subject to

$$x_1 + x_2 - x_3 = 4$$

$$3x_1 - x_2 = 0$$

$$x_i \geq 0, \qquad i = 1,2,3$$

4.39 A contractor has three sets of heavy construction equipment available at both New York and Los Angeles. He has construction jobs in Seattle, Houston, and Detroit that require two, three, and one set of equipment, respectively. The shipping costs per set between cities i and j (c_{ij}) are shown in Fig. 4.5. Formulate the problem of finding the shipping pattern that minimizes the cost.

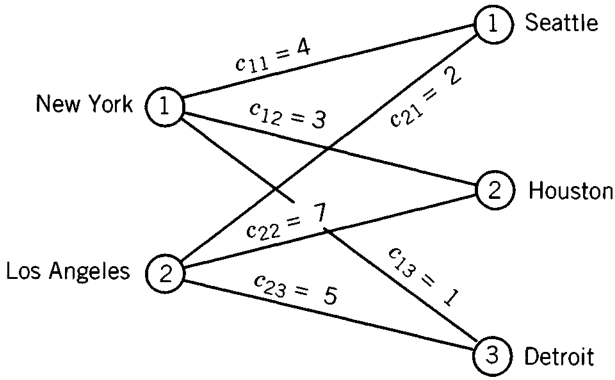

Figure 4.5 Shipping costs between cities.

4.40 Minimize $f(\mathbf{X}) = 3x_1^2 + 2x_2^2 + 5x_3^2 - 4x_1x_2 - 2x_1x_3 - 2x_2x_3$

subject to

$$3x_1 + 5x_2 + 2x_3 \geq 10$$
$$3x_1 + 5x_3 \leq 15$$
$$x_i \geq 0, \quad i = 1,2,3$$

by quadratic programming.

4.41 Find the solution of the quadratic programming problem stated in Example 1.5.

4.42 According to elastic–plastic theory, a frame structure fails (collapses) due to the formation of a plastic hinge mechanism. The various possible mechanisms in which a portal frame (Fig. 4.6) can fail are shown in

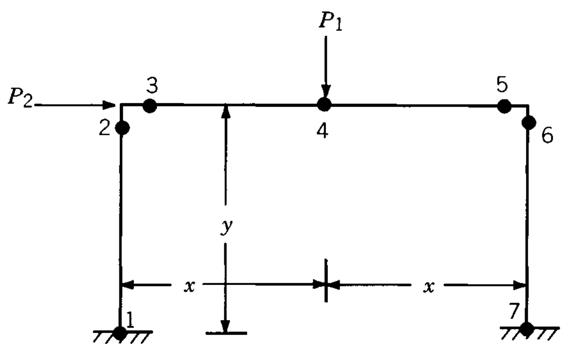

Figure 4.6 Plastic hinges in a frame.

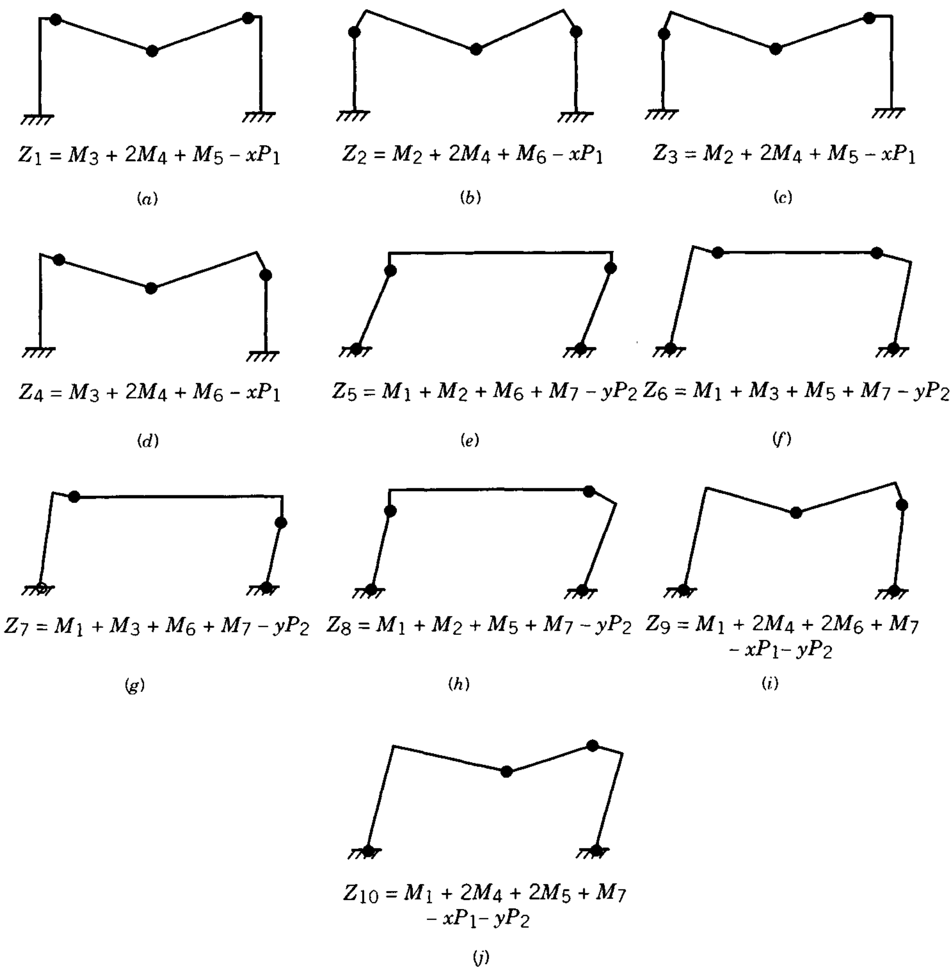

Figure 4.7 Possible failure mechanisms of a portal frame.

Fig. 4.7. The reserve strengths of the frame in various failure mechanisms (Z_i) can be expressed in terms of the plastic moment capacities of the hinges as indicated in Fig. 4.7. Assuming that the cost of the frame is proportional to 200 times each of the moment capacities M_1, M_2, M_6, and M_7, and 100 times each of the moment capacities M_3, M_4, and M_5, formulate the problem of minimizing the total cost to ensure nonzero reserve strength in each failure mechanism. Also, suggest a suitable technique for solving the problem. Assume that the moment capacities are restricted as $0 \leq M_i \leq 2 \times 10^5$ lb-in., $i = 1,2,\ldots,7$. Data: $x = 100$ in., $y = 150$ in., $P_1 = 1000$ lb, and $P_2 = 500$ lb.