3

LINEAR PROGRAMMING I: SIMPLEX METHOD

3.1 INTRODUCTION

Linear programming is an optimization method applicable for the solution of problems in which the objective function and the constraints appear as linear functions of the decision variables. The constraint equations in a linear programming problem may be in the form of equalities or inequalities. The linear programming type of optimization problem was first recognized in the 1930s by economists while developing methods for the optimal allocation of resources. During World War II the U.S. Air Force sought more effective procedures of allocating resources and turned to linear programming. George B. Dantzig, who was a member of the Air Force group, formulated the general linear programming problem and devised the simplex method of solution in 1947. This has become a significant step in bringing linear programming into wider use. Afterward, much progress has been made in the theoretical development and in the practical applications of linear programming. Among all the works, the theoretical contributions made by Kuhn and Tucker had a major impact in the development of the duality theory in LP. The works of Charnes and Cooper were responsible for industrial applications of LP.

Linear programming is considered a revolutionary development that permits us to make optimal decisions in complex situations. At least four Nobel Prizes were awarded for contributions related to linear programming. For example, when the Nobel Prize in Economics was awarded in 1975 jointly to L. V. Kantorovich of the former Soviet Union and T. C. Koopmans of the United States, the citation for the prize mentioned their contributions on the application of LP to the economic problem of allocating resources [3.1]. George Dantzig, the inventor of LP, was awarded the National Medal of Science by President Gerald Ford in 1976.

Although several other methods have been developed over the years for solving LP problems, the simplex method continues to be the most efficient and popular method for solving general LP problems. Among other methods, Karmarkar's method, developed in 1984, has been shown to be up to 50 times as fast as the simplex algorithm of Dantzig. In this chapter we present the theory, development, and applications of the simplex method for solving LP problems. Additional topics, such as the revised simplex method, duality theory, decomposition method, postoptimality analysis, and Karmarkar's method, are considered in Chapter 4.

3.2 APPLICATIONS OF LINEAR PROGRAMMING

The number of applications of linear programming has been so large that it is not possible to describe all of them here. Only the early applications are mentioned here and the exercises at the end of this chapter give additional example applications of linear programming. One of the early industrial applications of linear programming has been made in the petroleum refineries. In general, an oil refinery has a choice of buying crude oil from several different sources with differing compositions and at differing prices. It can manufacture different products, such as aviation fuel, diesel fuel, and gasoline, in varying quantities. The constraints may be due to the restrictions on the quantity of the crude oil available from a particular source, the capacity of the refinery to produce a particular product, and so on. A mix of the purchased crude oil and the manufactured products is sought that gives the maximum profit.

The optimal production plan in a manufacturing firm can also be decided using linear programming. Since the sales of a firm fluctuate, the company can have various options. It can build up an inventory of the manufactured products to carry it through the period of peak sales, but this involves an inventory holding cost. It can also pay overtime rates to achieve higher production during periods of higher demand. Finally, the firm need not meet the extra sales demand during the peak sales period, thus losing a potential profit. Linear programming can take into account the various cost and loss factors and arrive at the most profitable production plan.

In the food-processing industry, linear programming has been used to determine the optimal shipping plan for the distribution of a particular product from different manufacturing plants to various warehouses. In the iron and steel industry, linear programming was used to decide the types of products to be made in their rolling mills to maximize the profit. Metalworking industries use linear programming for shop loading and for determining the choice between producing and buying a part. Paper mills use it to decrease the amount of trim losses. The optimal routing of messages in a communication network and the routing of aircraft and ships can also be decided using linear programming.

Linear programming has also been applied to formulate and solve several types of engineering design problems, such as the plastic design of frame structures, as illustrated in the following example.

Example 3.1 In the limit design of steel frames, it is assumed that plastic hinges will be developed at points with peak moments. When a sufficient number of hinges develop, the structure becomes an unstable system referred to as a *collapse mechanism*. Thus a design will be safe if the energy-absorbing capacity of the frame (U) is greater than the energy imparted by the externally applied loads (E) in each of the deformed shapes as indicated by the various collapse mechanisms [3.9].

For the rigid frame shown in Fig. 3.1, plastic moments may develop at the points of peak moments (numbered 1 through 7 in Fig. 3.1). Four possible collapse mechanisms are shown in Fig. 3.2 for this frame. Assuming that the weight is a linear function of the plastic moment capacities, find the values of the ultimate moment capacities M_b and M_c for minimum weight. Assume that the two columns are identical and that $P_1 = 3$, $P_2 = 1$, $h = 8$, and $l = 10$.

SOLUTION The objective function can be expressed as

$$f(M_b,M_c) = \text{weight of beam} + \text{weight of columns}$$
$$= \alpha(2lM_b + 2hM_c)$$

where α is a constant indicating the weight per unit length of the member with a unit plastic moment capacity. Since a constant multiplication factor does not affect the result, f can be taken as

$$f = 2lM_b + 2hM_c = 20M_b + 16M_c \tag{E$_1$}$$

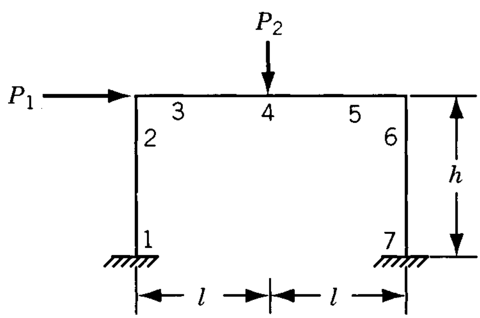

Figure 3.1 Rigid frame.

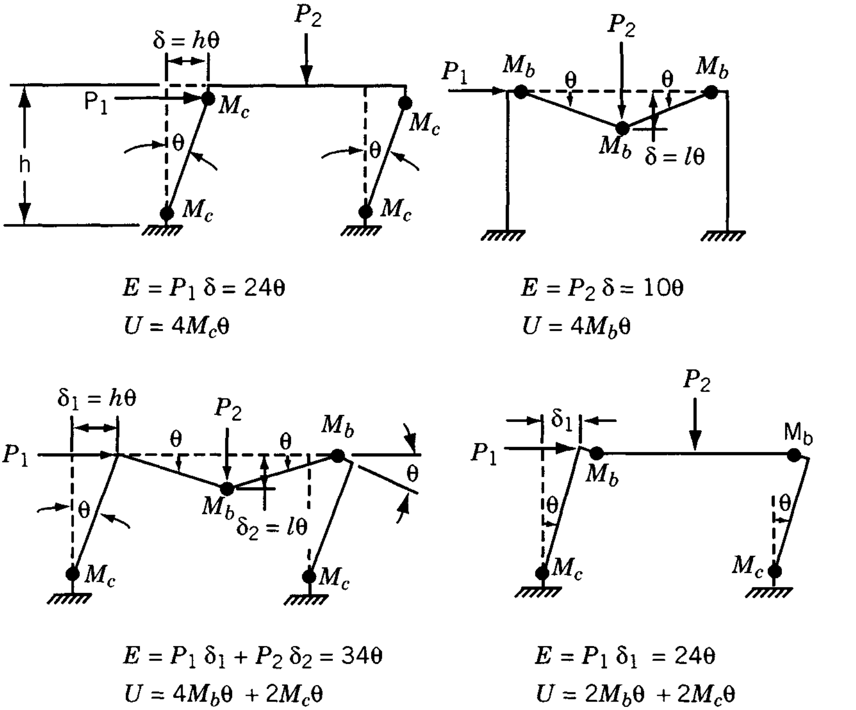

Figure 3.2 Collapse mechanisms of the frame. M_b, moment carrying capacity of beam; M_c, moment carrying capacity of column [3.9].

The constraints ($U \geq E$) from the four collapse mechanisms can be expressed as

$$\begin{aligned} M_c &\geq 6 \\ M_b &\geq 2.5 \\ 2M_b + M_c &\geq 17 \\ M_b + M_c &\geq 12 \end{aligned} \tag{E$_2$}$$

3.3 STANDARD FORM OF A LINEAR PROGRAMMING PROBLEM

The general linear programming problem can be stated in the following standard form:

1. *Scalar form*

$$\text{Minimize } f(x_1,x_2,\ldots,x_n) = c_1x_1 + c_2x_2 + \cdots + c_nx_n \tag{3.1a}$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{3.2a}$$

$$\begin{aligned} x_1 &\geq 0 \\ x_2 &\geq 0 \\ &\vdots \\ x_n &\geq 0 \end{aligned} \tag{3.3a}$$

where c_j, b_j, and a_{ij} ($i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$) are known constants, and x_j are the decision variables.

2. *Matrix form*

$$\text{Minimize } f(\mathbf{X}) = \mathbf{c}^T\mathbf{X} \tag{3.1b}$$

subject to the constraints

$$\mathbf{a}\mathbf{X} = \mathbf{b} \tag{3.2b}$$

$$\mathbf{X} \geq \mathbf{0} \tag{3.3b}$$

where

$$\mathbf{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}, \quad \mathbf{b} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{Bmatrix}, \quad \mathbf{c} = \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{Bmatrix},$$

$$\mathbf{a} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

The characteristics of a linear programming problem, stated in the standard form, are:

1. The objective function is of the minimization type.
2. All the constraints are of the equality type.
3. All the decision variables are nonnegative.

It is now shown that any linear programming problem can be expressed in the standard form by using the following transformations.

1. The maximization of a function $f(x_1, x_2, \ldots, x_n)$ is equivalent to the minimization of the negative of the same function. For example, the objective function

$$\text{minimize } f = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

is equivalent to

$$\text{maximize } f' = -f = -c_1x_1 - c_2x_2 - \cdots - c_nx_n$$

Consequently, the objective function can be stated in the minimization form in any linear programming problem.

2. In most engineering optimization problems, the decision variables represent some physical dimensions, and hence the variables x_j will be nonnegative. However, a variable may be unrestricted in sign in some problems. In such cases, an unrestricted variable (which can take a positive, negative, or zero value) can be written as the difference of two nonnegative variables. Thus if x_j is unrestricted in sign, it can be written as $x_j = x_j' - x_j''$, where

$$x_j' \geq 0 \quad \text{and} \quad x_j'' \geq 0$$

It can be seen that x_j will be negative, zero, or positive, depending on whether x_j'' is greater than, equal to, or less than x_j'.

3. If a constraint appears in the form of a "less than or equal to" type of inequality as

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n \leq b_k$$

it can be converted into the equality form by adding a nonnegative slack variable x_{n+1} as follows:

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n + x_{n+1} = b_k$$

Similarly, if the constraint is in the form of a "greater than or equal to" type of inequality as

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n \geq b_k$$

it can be converted into the equality form by subtracting a variable as

$$a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n - x_{n+1} = b_k$$

where x_{n+1} is a nonnegative variable known as a *surplus variable.*

It can be seen that there are m equations in n decision variables in a linear programming problem. We can assume that $m < n$; for if $m > n$, there would be $m - n$ redundant equations that could be eliminated. The case $n = m$ is of no interest, for then there is either a unique solution **X** that satisfies Eqs. (3.2) and (3.3) (in which case there can be no optimization) or no solution, in which case the constraints are inconsistent. The case $m < n$ corresponds to an underdetermined set of linear equations which, if they have one solution, have an infinite number of solutions. The problem of linear programming is to find one of these solutions that satisfies Eqs. (3.2) and (3.3) and yields the minimum of f.

3.4 GEOMETRY OF LINEAR PROGRAMMING PROBLEMS

A linear programming problem with only two variables presents a simple case for which the solution can be obtained by using a rather elementary graphical method. Apart from the solution, the graphical method gives a physical picture of certain geometrical characteristics of linear programming problems. The following example is considered to illustrate the graphical method of solution.

Example 3.2 A manufacturing firm produces two machine parts using lathes, milling machines, and grinding machines. The different machining times required for each part, the machining times available on different machines, and the profit on each machine part are given in the following table.

	Machining Time Required (min)		
Type of Machine	Machine Part I	Machine Part II	Maximum Time Available per Week (min)
Lathes	10	5	2500
Milling machines	4	10	2000
Grinding machines	1	1.5	450
Profit per unit	$50	$100	

Determine the number of parts I and II to be manufactured per week to maximize the profit.

SOLUTION Let the number of machine parts I and II manufactured per week be denoted by x and y, respectively. The constraints due to the maximum time limitations on the various machines are given by

$$10x + 5y \le 2500 \tag{E$_1$}$$

$$4x + 10y \le 2000 \tag{E$_2$}$$

$$x + 1.5y \le 450 \tag{E$_3$}$$

Since the variables x and y cannot take negative values, we have

$$\begin{aligned} x &\ge 0 \\ y &\ge 0 \end{aligned} \tag{E$_4$}$$

The total profit is given by

$$f(x,y) = 50x + 100y \tag{E$_5$}$$

Thus the problem is to determine the nonnegative values of x and y that satisfy the constraints stated in Eqs. (E_1) to (E_3) and maximize the objective function given by Eq. (E_5). The inequalities (E_1) to (E_4) can be plotted in the xy plane and the feasible region identified as shown in Fig. 3.3. Our objective is to find

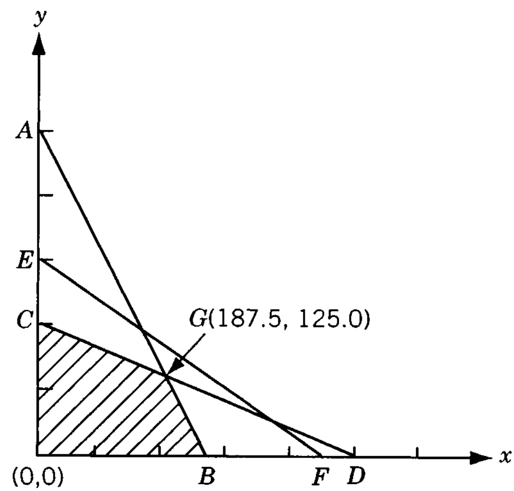

Figure 3.3 Feasible region given by Eqs. (E_1) to (E_4).

at least one point out of the infinite points in the shaded region of Fig. 3.3 which maximizes the profit function (E_5).

The contours of the objective function, f, are defined by the linear equation

$$50x + 100y = k = \text{constant}$$

As k is varied, the objective function line is moved parallel to itself. The maximum value of f is the largest k whose objective function line has at least one point in common with the feasible region. Such a point can be identified as point G in Fig. 3.4. The optimum solution corresponds to a value of $x^* = 187.5$, $y^* = 125.0$ and a profit of \$21,875.00.

In some cases, the optimum solution may not be unique. For example, if the profit rates for the machine parts I and II are \$40 and \$100 instead of \$50 and \$100, respectively, the contours of the profit function will be parallel to side CG of the feasible region as shown in Fig. 3.5. In this case, line $P''Q''$, which coincides with the boundary line CG, will correspond to the maximum (feasible) profit. Thus there is no unique optimal solution to the problem and any point between C and G on line $P''Q''$ can be taken as an optimum solution with a profit value of \$20,000. There are three other possibilities. In some problems, the feasible region may not be a closed convex polygon. In such a case, it may happen that the profit level can be increased to an infinitely large value without leaving the feasible region, as shown in Fig. 3.6. In this case the solution of the linear programming problem is said to be unbounded. On the other extreme, the constraint set may be empty in some problems. This could be due to the inconsistency of the constraints; or, sometimes, even though

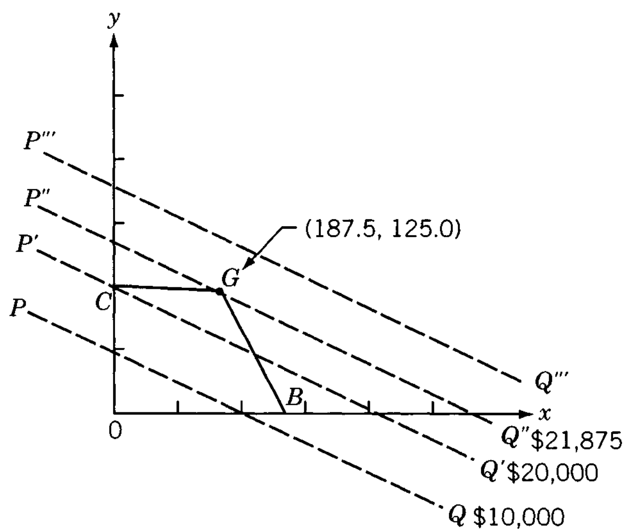

Figure 3.4 Contours of objective function.

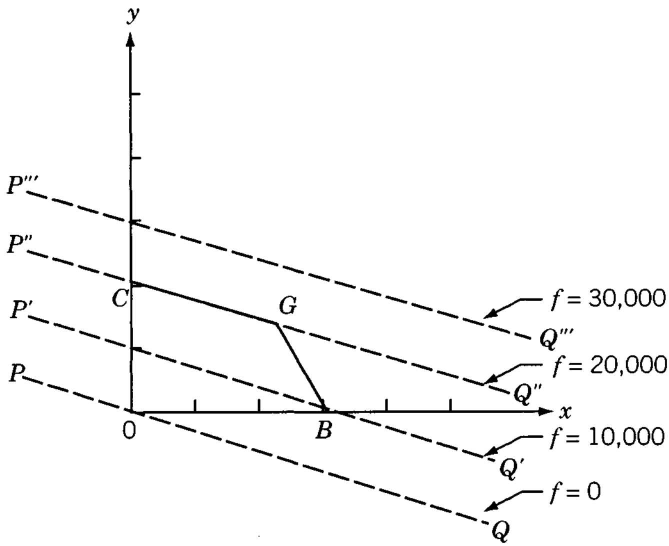

Figure 3.5 Infinite solutions.

the constraints may be consistent, no point satisfying the constraints may also satisfy the nonnegativity restrictions. The last possible case is when the feasible region consists of a single point. This can occur only if the number of constraints is at least equal to the number of variables. A problem of this kind is of no interest to us since there is only one feasible point and there is nothing to be optimized.

Thus a linear programming problem may have (1) a unique and finite optimum solution, (2) an infinite number of optimal solutions, (3) an unbounded

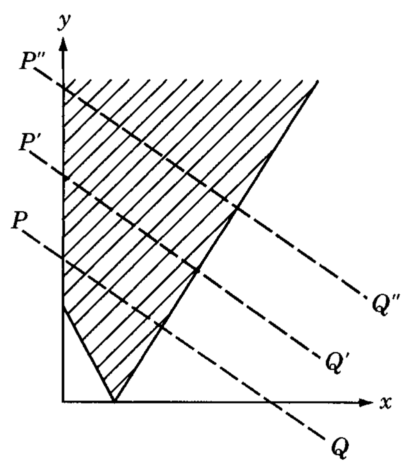

Figure 3.6 Unbounded solution.

solution, (4) no solution, or (5) a unique feasible point. Assuming that the linear programming problem is properly formulated, the following general geometrical characteristics can be noted from the graphical solution.

1. The feasible region is a convex polygon.†
2. The optimum value occurs at an extreme point or vertex of the feasible region.

3.5 DEFINITIONS AND THEOREMS

The geometrical characteristics of a linear programming problem stated in Section 3.4 can be proved mathematically. Some of the more powerful methods of solving linear programming problems take advantage of these characteristics. The terminology used in linear programming and some of the important theorems are presented in this section.

Definitions

1. Point in **n*****-Dimensional Space*** A point $\mathbf{X}$ in an n-dimensional space is characterized by an ordered set of n values or coordinates $(x_1, x_2, \ldots, x_n)$. The coordinates of $\mathbf{X}$ are also called the *components* of $\mathbf{X}$.

2. Line Segment in **n*****-Dimensions (L)*** If the coordinates of two points A and B are given by $x_j^{(1)}$ and $x_j^{(2)}$ $(j = 1, 2, \ldots, n)$, the line segment (L) joining these points is the collection of points $\mathbf{X}(\lambda)$ whose coordinates are given by $x_j = \lambda x_j^{(1)} + (1 - \lambda) x_j^{(2)}$, $j = 1, 2, \ldots, n$, with $0 \le \lambda \le 1$.

Thus

$$L = \{\mathbf{X} \mid \mathbf{X} = \lambda \mathbf{X}^{(1)} + (1 - \lambda)\mathbf{X}^{(2)}\} \tag{3.4}$$

In one dimension, for example, it is easy to see that the definition is in accordance with our experience (Fig. 3.7):

$$x^{(2)} - x(\lambda) = \lambda[x^{(2)} - x^{(1)}], \qquad 0 \le \lambda \le 1 \tag{3.5}$$

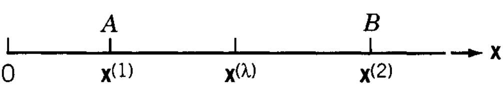

Figure 3.7 Line segment.

†A convex polygon consists of a set of points having the property that the line segment joining any two points in the set is entirely in the convex set. In problems having more than two decision variables, the feasible region is called a *convex polyhedron*, which is defined in the next section.

whence

$$x(\lambda) = \lambda x^{(1)} + (1 - \lambda)x^{(2)}, \qquad 0 \le \lambda \le 1 \tag{3.6}$$

3. Hyperplane In n-dimensional space, the set of points whose coordinates satisfy a linear equation

$$a_1x_1 + \cdots + a_nx_n = \mathbf{a}^T\mathbf{X} = b \tag{3.7}$$

is called a hyperplane.

A hyperplane, H, is represented as

$$H(\mathbf{a},b) = \{\mathbf{X}|\mathbf{a}^T\mathbf{X} = b\} \tag{3.8}$$

A hyperplane has $n - 1$ dimensions in an n-dimensional space. For example, in three-dimensional space it is a plane, and in two-dimensional space it is a line. The set of points whose coordinates satisfy a linear inequality like $a_1x_1 + \cdots + a_nx_n \le b$ is called a *closed half-space*, closed due to the inclusion of an equality sign in the inequality above. A hyperplane partitions the n-dimensional space (E^n) into two closed half-spaces, so that

$$H^+ = \{\mathbf{X}|\mathbf{a}^T\mathbf{X} \ge b\} \tag{3.9}$$

$$H^- = \{\mathbf{X}|\mathbf{a}^T\mathbf{X} \le b\} \tag{3.10}$$

This is illustrated in Fig. 3.8 in the case of a two-dimensional space (E^2).

4. Convex Set A convex set is a collection of points such that if $\mathbf{X}^{(1)}$ and $\mathbf{X}^{(2)}$ are any two points in the collection, the line segment joining them is also in the collection. A convex set, S, can be defined mathematically as follows:

$$\text{If} \quad \mathbf{X}^{(1)},\mathbf{X}^{(2)} \in S, \quad \text{then} \quad \mathbf{X} \in S$$

where

$$\mathbf{X} = \lambda\mathbf{X}^{(1)} + (1 - \lambda)\mathbf{X}^{(2)}, \qquad 0 \le \lambda \le 1$$

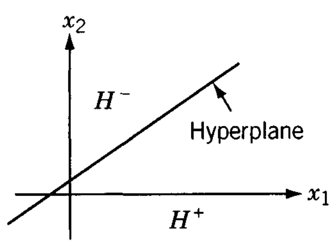

Figure 3.8 Hyperplane in two dimensions.

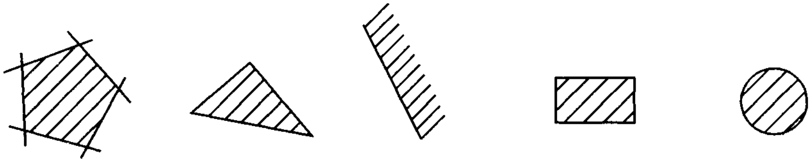

Figure 3.9 Convex sets.

A set containing only one point is always considered to be convex. Some examples of convex sets in two dimensions are shown shaded in Fig. 3.9. On the other hand, the sets depicted by the shaded region in Fig. 3.10 are not convex. The L-shaped region, for example, is not a convex set because it is possible to find two points a and b in the set such that not all points on the line joining them belong to the set.

5. Convex Polyhedron and Convex Polytope A convex polyhedron is a set of points common to one or more half-spaces. A convex polyhedron that is bounded is called a convex polytope.

Figure 3.11a and b represent convex polytopes in two and three dimensions, and Fig. 3.11c and d denote convex polyhedra in two and three dimensions. It can be seen that a convex polygon, shown in Fig. 3.11a and c, can be considered as the intersection of one or more half-planes.

6. Vertex or Extreme Point This is a point in the convex set that does not lie on a line segment joining two other points of the set. For example, every point on the circumference of a circle and each corner point of a polygon can be called a vertex or extreme point.

7. Feasible Solution In a linear programming problem, any solution that satisfies the constraints

$$\mathbf{aX} = \mathbf{b} \tag{3.2}$$

$$\mathbf{X} \geq \mathbf{0} \tag{3.3}$$

is called a *feasible solution*.

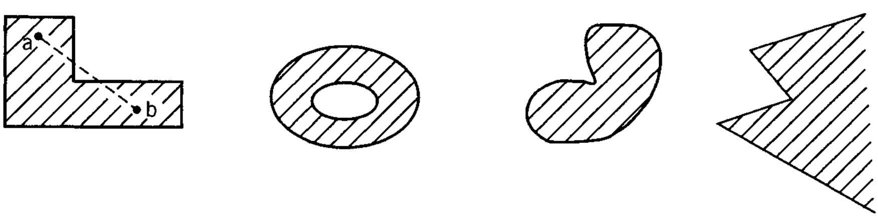

Figure 3.10 Nonconvex sets.

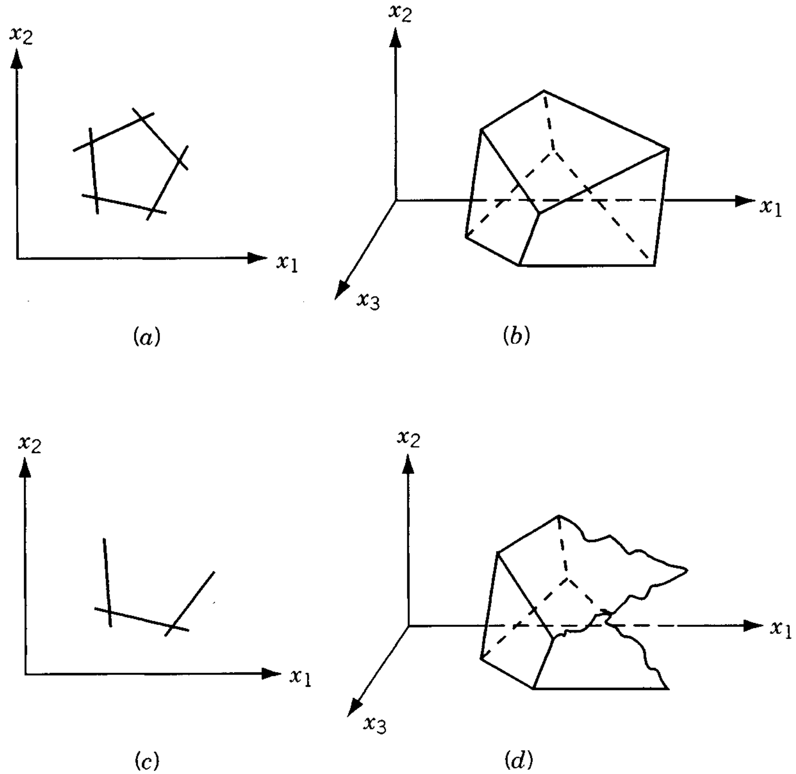

Figure 3.11 Convex polytopes in two and three dimensions (*a*, *b*) and convex polyhedra in two and three dimensions (*c*, *d*).

8. Basic Solution A basic solution is one in which $n - m$ variables are set equal to zero. A basic solution can be obtained by setting $n - m$ variables to zero and solving the constraint Eqs. (3.2) simultaneously.

9. Basis The collection of variables not set equal to zero to obtain the basic solution is called the basis.

10. Basic Feasible Solution This is a basic solution that satisfies the nonnegativity conditions of Eq. (3.3).

11. Nondegenerate Basic Feasible Solution This is a basic feasible solution that has got exactly m positive x_i.

12. Optimal Solution A feasible solution that optimizes the objective function is called an optimal solution.

13. Optimal Basic Solution This is a basic feasible solution for which the objective function is optimal.

Theorems The basic theorems of linear programming can now be stated and proved.†

Theorem 3.1 The intersection of any number of convex sets is also convex.

Proof: Let the given convex sets be represented as R_i $(i = 1,2,\ldots,K)$ and their intersection as R, so that‡

$$R = \bigcap_{i=1}^{K} R_i$$

If the points $\mathbf{X}^{(1)}, \mathbf{X}^{(2)} \in R$, then from the definition of intersection,

$$\mathbf{X} = \lambda\mathbf{X}^{(1)} + (1 - \lambda)\,\mathbf{X}^{(2)} \in R_i \qquad (i = 1,2,\ldots,K)$$
$$0 \le \lambda \le 1$$

Thus

$$\mathbf{X} \in R = \bigcap_{i=1}^{K} R_i$$

and the theorem is proved. Physically, the theorem states that if there are a number of convex sets represented by $R_1, R_2, \ldots$, the set of points R common to all these sets will also be convex. Figure 3.12 illustrates the meaning of this theorem for the case of two convex sets.

Theorem 3.2 The feasible region of a linear programming problem is convex.

†The proofs of the theorems are not needed for an understanding of the material presented in subsequent sections.
‡The symbol ∩ represents the intersection of sets.

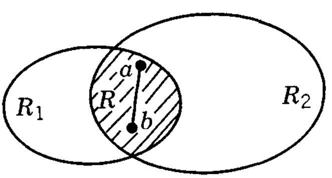

Figure 3.12 Intersection of two convex sets.

Proof: The feasible region S of a standard linear programming problem is defined as

$$S = \{\mathbf{X} | \mathbf{aX} = \mathbf{b}, \mathbf{X} \geq \mathbf{0}\} \tag{3.11}$$

Let the points $\mathbf{X}_1$ and $\mathbf{X}_2$ belong to the feasible set S so that

$$\mathbf{aX}_1 = \mathbf{b}, \qquad \mathbf{X}_1 \geq \mathbf{0} \tag{3.12}$$

$$\mathbf{aX}_2 = \mathbf{b}, \qquad \mathbf{X}_2 \geq \mathbf{0} \tag{3.13}$$

Multiply Eq. (3.12) by λ and Eq. (3.13) by $(1 - \lambda)$ and add them to obtain

$$\mathbf{a}[\lambda \mathbf{X}_1 + (1 - \lambda)\mathbf{X}_2] = \lambda \mathbf{b} + (1 - \lambda)\mathbf{b} = \mathbf{b}$$

that is,

$$\mathbf{aX}_\lambda = \mathbf{b}$$

where

$$\mathbf{X}_\lambda = \lambda \mathbf{X}_1 + (1 - \lambda)\mathbf{X}_2$$

Thus the point $\mathbf{X}_\lambda$ satisfies the constraints and if

$$0 \leq \lambda \leq 1, \qquad \mathbf{X}_\lambda \geq \mathbf{0}$$

Hence the theorem is proved.

Theorem 3.3 Any local minimum solution is global for a linear programming problem.

Proof: In the case of a function of one variable, the minimum (maximum) of a function $f(x)$ is obtained at a value x at which the derivative is zero. This may be a point like $A(x = x_1)$ in Fig. 3.13, where $f(x)$ is only a relative (local) minimum, or a point like $B(x = x_2)$, where $f(x)$ is a global minimum. Any solution that is a local minimum solution is also a global minimum solution for the linear programming problem. To see this, let A be the local minimum solution and assume that it is not a global minimum solution so that there is another point B at which $f_B < f_A$. Let the coordinates of A and B be given by $\begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$ and $\begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{Bmatrix}$, respectively. Then any point $C = \begin{Bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{Bmatrix}$ which lies on the

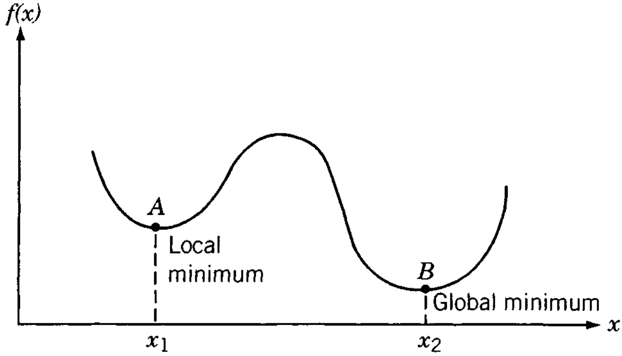

Figure 3.13 Local and global minima.

line segment joining the two points A and B is a feasible solution and $f_C = \lambda f_A + (1 - \lambda) f_B$. In this case, the value of f decreases uniformly from f_A to f_B, and thus all points on the line segment between A and B (including those in the neighborhood of A) have f values less than f_A and correspond to feasible solutions. Hence it is not possible to have a local minimum at A and at the same time another point B such that $f_A > f_B$. This means that for all B, $f_A \leq f_B$, so that f_A is the global minimum value.

The generalized version of this theorem is proved in Appendix A so that it can be applied to nonlinear programming problems also.

Theorem 3.4 Every basic feasible solution is an extreme point of the convex set of feasible solutions.

Theorem 3.5 Let S be a closed, bounded convex polyhedron with $\mathbf{X}_i^e$, $i = 1$ to p, as the set of its extreme points. Then any vector $\mathbf{X} \in S$ can be written as

$$\mathbf{X} = \sum_{i=1}^{p} \lambda_i \mathbf{X}_i^e$$

$$\lambda_i \geq 0$$

$$\sum_{i=1}^{p} \lambda_i = 1$$

Theorem 3.6 Let S be a closed convex polyhedron. Then the minimum of a linear function over S is attained at an extreme point of S.

The proofs of Theorems 3.4 to 3.6 can be found in Ref. [3.1].

3.6 SOLUTION OF A SYSTEM OF LINEAR SIMULTANEOUS EQUATIONS

Before studying the most general method of solving a linear programming problem, it will be useful to review the methods of solving a system of linear equations. Hence in the present section we review some of the elementary concepts of linear equations. Consider the following system of n equations in n unknowns.

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \qquad (E_1)\\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \qquad (E_2)\\
a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n &= b_3 \qquad (E_3)\\
&\;\vdots \qquad\quad \vdots\\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \qquad (E_n)
\end{aligned}
\tag{3.14}
$$

Assuming that this set of equations possesses a unique solution, a method of solving the system consists of reducing the equations to a form known as *canonical form*.

It is well known from elementary algebra that the solution of Eqs. (3.14) will not be altered under the following elementary operations: (1) any equation E_r is replaced by the equation kE_r, where k is a nonzero constant, and (2) any equation E_r is replaced by the equation $E_r + kE_s$, where E_s is any other equation of the system. By making use of these elementary operations, the system of Eqs. (3.14) can be reduced to a convenient equivalent form as follows. Let us select some variable x_i and try to eliminate it from all the equations except the jth one (for which a_{ji} is nonzero). This can be accomplished by dividing the jth equation by a_{ji} and subtracting a_{ki} times the result from each of the other equations, $k = 1,2,\ldots,j-1, j+1,\ldots,n$. The resulting system of equations can be written as

$$
\begin{aligned}
&a'_{11}x_1 + a'_{12}x_2 + \cdots + a'_{1,i-1}x_{i-1} + 0x_i + a'_{1,i+1}x_{i+1} + \cdots\\
&\qquad + a'_{1n}x_n = b'_1\\
&a'_{21}x_1 + a'_{22}x_2 + \cdots + a'_{2,i-1}x_{i-1} + 0x_i + a'_{2,i+1}x_{i+1} + \cdots\\
&\qquad + a'_{2n}x_n = b'_2\\
&\qquad \vdots\\
&a'_{j-1,1}x_1 + a'_{j-1,2}x_2 + \cdots + a'_{j-1,i-1} + 0x_i + a'_{j-1,i+1}x_{i+1}\\
&\qquad + \cdots + a'_{i-1,n}x_n = b'_{j-1}
\end{aligned}
$$

$$
\begin{aligned}
a'_{j1}x_1 &+ a'_{j2}x_2 + \cdots + a'_{j,i-1}x_{i-1} + 1x_i + a'_{j,i+1}x_{i+1} \\
&+ \cdots + a'_{jn}x_n = b'_j \\
a'_{j+1,1}x_1 &+ a'_{j+1,2}x_2 + \cdots + a'_{j+1,i-1}x_{i-1} + 0x_i + a'_{j+1,i+1}x_{i+1} \\
&+ \cdots + a'_{j+1,n}x_n = b'_{j+1} \\
&\vdots \\
a'_{n1}x_1 &+ a'_{n2}x_2 + \cdots + a'_{n,i-1}x_{i-1} + 0x_i + a'_{n,i+1}x_{i+1} + \cdots \\
&+ a'_{nn}x_n = b'_n
\end{aligned}
\tag{3.15}
$$

where the primes indicate that the a'_{ij} and b'_j are changed from the original system. This procedure of eliminating a particular variable from all but one equations is called a *pivot operation*. The system of Eqs. (3.15) produced by the pivot operation have exactly the same solution as the original set of Eqs. (3.14). That is, the vector **X** that satisfies Eqs. (3.14) satisfies Eqs. (3.15), and vice versa.

Next time, if we take the system of Eqs. (3.15) and perform a new pivot operation by eliminating x_s, $s \neq i$, in all the equations except the tth equation, $t \neq j$, the zeros or the 1 in the ith column will not be disturbed. The pivotal operations can be repeated by using a different variable and equation each time until the system of Eqs. (3.14) is reduced to the form

$$
\begin{aligned}
1x_1 + 0x_2 + 0x_3 + \cdots + 0x_n &= b''_1 \\
0x_1 + 1x_2 + 0x_3 + \cdots + 0x_n &= b''_2 \\
0x_1 + 0x_2 + 1x_3 + \cdots + 0x_n &= b''_3 \\
&\vdots \\
0x_1 + 0x_2 + 0x_3 + \cdots + 1x_n &= b''_n
\end{aligned}
\tag{3.16}
$$

This system of Eqs. (3.16) is said to be in canonical form and has been obtained after carrying out n pivot operations. From the canonical form, the solution vector can be directly obtained as

$$
x_i = b''_i, \qquad i = 1,2,\ldots,n \tag{3.17}
$$

Since the set of Eqs. (3.16) has been obtained from Eqs. (3.14) only through elementary operations, the system of Eqs. (3.16) is equivalent to the system of Eqs. (3.14). Thus the solution given by Eqs. (3.17) is the desired solution of Eqs. (3.14).

3.7 PIVOTAL REDUCTION OF A GENERAL SYSTEM OF EQUATIONS

Instead of a square system, let us consider a system of m equations in n variables with $n \geq m$. This system of equations is assumed to be consistent so that it will have at least one solution.

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{3.18}$$

The solution vector(s) $\mathbf{X}$ that satisfy Eqs. (3.18) are not evident from the equations. However, it is possible to reduce this system to an equivalent canonical system from which at least one solution can readily be deduced. If pivotal operations with respect to any set of m variables, say, $x_1, x_2, \ldots, x_m$, are carried, the resulting set of equations can be written as follows:

Canonical system with pivotal variables $x_1, x_2, \ldots, x_m$

$$\begin{aligned} 1x_1 + 0x_2 + \cdots + 0x_m + a''_{1,m+1}x_{m+1} + \cdots + a''_{1n}x_n &= b''_1 \\ 0x_1 + 1x_2 + \cdots + 0x_m + a''_{2,m+1}x_{m+1} + \cdots + a''_{2n}x_n &= b''_2 \\ &\vdots \\ 0x_1 + 0x_2 + \cdots + 1x_m + a''_{m,m+1}x_{m+1} + \cdots + a''_{mn}x_n &= b''_m \end{aligned} \tag{3.19}$$

Pivotal variables	Nonpivotal or independent variables	Constants

One special solution that can always be deduced from the system of Eqs. (3.19) is

$$\begin{cases} b''_i, & i = 1,2,\ldots,m \\ 0, & i = m+1, m+2, \ldots, n \end{cases} \tag{3.20}$$

This solution is called a *basic solution* since the solution vector contains no more than m nonzero terms. The pivotal variables x_i, $i = 1, 2, \ldots, m$, are called the *basic variables* and the other variables x_i, $i = m + 1, m + 2, \ldots, n$, are called the *nonbasic variables*. Of course, this is not the only solution, but it is the one most readily deduced from Eqs. (3.19). If all b''_i, $i = 1, 2, \ldots, m$, in the solution given by Eqs. (3.20) are nonnegative, it satisfies Eqs. (3.3) in addition to Eqs. (3.2), and hence it can be called a *basic feasible solution*.

It is possible to obtain the other basic solutions from the canonical system of Eqs. (3.19). We can perform an additional pivotal operation on the system after it is in canonical form, by choosing a''_{pq} (which is nonzero) as the pivot term, $q > m$, and using any row p (among $1,2,\ldots,m$). The new system will still be in canonical form but with x_q as the pivotal variable in place of x_p. The variable x_p, which was a basic variable in the original canonical form, will no longer be a basic variable in the new canonical form. This new canonical system yields a new basic solution (which may or may not be feasible) similar to that of Eqs. (3.20). It is to be noted that the values of all the basic variables change, in general, as we go from one basic solution to another, but only one zero variable (which is nonbasic in the original canonical form) becomes nonzero (which is basic in the new canonical system), and vice versa.

Example 3.3 Find all the basic solutions corresponding to the system of equations

$$2x_1 + 3x_2 - 2x_3 - 7x_4 = 1 \qquad (\mathrm{I}_0)$$

$$x_1 + x_2 + x_3 + 3x_4 = 6 \qquad (\mathrm{II}_0)$$

$$x_1 - x_2 + x_3 + 5x_4 = 4 \qquad (\mathrm{III}_0)$$

SOLUTION First we reduce the system of equations into a canonical form with x_1, x_2, and x_3 as basic variables. For this, first we pivot on the element $a_{11} = 2$ to obtain

$$\begin{aligned} x_1 + \tfrac{3}{2}x_2 - x_3 - \tfrac{7}{2}x_4 &= \tfrac{1}{2} \qquad & \mathrm{I}_1 &= \tfrac{1}{2}\mathrm{I}_0 \\ 0 - \tfrac{1}{2}x_2 + 2x_3 + \tfrac{13}{2}x_4 &= \tfrac{11}{2} & \mathrm{II}_1 &= \mathrm{II}_0 - \mathrm{I}_1 \\ 0 - \tfrac{5}{2}x_2 + 2x_3 + \tfrac{17}{2}x_4 &= \tfrac{7}{2} & \mathrm{III}_1 &= \mathrm{III}_0 - \mathrm{I}_1 \end{aligned}$$

Then we pivot on $a'_{22} = -\frac{1}{2}$, to obtain

$$\begin{aligned} x_1 + 0 + 5x_3 + 16x_4 &= 17 \qquad & \mathrm{I}_2 &= \mathrm{I}_1 - \tfrac{3}{2}\mathrm{II}_2 \\ 0 + x_2 - 4x_3 - 13x_4 &= -11 & \mathrm{II}_2 &= -2\,\mathrm{II}_1 \\ 0 + 0 - 8x_3 - 24x_4 &= -24 & \mathrm{III}_2 &= \mathrm{III}_1 + \tfrac{5}{2}\mathrm{II}_2 \end{aligned}$$

Finally we pivot on a'_{33} to obtain the required canonical form as

$$\begin{aligned} x_1 \qquad + x_4 &= 2 \qquad & \mathrm{I}_3 &= \mathrm{I}_2 - 5\,\mathrm{III}_3 \\ x_2 \qquad - x_4 &= 1 & \mathrm{II}_3 &= \mathrm{II}_2 + 4\,\mathrm{III}_3 \\ x_3 + 3x_4 &= 3 & \mathrm{III}_3 &= -\tfrac{1}{8}\mathrm{III}_2 \end{aligned}$$

From this canonical form, we can readily write the solution of x_1, x_2, and x_3 in terms of the other variable x_4 as

$$x_1 = 2 - x_4$$
$$x_2 = 1 + x_4$$
$$x_3 = 3 - 3x_4$$

If Eqs. (I_0), (II_0), and (III_0) are the constraints of a linear programming problem, the solution obtained by setting the independent variable equal to zero is called a basic solution. In the present case, the basic solution is given by

$$x_1 = 2, \quad x_2 = 1, \quad x_3 = 3 \quad \text{(basic variables)}$$

and $x_4 = 0$ (nonbasic or independent variable). Since this basic solution has all $x_j \geq 0$ ($j = 1,2,3,4$), it is a basic feasible solution.

If we want to move to a neighboring basic solution, we can proceed from the canonical form given by Eqs. (I_3), (II_3), and (III_3). Thus if a canonical form in terms of the variables x_1, x_2, and x_4 is required, we have to bring x_4 into the basis in place of the original basic variable x_3. Hence we pivot on a''_{34} in Eq. (III_3). This gives the desired canonical form as

$$\begin{array}{rcl} x_1 \qquad - \frac{1}{3}x_3 = 1 & & \mathrm{I}_4 = \mathrm{I}_3 - \mathrm{III}_4 \\ x_2 \quad + \frac{1}{3}x_3 = 2 & & \mathrm{II}_4 = \mathrm{II}_3 + \mathrm{III}_4 \\ x_4 + \frac{1}{3}x_3 = 1 & & \mathrm{III}_4 = \frac{1}{3}\,\mathrm{III}_3 \end{array}$$

This canonical system gives the solution of x_1, x_2, and x_4 in terms of x_3 as

$$x_1 = 1 + \tfrac{1}{3}x_3$$
$$x_2 = 2 - \tfrac{1}{3}x_3$$
$$x_4 = 1 - \tfrac{1}{3}x_3$$

and the corresponding basic solution is given by

$$x_1 = 1, \quad x_2 = 2, \quad x_4 = 1 \quad \text{(basic variables)}$$
$$x_3 = 0 \quad \text{(nonbasic variable)}$$

This basic solution can also be seen to be a basic feasible solution. If we want to move to the next basic solution with x_1, x_3, and x_4 as basic variables, we have to bring x_3 into the current basis in place of x_2. Thus we have to pivot

a''_{23} in Eq. (II_4). This leads to the following canonical system:

$$
\begin{aligned}
x_1 \quad + x_2 &= 3 \qquad & \mathrm{I}_5 &= \mathrm{I}_4 + \tfrac{1}{3}\mathrm{II}_5 \\
x_3 + 3x_2 &= 6 \qquad & \mathrm{II}_5 &= 3\mathrm{II}_4 \\
x_4 - x_2 &= -1 \qquad & \mathrm{III}_5 &= \mathrm{III}_4 - \tfrac{1}{3}\mathrm{II}_5
\end{aligned}
$$

The solution for x_1, x_3, and x_4 is given by

$$
\begin{aligned}
x_1 &= 3 - x_2 \\
x_3 &= 6 - 3x_2 \\
x_4 &= -1 + x_2
\end{aligned}
$$

from which the basic solution can be obtained as

$$
\begin{aligned}
&x_1 = 3, \quad x_3 = 6, \quad x_4 = -1 \quad \text{(basic variables)} \\
&x_2 = 0 \quad \text{(nonbasic variable)}
\end{aligned}
$$

Since all the x_j are not nonnegative, this basic solution is not feasible.

Finally, to obtain the canonical form in terms of the basic variables x_2, x_3, and x_4, we pivot on a''_{12} in Eq. (I_5), thereby bringing x_2 into the current basis in place of x_1. This gives

$$
\begin{aligned}
x_2 \quad + x_1 &= 3 \qquad & \mathrm{I}_6 &= \mathrm{I}_5 \\
x_3 - 3x_1 &= -3 \qquad & \mathrm{II}_6 &= \mathrm{II}_5 - 3\mathrm{I}_6 \\
x_4 + x_1 &= 2 \qquad & \mathrm{III}_6 &= \mathrm{III}_5 + \mathrm{I}_6
\end{aligned}
$$

This canonical form gives the solution for x_2, x_3, and x_4 in terms of x_1 as

$$
\begin{aligned}
x_2 &= 3 - x_1 \\
x_3 &= -3 + 3x_1 \\
x_4 &= 2 - x_1
\end{aligned}
$$

and the corresponding basic solution is

$$
\begin{aligned}
&x_2 = 3, \quad x_3 = -3, \quad x_4 = 2 \quad \text{(basic variables)} \\
&x_1 = 0 \quad \text{(nonbasic variable)}
\end{aligned}
$$

This basic solution can also be seen to be infeasible due to the negative value for x_3.

3.8 MOTIVATION OF THE SIMPLEX METHOD

Given a system in canonical form corresponding to a basic solution, we have seen how to move to a neighboring basic solution by a pivot operation. Thus one way to find the optimal solution of the given linear programming problem is to generate all the basic solutions and pick the one that is feasible and corresponds to the optimal value of the objective function. This can be done because the optimal solution, if one exists, always occurs at an extreme point or vertex of the feasible domain. If there are m equality constraints in n variables with $n \geq m$, a basic solution can be obtained by setting any of the $n - m$ variables equal to zero. The number of basic solutions to be inspected is thus equal to the number of ways in which m variables can be selected from a set of n variables, that is,

$$\binom{n}{m} = \frac{n!}{(n - m)!\, m!}$$

For example, if $n = 10$ and $m = 5$, we have 252 basic solutions, and if $n = 20$ and $m = 10$, we have 184,756 basic solutions. Usually, we do not have to inspect all these basic solutions since many of them will be infeasible. However, for large values of n and m, this is still a very large number to inspect one by one. Hence what we really need is a computational scheme that examines a sequence of basic feasible solutions, each of which corresponds to a lower value of f until a minimum is reached. The simplex method of Dantzig is a powerful scheme for obtaining a basic feasible solution; if the solution is not optimal, the method provides for finding a neighboring basic feasible solution that has a lower or equal value of f. The process is repeated until, in a finite number of steps, an optimum is found.

The first step involved in the simplex method is to construct an auxiliary problem by introducing certain variables known as *artificial variables* into the standard form of the linear programming problem. The primary aim of adding the artificial variables is to bring the resulting auxiliary problem into a canonical form from which its basic feasible solution can be obtained immediately. Starting from this canonical form, the optimal solution of the original linear programming problem is sought in two phases. The first phase is intended to find a basic feasible solution to the original linear programming problem. It consists of a sequence of pivot operations that produces a succession of different canonical forms from which the optimal solution of the auxiliary problem can be found. This also enables us to find a basic feasible solution, if one exists, of the original linear programming problem. The second phase is intended to find the optimal solution of the original linear programming problem. It consists of a second sequence of pivot operations that enables us to move from one basic feasible solution to the next of the original linear programming problem. In this process, the optimal solution of the problem, if one exists, will be identified. The sequence of different canonical forms that is necessary in both the phases of the simplex method is generated according to the simplex algorithm described in the next section. That is, the simplex algorithm forms the main subroutine of the simplex method.

3.9 SIMPLEX ALGORITHM

The starting point of the simplex algorithm is always a set of equations, which includes the objective function along with the equality constraints of the problem in canonical form. Thus the objective of the simplex algorithm is to find the vector $\mathbf{X} \geq 0$ that minimizes the function $f(\mathbf{X})$ and satisfies the equations:

$$
\begin{aligned}
&1x_1 + 0x_2 + \cdots + 0x_m + a''_{1,m+1}x_{m+1} + \cdots + a''_{1n}x_n = b''_1 \\
&0x_1 + 1x_2 + \cdots + 0x_m + a''_{2,m+1}x_{m+1} + \cdots + a''_{2n}x_n = b''_2 \\
&\vdots \\
&0x_1 + 0x_2 + \cdots + 1x_m + a''_{m,m+1}x_{m+1} + \cdots + a''_{mn}x_n = b''_m \\
&0x_1 + 0x_2 + \cdots + 0x_m - f \\
&\qquad\qquad\qquad + c''_{m+1}x_{m+1} + \cdots + c''_{mn}x_n = -f''_0
\end{aligned}
\tag{3.21}
$$

where a''_{ij}, c''_j, b''_i, and f''_0 are constants. Notice that $(-f)$ is treated as a basic variable in the canonical form of Eqs. (3.21). The basic solution which can readily be deduced from Eqs. (3.21) is

$$
\begin{aligned}
x_i &= b''_i, \qquad i = 1,2,\ldots,m \\
f &= f''_0 \\
x_i &= 0, \qquad i = m+1, m+2, \ldots, n
\end{aligned}
\tag{3.22}
$$

If the basic solution is also feasible, the values of x_i, $i = 1,2,\ldots,n$, are nonnegative and hence

$$
b''_i \geq 0, \qquad i = 1,2,\ldots,m \tag{3.23}
$$

In phase I of the simplex method, the basic solution corresponding to the canonical form obtained after the introduction of the artificial variables will be feasible for the auxiliary problem. As stated earlier, phase II of the simplex method starts with a basic feasible solution of the original linear programming problem. Hence the initial canonical form at the start of the simplex algorithm will always be a basic feasible solution.

We know from Theorem 3.6 that the optimal solution of a linear programming problem lies at one of the basic feasible solutions. Since the simplex algorithm is intended to move from one basic feasible solution to the other through pivotal operations, before moving to the next basic feasible solution, we have to make sure that the present basic feasible solution is not the optimal solution. By merely glancing at the numbers c_j'', $j = 1, 2, \ldots, n$, we can tell whether or not the present basic feasible solution is optimal. Theorem 3.7 provides a means of identifying the optimal point.

3.9.1 Identifying an Optimal Point

Theorem 3.7 A basic feasible solution is an optimal solution with a minimum objective function value of f_0'' if all the cost coefficients c_j'', $j = m + 1, m + 2, \ldots, n$, in Eqs. (3.21) are nonnegative.

Proof: From the last row of Eqs. (3.21), we can write that

$$f_0'' + \sum_{i=m+1}^{n} c_i'' x_i = f \tag{3.24}$$

Since the variables $x_{m+1}, x_{m+2}, \ldots, x_n$ are presently zero and are constrained to be nonnegative, the only way any one of them can change is to become positive. But if $c_i'' > 0$ for $i = m + 1, m + 2, \ldots, n$, then increasing any x_i cannot decrease the value of the objective function f. Since no change in the nonbasic variables can cause f to decrease, the present solution must be optimal with the optimal value of f equal to f_0''.

A glance over c_i'' can also tell us if there are multiple optima. Let all $c_i'' > 0$, $i = m + 1, m + 2, \ldots, k - 1, k + 1, \ldots, n$, and let $c_k'' = 0$ for some nonbasic variable x_k. Then if the constraints allow that variable to be made positive (from its present value of zero), no change in f results, and there are multiple optima. It is possible, however, that the variable may not be allowed by the constraints to become positive; this may occur in the case of degenerate solutions. Thus, as a corollary to the discussion above, we can state that a basic feasible solution is the unique optimal feasible solution if $c_j'' > 0$ for all nonbasic variables x_j, $j = m + 1, m + 2, \ldots, n$. If, after testing for optimality, the current basic feasible solution is found to be nonoptimal, an improved basic solution is obtained from the present canonical form as follows.

3.9.2 Improving a Nonoptimal Basic Feasible Solution

From the last row of Eqs. (3.21), we can write the objective function as

$$\begin{aligned} f &= f_0'' + \sum_{i=1}^{m} c_i'' x_i + \sum_{j=m+1}^{n} c_j'' x_j \\ &= f_0'' \qquad \text{for the solution given by Eqs. (3.22)} \end{aligned} \tag{3.25}$$

If at least one c_j'' is negative, the value of f can be reduced by making the corresponding $x_j > 0$. In other words, the nonbasic variable x_j, for which the cost coefficient c_j'' is negative, is to be made a basic variable in order to reduce the value of the objective function. At the same time, due to the pivotal operation, one of the current basic variables will become nonbasic and hence the values of the new basic variables are to be adjusted in order to bring the value of f less than f_0''. If there are more than one $c_j'' < 0$, the index s of the nonbasic variable x_s which is to be made basic is chosen such that

$$c_s'' = \text{minimum } c_j'' < 0 \tag{3.26}$$

Although this may not lead to the greatest possible decrease in f (since it may not be possible to increase x_s very far), this is intuitively at least a good rule for choosing the variable to become basic. It is the one generally used in practice because it is simple and it usually leads to fewer iterations than just choosing any $c_j'' < 0$. If there is a tie-in applying Eq. (3.26), (i.e., if more than one c_j'' has the same minimum value), we select one of them arbitrarily as c_s''.

Having decided on the variable x_s to become basic, we increase it from zero holding all other nonbasic variables zero and observe the effect on the current basic variables. From Eqs. (3.21), we can obtain

$$\begin{aligned}
x_1 &= b_1'' - a_{1s}''x_s, & b_1'' &\geq 0 \\
x_2 &= b_2'' - a_{2s}''x_s, & b_2'' &\geq 0 \\
&\vdots \\
x_m &= b_m'' - a_{ms}''x_s, & b_m'' &\geq 0
\end{aligned} \tag{3.27}$$

$$f = f_0'' + c_s''x_s, \qquad c_s'' < 0 \tag{3.28}$$

Since $c_s'' < 0$, Eq. (3.28) suggests that the value of x_s should be made as large as possible in order to reduce the value of f as much as possible. However, in the process of increasing the value of x_s, some of the variables x_i ($i = 1,2,\ldots,m$) in Eqs. (3.27) may become negative. It can be seen that if all the coefficients $a_{is}'' \leq 0$, $i = 1,2,\ldots,m$, then x_s can be made infinitely large without making any $x_i < 0$, $i = 1,2,\ldots,m$. In such a case, the minimum value of f is minus infinity and the linear programming problem is said to have an *unbounded solution*.

On the other hand, if at least one a_{is}'' is positive, the maximum value that x_s can take without making x_i negative is b_i''/a_{is}''. If there are more than one $a_{is}'' > 0$, the largest value x_s^* that x_s can take is given by the minimum of the ratios b_i''/a_{is}'' for which $a_{is}'' > 0$. Thus

$$x_s^* = \frac{b_r''}{a_{rs}''} = \underset{a_{is}''>0}{\text{minimum}} \left(\frac{b_i''}{a_{is}''}\right) \tag{3.29}$$

The choice of r in the case of a tie, assuming that all $b_i'' > 0$, is arbitrary. If any b_i'' for which $a_{is}'' > 0$ is zero in Eqs. (3.27), x_s cannot be increased by any amount. Such a solution is called a *degenerate solution*.

In the case of a nondegenerate basic feasible solution, a new basic feasible solution can be constructed with a lower value of the objective function as follows. By substituting the value of x_s^* given by Eq. (3.29) into Eqs. (3.27) and (3.28), we obtain

$$
\begin{aligned}
x_s &= x_s^* \\
x_i &= b_i'' - a_{is}''x_s^* \geq 0, \qquad i = 1,2,\ldots,m \quad \text{and} \quad i \neq r \qquad (3.30) \\
x_r &= 0 \\
x_j &= 0, \qquad j = m+1, m+2, \ldots, n \quad \text{and} \quad j \neq s
\end{aligned}
$$

$$f = f_0'' + c_s''x_s^* \leq f_0'' \tag{3.31}$$

which can readily be seen to be a feasible solution different from the previous one. Since $a_{rs}'' > 0$ in Eq. (3.29), a single pivot operation on the element a_{rs}'' in the system of Eqs. (3.21) will lead to a new canonical form from which the basic feasible solution of Eqs. (3.30) can easily be deduced. Also, Eq. (3.31) shows that this basic feasible solution corresponds to a lower objective function value compared to that of Eqs. (3.22). This basic feasible solution can again be tested for optimality by seeing whether all $c_i'' > 0$ in the new canonical form. If the solution is not optimal, the entire procedure of moving to another basic feasible solution from the present one has to be repeated. In the simplex algorithm, this procedure is repeated in an iterative manner until the algorithm finds either (1) a class of feasible solutions for which $f \to -\infty$ or (2) an optimal basic feasible solution with all $c_i'' \geq 0$, $i = 1,2,\ldots,n$. Since there are only a finite number of ways to choose a set of m basic variables out of n variables, the iterative process of the simplex algorithm will terminate in a finite number of cycles. The iterative process of the simplex algorithm is shown as a flow-chart in Fig. 3.14.

Example 3.4

$$\text{Maximize } F = x_1 + 2x_2 + x_3$$

subject to

$$
\begin{aligned}
2x_1 + x_2 - x_3 &\leq 2 \\
-2x_1 + x_2 - 5x_3 &\geq -6 \\
4x_1 + x_2 + x_3 &\leq 6 \\
x_i \geq 0, &\qquad i = 1,2,3
\end{aligned}
$$

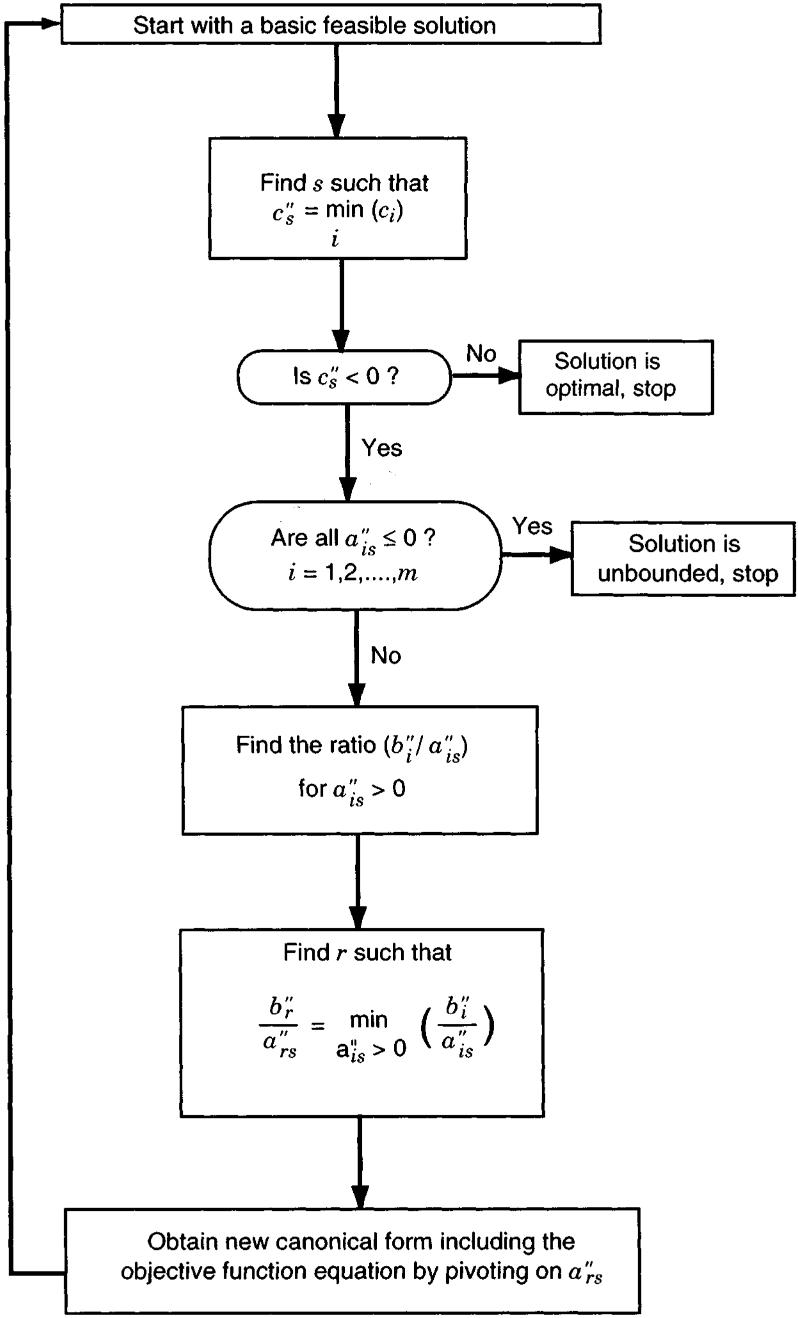

Figure 3.14 Flowchart for finding the optimal solution by the simplex algorithm.

SOLUTION We first change the sign of the objective function to convert it to a minimization problem and the signs of the inequalities (where necessary) so as to obtain nonnegative values of b_i (to see whether an initial basic feasible solution can be obtained readily). The resulting problem can be stated as:

$$\text{Minimize } f = -x_1 - 2x_2 - x_3$$

subject to

$$\begin{aligned} 2x_1 + x_2 - x_3 &\le 2 \\ 2x_1 - x_2 + 5x_3 &\le 6 \\ 4x_1 + x_2 + x_3 &\le 6 \\ x_i \ge 0, \quad i &= 1 \text{ to } 3 \end{aligned}$$

By introducing the slack variables $x_4 \ge 0$, $x_5 \ge 0$, and $x_6 \ge 0$, the system of equations can be stated in canonical form as

$$\begin{aligned} 2x_1 + x_2 - x_3 + x_4 \qquad &= 2 \\ 2x_1 - x_2 + 5x_3 + x_5 \qquad &= 6 \\ 4x_1 + x_2 + x_3 + x_6 \qquad &= 6 \\ -x_1 - 2x_2 - x_3 \qquad - f &= 0 \end{aligned} \tag{E$_1$}$$

where x_4, x_5, x_6, and $-f$ can be treated as basic variables. The basic solution corresponding to Eqs. (E$_1$) is given by

$$\begin{aligned} &x_4 = 2, \quad x_5 = 6, \quad x_6 = 6 \quad \text{(basic variables)} \\ &x_1 = x_2 = x_3 = 0 \quad \text{(nonbasic variables)} \\ &f = 0 \end{aligned} \tag{E$_2$}$$

which can be seen to be feasible.

Since the cost coefficients corresponding to nonbasic variables in Eqs. (E$_1$) are negative ($c_1'' = -1$, $c_2'' = -2$, $c_3'' = -1$), the present solution given by Eqs. (E$_2$) is not optimum. To improve the present basic feasible solution, we first decide the variable (x_s) to be brought into the basis as

$$c_s'' = \min(c_j'' < 0) = c_2'' = -2$$

Thus x_2 enters the next basic set. To obtain the new canonical form, we select the pivot element a_{rs}'' such that

$$\frac{b_r''}{a_{rs}''} = \min_{a_{is}'' > 0} \left(\frac{b_i''}{a_{is}''}\right)$$

In the present case, $s = 2$ and a''_{12} and a''_{32} are ≥ 0. Since $b''_1/a''_{12} = 2/1$ and $b''_3/a''_{32} = 6/1$, $x_r = x_1$. By pivoting an a''_{12}, the new system of equations can be obtained as

$$\begin{aligned}
2x_1 + 1x_2 - x_3 + x_4 \qquad\qquad &= 2 \\
4x_1 + 0x_2 + 4x_3 + x_4 + x_5 \qquad &= 8 \\
2x_1 + 0x_2 + 2x_3 - x_4 + x_6 \qquad &= 4 \\
3x_1 + 0x_2 - 3x_3 + 2x_4 \qquad - f &= 4
\end{aligned} \tag{E$_3$}$$

The basic feasible solution corresponding to this canonical form is

$$\begin{aligned}
&x_2 = 2, \quad x_5 = 8, \quad x_6 = 4 \quad \text{(basic variables)} \\
&x_1 = x_3 = x_4 = 0 \quad \text{(nonbasic variables)} \\
&f = -4
\end{aligned} \tag{E$_4$}$$

Since $c''_3 = -3$, the present solution is not optimum. As $c''_s = \min(c''_i < 0) = c''_3$, $x_s = x_3$ enters the next basis.

To find the pivot element a''_{rs}, we find the ratios b''_i/a''_{is} for $a''_{is} > 0$. In Eqs. (E$_3$), only a''_{23} and a''_{33} are > 0, and hence

$$\frac{b''_2}{a''_{23}} = \frac{8}{4} \quad \text{and} \quad \frac{b''_3}{a''_{33}} = \frac{4}{2}$$

Since both these ratios are same, we arbitrarily select a''_{23} as the pivot element. Pivoting on a''_{23} gives the following canonical system of equations:

$$\begin{aligned}
3x_1 + 1x_2 + 0x_3 + \tfrac{5}{4}x_4 + \tfrac{1}{4}x_5 \qquad\quad &= 4 \\
1x_1 + 0x_2 + 1x_3 + \tfrac{1}{4}x_4 + \tfrac{1}{4}x_5 \qquad\quad &= 2 \\
0x_1 + 0x_2 + 0x_3 - \tfrac{3}{2}x_4 - \tfrac{1}{2}x_5 + x_6 &= 0 \\
6x_1 + 0x_2 + 0x_3 + \tfrac{11}{4}x_4 + \tfrac{3}{4}x_5 - f &= 10
\end{aligned} \tag{E$_5$}$$

The basic feasible solution corresponding to this canonical system is given by

$$\begin{aligned}
&x_2 = 4, \quad x_3 = 2, \quad x_6 = 0 \quad \text{(basic variables)} \\
&x_1 = x_4 = x_5 = 0 \quad \text{(nonbasic variables)} \\
&f = -10
\end{aligned} \tag{E$_6$}$$

Since all c''_i are ≥ 0 in the present canonical form, the solution given in (E$_6$) will be optimum. Usually, starting with Eqs. (E$_1$), all the computations are done in a tableau form as shown below:

Basic Variables	Variables x_1	x_2	x_3	x_4	x_5	x_6	$-f$	b_i''	b_i''/a_{is}'' for $a_{is}'' > 0$
x_4	2	$\boxed{1}$ Pivot element	-1	1	0	0	0	2	2 ← Smaller one (x_4 drops from next basis)
x_5	2	-1	5	0	1	0	0	6	
x_6	4	1	1	0	0	1	0	6	6
$-f$	-1	-2 ↑	-1	0	0	0	1	0	

Most negative c_i'' (x_2 enters next basis)

Result of pivoting:

x_2	2	1	-1	1	0	0	0	2	
x_5	4	0	$\boxed{4}$ Pivot element	1	1	0	0	8	2 (Select this arbitrarily. x_5 drops from next basis)
x_6	2	0	2	-1	0	1	0	4	2
$-f$	3	0	-3	2	0	0	1	4	

↑
Most negative c_i'' (x_3 enters the next basis)

Result of pivoting:

x_2	3	1	0	$\frac{5}{4}$	$\frac{1}{4}$	0	0	4
x_3	1	0	1	$\frac{1}{4}$	$\frac{1}{4}$	0	0	2
x_6	0	0	0	$-\frac{3}{2}$	$-\frac{1}{2}$	1	0	0
$-f$	6	0	0	$\frac{11}{4}$	$\frac{3}{4}$	0	1	10

All c_i'' are ≥ 0 and hence the present solution is optimum.

Example 3.5: Unbounded Solution

$$\text{Minimize } f = -3x_1 - 2x_2$$

subject to

$$x_1 - x_2 \leq 1$$

$$3x_1 - 2x_2 \le 6$$
$$x_1 \ge 0, \quad x_2 \ge 0$$

SOLUTION Introducing the slack variables $x_3 \ge 0$ and $x_4 \ge 0$, the given system of equations can be written in canonical form as

$$\begin{aligned} x_1 - x_2 + x_3 \qquad\qquad &= 1 \\ 3x_1 - 2x_2 \qquad + x_4 \qquad &= 6 \\ -3x_1 - 2x_2 \qquad\qquad - f &= 0 \end{aligned} \tag{E$_1$}$$

The basic feasible solution corresponding to this canonical form is given by

$$\begin{aligned} &x_3 = 1, \quad x_4 = 6 \quad \text{(basic variables)} \\ &x_1 = x_2 = 0 \quad \text{(nonbasic variables)} \\ &f = 0 \end{aligned} \tag{E$_2$}$$

Since the cost coefficients corresponding to the nonbasic variables are negative, the solution given by Eq. (E_2) is not optimum. Hence the simplex procedure is applied to the canonical system of Eqs. (E_1) starting from the solution, Eqs. (E_2). The computations are done in tableau form as shown below:

Basic Variables	Variables						b_i''/a_{is}'' for
	x_1	x_2	x_3	x_4	$-f$	b_i''	$a_{is}'' > 0$
x_3	$\boxed{1}$ Pivot element	-1	1	0	0	1	1 ← Smaller value (x_3 leaves the basis)
x_4	3	-2	0	1	0	6	2
$-f$	-3	-2	0	0	1	0	
	↑ Most negative c_i'' (x_1 enters the next basis)						

Result of pivoting:

x_1	1	-1	1	0	0	1	
x_4	0	$\boxed{1}$ Pivot element	-3	1	0	3	3 (x_4 leaves the basis)
$-f$	0	-5	3	0	1	3	
		↑ Most negative c_i'' (x_2 enters the next basis)					

Result of pivoting:

x_1	1	0	-2	1	0	4	Both a''_{is} are negative (i.e., no variable leaves the basis)
x_2	0	1	-3	1	0	3	
$-f$	0	0	-12	5	1	18	
			↑ Most negative c''_i (x_3 enters the basis)				

At this stage we notice that x_3 has the most negative cost coefficient and hence it should be brought into the next basis. However, since all the coefficients a''_{i3} are negative, the value of f can be decreased indefinitely without violating any of the constraints if we bring x_3 into the basis. Hence the problem has no bounded solution.

In general, if all the coefficients of the entering variable x_s (a''_{is}) have negative or zero values at any iteration, we can conclude that the problem has an unbounded solution.

Example 3.6: Infinite Number of Solutions To demonstrate how a problem having infinite number of solutions can be solved, Example 3.2 is again considered with a modified objective function:

$$\text{Minimize } f = -40x_1 - 100x_2$$

subject to

$$10x_1 + 5x_2 \leq 2500$$

$$4x_1 + 10x_2 \leq 2000$$

$$2x_1 + 3x_2 \leq 900$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

SOLUTION By adding the slack variables $x_3 \geq 0$, $x_4 \geq 0$ and $x_5 \geq 0$, the equations can be written in canonical form as follows:

$$\begin{aligned}
10x_1 + 5x_2 + x_3 \qquad\qquad &= 2500 \\
4x_1 + 10x_2 \qquad + x_4 \qquad &= 2000 \\
2x_1 + 3x_2 \qquad\qquad + x_5 &= 900 \\
-40x_1 - 100x_2 \qquad\qquad - f &= 0
\end{aligned}$$

The computations can be done in tableau form as shown below:

Basic Variables	Variables x_1	x_2	x_3	x_4	x_5	$-f$	b_i''	b_i''/a_{is}'' for $a_{is}'' > 0$
x_3	10	5	1	0	0	0	2,500	500
x_4	4	$\boxed{10}$ Pivot element	0	1	0	0	2,000	200 ← Smaller value (x_4 leaves the basis)
x_5	2	3	0	0	1	0	900	300
$-f$	−40	−100	0	0	0	1	0	
		↑ Most negative c_i'' (x_2 enters the basis)						

Result of pivoting:

x_3	8	0	1	$-\frac{1}{2}$	0	0	1,500
x_2	$\frac{4}{10}$	1	0	$\frac{1}{10}$	0	0	200
x_5	$\frac{8}{10}$	0	0	$-\frac{3}{10}$	1	0	300
$-f$	0	0	0	10	0	1	20,000

Since all $c_i'' \geq 0$, the present solution is optimum. The optimum values are given by

$$x_2 = 200, \quad x_3 = 1500, \quad x_5 = 300 \quad \text{(basic variables)}$$

$$x_1 = x_4 = 0 \quad \text{(nonbasic variables)}$$

$$f_{\min} = -20{,}000$$

Important Note: It can be observed from the last row of the preceding tableau that the cost coefficient corresponding to the nonbasic variable x_1 (c_1'') is zero. This is an indication that an alternative solution exists. Here x_1 can be brought into the basis and the resulting new solution will also be an optimal basic feasible solution. For example, introducing x_1 into the basis in place of x_3 (i.e., by pivoting on a_{13}''), we obtain the new canonical system of equations as shown in the following tableau:

Basic Variables	Variables x_1	x_2	x_3	x_4	x_5	$-f$	b_i''	b_i''/a_{is}'' for $a_{is}'' > 0$
x_1	1	0	$\frac{1}{8}$	$-\frac{1}{16}$	0	0	$\frac{1500}{8}$	
x_2	0	1	$-\frac{1}{20}$	$\frac{1}{8}$	0	0	125	
x_5	0	0	$-\frac{1}{10}$	$-\frac{1}{4}$	1	0	150	
$-f$	0	0	0	10	0	1	20,000	

The solution corresponding to this canonical form is given by

$$x_1 = \tfrac{1500}{8}, \quad x_2 = 125, \quad x_5 = 150 \text{ (basic variables)}$$

$$x_3 = x_4 = 0 \quad \text{(nonbasic variables)}$$

$$f_{\min} = -20{,}000$$

Thus the value of f has not changed compared to the preceding value since x_1 has a zero cost coefficient in the last row of the preceding tableau. Once two basic (optimal) feasible solutions, namely,

$$\mathbf{X}_1 = \begin{Bmatrix} 0 \\ 200 \\ 1500 \\ 0 \\ 300 \end{Bmatrix} \quad \text{and} \quad \mathbf{X}_2 = \begin{Bmatrix} \frac{1500}{8} \\ 125 \\ 0 \\ 0 \\ 150 \end{Bmatrix}$$

are known, an infinite number of nonbasic (optimal) feasible solutions can be obtained by taking any weighted average of the two solutions as

$$\mathbf{X}^* = \lambda \mathbf{X}_1 + (1 - \lambda)\mathbf{X}_2$$

$$\mathbf{X}^* = \begin{Bmatrix} x_1^* \\ x_2^* \\ x_3^* \\ x_4^* \\ x_5^* \end{Bmatrix} = \begin{Bmatrix} (1-\lambda)\frac{1500}{8} \\ 200\lambda + (1-\lambda)125 \\ 1500\lambda \\ 0 \\ 300\lambda + (1-\lambda)150 \end{Bmatrix} = \begin{Bmatrix} (1-\lambda)\frac{1500}{8} \\ 125 + 75\lambda \\ 1500\lambda \\ 0 \\ 150 + 150\lambda \end{Bmatrix}$$

$$0 \le \lambda \le 1$$

It can be verified that the solution $\mathbf{X}^*$ will always give the same value of $-20{,}000$ for f for all $0 \le \lambda \le 1$.

3.10 TWO PHASES OF THE SIMPLEX METHOD

The problem is to find nonnegative values for the variables $x_1, x_2, \ldots, x_n$ that satisfy the equations

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m
\end{aligned}
\tag{3.32}
$$

and minimize the objective function given by

$$
c_1x_1 + c_2x_2 + \cdots + c_nx_n = f \tag{3.33}
$$

The general problems encountered in solving this problem are:

1. An initial feasible canonical form may not be readily available. This is the case when the linear programming problem does not have slack variables for some of the equations or when the slack variables have negative coefficients.
2. The problem may have redundancies and/or inconsistencies, and may not be solvable in nonnegative numbers.

The two-phase simplex method can be used to solve the problem.

Phase I of the simplex method uses the simplex algorithm itself to find whether the linear programming problem has a feasible solution. If a feasible solution exists, it provides a basic feasible solution in canonical form ready to initiate phase II of the method. Phase II, in turn, uses the simplex algorithm to find whether the problem has a bounded optimum. If a bounded optimum exists, it finds the basic feasible solution which is optimal. The simplex method is described in the following steps.

1. Arrange the original system of Eqs. (3.32) so that all constant terms b_i are positive or zero by changing, where necessary, the signs on both sides of any of the equations.
2. Introduce to this system a set of artificial variables $y_1, y_2, \ldots, y_m$ (which serve as basic variables in phase I), where each $y_i \geq 0$, so that it becomes

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 \qquad\qquad\qquad &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \qquad + y_2 \qquad\quad &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \qquad\qquad\quad + y_m &= b_m \\
b_i \geq 0 &
\end{aligned}
\tag{3.34}
$$

Note that in Eqs. (3.34), for a particular i, the a_{ij}'s and the b_i may be the negative of what they were in Eq. (3.32) because of step 1.

The objective function of Eq. (3.33) can be written as

$$c_1x_1 + c_2x_2 + \cdots + c_nx_n + (-f) = 0 \tag{3.35}$$

3. *Phase I of the Method.* Define a quantity w as the sum of the artificial variables

$$w = y_1 + y_2 + \cdots + y_m \tag{3.36}$$

and use the simplex algorithm to find $x_i \geq 0$ $(i = 1,2,\ldots,n)$ and $y_i \geq 0$ $(i = 1,2,\ldots,m)$ which minimize w and satisfy Eqs. (3.34) and (3.35). Consequently, consider the array

$$\begin{array}{llllll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 & & & & & = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & + y_2 & & & & = b_2 \\
\vdots & & & & & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & & + y_m & & & = b_m \\
c_1x_1 + c_2x_2 + \cdots + c_nx_n & & & + (-f) & & = 0 \\
 & y_1 + y_2 + \cdots & + y_m & & + (-w) & = 0
\end{array} \tag{3.37}$$

This array is not in canonical form; however, it can be rewritten as a canonical system with basic variables $y_1, y_2, \ldots, y_m, -f$, and $-w$ by subtracting the sum of the first m equations from the last to obtain the new system

$$\begin{array}{lllll}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + y_1 & & & & = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n & + y_2 & & & = b_2 \\
\vdots & & & & \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n & & + y_m & & = b_m \\
c_1x_1 + c_2x_2 + \cdots + c_nx_n & & & + (-f) & = 0 \\
d_1x_1 + d_2x_2 + \cdots + d_nx_n & & & + (-w) & = -w_0
\end{array} \tag{3.38}$$

where

$$d_i = -(a_{1i} + a_{2i} + \cdots + a_{mi}), \qquad i = 1,2,\ldots,n \tag{3.39}$$

$$-w_0 = -(b_1 + b_2 + \cdots + b_m) \tag{3.40}$$

Equations (3.38) provide the initial basic feasible solution that is necessary for starting phase I.

4. w is called the *infeasibility form* and has the property that if as a result of phase I, with a minimum of $w > 0$, no feasible solution exists for the original linear programming problem stated in Eqs. (3.32) and (3.33), and thus the procedure is terminated. On the other hand, if the minimum of $w = 0$, the resulting array will be in canonical form and hence initiate phase II by eliminating the w equation as well as the columns corresponding to each of the artificial variables $y_1, y_2, \ldots, y_m$ from the array.
5. *Phase II of the Method.* Apply the simplex algorithm to the adjusted canonical system at the end of phase I to obtain a solution, if a finite one exists, which optimizes the value of f.

The flowchart for the two-phase simplex method is given in Fig. 3.15.

Example 3.7

$$\text{Minimize } f = 2x_1 + 3x_2 + 2x_3 - x_4 + x_5$$

subject to the constraints

$$3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 = 0$$

$$x_1 + x_2 + x_3 + 3x_4 + x_5 = 2$$

$$x_i \geq 0, \quad i = 1 \text{ to } 5$$

SOLUTION

Step 1: As the constants on the right-hand side of the constraints are already nonnegative, the application of step 1 is unnecessary.

Step 2: Introducing the artificial variables $y_1 \geq 0$ and $y_2 \geq 0$, the equations can be written as follows:

$$\begin{aligned}
3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 + y_1 \qquad\quad &= 0 \\
x_1 + x_2 + x_3 + 3x_4 + x_5 \qquad + y_2 \quad &= 2 \qquad (\text{E}_1) \\
2x_1 + 3x_2 + 2x_3 - x_4 + x_5 \qquad\qquad - f &= 0
\end{aligned}$$

Step 3: By defining the infeasibility form w as

$$w = y_1 + y_2$$

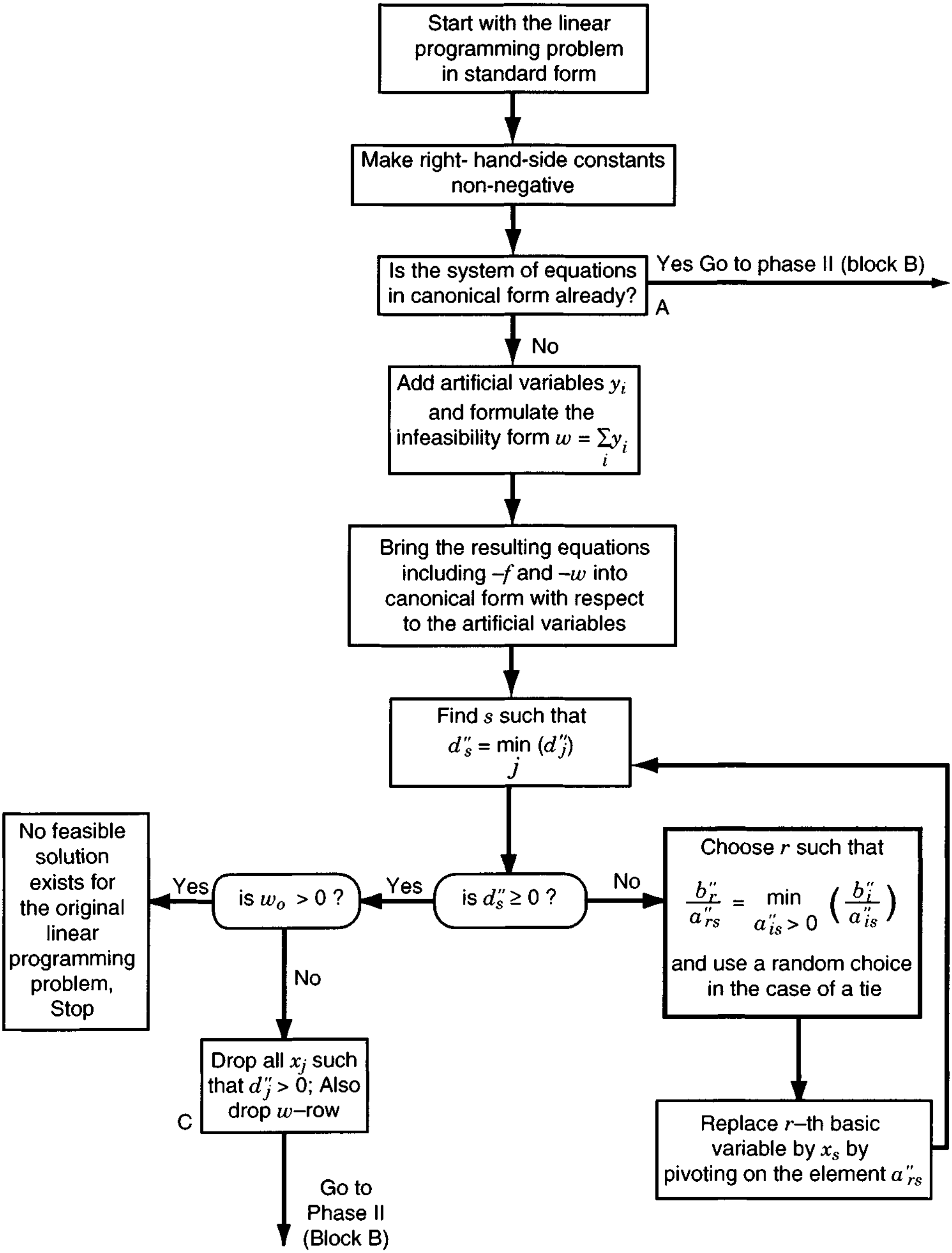

Figure 3.15 Flowchart for the two-phase simplex method.

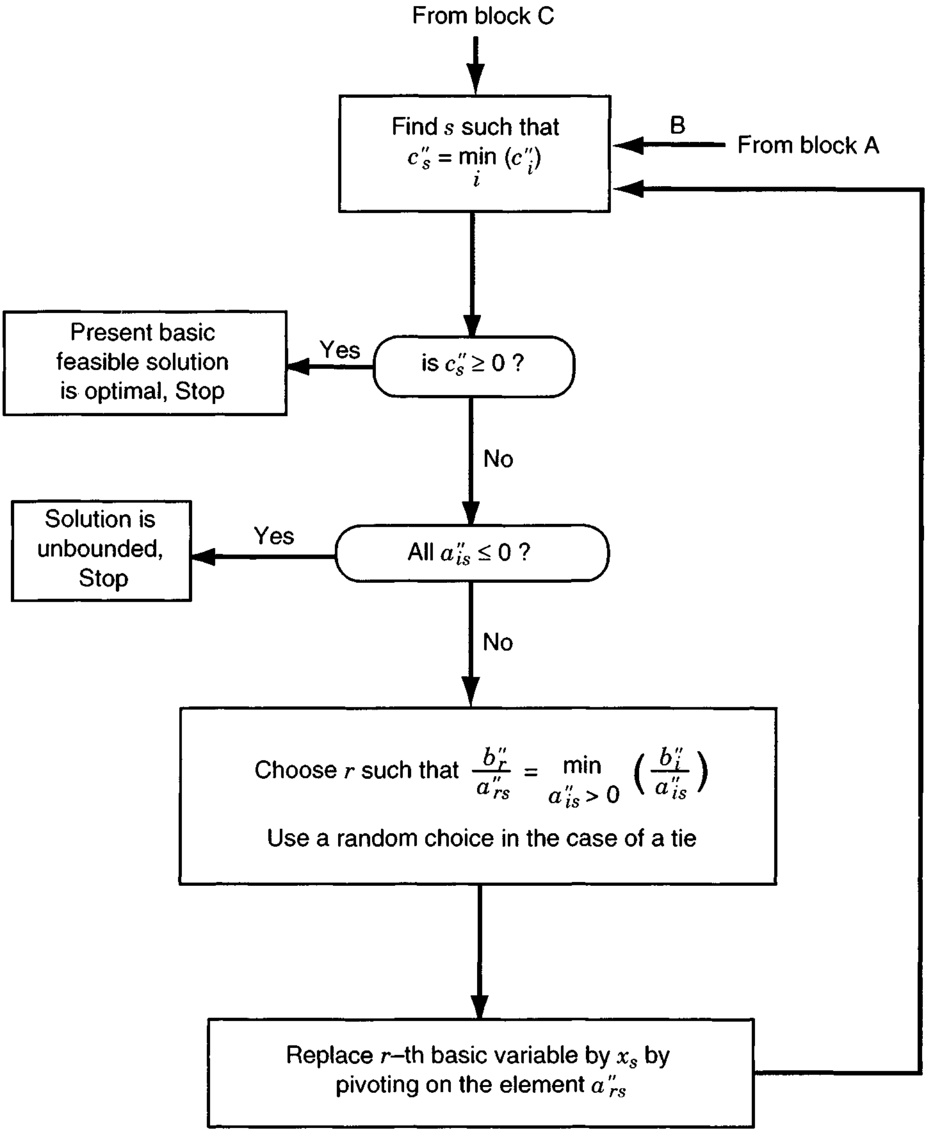

Figure 3.15 *(Continued)*

the complete array of equations can be written as

$$\begin{aligned}
3x_1 - 3x_2 + 4x_3 + 2x_4 - x_5 + y_1 \qquad\qquad &= 0 \\
x_1 + x_2 + x_3 + 3x_4 + x_5 \qquad + y_2 \qquad &= 2 \\
2x_1 + 3x_2 + 2x_3 - x_4 + x_5 \qquad\qquad - f &= 0 \\
y_1 + y_2 - w &= 0
\end{aligned} \tag{E$_2$}$$

This array can be rewritten as a canonical system with basic variables as y_1, y_2, $-f$, and $-w$ by subtracting the sum of the first two equations of (E_2) from the last equation of (E_2). Thus the last equation of (E_2) becomes

$$-4x_1 + 2x_2 - 5x_3 - 5x_4 + 0x_5 - w = -2 \qquad (E_3)$$

Since this canonical system [first three equations of (E_2), and (E_3)] provides an initial basic feasible solution, phase I of the simplex method can be started. The phase I computations are shown below in tableau form.

Basic Variables	Admissible Variables x_1	x_2	x_3	x_4	x_5	Artificial Variables y_1	y_2	b_i''	Value of b_i''/a_{is}'' for $a_{is}'' > 0$
y_1	3	-3	4	$\boxed{2}$ Pivot element	-1	1	0	0	0 ← Smaller value (y_1 drops from next basis)
y_2	1	1	1	3	1	0	1	2	$\frac{2}{3}$
$-f$	2	3	2	-1	1	0	0	0	
$-w$	-4	2	-5	-5	0	0	0	-2	
			↑	↑					
			Most negative						

Since there is a tie between d_3'' and d_4'', d_4'' is selected arbitrarily as the most negative d_i'' for pivoting (x_4 enters the next basis).

Result of pivoting:

x_4	$\frac{3}{2}$	$-\frac{3}{2}$	2	1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	
y_2	$-\frac{7}{2}$	$\boxed{\frac{11}{2}}$ Pivot element	-5	0	$\frac{5}{2}$	$-\frac{3}{2}$	1	2	$\frac{1}{11}$ ← y_2 drops from next basis
$-f$	$\frac{7}{2}$	$\frac{3}{2}$	4	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	
$-w$	$\frac{7}{2}$	$-\frac{11}{2}$	5	0	$-\frac{5}{2}$	$\frac{5}{2}$	0	-2	
		↑							
		Most negative d_i'' (x_2 enters next basis)							

Result of pivoting (since y_1 and y_2 are dropped from basis, the columns corresponding to them need not be filled):

x_4	$\frac{6}{11}$	0	$\frac{7}{11}$	1	$\frac{2}{11}$	Dropped	$\frac{6}{11}$	$\frac{6}{2}$
x_2	$-\frac{7}{11}$	1	$-\frac{10}{11}$	0	$\frac{5}{11}$		$\frac{4}{11}$	$\frac{4}{5}$
$-f$	$\frac{98}{22}$	0	$\frac{118}{22}$	0	$-\frac{4}{22}$		$-\frac{6}{11}$	
$-w$	0	0	0	0	0		0	

Step 4: At this stage we notice that the present basic feasible solution does not contain any of the artificial variables y_1 and y_2, and also the value of w is reduced to 0. This indicates that phase I is completed.

Step 5: Now we start phase II computations by dropping the w row from further consideration. The results of phase II are again shown in tableau form.

Basic Variables	Original Variables					Constant	Value of b_i''/a_{is}'' for
	x_1	x_2	x_3	x_4	x_5	b_i''	$a_{is}'' > 0$
x_4	$\frac{6}{11}$	0	$\frac{7}{11}$	1	$\frac{2}{11}$	$\frac{6}{11}$	$\frac{6}{2}$
x_2	$-\frac{7}{11}$	1	$-\frac{10}{11}$	0	$\boxed{\frac{5}{11}}$ Pivot element	$\frac{4}{11}$	$\frac{4}{5}$ ← Smaller value (x_2 drops from next basis)
$-f$	$\frac{98}{22}$	0	$\frac{118}{22}$	0	$-\frac{4}{22}$	$-\frac{6}{11}$	

↑ Most negative c_i'' (x_5 enters next basis)

Result of pivoting:

x_4	$\frac{4}{5}$	$-\frac{2}{5}$	1	1	0	$\frac{2}{5}$
x_5	$-\frac{7}{5}$	$\frac{11}{5}$	-2	0	1	$\frac{4}{5}$
$-f$	$\frac{21}{5}$	$\frac{2}{5}$	5	0	0	$-\frac{2}{5}$

Now, since all c_i'' are nonnegative, phase II is completed. The (unique) optimal solution is given by

$$x_1 = x_2 = x_3 = 0 \quad \text{(nonbasic variables)}$$

$$x_4 = \tfrac{2}{5}, \quad x_5 = \tfrac{4}{5} \quad \text{(basic variables)}$$

$$f_{\min} = \tfrac{2}{5}$$

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REVIEW QUESTIONS

3.1 Define a line segment in n-dimensional space.

3.2 What happens when $m = n$ in a (standard) LP problem?

3.3 How many basic solutions can an LP problem have?

3.4 State an LP problem in standard form.

3.5 State four applications of linear programming.

3.6 Why is linear programming important in several types of industries?

3.7 Define the following terms: point, hyperplane, convex set, extreme point.

3.8 What is a basis?

3.9 What is a pivot operation?

3.10 What is the difference between a convex polyhedron and a convex polytope?

3.11 What is a basic degenerate solution?

3.12 What is the difference between the simplex algorithm and the simplex method?

3.13 How do you identify the optimum solution in the simplex method?

3.14 Define the infeasibility form.

3.15 What is the difference between a slack and a surplus variable?

3.16 Can a slack variable be part of the basis at the optimum solution of an LP problem?

3.17 Can an artificial variable be in the basis at the optimum point of an LP problem?

3.18 How do you detect an unbounded solution in the simplex procedure?

3.19 How do you identify the presence of multiple optima in the simplex method?

3.20 What is a canonical form?

3.21 Answer true or false.

(a) The feasible region of an LP problem is always bounded.

(b) An LP problem will have infinite solutions whenever a constraint is redundant.

(c) The optimum solution of an LP problem always lies at a vertex.

(d) A linear function is always convex.

(e) The feasible space of some LP problems can be nonconvex.

(f) The variables must be nonnegative in a standard LP problem.

(g) The optimal solution of an LP problem can be called the optimal basic solution.

(h) Every basic solution represents an extreme point of the convex set of feasible solutions.

(i) We can generate all the basic solutions of an LP problem using pivot operations.

(j) The simplex algorithm permits us to move from one basic solution to another basic solution.

(k) The slack and surplus variables can be unrestricted in sign.

(l) An LP problem will have an infinite number of feasible solutions.

(m) An LP problem will have an infinite number of basic feasible solutions.

(n) The right-hand-side constants can assume negative values during the simplex procedure.

(o) All the right-hand-side constants can be zero in an LP problem.

(p) The cost coefficient corresponding to a nonbasic variable can be positive in a basic feasible solution.

(q) If all elements in the pivot column are negative, the LP problem will not have a feasible solution.

(r) A basic degenerate solution can have negative values for some of the variables.

(s) If a greater-than or equal-to type of constraint is active at the optimum point, the corresponding surplus variable must have a positive value.

(t) A pivot operation brings a nonbasic variable into the basis.

(u) The optimum solution of an LP problem cannot contain slack variables in the basis.

(v) If the infeasibility form has a nonzero value at the end of phase I, it indicates an unbounded solution to the LP problem.

(w) The solution of an LP problem can be a local optimum.

(x) In a standard LP problem, all the cost coefficients will be positive.

(y) In an standard LP problem, all the right-hand-side constants will be positive.

(z) In a LP problem, the number of inequality constraints cannot exceed the number of variables.

(aa) A basic feasible solution cannot have zero value for any of the variables.

PROBLEMS

3.1 State the following LP problem in standard form:

$$\text{Maximize } f = -2x_1 - x_2 + 5x_3$$

subject to

$$x_1 - 2x_2 + x_3 \le 8$$
$$3x_1 - 2x_2 \ge -18$$
$$2x_1 + x_2 - 2x_3 \le -4$$

3.2 State the following LP problem in standard form:

$$\text{Maximize } f = x_1 - 8x_2$$

subject to

$$3x_1 + 2x_2 \ge 6$$
$$9x_1 + 7x_2 \le 108$$
$$2x_1 - 5x_2 \ge -35$$
$$x_1, x_2 \text{ unrestricted in sign}$$

3.3 Solve the following system of equations using pivot operations:

$$6x_1 - 2x_2 + 3x_3 = 11$$
$$4x_1 + 7x_2 + x_3 = 21$$
$$5x_1 + 8x_2 + 9x_3 = 48$$

3.4 It is proposed to build a reservoir of capacity x_1 to better control the supply of water to an irrigation district [3.15, 3.17]. The inflow to the reservoir is expected to be 4.5×10^6 acre-ft during the wet (rainy) season and 1.1×10^6 acre-ft during the dry (summer) season. Between the reservoir and the irrigation district, one stream (A) adds water to and another stream (B) carries water away from the main stream, as shown in Fig. 3.16. Stream A adds 1.2×10^6 and 0.3×10^6 acre-ft of water during the wet and dry seasons, respectively. Stream B takes away 0.5×10^6 and 0.2×10^6 acre-ft of water during the wet and dry seasons, respectively. Of the total amount of water released to the irrigation district per year (x_2), 30% is to be released during the wet season and 70% during the dry season. The yearly cost of diverting the required amount of water from the main stream to the irrigation district is given by $18(0.3x_2) + 12(0.7x_2)$. The cost of building and maintaining the reservoir, reduced to an yearly basis, is given by $25x_1$. Determine the values of x_1 and x_2 to minimize the total yearly cost.

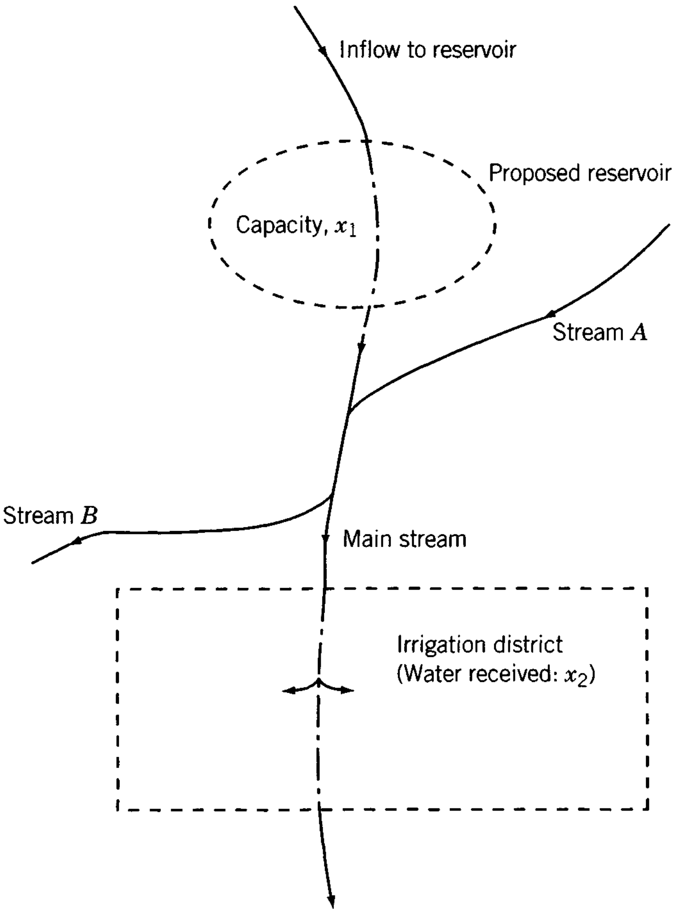

Figure 3.16 Reservoir in an irrigation district.

3.5 Solve the following system of equations using pivot operations:

$$4x_1 - 7x_2 + 2x_3 = -8$$

$$3x_1 + 4x_2 - 5x_3 = -8$$

$$5x_1 + x_2 - 8x_3 = -34$$

3.6 What elementary operations can be used to transform

$$2x_1 + x_2 + x_3 = 9$$

$$x_1 + x_2 + x_3 = 6$$

$$2x_1 + 3x_2 + x_3 = 13$$

into

$$x_1 = 3$$
$$x_2 = 2$$
$$x_1 + 3x_2 + x_3 = 10$$

Find the solution of this system by reducing into canonical form.

3.7 Find the solution of the following LP problem graphically:

$$\text{Maximize } f = 2x_1 + 6x_2$$

subject to

$$-x_1 + x_2 \le 1$$
$$2x_1 + x_2 \le 2$$
$$x_1 \ge 0, \quad x_2 \ge 0$$

3.8 Find the solution of the following LP problem graphically:

$$\text{Minimize } f = -3x_1 + 2x_2$$

subject to

$$0 \le x_1 \le 4$$
$$1 \le x_2 \le 6$$
$$x_1 + x_2 \le 5$$

3.9 Find the solution of the following LP problem graphically:

$$\text{Minimize } f = 3x_1 + 2x_2$$

subject to

$$8x_1 + x_2 \ge 8$$
$$2x_1 + x_2 \ge 6$$
$$x_1 + 3x_2 \ge 6$$
$$x_1 + 6x_2 \ge 8$$
$$x_1 \ge 0, \quad x_2 \ge 0$$

3.10 Find the solution of the following problem by the graphical method:

$$\text{Minimize } f = x_1^2 x_2^2$$

subject to

$$x_1^3 x_2^2 \geq e^3$$

$$x_1 x_2^4 \geq e^4$$

$$x_1^2 x_2^3 \leq e$$

$$x_1 \geq 0, \quad x_2 \geq 0$$

where e is the base of natural logarithms.

3.11 Prove Theorem 3.6.

For Problems 3.12 to 3.43, use a graphical procedure to identify (**a**) the feasible region, (**b**) the region where the slack (or surplus) variables are zero, and (**c**) the optimum solution.

3.12

$$\text{Maximize } f = 6x + 7y$$

subject to

$$7x + 6y \leq 42$$

$$5x + 9y \leq 45$$

$$x - y \leq 4$$

$$x \geq 0, \quad y \geq 0$$

3.13 Rework Problem 3.12 when x and y are unrestricted in sign.

3.14

$$\text{Maximize } f = 19x + 7y$$

subject to

$$7x + 6y \leq 42$$

$$5x + 9y \leq 45$$

$$x - y \leq 4$$

$$x \geq 0, \quad y \geq 0$$

3.15 Rework Problem 3.14 when x and y are unrestricted in sign.

3.16

$$\text{Maximize } f = x + 2y$$

subject to

$$\begin{aligned} x - y &\geq -8 \\ 5x - y &\geq 0 \\ x + y &\geq 8 \\ -x + 6y &\geq 12 \\ 5x + 2y &\leq 68 \\ x &\leq 10 \\ x \geq 0, \quad y &\geq 0 \end{aligned}$$

3.17 Rework Problem 3.16 by changing the objective to: Minimize $f = x - y$.

3.18

$$\text{Maximize } f = x + 2y$$

subject to

$$\begin{aligned} x - y &\geq -8 \\ 5x - y &\geq 0 \\ x + y &\geq 8 \\ -x + 6y &\geq 12 \\ 5x + 2y &\geq 68 \\ x &\leq 10 \\ x \geq 0, \quad y &\geq 0 \end{aligned}$$

3.19 Rework Problem 3.18 by changing the objective to: Minimize $f = x - y$.

3.20

$$\text{Maximize } f = x + 3y$$

subject to

$$\begin{aligned} -4x + 3y &\leq 12 \\ x + y &\leq 7 \\ x - 4y &\leq 2 \\ x \geq 0, \quad y &\geq 0 \end{aligned}$$

3.21 Minimize $f = x + 3y$

subject to

$$-4x + 3y \leq 12$$
$$x + y \leq 7$$
$$x - 4y \leq 2$$
$$x \text{ and } y \text{ are unrestricted in sign}$$

3.22 Rework Problem 3.20 by changing the objective to: Maximize $f = x + y$.

3.23 Maximize $f = x + 3y$

subject to

$$-4x + 3y \leq 12$$
$$x + y \leq 7$$
$$x - 4y \geq 2$$
$$x \geq 0, \quad y \geq 0$$

3.24 Minimize $f = x - 8y$

subject to

$$3x + 2y \geq 6$$
$$x - y \leq 6$$
$$9x + 7y \leq 108$$
$$3x + 7y \leq 70$$
$$2x - 5y \geq -35$$
$$x \geq 0, \quad y \geq 0$$

3.25 Rework Problem 3.24 by changing the objective to: Maximize $f = x - 8y$.

3.26 Maximize $f = x - 8y$

subject to

$$3x + 2y \geq 6$$

$$x - y \le 6$$
$$9x + 7y \le 108$$
$$3x + 7y \le 70$$
$$2x - 5y \ge -35$$
$$x \ge 0, \quad y \text{ is unrestricted in sign}$$

3.27 Maximize $f = 5x - 2y$

subject to

$$3x + 2y \ge 6$$
$$x - y \le 6$$
$$9x + 7y \le 108$$
$$3x + 7y \le 70$$
$$2x - 5y \ge -35$$
$$x \ge 0, \quad y \ge 0$$

3.28 Minimize $f = x - 4y$

subject to

$$x - y \ge -4$$
$$4x + 5y \le 45$$
$$5x - 2y \le 20$$
$$5x + 2y \le 10$$
$$x \ge 0, \quad y \ge 0$$

3.29 Maximize $f = x - 4y$

subject to

$$x - y \ge -4$$
$$4x + 5y \le 45$$
$$5x - 2y \le 20$$
$$5x + 2y \ge 10$$
$$x \ge 0, \quad y \text{ is unrestricted in sign}$$

3.30 $$\text{Minimize } f = x - 4y$$

subject to

$$\begin{aligned} x - y &\geq -4 \\ 4x + 5y &\leq 45 \\ 5x - 2y &\leq 20 \\ 5x + 2y &\geq 10 \\ x \geq 0, \quad y &\geq 0 \end{aligned}$$

3.31 Rework Problem 3.30 by changing the objective to: Maximize $f = x - 4y$.

3.32 $$\text{Minimize } f = 4x + 5y$$

subject to

$$\begin{aligned} 10x + y &\geq 10 \\ 5x + 4y &\geq 20 \\ 3x + 7y &\geq 21 \\ x + 12y &\geq 12 \\ x \geq 0, \quad y &\geq 0 \end{aligned}$$

3.33 Rework Problem 3.32 by changing the objective to: Maximize $f = 4x + 5y$.

3.34 Rework Problem 3.32 by changing the objective to: Minimize $f = 6x + 2y$.

3.35 $$\text{Minimize } f = 6x + 2y$$

subject to

$$\begin{aligned} 10x + y &\geq 10 \\ 5x + 4y &\geq 20 \\ 3x + 7y &\geq 21 \\ x + 12y &\geq 12 \end{aligned}$$

x and y are unrestricted in sign

3.36 $$\text{Minimize } f = 5x + 2y$$

subject to

$$3x + 4y \le 24$$
$$x - y \le 3$$
$$x + 4y \ge 4$$
$$3x + y \ge 3$$
$$x \ge 0, \quad y \ge 0$$

3.37 Rework Problem 3.36 by changing the objective to: Maximize $f = 5x + 2y$.

3.38 Rework Problem 3.36 when x is unrestricted in sign and $y \ge 0$.

3.39 $$\text{Maximize } f = 5x + 2y$$

subject to

$$3x + 4y \le 24$$
$$x - y \le 3$$
$$x + 4y \le 4$$
$$3x + y \ge 3$$
$$x \ge 0, \quad y \ge 0$$

3.40 $$\text{Maximize } f = 3x + 2y$$

subject to

$$9x + 10y \le 330$$
$$21x - 4y \ge -36$$
$$x + 2y \ge 6$$
$$6x - y \le 72$$
$$3x + y \le 54$$
$$x \ge 0, \quad y \ge 0$$

3.41 Rework Problem 3.40 by changing the constraint $x + 2y \ge 6$ to $x + 2y \le 6$.

3.42
$$\text{Maximize } f = 3x + 2y$$

subject to

$$9x + 10y \leq 330$$
$$21x - 4y \geq -36$$
$$x + 2y \leq 6$$
$$6x - y \leq 72$$
$$3x + y \geq 54$$
$$x \geq 0, \quad y \geq 0$$

3.43
$$\text{Maximize } f = 3x + 2y$$

subject to

$$21x - 4y \geq -36$$
$$x + 2y \geq 6$$
$$6x - y \leq 72$$
$$x \geq 0, \quad y \geq 0$$

3.44 Reduce the system of equations

$$2x_1 + 3x_2 - 2x_3 - 7x_4 = 2$$
$$x_1 + x_2 - x_3 + 3x_4 = 12$$
$$x_1 - x_2 + x_3 + 5x_4 = 8$$

into a canonical system with x_1, x_2 and x_3 as basic variables. From this derive all other canonical forms.

3.45
$$\text{Maximize } f = 240x_1 + 104x_2 + 60x_3 + 19x_4$$

subject to

$$20x_1 + 9x_2 + 6x_3 + x_4 \leq 20$$
$$10x_1 + 4x_2 + 2x_3 + x_4 \leq 10$$
$$x_i \geq 0, \quad i = 1 \text{ to } 4$$

Find all the basic feasible solutions of the problem and identify the optimal solution.

3.46 A progressive university has decided to keep its library open round the clock and gathered that the following number of attendants are required to reshelve the books:

Time of Day (hours)	Minimum Number of Attendants Required
0–4	4
4–8	7
8–12	8
12–16	9
16–20	14
20–24	3

If each attendant works eight consecutive hours per day, formulate the problem of finding the minimum number of attendants necessary to satisfy the requirements above as a LP problem.

3.47 A paper mill received an order for the supply of paper rolls of widths and lengths as indicated below.

Number of Rolls Ordered	Width of Roll (m)	Length (m)
1	6	100
1	8	300
1	9	200

The mill produces rolls only in two standard widths, 10 and 20 m. The mill cuts the standard rolls to size to meet the specifications of the orders. Assuming that there is no limit on the lengths of the standard rolls, find the cutting pattern that minimizes the trim losses while satisfying the order above.

3.48 Solve the LP problem stated in Example 1.6 for the following data: $l = 2$ m, $W_1 = 3000$ N, $W_2 = 2000$ N, $W_3 = 1000$ N, and $w_1 = w_2 = w_3 = 200$ N.

3.49 Find the solution of Problem 1.1 using the simplex method.

3.50 Find the solution of Problem 1.15 using the simplex method.

3.51 Find the solution of Example 3.1 using (**a**) the graphical method and (**b**) the simplex method.

3.52 In the scaffolding system shown in Fig. 3.17, loads x_1 and x_2 are applied on beams 2 and 3, respectively. Ropes A and B can carry a load of $W_1 = 300$ lb each, the middle ropes, C and D, can withstand a load

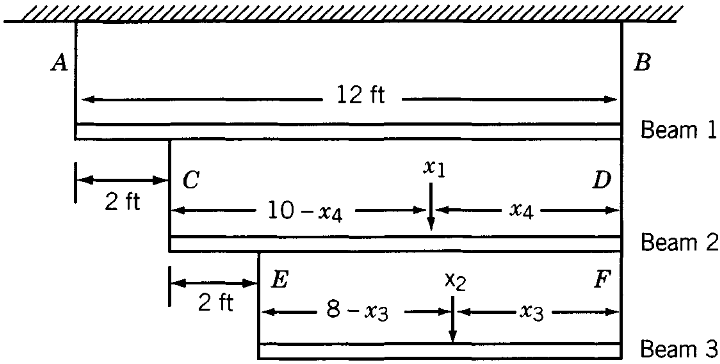

Figure 3.17 Scaffolding system with three beams.

of $W_2 = 200$ lb each, and ropes E and F are capable of supporting a load $W_3 = 100$ lb each. Formulate the problem of finding the loads x_1 and x_2 and their location parameters x_3 and x_4 to maximize the total load carried by the system, $x_1 + x_2$, by assuming that the beams and ropes are weightless.

3.53 A manufacturer produces three machine parts, A, B, and C. The raw material costs of parts A, B, and C are \$5, \$10, and \$15 per unit, and the corresponding prices of the finished parts are \$50, \$75, and \$100 per unit, respectively. Part A requires turning and drilling operations, while part B needs milling and drilling operations. Part C requires turning and milling operations. The number of parts that can be produced on various machines per day and the daily costs of running the machines are given below.

	Number of Parts That Can Be Produced on		
Machine Part	Turning Lathes	Drilling Machines	Milling Machines
A	15	15	
B		20	30
C	25		10
Cost of running the machines per day	\$250	\$200	\$300

Formulate the problem of maximizing the profit.

Solve each problem by the simplex method.

3.54 Problem 1.22

3.55 Problem 1.23

3.56 Problem 1.24

3.57 Problem 1.25

3.58 Problem 3.7

3.59 Problem 3.12

3.60 Problem 3.13

3.61 Problem 3.14

3.62 Problem 3.15

3.63 Problem 3.16

3.64 Problem 3.17

3.65 Problem 3.18

3.66 Problem 3.19

3.67 Problem 3.20

3.68 Problem 3.21

3.69 Problem 3.22

3.70 Problem 3.23

3.71 Problem 3.24

3.72 Problem 3.25

3.73 Problem 3.26

3.74 Problem 3.27

3.75 Problem 3.28

3.76 Problem 3.29

3.77 Problem 3.30

3.78 Problem 3.31

3.79 Problem 3.32

3.80 Problem 3.33

3.81 Problem 3.34

3.82 Problem 3.35

3.83 Problem 3.36

3.84 Problem 3.37

3.85 Problem 3.38

3.86 Problem 3.39

3.87 Problem 3.40

3.88 Problem 3.41

3.89 Problem 3.42

3.90 Problem 3.43

3.91 The temperatures measured at various points inside a heated wall are given below.

Distance from the heated surface as a percentage of wall thickness, x_i	0	20	40	60	80	100
Temperature, t_i (°C)	400	350	250	175	100	50

It is decided to use a linear model to approximate the measured values as

$$t = a + bx \tag{1}$$

where t is the temperature, x the percentage of wall thickness, and a and b the coefficients that are to be estimated. Obtain the best estimates of a and b using linear programming with the following objectives.

(a) Minimize the sum of absolute deviations between the measured values and those given by Eq. (1): $\Sigma_i \, |a + bx_i - t_i|$.

(b) Minimize the maximum absolute deviation between the measured values and those given by Eq. (1):

$$\underset{i}{\text{Max}} \, |a + bx_i - t_i|$$

3.92 A snack food manufacturer markets two kinds of mixed nuts, labeled A and B. Mixed nuts A contain 20% almonds, 10% cashew nuts, 15% walnuts, and 55% peanuts. Mixed nuts B contain 10% almonds, 20% cashew nuts, 25% walnuts, and 45% peanuts. A customer wants to use mixed nuts A and B to prepare a new mix that contains at least 4 lb of almonds, 5 lb of cashew nuts, and 6 lb of walnuts, for a party. If mixed nuts A and B cost \$2.50 and \$3.00 per pound, respectively, determine the amounts of mixed nuts A and B to be used to prepare the new mix at a minimum cost.

3.93 A company produces three types of bearings, B_1, B_2, and B_3, on two machines, A_1 and A_2. The processing times of the bearings on the two machines are indicated in the following table.

Machine	Processing Time (min) for Bearing: B_1	B_2	B_3
A_1	10	6	12
A_2	8	4	4

The times available on machines A_1 and A_2 per day are 1200 and 1000 minutes, respectively. The profits per unit of B_1, B_2, and B_3 are \$4, \$2, and \$3, respectively. The maximum number of units the company can sell are 500, 400, and 600 for B_1, B_2, and B_3, respectively. Formulate and solve the problem for maximizing the profit.

3.94 Two types of printed circuit boards A and B are produced in a computer manufacturing company. The component placement time, soldering time, and inspection time required in producing each unit of A and B are given below.

Circuit Board	Time Required per Unit (min) for: Component Placement	Soldering	Inspection
A	16	10	4
B	10	12	8

If the amounts of time available per day for component placement, soldering, and inspection are 1500, 1000, and 500 person-minutes, respectively, determine the number of units of A and B to be produced for maximizing the production. If each unit of A and B contributes a profit of \$10 and \$15, respectively, determine the number of units of A and B to be produced for maximizing the profit.

3.95 A paper mill produces paper rolls in two standard widths; one with width 20 in. and the other with width 50 in. It is desired to produce new rolls with different widths as indicated below.

Width (in.)	Number of Rolls Required
40	150
30	200
15	50
6	100

The new rolls are to be produced by cutting the rolls of standard widths to minimize the trim loss. Formulate the problem as an LP problem.

3.96 A manufacturer produces two types of machine parts, P_1 and P_2, using lathes and milling machines. The machining times required by each part on the lathe and the milling machine and the profit per unit of each part are given below.

	Machine Time (hr) Required by Each Unit on:		
Machine Part	Lathe	Milling Machine	Cost per Unit
P_1	5	2	\$200
P_2	4	4	\$300

If the total machining times available in a week are 500 hours on lathes and 400 hours on milling machines, determine the number of units of P_1 and P_2 to be produced per week to maximize the profit.

3.97 A bank offers four different types of certificates of deposits (CDs) as indicated below.

CD Type	Duration (yr)	Total Interest at Maturity (%)
1	0.5	5
2	1.0	7
3	2.0	10
4	4.0	15

If a customer wants to invest \$50,000 in various types of CDs, determine the plan that yields the maximum return at the end of the fourth year.

3.98 The production of two machine parts A and B requires operations on a lathe (L), a shaper (S), a drilling machine (D), a milling machine (M), and a grinding machine (G). The machining times required by A and B on various machines are given below.

	Machine Time Required (hours per unit) on:				
Machine Part	L	S	D	M	G
A	0.6	0.4	0.1	0.5	0.2
B	0.9	0.1	0.2	0.3	0.3

The number of machines of different types available is given by L: 10, S: 3, D: 4, M: 6, and G: 5. Each machine can be used for 8 hours a day for 30 days in a month.

(a) Determine the production plan for maximizing the output in a month

(b) If the number of units of A is to be equal to the number of units of B, find the optimum production plan.

3.99 A salesman sells two types of vacuum cleaners, A and B. He receives a commission of 20% on all sales provided that at least 10 units each of A and B are sold per month. The salesman needs to make telephone calls to make appointments with customers and demonstrate the products in order to sell the products. The selling price of the products, the average money to be spent on telephone calls, the time to be spent on demonstrations, and the probability of a potential customer buying the product are given below.

Vacuum Cleaner	Selling Price per Unit	Money to Be Spent on Telephone Calls to Find a Potential Customer	Time to Be Spent in Demonstrations to a Potential Customer (hr)	Probability of a Potential Customer Buying the Product
A	\$250	\$3	3	0.4
B	\$100	\$1	1	0.8

In a particular month, the salesman expects to sell at most 25 units of A and 45 units of B. If he plans to spend a maximum of 200 hours in the month, formulate the problem of determining the number of units of A and B to be sold to maximize his income.

3.100 An electric utility company operates two thermal power plants, A and B, using three different grades of coal, C_1, C_2, and C_3. The minimum power to be generated at plants A and B is 30 and 80 MWh, respectively. The quantities of various grades of coal required to generate 1 MWh of power at each power plant, the pollution caused by the various grades of coal at each power plant, and the costs of coal are given in the following table.

	Quantity of Coal Required to Generate 1 MWh at the Power Plant (tons)		Pollution Caused at Power Plant		Cost of Coal at Power Plant	
Coal Type	A	B	A	B	A	B
C_1	2.5	1.5	1.0	1.5	20	18
C_2	1.0	2.0	1.5	2.0	25	28
C_3	3.0	2.5	2.0	2.5	18	12

Formulate the problem of determining the amounts of different grades of coal to be used at each power plant to minimize (**a**) the total pollution level, and (**b**) the total cost of operation.

3.101 A grocery store wants to buy five different types of vegetables from four farms in a month. The prices of the vegetables at different farms, the capacities of the farms, and the minimum requirements of the grocery store are indicated in the following table.

Farm	Price ($/ton) of Vegetable Type 1 (Potato)	2 (Tomato)	3 (Okra)	4 (Eggplant)	5 (Spinach)	Maximum (of All Types Combined) They Can Supply
1	200	600	1600	800	1200	180
2	300	550	1400	850	1100	200
3	250	650	1500	700	1000	100
4	150	500	1700	900	1300	120
Minimum amount required (tons)	100	60	20	80	40	

Formulate the problem of determining the buying scheme that corresponds to a minimum cost.

3.102 A steel plant produces steel using four different types of processes. The iron ore, coal, and labor required, the amounts of steel and side products produced, the cost information, and the physical limitations on the system are given below.

Process Type	Iron Ore Required (tons/day)	Coal Required (tons/day)	Labor Required (person-days)	Steel Produced (tons/day)	Side Products Produced (tons/day)
1	5	3	6	4	1
2	8	5	12	6	2
3	3	2	5	2	1
4	10	7	12	6	4
Cost	$50/ton	$10/ton	$150/person-day	$350/ton	$100/ton
Limitations	600 tons available per month	250 tons available per month	No limitations on availability of labor	All steel produced can be sold	Only 200 tons can be sold per month

Assuming that a particular process can be employed for any number of days in a 30-day month, determine the operating schedule of the plant for maximizing the profit.