2007 Changchun Invitational World Youth Mathematics Intercity Competition

Individual Contest

Time limit: 120 minutes 2007/7/23 Changchun, China

Team:________________ Name:________________ Score:______________

Section I:

In this section, there are 12 questions, fill in the correct answers in the spaces provided at the end of each question. Each correct answer is worth 5 points.

1. Let A_n be the average of the multiples of n between 1 and 101. Which is the largest among A_2, A_3, A_4, A_5 and A_6?

Answer：________________

Solution

The smallest multiple of n is of course n. Denote by an the largest multiple of n not exceeding 101. Then $A_n=(n+a_n)/2$. Hence $A_2=A_3=51$, $A_4=52$, $A_5=52.5$ and $A_6=51$, and the largest one is A_5.

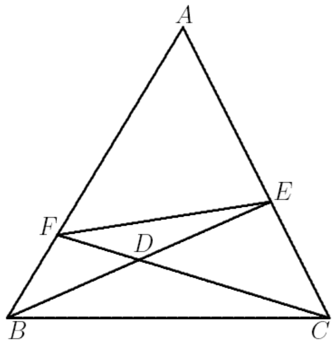
2. It is a dark and stormy night. Four people must evacuate from an island to the mainland. The only link is a narrow bridge which allows passage of two people at a time. Moreover, the bridge must be illuminated, and the four people have only one lantern among them. After each passage to the mainland, if there are still people on the island, someone must bring the lantern back. Crossing the bridge individually, the four people take 2, 4, 8 and 16 minutes respectively. Crossing the bridge in pairs, the slower speed is used. What is the minimum time for the whole evacuation?

Answer：________________

Solution

Exactly five passages are required, three pairs to the mainland and two individuals back to the island. Let the fastest two people cross first. One of them brings back the lantern. Then the slowest two people cross, and the fastest people on the mainland brings back the lantern, The final passage is the same as the first. The total time is 4+2+16+4+4=30 minutes. To show that this is minimum, note that the three passages in pairs take at least 16+4+4=24 minutes, and the two passages individually take at least 4+2=6 minutes.

3. In triangle ABC, E is a point on AC and F is a point on AB. BE and CF intersect at D. If the areas of triangles BDF, BCD and CDE are 3, 7 and 7 respectively, what is the area of the quadrilateral $AEDF$?



Answer：________________

Solution

Since triangles BCD and CDE have equal areas, $BD=DE$. Hence the area of triangle DEF is also 3. Let the area of triangle EFA be x. Then $\frac{x}{6}=\frac{AF}{BF}=\frac{x+3+7}{3+7}$. It follows that $10x=6x+60$ so that $x=15$. The area of the quadrilateral $AEDF$ is 15+3=18.

4. A regiment had 48 soldiers but only half of them had uniforms. During inspection, they form a 6×8 rectangle, and it was just enough to conceal in its interior everyone without a uniform. Later, some new soldiers joined the regiment, but again only half of them had uniforms. During the next inspection, they used a different rectangular formation, again just enough to conceal in its interior everyone without a uniform. How many new soldiers joined the regiment?

Answer：________________

Solution

Let the dimensions of the rectangle be x by y, with $x \leqq y$. Then the number of soldiers on the outside is $2x+2y-4$ while the number of those in the interior is $(x-2)(y-2)$. From $xy-2x-2y+4=2x+2y-4$, we have $(x-4)(y-4)=xy-4x-4y+16=8$. If $x-4=2$ and $y-4=4$, we obtain the original 6×8 rectangle. If $x-4=1$ and $y-4=8$, we obtain the new 5×12 rectangle. Thus the number of new soldiers is 5×12-6×8=12.

5. The sum of 2008 consecutive positive integers is a perfect square. What is the minimum value of the largest of these integers?

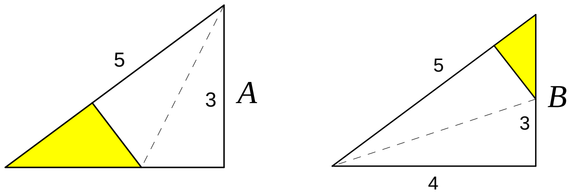
Answer：________________

Solution

Let a be the smallest of these integers.
Then $a+(a+1)+(a+2)+\ldots+(a+2007)=251\times(2a+2007)\times 2^2$. In order for this to be a perfect square, we must have $2a+2007=251n^2$ for some positive integer n. For $n=1$ or 2, a is negative. For $n=3$, we have $a=126$ so that $a+2007=2133$ is the desired minimum value.

6. The diagram shows two identical triangular pieces of paper A and B. The side lengths of each triangle are 3, 4 and 5. Each triangle is folded along a line through a vertex, so that the two sides meeting at this vertex coincide. The

regions not covered by the folded parts have respective areas S_A and S_B. If S_A+S_B=39 square centimetres, find the area of the original triangular piece of paper.



Answer：________________

Solution

In the first diagram, the ratio of the areas of the shaded triangle and one of the unshaded triangle is (5 - 3):3 so that S_A is one quarter of the area of the whole triangle. In the second diagram, the ratio of the areas of the shaded triangle and one of the unshaded triangle is (5 - 4):4 so that S_B is one ninth of the area of the whole triangle. Now 1/4+1/9=13/36. Hence the area of the whole triangle is (36/13)39=108 square centimetres.

7. Find the largest positive integer n such that 3^{1024}-1 is divisible by 2^n.

Answer：________________

Solution

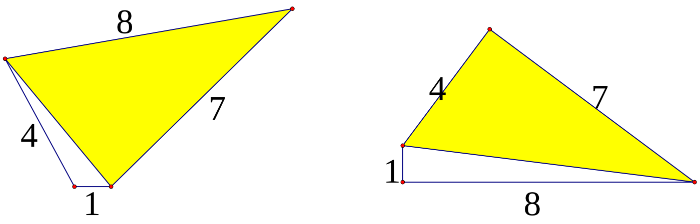
Note that $3^{1024}-1=(3^{512}+1)(3^{256}+1)(3^{128}+1)\ldots(3+1)(3-1)$. All 11 factors are even, and 3+1 is a multiple of 4. Clearly 3-1 is not divisible by 4. We claim that neither is any of the other 9. When the square of an odd number is divided by 4, the remainder is always 1. Adding 1 makes the remainder 2, justifying the claim. Hence the maximum value of n is 12.

8. A farmer has four straight fences, with respective lengths 1, 4, 7 and 8 metres. What is the maximum area of the quadrilateral the farmer can enclose?

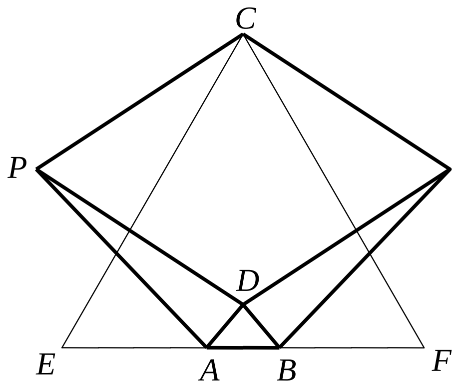
Answer：________________

Solution

We may assume that the sides of lengths 1 and 8 are adjacent sides of the quadrilateral, as otherwise we can flip over the shaded triangle in the first diagram. Now the quadrilateral may be divided into two triangles as shown in the second diagram. In each triangle, two sides have fixed length. Hence its area is maximum if these two sides are perpendicular to each other. Since 12+82=42+72, both maxima can be achieved simultaneously. In that case, the area of the unshaded triangle is 4 and the area of the shaded triangle is 14. Hence the maximum area of the quadrilateral is 18.



9. In the diagram, $PA=QB=PC=QC=PD=QD=1$, $CE=CF=EF$ and $EA=BF=2AB$. Determine BD.



Answer : ________________

Solution

Let M be the midpoint of EF. By symmetry, D lies on CM. Let $BM=x$. Then $FM=5x$, $CF=10x$, $CM=5\sqrt{5}\,x$ and $BC=2\sqrt{19}\,x$. It follows that $\frac{AB}{BC}=\frac{1}{\sqrt{19}}$. Now Q is the circumcentre of triangle BCD. Hence $\angle BQD=2\angle BCD=\angle BCA$. Since both QDB and CAB are isosceles triangles, they are similar to each other. It follows that $\frac{BD}{QB}=\frac{AB}{BC}=\frac{1}{\sqrt{19}}$, so that $BD=\frac{1}{\sqrt{19}}$.

10. Each of the numbers 2, 3, 4, 5, 6, 7, 8 and 9 is used once to fill in one of the boxes in the equation below to make it correct. Of the three fractions being added, what is the value of the largest one?

$$\frac{1}{\square\times\square}+\frac{\square}{\square\times\square}+\frac{\square}{\square\times\square}=1$$

Answer : ________________

Solution

We may assume that the second numerator is 5 and the third 7. If either 5 or 7 appears in a denominator, it can never be neutralized. Since the least common multiple of the two remaining numbers is $8\times9=72$, we use $\frac{1}{72}$ as the unit of measurement. Now one of the three fractions must be close to 1. This can only be $\frac{5}{2\times3}$ or $\frac{7}{2\times4}$. In the first case, we are short 12 units. Of this, 7 must come from the third fraction so that 5 must come from the first fraction. This is impossible because the first fraction has numerator 1 and 5 does not divide 72. In the second case, we are short 9 units. Of this, 5 must come from the second fraction so that 4 must come from the third. This can be achieved as shown in the equation below. Hence the largest of the three fractions has value $\frac{7}{8}$.

11. Let x be a positive number. Denote by $[x]$ the integer part of x and by $\{x\}$ the decimal part of x. Find the sum of all positive numbers satisfying $5\{x\}+0.2[x]=25$

Answer：________________

Solution

The given equation may be rewritten as $\{x\}=\frac{125-[x]}{25}$. From $0\leqq\{x\}<1$, We have $100<[x]\leqq125$. For each solution x, $x=[x]+\{x\}=5+\frac{24}{25}[x]$. It follows that the desired sum is $5(25)+(24/25)(101+102+103+\ldots+125)=2837$.

12. A positive integer n is said to be good if there exists a perfect square whose sum of digits in base 10 is equal to n. For instance, 13 is good because 72=49 and 4+9=13. How many good numbers are among 1, 2, 3, …, 2007?

Answer：________________

Solution

If a positive integer is a multiple of 3, then its square is a multiple of 9, and so is the sum of the digits of the square. If a positive integer is not a multiple of 3, then its square is 1 more than a multiple of 3, and so is the sum of the digits of the square. Now the square of 9…9 with m 9s is 9…980…01, with m - 1 9s and 0s. Its digit sum is $9m$. Hence all multiples of 9 are good, and there are 2007/9=223 of them not exceeding 2007. On the other hand, the square of 3…35 with m 3s is 1212..1225 with m sets of 12. Its digit sum is $3m+7$. Since 1 and 4 are also good, all numbers 1 more than a multiple are good, and there are 2007/3=669 of them. Hence there are altogether 223+669=992 good numbers not exceeding 2007.

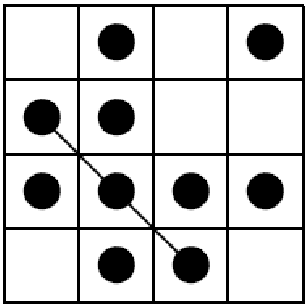
Section II:

Answer the following 3 questions, and show your detailed solution in the space provided after each question. Each question is worth 20 points.

1. A 4×4 table has 18 lines, consisting of the 4 rows, the 4 columns, 5 diagonals running from southwest to northeast, and 5 diagonals running from northwest to southeast. A diagonal may have 2, 3 or 4 squares. Ten counters are to be placed, one in each of ten of the sixteen cells. Each line which contains an even number of counters scores a point. What is the largest possible score?

Solution

The maximum score is 17, as shown in the placement in the diagram below. The only line not scoring a point is marked.



We now prove that a perfect score of 18 points leads to a contradiction. Note that the five diagonals in the same direction cover all but two opposite corner cells. These two cells must either be both vacant or both occupied. Note also that we must have a completely filled row, and a completely filled column. We consider three cases.

Case 1. All four corner cells are vacant.

We may assume by symmetry that the second row and the second column are completely filled. Then we must fill the remaining inner cells of the first row, the fourth row, the first column and the fourth column. These requires eleven counters.

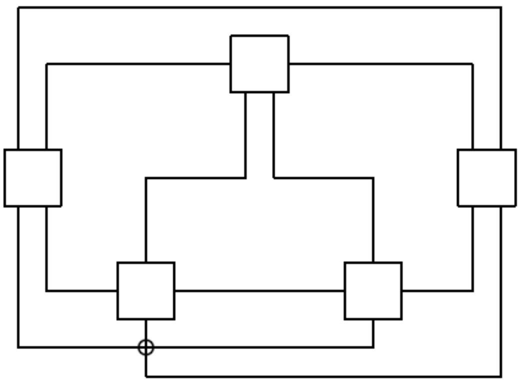
Case 2. Exactly two opposite corner cells are vacant.

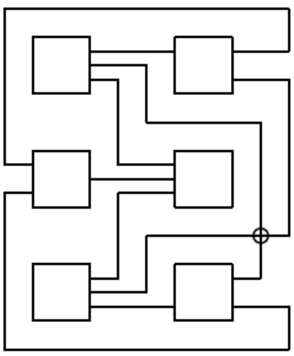
By symmetry, we may assume that one of them is on the first row and first column, and the other is on the fourth row and fourth column. Then we must have exactly one more occupied inner cell on each of the first row, the first column, the fourth row and the fourth column. This means that all four cells in the interior of the table are filled. By symmetry, we may assume that the completely filled row is the second. It is impossible to score both the diagonals of length 2 which intersect the second row.

Case 3. All four corner cells are occupied.

We claim that the completely filled row must be either the first or the fourth. Suppose to the contrary it is the second. Then we must fill the first column and the fourth column, thus using up all ten counters. Now there are several diagonals which do not yield scores. This justifies the claim. By symmetry, we may assume that the first row and the first column are completely filled. To score all rows and columns, the remaining two counters must be in the four interior cells. Again, some of the diagonals will not yield scores.

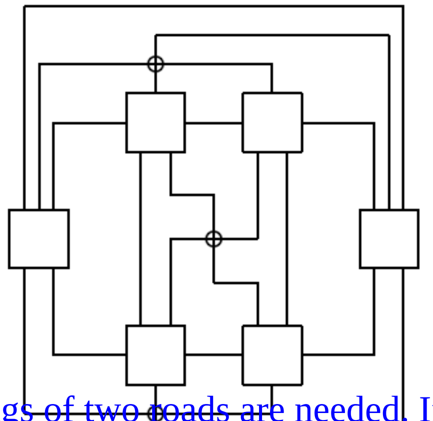
2. There are ten roads linking all possible pairs of five cities. It is known that there is at least one crossing of two roads, as illustrated in the diagram below on the left. There are nine roads linking each of three cities to each of three towns. It is known that there is also at least one crossing of two roads, as illustrated in the diagram below on the right. Of the fifteen roads linking all possible pairs of six cities, what is the minimum number of crossings of two roads?





Solution

The minimum number of crossing of two roads is three, as illustrated in the diagram below.



Suppose at most two crossings of two roads are needed. If we close one road from each crossing, the remaining ones can be drawn without any crossing. We consider two cases.

Case 1. The two roads closed meet at a city.

Consider the other five cities linked pairwise by ten roads, none of which has been closed. It is given that there must be a crossing of two roads, which is a contradiction.

Case 2. The two roads closed do not meet at a city.

Choose the two cities linked by one of the closed roads, and a third city not served by the other closed road. Call these three cities towns. Each is linked to each of the remaining three cities by a road. It is given that there must be a crossing of two roads, which is a contradiction.

3. A prime number is called an absolute prime if every permutation of its digits in base 10 is also a prime number. For example: 2, 3, 5, 7, 11, 13 (31), 17 (71), 37 (73) 79 (97), 113 (131, 311), 199 (919, 991) and 337 (373, 733) are absolute primes. Prove that no absolute prime contains all of the digits 1, 3, 7 and 9 in base 1

Solution

Let N be an absolute prime which contains all of the digits 1, 3, 7 and 9 in base 10. Let L be any number formed from the remaining digits. Consider the following seven permutations of N: $10000L+7931$, $10000L+1793$, $10000L+9137$, $10000L+7913$, $10000L+7193$, $10000L+1973$ and $10000L+7139$. They have different remainders when divided by 7. Therefore one of them is a multiple of 7, and is not a prime. Hence N is not an absolute prime.