Diophantine Equations Problems

Amir Hossein Parvardi *

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Dedicated to **Mahan Malihi**, **Goodarz Mehr**, **Behzad Behzadi**, and **Ahmad Reza Goodarzvand**, who are gold medalists of Iran mathematical olympiad 2011.

*email: ahpwsog@gmail.com, blog: http://math-olympiad.blogsky.com

1. Find all pairs (a, b) of positive integers that satisfy the equation $a^{b^2} = b^a$.

2. Find all pairs of positive integers (n, k) such that $n! = (n+1)^k - 1$.

3. Solve the following equation in integers (x, y, z):

$$2x^4 + 2x^2y^2 + y^4 = z^2.$$

4. Determine all triples (x, y, z) of positive integers such that

$$(x+1)^{y+1} + 1 = (x+2)^{z+1}.$$

5. Solve the diophantine equation: $x^a - 1 = y^b$ where $x, y, a, b > 1$ are integers and $x \equiv 1 \pmod{y}$.

6. Find all pairs (p, n) of positive integers which satisfy the equation $2^n = p + 3^p$.

7. Find all positive integer solutions to $2^x = 3^y + 5$.

8. Let p, q, r different primes. Prove that the diophantine equation: $z^r + x^p = 2y^q$ has an infinite set of solutions with $x > 1$, $y > 1$, $z > 1$ and $z^r \neq x^p$.

9. Prove that there do not exist solutions to $x^2y^2 = z^2(z^2 - x^2 - y^2)$ in positive integers.

10. Solve the equation $3^x = 2^x y + 1$ in positive integers.

11. Find all integer solutions to $x^5 = y^2 + 1$.

12. Prove that the equation

$$x^3 + y^3 = 9$$

has infinitely many rational solutions.

13. Find all positive integers x, y, z such that $xy^2 = z^3 + 1$ and x does not have prime factors of the form $6k + 1$.

14. Find all integer solutions to the equation $7^x = 3^y + 4$.

15. prove that the equation $x^3 + y^3 = z^4 - t^4$ has infinitely many solutions with $\gcd(x, y, z, t) = 1$.

16. Solve the following equation in rational numbers x, y, z:

$$x^2 + y^2 + z^2 + 3(x + y + z) + 5 = 0.$$

17. Let a, b, c some positive integers and x, y, z some integer numbers such that we have a) $ax^2 + by^2 + cz^2 = abc + 2xyz - 1$; b) $ab + bc + ca \geq x^2 + y^2 + z^2$. Prove that a, b, c are all sums of three squares of integer numbers.

18. Show that the number $x^8 + x^7 - x^5 - x^4 - x^3 + x + 1$ cannot be a perfect square whenever x is an integer greater than 1.

19. Find all integer solutions to the equation $x^2 + xy + y^2 = z^2$.

20. Prove that the equation $x^3 = y^2 + 3$ does not have any positive integer solutions.

21. Prove that the equation $x^n = y^2 + 3$ does not have any positive integer solutions for $n = 4k - 1$.

22. Let n be a positive integer. Prove that the equation

$$x + y + \frac{1}{x} + \frac{1}{y} = 3n$$

does not have solutions in positive rational numbers.

23. Prove that

$$k^4 + (k+1)^4 + (k+2)^4 = n^2 + (n+1)^2$$

has no integer solutions.

24. Solve the following diophantine equation in $\mathbb{N}$ (or more generally in $\mathbb{Z}$):

$$\sum_{k=1}^{n} a_k = \prod_{k=1}^{n} a_k.$$

25. Determine positive integers m, n such that $m^2 + mn + n^2$ is square.

26. Find all the solutions of the equation

$$a + b + c = abc$$

in the set of non-negative integers.

27. Determine all pairs (x, y) of positive integers such that $\frac{x^2y+x+y}{xy^2+y+11}$ is an integer.

28. Find all primes p, q such that $2^p = q^q + q + 2$.

29. Find all positive integers n and m such that

$$n! + 1 = (m! - 1)^2.$$

30. Solve in positive integers the following equation:

$$n^3 - 5n + 10 = 2^k.$$

31. Find all the solutions of the following equation in the set of positive integers

$$x^y = y^x - (x + y).$$

32. Find all positive integers x, y, z, n such that $x^3 + y^3 + z^3 = nx^2y^2z^2$.

33. Find all non-negative integers m, n, p, q such that

$$p^m q^n = (p+q)^2 + 1.$$

34. Find all positive integers m and n such that $1 + 5 \cdot 2^m = n^2$.

35. Let a, b, n be positive integers such that $2^n - 1 = ab$. Let $k \in \mathbb{N}$ such that $ab + a - b - 1 \equiv 0 \pmod{2^k}$ and $ab + a - b - 1 \neq 0 \pmod{2^{k+1}}$. Prove that k is even.

36. Find all positive integer solutions to $a^2 = b^3 + 1$.

37. Is it possible for a perfect square (which is not a power of 10) to contain only 0s and 1s in its decimal representation ?

38. Are there integers m and n such that $5m^2 - 6mn + 7n^2 = 1985$?

39. Prove that there are no integer solutions to the equation $x^6 = y^5 + 24$.

40. Find all integer solutions to $2^n - 7 = x^2$

41. Find all numbers triples (k, n, p) of positive integers such that

$$5^k - 3^n = p^2.$$

42. Find all positive integers x, y such that $x + x^2 = y + y^2 + y^3$.

43. Solve in positive integers the equation $(x^2+2)(y^2+3)(z^2+4) = 60xyz$.

44. Solve the equation

$$(a^2, b^2) + (a, bc) + (b, ac) + (c, ab) = 199.$$

in positive integers. (Here (x, y) denotes the greatest common divisor of x and y.)

45. Prove that the equation $m(m+1)(m+2)(m+3) = n(n+1)^2(n+2)^3(n+3)^4$ doesn't have any solution in positive integers.

46. Find all positive integers (a, b, c) satisfying

$$4a^3 + b + c = 4abc + 2a.$$

47. Find integral solutions to the equation

$$(m^2 - n^2)^2 = 16n + 1.$$

48. Show that the Diophantine equation

$$\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} + \frac{1}{x_1 x_2 \cdots x_n} = 1$$

has at least one solution for every positive integers n.

49. Solve in intergers the equation $x^3 + x^2 + x + 1 = y^2$.

50. Find all pairs (a, b, c, d) integer such that $a^2 + 5b^2 = 2c^2 + 2cd + 3d^2$.

Solutions

1. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=1238.

2. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=3452.

3. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=5944.

4. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=28543.

5. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=28571.

6. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=414069.

7. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=27477.

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15. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=48538.

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17. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=47030.

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19. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=81412

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29. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=103131

30. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=103239

31. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=100680

32. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=98325

33. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=88312

34. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=86541

35. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=85144

36. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=127208

37. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=121555

38. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=121391

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49. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=149349

50. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=180808