Analysis of Composite Structures

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Contents

PART II Multi-layer plates

Foreword

The aim of this book is to present the basis for calculation of composite structures, using continuum mechanics equations which enable the more elaborate theories to be treated.

The first part is devoted to study of materials constituting the layers of laminated composites. The constitutive equations for anisotropic and in particular orthotropic materials are presented, with temperature and hygrometry effects taken into account. Next the basic laws of mixtures are given, which enable the behaviour of unidirectional layers to be predicted from the characteristics of their fibres and matrix components.

The subject of the second part is multi-layer plates. We begin by presenting the general equations of thin plates in Kirchhoff-Love analysis. Later, symmetrical orthotropic plates are studied in detail for cases of bending, vibration and buckling. The thermo-elastic behaviour of multi-layers plates is considered separately. Then we tackle symmetric orthotropic moderately thick plates, using Reissner-Mindlin type analysis. Examples of asymmetrical plates in Kirchhoff-Love theory are analysed in detail. The cylindrical bending of laminated composites is treated in both Kirchhoff-Love and Reissner-Mindlin type analysis, with bending, vibration and buckling applications.

The third part of this book is devoted to beams. The first chapter of this part treats tension-compression loading. The following chapter treats bending with transverse shear deformations not taken in account. The last chapter of this part presents bending taking into account transverse shear. Examples of bending, vibration and buckling are considered for each case.

In the appendices, plate equations are developed by integrating local equations of motion. Global equations are obtained from variational formulae of continuum mechanics.

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PART I

Mechanical Behaviour of Composite Materials

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Chapter 1

Constitutive relations for anisotropic materials in linear elasticity

1.1. Introduction

Stress-strain or constitutive relations for anisotropic materials will be treated first, and in particular the behaviour of orthotropic and transversely isotropic materials, the latter including unidirectional composites.

Then, using tensor notation based on four indices and the conventional notation with two indices, the expressions for changing axes in terms of stresses, strains, stiffnesses and compliances will be detailed.

1.2. Four indices tensor notation

1.2.1. *Constitutive relations*

The reference state is a natural state without stress or strain, that is:

$\sigma_{ij} = 0$ and $\varepsilon_{ij} = 0$.

In linear elasticity theory the stress tensor $\boldsymbol{\sigma}$ is given as a function of the strain tensor $\boldsymbol{\varepsilon}$ by the tensorial relation:

$\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}$, or with the indices:

$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$, with $i, j, k, l = 1,2,3$.

The sign **:** indicates a tensor product. The C_{ijkl} elements are the 81 components of the elastic moduli tensor or stiffness tensor **C**. In a homogeneous medium the 81 elastic moduli C_{ijkl} are independent of the point considered.

Inversion of the constitutive relation provides the tensor expression:

$\boldsymbol{\varepsilon} = \mathbf{S} : \boldsymbol{\sigma}$, or using index notation:

$\varepsilon_{ij} = S_{ijkl}\sigma_{kl}$.

The S_{ijkl} are the 81 components of the elastic compliance in the compliance tensor **S**.

1.2.2. *Properties of* C_{ijkl} *and* S_{ijkl}

1.2.2.1. *Symmetry with respect to the last two indices*

The stresses are given by:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} = C_{ijlk}\varepsilon_{lk}.$$

As the strain tensor is symmetrical we have:

$\varepsilon_{kl} = \varepsilon_{lk}$, hence $\sigma_{ij} = C_{ijlk}\varepsilon_{kl}$.

By identification, the symmetry property is obtained:

$$C_{ijkl} = C_{ijlk}.$$

1.2.2.2. *Symmetry with respect to the first two indices*

The stresses are given by:

$\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$, or $\sigma_{ji} = C_{jikl}\varepsilon_{kl}$.

As the stress tensor is symmetrical we have:

$\sigma_{ij} = \sigma_{ji}$, hence $\sigma_{ij} = C_{jikl}\varepsilon_{kl}$.

By identification, the symmetry property is obtained:

$$C_{ijkl} = C_{jikl}.$$

1.2.2.3. *Symmetry of the first pair of indices with respect to the second pair*

From the application of the first law of thermodynamics to elasticity, the state function U, the internal energy, is identified with the strain energy W_d.

The elementary strain energy per volume:

$$\sigma_{ij}\, d\varepsilon_{ij},$$

is an exact derivative:

$$\frac{\partial \sigma_{ij}}{\partial \varepsilon_{kl}} = \frac{\partial \sigma_{kl}}{\partial \varepsilon_{ij}},$$

with $\sigma_{ij} = C_{ijkl}\varepsilon_{kl}$ and $\sigma_{kl} = C_{klij}\varepsilon_{ij}$. After introduction in the equality above, we obtain the symmetry property:

$$C_{ijkl} = C_{klij}.$$

Given this property, the elementary volume strain energy:

$$dw_d = \sigma_{ij} d\varepsilon_{ij} = C_{ijkl}\varepsilon_{kl} d\varepsilon_{ij},$$

leads to, by integration, the volume strain energy:

$$w_d = \tfrac{1}{2} C_{ijkl} \varepsilon_{ij} \varepsilon_{kl} = \tfrac{1}{2} \sigma_{ij} \varepsilon_{ij} = \tfrac{1}{2} S_{ijkl} \sigma_{ij} \sigma_{kl} .$$

The compliances have the same symmetry properties as the stiffnesses.

1.3. Conventional two indices Voigt notation

Given the two first symmetry properties:

$$C_{ijkl} = C_{jikl} = C_{ijlk} ,$$

the order of the first two indices (i, j) and the next two indices (k, l) does not affect the modulus of elasticity values. As there are six distinct values for the group (i, j) and six distinct values for the group (k, l), there remain 36 independent elastic moduli.

Given the third symmetry property:

$$C_{ijkl} = C_{klij} ,$$

the permutations of the (i, j) and (k, l) groups do not modify the elastic moduli values. The number of independent values is therefore reduced to 21.

Taking account of the previous remarks, we can propose:

$$C_{ijkl} = C_{IJ} \text{ , with } C_{IJ} = C_{JI} ,$$

where

$$I = i \text{ for } i = j, \; I = 9 - (i + j) \text{ for } i \neq j,$$

and

$$J = k \text{ for } k = l, \; J = 9 - (k + l) \text{ for } k \neq l.$$

The index relationship $\sigma_{ij} = C_{ijkl} \varepsilon_{kl}$ being written in the explicit form:

$$\sigma_{ij} = C_{ij11} \varepsilon_{11} + C_{ij22} \varepsilon_{22} + C_{ij33} \varepsilon_{33} + 2C_{ij23} \varepsilon_{23} + 2C_{ij31} \varepsilon_{31} + 2C_{ij12} \varepsilon_{12} ,$$

the constitutive relation can be written in the matrix form:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & C_{1123} & C_{1131} & C_{1112} \\ C_{1122} & C_{2222} & C_{2233} & C_{2223} & C_{2231} & C_{2212} \\ C_{1133} & C_{2233} & C_{3333} & C_{3323} & C_{3331} & C_{3312} \\ C_{1123} & C_{2223} & C_{3323} & C_{2323} & C_{2331} & C_{2312} \\ C_{1131} & C_{2231} & C_{3331} & C_{2331} & C_{3131} & C_{3112} \\ C_{1112} & C_{2212} & C_{3312} & C_{2312} & C_{3112} & C_{1212} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix} ,$$

or in the index form:

$$\sigma_I = C_{IJ} \varepsilon_J ,$$

with the convention:

$$\sigma_I = \sigma_{ij} \text{ and } I = i \text{ for } i = j,$$
$$\sigma_I = \sigma_{ij} \text{ and } I = 9 - (i + j) \text{ for } i \neq j,$$

and:

$$\varepsilon_J = \varepsilon_{kl} \text{ and } J = k \text{ for } k = l,$$
$$\varepsilon_J = 2\varepsilon_{kl} \text{ and } J = 9 - (k + l) \text{ for } k \neq l.$$

The constitutive relation can also be written in the form:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{12} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{13} & C_{23} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{14} & C_{24} & C_{34} & C_{44} & C_{45} & C_{46} \\ C_{15} & C_{25} & C_{35} & C_{45} & C_{55} & C_{56} \\ C_{16} & C_{26} & C_{36} & C_{46} & C_{56} & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}, \text{ or } \boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}.$$

Similarly we have:

$$\varepsilon_I = S_{IJ}\sigma_J, \text{ or } \boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma}, \text{ with } \mathbf{S} = \mathbf{C}^{-1}.$$

1.4. Anisotropic material

1.4.1. *Monoclinic material*

The monoclinic material studied has the plane $(M|\mathbf{x}_3, \mathbf{x}_1)$ as a plane of mirror symmetry.

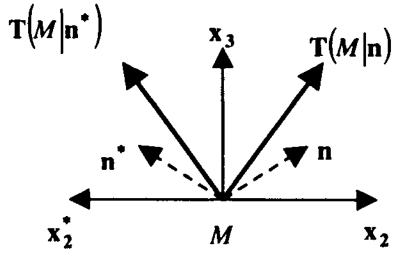

Figure 1.1. *Mirror symmetry axes*

The two axes in $(e) = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ and $(e^*) = (\mathbf{x}_1^*, \mathbf{x}_2^*, \mathbf{x}_3^*) = (\mathbf{x}_1, -\mathbf{x}_2, \mathbf{x}_3)$ are symmetrical about the plane $(M|\mathbf{x}_3, \mathbf{x}_1)$. The two vectors:

$\mathbf{n} = n_1\mathbf{x}_1 + n_2\mathbf{x}_2 + n_3\mathbf{x}_3$, and

$$\mathbf{n}^* = n_1\mathbf{x}_1 - n_2\mathbf{x}_2 + n_3\mathbf{x}_3 = n_1\mathbf{x}_1^* + n_2\mathbf{x}_2^* + n_3\mathbf{x}_3^* = n_1^*\mathbf{x}_1^* + n_2^*\mathbf{x}_2^* + n_3^*\mathbf{x}_3^*,$$

are symmetrical.

The components n_i^* of $\mathbf{n}^*$ in (e^*) are equal to the n_i components of $\mathbf{n}$ in (e). The mirror symmetry property of the two vectors $\mathbf{n}^*$ and $\mathbf{n}$ is written:

$$n_i^* = n_i \; .$$

For the two symmetric stress vectors $\mathbf{T}(M|\mathbf{n}^*)$ and $\mathbf{T}(M|\mathbf{n})$, we have the same:

$$T_i^* = T_i \; .$$

The relation $\mathbf{T}(M|\mathbf{n}) = (\boldsymbol{\sigma}(M))\mathbf{n}$ is written in index form:

$$T_i = \sigma_{ij} n_j \text{ in } (e),$$

and:

$$T_i^* = \sigma_{ij}^* n_j^* \text{ in } (e^*).$$

Given the preceding remarks, the second relation is written:

$$T_i = \sigma_{ij}^* n_j,$$

hence, by identification:

$$\sigma_{ij}^* = \sigma_{ij} \, .$$

The components with the same indices σ_{ij}^* and σ_{ij} of stress tensor in the two axes (e^*) and (e) are equal.

For the strain tensor we have the same:

$$\varepsilon_{ij}^* = \varepsilon_{ij} \, .$$

The constitutive relation $\boldsymbol{\sigma} = \mathbf{C} : \boldsymbol{\varepsilon}$ is written in index form:

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \text{ in the axes } (e),$$

and:

$$\sigma_{ij}^* = C_{ijkl}^*\varepsilon_{kl}^* \text{ in the axes } (e^*).$$

Given the previous properties, the second relation is written:

$$\sigma_{ij} = C_{ijkl}^*\varepsilon_{kl} \, ,$$

hence, by identification:

$$C_{ijkl}^* = C_{ijkl} \, .$$

The components with the same indices C^*_{ijkl} and C_{ijkl} of the stiffness tensor in the two axes (e^*) and (e) are equal.

Designating by **a**, the matrix of change of axes from (e) to (e^*), the expression for the change of axes for a fourth order tensor is written:

$$C^*_{ijkl} = a_{pi}a_{qj}a_{rk}a_{sl}C_{pqrs},$$

and the mirror symmetry property requires:

$$C^*_{ijkl} = C_{ijkl}.$$

The only non-zero components of the change of axes matrix:

$$\mathbf{a} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

are:

$$a_{11} = 1,\ a_{22} = -1,\ a_{33} = 1.$$

The elastic modulus C^*_{1111} is given by the expression for changing axes:

$$C^*_{1111} = a_{p1}a_{q1}a_{r1}a_{s1}C_{pqrs},$$

where the second index of a_{ij} is equal to 1, in the summation for p only $a_{11} = 1$ is not zero, therefore we obtain:

$$C^*_{1111} = C_{1111}.$$

The elastic modulus C^*_{1112} is, with the same axes change, equal to:

$$C^*_{1112} = a_{p1}a_{q1}a_{r1}a_{s2}C_{pqrs},$$

in the summation with a_{s2} only $a_{22} = -1$ is not zero, therefore we have:

$$C^*_{1112} = -C_{1112}.$$

The mirror symmetry property $C^*_{1112} = C_{1112}$ leads to

$$C_{1112} = -C_{1112},$$

hence:

$$C_{1112} = 0.$$

The elastic moduli which possess the index 2 an odd number of times are zero. The stiffness matrix in the monoclinic axes are thus of the form:

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & C_{1131} & 0 \\ C_{1122} & C_{2222} & C_{2233} & 0 & C_{2231} & 0 \\ C_{1133} & C_{2233} & C_{3333} & 0 & C_{3331} & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & C_{2312} \\ C_{1131} & C_{2231} & C_{3331} & 0 & C_{3131} & 0 \\ 0 & 0 & 0 & C_{2312} & 0 & C_{1212} \end{bmatrix}.$$

A monoclinic material is characterised by 13 elastic moduli.

With the two-index notation, for the mirror symmetry planer $(M|\mathbf{x}_3,\mathbf{x}_1)$, the stiffness matrix is written:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ C_{12} & C_{22} & C_{23} & 0 & C_{25} & 0 \\ C_{13} & C_{23} & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & C_{46} \\ C_{15} & C_{25} & C_{35} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & C_{46} & 0 & C_{66} \end{bmatrix}.$$

1.4.2. *Orthotropic material*

The orthotropic material studied has the two planes $(M|\mathbf{x}_3,\mathbf{x}_1)$ and $(M|\mathbf{x}_1,\mathbf{x}_2)$ as planes of mirror symmetry.

The non-zero components of the axes that change the matrix from $(e)=(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ to $(e^*)=(\mathbf{x}_1,\mathbf{x}_2,-\mathbf{x}_3)$:

$$\mathbf{a}=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

are:

$$a_{11}=1,\ a_{22}=1,\ a_{33}=-1.$$

According to the previous results, the elastic moduli with the index 3 an odd number of times are zero.

The stiffness matrix in the orthotropic axes has the form:

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1133} & 0 & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{1133} & C_{2233} & C_{3333} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{3131} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix}.$$

An orthotropic material is characterised by nine elastic moduli.

In addition, it is immediately obvious that the $(M|\mathbf{x}_2,\mathbf{x}_3)$ plane is also a plane of mirror symmetry.

With the two-index notation, the stiffness and compliance matrices are respectively equal to:

$$\mathbf{C} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}$$

and

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{13} & S_{23} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix}.$$

The inversion of the matrix **C** involves the calculation of the inverses of the two matrices:

$$\mathbf{a} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{12} & C_{22} & C_{23} \\ C_{13} & C_{23} & C_{33} \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} C_{44} & 0 & 0 \\ 0 & C_{55} & 0 \\ 0 & 0 & C_{66} \end{bmatrix}.$$

Putting:

$$\Delta = \det \mathbf{a} = C_{11}C_{22}C_{33} + 2C_{12}C_{23}C_{13} - C_{13}^2C_{22} - C_{12}^2C_{33} - C_{23}^2C_{11},$$

we obtain:

$$S_{11} = \frac{C_{22}C_{33} - C_{23}^2}{\Delta}, \quad S_{12} = \frac{C_{13}C_{23} - C_{12}C_{33}}{\Delta}, \quad S_{44} = \frac{1}{C_{44}},$$
$$S_{22} = \frac{C_{11}C_{33} - C_{13}^2}{\Delta}, \quad S_{13} = \frac{C_{12}C_{23} - C_{13}C_{22}}{\Delta}, \quad S_{55} = \frac{1}{C_{55}},$$
$$S_{33} = \frac{C_{11}C_{22} - C_{12}^2}{\Delta}, \quad S_{23} = \frac{C_{12}C_{13} - C_{23}C_{11}}{\Delta}, \quad S_{66} = \frac{1}{C_{66}}.$$

The strains are given as a function of the stresses by the matrix relation:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ S_{21} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{31} & S_{32} & S_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & S_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}.$$

The elastic compliances S_{ij} can be expressed as a function of Young's moduli E_i, the Poisson coefficients ν_{ij} and the shear moduli G_{ij}.

In order to reveal these different values, three simple tensile and three shear tests are proposed.

– In the case of a simple tensile loading in the direction $\mathbf{x}_1$, all the σ_i are zero except σ_1.

The strains ε_i, given by the constitutive relation are equal to:

$$\varepsilon_1 = S_{11}\sigma_1, \varepsilon_2 = S_{21}\sigma_1, \varepsilon_3 = S_{31}\sigma_1, \ \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 0.$$

Young's modulus E_1 in the $\mathbf{x}_1$ direction is defined by the relation:

$$\varepsilon_1 = \frac{\sigma_1}{E_1}.$$

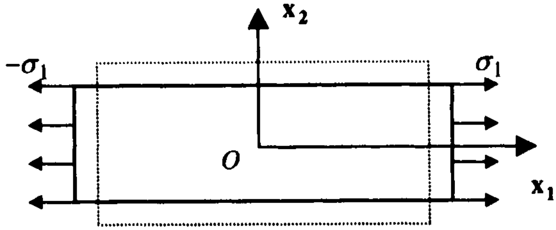

Figure 1.2. *Simple tension*

The Poisson coefficients ν_{12} and ν_{13} are given by:

$$\varepsilon_2 = -\nu_{12}\varepsilon_1 = -\frac{\nu_{12}}{E_1}\sigma_1,$$

and:

$$\varepsilon_3 = -\nu_{13}\varepsilon_1 = -\frac{\nu_{13}}{E_1}\sigma_1.$$

By identification, we obtain:

$$S_{11}=\frac{1}{E_1},\ S_{21}=-\frac{\nu_{12}}{E_1},\ S_{31}=-\frac{\nu_{13}}{E_1}.$$

– In the case of a simple tension loading in the $\mathbf{x}_2$ direction and for a simple tension in the $\mathbf{x}_3$ direction, we obtain:

$$S_{12}=-\frac{\nu_{21}}{E_2},\ S_{22}=\frac{1}{E_2},\ S_{32}=-\frac{\nu_{23}}{E_2} \text{ and } S_{13}=-\frac{\nu_{31}}{E_3},\ S_{23}=-\frac{\nu_{32}}{E_3},\ S_{33}=\frac{1}{E_3}.$$

– In the case of a simple shear in the $\mathbf{x}_2$ and $\mathbf{x}_3$ directions, all the σ_i are zero except σ_4.

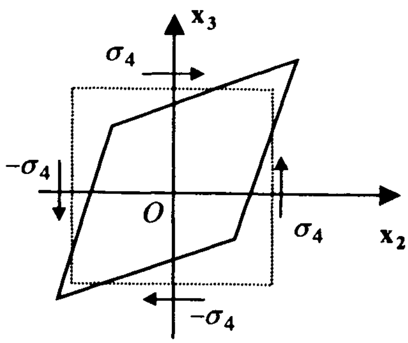

Figure 1.3. *Simple shear*

The strains ε_i given by the constitutive relation are equal to:

$$\varepsilon_4 = S_{44}\sigma_4,$$

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_5 = \varepsilon_6 = 0.$$

The shear modulus G_{23} is defined by:

$$\varepsilon_4 = \frac{\sigma_4}{G_{23}},$$

hence:

$$S_{44} = \frac{1}{G_{23}}.$$

– In the case of simple shear in the $\mathbf{x}_3$ and $\mathbf{x}_1$ directions, and in the $\mathbf{x}_1$ and $\mathbf{x}_2$ directions, we obtain:

$$S_{55} = \frac{1}{G_{31}} \text{ and } S_{66} = \frac{1}{G_{12}}.$$

– The compliance matrix is written in the orthotropic axes in the form:

$$\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix},$$

and from the symmetry properties of the elastic compliances we have the following relation between the Poisson coefficients and Young's moduli:

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \text{ (no summation).}$$

– The stiffness matrix is obtained by inversion of the compliance matrix and is written, in the axes of orthotropy, in the following form:

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix},$$

with:

$$C_{11} = \frac{1-\nu_{23}\nu_{32}}{\Delta}E_1, \quad C_{12} = \frac{\nu_{21}+\nu_{23}\nu_{31}}{\Delta}E_1, \quad C_{13} = \frac{\nu_{31}+\nu_{32}\nu_{21}}{\Delta}E_1,$$
$$C_{21} = \frac{\nu_{12}+\nu_{13}\nu_{32}}{\Delta}E_2, \quad C_{22} = \frac{1-\nu_{31}\nu_{13}}{\Delta}E_2, \quad C_{13} = \frac{\nu_{32}+\nu_{31}\nu_{12}}{\Delta}E_2,$$
$$C_{31} = \frac{\nu_{13}+\nu_{12}\nu_{23}}{\Delta}E_3, \quad C_{32} = \frac{\nu_{23}+\nu_{21}\nu_{13}}{\Delta}E_3, \quad C_{33} = \frac{1-\nu_{12}\nu_{21}}{\Delta}E_3,$$
$$C_{44} = G_{23}, \quad C_{55} = G_{31}, \quad C_{66} = G_{12},$$

and:

$$\Delta = 1 - 2\nu_{12}\nu_{23}\nu_{31} - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13}.$$

1.4.3. *Transversely isotropic material*

The transversely isotropic medium proposed here has the $(M|\mathbf{x}_2,\mathbf{x}_3)$ plane as the isotropic plane.

If the $(M|\mathbf{x}_2,\mathbf{x}_3)$ plane is an isotropic plane, the elastic moduli with the same indices in the two axes $(e)=(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ and $(e^*)=(\mathbf{x}_1^*,\mathbf{x}_2^*,\mathbf{x}_3^*)$ which are defined by the following relations:

$$\mathbf{x}_1^* = \mathbf{x}_1,$$
$$\mathbf{x}_2^* = \cos\alpha\mathbf{x}_2 + \sin\alpha\mathbf{x}_3,$$
$$\mathbf{x}_3^* = -\sin\alpha\mathbf{x}_2 + \cos\alpha\mathbf{x}_3,$$

have the same value whatever the angle α.

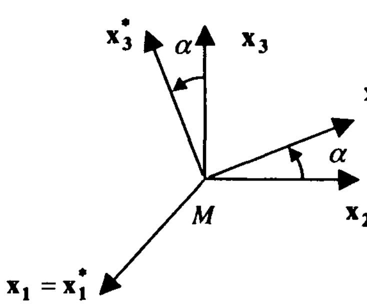

Figure 1.4. *Isotropic plane*

The matrix for changing the axes from (e) to (e^*):

$$\mathbf{a} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix},$$

has the following non-zero components:

$$a_{11} = 1,$$
$$a_{22} = a_{33} = \cos\alpha,$$
$$a_{32} = -a_{23} = \sin\alpha.$$

In the particular case where $\alpha = \frac{\pi}{2}$, the non-zero components of $\mathbf{a}$ are:

$$a_{11} = 1\,,\; a_{32} = 1\,,\; a_{23} = -1\,.$$

The formula for changing the axes $C^*_{ijkl} = a_{pi}a_{qj}a_{rk}a_{sl}C_{pqrs}$ leads to:

$$C^*_{1111} = a_{p1}a_{q1}a_{r1}a_{s1}C_{pqrs} = C_{1111}\,,$$
$$C^*_{2222} = a_{p2}a_{q2}a_{r2}a_{s2}C_{pqrs} = C_{3333}\,,$$
$$C^*_{3333} = a_{p3}a_{q3}a_{r3}a_{s3}C_{pqrs} = C_{2222}\,,$$
$$C^*_{1122} = C_{1133}\,, \qquad C^*_{2323} = C_{3232}\,,$$
$$C^*_{1133} = C_{1122}\,, \qquad C^*_{3131} = C_{2121}\,,$$
$$C^*_{2233} = C_{3322}\,, \qquad C^*_{1212} = C_{1313}\,.$$

The property $C^*_{ijkl} = C_{ijkl}$ provides the relations:

$$C_{2222} = C_{3333}\,,\; C_{1122} = C_{1133}\,,\; C_{3131} = C_{1212}\,.$$

Given these relations, the stiffness matrix:

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ C_{1122} & C_{2222} & C_{2233} & 0 & 0 & 0 \\ C_{1122} & C_{2233} & C_{2222} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1212} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix},$$

involves six independent components. With the two-index notation we have:

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}.$$

For any angle α we have:

$$a_{11} = 1,$$
$$a_{22} = \cos\alpha, \quad a_{23} = -\sin\alpha,$$
$$a_{32} = \sin\alpha, \quad a_{33} = \cos\alpha.$$

In the formula for changing axes $C^*_{2222} = a_{p2}a_{q2}a_{r2}a_{s2}C_{pqrs}$, the only non-zero a_{p2} are:

$$a_{22} = \cos\alpha \text{ and } a_{32} = \sin\alpha,$$

so we obtain:

$$\begin{aligned} C^*_{2222} &= \cos^4\alpha C_{2222} + \cos^3\alpha\sin\alpha\left(C_{2223} + C_{2232} + C_{2322} + C_{3222}\right)\ldots \\ &\ldots + \cos^2\alpha\sin^2\alpha\left(C_{2233} + C_{3322} + C_{2323} + C_{2332} + C_{3223} + C_{3232}\right)\ldots \\ &\ldots + \cos\alpha\sin^3\alpha\left(C_{2333} + C_{3332} + C_{3323} + C_{3233}\right) + \sin^4\alpha C_{3333}. \end{aligned}$$

With $C_{2222} = C_{3333}$, $C_{iiij} = 0$ for $i \neq j$ (no summation), and introducing the property $C^*_{ijkl} = C_{ijkl}$, we obtain:

$$C_{2222} = \cos^4\alpha C_{2222} + 2\cos^2\alpha\sin^2\alpha\left(C_{2233} + 2C_{2323}\right) + \sin^4\alpha C_{2222},$$

or

$$\left(1 - \cos^4\alpha - \sin^4\alpha\right)C_{2222} = 2\cos^2\alpha\sin^2\alpha\left(C_{2233} + 2C_{2323}\right),$$

$$\left[\left(\cos^2\alpha + \sin^2\alpha\right)^2 - \cos^4\alpha - \sin^4\alpha\right]C_{2222} = 2\cos^2\alpha\sin^2\alpha\left(C_{2233} + 2C_{2323}\right),$$

$$2\cos^2\alpha\sin^2\alpha C_{2222} = 2\cos^2\alpha\sin^2\alpha\left(C_{2233} + 2C_{2323}\right).$$

This relation is satisfied whatever the angle α if we have:

$$C_{2222} = C_{2233} + 2C_{2323}.$$

The application of the formula for changing axes to other elastic moduli does not result in new relations.

The stiffness matrix is written in the transverse isotropy axes in the following form:

$$\begin{bmatrix} C_{1111} & C_{1122} & C_{1122} & 0 & 0 & 0 \\ C_{1122} & C_{2233}+2C_{2323} & C_{2233} & 0 & 0 & 0 \\ C_{1122} & C_{2233} & C_{2233}+2C_{2323} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{2323} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{1212} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{1212} \end{bmatrix}.$$

A transversely isotropic medium is characterised by five independent elastic moduli. Certain unidirectional fibre composites can be considered as transversely isotropic materials. With the two-index notation we have:

$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{23}+2C_{44} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{23}+2C_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix}.$$

1.4.4. *Isotropic material*

In an isotropic material all directions are equivalent.

In axes $(e)=(\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$ and $(e^*)=(\mathbf{x}_2,\mathbf{x}_3,\mathbf{x}_1)$ the C^*_{ijkl} and C_{ijkl} components with the same indices have the same numerical value. The only non-zero terms of the change in axes matrix:

$$\mathbf{a}=\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix},$$

are:

$$a_{21}=1,\ a_{32}=1,\ a_{13}=1.$$

The formula for changing axes:

$$C^*_{ijkl}=a_{pi}a_{qj}a_{rk}a_{sl}C_{pqrs},$$

with the property:

$$C^*_{ijkl}=C_{ijkl},$$

providing the relations:

$$C_{1111} = C_{2222}, \quad C_{1122} = C_{2233}, \quad C_{2323} = C_{3131},$$
$$C_{2222} = C_{3333}, \quad C_{1133} = C_{2211}, \quad C_{3131} = C_{1212},$$
$$C_{3333} = C_{1111}, \quad C_{2233} = C_{3311}, \quad C_{1212} = C_{2323},$$

or:

$$C_{1111} = C_{2222} = C_{333},$$
$$C_{1122} = C_{2233} = C_{3311},$$
$$C_{2323} = C_{3131} = C_{1212}.$$

Putting $C_{2233} = \lambda$ and $C_{2323} = \mu$, the relation $C_{2222} = C_{2233} + 2C_{2323}$ gives:

$$C_{2222} = \lambda + 2\mu .$$

In all axes, the stiffness matrix is written in the following form:

$$\begin{bmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}.$$

An isotropic material is characterised by the two elastic moduli λ and μ termed the Lamé coefficient, or by Young's modulus E and by Poisson's coefficient ν. We recall the relations:

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad \mu = G = \frac{E}{2(1+\nu)},$$
$$E = \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, \quad \nu = \frac{\lambda}{2(\lambda+\mu)}.$$

Determination of the compliance matrix involves the following expressions

$$\Delta = (\lambda+2\mu)^3 + 2\lambda^3 - 3\lambda^2(\lambda+2\mu) = 4\mu^2(3\lambda+2\mu),$$
$$S_{11} = \frac{(\lambda+2\mu)^2-\lambda^2}{4\mu^2(3\lambda+2\mu)} = \frac{\lambda+\mu}{\mu(3\lambda+2\mu)} = \frac{1}{E},$$
$$S_{12} = -\frac{\lambda(\lambda+2\mu)-\lambda^2}{4\mu^2(3\lambda+2\mu)} = -\frac{\lambda}{2\mu(3\lambda+2\mu)} = -\frac{\nu}{E},$$
$$S_{44} = \frac{1}{\mu} = 2\frac{1+\nu}{E}.$$

The compliance matrix is written in all axes as:

$$\begin{bmatrix} \frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0 \\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\frac{1+\nu}{E} & 0 & 0 \\ 0 & 0 & 0 & 0 & 2\frac{1+\nu}{E} & 0 \\ 0 & 0 & 0 & 0 & 0 & 2\frac{1+\nu}{E} \end{bmatrix}.$$

1.4.5. *Influence of temperature and humidity on the constitutive relation of an orthotropic material*

The variation of temperature ΔT results in a strain field defined by:

$$\varepsilon_i = \alpha_i \Delta T \quad (i = 1,2,3), \quad \varepsilon_i = 0 \quad (i = 4,5,6),$$

or:

$$\varepsilon_i = \alpha_i \Delta T \quad (i = 1,2,\ldots,6), \text{ with } \alpha_4 = \alpha_5 = \alpha_6 = 0.$$

α_i being the coefficient of thermal expansion in the $\mathbf{x}_1$ direction.

The variation of humidity $\Delta\eta$, equal to the relative variation of the mass of the material, results in a strain field defined by:

$$\varepsilon_i = \beta_i \Delta\eta \quad (i = 1,2,3), \quad \varepsilon_i = 0 \quad (i = 4,5,6),$$

or:

$$\varepsilon_i = \beta_i \Delta\eta \quad (i = 1,2,\ldots,6), \text{ with } \beta_4 = \beta_5 = \beta_6 = 0.$$

β_i being the coefficient of hygroscopic expansion in the $\mathbf{x}_1$ direction.

In general the elastic moduli depend on the temperature and the humidity.

When the temperature and humidity variations are small, the elastic moduli can be assumed to be constant.

Taking into account the strains caused by the stress field due to the variations in temperature and humidity, the constitutive relation is written as:

$$\varepsilon_i = S_{ij}\sigma_j + \alpha_i \Delta T + \beta_i \Delta\eta \quad (i, j = 1,2,3),$$

$$\varepsilon_i = S_{ij}\sigma_j \quad (i, j = 4,5,6),$$

or:

$$\varepsilon_i = S_{ij}\sigma_j + \alpha_i \Delta T + \beta_i \Delta\eta \quad (i, j = 1,2,...,6),$$
$$\alpha_i = 0, \beta_i = 0 \quad (i = 4,5,6).$$

In the orthotropic axes the thermal and hygroscopic effects are revealed as expansions in the $\mathbf{x}_i$ direction, and by the absence of angular distortion in the $\mathbf{x}_i$ and $\mathbf{x}_j$ directions.

1.5. Matrix relations for a change of axes

In the direct orthotropic axes $(\bar{e}) = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3)$, the constitutive relations are written:

$$\bar{\boldsymbol{\sigma}} = \bar{\mathbf{C}}\bar{\boldsymbol{\varepsilon}} \text{ and } \bar{\boldsymbol{\varepsilon}} = \bar{\mathbf{S}}\bar{\boldsymbol{\sigma}} .$$

Whereas in the off-axis co-ordinates $(e) = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$, we have:

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} \text{ and } \boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma} .$$

The matrix for changing axes from $(\bar{e})$ to (e) is represented by $\mathbf{a}$:

$$\mathbf{a} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}.$$

1.5.1. *Change of axes for stress and strain matrices*

The following formula is proposed for the change of axes matrix:

$$\sigma_{ij} = a_{pi} a_{qj} \bar{\sigma}_{pq}$$

in matrix form:

$$\boldsymbol{\sigma} = \mathbf{M}\bar{\boldsymbol{\sigma}} ,$$

where $\mathbf{M}$ is a (6,6) matrix which will be determined.

The explicit expressions:

$$\sigma_{11} = a_{p1} a_{q1} \bar{\sigma}_{pq},$$
$$\sigma_{11} = a_{11}^2 \bar{\sigma}_{11} + a_{11} a_{21} \bar{\sigma}_{12} + a_{11} a_{31} \bar{\sigma}_{13} ...$$
$$... + a_{21} a_{11} \bar{\sigma}_{21} + a_{21}^2 \bar{\sigma}_{22} + a_{21} a_{31} \bar{\sigma}_{23} ...$$
$$... + a_{31} a_{11} \bar{\sigma}_{31} + a_{31} a_{21} \bar{\sigma}_{32} + a_{31}^2 \bar{\sigma}_{33},$$

and:

$$\sigma_{23} = a_{p2} a_{q3} \bar{\sigma}_{pq},$$

$$\sigma_{23} = a_{12}a_{13}\bar{\sigma}_{11} + a_{12}a_{23}\bar{\sigma}_{12} + a_{12}a_{33}\bar{\sigma}_{13}\ldots$$
$$\ldots + a_{22}a_{13}\bar{\sigma}_{21} + a_{22}a_{23}\bar{\sigma}_{22} + a_{22}a_{33}\bar{\sigma}_{23}\ldots$$
$$\ldots + a_{32}a_{13}\bar{\sigma}_{31} + a_{32}a_{23}\bar{\sigma}_{32} + a_{32}a_{33}\bar{\sigma}_{33},$$

enable the change of axes formula to be written in the matrix form:

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{23} \\ \sigma_{31} \\ \sigma_{12} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{21}^2 & a_{31}^2 & 2a_{21}a_{31} \\ a_{12}^2 & a_{22}^2 & a_{32}^2 & 2a_{22}a_{32} \\ a_{13}^2 & a_{23}^2 & a_{33}^2 & 2a_{23}a_{33} \\ a_{12}a_{13} & a_{22}a_{23} & a_{32}a_{33} & a_{22}a_{33} + a_{32}a_{23} \\ a_{13}a_{11} & a_{23}a_{21} & a_{33}a_{31} & a_{23}a_{31} + a_{33}a_{21} \\ a_{11}a_{12} & a_{21}a_{22} & a_{31}a_{32} & a_{21}a_{32} + a_{31}a_{22} \end{bmatrix} \ldots$$

$$\ldots \begin{bmatrix} 2a_{31}a_{11} & 2a_{11}a_{21} \\ 2a_{32}a_{12} & 2a_{12}a_{22} \\ 2a_{33}a_{13} & 2a_{13}a_{23} \\ a_{32}a_{13} + a_{12}a_{33} & a_{12}a_{23} + a_{22}a_{13} \\ a_{33}a_{11} + a_{13}a_{31} & a_{13}a_{21} + a_{23}a_{11} \\ a_{31}a_{12} + a_{11}a_{32} & a_{11}a_{22} + a_{21}a_{12} \end{bmatrix} \begin{bmatrix} \bar{\sigma}_{11} \\ \bar{\sigma}_{22} \\ \bar{\sigma}_{33} \\ \bar{\sigma}_{23} \\ \bar{\sigma}_{31} \\ \bar{\sigma}_{12} \end{bmatrix}.$$

The matrix **M** is made up of four sub-matrices (3,3) whose components can easily be found by writing the transposed change of axes matrix:

$$\mathbf{a}^T = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}.$$

Decomposing the matrix **M** in four sub-matrices (3,3):

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{11} & \mathbf{M}_{12} \\ \mathbf{M}_{21} & \mathbf{M}_{22} \end{bmatrix},$$

the following rules can be proposed:

– The (i, j) component of the sub-matrix $\mathbf{M}_{11}$ is equal to the square of the (i, j) component of $\mathbf{a}^T$.

– The (i, j) component of the sub-matrix $\mathbf{M}_{12}$ is equal to twice the product of the two other terms of the row i of $\mathbf{a}^T$.

– The (i, j) component of the sub-matrix $\mathbf{M}_{21}$ is equal to the product of the two other terms of the column j of $\mathbf{a}^T$.

– The (i,j) component of the sub-matrix $\mathbf{M}_{22}$ is equal to the sum of the cross products of the terms of the matrix obtained by removing row i and column j of $\mathbf{a}^T$.

In a similar manner we can develop the formula for changing the axes of the strains:

$$\varepsilon_{ij} = a_{pi}a_{qj}\bar{\varepsilon}_{pq},$$

in matrix form:

$$\boldsymbol{\varepsilon} = \mathbf{N}\bar{\boldsymbol{\varepsilon}},$$

with $\mathbf{N}$ being a (6,6) matrix which will be determined.

The explicit expressions:

$$\varepsilon_{11} = a_{p1}a_{q1}\bar{\varepsilon}_{pq},$$

$$\varepsilon_{11} = a_{11}^2\bar{\varepsilon}_{11} + a_{11}a_{21}\bar{\varepsilon}_{12} + a_{11}a_{31}\bar{\varepsilon}_{13}\ldots$$
$$\ldots + a_{21}a_{11}\bar{\varepsilon}_{21} + a_{21}^2\bar{\varepsilon}_{22} + a_{21}a_{31}\bar{\varepsilon}_{23}\ldots$$
$$\ldots + a_{31}a_{11}\bar{\varepsilon}_{31} + a_{31}a_{21}\bar{\varepsilon}_{32} + a_{31}^2\bar{\varepsilon}_{33},$$

and:

$$\varepsilon_{23} = a_{p2}a_{q3}\bar{\varepsilon}_{pq},$$

$$\varepsilon_{23} = a_{12}a_{13}\bar{\varepsilon}_{11} + a_{12}a_{23}\bar{\varepsilon}_{12} + a_{12}a_{33}\bar{\varepsilon}_{13}\ldots$$
$$\ldots + a_{22}a_{13}\bar{\varepsilon}_{21} + a_{22}a_{23}\bar{\varepsilon}_{22} + a_{22}a_{33}\bar{\varepsilon}_{23}\ldots$$
$$\ldots + a_{32}a_{13}\bar{\varepsilon}_{31} + a_{32}a_{23}\bar{\varepsilon}_{32} + a_{32}a_{33}\bar{\varepsilon}_{33},$$

enable the formula for changing axes to be put in the form:

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \\ 2\varepsilon_{12} \end{bmatrix} = \begin{bmatrix} a_{11}^2 & a_{21}^2 & a_{31}^2 & a_{21}a_{31} & a_{31}a_{11} & a_{11}a_{21} \\ a_{12}^2 & a_{22}^2 & a_{32}^2 & a_{22}a_{32} & a_{32}a_{12} & a_{12}a_{22} \\ a_{13}^2 & a_{23}^2 & a_{33}^2 & a_{23}a_{33} & a_{33}a_{13} & a_{13}a_{23} \\ 2a_{12}a_{13} & 2a_{22}a_{23} & 2a_{32}a_{33} & a_{22}a_{33}+a_{32}a_{23} & a_{32}a_{13}+a_{12}a_{33} & a_{12}a_{23}+a_{22}a_{13} \\ 2a_{13}a_{11} & 2a_{23}a_{21} & 2a_{33}a_{31} & a_{23}a_{31}+a_{33}a_{21} & a_{33}a_{11}+a_{13}a_{31} & a_{13}a_{21}+a_{23}a_{11} \\ 2a_{11}a_{12} & 2a_{21}a_{22} & 2a_{31}a_{32} & a_{21}a_{32}+a_{31}a_{22} & a_{31}a_{12}+a_{11}a_{32} & a_{11}a_{22}+a_{21}a_{12} \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_{11} \\ \bar{\varepsilon}_{22} \\ \bar{\varepsilon}_{33} \\ 2\bar{\varepsilon}_{23} \\ 2\bar{\varepsilon}_{31} \\ 2\bar{\varepsilon}_{12} \end{bmatrix}.$$

As for the matrix **M**, the matrix:

$$\mathbf{N} = \begin{bmatrix} \mathbf{N}_{11} & \mathbf{N}_{12} \\ \mathbf{N}_{21} & \mathbf{N}_{22} \end{bmatrix},$$

consists of four sub-matrices (3,3) whose components are found as follows:

– The component (i,j) of the sub-matrix $\mathbf{N}_{11}$ is equal to the square of the component (i,j) of $\mathbf{a}^T$.

– The component (i,j) of the sub-matrix $\mathbf{N}_{12}$ is equal to the product of the two other terms of the row i of $\mathbf{a}^T$.

– The component (i,j) of the sub-matrix $\mathbf{N}_{21}$ is equal to twice the product of the two other terms of the column j of $\mathbf{a}^T$.

– The component (i,j) of the sub-matrix $\mathbf{N}_{22}$ is equal to the sum of the cross products of the terms of the matrix obtained by removing row i and column j of $\mathbf{a}^T$.

From the matrix **a**, we construct the matrix **M** such that $\boldsymbol{\sigma} = \mathbf{M}\overline{\boldsymbol{\sigma}}$. Similarly from $\mathbf{a}^{-1}$, we can construct, in an analogous manner, the matrix $\mathbf{M}^{-1}$ such that:

$$\overline{\boldsymbol{\sigma}} = \mathbf{M}^{-1}\boldsymbol{\sigma}.$$

The matrix $\mathbf{a}^{-1} = \mathbf{a}^T$ is obtained by exchanging the indices i and j of the orthogonal matrix **a**. The matrix $\mathbf{M}^{-1}$ is obtained by exchanging the indices i and j of the terms a_{ij} which are involved in the matrix **M**, which results in:

$$\mathbf{M}^{-1} = \begin{bmatrix} a_{11}^2 & a_{12}^2 & a_{13}^2 & 2a_{12}a_{13} & 2a_{13}a_{11} & 2a_{11}a_{12} \\ a_{21}^2 & a_{22}^2 & a_{23}^2 & 2a_{22}a_{23} & 2a_{23}a_{21} & 2a_{21}a_{22} \\ a_{31}^2 & a_{32}^2 & a_{33}^2 & 2a_{32}a_{33} & 2a_{33}a_{31} & 2a_{31}a_{32} \\ a_{21}a_{31} & a_{22}a_{32} & a_{23}a_{33} & a_{22}a_{33}+a_{23}a_{32} & a_{23}a_{31}+a_{21}a_{33} & a_{21}a_{32}+a_{22}a_{31} \\ a_{31}a_{11} & a_{32}a_{12} & a_{33}a_{13} & a_{32}a_{13}+a_{33}a_{12} & a_{33}a_{11}+a_{31}a_{13} & a_{31}a_{12}+a_{32}a_{11} \\ a_{11}a_{21} & a_{12}a_{22} & a_{13}a_{23} & a_{12}a_{23}+a_{13}a_{22} & a_{13}a_{21}+a_{11}a_{23} & a_{11}a_{22}+a_{12}a_{21} \end{bmatrix}.$$

The matrix thus obtained is the transposed matrix **N**. We therefore have the relation:

$$\mathbf{M}^{-1} = \mathbf{N}^T.$$

An analogous calculation, performed on the **N** matrix gives the equation:

$$\mathbf{N}^{-1} = \mathbf{M}^T.$$

1.5.2. *Change of axes for stiffness and compliance matrices*

We have just obtained the following relations:

$$\boldsymbol{\sigma} = \mathbf{M}\overline{\boldsymbol{\sigma}} \quad [1], \quad \overline{\boldsymbol{\sigma}} = \mathbf{N}^T\boldsymbol{\sigma} \quad [2],$$

$$\boldsymbol{\varepsilon} = \mathbf{N}\overline{\boldsymbol{\varepsilon}} \quad [3], \quad \overline{\boldsymbol{\varepsilon}} = \mathbf{M}^T\boldsymbol{\varepsilon} \quad [4].$$

In addition, the constitutive relations in the othotropic axes and off-axis are written:

$$\overline{\boldsymbol{\sigma}} = \overline{\mathbf{C}}\overline{\boldsymbol{\varepsilon}} \quad [5], \quad \overline{\boldsymbol{\varepsilon}} = \overline{\mathbf{S}}\overline{\boldsymbol{\sigma}} \quad [7],$$

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} \quad [6], \quad \boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma} \quad [8].$$

From relations [1], [5] and [4], we obtain:

$$\boldsymbol{\sigma} = \mathbf{M}\overline{\boldsymbol{\sigma}} = \mathbf{M}\overline{\mathbf{C}}\overline{\boldsymbol{\varepsilon}} = \mathbf{M}\overline{\mathbf{C}}\mathbf{M}^T\boldsymbol{\varepsilon}.$$

Identification with [6] gives:

$$\mathbf{C} = \mathbf{M}\overline{\mathbf{C}}\mathbf{M}^T.$$

From relations [3], [7] and [2], we obtain in a similar way:

$$\boldsymbol{\varepsilon} = \mathbf{N}\overline{\boldsymbol{\varepsilon}} = \mathbf{N}\overline{\mathbf{S}}\overline{\boldsymbol{\sigma}} = \mathbf{N}\overline{\mathbf{S}}\mathbf{N}^T\boldsymbol{\sigma}.$$

Identification with [8] leads to:

$$\mathbf{S} = \mathbf{N}\overline{\mathbf{S}}\mathbf{N}^T.$$

Multiplication of the left hand part of $\mathbf{C} = \mathbf{M}\overline{\mathbf{C}}\mathbf{M}^T$ by $\mathbf{N}^T$ and the right hand part by **N** gives:

$$\mathbf{N}^T\mathbf{C}\mathbf{N} = \mathbf{N}^T\mathbf{M}\overline{\mathbf{C}}\mathbf{M}^T\mathbf{N},$$

or:

$$\overline{\mathbf{C}} = \mathbf{N}^T\mathbf{C}\mathbf{N}.$$

Multiplication of the members of $\mathbf{S} = \mathbf{N}\overline{\mathbf{S}}\mathbf{N}^T$ by $\mathbf{M}^T$ on the left and by **M** on the right gives the equation:

$$\mathbf{M}^T\mathbf{S}\mathbf{M} = \mathbf{M}^T\mathbf{N}\overline{\mathbf{S}}\mathbf{N}^T\mathbf{M},$$

hence:

$$\overline{\mathbf{S}} = \mathbf{M}^T\mathbf{S}\mathbf{M}.$$

Chapter 2

Orthotropic layer behaviour

2.1. Introduction

The expressions developed in the previous chapter for the change of axes will now be applied to an orthotropic ply of any orientation in a composite. In the plate theories which will be presented the assumption is made that the normal stress in the $\mathbf{x}_3$ direction is zero. This hypothesis leads to the introduction of column matrices of stresses and strains with only five terms. From these we will deduce the formulae for changing co-ordinates and the constitutive relations which will be used in the plate theories.

2.2. Stiffness and compliance matrices in orthotropic co-ordinates

The orthoptropic layer studied here has the co-ordinate axes $(O|\mathbf{X}_1, \mathbf{X}_2, \mathbf{x}_3)$. The layer of thickness h is limited by the two planes perpendicular to $(O|\mathbf{x}_3)$ which are defined by $x_3 = \frac{h}{2}$ and $x_3 = -\frac{h}{2}$.

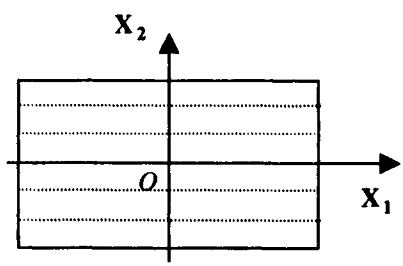

Figure 2.1. *Orthotropic axes*

In the following, all the quantities defined in the orthotropic axes will be overscored.

In the orthotropic axes $(\bar{e}) = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{x}_3)$ the constitutive relation is expressed as:

$$\bar{\boldsymbol{\varepsilon}} = \bar{\mathbf{S}} : \bar{\boldsymbol{\sigma}},$$

and in the matrix form as:

$$\bar{\boldsymbol{\varepsilon}} = \bar{\mathbf{S}}\bar{\boldsymbol{\sigma}},$$

or:

$$\begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \\ \bar{\varepsilon}_6 \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{13} & 0 & 0 & 0 \\ \bar{S}_{12} & \bar{S}_{22} & \bar{S}_{23} & 0 & 0 & 0 \\ \bar{S}_{13} & \bar{S}_{23} & \bar{S}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{S}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{S}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_3 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \\ \bar{\sigma}_6 \end{bmatrix}.$$

The compliance matrix can be written in terms of Young's moduli E_i, the Poisson coefficients ν_{ij} and the shear moduli G_{ij} in the form:

$$\bar{\mathbf{S}} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & 0 \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{G_{23}} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{G_{31}} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G_{12}} \end{bmatrix},$$

with:

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j} \quad \text{(no summation).}$$

If the plane $(\mathbf{X}_2, \mathbf{x}_3)$ is a plane of isotropy, then:

$$\bar{S}_{22} = \bar{S}_{33} = \bar{S}_{23} + \frac{1}{2}\bar{S}_{44}, \quad \bar{S}_{13} = \bar{S}_{12}, \quad \bar{S}_{66} = \bar{S}_{55},$$

or:

$$\frac{1}{E_2} = \frac{1}{E_3} = \frac{1}{2G_{23}} - \frac{\nu_{32}}{E_3}, \quad \nu_{21} = \nu_{31}, \quad \frac{1}{G_{12}} = \frac{1}{G_{31}}.$$

A unidirectional layer in which the fibres are parallel to $\mathbf{X}_1$ can, for a first approximation, be considered as transversely isotropic.

The relation $\bar{\boldsymbol{\varepsilon}} = \bar{\mathbf{S}}\bar{\boldsymbol{\sigma}}$ is then written as:

$$\begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \\ \bar{\varepsilon}_6 \end{bmatrix} = \begin{bmatrix} \bar{S}_{11} & \bar{S}_{12} & \bar{S}_{12} & 0 & 0 & 0 \\ \bar{S}_{12} & \bar{S}_{23} + \frac{1}{2}\bar{S}_{44} & \bar{S}_{23} & 0 & 0 & 0 \\ \bar{S}_{12} & \bar{S}_{23} & \bar{S}_{23} + \frac{1}{2}\bar{S}_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{S}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{S}_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{S}_{66} \end{bmatrix} \begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_3 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \\ \bar{\sigma}_6 \end{bmatrix}.$$

Similarly in the orthotropic axes $(\bar{e})$ the constitutive relation:

$$\bar{\boldsymbol{\sigma}} = \bar{\mathbf{C}} : \bar{\boldsymbol{\varepsilon}},$$

can be written in matrix form as:

$$\bar{\boldsymbol{\sigma}} = \bar{\mathbf{C}}\bar{\boldsymbol{\varepsilon}},$$

or:

$$\begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_3 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \\ \bar{\sigma}_6 \end{bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & 0 \\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & 0 \\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{C}_{66} \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \\ \bar{\varepsilon}_6 \end{bmatrix}.$$

In the particular case of the transversely isotropic unidirectional layer we then have:

$$\begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_3 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \\ \bar{\sigma}_6 \end{bmatrix} = \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{12} & 0 & 0 & 0 \\ \bar{C}_{12} & \bar{C}_{23} + 2\bar{C}_{44} & \bar{C}_{23} & 0 & 0 & 0 \\ \bar{C}_{12} & \bar{C}_{23} & \bar{C}_{23} + 2\bar{C}_{44} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{C}_{66} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{C}_{66} \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \\ \bar{\varepsilon}_6 \end{bmatrix}.$$

2.3. Conventional matrices for changing axes

The angle α between the vector $\mathbf{x}_1$ of the off-axes $(e) = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3)$ and the vector $\mathbf{X}_1$ of the orthotropic axes $(\bar{e}) = (\mathbf{X}_1, \mathbf{X}_2, \mathbf{x}_3)$, is measured on $\mathbf{x}_3$.

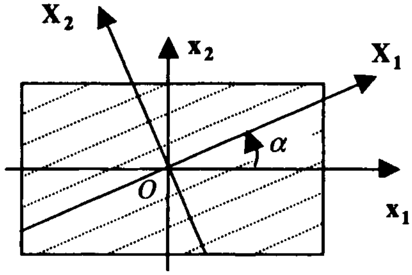

Figure 2.2. *Off-axis directions*

The matrix which enables us to go from the orthotropic directions $(\bar{e})$ to the off-axis directions (e):

$$\mathbf{a} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

is written in the form:

$$\mathbf{a} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

with:

$$c = \cos\alpha \text{ and } s = \sin\alpha .$$

From the transposed matrix of the axis change

$$\mathbf{a}^T = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

we obtain the following two matrices **M** and **N**:

$$\mathbf{M} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -2cs \\ s^2 & c^2 & 0 & 0 & 0 & 2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ cs & -cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}, \mathbf{N} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -cs \\ s^2 & c^2 & 0 & 0 & 0 & cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 2cs & -2cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}.$$

2.4. Stress and strain matrices

The formulae for changing the directions for $\boldsymbol{\sigma} = \mathbf{M}\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\sigma}} = \mathbf{N}^{\mathrm{T}}\boldsymbol{\sigma}$ can be written in the following forms:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -2cs \\ s^2 & c^2 & 0 & 0 & 0 & 2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ cs & -cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_3 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \\ \bar{\sigma}_6 \end{bmatrix},$$

and:

$$\begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_3 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \\ \bar{\sigma}_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & 2cs \\ s^2 & c^2 & 0 & 0 & 0 & -2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -cs & cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix},$$

similarly, the formulae for changing the strain axes for strains $\boldsymbol{\varepsilon} = \mathbf{N}\bar{\boldsymbol{\varepsilon}}$ and $\bar{\boldsymbol{\varepsilon}} = \mathbf{M}^T\boldsymbol{\varepsilon}$ can be written:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -cs \\ s^2 & c^2 & 0 & 0 & 0 & cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 2cs & -2cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \\ \bar{\varepsilon}_6 \end{bmatrix},$$

and:

$$\begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \\ \bar{\varepsilon}_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & cs \\ s^2 & c^2 & 0 & 0 & 0 & -cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -2cs & 2cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix}.$$

In the present case it may be noted that we go from **M** to its inverse $\mathbf{N}^T$ by replacing the angle α by $-\alpha$ in **M**. One can use the same transformation to go from **N** to its inverse $\mathbf{M}^T$.

2.5. Stiffness matrix in directions away from the orthotropic axes

The stiffness matrix **C** in the (e) space is given by the relationship:

$$\mathbf{C} = \mathbf{M}\overline{\mathbf{C}}\mathbf{M}^{\mathrm{T}},$$

which involves the matrix:

$$\mathbf{M} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -2cs \\ s^2 & c^2 & 0 & 0 & 0 & 2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ cs & -cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}.$$

The product of the latter two matrices is thus:

$$\overline{\mathbf{C}}\mathbf{M}^T = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & 0 & 0 & 0 \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{23} & 0 & 0 & 0 \\ \overline{C}_{13} & \overline{C}_{23} & \overline{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{C}_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & cs \\ s^2 & c^2 & 0 & 0 & 0 & -cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -2cs & 2cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix},$$

that is:

$$\overline{\mathbf{C}}\mathbf{M}^T = \begin{bmatrix} c^2\overline{C}_{11} + s^2\overline{C}_{12} & s^2\overline{C}_{11} + c^2\overline{C}_{12} & \overline{C}_{13} & 0 & 0 & cs\left(\overline{C}_{11} - \overline{C}_{12}\right) \\ c^2\overline{C}_{12} + s^2\overline{C}_{22} & s^2\overline{C}_{12} + c^2\overline{C}_{22} & \overline{C}_{23} & 0 & 0 & cs\left(\overline{C}_{12} - \overline{C}_{22}\right) \\ c^2\overline{C}_{13} + s^2\overline{C}_{23} & s^2\overline{C}_{13} + c^2\overline{C}_{23} & \overline{C}_{33} & 0 & 0 & cs\left(\overline{C}_{13} - \overline{C}_{23}\right) \\ 0 & 0 & 0 & c\overline{C}_{44} & -s\overline{C}_{44} & 0 \\ 0 & 0 & 0 & s\overline{C}_{55} & c\overline{C}_{55} & 0 \\ -2cs\overline{C}_{66} & 2cs\overline{C}_{66} & 0 & 0 & 0 & \left(c^2 - s^2\right)\overline{C}_{66} \end{bmatrix}.$$

After multiplying the left-hand side by **M** and identification, one obtains:

– the components of the first column of **C**:

$$
\begin{aligned}
C_{11} &= c^4\overline{C}_{11} + s^4\overline{C}_{22} + 2c^2s^2\left(\overline{C}_{12} + 2\overline{C}_{66}\right),\\
C_{21} &= c^2s^2\left(\overline{C}_{11} + \overline{C}_{22} - 4\overline{C}_{66}\right) + \left(c^4 + s^4\right)\overline{C}_{12},\\
C_{31} &= c^2\overline{C}_{13} + s^2\overline{C}_{23},\\
C_{41} &= 0,\\
C_{51} &= 0,\\
C_{61} &= cs\left[c^2\overline{C}_{11} - s^2\overline{C}_{22} - \left(c^2 - s^2\right)\left(\overline{C}_{12} + 2\overline{C}_{66}\right)\right],
\end{aligned}
$$

– the components of the second column of **C**:

$$
\begin{aligned}
C_{12} &= c^2s^2\left(\overline{C}_{11} + \overline{C}_{22} - 4\overline{C}_{66}\right) + \left(c^4 + s^4\right)\overline{C}_{12} = C_{21},\\
C_{22} &= s^4\overline{C}_{11} + c^4\overline{C}_{22} + 2c^2s^2\left(\overline{C}_{12} + 2\overline{C}_{66}\right),\\
C_{32} &= s^2\overline{C}_{13} + c^2\overline{C}_{23},\\
C_{42} &= 0,\\
C_{52} &= 0,\\
C_{62} &= cs\left[s^2\overline{C}_{11} - c^2\overline{C}_{22} + \left(c^2 - s^2\right)\left(\overline{C}_{12} + 2\overline{C}_{66}\right)\right],
\end{aligned}
$$

– the components of the third column of **C**:

$$
\begin{aligned}
C_{13} &= c^2\overline{C}_{13} + s^2\overline{C}_{23} = C_{31},\\
C_{23} &= s^2\overline{C}_{13} + c^2\overline{C}_{23} = C_{32},\\
C_{33} &= \overline{C}_{33},\\
C_{43} &= 0,\\
C_{53} &= 0,\\
C_{63} &= cs\left(\overline{C}_{13} - \overline{C}_{23}\right),
\end{aligned}
$$

– the components of the fourth column of **C**:

$$
\begin{aligned}
C_{14} &= 0,\\
C_{24} &= 0,\\
C_{34} &= 0,\\
C_{44} &= c^2\overline{C}_{44} + s^2\overline{C}_{55},\\
C_{54} &= cs\left(\overline{C}_{55} - \overline{C}_{44}\right),\\
C_{64} &= 0,
\end{aligned}
$$

– the components of the fifth column of **C**:

$$
\begin{aligned}
C_{15} &= 0,\\
C_{25} &= 0,\\
C_{35} &= 0,\\
C_{45} &= cs\left(\overline{C}_{55} - \overline{C}_{44}\right) = C_{54},\\
C_{55} &= s^2\overline{C}_{44} + c^2\overline{C}_{55},\\
C_{65} &= 0,
\end{aligned}
$$

– the components of the sixth column of **C**:

$$
\begin{aligned}
C_{16} &= cs\left[c^2\overline{C}_{11} - s^2\overline{C}_{22} - \left(c^2 - s^2\right)\left(\overline{C}_{12} + 2\overline{C}_{66}\right)\right] = C_{61},\\
C_{26} &= cs\left[s^2\overline{C}_{11} - c^2\overline{C}_{22} + \left(c^2 - s^2\right)\left(\overline{C}_{12} + 2\overline{C}_{66}\right)\right] = C_{62},\\
C_{36} &= cs\left(\overline{C}_{13} - \overline{C}_{23}\right) = C_{63},\\
C_{46} &= 0,\\
C_{56} &= 0,\\
C_{66} &= c^2s^2\left(\overline{C}_{11} + \overline{C}_{22} - 2\overline{C}_{12}\right) + \left(c^2 - s^2\right)^2\overline{C}_{66}.
\end{aligned}
$$

The constitutive relation $\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon}$ can then be written as:

$$
\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix}C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26}\\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36}\\ 0 & 0 & 0 & C_{44} & C_{45} & 0\\ 0 & 0 & 0 & C_{45} & C_{55} & 0\\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix}.
$$

Note that for $\alpha = 0$ or $\alpha = \frac{\pi}{2}$, the coefficients C_{16}, C_{26}, C_{36} and C_{45} are equal to zero (orthotropic materials whose directions of orthotropy coincide with the coordinate axes).

2.6. Compliance matrix in directions away from the orthotropic axes

The compliance matrix **S** in (e) is given by the relationship:

$$\mathbf{S} = \mathbf{N}\overline{\mathbf{S}}\mathbf{N}^{\mathrm{T}},$$

which involves the matrix:

$$\mathbf{N} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -cs \\ s^2 & c^2 & 0 & 0 & 0 & cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 2cs & -2cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}.$$

The product of these two matrices is:

$$\overline{\mathbf{S}}\mathbf{N}^T = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{13} & 0 & 0 & 0 \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{23} & 0 & 0 & 0 \\ \overline{S}_{13} & \overline{S}_{23} & \overline{S}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{S}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{S}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{S}_{66} \end{bmatrix} \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & 2cs \\ s^2 & c^2 & 0 & 0 & 0 & -2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -cs & cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix},$$

that is:

$$\overline{\mathbf{S}}\mathbf{N}^T = \begin{bmatrix} c^2\overline{S}_{11} + s^2\overline{S}_{12} & s^2\overline{S}_{11} + c^2\overline{S}_{12} & \overline{S}_{13} & 0 & 0 & 2cs\left(\overline{S}_{11} - \overline{S}_{12}\right) \\ c^2\overline{S}_{12} + s^2\overline{S}_{22} & s^2\overline{S}_{12} + c^2\overline{S}_{22} & \overline{S}_{23} & 0 & 0 & 2cs\left(\overline{S}_{12} - \overline{S}_{22}\right) \\ c^2\overline{S}_{13} + s^2\overline{S}_{23} & s^2\overline{S}_{13} + c^2\overline{S}_{23} & \overline{S}_{33} & 0 & 0 & 2cs\left(\overline{S}_{13} - \overline{S}_{23}\right) \\ 0 & 0 & 0 & c\overline{S}_{44} & -s\overline{S}_{44} & 0 \\ 0 & 0 & 0 & s\overline{S}_{55} & c\overline{S}_{55} & 0 \\ -cs\overline{S}_{66} & cs\overline{S}_{66} & 0 & 0 & 0 & \left(c^2 - s^2\right)\overline{S}_{66} \end{bmatrix}.$$

After multiplication of the left-hand side by **N** and identification, one obtains:

– the components of the first column of **S**:

$$\begin{aligned} S_{11} &= c^4\overline{S}_{11} + s^4\overline{S}_{22} + c^2s^2\left(2\overline{S}_{12} + \overline{S}_{66}\right), \\ S_{21} &= c^2s^2\left(\overline{S}_{11} + \overline{S}_{22} - \overline{S}_{66}\right) + \left(c^4 + s^4\right)\overline{S}_{12}, \\ S_{31} &= c^2\overline{S}_{13} + s^2\overline{S}_{23}, \\ S_{41} &= 0, \\ S_{51} &= 0, \\ S_{61} &= cs\left[2c^2\overline{S}_{11} - 2s^2\overline{S}_{22} - \left(c^2 - s^2\right)\left(2\overline{S}_{12} + \overline{S}_{66}\right)\right], \end{aligned}$$

– the components of the second column of **S**:

$$
\begin{aligned}
S_{12} &= c^2s^2\left(\bar{S}_{11}+\bar{S}_{22}-\bar{S}_{66}\right)+\left(c^4+s^4\right)\bar{S}_{12}=S_{21},\\
S_{22} &= s^4\bar{S}_{11}+c^4\bar{S}_{22}+c^2s^2\left(2\bar{S}_{12}+\bar{S}_{66}\right),\\
S_{32} &= s^2\bar{S}_{13}+c^2\bar{S}_{23},\\
S_{42} &= 0,\\
S_{52} &= 0,\\
S_{62} &= cs\left[2s^2\bar{S}_{11}-2c^2\bar{S}_{22}+\left(c^2-s^2\right)\left(2\bar{S}_{12}+\bar{S}_{66}\right)\right],
\end{aligned}
$$

– the components of the third column of **S**:

$$
\begin{aligned}
S_{13} &= c^2\bar{S}_{13}+s^2\bar{S}_{23}=S_{31},\\
S_{23} &= s^2\bar{S}_{13}+c^2\bar{S}_{23}=S_{32},\\
S_{33} &= \bar{S}_{33},\\
S_{43} &= 0,\\
S_{53} &= 0,\\
S_{63} &= 2cs\left(\bar{S}_{13}-\bar{S}_{23}\right),
\end{aligned}
$$

– the components of the fourth column of **S**:

$$
\begin{aligned}
S_{14} &= 0,\\
S_{24} &= 0,\\
S_{34} &= 0,\\
S_{44} &= c^2\bar{S}_{44}+s^2\bar{S}_{55},\\
S_{54} &= cs\left(\bar{S}_{55}-\bar{S}_{44}\right),\\
S_{64} &= 0,
\end{aligned}
$$

– the components of the fifth column of **S**:

$$
\begin{aligned}
S_{15} &= 0,\\
S_{25} &= 0,\\
S_{35} &= 0,\\
S_{45} &= cs\left(\bar{S}_{55}-\bar{S}_{44}\right)=S_{54},\\
S_{55} &= s^2\bar{S}_{44}+c^2\bar{S}_{55},\\
S_{65} &= 0,
\end{aligned}
$$

– the components of the sixth column of **S**:

$$S_{16} = cs\left[2c^2\overline{S}_{11} - 2s^2\overline{S}_{22} - \left(c^2 - s^2\right)\left(2\overline{S}_{12} + \overline{S}_{66}\right)\right] = S_{61},$$
$$S_{26} = cs\left[2s^2\overline{S}_{11} - 2c^2\overline{S}_{22} + \left(c^2 - s^2\right)\left(2\overline{S}_{12} + \overline{S}_{66}\right)\right] = S_{62},$$
$$S_{36} = 2cs\left(\overline{S}_{13} - \overline{S}_{23}\right) = S_{63},$$
$$S_{46} = 0,$$
$$S_{56} = 0,$$
$$S_{66} = 4c^2s^2\left(\overline{S}_{11} + \overline{S}_{22} - 2\overline{S}_{12}\right) + \left(c^2 - s^2\right)^2\overline{S}_{66}.$$

The constitutive relation $\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma}$ can then be written as:

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16}\\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26}\\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36}\\ 0 & 0 & 0 & S_{44} & S_{45} & 0\\ 0 & 0 & 0 & S_{45} & S_{55} & 0\\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66}\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix}.$$

As for the stiffnesses, note that for the compliances S_{16}, S_{26}, S_{36} and S_{45} are zero for $\alpha = 0$ or $\alpha = \frac{\pi}{2}$ (orthotropic axes coincide with the co-ordinate axes).

2.7. Orthotropic layer loaded in tension and in shear

2.7.1. *Simple tension*

For the case of a layer loaded in simple tension in the $\mathbf{x}_1$ direction, the only non-zero component of the matrix is σ_1.

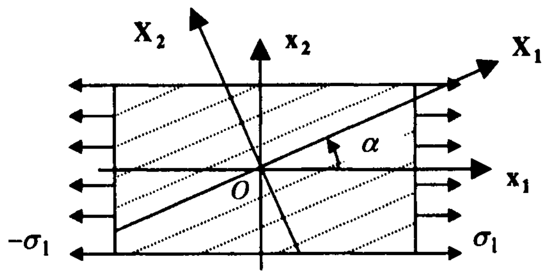

Figure 2.3. *Tension off-axis*

The constitutive relation $\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma}$ gives:

$$\varepsilon_1 = S_{11}\sigma_1,$$
$$\varepsilon_2 = S_{12}\sigma_1,$$
$$\varepsilon_3 = S_{13}\sigma_1,$$
$$\varepsilon_4 = 0,$$
$$\varepsilon_5 = 0,$$
$$\varepsilon_6 = S_{16}\sigma_1.$$

The layer undergoes three unit extensions in the $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ directions as well as an angular distortion in the two orthogonal directions $\mathbf{x}_1$ and $\mathbf{x}_2$.

For $\alpha = 0$ (fibres parallel to $\mathbf{x}_1$) or $\alpha = \frac{\pi}{2}$ (fibres parallel to $\mathbf{x}_2$), the elastic compliance S_{16} is zero. The angular distortion is then zero.

The strains obtained are shown in the figures below.

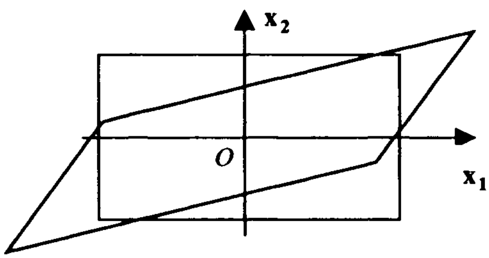

Figure 2.4. *Strains when* $S_{16} \neq 0$

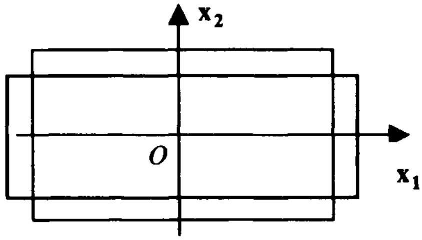

Figure 2.5. *Strains when* $S_{16} = 0$

2.7.2. *Simple shear*

When the layer described previously is loaded in simple shear in the two directions $\mathbf{x}_1$ and $\mathbf{x}_2$, the only component which is not zero in the stress matrix is σ_6.

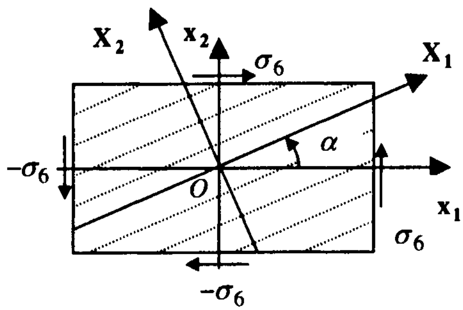

Figure 2.6. *Shear in off-axis directions*

Given the constitutive relation $\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma}$, we have:

$$\varepsilon_1 = S_{16}\sigma_6, \quad \varepsilon_4 = 0,$$
$$\varepsilon_2 = S_{26}\sigma_6, \quad \varepsilon_5 = 0,$$
$$\varepsilon_3 = S_{36}\sigma_6, \quad \varepsilon_6 = S_{66}\sigma_6.$$

The layer undergoes unit extensions in the $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ directions, and an angular distortion in the two directions $\mathbf{x}_1$ and $\mathbf{x}_2$.

For $\alpha = 0$ or $\alpha = \frac{\pi}{2}$, the elastic compliances S_{16}, S_{26} and S_{36} are zero. The linear expansions are therefore zero.

The corresponding strains are represented by the following figures.

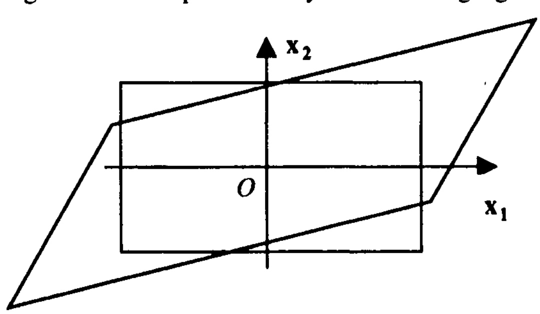

Figure 2.7. *Strains when* $S_{16} \neq 0, S_{26} \neq 0, S_{36} \neq 0$

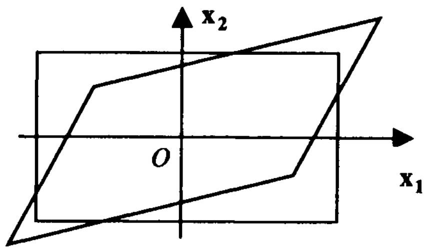

Figure 2.8. *Strains when* $S_{16} = S_{26} = S_{36} = 0$

2.8. Reduced stiffness matrix for the orthotropic layer

In this paragraph, we will describe the $\overline{\mathbf{Q}}$ and $\mathbf{Q}$ of an orthotropic layer with through thickness normal stress σ_3 zero. These matrices are introduced into the theorems of Kirchhoff-Love and Reissner-Mindlin for multi-layer materials.

2.8.1. *Reduced stiffness matrix* $\overline{\mathbf{Q}}$ *in orthotropic co-ordinates*

In the orthotropic co-ordinates $(\bar{e})$, the constitutive relation $\bar{\boldsymbol{\sigma}} = \overline{\mathbf{C}}\bar{\boldsymbol{\varepsilon}}$ is written as:

$$\begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_3 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \\ \bar{\sigma}_6 \end{bmatrix} = \begin{bmatrix} \overline{C}_{11} & \overline{C}_{12} & \overline{C}_{13} & 0 & 0 & 0 \\ \overline{C}_{12} & \overline{C}_{22} & \overline{C}_{23} & 0 & 0 & 0 \\ \overline{C}_{13} & \overline{C}_{23} & \overline{C}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \overline{C}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \overline{C}_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & \overline{C}_{66} \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_3 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \\ \bar{\varepsilon}_6 \end{bmatrix}.$$

When:

$\bar{\sigma}_3 = \overline{C}_{13}\bar{\varepsilon}_1 + \overline{C}_{23}\bar{\varepsilon}_2 + \overline{C}_{33}\bar{\varepsilon}_3 = 0$, the expansion $\bar{\varepsilon}_3$ is equal to:

$$\bar{\varepsilon}_3 = -\frac{\overline{C}_{13}\bar{\varepsilon}_1 + \overline{C}_{23}\bar{\varepsilon}_2}{\overline{C}_{33}}.$$

The normal stresses:

$$\bar{\sigma}_1 = \overline{C}_{11}\bar{\varepsilon}_1 + \overline{C}_{12}\bar{\varepsilon}_2 + \overline{C}_{13}\bar{\varepsilon}_3,$$

$$\bar{\sigma}_2 = \overline{C}_{12}\bar{\varepsilon}_1 + \overline{C}_{22}\bar{\varepsilon}_2 + \overline{C}_{23}\bar{\varepsilon}_3,$$

can then be expressed only in terms of $\bar{\varepsilon}_1$ and $\bar{\varepsilon}_2$, using:

$$\bar{\sigma}_1 = \left(\bar{C}_{11} - \frac{\bar{C}_{13}^2}{\bar{C}_{33}}\right)\bar{\varepsilon}_1 + \left(\bar{C}_{12} - \frac{\bar{C}_{13}\bar{C}_{23}}{\bar{C}_{33}}\right)\bar{\varepsilon}_2,$$

$$\bar{\sigma}_2 = \left(\bar{C}_{12} - \frac{\bar{C}_{13}\bar{C}_{23}}{\bar{C}_{33}}\right)\bar{\varepsilon}_1 + \left(\bar{C}_{22} - \frac{\bar{C}_{23}^2}{\bar{C}_{33}}\right)\bar{\varepsilon}_2.$$

By putting:

$$\bar{Q}_{11} = \bar{C}_{11} - \frac{\bar{C}_{13}^2}{\bar{C}_{33}}, \quad \bar{Q}_{12} = \bar{C}_{12} - \frac{\bar{C}_{13}\bar{C}_{23}}{\bar{C}_{33}},$$

$$\bar{Q}_{22} = \bar{C}_{22} - \frac{\bar{C}_{23}^2}{\bar{C}_{33}}, \quad \bar{Q}_{66} = \bar{C}_{66},$$

$$\bar{Q}_{44} = \bar{C}_{44}, \qquad \bar{Q}_{55} = \bar{C}_{55},$$

the constitutive relation can be expressed in one of the following two forms:

$$\begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 & 0 & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 & 0 & 0 \\ 0 & 0 & \bar{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \bar{Q}_{44} & 0 \\ 0 & 0 & 0 & 0 & \bar{Q}_{55} \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_6 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \end{bmatrix},$$

or:

$$\begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \end{bmatrix} = \begin{bmatrix} \bar{Q}_{11} & \bar{Q}_{12} & 0 \\ \bar{Q}_{12} & \bar{Q}_{22} & 0 \\ 0 & 0 & \bar{Q}_{66} \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_6 \end{bmatrix}, \quad \begin{bmatrix} \bar{\sigma}_4 \\ \bar{\sigma}_5 \end{bmatrix} = \begin{bmatrix} \bar{Q}_{44} & 0 \\ 0 & \bar{Q}_{55} \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \end{bmatrix},$$

with:

$$\bar{Q}_{ij} = \bar{C}_{ij} - \frac{\bar{C}_{i3}\bar{C}_{j3}}{\bar{C}_{33}} \quad (i, j = 1,2),$$

$$\bar{Q}_{ij} = \bar{C}_{ij} \qquad (i, j = 4,5,6).$$

In the plate theorems of Kirchhoff-Love and Reissner-Mindlin we keep the notations $\bar{\boldsymbol{\sigma}}$ and $\bar{\boldsymbol{\varepsilon}}$ to represent the matrix columns of stress components:

$$\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_6, \bar{\sigma}_4, \bar{\sigma}_5, \text{ or } \bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_6,$$

and strains:

$$\bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_6, \bar{\varepsilon}_4, \bar{\varepsilon}_5, \text{ or } \bar{\varepsilon}_1, \bar{\varepsilon}_2, \bar{\varepsilon}_6,$$

which have just been introduced, and designate by $\bar{\mathbf{Q}}$ the corresponding reduced stiffness matrix.

2.8.2. *Reduced stiffness matrix* **Q** *in the co-ordinates away from the orthotropic axes*

The change from the orthotropic axes $(\bar{e})$ to the off-axis co-ordinates (e) and its transposition are written as:

$$\mathbf{a}=\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{a}^T=\begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix axes change with respect to the stresses $\boldsymbol{\sigma}=\mathbf{M}\bar{\boldsymbol{\sigma}}$, involving the matrix:

$$\mathbf{M}=\begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -2cs \\ s^2 & c^2 & 0 & 0 & 0 & 2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ cs & -cs & 0 & 0 & 0 & c^2-s^2 \end{bmatrix},$$

can be written for this case in the form:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{bmatrix}=\begin{bmatrix} c^2 & s^2 & -2cs & 0 & 0 \\ s^2 & c^2 & 2cs & 0 & 0 \\ cs & -cs & c^2-s^2 & 0 & 0 \\ 0 & 0 & 0 & c & s \\ 0 & 0 & 0 & -s & c \end{bmatrix}\begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \end{bmatrix},$$

or:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix}=\begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2-s^2 \end{bmatrix}\begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \end{bmatrix}, \quad \begin{bmatrix} \sigma_4 \\ \sigma_5 \end{bmatrix}=\begin{bmatrix} c & s \\ -s & c \end{bmatrix}\begin{bmatrix} \bar{\sigma}_4 \\ \bar{\sigma}_5 \end{bmatrix},$$

keeping the notation **M** for the two new conventional change of axes matrices associated with the two matrices of the stress components $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_6, \bar{\sigma}_4, \bar{\sigma}_5$ and $\bar{\sigma}_1, \bar{\sigma}_2, \bar{\sigma}_6$:

$$\boldsymbol{\sigma}=\mathbf{M}\bar{\boldsymbol{\sigma}}.$$

The formula for changing the axes with respect to the strain matrices $\boldsymbol{\varepsilon}=\mathbf{N}\bar{\boldsymbol{\varepsilon}}$, with:

$$\mathbf{N} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -cs \\ s^2 & c^2 & 0 & 0 & 0 & cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 2cs & -2cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix},$$

can be written as:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -cs & 0 & 0 \\ s^2 & c^2 & cs & 0 & 0 \\ 2cs & -2cs & c^2 - s^2 & 0 & 0 \\ 0 & 0 & 0 & c & s \\ 0 & 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_6 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \end{bmatrix},$$

or:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_6 \end{bmatrix}, \quad \begin{bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \end{bmatrix}.$$

So, by designating as **N** the new conventional axis change matrix:

$$\boldsymbol{\varepsilon} = \mathbf{N}\bar{\boldsymbol{\varepsilon}}.$$

The formula for changing axes, established for the stiffness matrices $\mathbf{C} = \mathbf{M}\bar{\mathbf{C}}\mathbf{M}^T$, leads to the relation:

$$\mathbf{Q} = \mathbf{M}\bar{\mathbf{Q}}\mathbf{M}^T.$$

Given the calculations already performed for the change of axes the reduced stiffness matrix **Q** is of the form:

$$\mathbf{Q} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{26} & 0 & 0 \\ Q_{16} & Q_{26} & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix},$$

with:

$$Q_{11} = c^4\overline{Q}_{11} + s^4\overline{Q}_{22} + 2c^2s^2\left(\overline{Q}_{12} + 2\overline{Q}_{66}\right),$$
$$Q_{22} = s^4\overline{Q}_{11} + c^4\overline{Q}_{22} + 2c^2s^2\left(\overline{Q}_{12} + 2\overline{Q}_{66}\right),$$
$$Q_{12} = c^2s^2\left(\overline{Q}_{11} + \overline{Q}_{22} - 4\overline{Q}_{66}\right) + \left(c^4 + s^4\right)\overline{Q}_{12},$$
$$Q_{66} = c^2s^2\left(\overline{Q}_{11} + \overline{Q}_{22} - 2\overline{Q}_{12}\right) + \left(c^2 - s^2\right)^2\overline{Q}_{66},$$
$$Q_{16} = cs\left[c^2\overline{Q}_{11} - s^2\overline{Q}_{22} - \left(c^2 - s^2\right)\left(\overline{Q}_{12} + 2\overline{Q}_{66}\right)\right],$$
$$Q_{26} = cs\left[s^2\overline{Q}_{11} - c^2\overline{Q}_{22} + \left(c^2 - s^2\right)\left(\overline{Q}_{12} + 2\overline{Q}_{66}\right)\right],$$
$$Q_{44} = c^2\overline{Q}_{44} + s^2\overline{Q}_{55},$$
$$Q_{55} = s^2\overline{Q}_{44} + c^2\overline{Q}_{55},$$
$$Q_{45} = cs\left(\overline{Q}_{55} - \overline{Q}_{44}\right).$$

It should be noted that the reduced stiffnesses Q_{16}, Q_{26} and Q_{45} are zero for $\alpha = 0$ or $\alpha = \frac{\pi}{2}$. These relationships can be written in matrix form:

$$\begin{bmatrix} Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{66} \\ Q_{16} \\ Q_{26} \\ Q_{44} \\ Q_{55} \\ Q_{45} \end{bmatrix} = \begin{bmatrix} c^4 & s^4 & 2c^2s^2 & 4c^2s^2 & 0 & 0 \\ s^4 & c^4 & 2c^2s^2 & 4c^2s^2 & 0 & 0 \\ c^2s^2 & c^2s^2 & c^4 + s^4 & -4c^2s^2 & 0 & 0 \\ c^2s^2 & c^2s^2 & -2c^2s^2 & \left(c^2 - s^2\right)^2 & 0 & 0 \\ c^3s & -cs^3 & -cs\left(c^2 - s^2\right) & -2cs\left(c^2 - s^2\right) & 0 & 0 \\ cs^3 & -c^3s & cs\left(c^2 - s^2\right) & 2cs\left(c^2 - s^2\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & c^2 & s^2 \\ 0 & 0 & 0 & 0 & s^2 & c^2 \\ 0 & 0 & 0 & 0 & -cs & cs \end{bmatrix} \begin{bmatrix} \overline{Q}_{11} \\ \overline{Q}_{22} \\ \overline{Q}_{12} \\ \overline{Q}_{66} \\ \overline{Q}_{44} \\ \overline{Q}_{55} \end{bmatrix},$$

or:

$$\begin{bmatrix} Q_{11} \\ Q_{22} \\ Q_{12} \\ Q_{66} \\ Q_{16} \\ Q_{26} \end{bmatrix} = \begin{bmatrix} c^4 & s^4 & 2c^2s^2 & 4c^2s^2 \\ s^4 & c^4 & 2c^2s^2 & 4c^2s^2 \\ c^2s^2 & c^2s^2 & c^4 + s^4 & -4c^2s^2 \\ c^2s^2 & c^2s^2 & -2c^2s^2 & \left(c^2 - s^2\right)^2 \\ c^3s & -cs^3 & -cs\left(c^2 - s^2\right) & -2cs\left(c^2 - s^2\right) \\ cs^3 & -c^3s & cs\left(c^2 - s^2\right) & 2cs\left(c^2 - s^2\right) \end{bmatrix} \begin{bmatrix} \overline{Q}_{11} \\ \overline{Q}_{22} \\ \overline{Q}_{12} \\ \overline{Q}_{66} \end{bmatrix},$$

$$\begin{bmatrix} Q_{44} \\ Q_{55} \\ Q_{45} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 \\ s^2 & c^2 \\ -cs & cs \end{bmatrix} \begin{bmatrix} \overline{Q}_{44} \\ \overline{Q}_{55} \end{bmatrix}.$$

2.9. Reduced compliance matrices of an orthotropic layer

Having presented the reduced stiffness matrix $\overline{\mathbf{Q}}$ we will now derive the reduced compliance matrices $\overline{\mathbf{P}}$ and $\mathbf{P}$ for an orthotropic layer where the through-thickness normal stress σ_3 is zero.

2.9.1. *Reduced compliance matrix in orthotropic co-ordinates*

In the orthotropic co-ordinates we have:

$$\begin{bmatrix} \bar{\varepsilon}_1 \\ \bar{\varepsilon}_2 \\ \bar{\varepsilon}_6 \\ \bar{\varepsilon}_4 \\ \bar{\varepsilon}_5 \end{bmatrix} = \begin{bmatrix} \overline{P}_{11} & \overline{P}_{12} & 0 & 0 & 0 \\ \overline{P}_{12} & \overline{P}_{22} & 0 & 0 & 0 \\ 0 & 0 & \overline{P}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{P}_{44} & 0 \\ 0 & 0 & 0 & 0 & \overline{P}_{55} \end{bmatrix} \begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \\ \bar{\sigma}_4 \\ \bar{\sigma}_5 \end{bmatrix},$$

with:

$$\overline{P}_{11} = \frac{\overline{Q}_{22}}{\overline{Q}_{11}\overline{Q}_{22} - \overline{Q}_{12}^2}, \quad \overline{P}_{12} = -\frac{\overline{Q}_{12}}{\overline{Q}_{11}\overline{Q}_{22} - \overline{Q}_{12}^2},$$

$$\overline{P}_{22} = \frac{\overline{Q}_{11}}{\overline{Q}_{11}\overline{Q}_{22} - \overline{Q}_{12}^2}, \quad \overline{P}_{66} = \frac{1}{\overline{Q}_{66}},$$

$$\overline{P}_{44} = \frac{1}{\overline{Q}_{44}}, \quad \overline{P}_{55} = \frac{1}{\overline{Q}_{55}}.$$

2.9.2. *Reduced compliance matrix* P *in the direction away from the orthotropic axes*

The formula for the change in co-ordinates for the reduced compliance matrix is written as:

$$\mathbf{P} = \mathbf{N}\overline{\mathbf{P}}\mathbf{N}^T.$$

Given the preceding calculations for changing axes the reduced compliance matrix P may be written as:

$$\mathbf{P} = \begin{bmatrix} P_{11} & P_{12} & P_{16} & 0 & 0 \\ P_{12} & P_{22} & P_{26} & 0 & 0 \\ P_{16} & P_{26} & P_{66} & 0 & 0 \\ 0 & 0 & 0 & P_{44} & P_{45} \\ 0 & 0 & 0 & P_{45} & P_{55} \end{bmatrix},$$

with:

$$
\begin{aligned}
P_{11} &= c^4\overline{P}_{11} + s^4\overline{P}_{22} + c^2s^2\left(2\overline{P}_{12} + \overline{P}_{66}\right),\\
P_{22} &= s^4\overline{P}_{11} + c^4\overline{P}_{22} + c^2s^2\left(2\overline{P}_{12} + \overline{P}_{66}\right),\\
P_{12} &= c^2s^2\left(\overline{P}_{11} + \overline{P}_{22} - \overline{P}_{66}\right) + \left(c^4 + s^4\right)\overline{P}_{12},\\
P_{66} &= 4c^2s^2\left(\overline{P}_{11} + \overline{P}_{22} - 2\overline{P}_{12}\right) + \left(c^2 - s^2\right)^2\overline{P}_{66},\\
P_{16} &= cs\left[2c^2\overline{P}_{11} - 2s^2\overline{P}_{22} - \left(c^2 - s^2\right)\left(2\overline{P}_{12} + \overline{P}_{66}\right)\right],\\
P_{26} &= cs\left[2s^2\overline{P}_{11} - 2c^2\overline{P}_{22} + \left(c^2 - s^2\right)\left(2\overline{P}_{12} + \overline{P}_{66}\right)\right],\\
P_{44} &= c^2\overline{P}_{44} + s^2\overline{P}_{55},\\
P_{55} &= s^2\overline{P}_{44} + c^2\overline{P}_{55},\\
P_{45} &= cs\left(\overline{P}_{55} - \overline{P}_{44}\right).
\end{aligned}
$$

It may be noted that the terms P_{16}, P_{26} and P_{45} are zero for $\alpha = 0$ or $\alpha = \frac{\pi}{2}$. These relationships can be presented in the form:

$$
\begin{bmatrix} P_{11}\\ P_{22}\\ P_{12}\\ P_{66}\\ P_{16}\\ P_{26}\\ P_{44}\\ P_{55}\\ P_{45} \end{bmatrix}
=
\begin{bmatrix}
c^4 & s^4 & 2c^2s^2 & c^2s^2 & 0 & 0\\
s^4 & c^4 & 2c^2s^2 & c^2s^2 & 0 & 0\\
c^2s^2 & c^2s^2 & c^4+s^4 & -c^2s^2 & 0 & 0\\
4c^2s^2 & 4c^2s^2 & -8c^2s^2 & \left(c^2-s^2\right)^2 & 0 & 0\\
2c^3s & -2cs^3 & -2cs\left(c^2-s^2\right) & -cs\left(c^2-s^2\right) & 0 & 0\\
2cs^3 & -2c^3s & 2cs\left(c^2-s^2\right) & cs\left(c^2-s^2\right) & 0 & 0\\
0 & 0 & 0 & 0 & c^2 & s^2\\
0 & 0 & 0 & 0 & s^2 & c^2\\
0 & 0 & 0 & 0 & -cs & cs
\end{bmatrix}
\begin{bmatrix} \overline{P}_{11}\\ \overline{P}_{22}\\ \overline{P}_{12}\\ \overline{P}_{66}\\ \overline{P}_{44}\\ \overline{P}_{55} \end{bmatrix},
$$

or:

$$
\begin{bmatrix} P_{11}\\ P_{22}\\ P_{12}\\ P_{66}\\ P_{16}\\ P_{26} \end{bmatrix}
=
\begin{bmatrix}
c^4 & s^4 & 2c^2s^2 & c^2s^2\\
s^4 & c^4 & 2c^2s^2 & c^2s^2\\
c^2s^2 & c^2s^2 & c^4+s^4 & -c^2s^2\\
4c^2s^2 & 4c^2s^2 & -8c^2s^2 & \left(c^2-s^2\right)^2\\
2c^3s & -2cs^3 & -2cs\left(c^2-s^2\right) & -cs\left(c^2-s^2\right)\\
2cs^3 & -2c^3s & 2cs\left(c^2-s^2\right) & cs\left(c^2-s^2\right)
\end{bmatrix}
\begin{bmatrix} \overline{P}_{11}\\ \overline{P}_{22}\\ \overline{P}_{12}\\ \overline{P}_{66} \end{bmatrix},
$$

$$
\begin{bmatrix} P_{44}\\ P_{55}\\ P_{45} \end{bmatrix}
=
\begin{bmatrix}
c^2 & s^2\\
s^2 & c^2\\
-cs & cs
\end{bmatrix}
\begin{bmatrix} \overline{P}_{44}\\ \overline{P}_{55} \end{bmatrix}.
$$

Chapter 3

Elastic constants of a unidirectional composite

3.1. Introduction

When an orthotropic material is in a plane stress state the relationships between the stresses and strains involve the four elastic constants E_1, E_2, ν_{12} and G_{12}. In addition to these coefficients, when considering thermo-elasticity the coefficients of thermal expansion α_1 and α_2 are also required.

In this chapter we are able to obtain the characteristics of a unidirectional composite material as a function of the characteristics of the fibres and the matrix.

3.2. Density ρ

Designating as:

- M, M_f, M_m the masses of the composite, fibres and matrix,
- ρ, ρ_f, ρ_m the densities of the composite, fibres and matrix,
- v, v_f, v_m the volumes of the composite, fibres and matrix,

the mass of the composite is:

$$M = M_f + M_m,$$

or:

$$\rho v = \rho_f v_f + \rho_m v_m .$$

Then:

$$V_f = \frac{v_f}{v} \text{ the volume fraction of the fibres,}$$

$$V_m = \frac{v_m}{v} \text{ the volume fraction of the matrix,}$$

and noting that $V_f + V_m = 1$, the density of the composite is:

$$\rho = V_f \rho_f + V_m \rho_m = V_f \rho_f + (1 - V_f)\rho_m .$$

3.3. Longitudinal Young's modulus E_1

In the theory below we assume that, as shown in the figure, the fibres are concentrated in the lower part of the composite and that the matrix occupies the upper part. The composite is subjected to the longitudinal tensile force F.

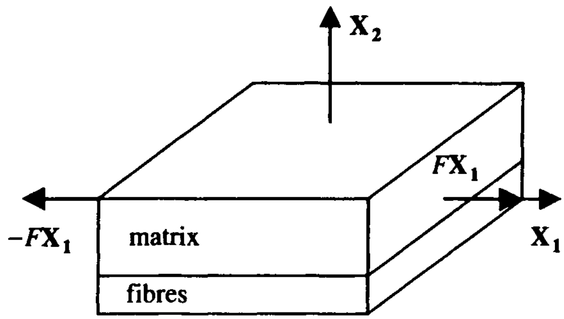

Figure 3.1. *Longitudinal tension*

In the model adopted here, we suppose that the longitudinal extensions resulting from the tensile force F are the same in the composite, fibres and matrix, which can be presented as:

$$\varepsilon_1 = \varepsilon_1^f = \varepsilon_1^m .$$

We designate as:

– E_f , E_m the Young's moduli of the fibres and matrix,

– S , S_f , S_m the surface areas, orthogonal to $\mathbf{X}_1$, occupied by the composite, fibres and matrix.

We note that, designating by l the length of the composite:

$$V_f = \frac{v_f}{v} = \frac{S_f l}{Sl} = \frac{S_f}{S},$$

and that:

$$V_m = \frac{S_m}{S}.$$

The tensile stresses in the fibres and in the matrix are equal to:

$$\sigma_1^f = E_f \varepsilon_1,$$

$$\sigma_1^m = E_m \varepsilon_1,$$

and the tensile force is given by:

$$F = S_f \sigma_1^f + S_m \sigma_1^m .$$

The tensile stress σ_1 in the composite is:

$$\sigma_1 = \frac{F}{S} = V_f \sigma_1^f + V_m \sigma_1^m = (V_f E_f + V_m E_m)\varepsilon_1 .$$

The longitudinal Young's modulus E_1 of the composite is defined by:

$$E_1 = \frac{\sigma_1}{\varepsilon_1},$$

and thus we obtain the relationship:

$$E_1 = V_f E_f + V_m E_m = V_f E_f + (1 - V_f)E_m,$$

which provides a satisfactory value of E_1. It should be noted that, for cases where $E_f >> E_m$, then as a first approximation:

$$E_1 = V_f E_f .$$

3.4. Poisson's coefficient ν_{12}

With the loading described above the transverse extensions of the fibres and the matrix are equal to:

$$\varepsilon_2^f = -\nu_f \varepsilon_1,$$

$$\varepsilon_2^m = -\nu_m \varepsilon_1,$$

where ν_f and ν_m represent Poisson's coefficients of the fibres and the matrix.

The change in thickness of the laminate is given by:

$$\Delta e = \Delta e_f + \Delta e_m = e_f \varepsilon_2^f + e_m \varepsilon_2^m,$$

where e_f and e_m are the thicknesses of the fibre and matrix parts.

Designating by b the width of the composite, it may be noted that:

$$V_f = \frac{v_f}{v} = \frac{e_f bl}{ebl} = \frac{e_f}{e},$$

and:

$$V_m = \frac{e_m}{e} .$$

The transverse expansion of the composite is:

$$\varepsilon_2 = \frac{\Delta e}{e} = V_f \varepsilon_2^f + V_m \varepsilon_2^m = -(V_f \nu_f + V_m \nu_m)\varepsilon_1 .$$

Poisson's coefficient ν_{12} of the composite is defined by:

$$\nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1}.$$

From this we obtain the expression:

$$\nu_{12} = V_f \nu_f + V_m \nu_m = V_f \nu_f + (1 - V_f)\nu_m,$$

which provides Poisson's coefficient ν_{12}.

3.5. Transverse Young's modulus E_2

The composite is subjected to the transverse tensile force F.

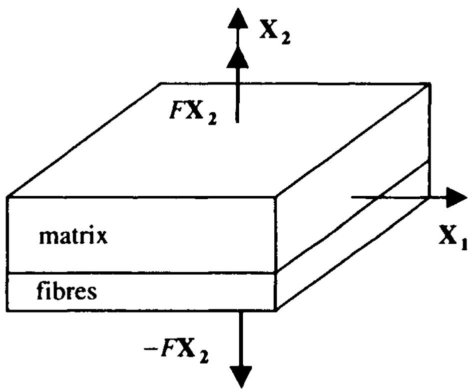

Figure 3.2. *Transverse tension*

With the same model it is assumed that the transverse strains in the fibres and in the matrix are equal:

$$\sigma_2 = \sigma_2^f = \sigma_2^m.$$

The transverse extensions in the fibres and in the matrix are:

$$\varepsilon_2^f = \frac{\sigma_2}{E_f},$$

$$\varepsilon_2^m = \frac{\sigma_2}{E_m}.$$

The thickness variation is equal to:

$$\Delta e = \Delta e_f + \Delta e_m = e_f \varepsilon_2^f + e_m \varepsilon_2^m ,$$

the transverse extension of the composite is:

$$\varepsilon_2 = \frac{\Delta e}{e} = V_f \varepsilon_2^f + V_m \varepsilon_2^m = \left(\frac{V_f}{E_f} + \frac{V_m}{E_m} \right) \sigma_2 .$$

The transverse Young's modulus E_2 of the composite is defined by the expression:

$$\frac{1}{E_2} = \frac{\varepsilon_2}{\sigma_2} ,$$

from which we obtain:

$$\frac{1}{E_2} = \frac{V_f}{E_f} + \frac{V_m}{E_m} .$$

The transverse Young's modulus E_2 is therefore equal to:

$$E_2 = \frac{E_f E_m}{V_m E_f + V_f E_m} = \frac{E_m}{V_m + V_f \dfrac{E_m}{E_f}} ,$$

or:

$$E_2 = \frac{E_m}{1 - V_f \left(1 - \dfrac{E_m}{E_f} \right)} .$$

The accuracy of the value of E_2 obtained by this expression is poorer than that for E_1. When $E_f >> E_m$, the expression for E_2 becomes:

$$E_2 = \frac{E_m}{1 - V_f} .$$

From the calculations above it may be noted that the two Young's moduli E_1 and E_2 are given by expressions analogous to those encountered when the equivalent stiffnesses of two springs in parallel and in series are calculated.

3.6. Shear modulus G_{12}

The composite is subjected to shear loading σ_6 as shown on the following figure:

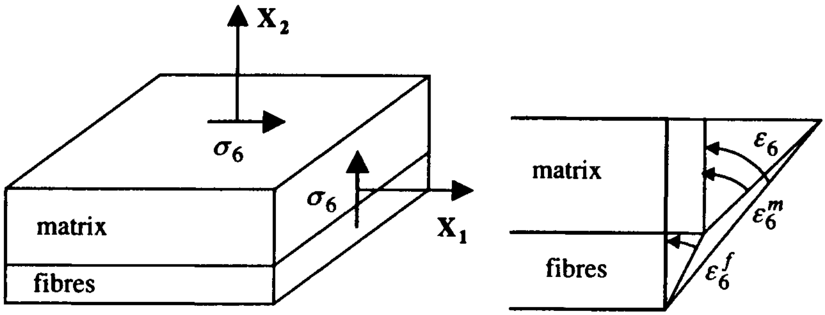

Figure 3.3. *Shear*

Again with the same model, it is assumed that the shear stresses in the fibres and in the matrix are equal:

$$\sigma_6 = \sigma_6^f = \sigma_6^m .$$

The angular distortions in the fibres and in the matrix are given by:

$$\varepsilon_6^f = \frac{\sigma_6}{G_f},$$

$$\varepsilon_6^m = \frac{\sigma_6}{G_m},$$

where G_f and G_m represent the shear moduli of the fibres and the matrix.

The displacement in the $\mathbf{X_1}$ direction of the upper plane of the composite with respect to the lower plane is:

$$u = u_f + u_m .$$

Given the displacements due to the fibre and matrix parts:

$$u_f = e_f \varepsilon_6^f ,$$

$$u_m = e_m \varepsilon_6^m ,$$

we obtain:

$$u = e_f \varepsilon_6^f + e_m \varepsilon_6^m = \left(\frac{e_f}{G_f} + \frac{e_m}{G_m} \right) \sigma_6 .$$

The angular distortion of the composite is:

$$\varepsilon_6 = \frac{u}{e} = \left(\frac{V_f}{G_f} + \frac{V_m}{G_m} \right) \sigma_6 ,$$

the G_{12} of the composite is given by:

$$\frac{1}{G_{12}} = \frac{\varepsilon_6}{\sigma_6},$$

so:

$$\frac{1}{G_{12}} = \frac{V_f}{G_f} + \frac{V_m}{G_m}.$$

The shear modulus is:

$$G_{12} = \frac{G_f G_m}{V_m G_f + V_f G_m} = \frac{G_m}{V_m + V_f \dfrac{G_m}{G_f}},$$

or:

$$G_{12} = \frac{G_m}{1 - V_f\left(1 - \dfrac{G_m}{G_f}\right)}.$$

Once again the accuracy of G_{12} is less than that for E_1. When $G_f >> G_m$, we have, as a first approximation:

$$G_{12} = \frac{G_m}{1 - V_f}.$$

3.7. Longitudinal thermal expansion coefficient α_1

The composite is not subjected to any external force but experiences a temperature variation of:

$$\Delta T = T - T_0.$$

In the model considered here we assume that the longitudinal extensions of the fibres and of the matrix are identical, that is:

$$\varepsilon_1 = \varepsilon_1^f = \varepsilon_1^m.$$

Only the longitudinal stresses due to the difference between the thermal coefficients of the fibres, α_f and of the matrix α_m are not zero.

The longitudinal expansions of the fibres and the matrix are equal to:

$$\varepsilon_1^f = \frac{\sigma_1^f}{E_f} + \alpha_f \Delta T,$$

$$\varepsilon_1^m = \frac{\sigma_1^m}{E_m} + \alpha_m \Delta T.$$

The longitudinal stresses in the fibres and the matrix are:

$$\sigma_1^f = E_f\left(\varepsilon_1 - \alpha_f \Delta T\right),$$

$$\sigma_1^m = E_m\left(\varepsilon_1 - \alpha_m \Delta T\right).$$

The resultant force of the tensile loads:

$$F = \sigma_1^f S_f + \sigma_1^m S_m,$$

being zero, we obtain the expression:

$$V_f \sigma_1^f + V_m \sigma_1^m = 0 .$$

Introducing in this expression the stress values σ_1^f and σ_1^m, we obtain:

$$V_f E_f\left(\varepsilon_1 - \alpha_f \Delta T\right) + V_m E_m\left(\varepsilon_1 - \alpha_m \Delta T\right) = 0,$$

then:

$$\left(V_f E_f + V_m E_m\right)\varepsilon_1 = \left(V_f \alpha_f E_f + V_m \alpha_m E_m\right)\Delta T .$$

The coefficient of longitudinal thermal expansion α_1 of the composite being defined by:

$$\alpha_1 = \frac{\varepsilon_1}{\Delta T},$$

we obtain the following expression:

$$\alpha_1 = \frac{V_f \alpha_f E_f + V_m \alpha_m E_m}{V_f E_f + V_m E_m} = \frac{V_f \alpha_f E_f + \left(1 - V_f\right)\alpha_m E_m}{V_f E_f + \left(1 - V_f\right)E_m}.$$

3.8. Transverse expansion coefficient α_2

The transverse expansions of the fibres and matrix are equal to:

$$\varepsilon_2^f = -\frac{\nu_f}{E_f}\sigma_1^f + \alpha_f \Delta T,$$

$$\varepsilon_2^m = -\frac{\nu_m}{E_m}\sigma_1^m + \alpha_m \Delta T,$$

replacing the longitudinal stresses by their values determined in the previous paragraph we obtain:

$$\varepsilon_2^f = -\nu_f\left(\varepsilon_1 - \alpha_f \Delta T\right) + \alpha_f \Delta T,$$

$$\varepsilon_2^m = -\nu_m\left(\varepsilon_1 - \alpha_m \Delta T\right) + \alpha_m \Delta T.$$

The thickness variation is given by:

$$\Delta e = \Delta e_f + \Delta e_m = e_f \varepsilon_2^f + e_m \varepsilon_2^m,$$

The transverse expansion of the composite is:

$$\varepsilon_2 = \frac{\Delta e}{e} = V_f \varepsilon_2^f + V_m \varepsilon_2^m .$$

Introducing the values of ε_2^f and ε_2^m, we obtain:

$$\varepsilon_2 = -\left(V_f \nu_f + V_m \nu_m\right)\varepsilon_1 + \left(V_f \alpha_f \nu_f + V_m \alpha_m \nu_m\right)\Delta T + \left(V_f \alpha_f + V_m \alpha_m\right)\Delta T .$$

As:

$$\varepsilon_1 = \frac{V_f \alpha_f E_f + V_m \alpha_m E_m}{V_f E_f + V_m E_m}\Delta T ,$$

we have:

$$\varepsilon_2 = \left\{ -\frac{\left(V_f \nu_f + V_m \nu_m\right)\left(V_f \alpha_f E_f + V_m \alpha_m E_m\right)}{V_f E_f + V_m E_m} \ldots \right.$$

$$\left. \ldots + \frac{\left(V_f \alpha_f \nu_f + V_m \alpha_m \nu_m\right)\left(V_f E_f + V_m E_m\right)}{V_f E_f + V_m E_m} \right\} \Delta T + \left(V_f \alpha_f + V_m \varepsilon_m\right)\Delta T,$$

or:

$$\varepsilon_2 = \left[\frac{V_f V_m \left(\nu_m E_f - \nu_f E_m\right)\left(\alpha_m - \alpha_f\right)}{V_f E_f + V_m E_m} + V_f \alpha_f + V_m \alpha_m \right] \Delta T .$$

The coefficient of transverse thermal expansion α_2 of the composite given by:

$$\alpha_2 = \frac{\varepsilon_2}{\Delta T},$$

is equal to:

$$\alpha_2 = V_f \alpha_f + V_m \alpha_m + \frac{\nu_m E_f - \nu_f E_m}{\dfrac{E_f}{V_m} + \dfrac{E_m}{V_f}} \left(\alpha_m - \alpha_f\right),$$

or:

$$\alpha_2 = V_f \alpha_f + \left(1 - V_f\right)\alpha_m + \frac{\nu_m E_f - \nu_f E_m}{\dfrac{E_f}{1 - V_f} + \dfrac{E_m}{V_f}} \left(\alpha_m - \alpha_f\right).$$

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Chapter 4

Failure criteria

4.1. Introduction

Having analysed the stresses and strains in composite materials we will now present the main failure criteria for these materials.

The degradation of the composites is characterised by one of the following local modes:

- damage dominated by fibre degradation (rupture, microbuckling, etc.),
- damage dominated by matrix degradation (crazing, etc.),
- damage dominated by singularities at the fibre-matrix interface (crack propagation, delamination, etc.).

Composite failure is a gradual process, as the damage in a layer results in a redistribution of stresses in the laminate.

4.2. Maximum stress theory

Failure of the composite occurs as soon as the stress field no longer satisfies the following relationships:

$$\begin{aligned}
&\bar{\sigma}_{1rc} < \bar{\sigma}_1 < \bar{\sigma}_{1rt}, \quad -\bar{\sigma}_{4r} < \bar{\sigma}_4 < \bar{\sigma}_{4r},\\
&\bar{\sigma}_{2rc} < \bar{\sigma}_2 < \bar{\sigma}_{2rt}, \quad -\bar{\sigma}_{5r} < \bar{\sigma}_5 < \bar{\sigma}_{5r},\\
&\bar{\sigma}_{3rc} < \bar{\sigma}_3 < \bar{\sigma}_{3rt}, \quad -\bar{\sigma}_{6r} < \bar{\sigma}_6 < \bar{\sigma}_{6r},
\end{aligned}$$

in which $\bar{\sigma}_{irt}$ $(i = 1,2,3)$ represents the failure stresses in tension, $\bar{\sigma}_{irc}$ $(i = 1,2,3)$ the failure stresses in compression, and $\bar{\sigma}_{ir}$ $(i = 4,5,6)$ the shear failure stresses.

In these expressions the failure stresses in tension and shear are positive, those in compression are negative.

When the stress state in the composite is expressed as a function of a single parameter which depends on the external loads, it is convenient to introduce for each expression the loading coefficient $F_{i\sigma}$ associated with the stress state leading to failure.

This stress state defined by:

$$\bar{\sigma}_i' = F_{i\sigma}\bar{\sigma}_i \text{ (no summation)},$$

leads to the loading coefficients:

$$F_{i\sigma} = \frac{\bar{\sigma}_{irt}}{\bar{\sigma}_i} \text{ or } F_{i\sigma} = \frac{\bar{\sigma}_{irc}}{\bar{\sigma}_i} \quad (i = 1,2,3),$$

according to whether the normal stresses $\bar{\sigma}_i$ are positive or negative and:

$$F_{i\sigma} = \frac{\bar{\sigma}_{ir}}{\bar{\sigma}_i} \text{ or } F_{i\sigma} = -\frac{\bar{\sigma}_{ir}}{\bar{\sigma}_i} \quad (i = 4,5,6),$$

according to whether the shear stresses $\bar{\sigma}_i$ are positive or negative.

Failure occurs for the smallest value of the loading coefficients $F_{i\sigma}$, calculated above, according to the failure mode given by the index i. For i = 1, 2 or 3, failure occurs in tension or compression in the directions $\mathbf{X}_1$, $\mathbf{X}_2$ or $\mathbf{X}_3$, whereas for i =4, 5 or 6, failure occurs by shearing in the planes $(\mathbf{X}_2, \mathbf{X}_3)$, $(\mathbf{X}_3, \mathbf{X}_1)$ or $(\mathbf{X}_1, \mathbf{X}_2)$.

4.3. Maximum strain theory

Failure of the composite occurs as soon as the strain field no longer satisfies the following relationships:

$$\bar{\varepsilon}_{1rc} < \bar{\varepsilon}_1 < \bar{\varepsilon}_{1rt}, \quad -\bar{\varepsilon}_{4r} < \bar{\varepsilon}_4 < \bar{\varepsilon}_{4r},$$
$$\bar{\varepsilon}_{2rc} < \bar{\varepsilon}_2 < \bar{\varepsilon}_{2rt}, \quad -\bar{\varepsilon}_{5r} < \bar{\varepsilon}_5 < \bar{\varepsilon}_{5r},$$
$$\bar{\varepsilon}_{3rc} < \bar{\varepsilon}_3 < \bar{\varepsilon}_{3rt}, \quad -\bar{\varepsilon}_{6r} < \bar{\varepsilon}_6 < \bar{\varepsilon}_{6r},$$

in which $\bar{\varepsilon}_{irt}$ $(i = 1,2,3)$ represent the failure strains in tension, $\bar{\varepsilon}_{irc}$ $(i = 1,2,3)$ the failure strains in compression and $\bar{\varepsilon}_{ir}$ $(i = 4,5,6)$ the shear failure strains.

The failure strains in tension and shear are positive, whereas in compression they are negative.

Just as for the maximum stress theory, if the strain state only depends on one parameter we can define for each expression the maximum strain coefficient $F_{i\varepsilon}$ which is associated with the strain state leading to failure.

This strain state:

$$\bar{\varepsilon}_i' = F_{i\varepsilon}\bar{\varepsilon}_i \text{ (no summation)},$$

gives the loading coefficients:

$$F_{i\varepsilon} = \frac{\bar{\varepsilon}_{irt}}{\bar{\varepsilon}_i} \text{ or } F_{i\varepsilon} = \frac{\bar{\varepsilon}_{irc}}{\bar{\varepsilon}_i} \quad (i = 1,2,3),$$

$$F_{i\varepsilon} = \frac{\overline{\varepsilon}_{ir}}{\overline{\varepsilon}_i} \text{ or } F_{i\varepsilon} = -\frac{\overline{\varepsilon}_{ir}}{\overline{\varepsilon}_i} \ (i = 4,5,6),$$

with, as for the maximum stress theory, the same remarks concerning the signs of the sign convention and the failure modes.

4.4. Polynomial failure criteria

Composite failure occurs when the stress field no longer satisfies the expression:

$$F_i\overline{\sigma}_i + F_{ij}\overline{\sigma}_i\overline{\sigma}_j + F_{ijk}\overline{\sigma}_i\overline{\sigma}_j\overline{\sigma}_k + \ldots < 1 \qquad (i, j, k, \ldots = 1, 2, \ldots, 6).$$

The coefficients F_i, F_{ij} and F_{ijk}, … in this criterion are found experimentally.

The different criteria described in the following paragraphs are of this type.

4.4.1. *Tsai-Hill criterion*

The Tsai-Hill criterion is of the form:

$$F_{ij}\overline{\sigma}_i\overline{\sigma}_j < 1 \quad (i, j = 1, 2, \ldots, 6),$$

with $F_{ij} = F_{ji}$.

As for the von Mises criterion, it is assumed that a change in hydrostatic pressure has no influence on the failure of the material.

For an orthotropic material the Tsai-Hill criterion is written as:

$$a(\overline{\sigma}_1 - \overline{\sigma}_2)^2 + b(\overline{\sigma}_2 - \overline{\sigma}_3)^2 + c(\overline{\sigma}_3 - \overline{\sigma}_1)^2 + d\overline{\sigma}_4^2 + e\overline{\sigma}_5^2 + f\overline{\sigma}_6^2 < 1.$$

The six constants a, b, c, d, e and f are determined from six independent loading cases.

For failure by extension in the $\mathbf{X}_1$ direction, then $\mathbf{X}_2$ and finally $\mathbf{X}_3$, we obtain the three following expressions:

$$(a + c)\overline{\sigma}_{1r}^2 = 1, \quad (a + b)\overline{\sigma}_{2r}^2 = 1, \quad (b + c)\overline{\sigma}_{3r}^2 = 1,$$

which leads to:

$$a + c = \frac{1}{\overline{\sigma}_{1r}^2}, \quad a + b = \frac{1}{\overline{\sigma}_{2r}^2}, \quad b + c = \frac{1}{\overline{\sigma}_{3r}^2},$$

for which the solution is:

$$a = \frac{1}{2}\left(\frac{1}{\bar{\sigma}_{1r}^2} + \frac{1}{\bar{\sigma}_{2r}^2} - \frac{1}{\bar{\sigma}_{3r}^2}\right),$$

$$b = \frac{1}{2}\left(\frac{1}{\bar{\sigma}_{2r}^2} + \frac{1}{\bar{\sigma}_{3r}^2} - \frac{1}{\bar{\sigma}_{1r}^2}\right),$$

$$c = \frac{1}{2}\left(\frac{1}{\bar{\sigma}_{3r}^2} + \frac{1}{\bar{\sigma}_{1r}^2} - \frac{1}{\bar{\sigma}_{2r}^2}\right).$$

For failure by shear in the plane $(\mathbf{X}_2,\mathbf{X}_3)$, then $(\mathbf{X}_3,\mathbf{X}_1)$ and finally $(\mathbf{X}_1,\mathbf{X}_2)$, we obtain the three expressions:

$$d\bar{\sigma}_{4r}^2 = 1, \quad e\bar{\sigma}_{5r}^2 = 1, \quad f\bar{\sigma}_{6r}^2 = 1,$$

which gives:

$$d = \frac{1}{\bar{\sigma}_{4r}^2}, \quad e = \frac{1}{\bar{\sigma}_{5r}^2}, \quad f = \frac{1}{\bar{\sigma}_{6r}^2}.$$

By identification of the polynomial, we obtain the following non-zero coefficients F_{ij}:

$$F_{11} = \frac{1}{\bar{\sigma}_{1r}^2}, \quad F_{22} = \frac{1}{\bar{\sigma}_{2r}^2}, \quad F_{33} = \frac{1}{\bar{\sigma}_{3r}^2},$$

$$F_{44} = \frac{1}{\bar{\sigma}_{4r}^2}, \quad F_{55} = \frac{1}{\bar{\sigma}_{5r}^2}, \quad F_{66} = \frac{1}{\bar{\sigma}_{6r}^2},$$

$$F_{12} = -\frac{1}{2}\left(\frac{1}{\bar{\sigma}_{1r}^2} + \frac{1}{\bar{\sigma}_{2r}^2} - \frac{1}{\bar{\sigma}_{3r}^2}\right),$$

$$F_{23} = -\frac{1}{2}\left(\frac{1}{\bar{\sigma}_{2r}^2} + \frac{1}{\bar{\sigma}_{3r}^2} - \frac{1}{\bar{\sigma}_{1r}^2}\right),$$

$$F_{31} = -\frac{1}{2}\left(\frac{1}{\bar{\sigma}_{3r}^2} + \frac{1}{\bar{\sigma}_{1r}^2} - \frac{1}{\bar{\sigma}_{2r}^2}\right).$$

In the expressions above the failure stresses, $\bar{\sigma}_{ir}$ $(i = 1, 2, 3)$ will be taken equal to the tensile failure stress $\bar{\sigma}_{irt}$ for $\bar{\sigma}_i$ positive, and in compression $\bar{\sigma}_{irc}$ for $\bar{\sigma}_i$ negative.

Composite failure occurs when the stress field no longer satisfies the expression:

$$F_{11}\bar{\sigma}_1^2 + F_{22}\bar{\sigma}_2^2 + F_{33}\bar{\sigma}_3^2 + 2F_{23}\bar{\sigma}_2\bar{\sigma}_3 + 2F_{31}\bar{\sigma}_3\bar{\sigma}_1 + 2F_{12}\bar{\sigma}_1\bar{\sigma}_2 \ldots$$

$$\ldots + F_{44}\bar{\sigma}_4^2 + F_{55}\bar{\sigma}_5^2 + F_{66}\bar{\sigma}_6^2 < 1.$$

The mode or modes of composite failure are determined from the dominant terms in the Tsai-Hill criterion.

When the external loads are such that the stress field is defined by a multiplying constant the Tsai-Hill loading coefficient F_{TH} is introduced for which failure occurs. The stress field leading to failure of the composite is written as:

$$\bar{\sigma}_i' = F_{TH}\bar{\sigma}_i .$$

This is introduced into the expression for the criterion:

$$(F_{TH})^2 F_{ij}\bar{\sigma}_i\bar{\sigma}_j < 1 ,$$

and the Tsai-Hill loading coefficient is obtained:

$$F_{TH} = \frac{1}{\sqrt{F_{ij}\bar{\sigma}_i\bar{\sigma}_j}}.$$

In a composite the first failure occurs in the layer with the lowest loading coefficient. In addition, the failure mode or modes correspond to the dominant terms in the criterion.

In the case of a plane stress field defined by:

$$\sigma_3 = \sigma_4 = \sigma_5 = 0 ,$$

the Tsai-Hill criterion is written as:

$$F_{11}\bar{\sigma}_1^2 + F_{22}\bar{\sigma}_2^2 + 2F_{12}\bar{\sigma}_1\bar{\sigma}_2 + F_{66}\bar{\sigma}_6^2 < 1 .$$

For a transverse isotropic material with the isotropic plane $(\mathbf{X}_2, \mathbf{X}_3)$, the two corresponding orthotropic directions are equivalent. The coefficients in the Tsai-Hill criterion are then equal to:

$$F_{11} = \frac{1}{\bar{\sigma}_{1r}^2}, \quad F_{22} = F_{33} = \frac{1}{\bar{\sigma}_{2r}^2},$$

$$F_{44} = \frac{1}{\bar{\sigma}_{4r}^2}, \quad F_{55} = F_{66} = \frac{1}{\bar{\sigma}_{6r}^2},$$

$$F_{23} = \frac{1}{2\bar{\sigma}_{1r}^2} - \frac{1}{\bar{\sigma}_{2r}^2}, \quad F_{12} = F_{31} = -\frac{1}{2\bar{\sigma}_{1r}^2},$$

and the criterion can be written in the form:

$$\frac{\bar{\sigma}_1^2}{\bar{\sigma}_{1r}^2} + \frac{\bar{\sigma}_2^2 + \bar{\sigma}_3^2}{\bar{\sigma}_{2r}^2} - \frac{\bar{\sigma}_1(\bar{\sigma}_2 + \bar{\sigma}_3)}{\bar{\sigma}_{1r}^2} + \left(\frac{1}{\bar{\sigma}_{1r}^2} - \frac{2}{\bar{\sigma}_{2r}^2}\right)\bar{\sigma}_2\bar{\sigma}_3 + \frac{\bar{\sigma}_4^2}{\bar{\sigma}_{4r}^2} + \frac{\bar{\sigma}_5^2 + \bar{\sigma}_6^2}{\bar{\sigma}_{6r}^2} < 1 .$$

When the stress state is plane stress $(\sigma_3 = \sigma_4 = \sigma_5 = 0)$, this reduces to:

$$\frac{\bar{\sigma}_1^2}{\bar{\sigma}_{1r}^2} + \frac{\bar{\sigma}_2^2}{\bar{\sigma}_{2r}^2} - \frac{\bar{\sigma}_1\bar{\sigma}_2}{\bar{\sigma}_{1r}^2} + \frac{\bar{\sigma}_6^2}{\bar{\sigma}_{6r}^2} < 1 .$$

In the latter case, designating the stress values at failure by $\bar{\sigma}_i' = F_{TH}\bar{\sigma}_i$ then the Tsai-Hill loading coefficients at failure are:

$$F_{TH} = \frac{1}{\sqrt{\left(\frac{\bar{\sigma}_1}{\bar{\sigma}_{1r}}\right)^2 + \left(\frac{\bar{\sigma}_2}{\bar{\sigma}_{2r}}\right)^2 - \frac{\bar{\sigma}_1 \bar{\sigma}_2}{\bar{\sigma}_{1r}^2} + \left(\frac{\bar{\sigma}_6}{\bar{\sigma}_{6r}}\right)^2}}.$$

4.4.2. *Tsai-Wu criterion*

The Tsai-Wu criterion is written as:

$$F_i \bar{\sigma}_i + F_{ij} \bar{\sigma}_i \bar{\sigma}_j < 1,$$

where the constants F_i and F_{ij} are determined from independent tests.

– In the case of tensile and compression failure in the direction $\mathbf{X}_1$, we obtain the equations:

$$F_1 \bar{\sigma}_{1rt} + F_{11} \bar{\sigma}_{1rt}^2 = 1,$$

$$F_1 \bar{\sigma}_{1rc} + F_{11} \bar{\sigma}_{1rc}^2 = 1,$$

in which the unknowns are F_1 and F_{11}.

The determinant of the system and the determinants associated with the two unknowns being equal to:

$$\Delta = \bar{\sigma}_{1rt} \bar{\sigma}_{1rc} \left(\bar{\sigma}_{1rc} - \bar{\sigma}_{1rt}\right),$$

$$\Delta_1 = \bar{\sigma}_{1rc}^2 - \bar{\sigma}_{1rt}^2,$$

$$\Delta_2 = \bar{\sigma}_{1rt} - \bar{\sigma}_{1rc},$$

we obtain:

$$F_1 = \frac{\bar{\sigma}_{1rc} + \bar{\sigma}_{1rt}}{\bar{\sigma}_{1rt} \bar{\sigma}_{1rc}} = \frac{1}{\bar{\sigma}_{1rt}} + \frac{1}{\bar{\sigma}_{1rc}},$$

$$F_{11} = -\frac{1}{\bar{\sigma}_{1rt} \bar{\sigma}_{1rc}}.$$

The constants F_2, F_{22}, F_3 and F_{33} are obtained from tensile and compression tests in the $\mathbf{X}_2$ and $\mathbf{X}_3$ directions.

– In the case of a shear failure in the plane $(\mathbf{X}_2, \mathbf{X}_3)$, there are two equations:

$$F_4 \bar{\sigma}_{4r} + F_{44} \bar{\sigma}_{4r}^2 = 1,$$

$$-F_4 \bar{\sigma}_{4r} + F_{44} \bar{\sigma}_{4r}^2 = 1,$$

with $\bar{\sigma}_{4r}$ and $-\bar{\sigma}_{4r}$ being the shear stress. From this:

$$F_4 = 0, \quad F_{44} = \frac{1}{\overline{\sigma}_{4r}^2} .$$

In an analogous manner we obtain the values of F_5, F_{55}, F_6 and F_{66}.

– The coupling term F_{12} can be obtained from a biaxial test but this is very difficult to perform and in the absence of experimental results the following expression for F_{12} can be used:

$$F_{12} = -\frac{1}{2\sqrt{\overline{\sigma}_{1rt}\overline{\sigma}_{1rc}\overline{\sigma}_{2rt}\overline{\sigma}_{2rc}}} .$$

The expressions for F_{23} and F_{31} are analogous to that for F_{12}.

– The other coupling terms F_{14}, F_{15}, ... are taken to be zero.

The non-zero coefficients in the Tsai-Wu criterion are therefore:

$$F_1 = \frac{1}{\overline{\sigma}_{1rt}} + \frac{1}{\overline{\sigma}_{1rc}}, \quad F_2 = \frac{1}{\overline{\sigma}_{2rt}} + \frac{1}{\overline{\sigma}_{2rc}}, \quad F_3 = \frac{1}{\overline{\sigma}_{3rt}} + \frac{1}{\overline{\sigma}_{3rc}},$$

$$F_{11} = -\frac{1}{\overline{\sigma}_{1rt}\overline{\sigma}_{1rc}}, \quad F_{22} = -\frac{1}{\overline{\sigma}_{2rt}\overline{\sigma}_{2rc}}, \quad F_{33} = -\frac{1}{\overline{\sigma}_{3rt}\overline{\sigma}_{3rc}},$$

$$F_{44} = \frac{1}{\overline{\sigma}_{4r}^2}, \quad F_{55} = \frac{1}{\overline{\sigma}_{5r}^2}, \quad F_{66} = \frac{1}{\overline{\sigma}_{6r}^2},$$

$$F_{12} = -\frac{1}{2\sqrt{\overline{\sigma}_{1rt}\overline{\sigma}_{1rc}\overline{\sigma}_{2rt}\overline{\sigma}_{2rc}}},$$

$$F_{23} = -\frac{1}{2\sqrt{\overline{\sigma}_{2rt}\overline{\sigma}_{2rc}\overline{\sigma}_{3rt}\overline{\sigma}_{3rc}}},$$

$$F_{31} = -\frac{1}{2\sqrt{\overline{\sigma}_{3rt}\overline{\sigma}_{3rc}\overline{\sigma}_{1rt}\overline{\sigma}_{1rc}}},$$

and the criterion is written in the form:

$$F_1\overline{\sigma}_1 + F_2\overline{\sigma}_2 + F_3\overline{\sigma}_3 + F_{11}\overline{\sigma}_1^2 + F_{22}\overline{\sigma}_2^2 + F_{33}\overline{\sigma}_3^2 + 2F_{12}\overline{\sigma}_1\overline{\sigma}_2 ...$$
$$... + 2F_{23}\overline{\sigma}_2\overline{\sigma}_3 + 2F_{31}\overline{\sigma}_3\overline{\sigma}_1 + F_{44}\overline{\sigma}_4^2 + F_{55}\overline{\sigma}_5^2 + F_{66}\overline{\sigma}_6^2 < 1.$$

If the stress field is defined by a multiplying constant we introduce the Tsai-Wu loading coefficient F_{TW} expressing the stresses leading to failure in the form:

$$\overline{\sigma}_i' = F_{TW}\overline{\sigma}_i .$$

Introducing this into the Tsai-Wu criterion we obtain the expression:

$$F_{TW} F_i\overline{\sigma}_i + F_{TW}^2 F_{ij}\overline{\sigma}_i\overline{\sigma}_j = 1,$$

which enables F_{TW} to be calculated.

The first composite failure occurs in the layer with the smallest Tsai-Wu loading coefficient.

For a plane stress state the criterion is written as:

$$F_1\bar{\sigma}_1 + F_2\bar{\sigma}_2 + F_{11}\bar{\sigma}_1^2 + F_{22}\bar{\sigma}_2^2 + 2F_{12}\bar{\sigma}_1\bar{\sigma}_2 + F_{66}\bar{\sigma}_6^2 < 1 .$$

For a transversely isotropic material with the isotropic plane $(\mathbf{X}_2, \mathbf{X}_3)$, the coefficients are:

$$F_1 = \frac{1}{\bar{\sigma}_{1rt}} + \frac{1}{\bar{\sigma}_{1rc}}, \quad F_2 = F_3 = \frac{1}{\bar{\sigma}_{2rt}} + \frac{1}{\bar{\sigma}_{2rc}},$$
$$F_{11} = -\frac{1}{\bar{\sigma}_{1rt}\bar{\sigma}_{1rc}}, \quad F_{22} = F_{33} = -\frac{1}{\bar{\sigma}_{2rt}\bar{\sigma}_{2rc}},$$
$$F_{44} = \frac{1}{\bar{\sigma}_{4r}^2}, \quad F_{55} = F_{66} = \frac{1}{\bar{\sigma}_{6r}^2},$$
$$F_{23} = \frac{1}{2\bar{\sigma}_{2rt}\bar{\sigma}_{2rc}}, \quad F_{12} = F_{31} = -\frac{1}{2\sqrt{\bar{\sigma}_{1rt}\bar{\sigma}_{1rc}\bar{\sigma}_{2rt}\bar{\sigma}_{2rc}}}.$$

The Tsai-Wu criterion is written as:

$$F_1\bar{\sigma}_1 + F_2(\bar{\sigma}_2 + \bar{\sigma}_3) + F_{11}\bar{\sigma}_1^2 + F_{22}\left(\bar{\sigma}_2^2 + \bar{\sigma}_3^2\right) + 2F_{12}\bar{\sigma}_1(\bar{\sigma}_2 + \bar{\sigma}_3)\ldots$$
$$\ldots + 2F_{23}\bar{\sigma}_2\bar{\sigma}_3 + F_{44}\bar{\sigma}_4^2 + F_{66}\left(\bar{\sigma}_5^2 + \bar{\sigma}_6^2\right) < 1.$$

and for the plane stress state this reduces to:

$$F_1\bar{\sigma}_1 + F_2\bar{\sigma}_2 + F_{11}\bar{\sigma}_1^2 + F_{22}\bar{\sigma}_2^2 + 2F_{12}\bar{\sigma}_1\bar{\sigma}_2 + F_{66}\bar{\sigma}_6^2 < 1 .$$

4.4.3. *Hoffman criterion*

The Hoffman criterion has the form:

$$F_i\bar{\sigma}_i + F_{ij}\bar{\sigma}_i\bar{\sigma}_j < 1,$$

with the non-zero coefficients:

$$F_1 = \frac{1}{\bar{\sigma}_{1rt}} + \frac{1}{\bar{\sigma}_{1rc}}, F_2 = \frac{1}{\bar{\sigma}_{2rt}} + \frac{1}{\bar{\sigma}_{2rc}}, F_3 = \frac{1}{\bar{\sigma}_{3rt}} + \frac{1}{\bar{\sigma}_{3rc}},$$
$$F_{11} = -\frac{1}{\bar{\sigma}_{1rt}\bar{\sigma}_{1rc}}, F_{22} = -\frac{1}{\bar{\sigma}_{2rt}\bar{\sigma}_{2rc}}, F_{33} = -\frac{1}{\bar{\sigma}_{3rt}\bar{\sigma}_{3rc}},$$

$$F_{44} = \frac{1}{\bar{\sigma}_{4r}^2},\ F_{55} = \frac{1}{\bar{\sigma}_{5r}^2},\ F_{66} = \frac{1}{\bar{\sigma}_{6r}^2},$$

$$F_{12} = -\frac{1}{2}\left(\frac{1}{\bar{\sigma}_{3rt}\bar{\sigma}_{3rc}} - \frac{1}{\bar{\sigma}_{1rt}\bar{\sigma}_{1rc}} - \frac{1}{\bar{\sigma}_{2rt}\bar{\sigma}_{2rc}}\right),$$

$$F_{23} = -\frac{1}{2}\left(\frac{1}{\bar{\sigma}_{1rt}\bar{\sigma}_{1rc}} - \frac{1}{\bar{\sigma}_{2rt}\bar{\sigma}_{2rc}} - \frac{1}{\bar{\sigma}_{3rt}\bar{\sigma}_{3rc}}\right),$$

$$F_{31} = -\frac{1}{2}\left(\frac{1}{\bar{\sigma}_{2rt}\bar{\sigma}_{2rc}} - \frac{1}{\bar{\sigma}_{3rt}\bar{\sigma}_{3rc}} - \frac{1}{\bar{\sigma}_{1rt}\bar{\sigma}_{1rc}}\right).$$

It may be noted that the coefficients F_1, F_2, F_3, F_{11}, F_{22}, F_{33}, F_{44}, F_{55} and F_{66} are identical to those in the Tsai-Wu criterion, but the coupling coefficients F_{12}, F_{23} and F_{31} are different.

Replacing $\bar{\sigma}_{irc}$ by $-\bar{\sigma}_{irt}$ in these coefficients, we obtain the coefficients of the Tsai-Hill criterion for the tensile stresses.

The Hoffman criterion has the same form as the Tsai-Wu criterion:

$$F_1\bar{\sigma}_1 + F_2\bar{\sigma}_2 + F_3\bar{\sigma}_3 + F_{11}\bar{\sigma}_1^2 + F_{22}\bar{\sigma}_2^2 + F_{33}\bar{\sigma}_3^2 + 2F_{12}\bar{\sigma}_1\bar{\sigma}_2 \ldots$$
$$\ldots + 2F_{23}\bar{\sigma}_2\bar{\sigma}_3 + 2F_{31}\bar{\sigma}_3\bar{\sigma}_1 + F_{44}\bar{\sigma}_4^2 + F_{55}\bar{\sigma}_5^2 + F_{66}\bar{\sigma}_6^2 < 1.$$

In the case where the stress field leading to failure can be written as:

$$\bar{\sigma}_i' = F_H\bar{\sigma}_i,$$

the Hoffman loading coefficient F_H is given by the expression:

$$F_H F_i\bar{\sigma}_i + F_H^2 F_{ij}\bar{\sigma}_i\bar{\sigma}_j = 1.$$

The composite layer where the first failure occurs is that with the smallest Hoffman loading coefficient.

For plane stress the Hoffman criterion is written as:

$$F_1\bar{\sigma}_1 + F_2\bar{\sigma}_2 + F_{11}\bar{\sigma}_1^2 + F_{22}\bar{\sigma}_2^2 + 2F_{12}\bar{\sigma}_1\bar{\sigma}_2 + F_{66}\bar{\sigma}_6^2 < 1.$$

For a transversely isotropic material with the isotropic plane $(\mathbf{X}_2, \mathbf{X}_3)$, we have the coefficients:

$$F_1 = \frac{1}{\bar{\sigma}_{1rt}} + \frac{1}{\bar{\sigma}_{1rc}},\ F_2 = F_3 = \frac{1}{\bar{\sigma}_{2rt}} + \frac{1}{\bar{\sigma}_{2rc}},$$

$$F_{11} = -\frac{1}{\bar{\sigma}_{1rt}\bar{\sigma}_{1rc}},\ F_{22} = F_{33} = -\frac{1}{\bar{\sigma}_{2rt}\bar{\sigma}_{2rc}},$$

$$F_{44} = \frac{1}{\overline{\sigma}_{4r}^2},\ F_{55} = F_{66} = \frac{1}{\overline{\sigma}_{6r}^2},$$

$$F_{23} = -\frac{1}{2}\left(\frac{1}{\overline{\sigma}_{1rt}\overline{\sigma}_{1rc}} - \frac{2}{\overline{\sigma}_{2rt}\overline{\sigma}_{2rc}}\right),\quad F_{12} = F_{31} = \frac{1}{2\overline{\sigma}_{1rt}\overline{\sigma}_{1rc}},$$

and the criterion:

$$F_1\overline{\sigma}_1 + F_2(\overline{\sigma}_2 + \overline{\sigma}_3) + F_{11}\overline{\sigma}_1^2 + F_{22}\left(\overline{\sigma}_2^2 + \overline{\sigma}_3^2\right) + 2F_{12}\overline{\sigma}_1(\overline{\sigma}_2 + \overline{\sigma}_3)\ldots$$
$$\ldots + 2F_{23}\overline{\sigma}_2\overline{\sigma}_{3_1} + F_{44}\overline{\sigma}_4^2 + F_{66}\left(\overline{\sigma}_5^2 + \overline{\sigma}_6^2\right) < 1,$$

which, for the special case of plane stress is written as:

$$F_1\overline{\sigma}_1 + F_2\overline{\sigma}_2 + F_{11}\overline{\sigma}_1^2 + F_{22}\overline{\sigma}_2^2 + 2F_{12}\overline{\sigma}_1\overline{\sigma}_2 + F_{66}\overline{\sigma}_6^2 < 1\,.$$

4.5. Tensile and shear strength of a unidirectional layer

4.5.1. *Tensile strength*

The unidirectional layer shown below with $0 < \alpha < \frac{\pi}{2}$, is subjected to a tensile load in the $\mathbf{x}_1$ direction, in an off-axis direction with respect to the orthotropic axes.

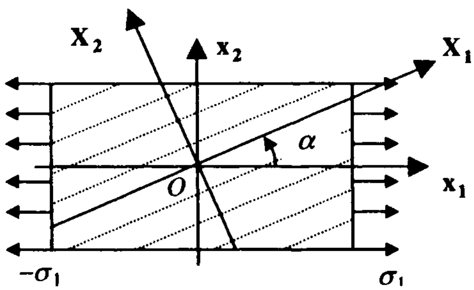

Figure 4.1. *Off-axis tension*

The co-ordinate change matrix and its transposed form are:

$$\mathbf{a} = \begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix},\ \mathbf{a}^T = \begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

the change of axes matrix $\mathbf{N}$ being written as:

$$\mathbf{N} = \begin{bmatrix} c^2 & s^2 & -cs \\ s^2 & c^2 & cs \\ 2cs & -2cs & c^2 - s^2 \end{bmatrix},$$

with the membrane stresses in the orthotropic axes given by:

$$\bar{\boldsymbol{\sigma}} = \mathbf{N}^T \boldsymbol{\sigma},$$

or:

$$\begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ 0 \\ 0 \end{bmatrix},$$

that is:

$$\bar{\sigma}_1 = c^2 \sigma_1,$$
$$\bar{\sigma}_2 = s^2 \sigma_1,$$
$$\bar{\sigma}_6 = -cs\sigma_1.$$

Introducing these values in the maximum stress criterion:

$$\bar{\sigma}_{1rc} < \bar{\sigma}_1 < \bar{\sigma}_{1rt},$$
$$\bar{\sigma}_{2rc} < \bar{\sigma}_2 < \bar{\sigma}_{2rt},$$
$$-\bar{\sigma}_{6r} < \bar{\sigma}_6 < \bar{\sigma}_{6r},$$

we obtain the following expressions:

$$\frac{\bar{\sigma}_{1rc}}{c^2} < \sigma_1 < \frac{\bar{\sigma}_{1rt}}{c^2},$$
$$\frac{\bar{\sigma}_{2rc}}{s^2} < \sigma_1 < \frac{\bar{\sigma}_{2rt}}{s^2},$$
$$-\frac{\bar{\sigma}_{6r}}{cs} < \sigma_1 < \frac{\bar{\sigma}_{6r}}{cs}.$$

The values of σ_1 for which there is no failure are located between the curves shown as continuous lines on figure 4.2. It may be noted that, in this case, failure occurs gradually as the angle α increases by tensile or compression failure of the fibres, then by composite shear failure, and finally by matrix failure in tension or compression.

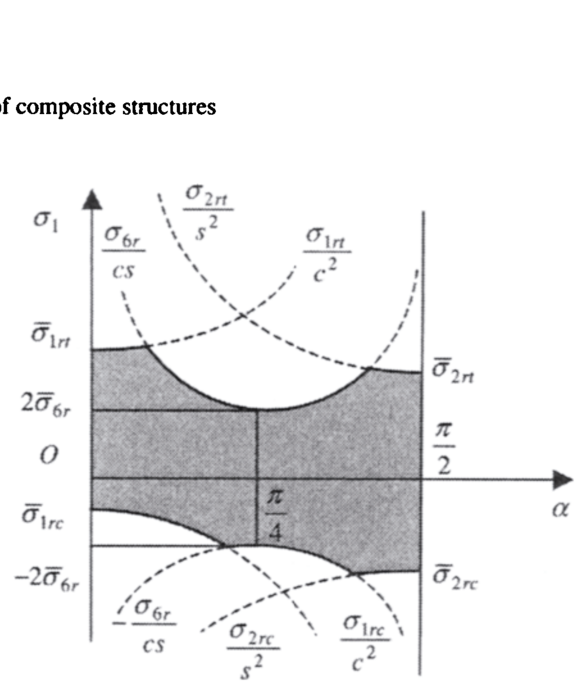

Figure 4.2. *Maximum stress criterion, simple tension*

The Tsai-Hill criterion for a transversely isotropic layer with the isotropic plane $(\vec{X}_2, \vec{X}_3)$:

$$\left(\frac{\sigma_1}{\bar{\sigma}_{1r}}\right)^2 + \left(\frac{\sigma_2}{\bar{\sigma}_{2r}}\right)^2 - \frac{\sigma_1 \sigma_2}{\bar{\sigma}_{1r}^2} + \left(\frac{\sigma_6}{\bar{\sigma}_{6r}}\right)^2 < 1,$$

may be written for this case as:

$$\left(\frac{c^4}{\bar{\sigma}_{1r}^2} + \frac{s^4}{\bar{\sigma}_{2r}^2} - \frac{c^2 s^2}{\bar{\sigma}_{1r}^2} + \frac{c^2 s^2}{\bar{\sigma}_{6r}^2}\right)\sigma_1^2 < 1 .$$

Using this criterion the tensile and compression failure stresses are equal to:

$$\sigma_{1rt} = \frac{1}{\sqrt{\frac{c^4}{\bar{\sigma}_{1rt}^2} + \frac{s^4}{\bar{\sigma}_{2rt}^2} - \frac{c^2 s^2}{\bar{\sigma}_{1rt}^2} + \frac{c^2 s^2}{\bar{\sigma}_{6r}^2}}}, \quad \sigma_{1rc} = -\frac{1}{\sqrt{\frac{c^4}{\bar{\sigma}_{1rc}^2} + \frac{s^4}{\bar{\sigma}_{2rc}^2} - \frac{c^2 s^2}{\bar{\sigma}_{1rc}^2} + \frac{c^2 s^2}{\bar{\sigma}_{6r}^2}}} .$$

Failure does not occur as long as the tensile-compression stress σ_1 is located, for a given value of α, between the two curves shown on the following figure:

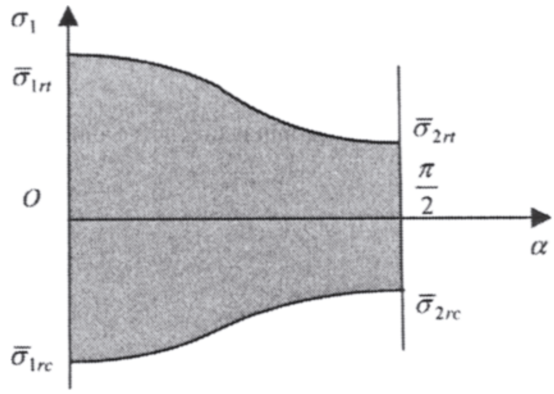

Figure 4.3. *Tsai-Hill criterion, simple tension*

4.5.2. *Shear strength*

The fibre orientation is chosen such that: $0 < \alpha < \frac{\pi}{2}$.

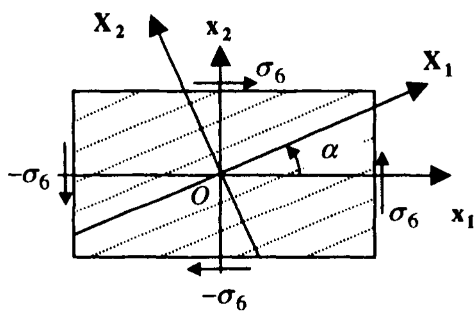

Figure 4.4. *Off-axis shear*

The membrane stresses in the orthotropic axes:

$$\begin{bmatrix} \bar{\sigma}_1 \\ \bar{\sigma}_2 \\ \bar{\sigma}_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2cs \\ s^2 & c^2 & -2cs \\ -cs & cs & c^2 - s^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \sigma_6 \end{bmatrix},$$

are equal to:

$$\overline{\sigma}_1 = 2cs\sigma_6,$$
$$\overline{\sigma}_2 = -2cs\sigma_6,$$
$$\overline{\sigma}_6 = \left(c^2 - s^2\right)\sigma_6.$$

Introducing these values in the maximum stress criterion we obtain:

$$\frac{\overline{\sigma}_{1rc}}{2cs} < \sigma_6 < \frac{\overline{\sigma}_{1rt}}{2cs},$$
$$-\frac{\overline{\sigma}_{2rt}}{2cs} < \sigma_6 < -\frac{\overline{\sigma}_{2rc}}{2cs},$$
$$-\frac{\overline{\sigma}_{6r}}{\left|c^2 - s^2\right|} < \sigma_6 < \frac{\overline{\sigma}_{6r}}{\left|c^2 - s^2\right|}.$$

For a positive shear stress σ_6, there is tension in the $\mathbf{X}_1$ direction and compression in the $\mathbf{X}_2$ direction.

Applying the maximum stress criterion there is no failure for values of σ_6 located in the grey zone represented in the figure below.

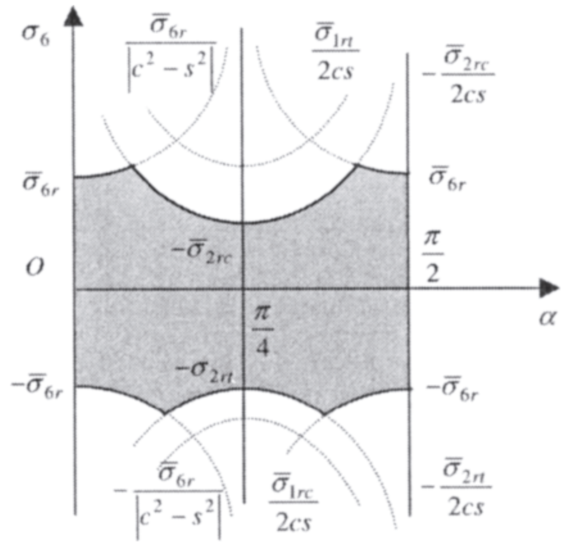

Figure 4.5. *Maximum stress criterion for pure shear*

When the shear stress is positive the Tsai-Hill criterion, for a transversely isotropic later with isotropic plane $(\mathbf{X}_2, \mathbf{X}_3)$, may be written, taking into account the signs of $\overline{\sigma}_1$ and $\overline{\sigma}_2$ as:

$$\left(\frac{4c^2s^2}{\overline{\sigma}_{1rt}^2}+\frac{4c^2s^2}{\overline{\sigma}_{2rc}^2}+\frac{4c^2s^2}{\overline{\sigma}_{1rt}^2}+\frac{\left(c^2-s^2\right)^2}{\overline{\sigma}_{6r}^2}\right)\sigma_6^2<1\,.$$

The shear stress, which leads to failure in the unidirectional layer, is:

$$\sigma_{6r}=\frac{1}{\sqrt{4c^2s^2\left(\dfrac{2}{\overline{\sigma}_{1rt}^2}+\dfrac{1}{\overline{\sigma}_{2rc}^2}\right)+\dfrac{\left(c^2-s^2\right)^2}{\overline{\sigma}_{6r}^2}}}\,.$$

When the shear stress is negative, there is compression in the $\mathbf{X}_1$ direction and tension in the $\mathbf{X}_2$ direction and the failure stress is then given by:

$$\sigma_{6r}=-\frac{1}{\sqrt{4c^2s^2\left(\dfrac{2}{\overline{\sigma}_{1rc}^2}+\dfrac{1}{\overline{\sigma}_{2rt}^2}\right)+\dfrac{\left(c^2-s^2\right)^2}{\overline{\sigma}_{6r}^2}}}\,.$$

As long as the shear stress is in the grey zone in the figure below there is no failure.

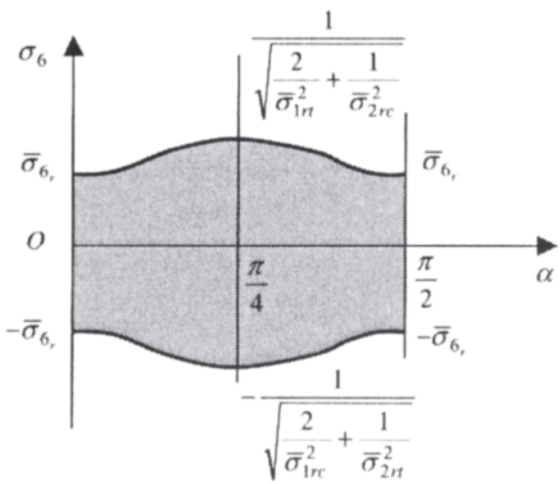

Figure 4.6. *Tsai-Hill criterion for pure shear*

4.6. Determination of failure stresses from three tension tests

The failure stress of a transversely isotropic layer, shown below, is obtained by a tensile test in the $\mathbf{x}_1$ direction.

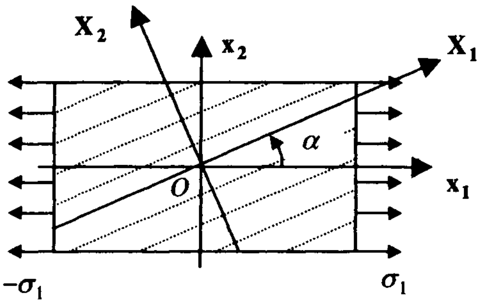

Figure 4.7. *Tensile test off-axis*

The tensile stress resulting in failure is, according to the Tsai-Hill criterion, given by:

$$\left(\frac{c^4}{\overline{\sigma}_{1rt}^2}+\frac{s^4}{\overline{\sigma}_{2rt}^2}-\frac{c^2s^2}{\overline{\sigma}_{1rt}^2}+\frac{c^2s^2}{\overline{\sigma}_{6r}^2}\right)\sigma_1^2=1\,.$$

For $\alpha=0$ and for $\alpha=\frac{\pi}{2}$, we obtain the tensile failure stresses in the $\mathbf{X}_1$ and $\mathbf{X}_2$ directions:

$$\overline{\sigma}_{1rt}=\sigma_0\,,\quad \overline{\sigma}_{2rt}=\sigma_{\frac{\pi}{2}}\,.$$

For an angle α between 0 and $\frac{\pi}{2}$, the failure stress in shear is given by the expression:

$$\left(\frac{c^4}{\sigma_0^2}+\frac{s^4}{\sigma_{\frac{\pi}{2}}^2}-\frac{c^2s^2}{\sigma_0^2}+\frac{c^2s^2}{\overline{\sigma}_{6r}^2}\right)\sigma_\alpha^2=1\,,$$

and equal to:

$$\overline{\sigma}_{6r}=\frac{cs}{\sqrt{\dfrac{1}{\sigma_\alpha^2}-\dfrac{c^2\left(c^2-s^2\right)}{\sigma_0^2}-\dfrac{s^4}{\sigma_{\frac{\pi}{2}}^2}}}\,.$$

PART II

Multi-Layer Plates

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Chapter 5

Multi-layer Kirchhoff-Love thin plates

5.1. Introduction

The Kirchhoff-Love theory, presented in the Appendices, which does not take transverse shear strains into account, is used in the study of thin plates for which the ratio of the thickness h to a characteristic dimension a of the mean surface is less than 1/20.

In this chapter the general expressions for multi-layer plates which conform to the Kirchhoff-Love theory will be presented.

The expressions for displacements and strains will first be recalled.

Then the plate equations will be presented for a specific case described below, for equilibrium and vibrations.

The strains and global loads will be introduced, with the global stiffness matrix of the composite and the classic decoupling.

Finally, the transverse shear stresses and composite strain energy will be determined.

5.2. Kirchhoff-Love hypotheses for thin plates

The rectangular plate represented below has thickness h and transverse dimensions a_1 and a_2. It is termed thin if h is small compared to a_1 and a_2. The mean plane of the plate is in the plane $(O|\mathbf{x}_1,\mathbf{x}_2)$ of the galileen reference $(g)=(O|\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$.

The plate is made up of N elastic, linear, homogeneous, orthotropic layers of constant thickness. For all these layers $\mathbf{x}_3$ is the direction of orthotropy. The interface between two successive layers is assumed to be perfect.

It should be noted that this plate is studied for small disturbances, i.e. small transformations and small displacements.

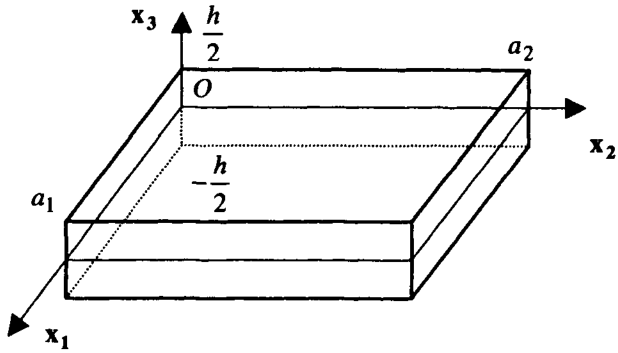

Figure 5.1. *Thin plate*

In the study of flexure and vibrations we assume that the displacements are small compared to the plate thickness.

The normal transverse stress σ_3 is ignored.

According to Kirchhoff-Love theory, the transverse shear strains $\varepsilon_5, \varepsilon_4$ and the rotational inertia are ignored.

5.3. Strain-displacement relationships

According to Kirchhoff-Love theory, the displacement field is given by the expressions:

$$u_1 = u_1^0(x_1, x_2|t) - x_3 \frac{\partial u_3^0}{\partial x_1}(x_1, x_2|t),$$

$$u_2 = u_2^0(x_1, x_2|t) - x_3 \frac{\partial u_3^0}{\partial x_2}(x_1, x_2|t),$$

$$u_3 = u_3^0(x_1, x_2|t),$$

and the strains by:

$$\varepsilon_1 = \frac{\partial u_1^0}{\partial x_1} - x_3 \frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$\varepsilon_2 = \frac{\partial u_2^0}{\partial x_2} - x_3 \frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$\varepsilon_6 = \frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1} - 2x_3 \frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

$$\varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0.$$

The strain field can then be written as:

$$\varepsilon_i = \varepsilon_i^0 + x_3\kappa_i \quad (i = 1, 2, 6),$$

with the membrane strains:

$$\varepsilon_1^0 = \frac{\partial u_1^0}{\partial x_1}, \quad \varepsilon_2^0 = \frac{\partial u_2^0}{\partial x_2}, \quad \varepsilon_6^0 = \frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1},$$

and the curvatures:

$$\kappa_1 = -\frac{\partial^2 u_3^0}{\partial x_1^2}, \quad \kappa_2 = -\frac{\partial^2 u_3^0}{\partial x_2^2}, \quad \kappa_6 = -2\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2}.$$

5.4. Global plate equations

In the absence of buckling, the global plate equations are written in the following form for the three sum equations:

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} + p_1 = I_0 \frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} + p_2 = I_0 \frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial N_5}{\partial x_1} + \frac{\partial N_4}{\partial x_2} + q_3 + p_3 = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

and for the two moment equations:

$$\frac{\partial M_1}{\partial x_1} + \frac{\partial M_6}{\partial x_2} - N_5 = 0,$$

$$\frac{\partial M_6}{\partial x_1} + \frac{\partial M_2}{\partial x_2} - N_4 = 0.$$

From these latter relationships we obtain the global equation:

$$\frac{\partial^2 M_1}{\partial x_1^2} + 2\frac{\partial^2 M_6}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} + q_3 + p_3 = I_0 \frac{\partial^2 u_3^0}{\partial t^2}.$$

Recalling the following expressions:

$$N_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i dx_3, \quad M_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i x_3 dx_3,$$

$$p_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_i dx_3, \quad I_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho dx_3.$$

5.5. Calculation of I_0

The plate being studied is made up of N orthotropic layers limited by planes parallel to the mid-plane of the plate. For all the layers $\mathbf{x}_3$ is the direction of orthotropy, the positions of the layers in the plate thickness are shown below:

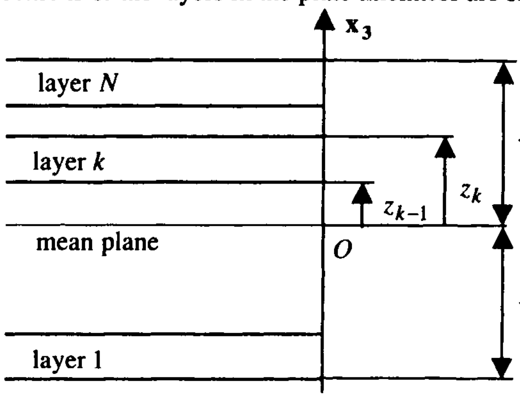

Figure 5.2. *Distribution of layers in the composite*

The layer k is limited by the two planes of the equations $x_3 = z_{k-1}$ and $x_3 = z_k$.

The expression which defines I_0 is:

$$I_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho dx_3 ,$$

or may be written, when applying layer by layer integration:

$$I_0 = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \rho^k dx_3 ,$$

where ρ^k is the density of the layer k, which is independent of x_3.

We thus obtain:

$$I_0 = \sum_{k=1}^{N} \rho^k \left(z_k - z_{k-1}\right).$$

In the particular cases of single layer plates or multi-layer plates with layers of the same density we have:

$$I_0 = \rho h .$$

5.6. Stress field

We now examine the distribution of layers presented in the previous section.

Designating by Q_{ij}^k the reduced stiffness matrix of the layer k, the components of the stress matrix are given by the expression:

$$\sigma_i^k = Q_{ij}^k \varepsilon_j \quad (i, j = 1, 2, 6),$$

which may be written as:

$$\sigma_i^k = Q_{ij}^k \left(\varepsilon_j^0 + x_3 \kappa_j\right), \text{ with } z_{k-1} < x_3 < z_k .$$

5.7. Global cohesive forces

The global cohesive loads are made up of global forces N_i and the global moments M_i which are defined as:

$$N_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i dx_3 \quad M_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i x_3 dx_3 \quad (i, j = 1, 2, 6).$$

The global force component N_i of the cohesion force is given by:

$$N_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i dx_3 \quad (i = 1, 2, 6),$$

which, by layer-by-layer integration, is written as:

$$N_i = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \sigma_i^k dx_3 .$$

Introducing into the expression the constitutive relation:

$$\sigma_i^k = Q_{ij}^k \left(\varepsilon_j^0 + x_3 \kappa_j\right),$$

we obtain:

$$N_i = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} Q_{ij}^k \left(\varepsilon_j^0 + x_3 \kappa_j\right) dx_3 ,$$

then:

$$N_i = \sum_{k=1}^{N} Q_{ij}^k \left[x_3 \varepsilon_j^0 + \frac{x_3^2}{2} \kappa_j \right]_{z_{k-1}}^{z_k} ,$$

or:

$$N_i = \sum_{k=1}^{N} Q_{ij}^k (z_k - z_{k-1})\varepsilon_j^0 + \frac{1}{2}\sum_{k=1}^{N} Q_{ij}^k \left(z_k^2 - z_{k-1}^2\right)\kappa_j \,.$$

For the global moment M_i analogous calculation leads to the following expressions:

$$M_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i \, x_3 dx_3 = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \sigma_i^k x_3 dx_3 \,,$$

$$M_i = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} Q_{ij}^k \left(\varepsilon_j^0 + x_3\kappa_j\right) x_3 dx_3 = \sum_{k=1}^{N} Q_{ij}^k \left[\frac{x_3^2}{2}\varepsilon_j^0 + \frac{x_3^3}{3}\kappa_j\right]_{z_{k-1}}^{z_k} \,,$$

or:

$$M_i = \frac{1}{2}\sum_{k=1}^{N} Q_{ij}^k \left(z_k^2 - z_{k-1}^2\right)\varepsilon_j^0 + \frac{1}{3}\sum_{k=1}^{N} Q_{ij}^k \left(z_k^3 - z_{k-1}^3\right)\kappa_j \,.$$

Putting:

$$A_{ij} = A_{ji} = \sum_{k=1}^{N} Q_{ij}^k (z_k - z_{k-1}),$$

$$B_{ij} = B_{ji} = \frac{1}{2}\sum_{k=1}^{N} Q_{ij}^k \left(z_k^2 - z_{k-1}^2\right),$$

$$D_{ij} = D_{ji} = \frac{1}{3}\sum_{k=1}^{N} Q_{ij}^k \left(z_k^3 - z_{k-1}^3\right),$$

the global cohesive forces N_i and M_i may be expressed as a function of the global strains ε_i^0 and κ_i by the expressions:

$$N_i = A_{ij}\varepsilon_j^0 + B_{ij}\kappa_j,$$

and:

$$M_i = B_{ij}\varepsilon_j^0 + D_{ij}\kappa_j \quad (i,\, j = 1, 2, 6).$$

From the definition of the global stiffnesses of the cohesive forces, the global stiffnesses A_{ij}, B_{ij} and D_{ij} are expressed respectively in $\mathrm{N.m^{-1}}$, N and N.m .

5.8. Composite global stiffness matrix

The six expressions giving the global cohesive forces N_i and M_i, as a function of the global strains ε_i^0 and κ_i, are written in the following matrix form:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix},$$

or in condensed form:

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix},$$

or:

$$\mathbf{N} = \mathbf{A}\boldsymbol{\varepsilon}^0 + \mathbf{B}\boldsymbol{\kappa},$$

$$\mathbf{M} = \mathbf{B}\boldsymbol{\varepsilon}^0 + \mathbf{D}\boldsymbol{\kappa}.$$

The sub-matrix **A** is the sub-matrix of the global membrane stiffness matrix which relates the global membrane forces **N** to the global membrane strains $\boldsymbol{\varepsilon}^0$.

The sub-matrix **D** is the sub-matrix of the global flexure stiffness matrix which relates the global flexural moments **M** to the global flexural strains $\boldsymbol{\kappa}$.

The sub-matrix **B** is the sub-matrix of global stiffness of membrane-flexure coupling which relates the global membrane forces **N** to the global flexural strains $\boldsymbol{\kappa}$ and the global flexural moments **M** to the global membrane strains $\boldsymbol{\varepsilon}^0$.

5.9. Decoupling

5.9.1. *Membrane-flexion decoupling*

The terms of the sub-matrix of global stiffness for membrane-flexion coupling:

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{N} Q_{ij}^k \left(z_k^2 - z_{k-1}^2\right)$$

indicate that the membrane forces cause flexural strains. In a similar way, flexural forces cause membrane strains.

This coupling does not exist when the B_{ij} are zero. If the plate shows mirror symmetry with respect to its mean plane the B_{ij} terms are zero.

As it may be noted from the figure below showing a multi-layer plate with mirror symmetry, for each layer k there is an associated symmetrical k' layer, with respect to the mean plane.

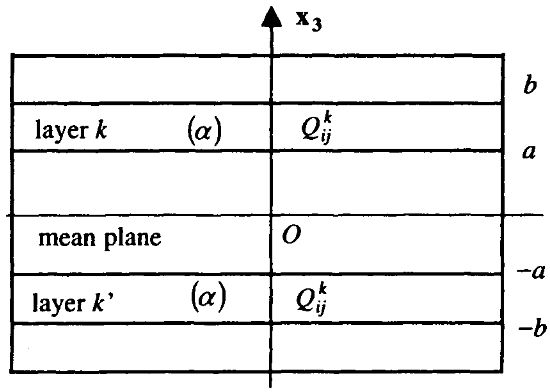

Figure 5.3. *Mirror symmetry*

The k and k' layers have the same reduced stiffnesses Q_{ij}^k.

The contribution of the layer k to the global membrane-flexure coupling stiffness is:

$$B_{ij}^k = \frac{1}{2} Q_{ij}^k \left(b^2 - a^2\right),$$

whereas the contribution of the layer k' is:

$$B_{ij}^{k'} = \frac{1}{2} Q_{ij}^k \left(a^2 - b^2\right).$$

By summation, the terms thus obtained cancel each other out in pairs.

The global coupling stiffnesses of membrane-flexure coupling of a symmetrical composite are zero:

$$B_{ij} = 0.$$

5.9.2. *Tension-shear decoupling*

The terms of the global membrane rigidity sub-matrix:

$$A_{16} = \sum_{k=1}^{N} Q_{16}^k \left(z_k - z_{k-1}\right), \quad A_{26} = \sum_{k=1}^{N} Q_{26}^k \left(z_k - z_{k-1}\right),$$

imply that the tension or shear loads respectively cause angular distortions or dilatations.

This coupling does not exist when the terms A_{16} and A_{26} are zero. These cases are known as *tension-shear decoupling* or *plane decoupling*. If the plate is made of layers with opposite reduced rigidities Q_{16}^{k} and Q_{26}^{k}, and their thicknesses are the same, the terms $Q_{16}^{k}(z_k - z_{k-1})$ and $Q_{26}^{k}(z_k - z_{k-1})$ cancel each other out in pairs.

In a balanced laminate, with each layer k of orientation α is associated a layer k' of orientation $-\alpha$. The stiffnesses Q_{16}^{k} and Q_{26}^{k} are respectively the opposite of the reduced stiffnesses $Q_{16}^{k'}$ and $Q_{26}^{k'}$.

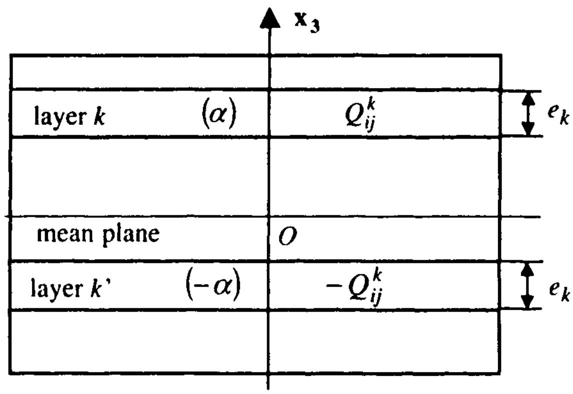

Figure 5.4. *Balanced laminate*

From the expressions:

$$Q_{16}^{k} = c^3 s\overline{Q}_{11}^{k} - cs^3\overline{Q}_{22}^{k} - cs\left(c^2 - s^2\right)\overline{Q}_{12}^{k} - 2cs\left(c^2 - s^2\right)\overline{Q}_{66}^{k},$$

$$Q_{26}^{k} = cs^3\overline{Q}_{11}^{k} - c^3 s\overline{Q}_{22}^{k} + cs\left(c^2 - s^2\right)\overline{Q}_{12}^{k} + 2cs\left(c^2 - s^2\right)\overline{Q}_{66}^{k},$$

for opposite angles the reduced stiffnesses Q_{16}^{k} and $Q_{16}^{k'}$, and the reduced stiffnesses Q_{26}^{k} and $Q_{26}^{k'}$ are opposites.

The contribution of the layer k, of orientation α to the tension-shear coupling is

$$A_{16}^{k} = Q_{16}^{k} e_k \quad \text{with } e_k = z_k - z_{k-1},$$

whereas the contribution of the layer k' of orientation $-\alpha$ is:

$$A_{16}^{k'} = -Q_{16}^{k} e_k .$$

By summation, the terms thus obtained cancel out in pairs. The position of the layers k and k' in the composite has no influence on the values of the global membrane stiffnesses.

The global tension-shear coupling stiffnesses in a balanced composite are zero:

$$A_{16} = A_{26} = 0\,.$$

5.9.3. *Membrane-flexion and tension-shear decoupling*

In the case of four identical layers with the stacking sequence $(\alpha,-\alpha,-\alpha,\alpha)$ represented on the figure below, there is both membrane-flexure decoupling and tension-shear decoupling. The laminate is termed *balanced symmetrical.*

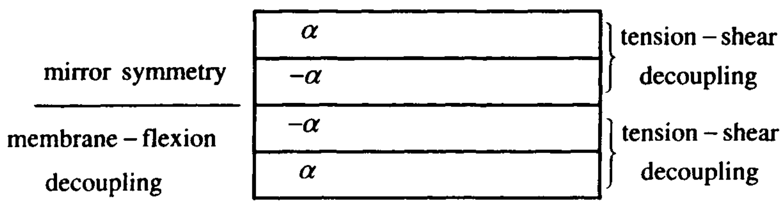

Figure 5.5. *Balanced symmetrical laminate*

In this case we have:

$$B_{ij} = 0, \quad A_{16} = A_{26} = 0\,.$$

The global stiffness matrix of the composite is written as:

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{16} & D_{26} & D_{66} \end{bmatrix}.$$

5.10. Global stiffnesses of a symmetrical composite

5.10.1. *Symmetrical laminate* $(\alpha,\beta)_{NS}$

The $4N$ orthotropic layers are identical and distributed in the composite by repeating N times the angular sequence (α,β) then N times the sequence (β,α).

The sequences are:

- $(\alpha,\beta,\beta,\alpha)$ for $N=1$,
- $(\alpha,\beta,\alpha,\beta,\beta,\alpha,\beta,\alpha)$ for $N=2$,
- $(\alpha,\beta,\alpha,\beta,\alpha,\beta,\beta,\alpha,\beta,\alpha,\beta,\alpha)$ for $N=3$,

and so on.

The global membrane stiffnesses are equal to:

$$A_{ij}=\sum_{k=1}^{4N}Q_{ij}^{k}\left(z_k-z_{k-1}\right),$$

with:

$$z_k=-2Ne+ke=(k-2N)e,$$

we obtain:

$$A_{ij}=e\sum_{k=1}^{4N}Q_{ij}^{k}\left[(k-2N)-(k-2N-1)\right]=e\sum_{k=1}^{4N}Q_{ij}^{k}=e\sum_{k=1}^{2N}\left(Q_{ij}^{\alpha}+Q_{ij}^{\beta}\right),$$

or:

$$A_{ij}=2Ne\left(Q_{ij}^{\alpha}+Q_{ij}^{\beta}\right).$$

The composite being symmetrical, the global membrane-flexure coupling terms are zero:

$$B_{ij}=0.$$

The global flexural stiffnesses are given by:

$$D_{ij}=\frac{1}{3}\sum_{k=1}^{4N}Q_{ij}^{k}\left(z_k^3-z_{k-1}^3\right),$$

or, with the preceding notation:

$$D_{ij}=\frac{e^3}{3}\sum_{k=1}^{4N}Q_{ij}^{k}\left[(k-2N)^3-(k-2N-1)^3\right].$$

By calling the α and β layers located on the x_3 side negative and the β and α layers on the x_3 side positive we obtain:

$$D_{ij}=\frac{e^3}{3}\left(\sum_{k=1,3,\ldots,2N-1}\left\{Q_{ij}^{\alpha}\left[(k-2N)^3-(k-2N-1)^3\right]+Q_{ij}^{\beta}\left[(k-2N+1)^3-(k-2N)^3\right]\right\}\ldots\right.$$

$$\left.\ldots+\sum_{k=2N+1,2N+3,\ldots,4N-1}\left\{Q_{ij}^{\beta}\left[(k-2N)^3-(k-2N-1)^3\right]+Q_{ij}^{\alpha}\left[(k-2N+1)^3-(k-2N)^3\right]\right\}\right).$$

Then putting $k = 2p-1$, D_{ij} is written as:

$$D_{ij} = \frac{e^3}{3}\left(\sum_{p=1}^{N}\left\{Q_{ij}^{\alpha}\left[(2p-2N-1)^3-(2p-2N-2)^3\right]\ldots\right.\right.$$
$$\left.\ldots+Q_{ij}^{\beta}\left[(2p-2N)^3-(2p-2N-1)^3\right]\right\}\ldots$$
$$\ldots+\sum_{p=N+1}^{2N}\left\{Q_{ij}^{\beta}\left[(2p-2N-1)^3-(2p-2N-2)^3\right]\ldots\right.$$
$$\left.\left.\ldots+Q_{ij}^{\alpha}\left[(2p-2N)^3-(2p-2N-1)^3\right]\right\}\right).$$

With $q = N - p + 1$ for the first summation, and $q = p - N$ for the second, we obtain:

$$D_{ij} = \frac{e^3}{3}\left(\sum_{q=N}^{1}\left\{Q_{ij}^{\alpha}\left[(1-2q)^3-(-2q)^3\right]+Q_{ij}^{\beta}\left[(2-2q)^3-(1-2q)^3\right]\right\}\ldots\right.$$
$$\left.\ldots+\sum_{q=1}^{N}\left\{Q_{ij}^{\beta}\left[(2q-1)^3-(2q-2)^3\right]+Q_{ij}^{\alpha}\left[(2q)^3-(2q-1)^3\right]\right\}\right).$$

Introducing $(1-2q)^3 = -(2q-1)^3$, $(-2q)^3 = -(2q)^3$, $(2-2q)^3 = -(2q-2)^3$, the previous expression is written as:

$$D_{ij} = \frac{2e^3}{3}\sum_{q=1}^{N}\left\{Q_{ij}^{\alpha}\left[(2q)^3-(2q-1)^3\right]+Q_{ij}^{\beta}\left[(2q-1)^3-(2q-2)^3\right]\right\}.$$

The coefficients of Q_{ij}^{α} and Q_{ij}^{β} are respectively equal to:

$$(2q)^3-(2q-1)^3 = (2q)^3-\left[(2q)^3-3(2q)^2+6q-1\right] = 12q^2-6q+1,$$
$$(2q-1)^3-(2q-2)^3 = (2q-1)^3-\left[(2q-1)^3 \; -3(2q-1)^2+3(2q-1)-1\right]\ldots$$
$$\ldots = 12q^2-18q+7,$$

from which we obtain the expression:

$$D_{ij} = \frac{2e^3}{3}\sum_{q=1}^{N}\left[Q_{ij}^{\alpha}\left(12q^2-6q+1\right)+Q_{ij}^{\beta}\left(12q^2-18q+7\right)\right].$$

Introducing:

$$\sum_{q=1}^{N}q^2 = \frac{N(N+1)(2N+1)}{6}, \text{ and } \sum_{q=1}^{N}q = \frac{N(N+1)}{2},$$

we obtain:

$$D_{ij} = \frac{2e^3}{3}\left\{Q_{ij}^{\alpha}[2N(N+1)(2N+1) - 3N(N+1) + N]\ldots\right.$$
$$\left.\ldots + Q_{ij}^{\beta}[2N(N+1)(2N+1) - 9N(N+1) + 7N]\right\},$$

or:

$$D_{ij} = \frac{2}{3}N^2 e^3\left[(4N+3)Q_{ij}^{\alpha} + (4N-3)Q_{ij}^{\beta}\right].$$

Designating by $h = 4Ne$ the composite thickness, we have:

$$A_{ij} = \frac{h}{2}\left(Q_{ij}^{\alpha} + Q_{ij}^{\beta}\right),$$
$$D_{ij} = \frac{h^3}{96N}\left[(4N+3)Q_{ij}^{\alpha} + (4N-3)Q_{ij}^{\beta}\right].$$

The membrane stiffnesses are independent of the number of layers, the flexural stiffnesses depend on the number of layers and when the number becomes very large they tend towards the value:

$$D_{ij} = \frac{h^3}{24}\left(Q_{ij}^{\alpha} + Q_{ij}^{\beta}\right),$$

which is independent of N.

5.10.2. *Symmetrical cross-ply laminate* $\left(0, \frac{\pi}{2}\right)_{NS}$

By introducing the expressions:

$$Q_{11}^{\frac{\pi}{2}} = Q_{22}^{0}, \quad Q_{22}^{\frac{\pi}{2}} = Q_{11}^{0}, \quad Q_{12}^{\frac{\pi}{2}} = Q_{12}^{0}, \quad Q_{66}^{\frac{\pi}{2}} = Q_{66}^{0},$$
$$Q_{16}^{\frac{\pi}{2}} = Q_{16}^{0} = Q_{26}^{\frac{\pi}{2}} = Q_{26}^{0} = 0,$$

we obtain the global membrane stiffnesses:

$$A_{11} = A_{22} = 2Ne\left(Q_{11}^{0} + Q_{22}^{0}\right),$$
$$A_{12} = 4NeQ_{12}^{0}, \quad A_{66} = 4NeQ_{66}^{0},$$
$$A_{16} = A_{26} = 0,$$

and the global flexural stiffnesses:

$$D_{11} = \frac{2}{3}N^2e^3\left[(4N+3)Q_{11}^0 + (4N-3)Q_{22}^0\right],$$

$$D_{22} = \frac{2}{3}N^2e^3\left[(4N+3)Q_{22}^0 + (4N-3)Q_{11}^0\right],$$

$$D_{12} = \frac{16}{3}N^3e^3Q_{12}^0, \quad D_{66} = \frac{16}{3}N^3e^3Q_{66}^0,$$

$$D_{16} = D_{26} = 0.$$

The global stiffness matrix is written:

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix}.$$

5.10.3. *Symmetrical balanced laminate* $(\alpha,-\alpha)_{NS}$

Since:

$$Q_{11}^{-\alpha} = Q_{11}^{\alpha}, \quad Q_{12}^{-\alpha} = Q_{12}^{\alpha}, \quad Q_{16}^{-\alpha} = -Q_{16}^{\alpha},$$
$$Q_{22}^{-\alpha} = Q_{22}^{\alpha}, \quad Q_{66}^{-\alpha} = Q_{66}^{\alpha}, \quad Q_{26}^{-\alpha} = -Q_{26}^{\alpha},$$

we obtain the global membrane stiffnesses:

$$A_{11} = 4NeQ_{11}^{\alpha}, \quad A_{22} = 4NeQ_{22}^{\alpha},$$
$$A_{12} = 4NeQ_{12}^{\alpha}, \quad A_{66} = 4NeQ_{66}^{\alpha},$$
$$A_{16} = A_{26} = 0,$$

and the global flexural stiffnesses:

$$D_{11} = \frac{16}{3}N^3e^3Q_{11}^{\alpha}, \quad D_{22} = \frac{16}{3}N^3e^3Q_{22}^{\alpha},$$
$$D_{12} = \frac{16}{3}N^3e^3Q_{12}^{\alpha}, \quad D_{66} = \frac{16}{3}N^3e^3Q_{66}^{\alpha},$$
$$D_{16} = 4N^2e^3Q_{16}^{\alpha}, \quad D_{26} = 4N^2e^3Q_{26}^{\alpha}.$$

The global stiffness matrix is then:

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{16} & D_{26} & D_{66} \end{bmatrix}.$$

5.11. Global stiffnesses for an asymmetrical laminate

5.11.1. *Asymmetrical laminate* $(\alpha,\beta)_N$

The $2N$ orthotropic layers are identical, of thickness e and with the stacking sequence (α,β) repeated N times.

The sequences are:
- (α,β) for $N=1$,
- $(\alpha,\beta,\alpha,\beta)$ for $N=2$,
- $(\alpha,\beta,\alpha,\beta,\alpha,\beta)$ for $N=3$,

and so on.

The global membrane stiffnesses are:

$$A_{ij} = \sum_{k=1}^{2N} Q_{ij}^k (z_k - z_{k-1}),$$

with $z_k = (k-N)e$, we obtain:

$$A_{ij} = e\sum_{k=1}^{2N} Q_{ij}^k [(k-N)-(k-N-1)] = e\sum_{k=1}^{N}\left(Q_{ij}^\alpha + Q_{ij}^\beta\right),$$

or:

$$A_{ij} = Ne\left(Q_{ij}^\alpha + Q_{ij}^\beta\right).$$

The global membrane-flexion coupling stiffnesses are equal to:

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{2N} Q_{ij}^k \left(z_k^2 - z_{k-1}^2\right),$$

or:

$$B_{ij} = \frac{e^2}{2}\sum_{k=1}^{2N} Q_{ij}^k \left[(k-N)^2 - (k-N-1)^2\right].$$

By dissociating the layers of orientation α and β, we obtain:

$$B_{ij} = \frac{e^2}{2} \sum_{k=1,3,\ldots,2N-1} \left\{ Q_{ij}^{\alpha} \left[(k-N)^2 - (k-N-1)^2 \right] + Q_{ij}^{\beta} \left[(k-N+1)^2 - (k-N)^2 \right] \right\}.$$

Posing $k = 2p - 1$, the previous expression becomes:

$$B_{ij} = \frac{e^2}{2} \sum_{p=1}^{N} \left\{ Q_{ij}^{\alpha} \left[(2p-N-1)^2 - (2p-N-2)^2 \right] + Q_{ij}^{\beta} \left[(2p-N)^2 - (2p-N-1)^2 \right] \right\}.$$

The coefficients of Q_{ij}^{α} *et* Q_{ij}^{β} being respectively equal to:

$$(2p-N-1)^2 - (2p-N-2)^2 = (2p-N-1)^2 - \left[(2p-N-1)^2 \ -2(2p-N-1)+1 \right],$$

$$(2p-N-1)^2 - (2p-N-2)^2 = 4p - 2N - 3,$$

and:

$$(2p-N)^2 - (2p-N-1)^2 = (2p-N)^2 - \left[(2p-N)^2 \ -2(2p-N)+1 \right],$$

$$(2p-N)^2 - (2p-N-1)^2 = 4p - 2N - 1,$$

we find:

$$B_{ij} = \frac{e^2}{2} \sum_{p=1}^{N} \left\{ Q_{ij}^{\alpha} (4p - 2N - 3) + Q_{ij}^{\beta} (4p - 2N - 1) \right\}.$$

Introducing:

$$\sum_{p=1}^{N} p = \frac{N(N+1)}{2},$$

we obtain:

$$B_{ij} = \frac{e^2}{2} \left\{ Q_{ij}^{\alpha} [2N(N+1) - N(2N+3)] + Q_{ij}^{\beta} [2N(N+1) - N(2N+1)] \right\},$$

or:

$$B_{ij} = -N \frac{e^2}{2} \left(Q_{ij}^{\alpha} - Q_{ij}^{\beta} \right).$$

The global flexural rigidities:

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{2N} Q_{ij}^{k} \left(z_k^3 - z_{k-1}^3 \right),$$

are, with the previous notation, given by:

$$D_{ij} = \frac{e^3}{3} \sum_{p=1}^{N} \left\{ Q_{ij}^{\alpha} \left[(2p-N-1)^3 - (2p-N-2)^3 \right] + Q_{ij}^{\beta} \left[(2p-N)^3 - (2p-N-1)^3 \right] \right\}.$$

The coefficients of Q_{ij}^{α} and Q_{ij}^{β} being equal to:

$$(2p-N-1)^3-(2p-N-2)^3=(2p-N-1)^3-\left[(2p-N-1)^3-3(2p-N-1)^2\ldots\right.$$
$$\left.\ldots+3(2p-N-1)-1\right],$$
$$(2p-N-1)^3-(2p-N-2)^3=12p^2-6p(2N+3)+3N^2+9N+7,$$

and:

$$(2p-N)^3-(2p-N-1)^3=(2p-N)^3-\left[(2p-N)^3-3(2p-N)^2+3(2p-N)-1\right],$$
$$(2p-N)^3-(2p-N-1)^3=12p^2-6p(2N+1)+3N^2+3N+1,$$

we have:

$$D_{ij}=\frac{e^3}{3}\sum_{p=1}^{N}\left\{Q_{ij}^{\alpha}\left[12p^2-6p(2N+3)+3N^2+9N+7\right]\ldots\right.$$
$$\left.\ldots+Q_{ij}^{\beta}\left[12p^2-6p(2N+1)+3N^2+3N+1\right]\right\}.$$

With:

$$\sum_{p=1}^{N}p^2=\frac{N(N+1)(2N+1)}{6},$$

we obtain:

$$D_{ij}=\frac{e^3}{3}\sum_{p=1}^{N}\left\{Q_{ij}^{\alpha}\left[2N(N+1)(2N+1)-3N(N+1)(2N+3)+N\left(3N^2+9N+7\right)\right]\ldots\right.$$
$$\left.\ldots+Q_{ij}^{\beta}\left[2N(N+1)(2N+1)-3N(N+1)(2N+1)+N\left(3N^2+3N+1\right)\right]\right\},$$

or:

$$D_{ij}=N^3\frac{e^3}{3}\left(Q_{ij}^{\alpha}+Q_{ij}^{\beta}\right).$$

Designating by $h=2Ne$ the composite thickness we obtain:

$$A_{ij}=\frac{h}{2}\left(Q_{ij}^{\alpha}+Q_{ij}^{\beta}\right),$$

$$B_{ij}=-\frac{h^2}{8N}\left(Q_{ij}^{\alpha}-Q_{ij}^{\beta}\right),$$

$$D_{ij}=\frac{h^3}{24}\left(Q_{ij}^{\alpha}+Q_{ij}^{\beta}\right).$$

For a given thickness h, the global membrane and flexural stiffnesses do not depend on the number of layers. When the number of layers $2N$ becomes very large the global membrane-flexure coupling terms B_{ij} tend to zero.

5.11.2. *Asymmetrical cross-ply laminate* $\left(0,\frac{\pi}{2}\right)_N$

Given the expressions:

$$Q_{11}^{\frac{\pi}{2}} = Q_{22}^{0}, \quad Q_{22}^{\frac{\pi}{2}} = Q_{11}^{0}, \quad Q_{12}^{\frac{\pi}{2}} = Q_{12}^{0}, \quad Q_{66}^{\frac{\pi}{2}} = Q_{66}^{0},$$
$$Q_{16}^{\frac{\pi}{2}} = Q_{16}^{0} = Q_{26}^{\frac{\pi}{2}} = Q_{26}^{0} = 0,$$

we obtain:

– the global membrane stiffnesses:

$$A_{11} = A_{22} = Ne\left(Q_{11}^{0} + Q_{22}^{0}\right),$$
$$A_{12} = 2NeQ_{12}^{0},$$
$$A_{66} = 2NeQ_{66}^{0},$$
$$A_{16} = A_{26} = 0,$$

– the global membrane-flexure coupling stiffnesses:

$$B_{11} = -B_{22} = -N\frac{e^2}{2}\left(Q_{11}^{0} - Q_{22}^{0}\right),$$
$$B_{12} = B_{66} = B_{16} = B_{26} = 0,$$

– the global flexural stiffnesses:

$$D_{11} = D_{22} = N^3\frac{e^3}{3}\left(Q_{11}^{0} + Q_{22}^{0}\right),$$
$$D_{12} = 2N^3\frac{e^3}{3}Q_{12}^{0},$$
$$D_{66} = 2N^3\frac{e^3}{3}Q_{66}^{0},$$
$$D_{16} = D_{26} = 0.$$

The global stiffness matrix is of the form :

$$\begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & 0 & 0 \\ A_{12} & A_{11} & 0 & 0 & -B_{11} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ B_{11} & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & -B_{11} & 0 & D_{12} & D_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix}.$$

5.11.3. *Asymmetrical balanced laminate* $(\alpha,-\alpha)_N$

Using the expressions:

$$Q_{11}^{-\alpha} = Q_{11}^{\alpha}, \quad Q_{12}^{-\alpha} = Q_{12}^{\alpha}, \quad Q_{16}^{-\alpha} = -Q_{16}^{\alpha},$$
$$Q_{22}^{-\alpha} = Q_{22}^{\alpha}, \quad Q_{66}^{-\alpha} = Q_{66}^{\alpha}, \quad Q_{26}^{-\alpha} = -Q_{26}^{\alpha},$$

leads to:

– global membrane stiffnesses:

$$A_{11} = 2NeQ_{11}^{\alpha}, \quad A_{66} = 2NeQ_{66}^{\alpha},$$
$$A_{22} = 2NeQ_{22}^{\alpha}, \quad A_{16} = 0,$$
$$A_{12} = 2NeQ_{12}^{\alpha}, \quad A_{26} = 0,$$

– global membrane-flexure coupling stiffnesses:

$$B_{16} = -Ne^2 Q_{16}^{\alpha}, \quad B_{26} = -Ne^2 Q_{26}^{\alpha},$$
$$B_{11} = B_{22} = B_{12} = B_{66} = 0,$$

– global flexural stiffnesses :

$$D_{11} = 2N^3 \frac{e^3}{3} Q_{11}^{\alpha}, \quad D_{66} = 2N^3 \frac{e^3}{3} Q_{66}^{\alpha},$$
$$D_{22} = 2N^3 \frac{e^3}{3} Q_{22}^{\alpha}, \quad D_{16} = 0,$$
$$D_{12} = 2N^3 \frac{e^3}{3} Q_{12}^{\alpha}, \quad D_{26} = 0.$$

The global stiffness matrix is therefore written as:

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\ A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\ 0 & 0 & B_{16} & D_{11} & D_{12} & 0 \\ 0 & 0 & B_{26} & D_{12} & D_{22} & 0 \\ B_{16} & B_{26} & 0 & 0 & 0 & D_{66} \end{bmatrix}.$$

5.12. Examples of global stiffness matrices

5.12.1. *Two layer plate*

For the laminate as shown in figure 5.6:

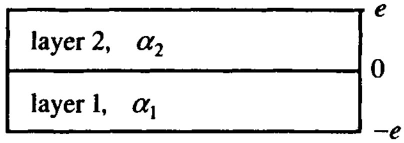

Figure 5.6. *Two layer laminate*

The components of the global stiffness matrix are:

$$A_{ij} = e\left(Q_{ij}^1 + Q_{ij}^2\right),$$

$$B_{ij} = -\frac{e^2}{2}\left(Q_{ij}^1 - Q_{ij}^2\right),$$

$$D_{ij} = \frac{e^3}{3}\left(Q_{ij}^1 + Q_{ij}^2\right).$$

5.12.2. *Three layer plate*

The components of the global stiffness matrix for the plate shown below:

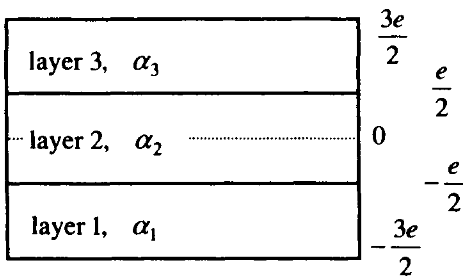

Figure 5.7. *Three layer plate*

are equal to:

$$A_{ij} = e\left(Q_{ij}^1 + Q_{ij}^2 + Q_{ij}^3\right),$$

$$B_{ij} = -e^2\left(Q_{ij}^1 - Q_{ij}^3\right),$$

$$D_{ij} = \frac{e^3}{12}\left[13\left(Q_{ij}^1 + Q_{ij}^3\right) + Q_{ij}^2\right].$$

5.12.3. *Four layer plate*

For the plate shown below:

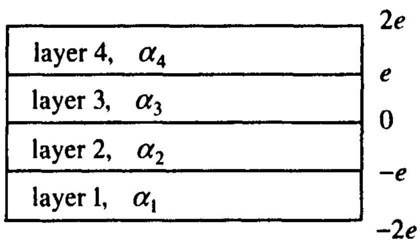

Figure 5.8. *Four layer plate*

the global stiffnesses are:

$$A_{ij} = e\left(Q_{ij}^1 + Q_{ij}^2 + Q_{ij}^3 + Q_{ij}^4\right),$$

$$B_{ij} = -\frac{e^2}{2}\left[3\left(Q_{ij}^1 - Q_{ij}^4\right) + Q_{ij}^2 - Q_{ij}^3\right],$$

$$D_{ij} = \frac{e^3}{3}\left[7\left(Q_{ij}^1 + Q_{ij}^4\right) + Q_{ij}^2 + Q_{ij}^3\right].$$

5.12.4. *Examples of decoupling*

The stiffness matrices of a multi-layer plate made up of identical layers takes particular forms for some special cases considered here.

– For membrane-flexure decoupling:

$$\begin{bmatrix} A_{11} & A_{12} & A_{16} & 0 & 0 & 0 \\ A_{12} & A_{22} & A_{26} & 0 & 0 & 0 \\ A_{16} & A_{26} & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{16} & D_{26} & D_{66} \end{bmatrix},$$

with the stacking sequence $(\alpha,-\alpha,\alpha)$: *a symmetrical laminate.*

– For membrane-flexure and tension-shear decoupling:

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{16} & D_{26} & D_{66} \end{bmatrix},$$

with the stacking sequence $(\alpha,-\alpha,-\alpha,\alpha)$: *a balanced, symmetrical laminate.*

– For tension-shear and flexure-torsion decoupling:

$$\begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & -B_{11} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ B_{11} & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & -B_{11} & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix},$$

with the stacking sequence $\left(0,\frac{\pi}{2}\right)$ or $\left(0,\frac{\pi}{2},0,\frac{\pi}{2}\right)$: *crossply asymmetrical,*

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\ A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\ 0 & 0 & B_{16} & D_{11} & D_{12} & 0 \\ 0 & 0 & B_{26} & D_{12} & D_{22} & 0 \\ B_{16} & B_{26} & 0 & 0 & 0 & D_{66} \end{bmatrix},$$

with the stacking sequence $(\alpha,-\alpha)$ or $(\alpha,-\alpha,\alpha,-\alpha)$: *balanced asymmetrical.*

– For membrane-flexure, tension-shear and flexure-torsion decoupling:

$$\begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix},$$

with the stacking sequence $\left(0,\frac{\pi}{2},\frac{\pi}{2},0\right)$: *symmetrical cross-ply.*

– For the unbalanced composite of stacking sequence $(\alpha,\alpha,-\alpha)$:

$$\begin{bmatrix} A_{11} & A_{12} & A_{16} & 0 & 0 & B_{16} \\ A_{12} & A_{22} & A_{26} & 0 & 0 & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & 0 \\ 0 & 0 & B_{16} & D_{11} & D_{12} & D_{16} \\ 0 & 0 & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & 0 & D_{16} & D_{26} & D_{66} \end{bmatrix}.$$

5.13. Boundary conditions

5.13.1. *Definition of boundary conditions*

It should be recalled that the so-called Kirchhoff boundary conditions for the edge $x_1 = a_1$ of a rectangular plate, in the most usual cases are:

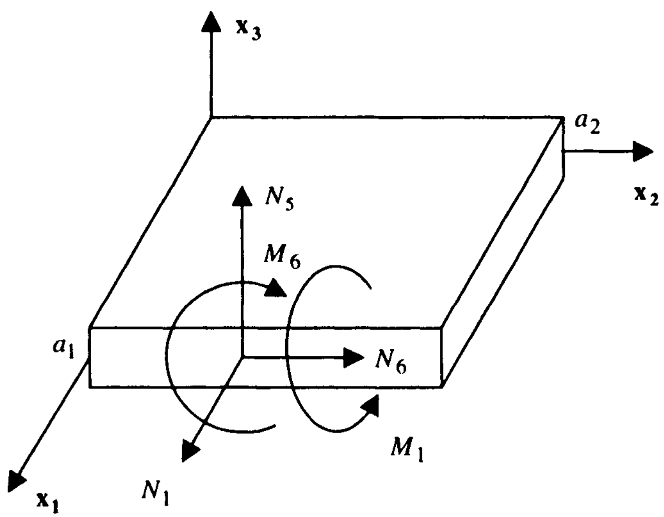

Figure 5.9. *Loading boundary conditions*

– simply supported edge:

$$u_3^0 = 0,$$
$$N_1 = N_6 = 0,$$
$$M_1 = 0,$$

– free edge in $\mathbf{x}_1$ direction:

$$u_2^0 = u_3^0 = 0,$$
$$N_1 = 0,$$
$$M_1 = 0,$$

– free edge in $\mathbf{x}_2$ direction:

$$u_1^0 = u_3^0 = 0,$$
$$N_6 = 0,$$
$$M_1 = 0,$$

– built-in edge:

$$u_1^0 = u_2^0 = u_3^0 = 0,$$
$$\frac{\partial u_3^0}{\partial x_1} = 0,$$

– free edge:

$$N_1 = N_6 = 0,$$
$$N_5 + \frac{\partial M_6}{\partial x_2} = 0,$$
$$M_1 = 0.$$

5.13.2. *Effective global transverse shear load*

The global cohesive forces provoke the five loadings N_1, N_5, N_6, M_1 and M_6 on the edge $x_1 = a_1$. The zero condition for these five terms leads to a greater number of conditions than those required by Kirchhoff-Love theory, which is four as in the case of a simply supported or a built-in edge.

For the edge considered we will calculate the global moment associated with the acting surface forces. On this edge we consider the point M^0 to belong to the mean plane, and the point M is defined by:

$$\mathbf{M^0M} = x_3\mathbf{x}_3.$$

The global moment in M^0 of the cohesion forces on the edge $x_1 = a_1$ is defined by:

$$\mathbf{M}_{M^0} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbf{M^0 M} \times \mathbf{T}(M|\mathbf{x}_1)dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\mathbf{x}_3 \times (\sigma_{11}\mathbf{x}_1 + \sigma_{12}\mathbf{x}_2 + \sigma_{13}\mathbf{x}_3)dx_3 \,,$$

$$\mathbf{M}_{M^0} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (-\sigma_{12}\mathbf{x}_1 + \sigma_{11}\mathbf{x}_2)x_3dx_3 = -\mathbf{x}_1\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{12}x_3dx_3 + \mathbf{x}_2\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{11}x_3dx_3 \,,$$

$$\mathbf{M}_{M^0} = -M_6\mathbf{x}_1 + M_1\mathbf{x}_2 .$$

Considering the rotation due to the global torsion moment $-M_6$ measured on $\mathbf{x}_1$, Kirchhoff proposed replacing the moment $-M_6 dx_2\mathbf{x}_1$, acting on the element of the edge of length dx_2, by the two forces $M_6\mathbf{x}_3$ and $-M_6\mathbf{x}_3$ acting on the ends of the element considered.

For the following element, of the same length dx_2, taken in the sense of increasing x_2, we replace the moment $-\left(M_6 + \dfrac{\partial M_6}{\partial x_2}dx_2\right)dx_2\mathbf{x}_1$ by the two forces $\left(M_6 + \dfrac{\partial M_6}{\partial x_2}dx_2\right)\mathbf{x}_3$ and $-\left(M_6 + \dfrac{\partial M_6}{\partial x_2}dx_2\right)\mathbf{x}_3$, and so on.

As shown in the following three figures, we can substitute from one to the next the global torsion moment $-M_6\mathbf{x}_1$ by the linear force density equal to $\dfrac{\partial M_6}{\partial x_2}\mathbf{x}_3$.

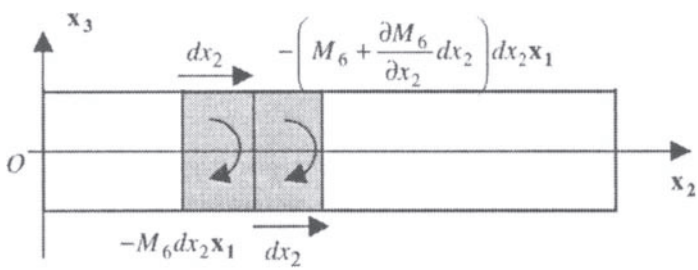

Figure 5.10. *Couples acting on two elements next to the edge* $x_1 = a_1$

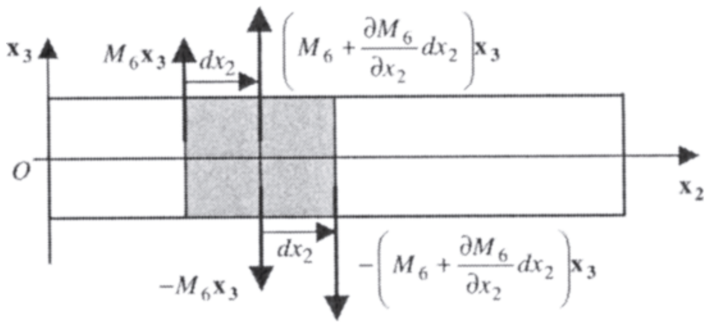

Figure 5.11. *Equivalent forces acting on two elements next to the edge* $x_1 = a_1$

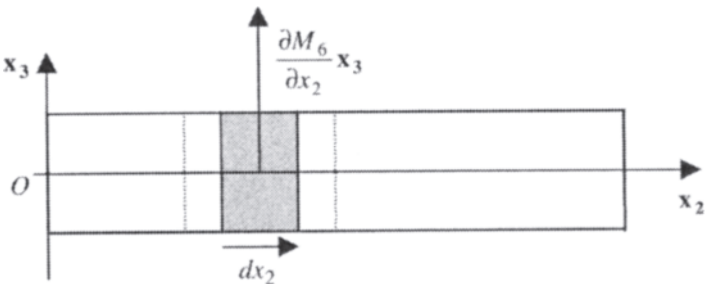

Figure 5.12. *Linear equivalent force density acting on two half-elements next to the edge* $x_1 = a_1$

The edge element shaded grey made up of the two preceding contiguous half elements is subjected to the force $\frac{\partial M_6}{\partial x_2}dx_2\mathbf{x}_3$ to which is added the force $N_5 dx_2 \mathbf{x}_3$ caused by transverse shear.

Following the Kirchhoff-Love theory, we replace the global forces N_5 and M_6 by the effective global transverse shear force defined by:

$$N_5^{eff} = N_5 + \frac{\partial M_6}{\partial x_2},$$

which leads, for the free edge case, to the condition:

$$N_5 + \frac{\partial M_6}{\partial x_2} = 0 .$$

For the edge $x_2 = a_2$, we have the same:

$$\mathbf{M}_{M^0} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \mathbf{M}^0\mathbf{M} \times \mathbf{T}(M|\mathbf{x}_2)dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} x_3\mathbf{x}_3 \wedge (\sigma_{12}\mathbf{x}_1 + \sigma_{22}\mathbf{x}_2 + \sigma_{23}\mathbf{x}_3)dx_3 \, ,$$

$$\mathbf{M}_{M^0} = \int_{-\frac{h}{2}}^{\frac{h}{2}} (-\sigma_{22}\mathbf{x}_1 + \sigma_{12}\mathbf{x}_2)x_3dx_3 = -\mathbf{x}_1\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{22}x_3dx_3 + \mathbf{x}_2\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{12}x_3dx_3 \, ,$$

$$\mathbf{M}_{M^0} = -M_2\mathbf{x}_1 + M_6\mathbf{x}_2 \, .$$

An analogous study to the preceding one leads, as shown in the following three figures, to the effective global transverse shear force:

$$N_4^{eff} = N_4 + \frac{\partial M_6}{\partial x_1} \, .$$

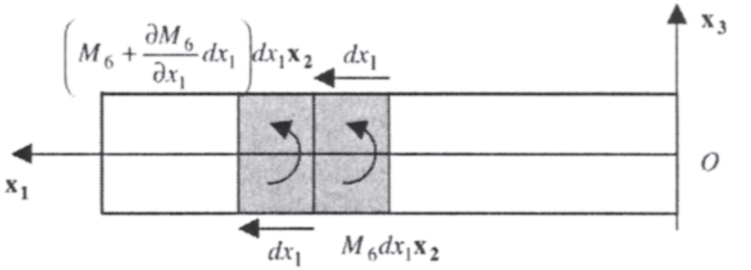

Figure 5.13. *Couples acting on two contiguous edge elements* $x_2 = a_2$

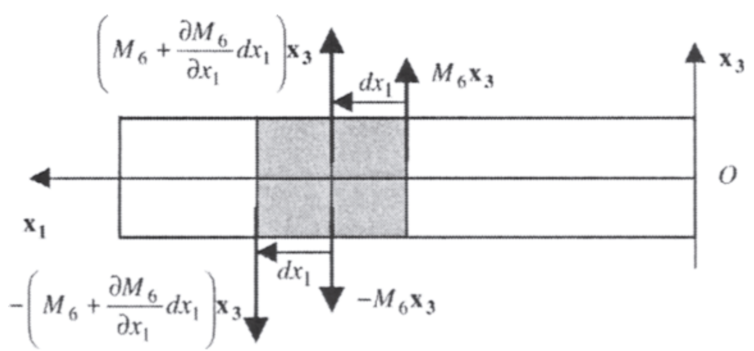

Figure 5.14. *Equivalent forces acting on two contiguous edge elements* $x_2 = a_2$

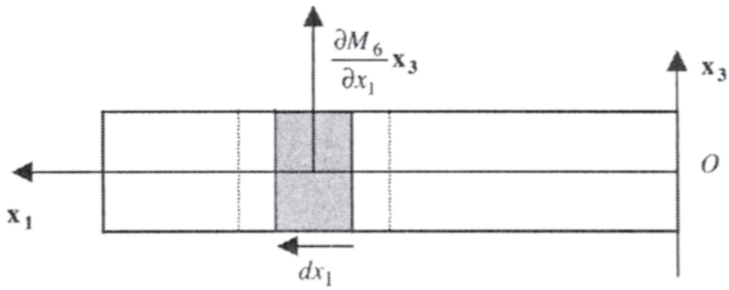

Figure 5.15. *Equivalent linear force density acting on two contiguous edge half elements* $x_2 = a_2$

5.14. Determination of transverse shear stresses

The transverse shear stresses σ_4^k and σ_5^k in the layer k can be obtained after determination of u_1, u_2 and u_3, from the two local equations of movement:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} = \rho \frac{\partial^2 u_1}{\partial t^2},$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} = \rho \frac{\partial^2 u_2}{\partial t^2},$$

which may be put in the form:

$$\frac{\partial \sigma_5}{\partial x_3} = -\frac{\partial \sigma_1}{\partial x_1} - \frac{\partial \sigma_6}{\partial x_2} + \rho \frac{\partial^2 u_1}{\partial t^2},$$

$$\frac{\partial \sigma_4}{\partial x_3} = -\frac{\partial \sigma_6}{\partial x_1} - \frac{\partial \sigma_2}{\partial x_2} + \rho \frac{\partial^2 u_2}{\partial t^2}.$$

The transverse shear stress σ_4^k, in the layer k, is given by the definite integral:

$$\sigma_4^k = -\int_{-\frac{h}{2}}^{x_3} \left(\frac{\partial \sigma_6}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} - \rho \frac{\partial^2 u_2}{\partial t^2} \right) d\zeta \text{ , with } z_{k-1} \le x_3 \le z_k .$$

Layer-by-layer integration leads to the expression:

$$\sigma_4^k = -\sum_{l=1}^{k-1} \int_{z_{l-1}}^{z_l} \left(\frac{\partial \sigma_6^l}{\partial x_1} + \frac{\partial \sigma_2^l}{\partial x_2} - \rho^l \frac{\partial^2 u_2}{\partial t^2} \right) d\zeta - \int_{z_k}^{x_3} \left(\frac{\partial \sigma_6^k}{\partial x_1} + \frac{\partial \sigma_2^k}{\partial x_2} - \rho^k \frac{\partial^2 u_2}{\partial t^2} \right) d\zeta .$$

The introduction of the displacements:

$$u_i = u_i^0 - x_3 \frac{\partial u_3^0}{\partial x_i} \quad (i = 1, 2),$$

and the stresses:

$$\sigma_i^k = Q_{ij}^k \left(\varepsilon_j^0 + x_3 \kappa_j\right) \qquad (i, j = 1, 2, 6),$$

enables us to write:

$$\sigma_4^k = -\sum_{l=1}^{k-1} \int_{z_{l-1}}^{z_l} \left[Q_{6j}^l \left(\frac{\partial \varepsilon_j^0}{\partial x_1} + \zeta \frac{\partial \kappa_j}{\partial x_1} \right) + Q_{2j}^l \left(\frac{\partial \varepsilon_j^0}{\partial x_2} + \zeta \frac{\partial \kappa_j}{\partial x_2} \right) \right. \ldots$$

$$\ldots \left. - \rho^l \left(\frac{\partial^2 u_2^0}{\partial t^2} - \zeta \frac{\partial^3 u_3^0}{\partial x_2 \partial t^2} \right) \right] d\zeta - \int_{z_k}^{x_3} \left[Q_{6j}^k \left(\frac{\partial \varepsilon_j^0}{\partial x_1} + \zeta \frac{\partial \kappa_j}{\partial x_1} \right) \right. \ldots$$

$$\ldots \left. + Q_{2j}^k \left(\frac{\partial \varepsilon_j^0}{\partial x_2} + \zeta \frac{\partial \kappa_j}{\partial x_2} \right) - \rho^k \left(\frac{\partial^2 u_2^0}{\partial t^2} - \zeta \frac{\partial^3 u_3^0}{\partial x_2 \partial t^2} \right) \right] d\zeta.$$

After integration, we obtain the following expression for the transverse shear stress σ_4^k:

$$\sigma_4^k = -\sum_{l=1}^{k-1} \left[\left(Q_{6j}^l \frac{\partial \varepsilon_j^0}{\partial x_1} + Q_{2j}^l \frac{\partial \varepsilon_j^0}{\partial x_2} - \rho^l \frac{\partial^2 u_2^0}{\partial t^2} \right) (z_l - z_{l-1}) \ldots \right.$$

$$\ldots \left. + \frac{1}{2} \left(Q_{6j}^l \frac{\partial \kappa_j}{\partial x_1} + Q_{2j}^l \frac{\partial \kappa_j}{\partial x_2} + \rho^l \frac{\partial^3 u_3^0}{\partial x_2 \partial t^2} \right) \left(z_l^2 - z_{l-1}^2\right) \right] \ldots$$

$$\ldots - \left(Q_{6j}^k \frac{\partial \varepsilon_j^0}{\partial x_1} + Q_{2j}^k \frac{\partial \varepsilon_j^0}{\partial x_2} - \rho^k \frac{\partial^2 u_2^0}{\partial t^2} \right) (x_3 - z_{k-1}) \ldots$$

$$\ldots - \frac{1}{2} \left(Q_{6j}^k \frac{\partial \kappa_j}{\partial x_1} + Q_{2j}^k \frac{\partial \kappa_j}{\partial x_2} + \rho^k \frac{\partial^3 u_3^0}{\partial x_2 \partial t^2} \right) \left(x_3^2 - z_{k-1}^2\right).$$

In a similar way from the expression:

$$\sigma_5^k = -\int_{-\frac{h}{2}}^{x_3} \left(\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \sigma_6}{\partial x_2} - \rho \frac{\partial^2 u_1}{\partial t^2} \right) d\zeta \text{ , with } z_{k-1} \le x_3 \le z_k \text{ ,}$$

we obtain:

$$\sigma_5^k = -\sum_{l=1}^{k-1} \int_{z_{l-1}}^{z_l} \left(\frac{\partial \sigma_1^l}{\partial x_1} + \frac{\partial \sigma_6^l}{\partial x_2} - \rho^l \frac{\partial^2 u_1}{\partial t^2} \right) d\zeta - \int_{z_k}^{x_3} \left(\frac{\partial \sigma_1^k}{\partial x_1} + \frac{\partial \sigma_6^k}{\partial x_2} - \rho^k \frac{\partial^2 u_1}{\partial t^2} \right) d\zeta,$$

then:

$$\sigma_5^k = -\sum_{l=1}^{k-1}\left[\left(Q_{1j}^l \frac{\partial \varepsilon_j^0}{\partial x_1} + Q_{6j}^l \frac{\partial \varepsilon_j^0}{\partial x_2} - \rho^l \frac{\partial^2 u_1^0}{\partial t^2}\right)(z_l - z_{l-1})\ldots\right.$$

$$\left.\ldots + \frac{1}{2}\left(Q_{1j}^l \frac{\partial \kappa_j}{\partial x_1} + Q_{6j}^l \frac{\partial \kappa_j}{\partial x_2} + \rho^l \frac{\partial^3 u_3^0}{\partial x_1 \partial t^2}\right)\left(z_l^2 - z_{l-1}^2\right)\right]\ldots$$

$$\ldots - \left(Q_{1j}^k \frac{\partial \varepsilon_j^0}{\partial x_1} + Q_{6j}^k \frac{\partial \varepsilon_j^0}{\partial x_2} - \rho^k \frac{\partial^2 u_1^0}{\partial t^2}\right)(x_3 - z_{k-1})\ldots$$

$$\ldots - \frac{1}{2}\left(Q_{1j}^k \frac{\partial \kappa_j}{\partial x_1} + Q_{6j}^k \frac{\partial \kappa_j}{\partial x_2} + \rho^k \frac{\partial^3 u_3^0}{\partial x_1 \partial t^2}\right)\left(x_3^2 - z_{k-1}^2\right).$$

The transverse shear stresses thus obtained, σ_4^k and σ_5^k, vary according to a parabolic law with the thickness of each layer.

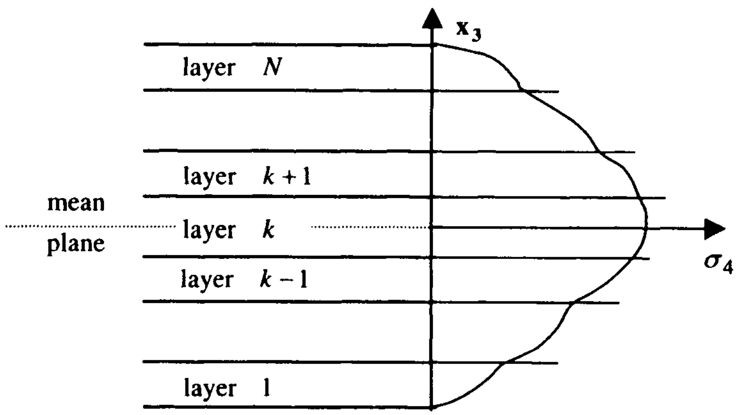

Figure 5.16. *Transverse shear stress distribution*

These transverse shear stresses satisfy the boundary conditions on the two outer faces of the plate and the continuity conditions at each of the $N-1$ interfaces:

$$\text{for } x_3 = \frac{h}{2} \qquad \sigma_4^N\left(x_1, x_2, \frac{h}{2}\middle| t\right) = \sigma_5^N\left(x_1, x_2, \frac{h}{2}\middle| t\right) = 0,$$

$$\text{for } x_3 = -\frac{h}{2} \qquad \sigma_4^1\left(x_1, x_2, -\frac{h}{2}\middle| t\right) = \sigma_5^1\left(x_1, x_2, -\frac{h}{2}\middle| t\right) = 0,$$

$$\text{for } x_3 = z_k \qquad \sigma_4^k(x_1, x_2, z_k|t) = \sigma_4^{k+1}(x_1, x_2, z_k|t),$$

$$\sigma_5^k(x_1, x_2, z_k|t) = \sigma_5^{k+1}(x_1, x_2, z_k|t).$$

5.15. Strain energy

The strain energy of the plate:

$$W_d = \frac{1}{2}\iiint_D \sigma_{ij}\varepsilon_{ij} d\Omega \quad (i,\ j = 1, 2, 3),$$

can be written in the form:

$$W_d = \frac{1}{2}\iiint_D \sigma_i\varepsilon_i d\Omega \quad (i = 1, 2, ..., 6),$$

with the usual notation.

By calculating the triple integral as the superposition of the single integral in the thickness and a double integral following the mean plane, we obtain:

$$W_d = \frac{1}{2}\iint_\Sigma \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i\varepsilon_i dx_3 \right) dx_1 dx_2 .$$

According to Kirchhoff-Love theory, the strains ε_3, ε_4 and ε_5 are ignored and the strain energy reduces to:

$$W_d = \frac{1}{2}\iint_\Sigma \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i\varepsilon_i dx_3 \right) dx_1 dx_2 \quad (i = 1, 2, 6).$$

Introducing the strains:

$$\varepsilon_i = \varepsilon_i^0 + x_3\kappa_i ,$$

and the stresses:

$$\sigma_i^k = Q_{ij}^k \left(\varepsilon_j^0 + x_3\kappa_j\right),$$

in the strain energy we obtain:

$$W_d = \frac{1}{2}\iint_\Sigma \left[\sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} Q_{ij}^k \left(\varepsilon_i^0 + x_3\kappa_i\right)\left(\varepsilon_j^0 + x_3\kappa_j\right) dx_3 \right] dx_1 dx_2 ,$$

then:

$$W_d = \frac{1}{2}\iint_\Sigma \left[\sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} Q_{ij}^k \left(\varepsilon_i^0\varepsilon_j^0 + x_3\varepsilon_i^0\kappa_j + x_3\varepsilon_j^0\kappa_i + x_3^2\kappa_i\kappa_j\right) dx_3 \right] dx_1 dx_2 .$$

Given the symmetry of the reduced stiffnesses Q_{ij}^k, we have:

$$Q_{ij}^k\varepsilon_j^0\kappa_i = Q_{ji}^k\varepsilon_j^0\kappa_i = Q_{ij}^k\varepsilon_i^0\kappa_j ,$$

from which:

$$W_d = \frac{1}{2}\iint_\Sigma \left[\sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} Q_{ij}^k \left(\varepsilon_i^0\varepsilon_j^0 + 2x_3\varepsilon_i^0\kappa_j + x_3^2\kappa_i\kappa_j\right) dx_3 \right] dx_1 dx_2 .$$

After integration through the thickness, we obtain:

$$W_d = \frac{1}{2}\iint_{\Sigma}\left[\sum_{k=1}^{N} Q_{ij}^{k}(z_k - z_{k-1})\varepsilon_i^0\varepsilon_j^0 + \sum_{k=1}^{N} Q_{ij}^{k}\left(z_k^2 - z_{k-1}^2\right)\varepsilon_i^0\kappa_j \ldots\right.$$

$$\left.\ldots + \frac{1}{3}\sum_{k=1}^{N} Q_{ij}^{k}\left(z_k^3 - z_{k-1}^3\right)\kappa_i\kappa_j\right]dx_1dx_2 \quad (i,\ j = 1, 2, 6).$$

The introduction of the global membrane A_{ij}, flexural D_{ij} and coupling B_{ij} stiffnesses leads to the expression:

$$W_d = \frac{1}{2}\iint_{\Sigma}\left(A_{ij}\varepsilon_i^0\varepsilon_j^0 + 2B_{ij}\varepsilon_i^0\kappa_j + D_{ij}\kappa_i\kappa_j\right)dx_1dx_2 \quad (i,\ j = 1, 2, 6).$$

In the case of a composite without membrane-flexure coupling, the strain energy expression reduces to:

$$W_d = \frac{1}{2}\iint_{\Sigma}\left(A_{ij}\varepsilon_i^0\varepsilon_j^0 + D_{ij}\kappa_i\kappa_j\right)dx_1dx_2 \quad (i,\ j = 1, 2, 6).$$

Chapter 6

Symmetrical orthotropic Kirchhoff-Love plates

6.1. Introduction

Since multi-layer plate calculations are complex we will limit ourselves, in the present chapter, to symmetrical orthotropic plates which conform to the Kirchhoff-Love theory.

In a symmetrical laminate the global coupling stiffnesses B_{ij} are zero.

In a cross-ply laminate the $\mathbf{x}_1$ and $\mathbf{x}_2$ directions are the directions of orthotropy of the different layers. The reduced stiffnesses Q_{16} and Q_{26} are zero, which results in the global tension-shear coupling stiffnesses A_{16} and A_{26} being zero, as well as the global flexure-torsion coupling terms D_{16} and D_{26}.

For a cross-ply laminate we always have:

$$B_{ij} = 0, \quad A_{16} = A_{26} = 0, \quad D_{16} = D_{26} = 0.$$

It is for this particular case of symmetrical orthotropic plates that the analytical methods of resolution are the easiest to apply. As indicated above, for a laminate with mirror symmetry there is decoupling between membrane and flexure so that the global constitutive relation for the composite:

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix},$$

can be decomposed into the two expressions:

$$\mathbf{N} = \mathbf{A}\boldsymbol{\varepsilon}^0,$$
$$\mathbf{M} = \mathbf{D}\boldsymbol{\kappa}.$$

It is therefore possible to study separately the plate loaded in its mean plane (*membrane load*) and the plate loaded transversely (*flexural loading*).

6.2. Global plate equations

The global plate equations according to Kirchhoff-Love theory are written as:

$$\frac{\partial N_1}{\partial x_1}+\frac{\partial N_6}{\partial x_2}+p_1=I_0\frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{\partial N_6}{\partial x_1}+\frac{\partial N_2}{\partial x_2}+p_2=I_0\frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial^2 M_1}{\partial x_1^2}+2\frac{\partial^2 M_6}{\partial x_1\partial x_2}+\frac{\partial^2 M_2}{\partial x_2^2}+\frac{\partial}{\partial x_1}\left(N_1\frac{\partial u_3^0}{\partial x_1}+N_6\frac{\partial u_3^0}{\partial x_2}\right)\dots$$

$$\dots+\frac{\partial}{\partial x_2}\left(N_6\frac{\partial u_3^0}{\partial x_1}+N_2\frac{\partial u_3^0}{\partial x_2}\right)+q_3+p_3=I_0\frac{\partial^2 u_3^0}{\partial t^2}.$$

6.3. Plate loaded in the mean plane

The global equations of motion of the plate are reduced when the volume effects are zero for the two equations:

$$\frac{\partial N_1}{\partial x_1}+\frac{\partial N_6}{\partial x_2}=I_0\frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{\partial N_6}{\partial x_1}+\frac{\partial N_2}{\partial x_2}=I_0\frac{\partial^2 u_2^0}{\partial t^2},$$

and the compatibility equations:

$$\frac{\partial^2 \varepsilon_1^0}{\partial x_2^2}+\frac{\partial^2 \varepsilon_2^0}{\partial x_1^2}-\frac{\partial^2 \varepsilon_6^0}{\partial x_1\partial x_2}=0.$$

The global constitutive relation for the composite is written, in this case, in the form:

$$\begin{bmatrix}N_1\\N_2\\N_6\end{bmatrix}=\begin{bmatrix}A_{11}&A_{12}&0\\A_{12}&A_{22}&0\\0&0&A_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_1^0\\\varepsilon_2^0\\\varepsilon_6^0\end{bmatrix},$$

the global membrane stiffnesses being equal to:

$$A_{ij}=\sum_{k=1}^{N}Q_{ij}^k\left(z_k-z_{k-1}\right).$$

The global compliance matrix of the composite is found by inversion of the constitutive relation:

$$\begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{bmatrix} = \begin{bmatrix} \dfrac{A_{22}}{A_{11}A_{22}-A_{12}^2} & -\dfrac{A_{12}}{A_{11}A_{22}-A_{12}^2} & 0 \\ -\dfrac{A_{12}}{A_{11}A_{22}-A_{12}^2} & \dfrac{A_{11}}{A_{11}A_{22}-A_{12}^2} & 0 \\ 0 & 0 & \dfrac{1}{A_{66}} \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_6 \end{bmatrix}.$$

Putting this expression in the form:

$$\begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{E_1} & -\dfrac{\nu_{21}}{E_2} & 0 \\ -\dfrac{\nu_{12}}{E_1} & \dfrac{1}{E_2} & 0 \\ 0 & 0 & \dfrac{1}{G_{12}} \end{bmatrix} \begin{bmatrix} \dfrac{N_1}{h} \\ \dfrac{N_2}{h} \\ \dfrac{N_6}{h} \end{bmatrix},$$

reveals the equivalent characteristics:

$$E_1 = \frac{A_{11}A_{22}-A_{12}^2}{A_{22}h}, \quad E_2 = \frac{A_{11}A_{22}-A_{12}^2}{A_{11}h},$$

$$\nu_{12} = \frac{A_{12}}{A_{22}}, \quad \nu_{12} = \frac{A_{12}}{A_{11}},$$

$$G_{12} = \frac{A_{66}}{h},$$

which correspond to the Young's moduli and shear and to Poisson's coefficients of a single layer, orthotropic plate which has the same membrane behaviour as the multi-layer plate considered here.

6.4. Plate loaded transversely

The global equation of the motion of plates loaded transversely is:

$$\frac{\partial^2 M_1}{\partial x_1^2} + 2\frac{\partial^2 M_6}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} + q = I_0 \frac{\partial^2 u_3^0}{\partial t^2}.$$

The constitutive relation of the composite:

$$\begin{bmatrix} M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix},$$

involves the global flexural stiffnesses:

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{N} Q_{ij}^{k}\left(z_k^3 - z_{k-1}^3\right).$$

Introducing the expressions:

$$M_1 = D_{11}\kappa_1 + D_{12}\kappa_2,$$
$$M_2 = D_{12}\kappa_1 + D_{22}\kappa_2,$$
$$M_6 = D_{66}\kappa_6,$$

and the curvatures by:

$$\kappa_1 = -\frac{\partial^2 u_3^0}{\partial x_1^2}, \quad \kappa_2 = -\frac{\partial^2 u_3^0}{\partial x_2^2}, \quad \kappa_6 = -2\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

we obtain the global cohesive moments:

$$M_1 = -D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_2 = -D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{22}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_6 = -2D_{66}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2}.$$

Introducing these in the plate global motion equation we obtain:

$$D_{11}\frac{\partial^4 u_3^0}{\partial x_1^4} + 2\left(D_{12} + 2D_{66}\right)\frac{\partial^4 u_3^0}{\partial x_1^2 \partial x_2^2} + D_{22}\frac{\partial^4 u_3^0}{\partial x_2^4} - q + I_0\frac{\partial^2 u_3^0}{\partial t^2} = 0.$$

In the following section we will consider the case of a single layer, elastic, linear, homogeneous, isotropic plate. When the transverse normal stress is zero the constitutive relation may be written as:

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{bmatrix}.$$

The stiffness matrix of the material being

$$Q = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu & 0\\ \nu & 1 & 0\\ 0 & 0 & \frac{1-\nu}{2}\end{bmatrix},$$

the global flexural stiffnesses are:

$$D_{ij} = \frac{1}{3} Q_{ij} \left[\left(\frac{h}{2} \right)^3 - \left(-\frac{h}{2} \right)^3 \right] = \frac{h^3}{12} Q_{ij} .$$

Introducing the flexural stiffness modulus of the plate:

$$D_0 = \frac{Eh^3}{12\left(1 - \nu^2\right)},$$

the global flexural stiffness matrix of the plate may be written as:

$$D = D_0 \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \dfrac{1-\nu}{2} \end{bmatrix} .$$

The equation of motion of plates loaded transversely becomes:

$$D_0 \frac{\partial^4 u_3^0}{\partial x_1^4} + 2[\nu D_0 + (1-\nu) D_0] \frac{\partial^4 u_3^0}{\partial x_1^2 \partial x_2^2} + D_0 \frac{\partial^4 u_3^0}{\partial x_2^4} - q + I_0 \frac{\partial^2 u_3^0}{\partial t^2} = 0 ,$$

or:

$$D_0 \left(\frac{\partial^4 u_3^0}{\partial x_1^4} + 2 \frac{\partial^4 u_3^0}{\partial x_1^2 \partial x_2^2} + \frac{\partial^4 u_3^0}{\partial x_2^4} \right) - q + I_0 \frac{\partial^2 u_3^0}{\partial t^2} = 0 ,$$

where:

$$D_0 \operatorname{lap}\left(\operatorname{lap} u_3^0\right) - q + I_0 \frac{\partial^2 u_3^0}{\partial t^2} = 0 .$$

We find, with $I_0 = \rho h$, the classical equation for isotropic plates.

6.5. Flexure of a rectangular plate simply supported around its edge

The rectangular plate represented in figure 6.1, with dimensions a_1 and a_2, is simply supported around its edge. It is only subjected to the surface force density $q(x_1, x_2)\vec{x}_3$.

The global equilibrium equation is written:

$$D_{11} \frac{\partial^4 u_3^0}{\partial x_1^4} + 2(D_{12} + 2D_{66}) \frac{\partial^4 u_3^0}{\partial x_1^2 \partial x_2^2} + D_{22} \frac{\partial^4 u_3^0}{\partial x_2^4} = q .$$

The edge conditions of displacement and load are:

– for $x_1 = 0$ and $x_1 = a_1$:

$$u_3^0 = 0,$$

$$M_1 = -D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0,$$

– for $x_2 = 0$ and $x_2 = a_2$:

$$u_3^0 = 0,$$

$$M_2 = -D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{22}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0.$$

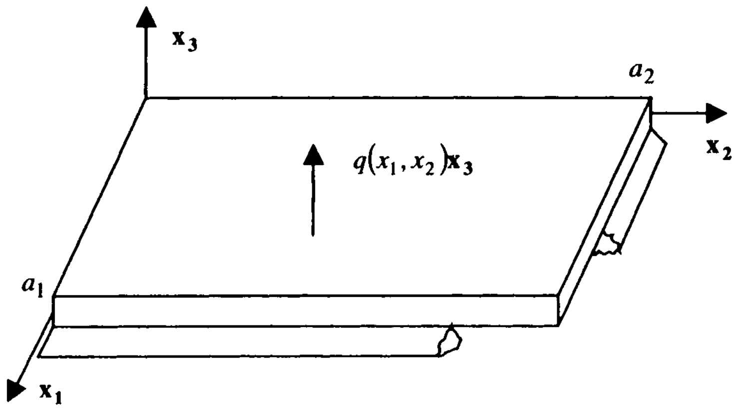

Figure 6.1. *Transversely loaded plate*

The load acting on the plate can be written using a double Fourier series:

$$q(x_1, x_2) = \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty} q_{m_1 m_2} \sin\frac{m_1 \pi x_1}{a_1}\sin\frac{m_2 \pi x_2}{a_2}.$$

Multiplying the two parts of this expression by:

$$\sin\frac{n_1 \pi x_1}{a_1}\sin\frac{n_2 \pi x_2}{a_2} dx_1 dx_2,$$

and integrating over the mean plane of the plate we obtain:

$$\int_0^{a_2}\int_0^{a_1} q(x_1, x_2)\sin\frac{n_1 \pi x_1}{a_1}\sin\frac{n_2 \pi x_2}{a_2} dx_1 dx_2 \ldots$$

$$\ldots = \sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty} q_{m_1 m_2}\int_0^{a_2}\int_0^{a_1} \sin\frac{m_1 \pi x_1}{a_1}\sin\frac{m_2 \pi x_2}{a_2}\sin\frac{n_1 \pi x_1}{a_1}\sin\frac{n_2 \pi x_2}{a_2} dx_1 dx_2.$$

Given the expression:

$$\int_0^{a_2}\int_0^{a_1} \sin\frac{m_1 \pi x_1}{a_1}\sin\frac{m_2 \pi x_2}{a_2}\sin\frac{n_1 \pi x_1}{a_1}\sin\frac{n_2 \pi x_2}{a_2} dx_1 dx_2 = \frac{a_1 a_2}{4}\delta_{m_1 n_1}\delta_{m_2 n_2},$$

the preceding equation becomes:

$$\int_0^{a_2}\int_0^{a_1} q(x_1,x_2)\sin\frac{n_1\pi x_1}{a_1}\sin\frac{n_2\pi x_2}{a_2}dx_1dx_2 = \frac{a_1a_2}{4}q_{n_1n_2}\ .$$

The coefficients $q_{m_1m_2}$ of the double Fourier series are equal to:

$$q_{m_1m_2} = \frac{4}{a_1a_2}\int_0^{a_2}\int_0^{a_1} q(x_1,x_2)\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}dx_1dx_2\ .$$

For fixed values of m_1 and m_2, that is for a given double-sinusoidal loading, the solution to the global equilibrium equation:

$$D_{11}\frac{\partial^4 u_3^0}{\partial x_1^4} + 2(D_{12}+2D_{66})\frac{\partial^4 u_3^0}{\partial x_1^2\partial x_2^2} + D_{22}\frac{\partial^4 u_3^0}{\partial x_2^4} = q_{m_1m_2}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

which satisfies the edge conditions for displacement and force is of the form:

$$u_{3m_1m_2}^0 = U_{m_1m_2}^3\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}\ .$$

Writing in the equilibrium equation and simplifying by the sine product we obtain:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2(D_{12}+2D_{66}) + \left(\frac{m_2\pi}{a_2}\right)^4 D_{22}\right]U_{m_1m_2}^3 = q_{m_1m_2}\ .$$

where:

$$U_{m_1m_2}^3 = \frac{q_{m_1m_2}}{\pi^4 D_{m_1m_2}^*},$$

with:

$$D_{m_1m_2}^* = \left(\frac{m_1}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2(D_{12}+2D_{66}) + \left(\frac{m_2}{a_2}\right)^4 D_{22}\ .$$

The transverse displacement is then:

$$u_{3m_1m_2}^0 = \frac{q_{m_1m_2}}{\pi^4 D_{m_1m_2}^*}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}\ .$$

For any loading it is given by:

$$u_3^0 = \frac{1}{\pi^4}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{q_{m_1m_2}}{D_{m_1m_2}^*}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}\ .$$

The non-transverse displacements given by:

$$u_1 = -x_3\frac{\partial u_3^0}{\partial x_1},$$

$$u_2 = -x_3 \frac{\partial u_3^0}{\partial x_2},$$

are equal to:

$$u_1 = -\frac{x_3}{\pi^3 a_1} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} m_1 \frac{q_{m_1m_2}}{D^*_{m_1m_2}} \cos\frac{m_1\pi x_1}{a_1} \sin\frac{m_2\pi x_2}{a_2},$$

$$u_2 = -\frac{x_3}{\pi^3 a_2} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} m_2 \frac{q_{m_1m_2}}{D^*_{m_1m_2}} \sin\frac{m_1\pi x_1}{a_1} \cos\frac{m_2\pi x_2}{a_2}.$$

The strains are equal to:

$$\varepsilon_1 = -x_3 \frac{\partial^2 u_3^0}{\partial x_1^2}, \quad \varepsilon_2 = -x_3 \frac{\partial^2 u_3^0}{\partial x_2^2}, \quad \varepsilon_6 = -2x_3 \frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

or:

$$\varepsilon_1 = \frac{x_3}{\pi^2 a_1^2} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} m_1^2 \frac{q_{m_1m_2}}{D^*_{m_1m_2}} \sin\frac{m_1\pi x_1}{a_1} \sin\frac{m_2\pi x_2}{a_2},$$

$$\varepsilon_2 = \frac{x_3}{\pi^2 a_2^2} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} m_2^2 \frac{q_{m_1m_2}}{D^*_{m_1m_2}} \sin\frac{m_1\pi x_1}{a_1} \sin\frac{m_2\pi x_2}{a_2},$$

$$\varepsilon_6 = -\frac{2x_3}{\pi^2 a_1 a_2} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} m_1 m_2 \frac{q_{m_1m_2}}{D^*_{m_1m_2}} \cos\frac{m_1\pi x_1}{a_1} \cos\frac{m_2\pi x_2}{a_2}.$$

The stresses are given by:

$$\sigma_i^k = Q_{ij}^k x_3 \kappa_j,$$

or:

$$\sigma_1^k = -x_3 \left(Q_{11}^k \frac{\partial^2 u_3^0}{\partial x_1^2} + Q_{12}^k \frac{\partial^2 u_3^0}{\partial x_2^2} \right),$$

$$\sigma_2^k = -x_3 \left(Q_{12}^k \frac{\partial^2 u_3^0}{\partial x_1^2} + Q_{22}^k \frac{\partial^2 u_3^0}{\partial x_2^2} \right),$$

$$\sigma_6^k = -2x_3 Q_{66}^k \frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

from which:

$$\sigma_1^k = \frac{x_3}{\pi^2} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \left[\left(\frac{m_1}{a_1}\right)^2 Q_{11}^k + \left(\frac{m_2}{a_2}\right)^2 Q_{12}^k \right] \frac{q_{m_1m_2}}{D^*_{m_1m_2}} \sin\frac{m_1\pi x_1}{a_1} \sin\frac{m_2\pi x_2}{a_2},$$

$$\sigma_2^k = \frac{x_3}{\pi^2}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\left[\left(\frac{m_1}{a_1}\right)^2 Q_{12}^k + \left(\frac{m_2}{a_2}\right)^2 Q_{22}^k\right]\frac{q_{m_1m_2}}{D^*_{m_1m_2}}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

$$\sigma_6^k = -\frac{2x_3}{\pi^2 a_1 a_2}Q_{66}^k\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty} m_1 m_2\frac{q_{m_1m_2}}{D^*_{m_1m_2}}\cos\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2}.$$

The transverse shear stresses obtained by integration through the thickness of the first two local equilibrium equations are given by:

$$\sigma_4^k = -\int_{-\frac{h}{2}}^{x_3}\left(\frac{\partial\sigma_6}{\partial x_1}+\frac{\partial\sigma_2}{\partial x_2}\right)d\zeta,\quad \sigma_5^k = -\int_{-\frac{h}{2}}^{x_3}\left(\frac{\partial\sigma_1}{\partial x_1}+\frac{\partial\sigma_6}{\partial x_2}\right)d\zeta,$$

with $z_{k-1} \leq x_3 \leq z_k$, from which:

$$\sigma_4^k = -\frac{1}{\pi a_2}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\left\{\int_{-\frac{h}{2}}^{x_3} m_2\left[\left(\frac{m_1}{a_1}\right)^2\left(Q_{12}+2Q_{66}\right)+\left(\frac{m_2}{a_2}\right)^2 Q_{22}\right]\right\}\zeta d\zeta \ldots$$

$$\ldots\times\frac{q_{m_1m_2}}{D^*_{m_1m_2}}\sin\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2},$$

$$\sigma_5^k = -\frac{1}{\pi a_1}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\left\{\int_{-\frac{h}{2}}^{x_3} m_1\left[\left(\frac{m_1}{a_1}\right)^2 Q_{11}+\left(\frac{m_2}{a_2}\right)^2\left(Q_{12}+2Q_{66}\right)\right]\right\}\zeta d\zeta \ldots$$

$$\ldots\times\frac{q_{m_1m_2}}{D^*_{m_1m_2}}\cos\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

so after integration:

$$\sigma_4^k = -\frac{1}{2\pi a_2}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\left\{\sum_{l=1}^{k-1} m_2\left[\left(\frac{m_1}{a_1}\right)^2\left(Q_{12}^l+2Q_{66}^l\right)+\left(\frac{m_2}{a_2}\right)^2 Q_{22}^l\right]\left(z_l^2-z_{l-1}^2\right)\ldots\right.$$

$$\left.\ldots+m_2\left[\left(\frac{m_1}{a_1}\right)^2\left(Q_{12}^k+2Q_{66}^k\right)+\left(\frac{m_2}{a_2}\right)^2 Q_{22}^k\right]\left(x_3^2-z_{k-1}^2\right)\right\}\frac{q_{m_1m_2}}{D^*_{m_1m_2}}\sin\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2},$$

$$\sigma_5^k = -\frac{1}{2\pi a_1}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\left\{\sum_{l=1}^{k-1} m_1\left[\left(\frac{m_1}{a_1}\right)^2 Q_{11}^l+\left(\frac{m_2}{a_2}\right)^2\left(Q_{12}^l+2Q_{66}^l\right)\right]\left(z_l^2-z_{l-1}^2\right)\ldots\right.$$

$$\left.\ldots+m_1\left[\left(\frac{m_1}{a_1}\right)^2 Q_{11}^k+\left(\frac{m_2}{a_2}\right)^2\left(Q_{12}^k+2Q_{66}^k\right)\right]\left(x_3^2-z_{k-1}^2\right)\right\}\frac{q_{m_1m_2}}{D^*_{m_1m_2}}\cos\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}.$$

The moments of the global cohesion loads are equal to:

$$M_1 = D_{11}\kappa_1 + D_{12}\kappa_2 = -\left(D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} + D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2}\right),$$

$$M_2 = D_{12}\kappa_1 + D_{22}\kappa_2 = -\left(D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} + D_{22}\frac{\partial^2 u_3^0}{\partial x_2^2}\right),$$

$$M_6 = D_{66}\kappa_6 = -2D_{66}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

or:

$$M_1 = \frac{1}{\pi^2}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\left[\left(\frac{m_1}{a_1}\right)^2 D_{11} + \left(\frac{m_2}{a_2}\right)^2 D_{12}\right]\frac{q_{m_1m_2}}{D^*_{m_1m_2}}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

$$M_2 = \frac{1}{\pi^2}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\left[\left(\frac{m_1}{a_1}\right)^2 D_{12} + \left(\frac{m_2}{a_2}\right)^2 D_{22}\right]\frac{q_{m_1m_2}}{D^*_{m_1m_2}}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

$$M_6 = -\frac{2}{\pi^2 a_1 a_2}D_{66}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty} m_1 m_2 \frac{q_{m_1m_2}}{D^*_{m_1m_2}}\cos\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2}.$$

In the particular case of a square orthotropic single layer plate subjected to a double sinusoidal loading:

$$q(x_1, x_2) = -\varpi\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}, \text{ with } \varpi > 0,$$

the previous results become:

– for the displacements:

$$u_1 = x_3\left(\frac{a}{\pi}\right)^3 \frac{\varpi}{D_{11} + 2(D_{12} + 2D_{66}) + D_{22}}\cos\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a},$$

$$u_2 = x_3\left(\frac{a}{\pi}\right)^3 \frac{\varpi}{D_{11} + 2(D_{12} + 2D_{66}) + D_{22}}\sin\frac{\pi x_1}{a}\cos\frac{\pi x_2}{a},$$

$$u_3 = -\left(\frac{a}{\pi}\right)^4 \frac{\varpi}{D_{11} + 2(D_{12} + 2D_{66}) + D_{22}}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a},$$

– for the strains:

$$\varepsilon_1 = -x_3\left(\frac{a}{\pi}\right)^2 \frac{\varpi}{D_{11} + 2(D_{12} + 2D_{66}) + D_{22}}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a},$$

$$\varepsilon_2 = -x_3\left(\frac{a}{\pi}\right)^2 \frac{\varpi}{D_{11} + 2(D_{12} + 2D_{66}) + D_{22}}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a},$$

$$\varepsilon_6 = 2x_3\left(\frac{a}{\pi}\right)^2 \frac{\varpi}{D_{11}+2(D_{12}+2D_{66})+D_{22}}\cos\frac{\pi x_1}{a}\cos\frac{\pi x_2}{a},$$

– for the stresses:

$$\sigma_1 = -x_3\left(\frac{a}{\pi}\right)^2 (Q_{11}+Q_{12})\frac{\varpi}{D_{11}+2(D_{12}+2D_{66})+D_{22}}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a},$$

$$\sigma_2 = -x_3\left(\frac{a}{\pi}\right)^2 (Q_{12}+Q_{22})\frac{\varpi}{D_{11}+2(D_{12}+2D_{66})+D_{22}}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a},$$

$$\sigma_6 = 2x_3\left(\frac{a}{\pi}\right)^2 Q_{66}\frac{\varpi}{D_{11}+2(D_{12}+2D_{66})+D_{22}}\cos\frac{\pi x_1}{a}\cos\frac{\pi x_2}{a},$$

$$\sigma_4 = \frac{a}{2\pi}\left(x_3^2-\frac{h^2}{4}\right)(Q_{12}+2Q_{66}+Q_{22})\frac{\varpi}{D_{11}+2(D_{12}+2D_{66})+D_{22}}\sin\frac{\pi x_1}{a}\cos\frac{\pi x_2}{a},$$

$$\sigma_5 = \frac{a}{2\pi}\left(x_3^2-\frac{h^2}{4}\right)(Q_{12}+2Q_{66}+Q_{11})\frac{\varpi}{D_{11}+2(D_{12}+2D_{66})+D_{22}}\cos\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}.$$

The maximum deflection at the centre of the plate is:

$$u_3^0\left(\frac{a}{2},\frac{a}{2}\right) = -\left(\frac{a}{\pi}\right)^4 \frac{\varpi}{D_{11}+2(D_{12}+2D_{66})+D_{22}}.$$

It may be noted that $u_1, u_2, \varepsilon_1, \varepsilon_2, \varepsilon_6, \sigma_1, \sigma_2$ and σ_6 vary linearly through the plate thickness, whereas u_3 is constant and σ_4 and σ_5 vary in a parabolic manner with the thickness.

Also, $u_3, \varepsilon_1, \varepsilon_2, \sigma_1$ and σ_2 are zero around the edge, whereas u_2 and σ_4 are zero for $x_1 = 0$, $x_1 = a$ and $x_2 = \frac{a}{2}$. We note that u_1 and σ_5 are zero for $x_2 = 0$, $x_2 = a$ and $x_1 = \frac{a}{2}$, and that ε_6 and σ_6 are zero for $x_1 = \frac{a}{2}$ and $x_2 = \frac{a}{2}$.

These conditions are illustrated in the following figures.

– For the displacements:

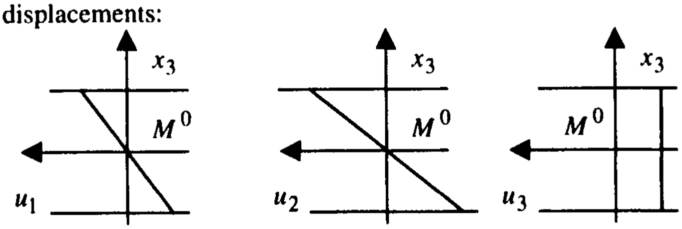

Figure 6.2. *Variations in displacements*

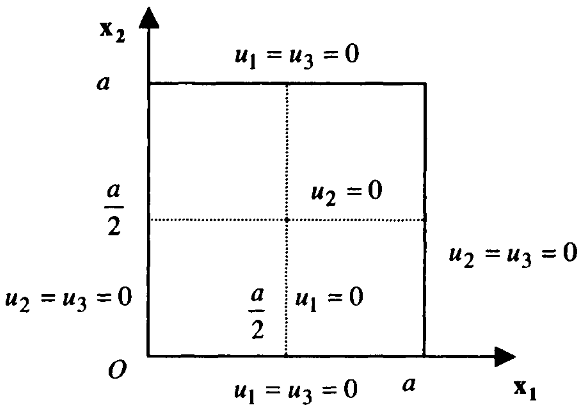

Figure 6.3. *Zero displacement*

– For the strains:

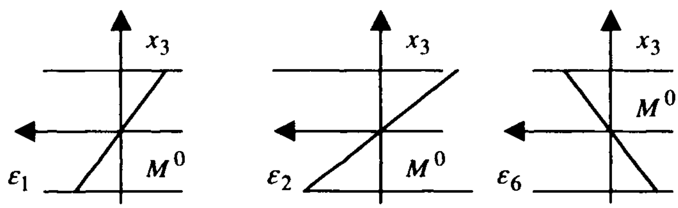

Figure 6.4. *Strain variations*

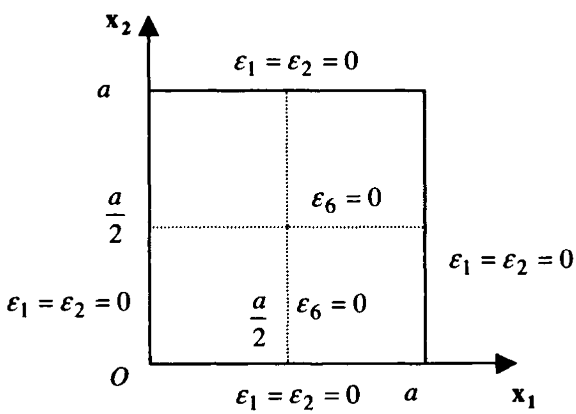

Figure 6.5. *Zero strain*

– For the stresses:

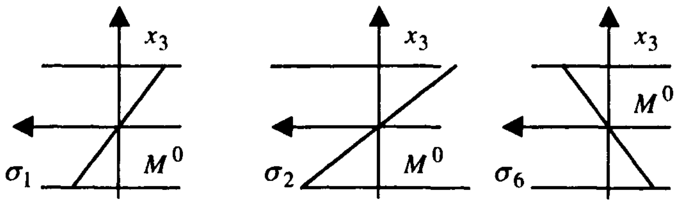

Figure 6.6. *Stress variations*

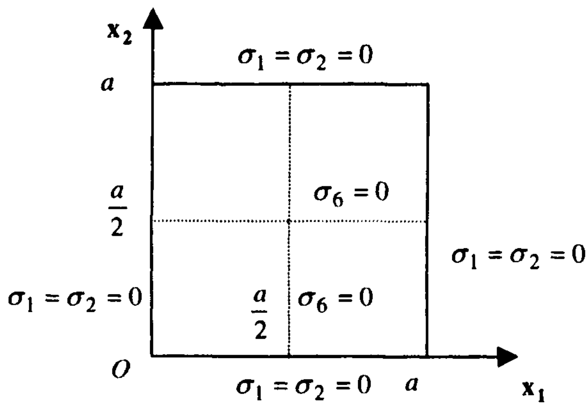

Figure 6.7. *Zero stresses*

– For the transverse shear stresses:

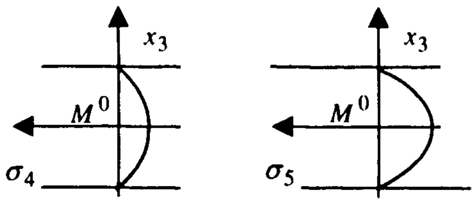

Figure 6.8. *Transverse shear stress variations*

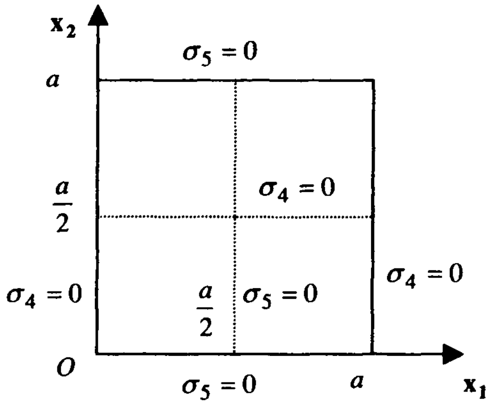

Figure 6.9. *Zero transverse shear stresses*

6.6. Free vibrations of a rectangular plate freely supported at its edge

The free vibrations of a symmetrical orthotropic plate are governed by the equation:

$$D_{11}\frac{\partial^4 u_3^0}{\partial x_1^4}+2(D_{12}+2D_{66})\frac{\partial^4 u_3^0}{\partial x_1^2 \partial x_2^2}+D_{22}\frac{\partial^4 u_3^0}{\partial x_2^4}+I_0\frac{\partial^2 u_3^0}{\partial t^2}=0 .$$

The solution to this equation, which satisfies the boundary conditions around the edge in displacement and load:

– for $x_1 = 0$ and $x_1 = a_1$:

$$u_3^0 = 0,$$

$$M_1 = -D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0,$$

– for $x_2 = 0$ and $x_2 = a_2$:

$$u_3^0 = 0,$$

$$M_2 = -D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{22}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0,$$

is of the form:

$$u_3^0 = U_{m_1 m_2}^3 \sin\frac{m_1 \pi x_1}{a_1}\sin\frac{m_2 \pi x_2}{a_2}\sin\left(\omega_{m_1 m_2} t + \varphi_{m_1 m_2}\right).$$

By introducing into the global vibration equation and after simplification by:

$$\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}\sin\left(\omega_{m_1m_2}t+\varphi_{m_1m_2}\right),$$

we obtain the equation:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^4 D_{11}+2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2\left(D_{12}+2D_{66}\right)\dots\right.$$

$$\left.\dots+\left(\frac{m_2\pi}{a_2}\right)^4 D_{22}-I_0\omega_{m_1m_2}^2\right]U_{m_1m_2}^3=0,$$

or:

$$\left(\pi^4 D_{m_1m_2}^*-I_0\omega_{m_1m_2}^2\right)U_{m_1m_2}^3=0,$$

with:

$$D_{m_1m_2}^*=\left(\frac{m_1}{a_1}\right)^4 D_{11}+2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2\left(D_{12}+2D_{66}\right)+\left(\frac{m_2}{a_2}\right)^4 D_{22},$$

which provides the natural frequencies:

$$\omega_{m_1m_2}=\pi^2\sqrt{\frac{\left(\frac{m_1}{a_1}\right)^4 D_{11}+2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2\left(D_{12}+2D_{66}\right)+\left(\frac{m_2}{a_2}\right)^4 D_{22}}{I_0}},$$

or:

$$\omega_{m_1m_2}=\pi^2\sqrt{\frac{D_{m_1m_2}^*}{I_0}}.$$

The natural frequencies of a square plate of side a are given by:

$$\omega_{m_1m_2}=\left(\frac{\pi}{a}\right)^2\sqrt{\frac{m_1^4D_{11}+2m_1^2m_2^2\left(D_{12}+2D_{66}\right)+m_2^4D_{22}}{I_0}}.$$

For a symmetrical orthotropic plate such that:

$$D_{11}=9D_{22},$$

$$D_{12}+2D_{66}=3D_{22},$$

the natural frequencies are equal to:

$$\omega_{m_1m_2}=k\left(\frac{\pi}{a}\right)^2\sqrt{\frac{D_{22}}{I_0}},$$

with:

$$k=\sqrt{9m_1^4+6m_1^2m_2^2+m_2^4}=3m_1^2+m_2^2.$$

For an isotropic plate, we have:

$$\omega'_{m_1 m_2} = k'\left(\frac{\pi}{a}\right)^2 \sqrt{\frac{D_0}{I_0}} ,$$

with:

$$k' = m_1^2 + m_2^2 .$$

The influence of orthotropy on the order of appearance of the first four modes of vibration is shown in the table below:

	Orthotropic plate			Isotropic plate		
Mode	m_1	m_2	k	m_1	m_2	k'
1	1	1	4	1	1	2
2	1	2	7	1	2	5
3	1	3	12	2	1	5
4	2	1	13	2	2	8

Figure 6.10. *Order of appearance of natural modes*

Its influence on the representation of the modes, and in particular the nodal lines (lines of zero transverse displacement), appears in the figure below:

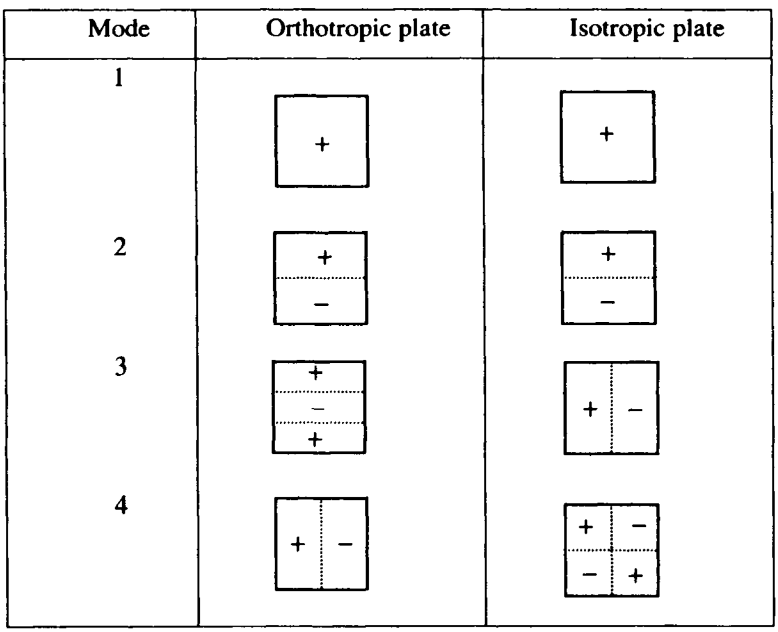

Figure 6.11. *Nodal lines*

In the case of a symmetrical, orthotropic plate, it may be noted that the mode $m_1 = 1, m_2 = 3$ appears before the mode $m_1 = 2, m_2 = 2$, the opposite order to that for an isotropic plate.

6.7. Buckling of a rectangular plate simply supported at its edge

6.7.1. *General case*

The rectangular plate of transverse dimensions a_1 and a_2 is simply supported at its edge. It is loaded in compression $-N_1^0$ and $-N_2^0$, with $N_1^0 > 0$ and $N_2^0 > 0$.

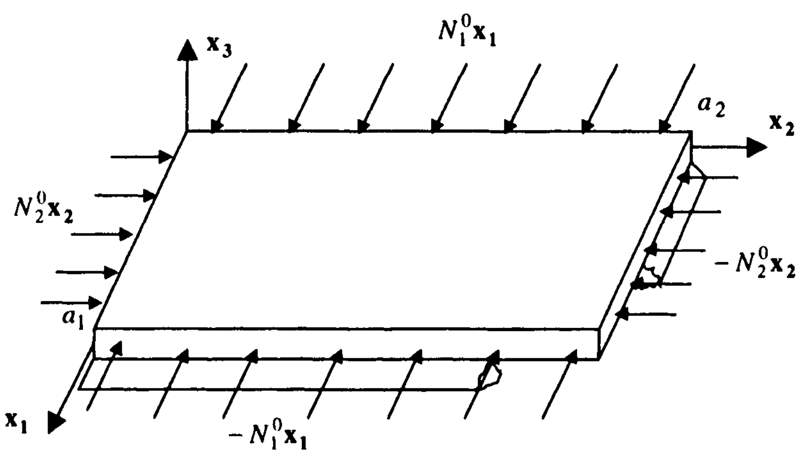

Figure 6.12. *Buckling loads*

The global buckling equation:

$$\frac{\partial^2 M_1}{\partial x_1^2} + 2\frac{\partial^2 M_6}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} - N_1^0 \frac{\partial^2 u_3^0}{\partial x_1^2} - N_2^0 \frac{\partial^2 u_3^0}{\partial x_2^2} = 0,$$

in which we have:

$$M_1 = -D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_2 = -D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{22}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_6 = -2D_{66}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

is written as:

$$D_{11}\frac{\partial^4 u_3^0}{\partial x_1^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 u_3^0}{\partial x_1^2 \partial x_2^2} + D_{22}\frac{\partial^4 u_3^0}{\partial x_2^4} + N_1^0\frac{\partial^2 u_3^0}{\partial x_1^2} + N_2^0\frac{\partial^2 u_3^0}{\partial x_2^2} = 0.$$

The boundary conditions for a simply supported plate are:

– for $x_1 = 0$ and $x_1 = a_1$:

$$u_3^0 = 0,$$

$$M_1 = -D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0,$$

– for $x_2 = 0$ and $x_2 = a_2$:

$$u_3^0 = 0,$$

$$M_2 = -D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{22}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0.$$

These conditions are satisfied by:

$$u_3^0 = U_{m_1 m_2}^3 \sin\frac{m_1 \pi x_1}{a_1}\sin\frac{m_2 \pi x_2}{a_2},$$

which, after introduction into the global buckling equation and simplification, gives:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2(D_{12} + 2D_{66}) + \left(\frac{m_2\pi}{a_2}\right)^4 D_{22}\cdots\right.$$

$$\left.\cdots - \left(\frac{m_1\pi}{a_1}\right)^2 N_1^0 - \left(\frac{m_2\pi}{a_2}\right)^2 N_2^0\right]U_{m_1 m_2}^3 = 0.$$

The critical buckling loads, corresponding to $U_{m_1 m_2}^3$ non-zero, are given by the expression:

$$\left(\frac{m_1}{a_1}\right)^2 N_1^0 + \left(\frac{m_1}{a_2}\right)^2 N_2^0 = \pi^2\left[\left(\frac{m_1}{a_1}\right)^4 D_{11}\cdots\right.$$

$$\left.\cdots + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2(D_{12} + 2D_{66}) + \left(\frac{m_2}{a_2}\right)^4 D_{22}\right],$$

which in the particular case where $N_2^0 = kN_1^0$, gives:

$$N_1^0 = \pi^2 \frac{\left(\frac{m_1}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 (D_{12}+2D_{66}) + \left(\frac{m_2}{a_2}\right)^4 D_{22}}{\left(\frac{m_1}{a_1}\right)^2 + k\left(\frac{m_2}{a_2}\right)^2}.$$

If, in addition the plate is square $a_1 = a_2 = a$, we have:

$$N_1^0 = \left(\frac{\pi}{a}\right)^2 \frac{m_1^4 D_{11} + 2m_1^2 m_2^2 (D_{12}+2D_{66}) + m_2^4 D_{22}}{m_1^2 + k m_2^2}.$$

We will now examine different special cases.

6.7.2. *Case of k=0*

The plate is subjected to a compression load $-N_1^0$ in the $\mathbf{x}_1$ direction.

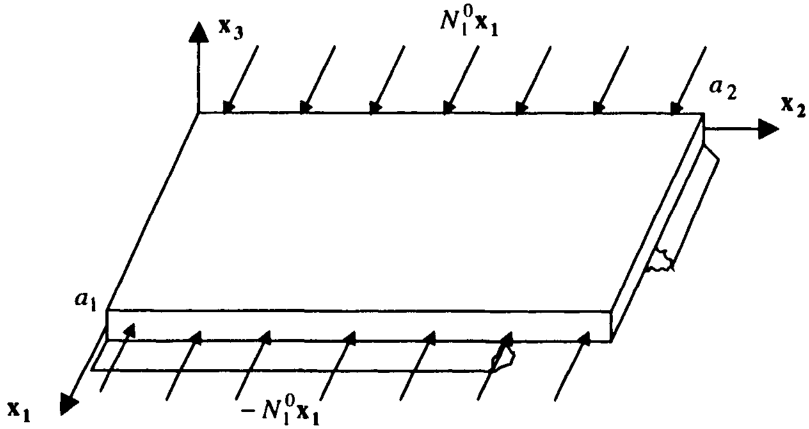

Figure 6.13. *Compression loading in one direction*

The critical buckling load is given by the expression:

$$N_1^0 = \left(\frac{\pi a_1}{m_1}\right)^2 \left[\left(\frac{m_1}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 (D_{12}+2D_{66}) + \left(\frac{m_2}{a_2}\right)^4 D_{22}\right],$$

or:

$$N_1^0 = \left(\frac{\pi}{a_2}\right)^2 \left[\left(\frac{m_1 a_2}{a_1}\right)^2 D_{11} + 2m_2^2 (D_{12}+2D_{66}) + m_2^4 \left(\frac{a_1}{m_1 a_2}\right)^2 D_{22}\right].$$

For $m_2 = 1$, this expression gives the series of curves plotted on the figure below.

For a given width a_2 and fixed values of m_1 and m_2 the value of the ratio $\frac{a_1}{a_2}$ which cancels the derivative of the critical buckling load:

$$\frac{\partial N_1^0}{\partial\left(\frac{a_1}{a_2}\right)} = -2\left(\frac{\pi}{a_2}\right)^2\left[m_1^2\left(\frac{a_2}{a_1}\right)^3 D_{11} - \frac{m_2^4}{m_1^2}\left(\frac{a_1}{a_2}\right)D_{22}\right],$$

is:

$$\frac{a_1}{a_2} = \frac{m_1}{m_2}\left(\frac{D_{11}}{D_{22}}\right)^{\frac{1}{4}}.$$

For this value, the critical buckling load is:

$$N_1^0 = 2\left(\frac{\pi m_2}{a_2}\right)^2\left(\sqrt{D_{11}D_{22}} + D_{12} + 2D_{66}\right).$$

The critical buckling load thus obtained is independent of m_1 and its minimum value is found for $m_2 = 1$.

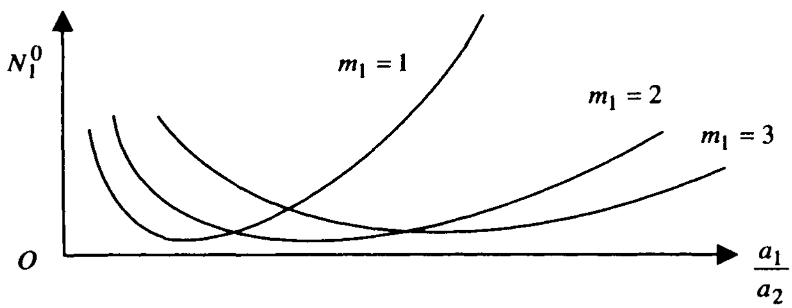

Figure 6.14. *Critical buckling loads*

The buckling of a plate loaded in compression in the $\mathbf{x}_1$ direction occurs in such a way that there can be several half-waves in the compression direction and only one in the perpendicular direction ($m_2 = 1$).

The critical buckling load is then given by the expression:

$$N_1^0 = \left(\frac{\pi}{a_2}\right)^2\left[\left(\frac{m_1 a_2}{a_1}\right)^2 D_{11} + 2(D_{12} + 2D_{66}) + \left(\frac{a_1}{m_1 a_2}\right)^2 D_{22}\right],$$

and the minimum obtained for:

$$\frac{a_1}{a_2} = m_1\left(\frac{D_{11}}{D_{22}}\right)^{\frac{1}{4}},$$

is equal to:

$$N_1^0 = 2\left(\frac{\pi}{a_2}\right)^2\left(\sqrt{D_{11}D_{22}} + D_{12} + 2D_{66}\right).$$

The intersection of the two plots of critical buckling load, relative to the two successive values m_1 and $m_1 + 1$, is found by equating:

$$N_{1_{m_1}}^0 = \left(\frac{\pi}{a_2}\right)^2\left[\left(\frac{m_1 a_2}{a_1}\right)^2 D_{11} + 2\left(D_{12} + 2D_{66}\right) + \left(\frac{a_1}{m_1 a_2}\right)^2 D_{22}\right],$$

$$N_{1_{m_1+1}}^0 = \left(\frac{\pi}{a_2}\right)^2\left[\left(\frac{(m_1+1)a_2}{a_1}\right)^2 D_{11} + 2\left(D_{12} + 2D_{66}\right) + \left(\frac{a_1}{(m_1+1)a_2}\right)^2 D_{22}\right],$$

which gives successively:

$$\left(\frac{m_1 a_2}{a_1}\right)^2 D_{11} + \left(\frac{a_1}{m_1 a_2}\right)^2 D_{22} = \left(\frac{(m_1+1)a_2}{a_1}\right)^2 D_{11} + \left(\frac{a_1}{(m_1+1)a_2}\right)^2 D_{22},$$

$$\left[(m_1+1)^2 - m_1^2\right]\left(\frac{a_2}{a_1}\right)^2 D_{11} = \left[\frac{1}{m_1^2} - \frac{1}{(m_1+1)^2}\right]\left(\frac{a_1}{a_2}\right)^2 D_{22},$$

$$\left(\frac{a_1}{a_2}\right)^4 = \frac{(m_1+1)^2 - m_1^2}{\dfrac{1}{m_1^2} - \dfrac{1}{(m_1+1)^2}}\frac{D_{11}}{D_{22}},$$

or:

$$\left(\frac{a_1}{a_2}\right)^2 = m_1(m_1+1)\sqrt{\frac{D_{11}}{D_{22}}}.$$

This expression enables the length a_1 to be calculated for a given value of a_2, for which the critical buckling load is identical for the modes m_1 and $m_1 + 1$.

When a plate is subjected to a compression load $-N_1^0$ in the $\mathbf{x}_1$ direction, the buckling of the plate occurs such that there exists:

- in the $\mathbf{x}_1$ direction: a single half-wave $(m_2 = 1)$,
- in the $\mathbf{x}_2$ direction:

  - a half-wave for: $0 < \dfrac{a_1}{a_2} < \left(4\dfrac{D_{11}}{D_{22}}\right)^{\frac{1}{4}}$,

- two half-waves for: $\left(4\frac{D_{11}}{D_{22}}\right)^{\frac{1}{4}} < \frac{a_1}{a_2} < \left(36\frac{D_{11}}{D_{22}}\right)^{\frac{1}{4}}$,

- m_1 half-waves for: $\left(m_1^2\frac{D_{11}}{D_{22}}\right)^{\frac{1}{4}} < \frac{a_1}{a_2} < \left(m_1^2(m_1^2+1)^2\frac{D_{11}}{D_{22}}\right)^{\frac{1}{4}}$.

The critical load corresponding to this intersection is given by:

$$N_1^0 = \left(\frac{\pi}{a_2}\right)^2\left[\frac{m_1}{m_1+1}\sqrt{D_{11}D_{22}} + 2(D_{12}+2D_{66}) + \frac{m_1+1}{m_1}\sqrt{D_{11}D_{22}}\right],$$

$$N_1^0 = \left(\frac{\pi}{a_2}\right)^2\left[\frac{m_1^2+(m_1+1)^2}{m_1(m_1+1)}\sqrt{D_{11}D_{22}} + 2(D_{12}+2D_{66})\right].$$

For the special case of a plate for which:

$$D_{11} = 9D_{22},$$

$$D_{12} + 2D_{66} = 3D_{22},$$

we have:

$$N_1^0 = \left(\frac{\pi}{a_2}\right)^2\left(3\frac{m_1a_2}{a_1} + m_2^2\frac{a_1}{m_1a_2}\right)^2 D_{22} ,$$

the critical buckling load, obtained for $m_2 = 1$, is equal to:

$$N_1^0 = \left(\frac{\pi}{a_2}\right)^2\left(3\frac{m_1a_2}{a_1} + \frac{a_1}{m_1a_2}\right)^2 D_{22} .$$

The minimum critical load:

$$N_1^0 = 12\left(\frac{\pi}{a_2}\right)^2 D_{22} ,$$

is obtained for:

$$\frac{a_1}{a_2} = \sqrt{3}m_1,$$

or, for the first two modes:

$$\frac{a_1}{a_2} = \sqrt{3} \text{ and } \frac{a_1}{a_2} = 2\sqrt{3} .$$

The intersection of the plots $m_1 = 1$ and $m_1 = 2$ is obtained for:

$$\frac{a_1}{a_2} = \sqrt{6},$$

and the corresponding critical load equal to:

$$N_1^0 = \frac{27}{2}\left(\frac{\pi}{a_2}\right)^2 D_{22} .$$

If the plate is square ($a_1 = a_2 = a$), the critical buckling load obtained for $m_1 = 1$, is:

$$N_1^0 = 16\left(\frac{\pi}{a}\right)^2 D_{22} .$$

6.7.3. *Case of k=1*

The plate is subjected to the same compression load $-N_1^0$ in the $\mathbf{x}_1$ and $\mathbf{x}_2$ directions.

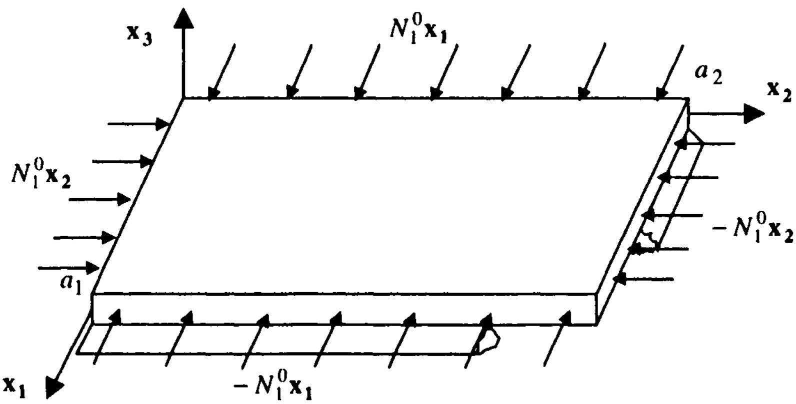

Figure 6.15. *Identical compression loads in two directions*

The critical buckling load is then given by the smallest value of:

$$N_1^0 = \pi^2 \frac{\left(\frac{m_1}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 (D_{12} + 2D_{66}) + \left(\frac{m_2}{a_2}\right)^4 D_{22}}{\left(\frac{m_1}{a_1}\right)^2 + \left(\frac{m_2}{a_2}\right)^2} .$$

In the case where:

$D_{11} = 9D_{22}$,

$D_{12} + 2D_{66} = 3D_{22}$,

we have:

$$N_1^0 = \pi^2 \frac{9\left(\frac{m_1}{a_1}\right)^4 + 6\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 + \left(\frac{m_2}{a_2}\right)^4}{\left(\frac{m_1}{a_1}\right)^2 + \left(\frac{m_2}{a_2}\right)^2} D_{22},$$

$$N_1^0 = \pi^2 \frac{\left[3\left(\frac{m_1}{a_1}\right)^2 + \left(\frac{m_2}{a_2}\right)^2\right]^2}{\left(\frac{m_1}{a_1}\right)^2 + \left(\frac{m_2}{a_2}\right)^2} D_{22},$$

and for a square plate:

$$N_1^0 = \left(\frac{\pi}{a}\right)^2 \frac{\left(3m_1^2 + m_2^2\right)^2}{m_1^2 + m_2^2} D_{22} .$$

The critical buckling load, obtained for $m_1 = m_2 = 1$, is equal to:

$$N_1^0 = 8\left(\frac{\pi}{a}\right)^2 D_{22} .$$

6.7.4. *Case of k= –1/2*

The plate is subjected in the $\mathbf{x}_1$ direction to the compression load $-N_1^0$ and in the $\mathbf{x}_2$ direction to the tensile load $\frac{1}{2}N_1^0$.

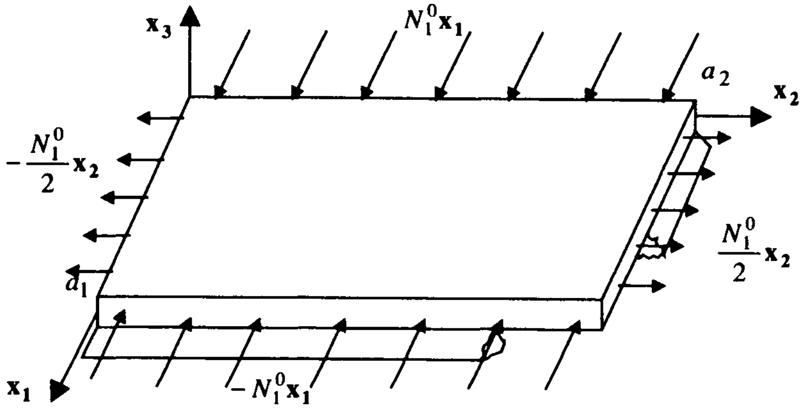

Figure 6.16. *Compression and tension loads*

The critical buckling load is equal to the smallest value of:

$$N_1^0 = \pi^2 \frac{\left(\frac{m_1}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 (D_{12} + 2D_{66}) + \left(\frac{m_2}{a_2}\right)^4 D_{22}}{\left(\frac{m_1}{a_1}\right)^2 - \frac{1}{2}\left(\frac{m_2}{a_2}\right)^2}.$$

For the particular case already described we have:

$$N_1^0 = \pi^2 \frac{\left[3\left(\frac{m_1}{a_1}\right)^2 + \left(\frac{m_2}{a_2}\right)^2\right]^2}{\left(\frac{m_1}{a_1}\right)^2 - \frac{1}{2}\left(\frac{m_2}{a_2}\right)^2} D_{22},$$

and for a square plate:

$$N_1^0 = 2\left(\frac{\pi}{a}\right)^2 \frac{\left(3m_1^2 + m_2^2\right)^2}{2m_1^2 - m_2^2} D_{22}.$$

The critical buckling load, obtained for $m_1 = m_2 = 1$, is:

$$N_1^0 = 32\left(\frac{\pi}{a}\right)^2 D_{22}.$$

The tensile load in the $\mathbf{x}_2$ direction increases the critical buckling load; with $k = -1/2$ it is twice as high as in the case of $k = 0$ and four times as high as the case of $k = 1$.

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Chapter 7

Thermo-elastic behaviour of composites

7.1. Introduction

The important roles played by a temperature variation ΔT and by the absorption of humidity $\Delta\eta$ were noted previously in the presentation of the laws of the behaviour of an orthotropic material. Both effects work in the same way, so the discussion here will be limited to a temperature variation ΔT.

In composites cured above room temperature there appear, on return to their normal temperature of use, residual stresses which may be large. These originate from the different values of the thermal expansion coefficients for the fibres and matrix.

After having presented the constitutive relations of an orthotropic material in its orthotropic axes and off-axis we examine the definition of these expressions in matrix form in plate theory for which the transverse normal stress is zero.

Then we introduce the global constitutive relation of the composite in thermo-elasticity which allows the study of the behaviour of a multi-layer plate in tension, flexure, vibration and buckling.

7.2. Constitutive relation for an orthotropic material

7.2.1. *Constitutive relation in orthotropic axes*

The thermal strains, resulting from the change in temperature ΔT and noted $\bar{\varepsilon}'_i$ in the orthotropic axes $(\bar{e})$, are:

$$\bar{\varepsilon}'_i = \bar{\alpha}_i \Delta T \quad (i = 1,2,\ldots,6) \text{ or } \bar{\boldsymbol{\varepsilon}}' = \bar{\boldsymbol{\alpha}} \Delta T,$$

with $\bar{\alpha}_4 = \bar{\alpha}_5 = \bar{\alpha}_6 = 0$.

The strains $\bar{\varepsilon}_i$ due to the stresses $\bar{\sigma}_i$ and to the thermal dilatation $\bar{\varepsilon}_i'$ are given by:

$$\bar{\varepsilon}_i = \bar{S}_{ij} \bar{\sigma}_j + \bar{\varepsilon}'_i \quad (i, j = 1,2,\ldots,6) \text{ or } \bar{\boldsymbol{\varepsilon}} = \bar{\mathbf{S}} \bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{\varepsilon}}'.$$

These may be expressed as a function of the variation in temperature ΔT using the expressions:

$$\bar{\varepsilon}_i = \bar{S}_{ij} \bar{\sigma}_j + \bar{\alpha}_i \Delta T \quad (i, j = 1,2,\ldots,6) \text{ or } \bar{\boldsymbol{\varepsilon}} = \bar{\mathbf{S}} \bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{\alpha}} \Delta T.$$

In an explicit manner we have:

$$\begin{bmatrix}\bar{\varepsilon}_1\\\bar{\varepsilon}_2\\\bar{\varepsilon}_3\\\bar{\varepsilon}_4\\\bar{\varepsilon}_5\\\bar{\varepsilon}_6\end{bmatrix}=\begin{bmatrix}\bar{S}_{11}&\bar{S}_{12}&\bar{S}_{13}&0&0&0\\\bar{S}_{12}&\bar{S}_{22}&\bar{S}_{23}&0&0&0\\\bar{S}_{13}&\bar{S}_{23}&\bar{S}_{33}&0&0&0\\0&0&0&\bar{S}_{44}&0&0\\0&0&0&0&\bar{S}_{55}&0\\0&0&0&0&0&\bar{S}_{66}\end{bmatrix}\begin{bmatrix}\bar{\sigma}_1\\\bar{\sigma}_2\\\bar{\sigma}_3\\\bar{\sigma}_4\\\bar{\sigma}_5\\\bar{\sigma}_6\end{bmatrix}+\begin{bmatrix}\bar{\alpha}_1\\\bar{\alpha}_2\\\bar{\alpha}_3\\0\\0\\0\end{bmatrix}\Delta T\,.$$

The stresses $\bar{\sigma}_i$ are given as a function of $\bar{\varepsilon}_i$ and of $\bar{\varepsilon}_i$' by:

$$\bar{\sigma}_i=\bar{C}_{ij}\left(\bar{\varepsilon}_j-\bar{\varepsilon}'_j\right)\quad(i,j=1,2,\ldots,6)\ \text{or}\ \bar{\boldsymbol{\sigma}}=\bar{\mathbf{C}}(\bar{\boldsymbol{\varepsilon}}-\bar{\boldsymbol{\varepsilon}}'),$$

and as a function of the variation in temperature ΔT by:

$$\bar{\sigma}_i=\bar{C}_{ij}\left(\bar{\varepsilon}_j-\bar{\alpha}_j\Delta T\right)\quad(i,j=1,2,\ldots,6)\ \text{or}\ \bar{\boldsymbol{\sigma}}=\bar{\mathbf{C}}(\bar{\boldsymbol{\varepsilon}}-\bar{\boldsymbol{\alpha}}\Delta T).$$

These two latter expressions can be written in the form:

$$\bar{\sigma}_i=\bar{C}_{ij}\bar{\varepsilon}_j-\bar{C}'_i\,\Delta T\quad(i,j=1,2,\ldots,6),\ \text{with}\ \bar{C}'_i=\bar{C}_{ij}\bar{\alpha}_j\,,$$

or:

$$\bar{\boldsymbol{\sigma}}=\bar{\mathbf{C}}\bar{\boldsymbol{\varepsilon}}-\bar{\mathbf{C}}'\Delta T\,,\ \text{with}\ \bar{\mathbf{C}}'=\bar{\mathbf{C}}\bar{\boldsymbol{\alpha}}\,.$$

The matrix $\bar{\mathbf{C}}'=\bar{\mathbf{C}}\bar{\boldsymbol{\alpha}}$ is equal to:

$$\bar{\mathbf{C}}'=\begin{bmatrix}\bar{C}_{11}&\bar{C}_{12}&\bar{C}_{13}&0&0&0\\\bar{C}_{12}&\bar{C}_{22}&\bar{C}_{23}&0&0&0\\\bar{C}_{13}&\bar{C}_{23}&\bar{C}_{33}&0&0&0\\0&0&0&\bar{C}_{44}&0&0\\0&0&0&0&\bar{C}_{55}&0\\0&0&0&0&0&\bar{C}_{66}\end{bmatrix}\begin{bmatrix}\bar{\alpha}_1\\\bar{\alpha}_2\\\bar{\alpha}_3\\0\\0\\0\end{bmatrix},$$

or:

$$\begin{bmatrix}\bar{C}'_1\\\bar{C}'_2\\\bar{C}'_3\\0\\0\\0\end{bmatrix}=\begin{bmatrix}\bar{C}_{11}\bar{\alpha}_1+\bar{C}_{12}\bar{\alpha}_2+\bar{C}_{13}\bar{\alpha}_3\\\bar{C}_{12}\bar{\alpha}_1+\bar{C}_{22}\bar{\alpha}_2+\bar{C}_{23}\bar{\alpha}_3\\\bar{C}_{13}\bar{\alpha}_1+\bar{C}_{23}\bar{\alpha}_2+\bar{C}_{33}\bar{\alpha}_3\\0\\0\\0\end{bmatrix}.$$

The constitutive relation may be written in the form:

$$\begin{bmatrix}\bar{\sigma}_1\\ \bar{\sigma}_2\\ \bar{\sigma}_3\\ \bar{\sigma}_4\\ \bar{\sigma}_5\\ \bar{\sigma}_6\end{bmatrix}=\begin{bmatrix}\bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & 0 & 0 & 0\\ \bar{C}_{12} & \bar{C}_{22} & \bar{C}_{23} & 0 & 0 & 0\\ \bar{C}_{13} & \bar{C}_{23} & \bar{C}_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & \bar{C}_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & \bar{C}_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & \bar{C}_{66}\end{bmatrix}\begin{bmatrix}\bar{\varepsilon}_1\\ \bar{\varepsilon}_2\\ \bar{\varepsilon}_3\\ \bar{\varepsilon}_4\\ \bar{\varepsilon}_5\\ \bar{\varepsilon}_6\end{bmatrix}-\begin{bmatrix}\bar{C}'_1\\ \bar{C}'_2\\ \bar{C}'_3\\ 0\\ 0\\ 0\end{bmatrix}\Delta T .$$

7.2.2. *Constitutive relation in orthotropic off-axes*

By introducing the conventional matrices for changing the axes $\mathbf{M}$ and $\mathbf{N}$, the strains in the base (e) off-axis of orthotropy are:

$$\boldsymbol{\varepsilon} = \mathbf{N}\bar{\boldsymbol{\varepsilon}} = \mathbf{N}\left(\bar{\mathbf{S}}\bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{\varepsilon}}'\right) = \mathbf{N}\left(\bar{\mathbf{S}}\bar{\boldsymbol{\sigma}} + \bar{\boldsymbol{\alpha}}\Delta T\right),$$

or:

$$\boldsymbol{\varepsilon} = \mathbf{N}\bar{\mathbf{S}}\mathbf{N}^{\mathrm{T}}\boldsymbol{\sigma} + \mathbf{N}\bar{\boldsymbol{\alpha}}\Delta T .$$

The strains can be written as:

$$\boldsymbol{\varepsilon} = \mathbf{S}\boldsymbol{\sigma} + \mathbf{S}'\Delta T ,$$

with:

$$\mathbf{S}' = \mathbf{N}\bar{\boldsymbol{\alpha}} .$$

The matrix $\mathbf{S}'$ is given by:

$$\mathbf{S}'=\begin{bmatrix}c^2 & s^2 & 0 & 0 & 0 & -cs\\ s^2 & c^2 & 0 & 0 & 0 & cs\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & c & s & 0\\ 0 & 0 & 0 & -s & c & 0\\ 2cs & -2cs & 0 & 0 & 0 & c^2-s^2\end{bmatrix}\begin{bmatrix}\bar{\alpha}_1\\ \bar{\alpha}_2\\ \bar{\alpha}_3\\ 0\\ 0\\ 0\end{bmatrix},$$

or:

$$\begin{bmatrix}\bar{S}'_1\\ \bar{S}'_2\\ \bar{S}'_3\\ 0\\ 0\\ \bar{S}'_6\end{bmatrix}=\begin{bmatrix}c^2\bar{\alpha}_1 + s^2\bar{\alpha}_2\\ s^2\bar{\alpha}_1 + c^2\bar{\alpha}_2\\ \bar{\alpha}_3\\ 0\\ 0\\ 2cs\left(\bar{\alpha}_1 - \bar{\alpha}_2\right)\end{bmatrix},$$

in an explicit form we have:

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} + \begin{bmatrix} \overline{S}'_1 \\ \overline{S}'_2 \\ \overline{S}'_3 \\ 0 \\ 0 \\ \overline{S}'_6 \end{bmatrix} \Delta T \, .$$

The stresses are given by:

$$\boldsymbol{\sigma} = \mathbf{M}\overline{\boldsymbol{\sigma}} = \mathbf{M}\left(\overline{\mathbf{C}}\overline{\boldsymbol{\varepsilon}} - \overline{\mathbf{C}}'\Delta T\right) = \mathbf{M}\overline{\mathbf{C}}\mathbf{M}^{\mathbf{T}}\boldsymbol{\varepsilon} - \mathbf{M}\overline{\mathbf{C}}'\Delta T \, ,$$

or:

$$\boldsymbol{\sigma} = \mathbf{C}\boldsymbol{\varepsilon} - \mathbf{C}'\Delta T \, ,$$

with:

$$\mathbf{C}' = \mathbf{M}\overline{\mathbf{C}}' .$$

The matrix $\mathbf{C}'$ given by the product:

$$\mathbf{M}\overline{\mathbf{C}}' = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -2cs \\ s^2 & c^2 & 0 & 0 & 0 & 2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ cs & -cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \overline{C}_{11}\overline{\alpha}_1 + \overline{C}_{12}\overline{\alpha}_2 + \overline{C}_{13}\overline{\alpha}_3 \\ \overline{C}_{12}\overline{\alpha}_1 + \overline{C}_{22}\overline{\alpha}_2 + \overline{C}_{23}\overline{\alpha}_3 \\ \overline{C}_{13}\overline{\alpha}_1 + \overline{C}_{23}\overline{\alpha}_2 + \overline{C}_{33}\overline{\alpha}_3 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

is equal to:

$$\begin{bmatrix} C'_1 \\ C'_2 \\ C'_3 \\ 0 \\ 0 \\ C'_6 \end{bmatrix} = \begin{bmatrix} \left(c^2\overline{C}_{11} + s^2\overline{C}_{12}\right)\overline{\alpha}_1 + \left(c^2\overline{C}_{12} + s^2\overline{C}_{22}\right)\overline{\alpha}_2 + \left(c^2\overline{C}_{13} + s^2\overline{C}_{23}\right)\overline{\alpha}_3 \\ \left(s^2\overline{C}_{11} + c^2\overline{C}_{12}\right)\overline{\alpha}_1 + \left(s^2\overline{C}_{12} + c^2\overline{C}_{22}\right)\overline{\alpha}_2 + \left(s^2\overline{C}_{13} + c^2\overline{C}_{23}\right)\overline{\alpha}_3 \\ \overline{C}_{13}\overline{\alpha}_1 + \overline{C}_{23}\overline{\alpha}_2 + \overline{C}_{33}\overline{\alpha} \\ 0 \\ 0 \\ cs\left(\overline{C}_{11} - \overline{C}_{12}\right)\overline{\alpha}_1 + cs\left(\overline{C}_{12} - \overline{C}_{13}\right)\overline{\alpha}_2 + cs\left(\overline{C}_{13} - \overline{C}_{23}\right)\overline{\alpha}_3 \end{bmatrix} .$$

The constitutive relation is written as:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} - \begin{bmatrix} C'_1 \\ C'_2 \\ C'_3 \\ 0 \\ 0 \\ C'_6 \end{bmatrix} \Delta T \, .$$

7.3. Constitutive relation when the normal transverse stress is zero

7.3.1. *Constitutive relation in orthotropic axes*

Given the notations introduced above we have the expression:

$$\overline{\boldsymbol{\sigma}} = \overline{\mathbf{Q}}\left(\overline{\boldsymbol{\varepsilon}} - \overline{\boldsymbol{\alpha}}\Delta T\right),$$

or:

$$\overline{\boldsymbol{\sigma}} = \overline{\mathbf{Q}}\overline{\boldsymbol{\varepsilon}} - \overline{\mathbf{Q}}'\Delta T \text{ , with } \overline{\mathbf{Q}}' = \overline{\mathbf{Q}}\overline{\boldsymbol{\alpha}} \text{ ,}$$

which is written as :

$$\overline{\mathbf{Q}}' = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & 0 & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & 0 & 0 & 0 \\ 0 & 0 & \overline{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{Q}_{44} & 0 \\ 0 & 0 & 0 & 0 & \overline{Q}_{55} \end{bmatrix} \begin{bmatrix} \overline{\alpha}_1 \\ \overline{\alpha}_2 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

or:

$$\begin{bmatrix} \overline{Q}'_1 \\ \overline{Q}'_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \overline{Q}_{11}\overline{\alpha}_1 + \overline{Q}_{12}\overline{\alpha}_2 \\ \overline{Q}_{12}\overline{\alpha}_1 + \overline{Q}_{22}\overline{\alpha}_2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

The constitutive relation is written in the form:

$$\begin{bmatrix} \overline{\sigma}_1 \\ \overline{\sigma}_2 \\ \overline{\sigma}_6 \\ \overline{\sigma}_4 \\ \overline{\sigma}_5 \end{bmatrix} = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & 0 & 0 & 0 \\ \overline{Q}_{12} & \overline{Q}_{22} & 0 & 0 & 0 \\ 0 & 0 & \overline{Q}_{66} & 0 & 0 \\ 0 & 0 & 0 & \overline{Q}_{44} & 0 \\ 0 & 0 & 0 & 0 & \overline{Q}_{55} \end{bmatrix} \begin{bmatrix} \overline{\varepsilon}_1 \\ \overline{\varepsilon}_2 \\ \overline{\varepsilon}_6 \\ \overline{\varepsilon}_4 \\ \overline{\varepsilon}_5 \end{bmatrix} - \begin{bmatrix} \overline{Q}'_1 \\ \overline{Q}'_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta T .$$

7.3.2. *Constitutive relation in orthotropic off-axes*

The conventional formulae for changing axes and the preceding expressions allow us to write:

$$\boldsymbol{\sigma} = \mathbf{M}\overline{\boldsymbol{\sigma}} = \mathbf{M}\left(\overline{\mathbf{Q}}\overline{\boldsymbol{\varepsilon}} - \overline{\mathbf{Q}}'\Delta T\right) = \mathbf{M}\overline{\mathbf{Q}}\mathbf{M}^{\mathrm{T}}\boldsymbol{\varepsilon} - \mathbf{M}\overline{\mathbf{Q}}'\Delta T .$$

The constitutive relation is now in the form:

$$\boldsymbol{\sigma} = \mathbf{Q}\boldsymbol{\varepsilon} - \mathbf{Q}'\Delta T ,$$

where the matrix:

$$\mathbf{Q'} = \mathbf{M}\overline{\mathbf{Q}}',$$

is equal to:

$$\mathbf{Q'} = \begin{bmatrix} c^2 & s^2 & -2cs & 0 & 0 \\ s^2 & c^2 & 2cs & 0 & 0 \\ cs & -cs & c^2 - s^2 & 0 & 0 \\ 0 & 0 & 0 & c & s \\ 0 & 0 & 0 & -s & c \end{bmatrix} \begin{bmatrix} \overline{Q}_{11}\overline{\alpha}_1 + \overline{Q}_{12}\overline{\alpha}_2 \\ \overline{Q}_{12}\overline{\alpha}_1 + \overline{Q}_{22}\overline{\alpha}_2 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

or:

$$\begin{bmatrix} Q'_1 \\ Q'_2 \\ Q'_6 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \left(c^2\overline{Q}_{11} + s^2\overline{Q}_{12}\right)\overline{\alpha}_1 + \left(c^2\overline{Q}_{12} + s^2\overline{Q}_{22}\right)\overline{\alpha}_2 \\ \left(s^2\overline{Q}_{11} + c^2\overline{Q}_{12}\right)\overline{\alpha}_1 + \left(s^2\overline{Q}_{12} + c^2\overline{Q}_{22}\right)\overline{\alpha}_2 \\ cs\left(\overline{Q}_{11} - \overline{Q}_{12}\right)\overline{\alpha}_1 + cs\left(\overline{Q}_{12} - \overline{Q}_{22}\right)\overline{\alpha}_2 \\ 0 \\ 0 \end{bmatrix}.$$

The constitutive relation is written as:

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ Q_{12} & Q_{22} & Q_{26} & 0 & 0 \\ Q_{16} & Q_{26} & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{45} & Q_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} - \begin{bmatrix} Q'_1 \\ Q'_2 \\ Q'_6 \\ 0 \\ 0 \end{bmatrix} \Delta T.$$

7.4. Global cohesion forces

The resultants of the global cohesion forces are equal to:

$$N_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i dx_3, \quad (i = 1, 2, 6),$$

or:

$$N_i = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \left[Q_{ij}^k\left(\varepsilon_j^0 + x_3\kappa_j\right) - Q'^k_i \,\Delta T\right] dx_3,$$

$$N_i = \sum_{k=1}^{N} Q_{ij}^k (z_k - z_{k-1})\varepsilon_j^0 + \frac{1}{2}\sum_{k=1}^{N} Q_{ij}^k \left(z_k^2 - z_{k-1}^2\right)\kappa_j \ldots$$

$$\ldots - \sum_{k=1}^{N} Q'^k_i \,(z_k - z_{k-1})\Delta T.$$

Introducing:

$$A'_i = \sum_{k=1}^{N} Q'^k_i \left(z_k - z_{k-1}\right),$$

the resultant of the global cohesion forces is written as:

$$N_i = A_{ij}\varepsilon_j^0 + B_{ij}\kappa_j - A'_i \,\Delta T \quad \left(i,\, j = 1, 2, 6\right).$$

The moments of the global cohesion loads are given by:

$$M_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i x_3 dx_3 \,,$$

from which:

$$M_i = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \left[Q_{ij}^k\left(\varepsilon_j^0 + x_3\kappa_j\right) - Q'^k_i \,\Delta T\right] x_3 dx_3 \,,$$

$$M_i = \frac{1}{2}\sum_{k=1}^{N} Q_{ij}^k \left(z_k^2 - z_{k-1}^2\right)\varepsilon_j^0 + \frac{1}{3}\sum_{k=1}^{N} Q_{ij}^k \left(z_k^3 - z_{k-1}^3\right)\kappa_j \ldots$$

$$\ldots - \frac{1}{2}\sum_{k=1}^{N} Q'^k_i \left(z_k^2 - z_{k-1}^2\right)\Delta T.$$

By putting:

$$B'_i = \frac{1}{2}\sum_{k=1}^{N} Q'^k_i \left(z_k^2 - z_{k-1}^2\right),$$

the moments of the global cohesion loads may be written in the form:

$$M_i = B_{ij}\varepsilon_j^0 + D_{ij}\kappa_j - B'_i \Delta T \quad \left(i,\, j = 1, 2, 6\right).$$

7.5. Global composite constitutive relation

The global composite constitutive relation is written in explicit matrix form as:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} - \begin{bmatrix} A'_1 \\ A'_2 \\ A'_6 \\ B'_1 \\ B'_2 \\ B'_6 \end{bmatrix} \Delta T \,,$$

and in conventional form as:

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{M} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix} - \begin{bmatrix} \mathbf{A'} \\ \mathbf{B'} \end{bmatrix} \Delta T,$$

or:

$$\mathbf{N} = \mathbf{A}\boldsymbol{\varepsilon}^0 + \mathbf{B}\boldsymbol{\kappa} - \mathbf{A'}\Delta T,$$

$$\mathbf{M} = \mathbf{B}\boldsymbol{\varepsilon}^0 + \mathbf{D}\boldsymbol{\kappa} - \mathbf{B'}\Delta T,$$

with:

$$A'_i = \sum_{k=1}^{N} Q'^k_i \left(z_k - z_{k-1}\right), \quad B'_i = \frac{1}{2}\sum_{k=1}^{N} Q'^k_i \left(z_k^2 - z_{k-1}^2\right) \quad (i = 1,2,6).$$

7.6. Decoupling

7.6.1. *Composite with mirror symmetry*

In the case of mirror symmetry the terms:

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{N} Q^k_{ij} \left(z_k^2 - z_{k-1}^2\right), B'_i = \frac{1}{2}\sum_{k=1}^{N} Q'^k_i \left(z_k^2 - z_{k-1}^2\right),$$

of the global composite constitutive relation are zero.

This, which may then be written as:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & 0 & 0 & 0 \\ A_{12} & A_{22} & A_{26} & 0 & 0 & 0 \\ A_{16} & A_{26} & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} - \begin{bmatrix} A'_1 \\ A'_2 \\ A'_6 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta T,$$

or:

$$\mathbf{N} = \mathbf{A}\boldsymbol{\varepsilon}^0 - \mathbf{A'}\Delta T,$$

$$\mathbf{M} = \mathbf{D}\boldsymbol{\kappa},$$

shows that a plate, which is not loaded by external forces, will only show membrane strains.

7.6.2. *Balanced composite*

In the case of a balanced composite we have seen previously that the global stiffnesses A_{16} and A_{26} are zero.

For two layers of the same thickness and with opposite angles the values of:

$$Q'^{k}_{6} = cs\left(\overline{Q}_{11} - \overline{Q}_{12}\right)\overline{\alpha}_1 + cs\left(\overline{Q}_{12} - \overline{Q}_{22}\right)\overline{\alpha}_2 ,$$

are opposite and the term:

$$A'_6 = \sum_{k=1}^{N} Q'^{k}_{6} \left(z_k - z_{k-1}\right),$$

of the global composite constitutive relation is zero.

This is written as:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} - \begin{bmatrix} A'_1 \\ A'_2 \\ 0 \\ B'_1 \\ B'_2 \\ B'_6 \end{bmatrix} \Delta T .$$

7.6.3. *Balanced symmetrical composite*

Given that $A_{16} = A_{26} = 0$, $B_{ij} = 0$ and $A'_6 = 0$, $B'_i = 0$, the constitutive relation is written as:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} - \begin{bmatrix} A'_1 \\ A'_2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Delta T .$$

7.7. Balanced symmetrical composite loaded in the mean plane

For such a composite the global constitutive relation which is written as:

$$\begin{bmatrix} \mathbf{N} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}^0 \\ \boldsymbol{\kappa} \end{bmatrix} - \begin{bmatrix} \mathbf{A'} \\ \mathbf{0} \end{bmatrix} \Delta T ,$$

gives:

$$\boldsymbol{\kappa} = \mathbf{0},$$

$$\mathbf{N} = \mathbf{A}\boldsymbol{\varepsilon}^0 - \mathbf{A'}\Delta T,$$

or:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{bmatrix} - \begin{bmatrix} A'_1 \\ A'_2 \\ 0 \end{bmatrix} \Delta T .$$

The membrane strains are written as:

$$\boldsymbol{\varepsilon}^0 = \mathbf{A}^{-1}\mathbf{N} + \mathbf{A}^{-1}\mathbf{A'}\Delta T .$$

Designating by A_{ij}^* the term of row i and column j of $\mathbf{A}^{-1}$, the preceding expression is written as:

$$\begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & 0 \\ A_{12}^* & A_{22}^* & 0 \\ 0 & 0 & A_{66}^* \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_6 \end{bmatrix} + \begin{bmatrix} A_{11}^* A'_1 + A_{12}^* A'_2 \\ A_{12}^* A'_1 + A_{22}^* A'_2 \\ 0 \end{bmatrix} \Delta T .$$

The global constitutive relation for the composite is written as:

$$\boldsymbol{\varepsilon}^0 = h\mathbf{A}^{-1}\frac{\mathbf{N}}{h} + \mathbf{A}^{-1}\mathbf{A'}\Delta T ,$$

or:

$$\boldsymbol{\varepsilon}^0 = h\mathbf{A}^{-1}\boldsymbol{\sigma} + \mathbf{A}^{-1}\mathbf{A'}\Delta T .$$

This expression can be written in the form:

$$\begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{E_1^0} & -\dfrac{\nu_{21}^0}{E_2^0} & 0 \\ -\dfrac{\nu_{12}^0}{E_1^0} & \dfrac{1}{E_2^0} & 0 \\ 0 & 0 & \dfrac{1}{G_{12}^0} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} + \begin{bmatrix} \alpha_1^0 \\ \alpha_2^0 \\ 0 \end{bmatrix} \Delta T ,$$

with the equivalent characteristics:

$$E_1^0 = \frac{1}{hA_{11}^*}, \quad E_2^0 = \frac{1}{hA_{22}^*}, \quad G_{12}^0 = \frac{1}{hA_{66}^*},$$

$$\nu_{12}^0 = -\frac{A_{12}^*}{A_{11}^*}, \quad \nu_{21}^0 = -\frac{A_{12}^*}{A_{22}^*},$$

$$\alpha_1^0 = A_{11}^* A'_1 + A_{12}^* A'_2 , \quad \alpha_2^0 = A_{12}^* A'_1 + A_{22}^* A'_2 .$$

Chapter 8

Symmetrical orthotropic Reissner-Mindlin plates

8.1. Introduction

Exact theories for a multi-layer rectangular plate, simply supported around its edge, loaded in flexure, vibration and buckling, were developed by N. J. Pagano and S. Srinivas. They enable the areas of application of plate theories to be defined. The Kirchhoff-Love theory only provides acceptable deflections, natural frequencies and critical buckling loads for thin plates whose ratio of thickness to the characteristic dimension of the mean surface is less than 1/20. Reissner-Mindlin theory, in which the transverse shear strains are constant through the plate thickness, gives satisfactory results for flexure, vibration and buckling of moderately thick plates whose ratio of thickness to the characteristic dimension of the mean surface is between 1/5 and 1/20.

8.2. Moderately thick plate, Reissner-Mindlin assumptions

As indicated in the Appendix, in Reissner-Mindlin theory the hypotheses of Kirchhoff-Love theory are used without ignoring the transverse shear strains.

8.3. Displacements, strains and stresses

The displacement field:

$$u_1 = u_1^0(x_1, x_2|t) + x_3\psi_1(x_1, x_2|t),$$

$$u_2 = u_2^0(x_1, x_2|t) + x_3\psi_2(x_1, x_2|t),$$

$$u_3 = u_3^0(x_1, x_2|t),$$

leads to the strain field:

$$\varepsilon_1 = \frac{\partial u_1^0}{\partial x_1} + x_3\frac{\partial \psi_1}{\partial x_1},\ \varepsilon_2 = \frac{\partial u_2^0}{\partial x_2} + x_3\frac{\partial \psi_2}{\partial x_2},\ \varepsilon_3 = 0,$$

$$\varepsilon_6 = \frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1} + x_3\left(\frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1}\right),\ \varepsilon_4 = \psi_2 + \frac{\partial u_3^0}{\partial x_2},\ \varepsilon_5 = \psi_1 + \frac{\partial u_3^0}{\partial x_1},$$

which given the notations introduced above can be written in the form:

$$\varepsilon_i = \varepsilon_i^0 + x_3\kappa_i \quad (i = 1,2,6),$$

$$\varepsilon_4 = \psi_2 + \frac{\partial u_3^0}{\partial x_2},$$

$$\varepsilon_5 = \psi_1 + \frac{\partial u_3^0}{\partial x_1}.$$

The stresses in the k layer are given by the expression:

$$\sigma_i^k = Q_{ij}^k \varepsilon_j \quad (i,\ j = 1, 2, 4, 5, 6).$$

8.4. Global plate equations

The global plate equations according to Reissner-Mindlin theory are written as:

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} + p_1 = I_0 \frac{\partial^2 u_1^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} + p_2 = I_0 \frac{\partial^2 u_2^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_2}{\partial t^2},$$

$$\frac{\partial N_5}{\partial x_1} + \frac{\partial N_4}{\partial x_2} + \frac{\partial}{\partial x_1}\left(N_1 \frac{\partial u_3^0}{\partial x_1} + N_6 \frac{\partial u_3^0}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(N_6 \frac{\partial u_3^0}{\partial x_1} + N_2 \frac{\partial u_3^0}{\partial x_2}\right)\ldots$$

$$\ldots + q_3 + p_3 = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

$$\frac{\partial M_1}{\partial x_1} + \frac{\partial M_6}{\partial x_2} - N_5 = I_1 \frac{\partial^2 u_1^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial M_6}{\partial x_1} + \frac{\partial M_2}{\partial x_2} - N_4 = I_1 \frac{\partial^2 u_2^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_2}{\partial t^2}.$$

8.5. Calculation of I_1 and I_2

For these calculations, we retain the layer distribution in the plate thickness adopted previously. With this distribution we obtained:

$$I_0 = \sum_{k=1}^{N} \rho^k \left(z_k - z_{k-1}\right).$$

In addition, we have the rotational inertias:

$$I_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho x_3 dx_3 ,$$

or:

$$I_1 = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \rho^k x_3 dx_3 = \frac{1}{2} \sum_{k=1}^{N} \rho^k \left[x_3^2\right]_{z_{k-1}}^{z_k} ,$$

from which:

$$I_1 = \frac{1}{2} \sum_{k=1}^{N} \rho^k \left(z_k^2 - z_{k-1}^2\right),$$

and:

$$I_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho x_3^2 dx_3 ,$$

then:

$$I_2 = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \rho^k x_3^2 dx_3 = \frac{1}{3} \sum_{k=1}^{N} \rho^k \left[x_3^3\right]_{z_{k-1}}^{z_k} ,$$

or:

$$I_2 = \frac{1}{3} \sum_{k=1}^{N} \rho^k \left(z_k^3 - z_{k-1}^3\right).$$

In the case where the plate is a single layer or multi-layer with layers of the same density, the preceding expressions are equal to:

$$I_0 = \rho h, \quad I_1 = 0, \quad I_2 = \frac{\rho h^3}{12} = \frac{h^2}{12} I_0.$$

8.6. Global cohesive forces

As with Kirchhoff-Love theory we have the expressions:

$$N_i = A_{ij} \varepsilon_j^0 + B_{ij} \kappa_j ,$$

$$M_i = B_{ij} \varepsilon_j^0 + D_{ij} \kappa_j \quad (i, j = 1, 2, 6),$$

to which we add:

$$N_i = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_i dx_3 \quad (i = 4, 5).$$

The transverse shear stresses in the k layer are given by:

$$\sigma_i^k = Q_{ij}^k \varepsilon_j \quad (i, j = 4, 5),$$

where ε_j is constant throughout the plate, and the transverse shear stresses are therefore constant through the thickness of the k layer.

In order to take account of the variation in transverse shear stresses throughout the thickness the global transverse shear force is taken to be equal to:

$$N_i = K_{\underline{ij}} \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} Q_{ij}^k \varepsilon_j dx_3 = K_{\underline{ij}} \sum_{k=1}^{N} Q_{ij}^k (z_k - z_{k-1}) \varepsilon_j ,$$

the summation convention does not apply to the underlined indices i and j, they have the same values as the indices i and j not underlined.

The global transverse shear loads are written as:

$$N_i = K_{\underline{ij}} A_{ij} \varepsilon_j \quad (i, j = 4, 5),$$

with:

$$A_{ij} = A_{ji} = \sum_{k=1}^{N} Q_{ij}^k (z_k - z_{k-1}),$$

where the K_{ij} are correction coefficients for the transverse shear, the effective calculation of which will be described in a later paragraph.

8.7. Global stiffness matrix of the composite

The global constitutive relation for the composite is written in the following matrix form:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \\ N_4 \\ N_5 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} & 0 & 0 \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} & 0 & 0 \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} & 0 & 0 \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} & 0 & 0 \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} & 0 & 0 \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{44}A_{44} & K_{45}A_{45} \\ 0 & 0 & 0 & 0 & 0 & 0 & K_{45}A_{45} & K_{55}A_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix},$$

where:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix},$$

$$\begin{bmatrix} N_4 \\ N_5 \end{bmatrix} = \begin{bmatrix} K_{44}A_{44} & K_{45}A_{45} \\ K_{45}A_{45} & K_{55}A_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{bmatrix},$$

in the case where $K_{44} = K_{45} = K_{55} = K$, this latter expression becomes:

$$\begin{bmatrix} N_4 \\ N_5 \end{bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{bmatrix}.$$

8.8. Transverse shear correction coefficient

In the case of an orthotropic monolayer plate Uflyand, Reissner and Mindlin respectively proposed for K the values $\frac{2}{3}$, $\frac{5}{6}$ and $\frac{\pi^2}{12}$.

8.8.1. *Uflyand coefficient*

The transverse shear stress σ_4 from the global equilibrium equation:

$$\frac{\partial \sigma_6}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \sigma_4}{\partial x_3} = 0,$$

is equal to

$$\sigma_4 = -\int_{-\frac{h}{2}}^{x_3} \left(\frac{\partial \sigma_6}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} \right) d\zeta .$$

In the case of a single layer orthotropic plate, loaded in flexure, the expression:

$$\sigma_i = Q_{ij} x_3 \kappa_j,$$

gives:

$$\sigma_1 = (Q_{11}\kappa_1 + Q_{12}\kappa_2)x_3,$$
$$\sigma_2 = (Q_{12}\kappa_1 + Q_{22}\kappa_2)x_3,$$
$$\sigma_6 = Q_{66}\kappa_6 x_3.$$

Included in the expression for σ_4 , we have:

$$\sigma_4 = -\left[\frac{\partial}{\partial x_1}(Q_{66}\kappa_6) + \frac{\partial}{\partial x_2}(Q_{12}\kappa_1 + Q_{22}\kappa_2) \right] \int_{-\frac{h}{2}}^{x_3} \zeta \, d\zeta ,$$

since:

$$\int_{-\frac{h}{2}}^{x_3} \zeta \, d\zeta = \left[\frac{\zeta^2}{2}\right]_{-\frac{h}{2}}^{x_3} = \frac{1}{2}\left(x_3^2 - \frac{h^2}{4}\right) = \frac{h^2}{8}\left(\frac{4x_3^2}{h^2} - 1\right),$$

we obtain:

$$\sigma_4 = \frac{h^2}{8}\left[\frac{\partial}{\partial x_1}(Q_{66}\kappa_6) + \frac{\partial}{\partial x_2}(Q_{12}\kappa_1 + Q_{22}\kappa_2)\right]\left(1 - \frac{4x_3^2}{h^2}\right).$$

From the global constitutive relation of the orthotropic plate we have:

$$M_1 = D_{11}\kappa_1 + D_{12}\kappa_2,$$
$$M_2 = D_{12}\kappa_1 + D_{22}\kappa_2,$$
$$M_6 = D_{66}\kappa_6,$$

with:

$$D_{ij} = \frac{1}{3}Q_{ij}\left[\left(\frac{h}{2}\right)^3 - \left(-\frac{h}{2}\right)^3\right] = \frac{h^3}{12}Q_{ij},$$

we obtain:

$$M_1 = \frac{h^3}{12}(Q_{11}\kappa_1 + Q_{12}\kappa_2),$$

$$M_2 = \frac{h^3}{12}(Q_{12}\kappa_1 + Q_{22}\kappa_2),$$

$$M_6 = \frac{h^3}{12}Q_{66}\kappa_6.$$

The stress σ_4 is written as:

$$\sigma_4 = \frac{3}{2h}\left(\frac{\partial M_6}{\partial x_1} + \frac{\partial M_2}{\partial x_2}\right)\left(1 - \frac{4x_3^2}{h^2}\right).$$

Given the global equilibrium equation:

$$\frac{\partial M_6}{\partial x_1} + \frac{\partial M_2}{\partial x_2} - N_4 = 0,$$

the transverse shear stress σ_4 is equal to:

$$\sigma_4 = \frac{3N_4}{2h}\left(1 - \frac{4x_3^2}{h^2}\right),$$

similarly:

$$\sigma_5 = \frac{3N_5}{2h}\left(1 - \frac{4x_3^2}{h^2}\right).$$

The maximum transverse shear stresses, obtained for $x_3 = 0$, are equal to:

$$\sigma_{4_{\max}} = \frac{3N_4}{2h}, \quad \sigma_{5_{\max}} = \frac{3N_5}{2h}.$$

From the global composite constitutive relation we have:

$$N_4 = KA_{44}\varepsilon_4 = KhQ_{44}\varepsilon_4,$$

and from the material constitutive relation we have:

$$\sigma_4 = Q_{44}\varepsilon_4.$$

The transverse shear stresses are thus given by:

$$\sigma_4 = \frac{N_4}{Kh}, \quad \sigma_5 = \frac{N_5}{Kh}.$$

By identification with the maximum transverse shear stress calculated above we obtain the Uflyand transverse shear correction coefficient:

$$K = \frac{2}{3}.$$

8.8.2. *Reissner coefficient*

The transverse shear strain energy is equal to:

$$W_d = \frac{1}{2}\int_0^{a_1}\int_0^{a_2}\int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_4\varepsilon_4 + \sigma_5\varepsilon_5)dx_3dx_2dx_1,$$

introducing:

$$\varepsilon_4 = \frac{\sigma_4}{Q_{44}}, \quad \varepsilon_5 = \frac{\sigma_5}{Q_{55}},$$

we have:

$$W_d = \frac{1}{2}\int_0^{a_1}\int_0^{a_2}\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\frac{\sigma_4^2}{Q_{44}} + \frac{\sigma_5^2}{Q_{55}}\right)dx_3dx_2dx_1.$$

With the values of the transverse shear stresses already obtained:

$$\sigma_4 = \frac{3N_4}{2h}\left(1 - \frac{4x_3^2}{h^2}\right) \text{ and } \sigma_5 = \frac{3N_5}{2h}\left(1 - \frac{4x_3^2}{h^2}\right),$$

we obtain:

$$W_d = \frac{1}{2}\int_0^{a_1}\int_0^{a_2}\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{9}{4h^2}\left(\frac{N_4^2}{Q_{44}} + \frac{N_5^2}{Q_{55}}\right)\left(1 - \frac{4x_3^2}{h^2}\right)^2 dx_3dx_2dx_1,$$

with:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(1-\frac{4x_3^2}{h^2}\right)^2 dx_3=\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(1-\frac{8x_3^2}{h^2}+\frac{16x_3^4}{h^4}\right)dx_3=\left[x_3-\frac{8x_3^3}{3h^2}+\frac{16x_3^5}{5h^4}\right]_{-\frac{h}{2}}^{\frac{h}{2}},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(1-\frac{4x_3^2}{h^2}\right)^2 dx_3=2\left[\frac{h}{2}-\frac{8\left(\frac{h}{2}\right)^3}{3h^2}+\frac{16\left(\frac{h}{2}\right)^5}{5h^4}\right]=\frac{8h}{15},$$

we find:

$$W_d=\frac{1}{2}\int_0^{a_1}\int_0^{a_2}\frac{9}{4h^2}\frac{8h}{15}\left(\frac{N_4^2}{Q_{44}}+\frac{N_5^2}{Q_{55}}\right)dx_2dx_1.$$

Introducing $A_{44}=hQ_{44}$ and $A_{55}=hQ_{55}$, the transverse shear strain energy is equal to:

$$W_d=\frac{3}{5}\int_0^{a_1}\int_0^{a_2}\left(\frac{N_4^2}{A_{44}}+\frac{N_5^2}{A_{55}}\right)dx_2dx_1.$$

The global constitutive relation of the composite supplies the transverse shear strains:

$$\varepsilon_4=\frac{N_4}{KA_{44}} \text{ and } \varepsilon_5=\frac{N_5}{KA_{55}}.$$

Putting this into the general formula for the transverse shear strain energy we obtain:

$$W_d=\frac{1}{2K}\int_0^{a_1}\int_0^{a_2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_4\frac{N_4}{A_{44}}+\sigma_5\frac{N_5}{A_{55}}\right)dx_3dx_2dx_1.$$

The global transverse shear forces being equal to:

$$N_4=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_4dx_3 \text{ and } N_5=\int_{-\frac{h}{2}}^{\frac{h}{2}}\sigma_5dx_3,$$

the transverse shear strain energy is written as:

$$W_d=\frac{1}{2K}\int_0^{a_1}\int_0^{a_2}\left(\frac{N_4^2}{A_{44}}+\frac{N_5^2}{A_{55}}\right)dx_2dx_1.$$

The transverse shear correction Reissner coefficient, obtained by equating the two strain energies calculated above, is equal to:

$$K=\frac{5}{6}.$$

8.9. Boundary conditions

The Kirchhoff boundary conditions, for the edge $x_1 = a_1$ of a rectangular plate, are:

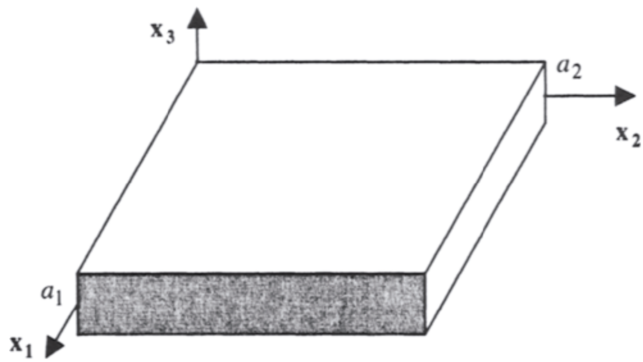

Figure 8.1. $x_1 = a_1$ *edge*

– for a simply supported edge:

$$N_1 = N_6 = 0, \quad M_1 = 0, \quad \psi_2 = 0, \quad u_3^0 = 0,$$

– for a simply supported edge in $\mathbf{x}_1$ direction:

$$N_1 = 0, \quad M_1 = 0, \quad \psi_2 = 0, \quad u_2^0 = u_3^0 = 0,$$

– for a simply supported edge in $\mathbf{x}_2$ direction:

$$N_6 = 0, \quad M_1 = 0, \quad \psi_2 = 0, \quad u_1^0 = u_3^0 = 0,$$

– for a built-in edge:

$$\psi_1 = \psi_2 = 0, \quad u_1^0 = u_2^0 = u_3^0 = 0,$$

– for a free edge:

$$N_1 = N_5 = N_6 = 0, \quad M_1 = M_6 = 0.$$

8.10. Symmetrical orthotropic plate

The global constitutive relation of a symmetrical orthotropic plate is written as:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix}.$$

For such a plate we therefore have:

$$\begin{aligned} N_1 &= A_{11}\varepsilon_1^0 + A_{12}\varepsilon_2^0, & M_1 &= D_{11}\kappa_1 + D_{12}\kappa_2, & N_4 &= KA_{44}\varepsilon_4, \\ N_2 &= A_{12}\varepsilon_1^0 + A_{22}\varepsilon_2^0, & M_2 &= D_{12}\kappa_1 + D_{22}\kappa_2, & N_5 &= KA_{55}\varepsilon_5, \\ N_6 &= A_{66}\varepsilon_6^0, & M_6 &= D_{66}\kappa_6. \end{aligned}$$

8.11. Flexure of a rectangular orthotropic symmetrical plate simply supported around its edge

The multi-layer plate studied here, of dimensions a_1 and a_2 is subjected to the surface force density of $q(x_1, x_2)\mathbf{x}_3$.

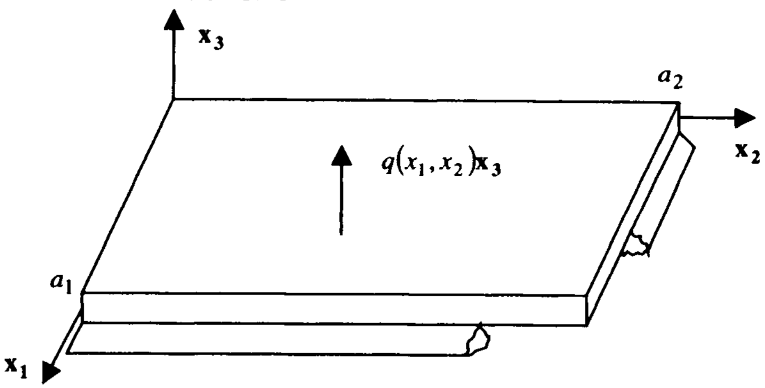

Figure 8.2. *Transversely loaded plate*

The equilibrium equations are:

$$\frac{\partial M_1}{\partial x_1} + \frac{\partial M_6}{\partial x_2} - N_5 = 0,$$

$$\frac{\partial M_6}{\partial x_1} + \frac{\partial M_2}{\partial x_2} - N_4 = 0,$$

$$\frac{\partial N_5}{\partial x_1}+\frac{\partial N_4}{\partial x_2}+q=0,$$

with the global cohesion forces:

$$M_1=D_{11}\frac{\partial \psi_1}{\partial x_1}+D_{12}\frac{\partial \psi_2}{\partial x_2},\qquad N_4=KA_{44}\left(\psi_2+\frac{\partial u_3^0}{\partial x_2}\right),$$

$$M_2=D_{12}\frac{\partial \psi_1}{\partial x_1}+D_{22}\frac{\partial \psi_2}{\partial x_2},\qquad N_5=KA_{55}\left(\psi_1+\frac{\partial u_3^0}{\partial x_1}\right),$$

$$M_6=D_{66}\left(\frac{\partial \psi_1}{\partial x_2}+\frac{\partial \psi_2}{\partial x_1}\right).$$

For a double sinusoidal loading:

$$q(x_1,x_2)=q_{m_1m_2}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

the solution which satisfies the equilibrium equations:

$$D_{11}\frac{\partial^2\psi_1}{\partial x_1^2}+D_{12}\frac{\partial^2\psi_2}{\partial x_1\partial x_2}+D_{66}\left(\frac{\partial^2\psi_1}{\partial x_2^2}+\frac{\partial^2\psi_2}{\partial x_1\partial x_2}\right)-KA_{55}\left(\psi_1+\frac{\partial u_3^0}{\partial x_1}\right)=0,$$

$$D_{66}\left(\frac{\partial^2\psi_1}{\partial x_1\partial x_2}+\frac{\partial^2\psi_2}{\partial x_1^2}\right)+D_{12}\frac{\partial^2\psi_2}{\partial x_1\partial x_2}+D_{22}\frac{\partial^2\psi_2}{\partial x_2^2}-KA_{44}\left(\psi_2+\frac{\partial u_3^0}{\partial x_2}\right)=0,$$

$$KA_{55}\left(\frac{\partial \psi_1}{\partial x_1}+\frac{\partial^2 u_3^0}{\partial x_1^2}\right)+KA_{44}\left(\frac{\partial \psi_2}{\partial x_2}+\frac{\partial^2 u_3^0}{\partial x_2^2}\right)+q=0,$$

and the boundary conditions:

– for $x_1=0$ and $x_1=a_1$:

$$u_3^0=0,\quad \psi_2=0,$$

$$M_1=D_{11}\frac{\partial \psi_1}{\partial x_1}+D_{12}\frac{\partial \psi_2}{\partial x_2}=0,$$

– for $x_2=0$ and $x_2=a_2$:

$$u_3^0=0,\quad \psi_1=0,$$

$$M_2=D_{12}\frac{\partial \psi_1}{\partial x_1}+D_{22}\frac{\partial \psi_2}{\partial x_2}=0,$$

is of the form:

$$\psi_1=\Psi^1_{m_1m_2}\cos\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

$$\psi_2 = \Psi^2_{m_1m_2} \sin\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2},$$

$$u_3^0 = U^3_{m_1m_2} \sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}.$$

Including this in the equilibrium equations and changing the sign, we obtain:

$$\left\{\left[\left(\frac{m_1\pi}{a_1}\right)^2 D_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 D_{66} + KA_{55}\right]\Psi^1_{m_1m_2} + \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(D_{12}+D_{66})\Psi^2_{m_1m_2}\cdots\right.$$

$$\left.\cdots + \frac{m_1\pi}{a_1}KA_{55}U^3_{m_1m_2}\right\}\cos\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2} = 0,$$

$$\left\{\frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(D_{12}+D_{66})\Psi^1_{m_1m_2} + \left[\left(\frac{m_2\pi}{a_2}\right)^2 D_{22} + \left(\frac{m_1\pi}{a_1}\right)^2 D_{66} + KA_{44}\right]\Psi^2_{m_1m_2}\cdots\right.$$

$$\left.\cdots + \frac{m_2\pi}{a_2}KA_{44}U^3_{m_1m_2}\right\}\sin\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2} = 0,$$

$$\left\{\frac{m_1\pi}{a_1}KA_{55}\Psi^1_{m_1m_2} + \frac{m_2\pi}{a_2}KA_{44}\Psi^2_{m_1m_2} + \left[\left(\frac{m_1\pi}{a_1}\right)^2 KA_{55} + \left(\frac{m_2\pi}{a_2}\right)^2 KA_{44}\right]U^3_{m_1m_2}\cdots\right.$$

$$\left.\cdots - q_{m_1m_2}\right\}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2} = 0.$$

The three coefficients $\Psi^1_{m_1m_2}$, $\Psi^2_{m_1m_2}$ and $U^3_{m_1m_2}$ are solutions to the system:

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} \end{bmatrix}\begin{bmatrix} \Psi^1_{m_1m_2} \\ \Psi^2_{m_1m_2} \\ U^3_{m_1m_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q_{m_1m_2} \end{bmatrix},$$

with:

$$H_{11} = \left(\frac{m_1\pi}{a_1}\right)^2 D_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 D_{66} + KA_{55}, \quad H_{12} = \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(D_{12}+D_{66}),$$

$$H_{22} = \left(\frac{m_2\pi}{a_2}\right)^2 D_{22} + \left(\frac{m_1\pi}{a_1}\right)^2 D_{66} + KA_{44}, \quad H_{13} = \frac{m_1\pi}{a_1}KA_{55},$$

$$H_{33} = \left(\frac{m_1\pi}{a_1}\right)^2 KA_{55} + \left(\frac{m_2\pi}{a_2}\right)^2 KA_{44}, \qquad H_{23} = \frac{m_2\pi}{a_2}KA_{44}.$$

For the loading:

$$q(x_1,x_2)=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty} q_{m_1m_2}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

we obtain the solutions:

$$\psi_1=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\Psi^1_{m_1m_2}\cos\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

$$\psi_2=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\Psi^2_{m_1m_2}\sin\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2},$$

$$u_3^0=\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}U^3_{m_1m_2}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

with:

$$\Psi^1_{m_1m_2}=\frac{H_{12}H_{23}-H_{13}H_{22}}{D}q_{m_1m_2},$$

$$\Psi^2_{m_1m_2}=\frac{H_{12}H_{13}-H_{11}H_{23}}{D}q_{m_1m_2},$$

$$U^3_{m_1m_2}=\frac{H_{11}H_{22}-H_{12}^2}{D}q_{m_1m_2},$$

$$D=\left(H_{11}H_{22}-H_{12}^2\right)H_{33}+2H_{12}H_{13}H_{23}-H_{11}H_{23}^2-H_{22}H_{13}^2.$$

8.12. Transverse vibration of a rectangular orthotropic symmetrical plate simply supported around its edge

The rectangular plate of dimensions a_1 and a_2 is simply supported around its edge and not subjected to any given force.

In the case of a symmetrical plate, we have $I_1=0$. The equations of global vibration are written as:

$$\frac{\partial M_1}{\partial x_1}+\frac{\partial M_6}{\partial x_2}-N_5=I_2\frac{\partial^2\psi_1}{\partial t^2},$$

$$\frac{\partial M_6}{\partial x_1}+\frac{\partial M_2}{\partial x_2}-N_4=I_2\frac{\partial^2\psi_2}{\partial t^2},$$

$$\frac{\partial N_5}{\partial x_1}+\frac{\partial N_4}{\partial x_2}=I_0\frac{\partial^2 u_3^0}{\partial t^2},$$

with the global cohesion forces:

$$M_1 = D_{11}\frac{\partial \psi_1}{\partial x_1} + D_{12}\frac{\partial \psi_2}{\partial x_2}, \quad N_4 = KA_{44}\left(\psi_2 + \frac{\partial u_3^0}{\partial x_2}\right),$$

$$M_2 = D_{12}\frac{\partial \psi_1}{\partial x_1} + D_{22}\frac{\partial \psi_2}{\partial x_2}, \quad N_5 = KA_{55}\left(\psi_1 + \frac{\partial u_3^0}{\partial x_1}\right),$$

$$M_6 = D_{66}\left(\frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1}\right).$$

Including the latter in the global equations we obtain the three expressions:

$$D_{11}\frac{\partial^2 \psi_1}{\partial x_1^2} + D_{12}\frac{\partial^2 \psi_2}{\partial x_1 \partial x_2} + D_{66}\left(\frac{\partial^2 \psi_1}{\partial x_2^2} + \frac{\partial^2 \psi_2}{\partial x_1 \partial x_2}\right) - KA_{55}\left(\psi_1 + \frac{\partial u_3^0}{\partial x_1}\right) = I_2\frac{\partial^2 \psi_1}{\partial t^2},$$

$$D_{66}\left(\frac{\partial^2 \psi_1}{\partial x_1 \partial x_2} + \frac{\partial^2 \psi_2}{\partial x_1^2}\right) + D_{12}\frac{\partial^2 \psi_2}{\partial x_1 \partial x_2} + D_{22}\frac{\partial^2 \psi_2}{\partial x_2^2} - KA_{44}\left(\psi_2 + \frac{\partial u_3^0}{\partial x_2}\right) = I_2\frac{\partial^2 \psi_2}{\partial t^2},$$

$$KA_{55}\left(\frac{\partial \psi_1}{\partial x_1} + \frac{\partial^2 u_3^0}{\partial x_1^2}\right) + KA_{44}\left(\frac{\partial \psi_2}{\partial x_2} + \frac{\partial^2 u_3^0}{\partial x_2^2}\right) = I_0\frac{\partial^2 u_3^0}{\partial t^2}.$$

The solution which satisfies these conditions at the edges:

– for $x_1 = 0$ and $x_1 = a_1$:

$$u_3^0 = 0, \quad \psi_2 = 0,$$

$$M_1 = D_{11}\frac{\partial \psi_1}{\partial x_1} + D_{12}\frac{\partial \psi_2}{\partial x_2} = 0,$$

– for $x_2 = 0$ and $x_2 = a_2$:

$$u_3^0 = 0, \quad \psi_1 = 0,$$

$$M_2 = D_{12}\frac{\partial \psi_1}{\partial x_1} + D_{22}\frac{\partial \psi_2}{\partial x_2} = 0,$$

is of the form:

$$\psi_1 = \Psi^1_{m_1 m_2}\cos\frac{m_1 \pi x_1}{a_1}\sin\frac{m_2 \pi x_2}{a_2}\sin\left(\omega_{m_1 m_2} t + \varphi_{m_1 m_2}\right),$$

$$\psi_2 = \Psi^2_{m_1 m_2}\sin\frac{m_1 \pi x_1}{a_1}\cos\frac{m_2 \pi x_2}{a_2}\sin\left(\omega_{m_1 m_2} t + \varphi_{m_1 m_2}\right),$$

$$u_3^0 = U^3_{m_1 m_2}\sin\frac{m_1 \pi x_1}{a_1}\sin\frac{m_2 \pi x_2}{a_2}\sin\left(\omega_{m_1 m_2} t + \varphi_{m_1 m_2}\right).$$

Putting these values into the global vibration equations and after simplifying we obtain the system:

$$\begin{bmatrix} H_{11} - I_2\omega^2_{m_1m_2} & H_{12} & H_{13} \\ H_{12} & H_{22} - I_2\omega^2_{m_1m_2} & H_{23} \\ H_{13} & H_{23} & H_{33} - I_0\omega^2_{m_1m_2} \end{bmatrix} \begin{bmatrix} \Psi^1_{m_1m_2} \\ \Psi^2_{m_1m_2} \\ U^3_{m_1m_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

with:

$$H_{11} = \left(\frac{m_1\pi}{a_1}\right)^2 D_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 D_{66} + KA_{55}, \quad H_{12} = \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(D_{12} + D_{66}),$$

$$H_{22} = \left(\frac{m_2\pi}{a_2}\right)^2 D_{22} + \left(\frac{m_1\pi}{a_1}\right)^2 D_{66} + KA_{44}, \quad H_{13} = \frac{m_1\pi}{a_1} KA_{55},$$

$$H_{33} = \left(\frac{m_1\pi}{a_1}\right)^2 KA_{55} + \left(\frac{m_2\pi}{a_2}\right)^2 KA_{44}, \qquad H_{23} = \frac{m_2\pi}{a_2} KA_{44}.$$

This homogeneous algebraic system in $\Psi^1_{m_1m_2}$, $\Psi^2_{m_1m_2}$ and $U^3_{m_1m_2}$ has a solution other than the trivial solution for the $\omega_{m_1m_2}$ solution of the third degree equation in $\omega^2_{m_1m_2}$:

$$-A\omega^6_{m_1m_2} + B\omega^4_{m_1m_2} - C\omega^2_{m_1m_2} + D = 0,$$

with:

$$A = I_2^2 I_0,$$

$$B = (H_{11} + H_{22})I_2 I_0 + H_{33} I_2^2,$$

$$C = \left(H_{11}H_{22} - H_{12}^2\right)I_0 + \left[(H_{11} + H_{22})H_{33} - H_{13}^2 - H_{23}^2\right]I_2,$$

$$D = \left(H_{11}H_{22} - H_{12}^2\right)H_{33} + 2H_{12}H_{13}H_{23} - H_{11}H_{23}^2 - H_{22}H_{13}^2.$$

For the fixed values of m_1 and m_2, we obtain three antisymmetric modes of vibration.

8.13. Buckling of a rectangular orthotropic symmetrical plate simply supported around its edge

The rectangular plate of dimensions a_1 and a_2 is simply supported around its edge. It is subjected to the membrane loads $-N_1^0$ and $-N_2^0$.

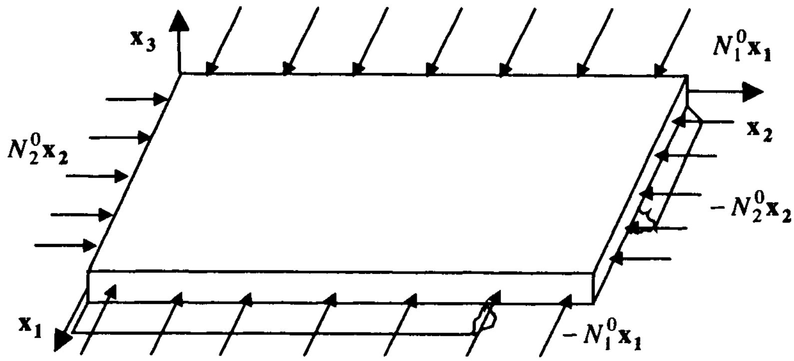

Figure 8.3. *Buckling loading*

The global buckling equations:

$$\frac{\partial M_1}{\partial x_1}+\frac{\partial M_6}{\partial x_2}-N_5=0,$$

$$\frac{\partial M_6}{\partial x_1}+\frac{\partial M_2}{\partial x_2}-N_4=0,$$

$$\frac{\partial N_5}{\partial x_1}+\frac{\partial N_4}{\partial x_2}-N_1^0\frac{\partial^2 u_3^0}{\partial x_1^2}-N_2^0\frac{\partial^2 u_3^0}{\partial x_2^2}=0,$$

with:

$$M_1=D_{11}\frac{\partial \psi_1}{\partial x_1}+D_{12}\frac{\partial \psi_2}{\partial x_2},\qquad N_4=KA_{44}\left(\psi_2+\frac{\partial u_3^0}{\partial x_2}\right),$$

$$M_2=D_{12}\frac{\partial \psi_1}{\partial x_1}+D_{22}\frac{\partial \psi_2}{\partial x_2},\qquad N_5=KA_{55}\left(\psi_1+\frac{\partial u_3^0}{\partial x_1}\right),$$

$$M_6=D_{66}\left(\frac{\partial \psi_1}{\partial x_2}+\frac{\partial \psi_2}{\partial x_1}\right),$$

are written:

$$D_{11}\frac{\partial^2 \psi_1}{\partial x_1^2}+D_{12}\frac{\partial^2 \psi_2}{\partial x_1 \partial x_2}+D_{66}\left(\frac{\partial^2 \psi_1}{\partial x_2^2}+\frac{\partial^2 \psi_2}{\partial x_1 \partial x_2}\right)-KA_{55}\left(\psi_1+\frac{\partial u_3^0}{\partial x_1}\right)=0,$$

$$D_{66}\left(\frac{\partial^2 \psi_1}{\partial x_1 \partial x_2}+\frac{\partial^2 \psi_2}{\partial x_1^2}\right)+D_{12}\frac{\partial^2 \psi_2}{\partial x_1 \partial x_2}+D_{22}\frac{\partial^2 \psi_2}{\partial x_2^2}-KA_{44}\left(\psi_2+\frac{\partial u_3^0}{\partial x_2}\right)=0,$$

$$KA_{55}\left(\frac{\partial \psi_1}{\partial x_1}+\frac{\partial^2 u_3^0}{\partial x_1^2}\right)+KA_{44}\left(\frac{\partial \psi_2}{\partial x_2}+\frac{\partial^2 u_3^0}{\partial x_2^2}\right)-N_1^0\frac{\partial^2 u_3^0}{\partial x_1^2}-N_2^0\frac{\partial^2 u_3^0}{\partial x_2^2}=0.$$

The boundary conditions, for a simply supported plate are:

– for $x_1 = 0$ and $x_1 = a_1$:

$$u_3^0 = 0, \quad \psi_2 = 0,$$

$$M_1 = D_{11}\frac{\partial \psi_1}{\partial x_1} + D_{12}\frac{\partial \psi_2}{\partial x_2} = 0,$$

– for $x_2 = 0$ and $x_2 = a_2$:

$$u_3^0 = 0, \quad \psi_1 = 0,$$

$$M_2 = D_{12}\frac{\partial \psi_1}{\partial x_1} + D_{22}\frac{\partial \psi_2}{\partial x_2} = 0.$$

These conditions are satisfied by:

$$\psi_1 = \Psi^1_{m_1 m_2} \cos\frac{m_1 \pi x_1}{a_1} \sin\frac{m_2 \pi x_2}{a_2},$$

$$\psi_2 = \Psi^2_{m_1 m_2} \sin\frac{m_1 \pi x_1}{a_1} \cos\frac{m_2 \pi x_2}{a_2},$$

$$u_3^0 = U^3_{m_1 m_2} \sin\frac{m_1 \pi x_1}{a_1} \sin\frac{m_2 \pi x_2}{a_2}.$$

Putting these into the global buckling equations and changing the sign we obtain:

$$\left\{\left[\left(\frac{m_1\pi}{a_1}\right)^2 D_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 D_{66} + KA_{55}\right]\Psi^1_{m_1 m_2} + \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(D_{12}+D_{66})\Psi^2_{m_1 m_2} \cdots\right.$$

$$\left.\cdots + \frac{m_1\pi}{a_1}KA_{55}U^3_{m_1 m_2}\right\}\cos\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2} = 0,$$

$$\left\{\frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(D_{12}+D_{66})\Psi^1_{m_1 m_2} + \left[\left(\frac{m_2\pi}{a_2}\right)^2 D_{22} + \left(\frac{m_1\pi}{a_1}\right)^2 D_{66} + KA_{44}\right]\Psi^2_{m_1 m_2} \cdots\right.$$

$$\left.\cdots + \frac{m_2\pi}{a_2}KA_{44}U^3_{m_1 m_2}\right\}\sin\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2} = 0,$$

$$\left\{\frac{m_1\pi}{a_1}KA_{55}\Psi^1_{m_1 m_2} + \frac{m_2\pi}{a_2}KA_{44}\Psi^2_{m_1 m_2} + \left[\left(\frac{m_1\pi}{a_1}\right)^2 KA_{55} + \left(\frac{m_2\pi}{a_2}\right)^2 KA_{44} \cdots\right.\right.$$

$$\left.\left.\cdots - \left(\frac{m_1\pi}{a_1}\right)^2 N_1^0 - \left(\frac{m_2\pi}{a_2}\right)^2 N_2^0\right]U^3_{m_1 m_2}\right\}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2} = 0.$$

The algebraic system in $\Psi^1_{m_1m_2}$, $\Psi^2_{m_1m_2}$ and $U^3_{m_1m_2}$ thus obtained is written in the form:

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33}-\left(\frac{m_1\pi}{a_1}\right)^2 N_1^0-\left(\frac{m_2\pi}{a_2}\right)^2 N_2^0 \end{bmatrix}\begin{bmatrix}\Psi^1_{m_1m_2}\\ \Psi^2_{m_1m_2}\\ U^3_{m_1m_2}\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix},$$

with:

$$H_{11}=\left(\frac{m_1\pi}{a_1}\right)^2 D_{11}+\left(\frac{m_2\pi}{a_2}\right)^2 D_{66}+KA_{55},\quad H_{12}=\frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}\left(D_{12}+D_{66}\right),$$

$$H_{22}=\left(\frac{m_2\pi}{a_2}\right)^2 D_{22}+\left(\frac{m_1\pi}{a_1}\right)^2 D_{66}+KA_{44},\quad H_{13}=\frac{m_1\pi}{a_1}KA_{55},$$

$$H_{33}=\left(\frac{m_1\pi}{a_1}\right)^2 KA_{55}+\left(\frac{m_2\pi}{a_2}\right)^2 KA_{44},\qquad H_{23}=\frac{m_2\pi}{a_2}KA_{44}.$$

In the particular case where $N_2^0=kN_1^0$, this system has a solution other than the trivial solution for the N_1^0 solution of the equation:

$$\begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33}-\left[\left(\frac{m_1\pi}{a_1}\right)^2+k\left(\frac{m_2\pi}{a_2}\right)^2\right]N_1^0 \end{vmatrix}=0,$$

or:

$$H_{13}\left(H_{12}H_{23}-H_{22}H_{13}\right)-H_{23}\left(H_{11}H_{23}-H_{12}H_{13}\right)\ldots$$
$$\ldots+\left\{H_{33}-\left[\left(\frac{m_1\pi}{a_1}\right)^2+k\left(\frac{m_2\pi}{a_2}\right)^2\right]N_1^0\right\}\left(H_{11}H_{22}-H_{12}^2\right)=0.$$

The critical global buckling force is given by:

$$N_1^0=\frac{1}{\left(\frac{m_1\pi}{a_1}\right)^2+k\left(\frac{m_2\pi}{a_2}\right)^2}\left[H_{33}\ldots\right.$$
$$\left.\ldots+\frac{H_{13}\left(H_{12}H_{23}-H_{22}H_{13}\right)-H_{23}\left(H_{11}H_{23}-H_{12}H_{13}\right)}{H_{11}H_{22}-H_{12}^2}\right].$$

Chapter 9

Asymmetrical multi-layer Kirchhoff-Love plates

9.1. Introduction

In a previous chapter we studied symmetric multi-layer plates using the Kirchhoff-Love theory. In this chapter we will look at the behaviour of asymmetric plates in flexure, vibration and buckling. We will limit the discussion to cross-ply or balanced antisymmetric plates.

9.2. Flexure of a cross-ply asymmetrical plate

The rectangular plate of dimensions a_1 and a_2 is freely supported in the direction orthogonal to its perimeter. It is only subjected to the distributed surface force $q(x_1, x_2)\mathbf{x}_3$ on its upper face.

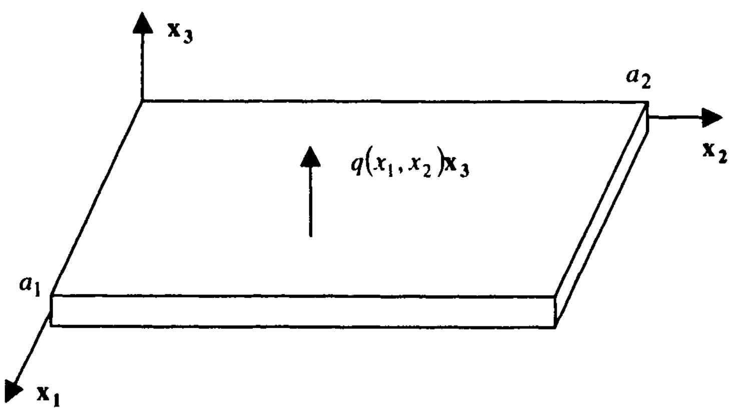

Figure 9.1. *Cross-ply asymmetrical plate in flexure*

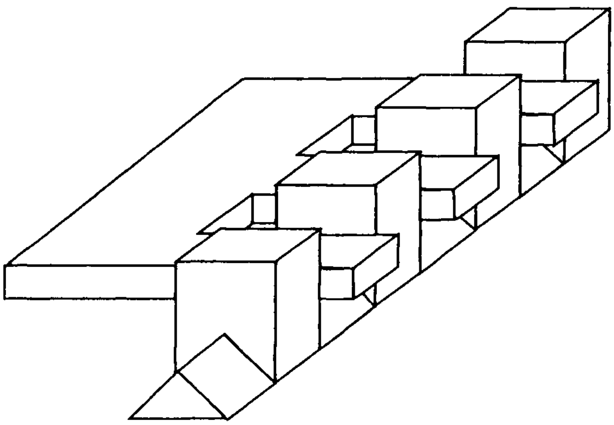

Figure 9.2. *Schematic diagram of edge of a cross-ply asymmetrical plate*

The global stiffnesses of the composite satisfy the expressions:

$$A_{16} = A_{26} = 0, \quad D_{16} = D_{26} = 0,$$
$$B_{12} = B_{66} = B_{16} = B_{26} = 0,$$
$$A_{22} = A_{11}, \quad D_{22} = D_{11}, \quad B_{22} = -B_{11}.$$

The global constitutive relation for the composite:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & B_{11} & 0 & 0 \\ A_{12} & A_{11} & 0 & 0 & -B_{11} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ B_{11} & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & -B_{11} & 0 & D_{12} & D_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix},$$

with, according to Kirchhoff-Love theory:

$$\varepsilon_1^0 = \frac{\partial u_1^0}{\partial x_1}, \qquad \varepsilon_2^0 = \frac{\partial u_2^0}{\partial x_2}, \qquad \varepsilon_6^0 = \frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1},$$
$$\kappa_1 = -\frac{\partial^2 u_3^0}{\partial x_1^2}, \quad \kappa_2 = -\frac{\partial^2 u_3^0}{\partial x_2^2}, \quad \kappa_6 = -2\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

provides the global membrane cohesion forces:

$$N_1 = A_{11}\frac{\partial u_1^0}{\partial x_1} + A_{12}\frac{\partial u_2^0}{\partial x_2} - B_{11}\frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$N_2 = A_{12}\frac{\partial u_1^0}{\partial x_1} + A_{11}\frac{\partial u_2^0}{\partial x_2} + B_{11}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$N_6 = A_{66}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right),$$

and flexure:

$$M_1 = B_{11}\frac{\partial u_1^0}{\partial x_1} - D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_2 = -B_{11}\frac{\partial u_2^0}{\partial x_2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{11}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_6 = -2D_{66}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

which, introduced into the global equilibrium equations:

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} = 0,$$

$$\frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} = 0,$$

$$\frac{\partial^2 M_1}{\partial x_1^2} + 2\frac{\partial^2 M_6}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} + q = 0,$$

gives the expressions:

$$A_{11}\frac{\partial^2 u_1^0}{\partial x_1^2} + A_{66}\frac{\partial^2 u_1^0}{\partial x_2^2} + (A_{12} + A_{66})\frac{\partial^2 u_2^0}{\partial x_1 \partial x_2} - B_{11}\frac{\partial^3 u_3^0}{\partial x_1^3} = 0,$$

$$(A_{12} + A_{66})\frac{\partial^2 u_1^0}{\partial x_1 \partial x_2} + A_{66}\frac{\partial^2 u_2^0}{\partial x_1^2} + A_{11}\frac{\partial^2 u_2^0}{\partial x_2^2} + B_{11}\frac{\partial^3 u_3^0}{\partial x_2^3} = 0,$$

$$D_{11}\left(\frac{\partial^4 u_3^0}{\partial x_1^4} + \frac{\partial^4 u_3^0}{\partial x_2^4}\right) + 2(D_{12} + 2D_{66})\frac{\partial^4 u_3^0}{\partial x_1^2 \partial x_2^2} - B_{11}\left(\frac{\partial^3 u_1^0}{\partial x_1^3} - \frac{\partial^3 u_2^0}{\partial x_2^3}\right) = q.$$

The boundary conditions for a freely supported plate in the direction at right angles to the edge are written:

– for $x_1 = 0$ and $x_1 = a_1$:

$$u_3^0 = 0, \quad M_1 = B_{11}\frac{\partial u_1^0}{\partial x_1} - D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0,$$

$$u_2^0 = 0, \quad N_1 = A_{11}\frac{\partial u_1^0}{\partial x_1} + A_{12}\frac{\partial u_2^0}{\partial x_2} - B_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} = 0,$$

– for $x_2 = 0$ and $x_2 = a_2$:

$$u_3^0 = 0, \quad M_2 = -B_{11}\frac{\partial u_2^0}{\partial x_2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{11}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0,$$

$$u_1^0 = 0, \quad N_2 = A_{12}\frac{\partial u_1^0}{\partial x_1} + A_{11}\frac{\partial u_2^0}{\partial x_2} + B_{11}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0.$$

The displacement field, which satisfies both the boundary conditions and the global equilibrium equations of the plate, subjected to the double sinusoidal loading:

$$q_{m_1m_2} \sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

is of the form:

$$u_1^0 = U^1_{m_1m_2} \cos\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

$$u_2^0 = U^2_{m_1m_2} \sin\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2},$$

$$u_3^0 = U^3_{m_1m_2} \sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}.$$

Introducing these into the global equilibrium equations we obtain, after simplification, the following system for $U^1_{m_1m_2}$, $U^2_{m_1m_2}$ and $U^3_{m_1m_2}$:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{66}\right]U^1_{m_1m_2} + \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12} + A_{66})U^2_{m_1m_2} \cdots$$

$$\cdots - \left(\frac{m_1\pi}{a_1}\right)^3 B_{11}U^3_{m_1m_2} = 0,$$

$$\frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12} + A_{66})U^1_{m_1m_2} + \left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{66} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{11}\right]U^2_{m_1m_2} \cdots$$

$$\cdots + \left(\frac{m_2\pi}{a_2}\right)^3 B_{11}U^3_{m_1m_2} = 0,$$

$$-\left(\frac{m_1\pi}{a_1}\right)^3 B_{11}U^1_{m_1m_2} + \left(\frac{m_2\pi}{a_2}\right)^3 B_{11}U^2_{m_1m_2} + \left\{\left[\left(\frac{m_1\pi}{a_1}\right)^4 + \left(\frac{m_2\pi}{a_2}\right)^4\right]D_{11} \cdots\right.$$

$$\left.\cdots + 2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2 (D_{12} + 2D_{66})\right\}U^3_{m_1m_2} = q_{m_1m_2},$$

which may be written in the form:

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} \end{bmatrix} \begin{bmatrix} U^1_{m_1m_2} \\ U^2_{m_1m_2} \\ U^3_{m_1m_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q_{m_1m_2} \end{bmatrix},$$

with:

$$H_{11} = \left(\frac{m_1\pi}{a_1}\right)^2 A_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{66},$$

$$H_{22} = \left(\frac{m_1\pi}{a_1}\right)^2 A_{66} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{11},$$

$$H_{33} = \left[\left(\frac{m_1\pi}{a_1}\right)^4 + \left(\frac{m_2\pi}{a_2}\right)^4\right] D_{11} + 2\left(\frac{m_1\pi}{a_1}\right)^2 \left(\frac{m_2\pi}{a_2}\right)^2 (D_{12} + 2D_{66}),$$

$$H_{12} = \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12} + A_{66}),$$

$$H_{13} = -\left(\frac{m_1\pi}{a_1}\right)^3 B_{11},$$

$$H_{23} = \left(\frac{m_2\pi}{a_2}\right)^3 B_{11}.$$

The determinant of the system and the determinants associated with the three unknowns are equal to:

$$\Delta_{m_1m_2} = \left(H_{11}H_{22} - H_{12}^2\right)H_{33} + 2H_{12}H_{13}H_{23} - H_{11}H_{23}^2 - H_{22}H_{13}^2,$$

$$\Delta^1_{m_1m_2} = (H_{12}H_{23} - H_{13}H_{22})q_{m_1m_2},$$

$$\Delta^2_{m_1m_2} = (H_{12}H_{13} - H_{11}H_{23})q_{m_1m_2},$$

$$\Delta^3_{m_1m_2} = \left(H_{11}H_{22} - H_{12}^2\right)q_{m_1m_2},$$

or:

$$\Delta_{m_1m_2} = \pi^8\left(\left\{\left[\left(\frac{m_1}{a_1}\right)^4 + \left(\frac{m_2}{a_2}\right)^4\right]A_{11}A_{66} + \left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2\left[A_{11}^2 - A_{12}(A_{12}\ldots\right.\right.\right.$$

$$\left.\left.\left.\ldots + 2A_{66})\right]\right\}\left\{\left[\left(\frac{m_1}{a_1}\right)^4 + \left(\frac{m_2}{a_2}\right)^4\right]D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2(D_{12} + 2D_{66})\right\}\ldots$$

$$\ldots - \left\{ 2\left(\frac{m_1}{a_1}\right)^4 \left(\frac{m_2}{a_2}\right)^4 (A_{12} + A_{66}) + \left(\frac{m_1}{a_1}\right)^2 \left(\frac{m_2}{a_2}\right)^2 \left[\left(\frac{m_1}{a_1}\right)^4 + \left(\frac{m_2}{a_2}\right)^4 \right] A_{11} \ldots \right.$$

$$\left. \ldots + \left[\left(\frac{m_1}{a_1}\right)^8 + \left(\frac{m_2}{a_2}\right)^8 \right] A_{66} \right\} B_{11}^2 \Bigg),$$

$$\Delta^1_{m_1 m_2} = \pi^5 \frac{m_1}{a_1} \left[\left(\frac{m_2}{a_2}\right)^4 (A_{12} + A_{66}) + \left(\frac{m_1}{a_1}\right)^4 A_{66} + \left(\frac{m_1}{a_1}\right)^2 \left(\frac{m_2}{a_2}\right)^2 A_{11} \right] B_{11} q_{m_1 m_2},$$

$$\Delta^2_{m_1 m_2} = -\pi^5 \frac{m_2}{a_2} \left[\left(\frac{m_1}{a_1}\right)^4 (A_{12} + A_{66}) + \left(\frac{m_2}{a_2}\right)^4 A_{66} + \left(\frac{m_1}{a_1}\right)^2 \left(\frac{m_2}{a_2}\right)^2 A_{11} \right] B_{11} q_{m_1 m_2},$$

$$\Delta^3_{m_1 m_2} = \pi^4 \left\{ \left[\left(\frac{m_1}{a_1}\right)^4 + \left(\frac{m_2}{a_2}\right)^4 \right] A_{11} A_{66} + \left(\frac{m_1}{a_1}\right)^2 \left(\frac{m_2}{a_2}\right)^2 \left[A_{11}^2 \ldots \right. \right.$$

$$\left. \left. \ldots - A_{12} \left(A_{12} + 2 A_{66} \right) \right] \right\} q_{m_1 m_2}.$$

We therefore obtain:

$$U^1_{m_1 m_2} = \frac{\Delta^1_{m_1 m_2}}{\Delta_{m_1 m_2}}, \quad U^2_{m_1 m_2} = \frac{\Delta^2_{m_1 m_2}}{\Delta_{m_1 m_2}}, \quad U^3_{m_1 m_2} = \frac{\Delta^3_{m_1 m_2}}{\Delta_{m_1 m_2}}.$$

For a loading of the form:

$$q(x_1, x_2) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} q_{m_1 m_2} \sin \frac{m_1 \pi x_1}{a_1} \sin \frac{m_2 \pi x_2}{a_2},$$

we obtain the displacements:

$$u_1^0 = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} U^1_{m_1 m_2} \cos \frac{m_1 \pi x_1}{a_1} \sin \frac{m_2 \pi x_2}{a_2},$$

$$u_2^0 = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} U^2_{m_1 m_2} \sin \frac{m_1 \pi x_1}{a_1} \cos \frac{m_2 \pi x_2}{a_2},$$

$$u_3^0 = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} U^3_{m_1 m_2} \sin \frac{m_1 \pi x_1}{a_1} \sin \frac{m_2 \pi x_2}{a_2},$$

from which we can determine the strains and stresses.

When the number of layers is large, we can take $B_{11} = 0$. The previous expressions become:

$$u_1^0 = u_2^0 = 0,$$

$$u_3^0 = \frac{1}{\pi^4}\sum_{m_1=1}^{\infty}\sum_{m_2=1}^{\infty}\frac{q_{m_1m_2}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}}{\left[\left(\frac{m_1}{a_1}\right)^4+\left(\frac{m_2}{a_2}\right)^4\right]D_{11}+2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2\left(D_{12}+2D_{66}\right)}.$$

In the case of a square plate of edge a subjected to the loading:

$$q_{11}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a},$$

we obtain:

$$u_1^0 = \frac{a^3B_{11}q_{11}\cos\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}}{2\pi^3\left[\left(A_{11}-A_{12}\right)\left(D_{11}+D_{12}+2D_{66}\right)-B_{11}^2\right]},$$

$$u_2^0 = -\frac{a^3B_{11}q_{11}\sin\frac{\pi x_1}{a}\cos\frac{\pi x_2}{a}}{2\pi^3\left[\left(A_{11}-A_{12}\right)\left(D_{11}+D_{12}+2D_{66}\right)-B_{11}^2\right]},$$

$$u_3^0 = \frac{a^4\left(A_{11}-A_{12}\right)q_{11}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}}{2\pi^4\left[\left(A_{11}-A_{12}\right)\left(D_{11}+D_{12}+2D_{66}\right)-B_{11}^2\right]}.$$

When we take $B_{11} = 0$, we have:

$$u_1^0 = u_2^0 = 0,$$

$$u_3^0 = \frac{a^4q_{11}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}}{2\pi^4\left(D_{11}+D_{12}+2D_{66}\right)}.$$

9.3. Vibration of a cross-ply asymmetrical plate

The rectangular plate of dimensions a_1 and a_2 is freely supported in the direction orthogonal to its perimeter. It is not subjected to any given volume or surface loading.

Introducing the global membrane loads:

$$N_1 = A_{11}\frac{\partial u_1^0}{\partial x_1} + A_{12}\frac{\partial u_2^0}{\partial x_2} - B_{11}\frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$N_2 = A_{12}\frac{\partial u_1^0}{\partial x_1} + A_{11}\frac{\partial u_2^0}{\partial x_2} + B_{11}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$N_6 = A_{66}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right),$$

and the flexural moments:

$$M_1 = B_{11}\frac{\partial u_1^0}{\partial x_1} - D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_2 = -B_{11}\frac{\partial u_2^0}{\partial x_2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{11}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_6 = -2D_{66}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

into the equations for global vibration:

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} = I_0\frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} = I_0\frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial^2 M_1}{\partial x_1^2} + 2\frac{\partial^2 M_6}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} = I_0\frac{\partial^2 u_3^0}{\partial t^2},$$

gives the three equations:

$$A_{11}\frac{\partial^2 u_1^0}{\partial x_1^2} + A_{66}\frac{\partial^2 u_1^0}{\partial x_2^2} + (A_{12} + A_{66})\frac{\partial^2 u_2^0}{\partial x_1 \partial x_2} - B_{11}\frac{\partial^3 u_3^0}{\partial x_1^3} = I_0\frac{\partial^2 u_1^0}{\partial t^2},$$

$$(A_{12} + A_{66})\frac{\partial^2 u_1^0}{\partial x_1 \partial x_2} + A_{66}\frac{\partial^2 u_2^0}{\partial x_1^2} + A_{11}\frac{\partial^2 u_2^0}{\partial x_2^2} + B_{11}\frac{\partial^3 u_3^0}{\partial x_2^3} = I_0\frac{\partial^2 u_2^0}{\partial t^2},$$

$$-D_{11}\left(\frac{\partial^4 u_3^0}{\partial x_1^4} + \frac{\partial^4 u_3^0}{\partial x_2^4}\right) - 2(D_{12} + 2D_{66})\frac{\partial^4 u_3^0}{\partial x_1^2 \partial x_2^2} + B_{11}\left(\frac{\partial^3 u_1^0}{\partial x_1^3} - \frac{\partial^3 u_2^0}{\partial x_2^3}\right) = I_0\frac{\partial^2 u_3^0}{\partial t^2}.$$

The boundary conditions for a supported plate free to move in the direction normal to the edges are:

– for $x_1 = 0$ and $x_1 = a_1$:

$$u_3^0 = 0, \quad M_1 = B_{11}\frac{\partial u_1^0}{\partial x_1} - D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0,$$

$$u_2^0 = 0, \quad N_1 = A_{11}\frac{\partial u_1^0}{\partial x_1} + A_{12}\frac{\partial u_2^0}{\partial x_2} - B_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} = 0,$$

– for $x_2 = 0$ and $x_2 = a_2$:

$$u_3^0 = 0, \quad M_2 = -B_{11}\frac{\partial u_2^0}{\partial x_2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{11}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0,$$

$$u_1^0 = 0, \quad N_2 = A_{12}\frac{\partial u_1^0}{\partial x_1} + A_{11}\frac{\partial u_2^0}{\partial x_2} + B_{11}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0.$$

The displacement field defined by:

$$u_1^0 = U_{m_1m_2}^1 \cos\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}\sin\left(\omega_{m_1m_2}t + \varphi_{m_1m_2}\right),$$

$$u_2^0 = U_{m_1m_2}^2 \sin\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2}\sin\left(\omega_{m_1m_2}t + \varphi_{m_1m_2}\right),$$

$$u_3^0 = U_{m_1m_2}^3 \sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}\sin\left(\omega_{m_1m_2}t + \varphi_{m_1m_2}\right),$$

satisfies the boundary conditions at the edges and the global vibration equations which, after simplification, lead to the system:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{66} - I_0\omega_{m_1m_2}^2\right]U_{m_1m_2}^1 + \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12} + A_{66})U_{m_1m_2}^2 \cdots$$

$$\cdots - \left(\frac{m_1\pi}{a_1}\right)^3 B_{11}U_{m_1m_2}^3 = 0,$$

$$\frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12} + A_{66})U_{m_1m_2}^1 + \left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{66} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{11} - I_0\omega_{m_1m_2}^2\right]U_{m_1m_2}^2 \cdots$$

$$\cdots + \left(\frac{m_2\pi}{a_2}\right)^3 B_{11}U_{m_1m_2}^3 = 0,$$

$$-\left(\frac{m_1\pi}{a_1}\right)^3 B_{11}U_{m_1m_2}^1 + \left(\frac{m_2\pi}{a_2}\right)^3 B_{11}U_{m_1m_2}^2 + \left\{\left[\left(\frac{m_1\pi}{a_1}\right)^4 + \left(\frac{m_2\pi}{a_2}\right)^4\right]D_{11} \cdots\right.$$

$$\left.\cdots + 2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2 (D_{12} + 2D_{66}) - I_0\omega_{m_1m_2}^2\right\}U_{m_1m_2}^3 = 0,$$

which can be written in the following matrix form:

$$\begin{bmatrix} H_{11} - I_0\omega_{m_1m_2}^2 & H_{12} & H_{13} \\ H_{12} & H_{22} - I_0\omega_{m_1m_2}^2 & H_{23} \\ H_{13} & H_{23} & H_{33} - I_0\omega_{m_1m_2}^2 \end{bmatrix}\begin{bmatrix} U_{m_1m_2}^1 \\ U_{m_1m_2}^2 \\ U_{m_1m_2}^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

with:

$$H_{11} = \left(\frac{m_1\pi}{a_1}\right)^2 A_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{66},$$

$$H_{22} = \left(\frac{m_1\pi}{a_1}\right)^2 A_{66} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{11},$$

$$H_{33} = \left[\left(\frac{m_1\pi}{a_1}\right)^4 + \left(\frac{m_2\pi}{a_2}\right)^4\right] D_{11} + 2\left(\frac{m_1\pi}{a_1}\right)^2 \left(\frac{m_2\pi}{a_2}\right)^2 (D_{12} + 2D_{66}),$$

$$H_{12} = \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12} + A_{66}),$$

$$H_{13} = -\left(\frac{m_1\pi}{a_1}\right)^3 B_{11},$$

$$H_{23} = \left(\frac{m_2\pi}{a_2}\right)^3 B_{11}.$$

This algebraic system has a solution other than the trivial solution $U^1_{m_1m_2} = U^2_{m_1m_2} = U^3_{m_1m_2} = 0$ for the values of $\omega_{m_1m_2}$ which cancel the determinant:

$$\begin{vmatrix} H_{11} - I_0\omega^2_{m_1m_2} & H_{12} & H_{13} \\ H_{12} & H_{22} - I_0\omega^2_{m_1m_2} & H_{23} \\ H_{13} & H_{23} & H_{33} - I_0\omega^2_{m_1m_2} \end{vmatrix} = 0,$$

$$\left(H_{11} - I_0\omega^2_{m_1m_2}\right)\left(H_{22} - I_0\omega^2_{m_1m_2}\right)\left(H_{33} - I_0\omega^2_{m_1m_2}\right) + 2H_{12}H_{13}H_{23}\ldots$$
$$\ldots - \left(H_{11} - I_0\omega^2_{m_1m_2}\right)H^2_{23} - \left(H_{22} - I_0\omega^2_{m_1m_2}\right)H^2_{13} - \left(H_{33} - I_0\omega^2_{m_1m_2}\right)H^2_{12} = 0,$$

which is written in the form:

$$-A\omega^6_{m_1m_2} + B\omega^4_{m_1m_2} - C\omega^2_{m_1m_2} + D = 0,$$

with:

$$A = I_0^3,$$

$$B = (H_{11} + H_{22} + H_{33})I_0^2,$$

$$C = \left(H_{11}H_{22} + H_{22}H_{33} + H_{33}H_{11} - H^2_{12} - H^2_{13} - H^2_{23}\right)I_0,$$

$$D = \left(H_{11}H_{22} - H^2_{12}\right)H_{33} + 2H_{12}H_{13}H_{23} - H_{11}H^2_{23} - H_{22}H^2_{13},$$

or:

$$B=\left\{\left[\left(\frac{m_1\pi}{a_1}\right)^2+\left(\frac{m_2\pi}{a_2}\right)^2\right](A_{11}+A_{66})+\left[\left(\frac{m_1\pi}{a_1}\right)^4+\left(\frac{m_2\pi}{a_2}\right)^4\right]D_{11}\dots\right.$$

$$\left.\dots+2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2(D_{12}+2D_{66})\right\}I_0^2,$$

$$C=\left\{\left[\left(\frac{m_1\pi}{a_1}\right)^2+\left(\frac{m_2\pi}{a_2}\right)^2\right](A_{11}+A_{66})\left\{\left[\left(\frac{m_1\pi}{a_1}\right)^4+\left(\frac{m_2\pi}{a_2}\right)^4\right]D_{11}\dots\right.\right.$$

$$\dots+2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2(D_{12}+2D_{66})\right\}+\left[\left(\frac{m_1\pi}{a_1}\right)^4+\left(\frac{m_2\pi}{a_2}\right)^4\right]A_{11}A_{66}\dots$$

$$\left.\dots+\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2\left[A_{11}^2-A_{12}(A_{12}+2A_{66})\right]-\left[\left(\frac{m_1\pi}{a_1}\right)^6+\left(\frac{m_2\pi}{a_2}\right)^6\right]B_{11}^2\right\}I_0,$$

$$D=\left\{\left[\left(\frac{m_1\pi}{a_1}\right)^4+\left(\frac{m_2\pi}{a_2}\right)^4\right]A_{11}A_{66}+\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2\left[A_{11}^2-A_{12}(A_{12}+2A_{66})\right]\right\}\dots$$

$$\dots\times\left\{\left[\left(\frac{m_1\pi}{a_1}\right)^4+\left(\frac{m_2\pi}{a_2}\right)^4\right]D_{11}+2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2(D_{12}+2D_{66})\right\}\dots$$

$$\dots-\left\{2\left(\frac{m_1\pi}{a_1}\right)^4\left(\frac{m_2\pi}{a_2}\right)^4(A_{12}+A_{66})+\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2\left[\left(\frac{m_1\pi}{a_1}\right)^4+\left(\frac{m_2\pi}{a_2}\right)^4\right]A_{11}\dots\right.$$

$$\left.\dots+\left[\left(\frac{m_1\pi}{a_1}\right)^8+\left(\frac{m_2\pi}{a_2}\right)^8\right]A_{66}\right\}B_{11}^2.$$

For each couple (m_1,m_2), we obtain three natural frequencies.

In the case when the membrane inertias $I_0\frac{\partial^2 u_1^0}{\partial t^2}$ and $I_0\frac{\partial^2 u_2^0}{\partial t^2}$ are neglected, the previous system can be written, with the same notation, in the form:

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33}-I_0\omega_{m_1m_2}^2 \end{bmatrix}\begin{bmatrix} U_{m_1m_2}^1 \\ U_{m_1m_2}^2 \\ U_{m_1m_2}^3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

and the determinant becomes:

$$\begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} - I_0 \omega^2_{m_1 m_2} \end{vmatrix} = 0,$$

$$\left(H_{11}H_{22} - H_{12}^2\right)\left(H_{33} - I_0\omega^2_{m_1m_2}\right) + 2H_{12}H_{13}H_{23} - H_{11}H_{23}^2 - H_{22}H_{13}^2 = 0.$$

For each couple (m_1, m_2), we obtain the natural frequency:

$$\omega^2_{m_1m_2} = \frac{1}{I_0}\left(H_{33} + \frac{2H_{12}H_{13}H_{23} - H_{11}H_{23}^2 - H_{22}H_{13}^2}{H_{11}H_{22} - H_{12}^2}\right),$$

or:

$$\omega^2_{m_1m_2} = \frac{\pi^4}{I_0}\left(P_1 - \frac{P_2}{P_3}\right),$$

with:

$$P_1 = \left[\left(\frac{m_1}{a_1}\right)^4 + \left(\frac{m_2}{a_2}\right)^4\right]D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2(D_{12} + 2D_{66}),$$

$$P_2 = \left\{2\left(\frac{m_1}{a_1}\right)^4\left(\frac{m_2}{a_2}\right)^4(A_{12} + A_{66}) + \left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2\left[\left(\frac{m_1}{a_1}\right)^4 + \left(\frac{m_2}{a_2}\right)^4\right]A_{11}\ldots\right.$$

$$\left.\ldots + \left[\left(\frac{m_1}{a_1}\right)^8 + \left(\frac{m_2}{a_2}\right)^8\right]A_{66}\right\}B_{11}^2,$$

$$P_3 = \left[\left(\frac{m_1}{a_1}\right)^4 + \left(\frac{m_2}{a_2}\right)^4\right]A_{11}A_{66} + \left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2\left[A_{11}^2 - A_{12}(A_{12} + 2A_{66})\right].$$

When B_{11} is negligible, we have:

$$\omega^2_{m_1m_2} = \frac{\pi^4}{I_0}\left\{\left[\left(\frac{m_1}{a_1}\right)^4 + \left(\frac{m_2}{a_2}\right)^4\right]D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2(D_{12} + 2D_{66})\right\}.$$

9.4. Buckling of a cross-ply asymmetrical plate

The rectangular plate of dimensions a_1 and a_2 is supported freely in the direction orthogonal to its perimeter. The edges $x_1 = 0$, $x_1 = a_1$, $x_2 = 0$ and $x_2 = a_2$ are respectively subjected to the loads N_1^0, $-N_1^0$, N_2^0 and $-N_2^0$, with N_1^0 and N_2^0 being positive.

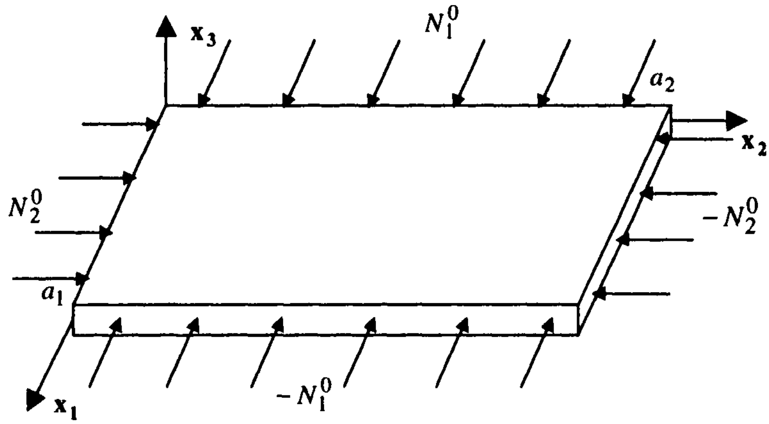

Figure 9.3. *Buckling asymmetrical cross-ply plate*

By introducing the global membrane loads:

$$N_1 = A_{11}\frac{\partial u_1^0}{\partial x_1} + A_{12}\frac{\partial u_2^0}{\partial x_2} - B_{11}\frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$N_2 = A_{12}\frac{\partial u_1^0}{\partial x_1} + A_{11}\frac{\partial u_2^0}{\partial x_2} + B_{11}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$N_6 = A_{66}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right),$$

and the flexural loads:

$$M_1 = B_{11}\frac{\partial u_1^0}{\partial x_1} - D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_2 = -B_{11}\frac{\partial u_2^0}{\partial x_2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{11}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_6 = -2D_{66}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

into the global buckling equations:

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} = 0,$$

$$\frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} = 0,$$

$$\frac{\partial^2 M_1}{\partial x_1^2}+2\frac{\partial^2 M_6}{\partial x_1 \partial x_2}+\frac{\partial^2 M_2}{\partial x_2^2}-N_1^0\frac{\partial^2 u_3^0}{\partial x_1^2}-N_2^0\frac{\partial^2 u_3^0}{\partial x_2^2}=0,$$

we obtain:

$$A_{11}\frac{\partial^2 u_1^0}{\partial x_1^2}+A_{66}\frac{\partial^2 u_1^0}{\partial x_2^2}+(A_{12}+A_{66})\frac{\partial^2 u_2^0}{\partial x_1 \partial x_2}-B_{11}\frac{\partial^3 u_3^0}{\partial x_1^3}=0,$$

$$(A_{12}+A_{66})\frac{\partial^2 u_1^0}{\partial x_1 \partial x_2}+A_{66}\frac{\partial^2 u_2^0}{\partial x_1^2}+A_{11}\frac{\partial^2 u_2^0}{\partial x_2^2}+B_{11}\frac{\partial^3 u_3^0}{\partial x_2^3}=0,$$

$$D_{11}\left(\frac{\partial^4 u_3^0}{\partial x_1^4}+\frac{\partial^4 u_3^0}{\partial x_2^4}\right)+2(D_{12}+2D_{66})\frac{\partial^4 u_3^0}{\partial x_1^2 \partial x_2^2}-B_{11}\left(\frac{\partial^3 u_1^0}{\partial x_1^3}-\frac{\partial^3 u_2^0}{\partial x_2^3}\right)\ldots$$

$$\ldots+N_1^0\frac{\partial^2 u_3^0}{\partial x_1^2}+N_2^0\frac{\partial^2 u_3^0}{\partial x_2^2}=0.$$

The boundary conditions for a plate freely supported in the direction normal to its edges are:

– for $x_1=0$ and $x_1=a_1$:

$$u_3^0=0,\quad M_1=B_{11}\frac{\partial u_1^0}{\partial x_1}-D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2}-D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2}=0,$$

$$u_2^0=0,\quad N_1=A_{11}\frac{\partial u_1^0}{\partial x_1}+A_{12}\frac{\partial u_2^0}{\partial x_2}-B_{11}\frac{\partial^2 u_3^0}{\partial x_1^2}=0,$$

– for $x_2=0$ and $x_2=a_2$:

$$u_3^0=0,\quad M_2=-B_{11}\frac{\partial u_2^0}{\partial x_2}-D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2}-D_{11}\frac{\partial^2 u_3^0}{\partial x_2^2}=0,$$

$$u_1^0=0,\quad N_2=A_{12}\frac{\partial u_1^0}{\partial x_1}+A_{11}\frac{\partial u_2^0}{\partial x_2}+B_{11}\frac{\partial^2 u_3^0}{\partial x_2^2}=0.$$

The displacement field, which satisfies the boundary conditions and the global equations, has the form:

$$u_1^0=U^1_{m_1m_2}\cos\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

$$u_2^0=U^2_{m_1m_2}\sin\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2},$$

$$u_3^0=U^3_{m_1m_2}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}.$$

By introducing this into the global buckling equations we obtain, after simplification, the system:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{66}\right]U^1_{m_1m_2} + \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12}+A_{66})U^2_{m_1m_2}\cdots$$

$$\cdots - \left(\frac{m_1\pi}{a_1}\right)^3 B_{11}U^3_{m_1m_2} = 0,$$

$$\frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12}+A_{66})U^1_{m_1m_2} + \left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{66} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{11}\right]U^2_{m_1m_2}\cdots$$

$$\cdots + \left(\frac{m_2\pi}{a_2}\right)^3 B_{11}U^3_{m_1m_2} = 0,$$

$$-\left(\frac{m_1\pi}{a_1}\right)^3 B_{11}U^1_{m_1m_2} + \left(\frac{m_2\pi}{a_2}\right)^3 B_{11}U^2_{m_1m_2} + \left\{\left[\left(\frac{m_1\pi}{a_1}\right)^4 + \left(\frac{m_2\pi}{a_2}\right)^4\right]D_{11}\cdots\right.$$

$$\left.\cdots + 2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2(D_{12}+2D_{66}) - \left(\frac{m_1\pi}{a_1}\right)^2 N_1^0 - \left(\frac{m_2\pi}{a_2}\right)^2 N_2^0\right\}U^3_{m_1m_2} = 0,$$

of the form:

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} - \left(\frac{m_1\pi}{a_1}\right)^2 N_1^0 - \left(\frac{m_2\pi}{a_2}\right)^2 N_2^0 \end{bmatrix}\begin{bmatrix} U^1_{m_1m_2} \\ U^2_{m_1m_2} \\ U^3_{m_1m_2}\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},$$

with:

$$H_{11} = \left(\frac{m_1\pi}{a_1}\right)^2 A_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{66},$$

$$H_{22} = \left(\frac{m_1\pi}{a_1}\right)^2 A_{66} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{11},$$

$$H_{33} = \left[\left(\frac{m_1\pi}{a_1}\right)^4 + \left(\frac{m_2\pi}{a_2}\right)^4\right]D_{11} + 2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2(D_{12}+2D_{66}),$$

$$H_{12} = \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12}+A_{66}),$$

$$H_{13} = -\left(\frac{m_1\pi}{a_1}\right)^3 B_{11},$$

$$H_{23}=\left(\frac{m_2\pi}{a_2}\right)^3 B_{11}.$$

The critical buckling loads are the values of N_1^0 and N_2^0 for which $U^1_{m_1m_2}$, $U^2_{m_1m_2}$ and $U^3_{m_1m_2}$ are not simultaneously zero, i.e. the values which cancel the determinant of the system:

$$\begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33}-\left(\frac{m_1\pi}{a_1}\right)^2 N_1^0-\left(\frac{m_2\pi}{a_2}\right)^2 N_2^0 \end{vmatrix}=0,$$

or:

$$\left(H_{11}H_{22}-H_{12}^2\right)\left(H_{33}-\left(\frac{m_1\pi}{a_1}\right)^2 N_1^0-\left(\frac{m_2\pi}{a_2}\right)^2 N_2^0\right)+2H_{12}H_{13}H_{23}\ldots$$

$$\ldots-H_{11}H_{23}^2-H_{22}H_{13}^2=0,$$

from which:

$$\left(\frac{m_1\pi}{a_1}\right)^2 N_1^0+\left(\frac{m_2\pi}{a_2}\right)^2 N_2^0=H_{33}+\frac{2H_{12}H_{13}H_{23}-H_{11}H_{23}^2-H_{22}H_{13}^2}{H_{11}H_{22}-H_{12}^2}.$$

The critical buckling loads are given by:

$$\left(\frac{m_1\pi}{a_1}\right)^2 N_1^0+\left(\frac{m_2\pi}{a_2}\right)^2 N_2^0=H_{33}-\frac{R_1}{R_2},$$

with:

$$H_{33}=\left[\left(\frac{m_1\pi}{a_1}\right)^4+\left(\frac{m_2\pi}{a_2}\right)^4\right]D_{11}+2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2\left(D_{12}+2D_{66}\right),$$

$$R_1=\left\{2\left(\frac{m_1\pi}{a_1}\right)^4\left(\frac{m_2\pi}{a_2}\right)^4\left(A_{12}+A_{66}\right)\ldots\right.$$

$$\left.\ldots+\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2\left[\left(\frac{m_1\pi}{a_1}\right)^4+\left(\frac{m_2\pi}{a_2}\right)^4\right]A_{11}+\left[\left(\frac{m_1\pi}{a_1}\right)^8+\left(\frac{m_2\pi}{a_2}\right)^8\right]A_{66}\right\}B_{11}^2,$$

$$R_2 = \left[\left(\frac{m_1\pi}{a_1}\right)^4 + \left(\frac{m_2\pi}{a_2}\right)^4\right] A_{11}A_{66} + \left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2 \left[A_{11}^2 - A_{12}\left(A_{12} + 2A_{66}\right)\right].$$

In the case when $N_2^0 = kN_1^0$, we obtain the critical buckling loads:

$$N_1^0 = \frac{\pi^2}{\left(\frac{m_1}{a_1}\right)^2 + k\left(\frac{m_2}{a_2}\right)^2}\left(P_1 - \frac{P_2}{P_2}\right),$$

with:

$$P_1 = \left[\left(\frac{m_1}{a_1}\right)^4 + \left(\frac{m_2}{a_2}\right)^4\right] D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 \left(D_{12} + 2D_{66}\right),$$

$$P_2 = \left\{2\left(\frac{m_1}{a_1}\right)^4\left(\frac{m_2}{a_2}\right)^4 \left(A_{12} + A_{66}\right) + \left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2\left[\left(\frac{m_1}{a_1}\right)^4 + \left(\frac{m_2}{a_2}\right)^4\right] A_{11} \ldots\right.$$

$$\left.\ldots + \left[\left(\frac{m_1}{a_1}\right)^8 + \left(\frac{m_2}{a_2}\right)^8\right] A_{66}\right\} B_{11}^2,$$

$$P_3 = \left[\left(\frac{m_1}{a_1}\right)^4 + \left(\frac{m_2}{a_2}\right)^4\right] A_{11}A_{66} + \left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 \left[A_{11}^2 - A_{12}\left(A_{12} + 2A_{66}\right)\right].$$

When the number of layers is large, B_{11} is negligible and the critical buckling loads are given by:

$$N_1^0 = \pi^2 \frac{\left[\left(\frac{m_1}{a_1}\right)^4 + \left(\frac{m_2}{a_2}\right)^4\right] D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 \left(D_{12} + 2D_{66}\right)}{\left(\frac{m_1}{a_1}\right)^2 + k\left(\frac{m_2}{a_2}\right)^2}.$$

9.5. Flexure of a balanced asymmetrical plate

The rectangular plate of dimensions a_1 and a_2 is freely supported in the direction of its perimeter. It is only subjected on its upper face to the surface force $q(x_1, x_2)\mathbf{x}_3$.

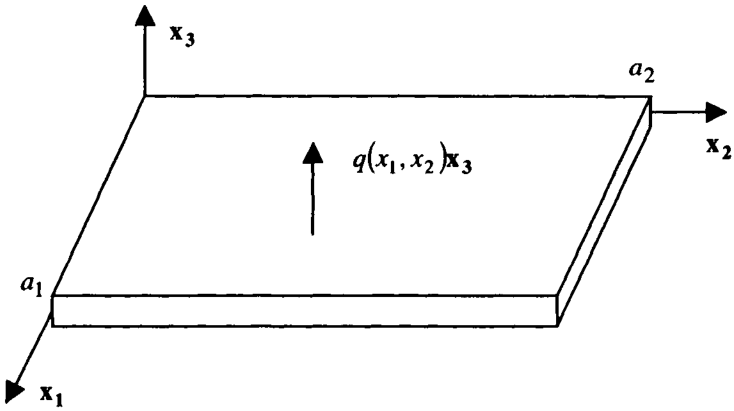

Figure 9.4. *Asymmetrically balanced plate in flexure*

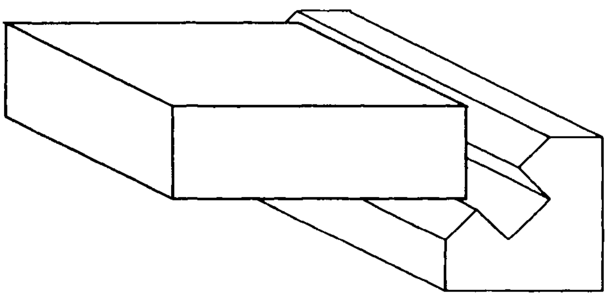

Figure 9.5. *Schematic representation of edges of a balanced asymmetrical plate*

The global stiffnesses of the composite are such that:

$$A_{16} = A_{26} = 0, \quad D_{16} = D_{26} = 0,$$
$$B_{11} = B_{22} = B_{12} = B_{66} = 0.$$

The constitutive relation of the composite is written:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\ A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\ 0 & 0 & B_{16} & D_{11} & D_{12} & 0 \\ 0 & 0 & B_{26} & D_{12} & D_{22} & 0 \\ B_{16} & B_{26} & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix},$$

with:

$$\varepsilon_1^0 = \frac{\partial u_1^0}{\partial x_1}, \qquad \varepsilon_2^0 = \frac{\partial u_2^0}{\partial x_2}, \qquad \varepsilon_6^0 = \frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1},$$

$$\kappa_1 = -\frac{\partial^2 u_3^0}{\partial x_1^2}, \quad \kappa_2 = -\frac{\partial^2 u_3^0}{\partial x_2^2}, \quad \kappa_6 = -2\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2}.$$

By introducing the global cohesion loads:

$$N_1 = A_{11}\frac{\partial u_1^0}{\partial x_1} + A_{12}\frac{\partial u_2^0}{\partial x_2} - 2B_{16}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

$$N_2 = A_{12}\frac{\partial u_1^0}{\partial x_1} + A_{22}\frac{\partial u_2^0}{\partial x_2} - 2B_{26}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

$$N_6 = A_{66}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - B_{16}\frac{\partial^2 u_3^0}{\partial x_1^2} - B_{26}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

and:

$$M_1 = B_{16}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_2 = B_{26}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{22}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_6 = B_{16}\frac{\partial u_1^0}{\partial x_1} + B_{26}\frac{\partial u_2^0}{\partial x_2} - 2D_{66}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

into the global equilibrium equations:

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} = 0,$$

$$\frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} = 0,$$

$$\frac{\partial^2 M_1}{\partial x_1^2} + 2\frac{\partial^2 M_6}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} + q = 0,$$

we obtain:

$$A_{11}\frac{\partial^2 u_1^0}{\partial x_1^2} + A_{66}\frac{\partial^2 u_1^0}{\partial x_2^2} + (A_{12} + A_{66})\frac{\partial^2 u_2^0}{\partial x_1 \partial x_2} - 3B_{16}\frac{\partial^3 u_3^0}{\partial x_1^2 \partial x_2} - B_{26}\frac{\partial^3 u_3^0}{\partial x_2^3} = 0,$$

$$(A_{12} + A_{66})\frac{\partial^2 u_1^0}{\partial x_1 \partial x_2} + A_{66}\frac{\partial^2 u_2^0}{\partial x_1^2} + A_{22}\frac{\partial^2 u_2^0}{\partial x_2^2} - B_{16}\frac{\partial^3 u_3^0}{\partial x_1^3} - 3B_{26}\frac{\partial^3 u_3^0}{\partial x_1 \partial x_2^2} = 0,$$

$$D_{11}\frac{\partial^4 u_3^0}{\partial x_1^4}+2(D_{12}+2D_{66})\frac{\partial^4 u_3^0}{\partial x_1^2\partial x_2^2}+D_{22}\frac{\partial^4 u_3^0}{\partial x_2^4}-3B_{16}\frac{\partial^3 u_1^0}{\partial x_1^2\partial x_2}-B_{26}\frac{\partial^3 u_1^0}{\partial x_2^3}\dots$$

$$\dots-B_{16}\frac{\partial^3 u_2^0}{\partial x_1^3}-3B_{26}\frac{\partial^3 u_2^0}{\partial x_1\partial x_2^2}=q.$$

The boundary conditions for an edge freely supported in the edge direction are:
– for $x_1=0$ are $x_1=a_1$:

$$u_3^0=0,\quad M_1=B_{16}\left(\frac{\partial u_1^0}{\partial x_2}+\frac{\partial u_2^0}{\partial x_1}\right)-D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2}-D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2}=0,$$

$$u_1^0=0,\quad N_6=A_{66}\left(\frac{\partial u_1^0}{\partial x_2}+\frac{\partial u_2^0}{\partial x_1}\right)-B_{16}\frac{\partial^2 u_3^0}{\partial x_1^2}-B_{26}\frac{\partial^2 u_3^0}{\partial x_2^2}=0,$$

– for $x_2=0$ and $x_2=a_2$:

$$u_3^0=0,\quad M_2=B_{26}\left(\frac{\partial u_1^0}{\partial x_2}+\frac{\partial u_2^0}{\partial x_1}\right)-D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2}-D_{22}\frac{\partial^2 u_3^0}{\partial x_2^2}=0,$$

$$u_2^0=0,\quad N_6=A_{66}\left(\frac{\partial u_1^0}{\partial x_2}+\frac{\partial u_2^0}{\partial x_1}\right)-B_{16}\frac{\partial^2 u_3^0}{\partial x_1^2}-B_{26}\frac{\partial^2 u_3^0}{\partial x_2^2}=0.$$

The displacement field for the plate subjected to a double sinusoidal load:

$$q_{m_1m_2}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

which satisfies the global equilibrium equations and the boundary conditions is of the form:

$$u_1^0=U^1_{m_1m_2}\sin\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2},$$

$$u_2^0=U^2_{m_1m_2}\cos\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

$$u_3^0=U^3_{m_1m_2}\sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}.$$

Introducing these expressions in the global equilibrium equations we obtain, after simplification, the system for $U^1_{m_1m_2}$, $U^2_{m_1m_2}$ and $U^3_{m_1m_2}$:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11}+\left(\frac{m_2\pi}{a_2}\right)^2 A_{66}\right]U^1_{m_1m_2}+\frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12}+A_{66})U^2_{m_1m_2}\dots$$

$$\ldots - \frac{m_2\pi}{a_2}\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + \left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right] U^3_{m_1m_2} = 0,$$

$$\frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12} + A_{66})U^1_{m_1m_2} + \left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{66} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{11}\right] U^2_{m_1m_2} \ldots$$

$$\ldots - \frac{m_1\pi}{a_1}\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + 3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right] U^3_{m_1m_2} = 0,$$

$$-\frac{m_2\pi}{a_2}\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + \left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right] U^1_{m_1m_2} - \frac{m_1\pi}{a_1}\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} \ldots\right.$$

$$\left.\ldots + 3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right] U^2_{m_1m_2} + \left\{\left(\frac{m_1\pi}{a_1}\right)^4 D_{11} \ldots\right.$$

$$\left.\ldots + 2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2 (D_{12} + 2D_{66}) + \left(\frac{m_2\pi}{a_2}\right)^4 D_{22}\right\} U^3_{m_1m_2} = q_{m_1m_2},$$

which can be written as:

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} \end{bmatrix} \begin{bmatrix} U^1_{m_1m_2} \\ U^2_{m_1m_2} \\ U^3_{m_1m_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q_{m_1m_2} \end{bmatrix},$$

with:

$$H_{11} = \left(\frac{m_1\pi}{a_1}\right)^2 A_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{66},$$

$$H_{22} = \left(\frac{m_1\pi}{a_1}\right)^2 A_{66} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{22},$$

$$H_{33} = \left(\frac{m_1\pi}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2 (D_{12} + 2D_{66}) + \left(\frac{m_2\pi}{a_2}\right)^4 D_{22},$$

$$H_{12} = \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12} + A_{66}),$$

$$H_{13} = -\frac{m_2\pi}{a_2}\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + \left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right],$$

$$H_{23} = -\frac{m_1\pi}{a_1}\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + 3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right].$$

The determinant of the system and the determinants associated with the three unknowns are:

$$\Delta_{m_1m_2} = \left(H_{11}H_{22} - H_{12}^2\right)H_{33} + 2H_{12}H_{13}H_{23} - H_{11}H_{23}^2 - H_{22}H_{13}^2,$$

$$\Delta^1_{m_1m_2} = \left(H_{12}H_{23} - H_{13}H_{22}\right)q_{m_1m_2},$$

$$\Delta^2_{m_1m_2} = \left(H_{12}H_{13} - H_{11}H_{23}\right)q_{m_1m_2},$$

$$\Delta^3_{m_1m_2} = \left(H_{11}H_{22} - H_{12}^2\right)q_{m_1m_2},$$

or:

$$\Delta_{m_1m_2} = \pi^8\left(\left\{\left[\left(\frac{m_1}{a_1}\right)^2 A_{11} + \left(\frac{m_2}{a_2}\right)^2 A_{66}\right]\left[\left(\frac{m_1}{a_1}\right)^2 A_{66} + \left(\frac{m_2}{a_2}\right)^2 A_{22}\right]\ldots\right.\right.$$

$$\ldots - \left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 (A_{12} + A_{66})^2\right\}\left[\left(\frac{m_1}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 (D_{12} + 2D_{66})\ldots$$

$$\ldots + \left(\frac{m_2}{a_2}\right)^4 D_{22}\right] + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 (A_{12} + A_{66})\left[3\left(\frac{m_1}{a_1}\right)^2 B_{16} + \left(\frac{m_2}{a_2}\right)^2 B_{26}\right]\ldots$$

$$\ldots\times\left[\left(\frac{m_1}{a_1}\right)^2 B_{16} + 3\left(\frac{m_2}{a_2}\right)^2 B_{26}\right]\ldots$$

$$\ldots - \left(\frac{m_2}{a_2}\right)^2\left[\left(\frac{m_1}{a_1}\right)^2 A_{66} + \left(\frac{m_2}{a_2}\right)^2 A_{22}\right]\left[3\left(\frac{m_1}{a_1}\right)^2 B_{16} + \left(\frac{m_2}{a_2}\right)^2 B_{26}\right]^2\ldots$$

$$\left.\ldots - \left(\frac{m_1}{a_1}\right)^2\left[\left(\frac{m_1}{a_1}\right)^2 A_{11} + \left(\frac{m_2}{a_2}\right)^2 A_{66}\right]\left[\left(\frac{m_1}{a_1}\right)^2 B_{16} + 3\left(\frac{m_2}{a_2}\right)^2 B_{26}\right]^2\right),$$

$$\Delta^1_{m_1m_2} = \pi^5\frac{m_2}{a_2}\left\{\left[\left(\frac{m_1}{a_1}\right)^2 A_{66} + \left(\frac{m_2}{a_2}\right)^2 A_{11}\right]\left[3\left(\frac{m_1}{a_1}\right)^2 B_{16} + \left(\frac{m_2}{a_2}\right)^2 B_{26}\right]\ldots\right.$$

$$\left.\ldots - \left(\frac{m_1}{a_1}\right)^2 (A_{12} + A_{66})\left[\left(\frac{m_1}{a_1}\right)^2 B_{16} + 3\left(\frac{m_2}{a_2}\right)^2 B_{26}\right]\right\}q_{m_1m_2},$$

$$\Delta^2_{m_1 m_2} = \pi^5 \frac{m_1}{a_1}\left\{\left[\left(\frac{m_1}{a_1}\right)^2 A_{11} + \left(\frac{m_2}{a_2}\right)^2 A_{66}\right]\left[\left(\frac{m_1}{a_1}\right)^2 B_{16} + 3\left(\frac{m_2}{a_2}\right)^2 B_{26}\right]\ldots\right.$$

$$\left.\ldots - \left(\frac{m_2}{a_2}\right)^2 (A_{12} + A_{66})\left[3\left(\frac{m_1}{a_1}\right)^2 B_{16} + \left(\frac{m_2}{a_2}\right)^2 B_{26}\right]\right\} q_{m_1 m_2},$$

$$\Delta^3_{m_1 m_2} = \pi^4 \left\{\left[\left(\frac{m_1}{a_1}\right)^2 A_{11} + \left(\frac{m_2}{a_2}\right)^2 A_{66}\right]\left[\left(\frac{m_1}{a_1}\right)^2 A_{66} + \left(\frac{m_2}{a_2}\right)^2 A_{22}\right]\ldots\right.$$

$$\left.\ldots - \left(\frac{m_1}{a_1}\right)^2 \left(\frac{m_2}{a_2}\right)^2 (A_{12} + A_{66})^2\right\} q_{m_1 m_2}.$$

The displacements are given by:

$$U^1_{m_1 m_2} = \frac{\Delta^1_{m_1 m_2}}{\Delta_{m_1 m_2}}, \quad U^2_{m_1 m_2} = \frac{\Delta^2_{m_1 m_2}}{\Delta_{m_1 m_2}}, \quad U^3_{m_1 m_2} = \frac{\Delta^3_{m_1 m_2}}{\Delta_{m_1 m_2}}.$$

For a load of the type:

$$q(x_1, x_2) = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} q_{m_1 m_2} \sin\frac{m_1 \pi x_1}{a_1} \sin\frac{m_2 \pi x_2}{a_2},$$

we have the displacements:

$$u_1^0 = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} U^1_{m_1 m_2} \sin\frac{m_1 \pi x_1}{a_1} \cos\frac{m_2 \pi x_2}{a_2},$$

$$u_2^0 = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} U^2_{m_1 m_2} \cos\frac{m_1 \pi x_1}{a_1} \sin\frac{m_2 \pi x_2}{a_2},$$

$$u_3^0 = \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} U^3_{m_1 m_2} \sin\frac{m_1 \pi x_1}{a_1} \sin\frac{m_2 \pi x_2}{a_2},$$

from which we obtain the strains and stresses.

When the number of layers is large, we can take $B_{16} = B_{26} = 0$. The previous expressions then give:

$$u_1^0 = u_2^0 = 0,$$

$$u_3^0 = \frac{1}{\pi^4} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \frac{q_{m_1 m_2} \sin\frac{m_1 \pi x_1}{a_1} \sin\frac{m_2 \pi x_2}{a_2}}{\left(\frac{m_1}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1}{a_1}\right)^2 \left(\frac{m_2}{a_2}\right)^2 (D_{12} + 2D_{66}) + \left(\frac{m_2}{a_2}\right)^4 D_{22}}.$$

The previous results for a square plate of side a and for a load:

$$q_{11}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a},$$

become:

$$\Delta_{11}=\left(\frac{\pi}{a}\right)^8\left\{\left[(A_{11}+A_{66})(A_{66}+A_{22})-(A_{12}+A_{66})^2\right]\left[D_{11}+2(D_{12}+2D_{66})+D_{22}\right]\ldots\right.$$

$$\ldots+2(A_{12}+A_{66})(3B_{16}+B_{26})(B_{16}+3B_{26})-(A_{66}+A_{22})(3B_{16}+B_{26})^2\ldots$$

$$\left.\ldots-(A_{11}+A_{66})(B_{16}+3B_{26})^2\right\},$$

$$\Delta_{11}^1=\left(\frac{\pi}{a}\right)^5\left[(A_{66}+A_{22})(3B_{16}+B_{26})-(A_{12}+A_{66})(B_{16}+3B_{26})\right]q_{11},$$

$$\Delta_{11}^2=\left(\frac{\pi}{a}\right)^5\left[(A_{11}+A_{66})(B_{16}+3B_{26})-(A_{12}+A_{66})(3B_{16}+B_{26})\right]q_{11},$$

$$\Delta_{11}^3=\left(\frac{\pi}{a}\right)^4\left[(A_{11}+A_{66})(A_{66}+A_{22})-(A_{12}+A_{66})^2\right]q_{11},$$

and:

$$u_1^0=\frac{\Delta_{11}^1}{\Delta_{11}}\sin\frac{\pi x_1}{a}\cos\frac{\pi x_2}{a},$$

$$u_2^0=\frac{\Delta_{11}^2}{\Delta_{11}}\cos\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a},$$

$$u_3^0=\frac{\Delta_{11}^3}{\Delta_{11}}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}.$$

When B_{16} and B_{26} are very small we have:

$$u_1^0=u_2^0=0,$$

$$u_3^0=\frac{a^4 q_{11}\sin\frac{\pi x_1}{a}\sin\frac{\pi x_2}{a}}{\pi^4\left[D_{11}+2(D_{12}+2D_{66})+D_{22}\right]}.$$

9.6. Vibration of a balanced asymmetrical plate

The rectangular plate of dimensions a_1 and a_2 is freely supported in the direction of its perimeter. It is not subjected to any given external loading.

By introducing the global membrane loads:

$$N_1 = A_{11}\frac{\partial u_1^0}{\partial x_1} + A_{12}\frac{\partial u_2^0}{\partial x_2} - 2B_{16}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

$$N_2 = A_{12}\frac{\partial u_1^0}{\partial x_1} + A_{22}\frac{\partial u_2^0}{\partial x_2} - 2B_{26}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

$$N_6 = A_{66}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - B_{16}\frac{\partial^2 u_3^0}{\partial x_1^2} - B_{26}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

and flexural loads:

$$M_1 = B_{16}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_2 = B_{26}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{22}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_6 = B_{16}\frac{\partial u_1^0}{\partial x_1} + B_{26}\frac{\partial u_2^0}{\partial x_2} - 2D_{66}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

into the global vibration equations:

$$\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} = I_0\frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} = I_0\frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial^2 M_1}{\partial x_1^2} + 2\frac{\partial^2 M_6}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} = I_0\frac{\partial^2 u_3^0}{\partial t^2},$$

we obtain the three equations:

$$A_{11}\frac{\partial^2 u_1^0}{\partial x_1^2} + A_{66}\frac{\partial^2 u_1^0}{\partial x_2^2} + (A_{12} + A_{66})\frac{\partial^2 u_2^0}{\partial x_1 \partial x_2} - 3B_{16}\frac{\partial^3 u_3^0}{\partial x_1^2 \partial x_2} - B_{26}\frac{\partial^3 u_3^0}{\partial x_2^3} = I_0\frac{\partial^2 u_1^0}{\partial t^2},$$

$$(A_{12} + A_{66})\frac{\partial^2 u_1^0}{\partial x_1 \partial x_2} + A_{66}\frac{\partial^2 u_2^0}{\partial x_1^2} + A_{22}\frac{\partial^2 u_2^0}{\partial x_2^2} - B_{16}\frac{\partial^3 u_3^0}{\partial x_1^3} - 3B_{26}\frac{\partial^3 u_3^0}{\partial x_1 \partial x_2^2} = I_0\frac{\partial^2 u_2^0}{\partial t^2},$$

$$D_{11}\frac{\partial^4 u_3^0}{\partial x_1^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 u_3^0}{\partial x_1^2 \partial x_2^2} + D_{22}\frac{\partial^4 u_3^0}{\partial x_2^4} - 3B_{16}\frac{\partial^3 u_1^0}{\partial x_1^2 \partial x_2} - B_{26}\frac{\partial^3 u_1^0}{\partial x_2^3}\ldots$$

$$\ldots - B_{16}\frac{\partial^3 u_2^0}{\partial x_1^3} - 3B_{26}\frac{\partial^3 u_2^0}{\partial x_1 \partial x_2^2} + I_0\frac{\partial^2 u_3^0}{\partial t^2} = 0.$$

The boundary conditions for a plate freely supported in the direction of the perimeter are:

– for $x_1 = 0$ and $x_1 = a_1$:

$$u_3^0 = 0, \quad M_1 = B_{16}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0,$$

$$u_1^0 = 0, \quad N_6 = A_{66}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - B_{16}\frac{\partial^2 u_3^0}{\partial x_1^2} - B_{26}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0,$$

– for $x_2 = 0$ and $x_2 = a_2$:

$$u_3^0 = 0, \quad M_2 = B_{26}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{22}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0,$$

$$u_2^0 = 0, \quad N_6 = A_{66}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - B_{16}\frac{\partial^2 u_3^0}{\partial x_1^2} - B_{26}\frac{\partial^2 u_3^0}{\partial x_2^2} = 0.$$

The displacement fields defined by:

$$u_1^0 = U_{m_1m_2}^1 \sin\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2}\sin\left(\omega_{m_1m_2}t + \varphi_{m_1m_2}\right),$$

$$u_2^0 = U_{m_1m_2}^2 \cos\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}\sin\left(\omega_{m_1m_2}t + \varphi_{m_1m_2}\right),$$

$$u_3^0 = U_{m_1m_2}^3 \sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}\sin\left(\omega_{m_1m_2}t + \varphi_{m_1m_2}\right),$$

satisfy the boundary conditions and the equations of motion.

Introducing these expressions in the global vibration equations, we obtain, after simplification, the following system in $U_{m_1m_2}^1$, $U_{m_1m_2}^2$ and $U_{m_1m_2}^3$:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{66} - I_0\omega_{m_1m_2}^2\right]U_{m_1m_2}^1 + \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12} + A_{66})U_{m_1m_2}^2 \cdots$$

$$\cdots - \frac{m_2\pi}{a_2}\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + \left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]U_{m_1m_2}^3 = 0,$$

$$\frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12} + A_{66})U_{m_1m_2}^1 + \left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{66} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{22} - I_0\omega_{m_1m_2}^2\right]U_{m_1m_2}^2 \cdots$$

$$\cdots - \frac{m_1\pi}{a_1}\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + 3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]U_{m_1m_2}^3 = 0,$$

$$-\frac{m_2\pi}{a_2}\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16}+\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]U^1_{m_1m_2}-\frac{m_1\pi}{a_1}\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16}\cdots\right.$$

$$\left.\cdots+3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]U^2_{m_1m_2}+\left\{\left(\frac{m_1\pi}{a_1}\right)^4 D_{11}\cdots\right.$$

$$\left.\cdots+2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2(D_{12}+2D_{66})+\left(\frac{m_2\pi}{a_2}\right)^4 D_{22}-I_0\omega^2_{m_1m_2}\right\}U^3_{m_1m_2}=0,$$

which are written as:

$$\begin{bmatrix} H_{11}-I_0\omega^2_{m_1m_2} & H_{12} & H_{13} \\ H_{12} & H_{22}-I_0\omega^2_{m_1m_2} & H_{23} \\ H_{13} & H_{23} & H_{33}-I_0\omega^2_{m_1m_2} \end{bmatrix}\begin{bmatrix} U^1_{m_1m_2} \\ U^2_{m_1m_2} \\ U^3_{m_1m_2} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

with:

$$H_{11}=\left(\frac{m_1\pi}{a_1}\right)^2 A_{11}+\left(\frac{m_2\pi}{a_2}\right)^2 A_{66},$$

$$H_{22}=\left(\frac{m_1\pi}{a_1}\right)^2 A_{66}+\left(\frac{m_2\pi}{a_2}\right)^2 A_{22},$$

$$H_{33}=\left(\frac{m_1\pi}{a_1}\right)^4 D_{11}+2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2(D_{12}+2D_{66})+\left(\frac{m_2\pi}{a_2}\right)^4 D_{22},$$

$$H_{12}=\frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12}+A_{66}),$$

$$H_{13}=-\frac{m_2\pi}{a_2}\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16}+\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right],$$

$$H_{23}=-\frac{m_1\pi}{a_1}\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16}+3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right].$$

This algebraic system has a solution other than the trivial solution $U^1_{m_1m_2}=U^2_{m_1m_2}=U^3_{m_1m_2}=0$ for the values of $\omega_{m_1m_2}$ which cancel the determinant:

$$\begin{vmatrix} H_{11} - I_0\omega^2_{m_1m_2} & H_{12} & H_{13} \\ H_{12} & H_{22} - I_0\omega^2_{m_1m_2} & H_{23} \\ H_{13} & H_{23} & H_{33} - I_0\omega^2_{m_1m_2} \end{vmatrix} = 0,$$

$$\left(H_{11} - I_0\omega^2_{m_1m_2}\right)\left(H_{22} - I_0\omega^2_{m_1m_2}\right)\left(H_{33} - I_0\omega^2_{m_1m_2}\right) + 2H_{12}H_{13}H_{23}\ldots$$
$$\ldots - \left(H_{11} - I_0\omega^2_{m_1m_2}\right)H^2_{23} - \left(H_{22} - I_0\omega^2_{m_1m_2}\right)H^2_{13} - \left(H_{33} - I_0\omega^2_{m_1m_2}\right)H^2_{12} = 0,$$

which is written in the form:

$$-A\omega^6_{m_1m_2} + B\omega^4_{m_1m_2} - C\omega^2_{m_1m_2} + D = 0,$$

with:

$$A = I_0^3,$$
$$B = \left(H_{11} + H_{22} + H_{33}\right)I_0^2,$$
$$C = \left(H_{11}H_{22} + H_{22}H_{33} + H_{33}H_{11} - H^2_{12} - H^2_{13} - H^2_{23}\right)I_0,$$
$$D = \left(H_{11}H_{22} - H^2_{12}\right)H_{33} + 2H_{12}H_{13}H_{23} - H_{11}H^2_{23} - H_{22}H^2_{13},$$

or:

$$B = \left\{\left(\frac{m_1\pi}{a_1}\right)^2 (A_{11} + A_{66}) + \left(\frac{m_2\pi}{a_2}\right)^2 (A_{22} + A_{66}) + \left(\frac{m_1\pi}{a_1}\right)^4 D_{11}\ldots\right.$$
$$\left.\ldots + 2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2 (D_{12} + 2D_{66}) + \left(\frac{m_2\pi}{a_2}\right)^4 D_{22}\right\}I_0^2,$$

$$C = \left(\left\{\left(\frac{m_1\pi}{a_1}\right)^2 (A_{11} + A_{66}) + \left(\frac{m_2\pi}{a_2}\right)^2 (A_{22} + A_{66})\right\}\left\{\left(\frac{m_1\pi}{a_1}\right)^4 D_{11}\ldots\right.\right.$$
$$\left.\ldots + 2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2 (D_{12} + 2D_{66}) + \left(\frac{m_2\pi}{a_2}\right)^4 D_{22}\right\} + \left[\left(\frac{m_1\pi}{a_1}\right)^4 A_{11}\ldots\right.$$
$$\left.\ldots + \left(\frac{m_2\pi}{a_2}\right)^4 A_{22}\right]A_{66} + \left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2 \left[A_{11}A_{22} - A_{12}(A_{12} + 2A_{66})\right]\ldots$$
$$\ldots - \left(\frac{m_2\pi}{a_2}\right)^2\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + \left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]^2\ldots$$

$$\ldots-\left(\frac{m_1\pi}{a_1}\right)^2\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16}+3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]^2\Bigg\} I_0,$$

$$D=\left\{\left[\left(\frac{m_1\pi}{a_1}\right)^4 A_{11}+\left(\frac{m_2\pi}{a_2}\right)^4 A_{22}\right]A_{66}\ldots\right.$$

$$\left.\ldots+\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2\left[A_{11}A_{22}-A_{12}\left(A_{12}+2A_{66}\right)\right]\right\}\ldots$$

$$\ldots\times\left\{\left(\frac{m_1\pi}{a_1}\right)^4 D_{11}+2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2\left(D_{12}+2D_{66}\right)+\left(\frac{m_2\pi}{a_2}\right)^4 D_{22}\right\}\ldots$$

$$\ldots+2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2\left(A_{12}+A_{66}\right)\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16}+\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]\ldots$$

$$\ldots\times\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16}+3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]\ldots$$

$$\ldots-\left(\frac{m_1\pi}{a_1}\right)^2\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11}+\left(\frac{m_2\pi}{a_2}\right)^2 A_{66}\right]\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16}+3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]^2\ldots$$

$$\ldots-\left(\frac{m_2\pi}{a_2}\right)^2\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{66}+\left(\frac{m_2\pi}{a_2}\right)^2 A_{22}\right]\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16}+\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]^2.$$

For each value of the couple (m_1, m_2), we obtain the three natural frequencies.

In the case where the membrane inertias $I_0\dfrac{\partial^2 u_1^0}{\partial t^2}$ and $I_0\dfrac{\partial^2 u_2^0}{\partial t^2}$ are negligible, the previous system is written as:

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33}-I_0\omega^2_{m_1m_2} \end{bmatrix}\begin{bmatrix} U^1_{m_1m_2} \\ U^2_{m_1m_2} \\ U^3_{m_1m_2} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

with the previous values of H_{ij}. The determinant becomes:

$$\begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} - I_0\omega^2_{m_1m_2} \end{vmatrix} = 0,$$

$$\left(H_{11}H_{22} - H_{12}^2\right)\left(H_{33} - I_0\omega^2_{m_1m_2}\right) + 2H_{12}H_{13}H_{23} - H_{11}H_{23}^2 - H_{22}H_{13}^2 = 0.$$

For each couple (m_1, m_2), we obtain the natural frequency:

$$\omega^2_{m_1m_2} = \frac{1}{I_0}\left(H_{33} + \frac{2H_{12}H_{13}H_{23} - H_{11}H_{23}^2 - H_{22}H_{13}^2}{H_{11}H_{22} - H_{12}^2}\right),$$

or:

$$\omega^2_{m_1m_2} = \frac{\pi^4}{I_0}\left(P_1 + \frac{P_2}{P_3}\right),$$

with:

$$P_1 = \left(\frac{m_1}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 (D_{12} + 2D_{66}) + \left(\frac{m_2}{a_2}\right)^4 D_{22},$$

$$P_2 = 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 (A_{12} + A_{66})\left[3\left(\frac{m_1}{a_1}\right)^2 B_{16} + \left(\frac{m_2}{a_2}\right)^2 B_{26}\right]\ldots$$

$$\ldots\times\left[\left(\frac{m_1}{a_1}\right)^2 B_{16} + 3\left(\frac{m_2}{a_2}\right)^2 B_{26}\right]\ldots$$

$$\ldots - \left(\frac{m_1}{a_1}\right)^2\left[\left(\frac{m_1}{a_1}\right)^2 A_{11} + \left(\frac{m_2}{a_2}\right)^2 A_{66}\right]\left[\left(\frac{m_1}{a_1}\right)^2 B_{16} + 3\left(\frac{m_2}{a_2}\right)^2 B_{26}\right]^2\ldots$$

$$\ldots - \left(\frac{m_2}{a_2}\right)^2\left[\left(\frac{m_1}{a_1}\right)^2 A_{11} + \left(\frac{m_2}{a_2}\right)^2 A_{66}\right]\left[3\left(\frac{m_1}{a_1}\right)^2 B_{16} + \left(\frac{m_2}{a_2}\right)^2 B_{26}\right]^2,$$

$$P_3 = \left[\left(\frac{m_1}{a_1}\right)^4 A_{11} + \left(\frac{m_2}{a_2}\right)^4 A_{22}\right] A_{66} + \left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 \left[A_{11}A_{22} - A_{12}(A_{12} + 2A_{66})\right].$$

When B_{16} and B_{26} are negligible, the previous expression reduces to:

$$\omega^2_{m_1m_2} = \pi^4 \frac{\left(\frac{m_1}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 (D_{12} + 2D_{66}) + \left(\frac{m_2}{a_2}\right)^4 D_{22}}{I_0}.$$

9.7. Buckling of a balanced asymmetrical plate

The rectangular plate of dimensions a_1 and a_2 is freely supported in the direction of its perimeter. The edges $x_1 = 0$, $x_1 = a_1$, $x_2 = 0$ and $x_2 = a_2$ are respectively subjected to the loads N_1^0, $-N_1^0$, N_2^0 and $-N_2^0$, with N_1^0 and N_2^0 positive.

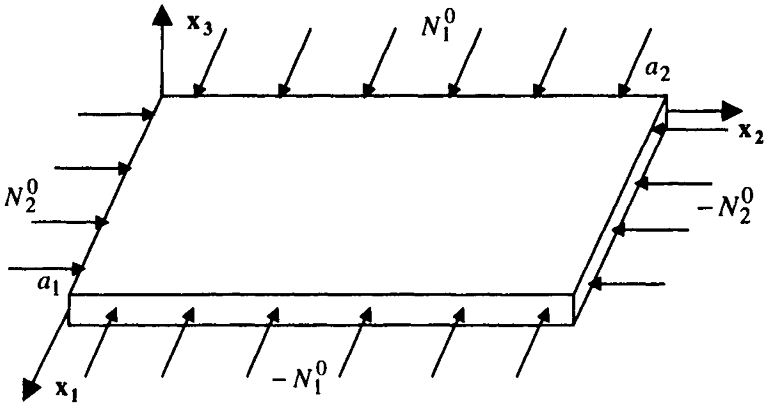

Figure 9.6. *Buckling of balanced, asymmetrical plate*

By introducing the global membrane loads:

$$N_1 = A_{11}\frac{\partial u_1^0}{\partial x_1} + A_{12}\frac{\partial u_2^0}{\partial x_2} - 2B_{16}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

$$N_2 = A_{12}\frac{\partial u_1^0}{\partial x_1} + A_{22}\frac{\partial u_2^0}{\partial x_2} - 2B_{26}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

$$N_6 = A_{66}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - B_{16}\frac{\partial^2 u_3^0}{\partial x_1^2} - B_{26}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

and flexural loads:

$$M_1 = B_{16}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_2 = B_{26}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2} - D_{22}\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$M_6 = B_{16}\frac{\partial u_1^0}{\partial x_1} + B_{26}\frac{\partial u_2^0}{\partial x_2} - 2D_{66}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

into the global buckling equations:

$$\frac{\partial N_1}{\partial x_1}+\frac{\partial N_6}{\partial x_2}=0,$$
$$\frac{\partial N_6}{\partial x_1}+\frac{\partial N_2}{\partial x_2}=0,$$
$$\frac{\partial^2 M_1}{\partial x_1^2}+2\frac{\partial^2 M_6}{\partial x_1\partial x_2}+\frac{\partial^2 M_2}{\partial x_2^2}-N_1^0\frac{\partial^2 u_3^0}{\partial x_1^2}-N_2^0\frac{\partial^2 u_3^0}{\partial x_2^2}=0,$$

we obtain:

$$A_{11}\frac{\partial^2 u_1^0}{\partial x_1^2}+A_{66}\frac{\partial^2 u_1^0}{\partial x_2^2}+(A_{12}+A_{66})\frac{\partial^2 u_2^0}{\partial x_1\partial x_2}-3B_{16}\frac{\partial^3 u_3^0}{\partial x_1^2\partial x_2}-B_{26}\frac{\partial^3 u_3^0}{\partial x_2^3}=0,$$

$$(A_{12}+A_{66})\frac{\partial^2 u_1^0}{\partial x_1\partial x_2}+A_{66}\frac{\partial^2 u_2^0}{\partial x_1^2}+A_{22}\frac{\partial^2 u_2^0}{\partial x_2^2}-B_{16}\frac{\partial^3 u_3^0}{\partial x_1^3}-3B_{26}\frac{\partial^3 u_3^0}{\partial x_1\partial x_2^2}=0,$$

$$D_{11}\frac{\partial^4 u_3^0}{\partial x_1^4}+2(D_{12}+2D_{66})\frac{\partial^4 u_3^0}{\partial x_1^2\partial x_2^2}+D_{22}\frac{\partial^4 u_3^0}{\partial x_2^4}-3B_{16}\frac{\partial^3 u_1^0}{\partial x_1^2\partial x_2}-B_{26}\frac{\partial^3 u_1^0}{\partial x_2^3}\dots$$
$$\dots-B_{16}\frac{\partial^3 u_2^0}{\partial x_1^3}-3B_{26}\frac{\partial^3 u_2^0}{\partial x_1\partial x_2^2}+N_1^0\frac{\partial^2 u_3^0}{\partial x_1^2}+N_2^0\frac{\partial^2 u_3^0}{\partial x_2^2}=0.$$

The boundary conditions for an edge freely supported in the direction of the perimeter are:

– for $x_1=0$ and $x_1=a_1$:

$$u_3^0=0,\quad M_1=B_{16}\left(\frac{\partial u_1^0}{\partial x_2}+\frac{\partial u_2^0}{\partial x_1}\right)-D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2}-D_{12}\frac{\partial^2 u_3^0}{\partial x_2^2}=0,$$

$$u_1^0=0,\quad N_6=A_{66}\left(\frac{\partial u_1^0}{\partial x_2}+\frac{\partial u_2^0}{\partial x_1}\right)-B_{16}\frac{\partial^2 u_3^0}{\partial x_1^2}-B_{26}\frac{\partial^2 u_3^0}{\partial x_2^2}=0,$$

– for $x_2=0$ and $x_2=a_2$:

$$u_3^0=0,\quad M_2=B_{26}\left(\frac{\partial u_1^0}{\partial x_2}+\frac{\partial u_2^0}{\partial x_1}\right)-D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2}-D_{22}\frac{\partial^2 u_3^0}{\partial x_2^2}=0,$$

$$u_2^0=0,\quad N_6=A_{66}\left(\frac{\partial u_1^0}{\partial x_2}+\frac{\partial u_2^0}{\partial x_1}\right)-B_{16}\frac{\partial^2 u_3^0}{\partial x_1^2}-B_{26}\frac{\partial^2 u_3^0}{\partial x_2^2}=0.$$

The displacement fields which satisfy the boundary conditions and the global equations are of the form:

$$u_1^0 = U_{m_1m_2}^1 \sin\frac{m_1\pi x_1}{a_1}\cos\frac{m_2\pi x_2}{a_2},$$

$$u_2^0 = U_{m_1m_2}^2 \cos\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2},$$

$$u_3^0 = U_{m_1m_2}^3 \sin\frac{m_1\pi x_1}{a_1}\sin\frac{m_2\pi x_2}{a_2}.$$

By introducing these into the global buckling equations, we obtain after simplification, the system:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{66}\right]U_{m_1m_2}^1 + \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12} + A_{66})U_{m_1m_2}^2 \cdots$$

$$\cdots - \frac{m_2\pi}{a_2}\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + \left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]U_{m_1m_2}^3 = 0,$$

$$\frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12} + A_{66})U_{m_1m_2}^1 + \left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{66} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{11}\right]U_{m_1m_2}^2 \cdots$$

$$\cdots - \frac{m_1\pi}{a_1}\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + 3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]U_{m_1m_2}^3 = 0,$$

$$-\frac{m_2\pi}{a_2}\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + \left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]U_{m_1m_2}^1 - \frac{m_1\pi}{a_1}\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} \cdots\right.$$

$$\left.\cdots + 3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]U_{m_1m_2}^2 + \left\{\left(\frac{m_1\pi}{a_1}\right)^4 D_{11} \cdots\right.$$

$$\cdots + 2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2 (D_{12} + 2D_{66}) + \left(\frac{m_2\pi}{a_2}\right)^4 D_{22} \cdots$$

$$\left.\cdots - \left(\frac{m_1\pi}{a_1}\right)^2 N_1^0 - \left(\frac{m_2\pi}{a_2}\right)^2 N_2^0\right\}U_{m_1m_2}^3 = 0,$$

which is written in the form:

$$\begin{bmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} - \left(\frac{m_1\pi}{a_1}\right)^2 N_1^0 - \left(\frac{m_2\pi}{a_2}\right)^2 N_2^0 \end{bmatrix} \begin{bmatrix} U^1_{m_1m_2} \\ U^2_{m_1m_2} \\ U^3_{m_1m_2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

with:

$$H_{11} = \left(\frac{m_1\pi}{a_1}\right)^2 A_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{66},$$

$$H_{22} = \left(\frac{m_1\pi}{a_1}\right)^2 A_{66} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{22},$$

$$H_{33} = \left(\frac{m_1\pi}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1\pi}{a_1}\right)^2 \left(\frac{m_2\pi}{a_2}\right)^2 (D_{12} + 2D_{66}) + \left(\frac{m_2\pi}{a_2}\right)^4 D_{22},$$

$$H_{12} = \frac{m_1\pi}{a_1}\frac{m_2\pi}{a_2}(A_{12} + A_{66}),$$

$$H_{13} = -\frac{m_2\pi}{a_2}\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + \left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right],$$

$$H_{23} = -\frac{m_1\pi}{a_1}\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + 3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right].$$

The critical buckling loads are the values of N_1^0 and N_2^0 for which $U^1_{m_1m_2}$, $U^2_{m_1m_2}$ and $U^3_{m_1m_2}$ are not simultaneously zero, i.e. for values which cancel the determinant of the system:

$$\begin{vmatrix} H_{11} & H_{12} & H_{13} \\ H_{12} & H_{22} & H_{23} \\ H_{13} & H_{23} & H_{33} - \left(\frac{m_1\pi}{a_1}\right)^2 N_1^0 - \left(\frac{m_2\pi}{a_2}\right)^2 N_2^0 \end{vmatrix} = 0,$$

or:

$$\left(H_{11}H_{22} - H_{12}^2\right)\left[H_{33} - \left(\frac{m_1\pi}{a_1}\right)^2 N_1^0 - \left(\frac{m_2\pi}{a_2}\right)^2 N_2^0\right] + 2H_{12}H_{13}H_{23}\ldots$$

$$\ldots - H_{11}H_{23}^2 - H_{22}H_{13}^2 = 0,$$

from which:

$$\left(\frac{m_1\pi}{a_1}\right)^2 N_1^0 + \left(\frac{m_2\pi}{a_2}\right)^2 N_2^0 = H_{33} + \frac{2H_{12}H_{13}H_{23} - H_{11}H_{23}^2 - H_{22}H_{13}^2}{H_{11}H_{22} - H_{12}^2}.$$

The critical buckling loads are given by:

$$\left(\frac{m_1\pi}{a_1}\right)^2 N_1^0 + \left(\frac{m_2\pi}{a_2}\right)^2 N_2^0 = H_{33} + \frac{R_1}{R_2},$$

with:

$$H_{33} = \left(\frac{m_1\pi}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2 (D_{12} + 2D_{66}) + \left(\frac{m_2\pi}{a_2}\right)^4 D_{22},$$

$$R_1 = 2\left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2 (A_{12} + A_{66})\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + \left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]\ldots$$

$$\ldots\times\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + 3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]\ldots$$

$$\ldots-\left(\frac{m_1\pi}{a_1}\right)^2\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{66}\right]\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + 3\left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]^2\ldots$$

$$\ldots-\left(\frac{m_2\pi}{a_2}\right)^2\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11} + \left(\frac{m_2\pi}{a_2}\right)^2 A_{66}\right]\left[3\left(\frac{m_1\pi}{a_1}\right)^2 B_{16} + \left(\frac{m_2\pi}{a_2}\right)^2 B_{26}\right]^2,$$

$$R_2 = \left[\left(\frac{m_1\pi}{a_1}\right)^4 A_{11} + \left(\frac{m_2\pi}{a_2}\right)^4 A_{22}\right]A_{66} + \left(\frac{m_1\pi}{a_1}\right)^2\left(\frac{m_2\pi}{a_2}\right)^2 [A_{11}A_{22} - A_{12}(A_{12} + 2A_{66})].$$

In the particular case where $N_2^0 = kN_1^0$, we obtain the critical buckling loads:

$$N_1^0 = \frac{\pi^2}{\left(\frac{m_1}{a_1}\right)^2 + k\left(\frac{m_2}{a_2}\right)^2}\left(P_1 + \frac{P_2}{P_2}\right),$$

with:

$$P_1 = \left(\frac{m_1}{a_1}\right)^4 D_{11} + 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2 (D_{12} + 2D_{66}) + \left(\frac{m_2}{a_2}\right)^4 D_{22},$$

$$P_2 = 2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2(A_{12}+A_{66})\left[3\left(\frac{m_1}{a_1}\right)^2 B_{16}+\left(\frac{m_2}{a_2}\right)^2 B_{26}\right]\ldots$$

$$\ldots\times\left[\left(\frac{m_1}{a_1}\right)^2 B_{16}+3\left(\frac{m_2}{a_2}\right)^2 B_{26}\right]\ldots$$

$$\ldots-\left(\frac{m_1}{a_1}\right)^2\left[\left(\frac{m_1}{a_1}\right)^2 A_{11}+\left(\frac{m_2}{a_2}\right)^2 A_{66}\right]\left[\left(\frac{m_1}{a_1}\right)^2 B_{16}+3\left(\frac{m_2}{a_2}\right)^2 B_{26}\right]^2\ldots$$

$$\ldots-\left(\frac{m_2}{a_2}\right)^2\left[\left(\frac{m_1}{a_1}\right)^2 A_{11}+\left(\frac{m_2}{a_2}\right)^2 A_{66}\right]\left[3\left(\frac{m_1}{a_1}\right)^2 B_{16}+\left(\frac{m_2}{a_2}\right)^2 B_{26}\right]^2,$$

$$P_3 = \left[\left(\frac{m_1}{a_1}\right)^4 A_{11}+\left(\frac{m_2}{a_2}\right)^4 A_{22}\right]A_{66}+\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2\left[A_{11}A_{22}-A_{12}(A_{12}+2A_{66})\right].$$

When B_{16} and B_{26} are very small, the critical buckling loads are given by the expression:

$$N_1^0 = \pi^2\frac{\left(\frac{m_1}{a_1}\right)^4 D_{11}+2\left(\frac{m_1}{a_1}\right)^2\left(\frac{m_2}{a_2}\right)^2(D_{12}+2D_{66})+\left(\frac{m_2}{a_2}\right)^4 D_{22}}{\left(\frac{m_1}{a_1}\right)^2+k\left(\frac{m_2}{a_2}\right)^2}.$$

Chapter 10

Cylindrical flexure of multi-layer Kirchhoff-Love plates

10.1. Introduction

An infinitely wide plate is said to be in *cylindrical flexure* when the displacements, strains and stresses are independent of the cartesian co-ordinate x_2.

In this chapter we will study, based on the Kirchhoff-Love theory for which transverse shear strains are neglected, the static, vibration and buckling behaviour of an infinitely wide plate.

10.2. Strain-displacement relationship

In cylindrical flexure the displacement field is of the form:

$$u_1 = u_1^0(x_1|t) + x_3\psi_1(x_1|t),$$
$$u_2 = u_2^0(x_1|t),$$
$$u_3 = u_3^0(x_1|t).$$

and the strain field is written:

$$\varepsilon_1 = \frac{\partial u_1^0}{\partial x_1} + x_3\frac{\partial \psi_1}{\partial x_1}, \quad \varepsilon_6 = \frac{\partial u_2^0}{\partial x_1},$$
$$\varepsilon_2 = 0, \quad \varepsilon_4 = 0,$$
$$\varepsilon_3 = 0, \quad \varepsilon_5 = \psi_1 + \frac{\partial u_3^0}{\partial x_1}.$$

In Kirchhoff-Love theory the transverse shear strain ε_5 is zero, which gives:

$$\psi_1 = -\frac{\partial u_3^0}{\partial x_1}.$$

The displacements are given by the expressions:

$$u_1 = u_1^0(x_1|t) - x_3 \frac{\partial u_3^0}{\partial x_1},$$
$$u_2 = u_2^0(x_1|t),$$
$$u_3 = u_3^0(x_1|t).$$

the strains:

$$\varepsilon_1 = \varepsilon_1^0 + x_3\kappa_1,$$
$$\varepsilon_6 = \varepsilon_6^0,$$
$$\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_5 = 0,$$

involve, as non-zero terms, the membrane strains:

$$\varepsilon_1^0 = \frac{\partial u_1^0}{\partial x_1},$$
$$\varepsilon_6^0 = \frac{\partial u_2^0}{\partial x_1},$$

and the curvature:

$$\kappa_1 = -\frac{\partial^2 u_3^0}{\partial x_1^2}.$$

10.3. Global constitutive relation

This is written:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \frac{\partial u_1^0}{\partial x_1} \\ 0 \\ \frac{\partial u_2^0}{\partial x_1} \\ -\frac{\partial^2 u_3^0}{\partial x_1^2} \\ 0 \\ 0 \end{bmatrix},$$

and gives the global membrane loads:

$$N_1 = A_{11}\frac{\partial u_1^0}{\partial x_1} + A_{16}\frac{\partial u_2^0}{\partial x_1} - B_{11}\frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$N_2 = A_{12}\frac{\partial u_1^0}{\partial x_1} + A_{26}\frac{\partial u_2^0}{\partial x_1} - B_{12}\frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$N_6 = A_{16}\frac{\partial u_1^0}{\partial x_1} + A_{66}\frac{\partial u_2^0}{\partial x_1} - B_{16}\frac{\partial^2 u_3^0}{\partial x_1^2},$$

and flexure:

$$M_1 = B_{11}\frac{\partial u_1^0}{\partial x_1} + B_{16}\frac{\partial u_2^0}{\partial x_1} - D_{11}\frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$M_2 = B_{12}\frac{\partial u_1^0}{\partial x_1} + B_{26}\frac{\partial u_2^0}{\partial x_1} - D_{12}\frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$M_6 = B_{16}\frac{\partial u_1^0}{\partial x_1} + B_{66}\frac{\partial u_2^0}{\partial x_1} - D_{16}\frac{\partial^2 u_3^0}{\partial x_1^2}.$$

10.4. Global plate equations

In the case of cylindrical flexure the global equations:

$$\frac{\partial N_1}{\partial x_1} + p_1 = I_0\frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{\partial N_6}{\partial x_1} + p_2 = I_0\frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial^2 M_1}{\partial x_1^2} + \frac{\partial}{\partial x_1}\left(N_1\frac{\partial u_3^0}{\partial x_1}\right) + q_3 + p_3 = I_0\frac{\partial^2 u_3^0}{\partial t^2},$$

give the following expressions:

– static:

$$\frac{dN_1}{dx_1} + p_1 = 0,$$

$$\frac{dN_6}{dx_1} + p_2 = 0,$$

$$\frac{d^2 M_1}{dx_1^2} + q_3 + p_3 = 0,$$

–vibration:

$$\frac{\partial N_1}{\partial x_1} = I_0\frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{\partial N_6}{\partial x_1} = I_0 \frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial^2 M_1}{\partial x_1^2} = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

– buckling:

$$\frac{dN_1}{dx_1} = 0,$$

$$\frac{dN_6}{dx_1} = 0,$$

$$\frac{d^2 M_1}{dx_1^2} - N_1^0 \frac{d^2 u_3^0}{dx_1^2} = 0.$$

10.5. Flexure

In the particular case where the volume loads are zero, by putting into the global equilibrium equations:

$$\frac{dN_1}{dx_1} = 0,$$

$$\frac{dN_6}{dx_1} = 0,$$

$$\frac{d^2 M_1}{dx_1^2} + q = 0,$$

the global cohesion loads:

$$N_1 = A_{11} \frac{du_1^0}{dx_1} + A_{16} \frac{du_2^0}{dx_1} - B_{11} \frac{d^2 u_3^0}{dx_1^2},$$

$$N_6 = A_{16} \frac{du_1^0}{dx_1} + A_{66} \frac{du_2^0}{dx_1} - B_{16} \frac{d^2 u_3^0}{dx_1^2},$$

$$M_1 = B_{11} \frac{du_1^0}{dx_1} + B_{16} \frac{du_2^0}{dx_1} - D_{11} \frac{d^2 u_3^0}{dx_1^2},$$

we obtain the three equations:

$$A_{11} \frac{d^2 u_1^0}{dx_1^2} + A_{16} \frac{d^2 u_2^0}{dx_1^2} - B_{11} \frac{d^3 u_3^0}{dx_1^3} = 0,$$

$$A_{16}\frac{d^2u_1^0}{dx_1^2}+A_{66}\frac{d^2u_2^0}{dx_1^2}-B_{16}\frac{d^3u_3^0}{dx_1^3}=0,$$

$$D_{11}\frac{d^4u_3^0}{dx_1^4}-B_{11}\frac{d^3u_1^0}{dx_1^3}-B_{16}\frac{d^3u_2^0}{dx_1^3}=q\,.$$

10.5.1. *Elimination of u_1^0 and u_2^0*

The first two equations, which have just been written, enable $\dfrac{d^2u_1^0}{dx_1^2}$ and $\dfrac{d^2u_2^0}{dx_1^2}$ to be expressed as a function of $\dfrac{d^3u_3^0}{dx_1^3}$, with the help of the two expressions:

$$\frac{d^2u_1^0}{dx_1^2}=\frac{A_{66}B_{11}-A_{16}B_{16}}{A_{11}A_{66}-A_{16}^2}\frac{d^3u_3^0}{dx_1^3},$$

$$\frac{d^2u_2^0}{dx_2^2}=\frac{A_{11}B_{16}-A_{16}B_{11}}{A_{11}A_{66}-A_{16}^2}\frac{d^3u_3^0}{dx_1^3}.$$

Putting these into the third equation, we obtain the expression:

$$\left(D_{11}-B_{11}\frac{A_{66}B_{11}-A_{16}B_{16}}{A_{11}A_{66}-A_{16}^2}-B_{16}\frac{A_{11}B_{16}-A_{16}B_{11}}{A_{11}A_{66}-A_{16}^2}\right)\frac{d^4u_3^0}{dx_1^4}=q,$$

which has the form:

$$D\frac{d^4u_3^0}{dx_1^4}=q,$$

with:

$$D=D_{11}-\frac{A_{66}B_{11}^2+A_{11}B_{16}^2-2A_{16}B_{11}B_{16}}{A_{11}A_{66}-A_{16}^2}.$$

The integration of this equation and taking into account the boundary conditions allows us to determine u_3^0. u_1^0 and u_2^0 are then found by integration of:

$$\frac{d^2u_1^0}{dx_1^2}=\frac{A_{66}B_{11}-A_{16}B_{16}}{A_{11}A_{66}-A_{16}^2}\frac{d^3u_3^0}{dx_1^3},$$

$$\frac{d^2u_2^0}{dx_2^2}=\frac{A_{11}B_{16}-A_{16}B_{11}}{A_{11}A_{66}-A_{16}^2}\frac{d^3u_3^0}{dx_1^3}.$$

10.5.2. *Simply supported plate subjected to a sinusoidal load*

10.5.2.1. *General case*

The plate is simply supported on two edges $x_1 = 0$ and $x_1 = a_1$, and is subjected to the loading:

$$q(x_1) = q_{m_1} \sin \frac{m_1 \pi x_1}{a_1}.$$

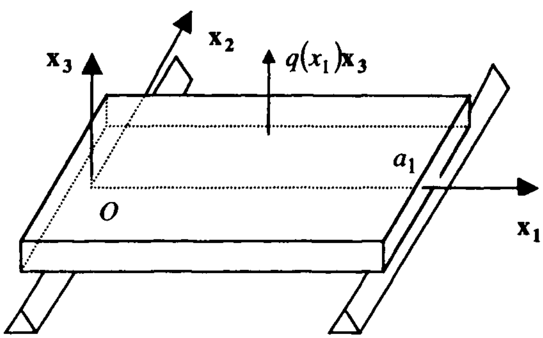

Figure 10.1. *Plate under cylindrical flexure*

The boundary conditions at the edges $x_1 = 0$ and $x_1 = a_1$ are:

$$u_3^0 = 0,$$

$$N_1 = A_{11} \frac{du_1^0}{dx_1} + A_{16} \frac{du_2^0}{dx_1} - B_{11} \frac{d^2 u_3^0}{dx_1^2} = 0,$$

$$N_6 = A_{16} \frac{du_1^0}{dx_1} + A_{66} \frac{du_2^0}{dx_1} - B_{16} \frac{d^2 u_3^0}{dx_1^2} = 0,$$

$$M_1 = B_{11} \frac{du_1^0}{dx_1} + B_{16} \frac{du_2^0}{dx_1} - D_{11} \frac{d^2 u_3^0}{dx_1^2} = 0.$$

The displacement field defined by:

$$u_1^0 = U_{m_1}^1 \cos \frac{m_1 \pi x_1}{a_1},$$

$$u_2^0 = U_{m_1}^2 \cos \frac{m_1 \pi x_1}{a_1},$$

$$u_3^0 = U_{m_1}^3 \sin \frac{m_1 \pi x_1}{a_1},$$

satisfies the previous conditions as well as the global equilibrium equations. By introducing these into the latter we obtain the system:

$$\left[-\left(\frac{m_1\pi}{a_1}\right)^2 A_{11}U^1_{m_1} - \left(\frac{m_1\pi}{a_1}\right)^2 A_{16}U^2_{m_1} + \left(\frac{m_1\pi}{a_1}\right)^3 B_{11}U^3_{m_1}\right]\cos\frac{m_1\pi x_1}{a_1} = 0,$$

$$\left[-\left(\frac{m_1\pi}{a_1}\right)^2 A_{16}U^1_{m_1} - \left(\frac{m_1\pi}{a_1}\right)^2 A_{66}U^2_{m_1} + \left(\frac{m_1\pi}{a_1}\right)^3 B_{16}U^3_{m_1}\right]\cos\frac{m_1\pi x_1}{a_1} = 0,$$

$$\left[-\left(\frac{m_1\pi}{a_1}\right)^3 B_{11}U^1_{m_1} - \left(\frac{m_1\pi}{a_1}\right)^3 B_{16}U^2_{m_1} + \left(\frac{m_1\pi}{a_1}\right)^4 D_{11}U^3_{m_1} - q_{m_1}\right]\sin\frac{m_1\pi x_1}{a_1} = 0,$$

which, after simplification, can be written in matrix form:

$$\begin{bmatrix} \left(\frac{m_1\pi}{a_1}\right)^2 A_{11} & \left(\frac{m_1\pi}{a_1}\right)^2 A_{16} & -\left(\frac{m_1\pi}{a_1}\right)^3 B_{11} \\ \left(\frac{m_1\pi}{a_1}\right)^2 A_{16} & \left(\frac{m_1\pi}{a_1}\right)^2 A_{66} & -\left(\frac{m_1\pi}{a_1}\right)^3 B_{16} \\ -\left(\frac{m_1\pi}{a_1}\right)^3 B_{11} & -\left(\frac{m_1\pi}{a_1}\right)^3 B_{16} & \left(\frac{m_1\pi}{a_1}\right)^4 D_{11} \end{bmatrix} \begin{bmatrix} U^1_{m_1} \\ U^2_{m_1} \\ U^3_{m_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ q_{m_1} \end{bmatrix}.$$

The determinant of this system:

$$\Delta = \left(\frac{m_1\pi}{a_1}\right)^8 \left(A_{11}A_{66}D_{11} + 2A_{16}B_{11}B_{16} - A_{66}B_{11}^2 - A_{11}B_{16}^2 - A_{16}^2 D_{11}\right),$$

as well as the determinants:

$$\Delta_1 = \left(\frac{m_1\pi}{a_1}\right)^5 \left(A_{66}B_{11} - A_{16}B_{16}\right)q_{m_1},$$

$$\Delta_2 = \left(\frac{m_1\pi}{a_1}\right)^5 \left(A_{11}B_{16} - A_{16}B_{11}\right)q_{m_1},$$

$$\Delta_3 = \left(\frac{m_1\pi}{a_1}\right)^4 \left(A_{11}A_{66} - A_{16}^2\right)q_{m_1},$$

give the values of the constants $U^i_{m_1}$:

$$U^1_{m_1} = \left(\frac{a_1}{m_1\pi}\right)^3 \frac{\left(A_{66}B_{11} - A_{16}B_{16}\right)q_{m_1}}{\left(A_{11}A_{66} - A_{16}^2\right)D_{11} + 2A_{16}B_{11}B_{16} - A_{66}B_{11}^2 - A_{11}B_{16}^2},$$

$$U_{m_1}^2 = \left(\frac{a_1}{m_1\pi}\right)^3 \frac{\left(A_{11}B_{16} - A_{16}B_{11}\right)q_{m_1}}{\left(A_{11}A_{66} - A_{16}^2\right)D_{11} + 2A_{16}B_{11}B_{16} - A_{66}B_{11}^2 - A_{11}B_{16}^2},$$

$$U_{m_1}^3 = \left(\frac{a_1}{m_1\pi}\right)^4 \frac{\left(A_{11}A_{66} - A_{16}^2\right)q_{m_1}}{\left(A_{11}A_{66} - A_{16}^2\right)D_{11} + 2A_{16}B_{11}B_{16} - A_{66}B_{11}^2 - A_{11}B_{16}^2}.$$

the maximum deflection, obtained at $x_1 = \frac{a_1}{2}$, is equal to:

$$u_3^0\left(\frac{a_1}{2}\right) = \left(\frac{a_1}{m_1\pi}\right)^4 \frac{\left(A_{11}A_{66} - A_{16}^2\right)q_{m_1}}{\left(A_{11}A_{66} - A_{16}^2\right)D_{11} + 2A_{16}B_{11}B_{16} - A_{66}B_{11}^2 - A_{11}B_{16}^2}.$$

10.5.2.2. *Asymmetrical cross-ply composite* $\left(0, \frac{\pi}{2}\right)_N$.

The following global stiffnesses being zero:

$$A_{16} = A_{26} = 0, \quad D_{16} = D_{26} = 0,$$
$$B_{16} = B_{26} = B_{12} = B_{66} = 0,$$

we obtain:

$$U_{m_1}^1 = \left(\frac{a_1}{m_1\pi}\right)^3 \frac{B_{11}q_{m_1}}{A_{11}D_{11} - B_{11}^2},$$

$$U_{m_1}^2 = 0,$$

$$U_{m_1}^3 = \left(\frac{a_1}{m_1\pi}\right)^4 \frac{A_{11}q_{m_1}}{A_{11}D_{11} - B_{11}^2}.$$

10.5.2.3. *Asymmetrical balanced composite* $(\alpha, -\alpha)_N$.

The following global stiffnesses being zero:

$$A_{16} = A_{26} = 0, \quad D_{16} = D_{26} = 0,$$
$$B_{11} = B_{12} = B_{22} = B_{66} = 0,$$

we have:

$$U_{m_1}^1 = 0,$$

$$U_{m_1}^2 = \left(\frac{a_1}{m_1\pi}\right)^3 \frac{B_{16}q_{m_1}}{A_{66}D_{11} - B_{16}^2},$$

$$U_{m_1}^3 = \left(\frac{a_1}{m_1\pi}\right)^4 \frac{A_{66}q_{m_1}}{A_{66}D_{11} - B_{16}^2}.$$

10.5.2.4. *Symmetrical composite*

As the membrane-flexure global coupling stiffnesses are zero:

$$B_{ij} = 0,$$

we obtain:

$$U_{m_1}^1 = 0,$$

$$U_{m_1}^2 = 0,$$

$$U_{m_1}^3 = \left(\frac{a_1}{m_1\pi}\right)^4 \frac{q_{m_1}}{D_{11}}.$$

10.5.2.5. *Displacement field*

Given the values of $U_{m_1}^1$, $U_{m_1}^2$ and $U_{m_1}^3$ obtained previously, we have the following expressions for the displacements:

$$u_1 = \left(U_{m_1}^1 - x_3\frac{m_1\pi}{a_1}U_{m_1}^3\right)\cos\frac{m_1\pi x_1}{a_1},$$

$$u_2 = U_{m_1}^2 \cos\frac{m_1\pi x_1}{a_1},$$

$$u_3 = U_{m_1}^3 \sin\frac{m_1\pi x_1}{a_1}.$$

10.5.2.6. *Strain field*

The non-zero strains are then given by the following expressions:

$$\varepsilon_1 = -\frac{m_1\pi}{a_1}\left(U_{m_1}^1 - x_3\frac{m_1\pi}{a_1}U_{m_1}^3\right)\sin\frac{m_1\pi x_1}{a_1},$$

$$\varepsilon_6 = -\frac{m_1\pi}{a_1}U_{m_1}^2 \sin\frac{m_1\pi x_1}{a_1}.$$

10.5.2.7. *Stress field*

The stresses in layer k are given by:

$$\sigma_i^k = Q_{ij}^k \varepsilon_j \quad (i, j = 1, 2, 6),$$

where the Q_{ij}^k are the reduced stiffnesses of layer k. We then obtain:

$$\sigma_i^k = -\frac{m_1 \pi}{a_1}\left[Q_{i1}^k\left(U_{m_1}^1 - x_3 \frac{m_1 \pi}{a_1} U_{m_1}^3\right) + Q_{i6}^k U_{m_1}^2\right]\sin\frac{m_1 \pi x_1}{a_1} \quad (i = 1, 2, 6).$$

10.6. Vibrations

10.6.1. *General case*

The plate studied is supported on two supports separated by a_1.

Putting into the global vibration equations:

$$\frac{\partial N_1}{\partial x_1} = I_0 \frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{\partial N_6}{\partial x_1} = I_0 \frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial^2 M_1}{\partial x_1^2} = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

the global cohesion loads:

$$N_1 = A_{11} \frac{\partial u_1^0}{\partial x_1} + A_{16} \frac{\partial u_2^0}{\partial x_1} - B_{11} \frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$N_6 = A_{16} \frac{\partial u_1^0}{\partial x_1} + A_{66} \frac{\partial u_2^0}{\partial x_1} - B_{16} \frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$M_1 = B_{11} \frac{\partial u_1^0}{\partial x_1} + B_{16} \frac{\partial u_2^0}{\partial x_1} - D_{11} \frac{\partial^2 u_3^0}{\partial x_1^2},$$

we have three equations of motion:

$$A_{11} \frac{\partial^2 u_1^0}{\partial x_1^2} + A_{16} \frac{\partial^2 u_2^0}{\partial x_1^2} - B_{11} \frac{\partial^3 u_3^0}{\partial x_1^3} = I_0 \frac{\partial^2 u_1^0}{\partial t^2},$$

$$A_{16} \frac{\partial^2 u_1^0}{\partial x_1^2} + A_{66} \frac{\partial^2 u_2^0}{\partial x_1^2} - B_{16} \frac{\partial^3 u_3^0}{\partial x_1^3} = I_0 \frac{\partial^2 u_2^0}{\partial t^2},$$

$$B_{11}\frac{\partial^3 u_1^0}{\partial x_1^3}+B_{16}\frac{\partial^3 u_2^0}{\partial x_1^3}-D_{11}\frac{\partial^4 u_3^0}{\partial x_1^4}=I_0\frac{\partial^2 u_3^0}{\partial t^2}.$$

The solution, which satisfies the boundary conditions at $x_1=0$ and $x_1=a_1$:

$$u_3^0=0,$$

$$N_1=A_{11}\frac{du_1^0}{dx_1}+A_{16}\frac{du_2^0}{dx_1}-B_{11}\frac{d^2u_3^0}{dx_1^2}=0,$$

$$N_6=A_{16}\frac{du_1^0}{dx_1}+A_{66}\frac{du_2^0}{dx_1}-B_{16}\frac{d^2u_3^0}{dx_1^2}=0,$$

$$M_1=B_{11}\frac{du_1^0}{dx_1}+B_{16}\frac{du_2^0}{dx_1}-D_{11}\frac{d^2u_3^0}{dx_1^2}=0.$$

is of the form:

$$u_1^0=U_{m_1}^1\cos\frac{m_1\pi x_1}{a_1}\sin\left(\omega_{m_1}t+\varphi_{m_1}\right),$$

$$u_2^0=U_{m_1}^2\cos\frac{m_1\pi x_1}{a_1}\sin\left(\omega_{m_1}t+\varphi_{m_1}\right),$$

$$u_3^0=U_{m_1}^3\sin\frac{m_1\pi x_1}{a_1}\sin\left(\omega_{m_1}t+\varphi_{m_1}\right).$$

Putting these expressions into the global equations of motion and after simplification, we obtain the system:

$$-\left(\frac{m_1\pi}{a_1}\right)^2A_{11}U_{m_1}^1-\left(\frac{m_1\pi}{a_1}\right)^2A_{16}U_{m_1}^2+\left(\frac{m_1\pi}{a_1}\right)^3B_{11}U_{m_1}^3+I_0\omega_{m_1}^2U_{m_1}^1=0,$$

$$-\left(\frac{m_1\pi}{a_1}\right)^2A_{16}U_{m_1}^1-\left(\frac{m_1\pi}{a_1}\right)^2A_{66}U_{m_1}^2+\left(\frac{m_1\pi}{a_1}\right)^3B_{16}U_{m_1}^3+I_0\omega_{m_1}^2U_{m_1}^2=0,$$

$$-\left(\frac{m_1\pi}{a_1}\right)^3B_{11}U_{m_1}^1-\left(\frac{m_1\pi}{a_1}\right)^3B_{16}U_{m_1}^2+\left(\frac{m_1\pi}{a_1}\right)^4D_{11}U_{m_1}^3-I_0\omega_{m_1}^2U_{m_1}^3=0,$$

which is written in the following matrix form:

$$\begin{bmatrix}\left(\frac{m_1\pi}{a_1}\right)^2A_{11}-I_0\omega_{m_1}^2 & \left(\frac{m_1\pi}{a_1}\right)^2A_{16} & -\left(\frac{m_1\pi}{a_1}\right)^3B_{11}\\ \left(\frac{m_1\pi}{a_1}\right)^2A_{16} & \left(\frac{m_1\pi}{a_1}\right)^2A_{66}-I_0\omega_{m_1}^2 & -\left(\frac{m_1\pi}{a_1}\right)^3B_{16}\\ -\left(\frac{m_1\pi}{a_1}\right)^3B_{11} & -\left(\frac{m_1\pi}{a_1}\right)^3B_{16} & \left(\frac{m_1\pi}{a_1}\right)^4D_{11}-I_0\omega_{m_1}^2\end{bmatrix}\begin{bmatrix}U_{m_1}^1\\U_{m_1}^2\\U_{m_1}^3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

This system has a solution other than the trivial solution $U^1_{m_1} = U^2_{m_1} = U^3_{m_1} = 0$ for the values of ω_{m_1} which cancel its determinant, that is for the ω_{m_1} solution to the equation:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11} - I_0\omega^2_{m_1}\right]\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{66} - I_0\omega^2_{m_1}\right]\left[\left(\frac{m_1\pi}{a_1}\right)^4 D_{11} - I_0\omega^2_{m_1}\right]\ldots$$

$$\ldots + 2\left(\frac{m_1\pi}{a_1}\right)^8 A_{16}B_{11}B_{16} - \left(\frac{m_1\pi}{a_1}\right)^4 A^2_{16}\left[\left(\frac{m_1\pi}{a_1}\right)^4 D_{11} - I_0\omega^2_{m_1}\right]\ldots$$

$$\ldots - \left(\frac{m_1\pi}{a_1}\right)^6 B^2_{11}\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{66} - I_0\omega^2_{m_1}\right] - \left(\frac{m_1\pi}{a_1}\right)^6 B^2_{16}\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11} - I_0\omega^2_{m_1}\right] = 0,$$

of the form:

$$-A\omega^6_{m_1m_2} + B\omega^4_{m_1m_2} - C\omega^2_{m_1m_2} + D = 0,$$

with:

$$A = I_0^3,$$

$$B = \left(\frac{m_1\pi}{a_1}\right)^2\left[A_{11} + A_{66} + \left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right]I_0^2,$$

$$C = \left(\frac{m_1\pi}{a_1}\right)^4\left\{A_{11}A_{66} - A^2_{16} + \left(\frac{m_1\pi}{a_1}\right)^2\left[(A_{11} + A_{66})D_{11} - B^2_{11} - B^2_{16}\right]\right\}I_0,$$

$$D = \left(\frac{m_1\pi}{a_1}\right)^8\left[\left(A_{11}A_{66} - A^2_{16}\right)D_{11} + 2A_{16}B_{11}B_{16} - A_{66}B^2_{11} - A_{11}B^2_{16}\right].$$

For each value of m_1, we obtain three natural frequencies.

10.6.2. *Asymmetrical cross-ply composite* $\left(0, \frac{\pi}{2}\right)_N$.

We have:

$$A_{16} = A_{26} = 0, \quad D_{16} = D_{26} = 0,$$

$$B_{16} = B_{26} = B_{12} = B_{66} = 0.$$

The coefficients A, B, C and D are equal to:

$$A = I_0^3,$$

$$B=\left(\frac{m_1\pi}{a_1}\right)^2\left[A_{11}+A_{66}+\left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right]I_0^2,$$

$$C=\left(\frac{m_1\pi}{a_1}\right)^4\left\{A_{11}A_{66}+\left(\frac{m_1\pi}{a_1}\right)^2\left[(A_{11}+A_{66})D_{11}-B_{11}^2\right]\right\}I_0,$$

$$D=\left(\frac{m_1\pi}{a_1}\right)^8\left(A_{11}D_{11}-B_{11}^2\right)A_{66}.$$

In this particular case, the equation for the natural frequencies is:

$$\begin{vmatrix} \left(\frac{m_1\pi}{a_1}\right)^2 A_{11}-I_0\omega_{m_1}^2 & 0 & -\left(\frac{m_1\pi}{a_1}\right)^3 B_{11} \\ 0 & \left(\frac{m_1\pi}{a_1}\right)^2 A_{66}-I_0\omega_{m_1}^2 & 0 \\ -\left(\frac{m_1\pi}{a_1}\right)^3 B_{11} & 0 & \left(\frac{m_1\pi}{a_1}\right)^4 D_{11}-I_0\omega_{m_1}^2 \end{vmatrix}=0,$$

or:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{66}-I_0\omega_{m_1}^2\right]\left\{I_0^2\omega_{m_1}^4-\left(\frac{m_1\pi}{a_1}\right)^2\left[A_{11}+\left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right]I_0\omega_{m_1}^2\dots\right.$$

$$\left.\dots+\left(\frac{m_1\pi}{a_1}\right)^6\left(A_{11}D_{11}-B_{11}^2\right)\right\}=0.$$

The discriminant of the equation in $I_0\omega_{m_1}^2$:

$$\Delta=\left(\frac{m_1\pi}{a_1}\right)^4\left\{\left[A_{11}+\left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right]^2-4\left(\frac{m_1\pi}{a_1}\right)^2\left(A_{11}D_{11}-B_{11}^2\right)\right\},$$

$$\Delta=\left(\frac{m_1\pi}{a_1}\right)^4\left\{\left[A_{11}-\left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right]^2+4\left(\frac{m_1\pi}{a_1}\right)^2 B_{11}^2\right\},$$

is always positive.

So we obtain, for each value of m_1, the following three natural frequencies:

$$\omega_{m_1,1}^2=\frac{1}{2I_0}\left(\frac{m_1\pi}{a_1}\right)^2\left\{A_{11}+\left(\frac{m_1\pi}{a_1}\right)^2 D_{11}-\sqrt{\left[A_{11}-\left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right]^2+4\left(\frac{m_1\pi}{a_1}\right)^2 B_{11}^2}\right\},$$

$$\omega_{m_1,2}^2=\frac{1}{2I_0}\left(\frac{m_1\pi}{a_1}\right)^2\left\{A_{11}+\left(\frac{m_1\pi}{a_1}\right)^2 D_{11}+\sqrt{\left[A_{11}-\left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right]^2+4\left(\frac{m_1\pi}{a_1}\right)^2 B_{11}^2}\right\},$$

$$\omega_{m_1,3}^2=\frac{1}{I_0}\left(\frac{m_1\pi}{a_1}\right)^2 A_{66}.$$

10.6.3. *Balanced asymmetrical composite* $(\alpha,-\alpha)_N$.

We have:

$$A_{16}=A_{26}=0,\quad D_{16}=D_{26}=0,$$
$$B_{11}=B_{12}=B_{22}=B_{66}=0.$$

The coefficients A, B, C and D are equal to:

$$A=I_0^3,$$

$$B=\left(\frac{m_1\pi}{a_1}\right)^2\left[A_{11}+A_{66}+\left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right]I_0^2,$$

$$C=\left(\frac{m_1\pi}{a_1}\right)^4\left\{A_{11}A_{66}+\left(\frac{m_1\pi}{a_1}\right)^2\left[(A_{11}+A_{66})D_{11}-B_{16}^2\right]\right\}I_0,$$

$$D=\left(\frac{m_1\pi}{a_1}\right)^8\left(A_{66}D_{11}-B_{16}^2\right)A_{11}.$$

In this case the determinant is written:

$$\begin{vmatrix}\left(\frac{m_1\pi}{a_1}\right)^2 A_{11}-I_0\omega_{m_1}^2 & 0 & 0\\ 0 & \left(\frac{m_1\pi}{a_1}\right)^2 A_{66}-I_0\omega_{m_1}^2 & -\left(\frac{m_1\pi}{a_1}\right)^3 B_{16}\\ 0 & -\left(\frac{m_1\pi}{a_1}\right)^3 B_{16} & \left(\frac{m_1\pi}{a_1}\right)^4 D_{11}-I_0\omega_{m_1}^2\end{vmatrix}=0,$$

from which:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11} - I_0\omega_{m_1}^2\right]\left\{I_0^2\omega_{m_1}^4 - \left(\frac{m_1\pi}{a_1}\right)^2\left[A_{66} + \left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right]I_0\omega_{m_1}^2 \ldots\right.$$

$$\left.\ldots + \left(\frac{m_1\pi}{a_1}\right)^6\left(A_{66}D_{11} - B_{16}^2\right)\right\} = 0.$$

For each value of m_1, we obtain the following three natural frequencies:

$$\omega_{m_1,1}^2 = \frac{1}{2I_0}\left(\frac{m_1\pi}{a_1}\right)^2\left\{A_{66} + \left(\frac{m_1\pi}{a_1}\right)^2 D_{11} - \sqrt{\left[A_{66} - \left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right]^2 + 4\left(\frac{m_1\pi}{a_1}\right)^2 B_{16}^2}\right\},$$

$$\omega_{m_1,2}^2 = \frac{1}{2I_0}\left(\frac{m_1\pi}{a_1}\right)^2\left\{A_{66} + \left(\frac{m_1\pi}{a_1}\right)^2 D_{11} + \sqrt{\left[A_{66} - \left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right]^2 + 4\left(\frac{m_1\pi}{a_1}\right)^2 B_{16}^2}\right\},$$

$$\omega_{m_1,3}^2 = \frac{1}{I_0}\left(\frac{m_1\pi}{a_1}\right)^2 A_{11}.$$

10.6.4. *Symmetrical composite*

We have:

$$B_{ij} = 0.$$

The coefficients A, B, C and D have the values:

$$A = I_0^3,$$

$$B = \left(\frac{m_1\pi}{a_1}\right)^2\left[A_{11} + A_{66} + \left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right]I_0^2,$$

$$C = \left(\frac{m_1\pi}{a_1}\right)^4\left\{A_{11}A_{66} - A_{16}^2 + \left(\frac{m_1\pi}{a_1}\right)^2\left(A_{11} + A_{66}\right)D_{11}\right\}I_0,$$

$$D = \left(\frac{m_1\pi}{a_1}\right)^8\left(A_{11}A_{66} - A_{16}^2\right)D_{11}.$$

and the equation for the natural frequencies is written :

$$\begin{vmatrix} \left(\frac{m_1\pi}{a_1}\right)^2 A_{11} - I_0\omega_{m_1}^2 & \left(\frac{m_1\pi}{a_1}\right)^2 A_{16} & 0 \\ \left(\frac{m_1\pi}{a_1}\right)^2 A_{16} & \left(\frac{m_1\pi}{a_1}\right)^2 A_{66} - I_0\omega_{m_1}^2 & 0 \\ 0 & 0 & \left(\frac{m_1\pi}{a_1}\right)^4 D_{11} - I_0\omega_{m_1}^2 \end{vmatrix} = 0,$$

or:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^4 D_{11} - I_0\omega_{m_1}^2\right]\left\{ I_0^2\omega_{m_1}^4 - \left(\frac{m_1\pi}{a_1}\right)^2 (A_{11} + A_{66}) I_0\omega_{m_1}^2 \dots\right.$$

$$\left.\dots + \left(\frac{m_1\pi}{a_1}\right)^4 \left(A_{11}A_{66} - A_{16}^2\right)\right\} = 0.$$

For fixed m_1, the three natural frequencies are:

$$\omega_{m_1,1}^2 = \frac{1}{2I_0}\left(\frac{m_1\pi}{a_1}\right)^2 \left[A_{11} + A_{66} + \sqrt{(A_{11} - A_{66})^2 + 4A_{16}^2}\right],$$

$$\omega_{m_1,2}^2 = \frac{1}{2I_0}\left(\frac{m_1\pi}{a_1}\right)^2 \left[A_{11} + A_{66} - \sqrt{(A_{11} - A_{66})^2 + 4A_{16}^2}\right],$$

$$\omega_{m_1,3}^2 = \frac{1}{I_0}\left(\frac{m_1\pi}{a_1}\right)^4 D_{11}.$$

If the composite shows tension-shear decoupling we have:

$$A_{16} = A_{26} = 0,$$

and:

$$\omega_{m_1,1}^2 = \left(\frac{m_1\pi}{a_1}\right)^2 \frac{A_{11}}{I_0},$$

$$\omega_{m_1,2}^2 = \left(\frac{m_1\pi}{a_1}\right)^2 \frac{A_{66}}{I_0},$$

$$\omega_{m_1,3}^2 = \left(\frac{m_1\pi}{a_1}\right)^4 \frac{D_{11}}{I_0}.$$

For an isotropic mono-layer plate we have the following particular values:

$$A_{11} = Q_{11}h = \frac{Eh}{1-\nu^2},$$

$$A_{66} = Q_{66}h = \frac{Eh}{2(1+\nu)},$$

$$D_{11} = \frac{Eh^3}{12\left(1-\nu^2\right)},$$

$$I_0 = \rho h.$$

The natural frequencies are written:

$$\omega_{m_1,1} = \frac{m_1\pi}{a_1}\sqrt{\frac{E}{\rho\left(1-\nu^2\right)}},$$

$$\omega_{m_1,2} = \frac{m_1\pi}{a_1}\sqrt{\frac{E}{2\rho(1+\nu)}},$$

$$\omega_{m_1,3} = h\left(\frac{m_1\pi}{a_1}\right)^2\sqrt{\frac{E}{12\rho\left(1-\nu^2\right)}}.$$

10.7. Buckling

10.7.1. *General case*

The plate studied lies on two simple supports $x_1 = 0$ and $x_1 = a_1$, and is only subjected to compression loading $-N_1^0$, with $N_1^0 > 0$.

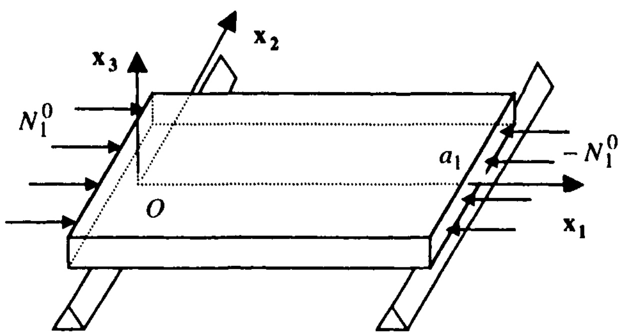

Figure 10.2. *Plate subjected to buckling*

Introducing into the global buckling equations:

$$\frac{dN_1}{dx_1} = 0,$$

$$\frac{dN_6}{dx_1}=0,$$

$$\frac{d^2M_1}{dx_1^2}-N_1^0\frac{d^2u_3^0}{dx_1^2}=0,$$

the global cohesion loads:

$$N_1=A_{11}\frac{du_1^0}{dx_1}+A_{16}\frac{du_2^0}{dx_1}-B_{11}\frac{d^2u_3^0}{dx_1^2},$$

$$N_6=A_{16}\frac{du_1^0}{dx_1}+A_{66}\frac{du_2^0}{dx_1}-B_{16}\frac{d^2u_3^0}{dx_1^2},$$

$$M_1=B_{11}\frac{du_1^0}{dx_1}+B_{16}\frac{du_2^0}{dx_1}-D_{11}\frac{d^2u_3^0}{dx_1^2},$$

we arrive at the three equations:

$$A_{11}\frac{d^2u_1^0}{dx_1^2}+A_{16}\frac{d^2u_2^0}{dx_1^2}-B_{11}\frac{d^3u_3^0}{dx_1^3}=0,$$

$$A_{16}\frac{d^2u_1^0}{dx_1^2}+A_{66}\frac{d^2u_2^0}{dx_1^2}-B_{16}\frac{d^3u_3^0}{dx_1^3}=0,$$

$$B_{11}\frac{d^3u_1^0}{dx_1^3}+B_{16}\frac{d^3u_2^0}{dx_1^3}-D_{11}\frac{d^4u_3^0}{dx_1^4}-N_1^0\frac{d^2u_3^0}{dx_1^2}=0.$$

The displacement field:

$$u_1^0=U_{m_1}^1\cos\frac{m_1\pi x_1}{a_1},$$

$$u_2^0=U_{m_1}^2\cos\frac{m_1\pi x_1}{a_1},$$

$$u_3^0=U_{m_1}^3\sin\frac{m_1\pi x_1}{a_1},$$

satisfies the previous equations as well as the boundary conditions at $x_1=0$ and $x_1=a_1$:

$$u_3^0=0,$$

$$N_1=A_{11}\frac{du_1^0}{dx_1}+A_{16}\frac{du_2^0}{dx_1}-B_{11}\frac{d^2u_3^0}{dx_1^2}=0,$$

$$N_6=A_{16}\frac{du_1^0}{dx_1}+A_{66}\frac{du_2^0}{dx_1}-B_{16}\frac{d^2u_3^0}{dx_1^2}=0,$$

$$M_1 = B_{11}\frac{du_1^0}{dx_1} + B_{16}\frac{du_2^0}{dx_1} - D_{11}\frac{d^2u_3^0}{dx_1^2} = 0.$$

Introducing this into the global buckling equations we obtain the set of equations:

$$\left[-\left(\frac{m_1\pi}{a_1}\right)^2 A_{11}U^1_{m_1} - \left(\frac{m_1\pi}{a_1}\right)^2 A_{16}U^2_{m_1} + \left(\frac{m_1\pi}{a_1}\right)^3 B_{11}U^3_{m_1}\right]\cos\frac{m_1\pi x_1}{a_1} = 0,$$

$$\left[-\left(\frac{m_1\pi}{a_1}\right)^2 A_{16}U^1_{m_1} - \left(\frac{m_1\pi}{a_1}\right)^2 A_{66}U^2_{m_1} + \left(\frac{m_1\pi}{a_1}\right)^3 B_{16}U^3_{m_1}\right]\cos\frac{m_1\pi x_1}{a_1} = 0,$$

$$\left[\left(\frac{m_1\pi}{a_1}\right)^3 B_{11}U^1_{m_1} + \left(\frac{m_1\pi}{a_1}\right)^3 B_{16}U^2_{m_1} - \left(\frac{m_1\pi}{a_1}\right)^4 D_{11}U^3_{m_1} \cdots\right.$$
$$\left. \cdots + \left(\frac{m_1\pi}{a_1}\right)^2 N_1^0 U^3_{m_1}\right]\sin\frac{m_1\pi x_1}{a_1} = 0,$$

which, after simplification, can be presented in matrix form:

$$\begin{bmatrix} A_{11} & A_{16} & -\frac{m_1\pi}{a_1}B_{11} \\ A_{16} & A_{66} & -\frac{m_1\pi}{a_1}B_{16} \\ -\frac{m_1\pi}{a_1}B_{11} & -\frac{m_1\pi}{a_1}B_{16} & \left(\frac{m_1\pi}{a_1}\right)^2 D_{11} - N_1^0 \end{bmatrix} \begin{bmatrix} U^1_{m_1} \\ U^2_{m_1} \\ U^3_{m_1} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The critical buckling loads, which correspond to the out-of-plane equilibrium configuration, are the values of N_1^0 which cancel the determinant of the previous system:

$$-\left(\frac{m_1\pi}{a_1}\right)^2 (A_{66}B_{11} - A_{16}B_{16})B_{11} + \left(\frac{m_1\pi}{a_1}\right)^2 (A_{16}B_{11} - A_{11}B_{16})B_{16} \cdots$$
$$\cdots + \left[\left(\frac{m_1\pi}{a_1}\right)^2 D_{11} - N_1^0\right]\left(A_{11}A_{66} - A_{16}^2\right) = 0.$$

For each value of m_1, the critical buckling load $N^0_{1_{m_1}}$ of the mode m_1 is given by:

$$N^0_{1_{m_1}} = \left(\frac{m_1\pi}{a_1}\right)^2 \left(D_{11} - \frac{A_{66}B_{11}^2 + A_{11}B_{16}^2 - 2A_{16}B_{11}B_{16}}{A_{11}A_{66} - A_{16}^2}\right).$$

The critical buckling load for the first mode 1 is:

$$N^0_{1_1} = \left(\frac{\pi}{a_1}\right)^2 \left(D_{11} - \frac{A_{66}B_{11}^2 + A_{11}B_{16}^2 - 2A_{16}B_{11}B_{16}}{A_{11}A_{66} - A_{16}^2}\right).$$

10.7.2. *Cross-ply asymmetrical composite* $\left(0, \frac{\pi}{2}\right)_N$.

We have:

$$A_{16} = A_{26} = 0, \quad D_{16} = D_{26} = 0,$$

$$B_{16} = B_{26} = B_{12} = B_{66} = 0.$$

The critical buckling load is:

$$N^0_{1_{m_1}} = \left(\frac{m_1\pi}{a_1}\right)^2 \left(D_{11} - \frac{B_{11}^2}{A_{11}}\right).$$

10.7.3. *Balanced asymmetrical composite* $(\alpha, -\alpha)_N$.

We have:

$$A_{16} = A_{26} = 0, \quad D_{16} = D_{26} = 0,$$

$$B_{11} = B_{12} = B_{22} = B_{66} = 0.$$

The critical buckling load is given by:

$$N^0_{1_{m_1}} = \left(\frac{m_1\pi}{a_1}\right)^2 \left(D_{11} - \frac{B_{16}^2}{A_{66}}\right).$$

10.7.4. *Symmetrical composite*

We have:

$$B_{ij} = 0.$$

In this case the critical buckling load value is:

$$N^0_{1_{m_1}} = \left(\frac{m_1\pi}{a_1}\right)^2 D_{11}.$$

For a single layer isotropic plate we have:

$$N^0_{1_{m_1}} = \left(\frac{m_1\pi}{a_1}\right)^2 \frac{Eh^3}{12\left(1 - \nu^2\right)}.$$

Chapter 11

Cylindrical flexure of multi-layer Reissner-Mindlin plates

11.1. Introduction

After having studied one-dimensional cylindrical flexure according to the Kirchhoff-Love theory, we will now examine the use of Reissner-Mindlin theory, in which the transverse shear strains are taken into account, for the study of the cylindrical flexure of an infinitely wide plate in flexure, vibration and buckling.

11.2. Strain-displacement relationship

In cylindrical flexure we have a displacement field of the form:

$$u_1 = u_1^0(x_1|t) + x_3\psi_1(x_1|t),$$

$$u_2 = u_2^0(x_1|t),$$

$$u_3 = u_3^0(x_1|t),$$

which leads to the strains:

$$\varepsilon_1 = \frac{\partial u_1^0}{\partial x_1} + x_3\frac{\partial \psi_1}{\partial x_1},$$

$$\varepsilon_6 = \frac{\partial u_2^0}{\partial x_1},$$

$$\varepsilon_5 = \psi_1 + \frac{\partial u_3^0}{\partial x_1},$$

$$\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = 0.$$

11.3. Global constitutive relation

From the expressions:

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \dfrac{\partial u_1^0}{\partial x_1} \\ 0 \\ \dfrac{\partial u_2^0}{\partial x_1} \\ \dfrac{\partial \psi_1}{\partial x_1} \\ 0 \\ 0 \end{bmatrix},$$

$$\begin{bmatrix} N_4 \\ N_5 \end{bmatrix} = \begin{bmatrix} KA_{44} & KA_{45} \\ KA_{45} & KA_{55} \end{bmatrix} \begin{bmatrix} 0 \\ \psi_1 + \dfrac{\partial u_3^0}{\partial x_1} \end{bmatrix},$$

we obtain the global membrane loads:

$$N_1 = A_{11}\frac{\partial u_1^0}{\partial x_1} + A_{16}\frac{\partial u_2^0}{\partial x_1} + B_{11}\frac{\partial \psi_1}{\partial x_1},$$

$$N_2 = A_{12}\frac{\partial u_1^0}{\partial x_1} + A_{26}\frac{\partial u_2^0}{\partial x_1} + B_{12}\frac{\partial \psi_1}{\partial x_1},$$

$$N_6 = A_{16}\frac{\partial u_1^0}{\partial x_1} + A_{66}\frac{\partial u_2^0}{\partial x_1} + B_{16}\frac{\partial \psi_1}{\partial x_1},$$

with flexure:

$$M_1 = B_{11}\frac{\partial u_1^0}{\partial x_1} + B_{16}\frac{\partial u_2^0}{\partial x_1} + D_{11}\frac{\partial \psi_1}{\partial x_1},$$

$$M_2 = B_{12}\frac{\partial u_1^0}{\partial x_1} + B_{26}\frac{\partial u_2^0}{\partial x_1} + D_{12}\frac{\partial \psi_1}{\partial x_1},$$

$$M_6 = B_{16}\frac{\partial u_1^0}{\partial x_1} + B_{66}\frac{\partial u_2^0}{\partial x_1} + D_{16}\frac{\partial \psi_1}{\partial x_1},$$

and transverse shear:

$$N_4 = KA_{45}\left(\psi_1 + \frac{\partial u_3^0}{\partial x_1}\right), \quad N_5 = KA_{55}\left(\psi_1 + \frac{\partial u_3^0}{\partial x_1}\right).$$

11.4. Global plate equations

In the case considered here for cylindrical flexure the global equations:

$$\frac{\partial N_1}{\partial x_1} + p_1 = I_0 \frac{\partial^2 u_1^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial N_6}{\partial x_1} + p_2 = I_0 \frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial M_1}{\partial x_1} - N_5 = I_1 \frac{\partial^2 u_1^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial M_6}{\partial x_1} - N_4 = I_1 \frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial N_5}{\partial x_1} + \frac{\partial}{\partial x_1}\left(N_1 \frac{\partial u_3^0}{\partial x_1} \right) + q_3 + p_3 = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

lead to the following expressions:

– static:

$$\frac{dN_1}{dx_1} + p_1 = 0, \qquad \frac{dM_1}{dx_1} - N_5 = 0,$$

$$\frac{dN_6}{dx_1} + p_2 = 0, \qquad \frac{dM_6}{dx_1} - N_4 = 0,$$

$$\frac{dN_5}{dx_1} + q_3 + p_3 = 0,$$

– vibration:

$$\frac{\partial N_1}{\partial x_1} = I_0 \frac{\partial^2 u_1^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_1}{\partial t^2}, \qquad \frac{\partial M_1}{\partial x_1} - N_5 = I_1 \frac{\partial^2 u_1^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial N_6}{\partial x_1} = I_0 \frac{\partial^2 u_2^0}{\partial t^2}, \qquad \frac{\partial M_6}{\partial x_1} - N_4 = I_1 \frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial N_5}{\partial x_1} = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

– buckling:

$$\frac{dN_1}{dx_1} = 0, \qquad \frac{dM_1}{dx_1} - N_5 = 0,$$

$$\frac{dN_6}{dx_1} = 0, \qquad \frac{dM_6}{dx_1} - N_4 = 0,$$

$$\frac{dN_5}{dx_1} - N_1^0 \frac{d^2 u_3^0}{dx_1^2} = 0.$$

In this chapter we will limit ourselves to the case of an asymmetric cross-ply laminate $\left(0, \frac{\pi}{2}\right)_N$, for which the following global stiffnesses are zero:

$$A_{16} = A_{26} = A_{45} = 0, \quad D_{16} = D_{26} = 0, \quad B_{16} = B_{26} = B_{12} = B_{66} = 0.$$

11.5. Flexure

The plate rests on two simple supports at $x_1 = 0$ and $x_1 = a_1$, the volume forces are zero and the loading is defined by:

$$q(x_1) = q_{m_1} \sin \frac{m_1 \pi x_1}{a_1}.$$

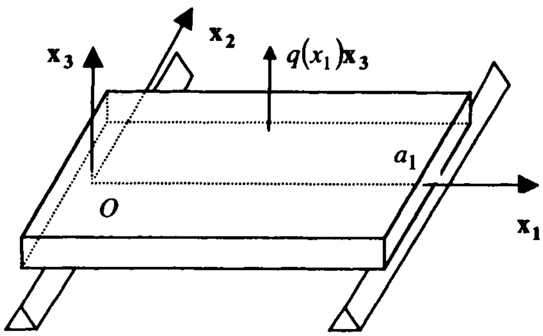

Figure 11.1. *Plate in cylindrical flexure*

Introducing:

$$N_1 = A_{11} \frac{du_1^0}{dx_1} + B_{11} \frac{d\psi_1}{dx_1},$$

$$N_2 = A_{12} \frac{du_1^0}{dx_1},$$

$$N_6 = A_{66} \frac{du_2^0}{dx_1},$$

as well as:

$$M_1 = B_{11} \frac{du_1^0}{dx_1} + D_{11} \frac{d\psi_1}{dx_1},$$

$$M_2 = D_{12} \frac{d\psi_1}{dx_1},$$

$$M_6 = 0,$$

and:

$$N_4 = 0,$$

$$N_5 = KA_{55} \left(\psi_1 + \frac{du_3^0}{dx_1} \right),$$

in the global equations:

$$\frac{dN_1}{dx_1}=0,$$

$$\frac{dN_6}{dx_1}=0,$$

$$\frac{dM_1}{dx_1}-N_5=0,$$

$$\frac{dM_6}{dx_1}-N_4=0,$$

$$\frac{dN_5}{dx_1}+q=0,$$

we obtain the four equations:

$$A_{11}\frac{d^2u_1^0}{dx_1^2}+B_{11}\frac{d^2\psi_1}{dx_1^2}=0,$$

$$A_{66}\frac{d^2u_2^0}{dx_1^2}=0,$$

$$B_{11}\frac{d^2u_1^0}{dx_1^2}+D_{11}\frac{d^2\psi_1}{dx_1^2}-KA_{55}\left(\psi_1+\frac{du_3^0}{dx_1}\right)=0,$$

$$KA_{55}\left(\frac{d\psi_1}{dx_1}+\frac{d^2u_3^0}{dx_1^2}\right)+q=0.$$

The boundary conditions at $x_1=0$ and $x_1=a_1$ are:

$$N_1=A_{11}\frac{du_1^0}{dx_1}+B_{11}\frac{d\psi_1}{dx_1}=0,\quad N_6=A_{66}\frac{du_2^0}{dx_1}=0,$$

$$M_1=B_{11}\frac{du_1^0}{dx_1}+D_{11}\frac{d\psi_1}{dx_1}=0,\quad u_3^0=0.$$

The displacement field:

$$u_1^0=U_{m_1}^1\cos\frac{m_1\pi x_1}{a_1},$$

$$u_2^0=U_{m_1}^2\cos\frac{m_1\pi x_1}{a_1},$$

$$u_3^0=U_{m_1}^3\sin\frac{m_1\pi x_1}{a_1},$$

$$\psi_1=\Psi_{m_1}^1\cos\frac{m_1\pi x_1}{a_1},$$

satisfies the boundary conditions and the global equilibrium equations. Introducing the displacements in the global equilibrium equations, we obtain the system:

$$\left[-\left(\frac{m_1\pi}{a_1}\right)^2 A_{11}U^1_{m_1} - \left(\frac{m_1\pi}{a_1}\right)^2 B_{11}\Psi^1_{m_1}\right]\cos\frac{m_1\pi x_1}{a_1} = 0,$$

$$-\left(\frac{m_1\pi}{a_1}\right)^2 A_{66}U^2_{m_1}\cos\frac{m_1\pi x_1}{a_1} = 0,$$

$$\left[-\left(\frac{m_1\pi}{a_1}\right)^2 B_{11}U^1_{m_1} - \left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\Psi^1_{m_1} - KA_{55}\left(\Psi^1_{m_1} + \frac{m_1\pi}{a_1}U^3_{m_1}\right)\right]\cos\frac{m_1\pi x_1}{a_1} = 0,$$

$$\left\{-KA_{55}\left[\frac{m_1\pi}{a_1}\Psi^1_{m_1} + \left(\frac{m_1\pi}{a_1}\right)^2 U^3_{m_1}\right] + q_{m_1}\right\}\sin\frac{m_1\pi x_1}{a_1} = 0.$$

The second equation immediately gives $U^2_{m_1} = 0$, from which $u^0_2 = 0$.

After simplification, the three remaining equations lead to a matrix set:

$$\begin{bmatrix} \left(\frac{m_1\pi}{a_1}\right)^2 A_{11} & 0 & \left(\frac{m_1\pi}{a_1}\right)^2 B_{11} \\ 0 & \left(\frac{m_1\pi}{a_1}\right)^2 KA_{55} & \frac{m_1\pi}{a_1}KA_{55} \\ \left(\frac{m_1\pi}{a_1}\right)^2 B_{11} & \frac{m_1\pi}{a_1}KA_{55} & KA_{55} + \left(\frac{m_1\pi}{a_1}\right)^2 D_{11} \end{bmatrix} \begin{bmatrix} U^1_{m_1} \\ U^3_{m_1} \\ \Psi^1_{m_1} \end{bmatrix} = \begin{bmatrix} 0 \\ q_{m_1} \\ 0 \end{bmatrix}.$$

The determinant of the set:

$$\Delta = \left(\frac{m_1\pi}{a_1}\right)^4 A_{11}KA_{55}\left[KA_{55} + \left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right] - \left(\frac{m_1\pi}{a_1}\right)^6 B^2_{11}KA_{55}\ldots$$

$$\ldots - \left(\frac{m_1\pi}{a_1}\right)^4 A_{11}K^2A^2_{55} = \left(\frac{m_1\pi}{a_1}\right)^6 KA_{55}\left(A_{11}D_{11} - B^2_{11}\right),$$

and the determinants:

$$\Delta_1 = \left(\frac{m_1\pi}{a_1}\right)^3 B_{11}KA_{55}q_{m_1},$$

$$\Delta_2 = \left(\frac{m_1\pi}{a_1}\right)^2 \left\{\left[KA_{55} + \left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\right]A_{11} - \left(\frac{m_1\pi}{a_1}\right)^2 B^2_{11}\right\}q_{m_1},$$

$$\Delta_3 = -\left(\frac{m_1\pi}{a_1}\right)^3 A_{11}KA_{55}q_{m_1},$$

provide the solution:

$$U^1_{m_1} = \left(\frac{a_1}{m_1\pi}\right)^3 \frac{B_{11}q_{m_1}}{A_{11}D_{11} - B_{11}^2},$$

$$U^3_{m_1} = \left(\frac{a_1}{m_1\pi}\right)^4 \frac{\left[\left(\frac{m_1\pi}{a_1}\right)^2 \left(A_{11}D_{11} - B_{11}^2\right) + A_{11}KA_{55}\right]q_{m_1}}{KA_{55}\left(A_{11}D_{11} - B_{11}^2\right)},$$

$$\Psi^1_{m_1} = -\left(\frac{a_1}{m_1\pi}\right)^3 \frac{A_{11}q_{m_1}}{A_{11}D_{11} - B_{11}^2}.$$

We then obtain:

– for the displacements:

$$u_1 = \left(U^1_{m_1} + x_3\Psi^1_{m_1}\right)\cos\frac{m_1\pi x_1}{a_1},$$

$$u_2 = 0,$$

$$u_3 = U^3_{m_1} \sin\frac{m_1\pi x_1}{a_1},$$

– for the strains:

$$\varepsilon_1 = -\frac{m_1\pi}{a_1}\left(U^1_{m_1} + x_3\Psi^1_{m_1}\right)\sin\frac{m_1\pi x_1}{a_1},$$

$$\varepsilon_5 = \left(\Psi^1_{m_1} + \frac{m_1\pi}{a_1}U^3_{m_1}\right)\cos\frac{m_1\pi x_1}{a_1},$$

$$\varepsilon_2 = \varepsilon_3 = \varepsilon_4 = \varepsilon_6 = 0,$$

– for the stresses in layer k:

$$\sigma_i^k = Q_{i1}^k\varepsilon_1 = -\frac{m_1\pi}{a_1}Q_{i1}^k\left(U^1_{m_1} + x_3\Psi^1_{m_1}\right)\sin\frac{m_1\pi x_1}{a_1} \quad (i = 1,2),$$

$$\sigma_5^k = Q_{55}^k\varepsilon_5 = Q_{55}^k\left(\Psi^1_{m_1} + \frac{m_1\pi}{a_1}U^3_{m_1}\right)\cos\frac{m_1\pi x_1}{a_1},$$

$$\sigma_3^k = \sigma_4^k = \sigma_6^k = 0.$$

11.6. Vibrations

The plate studied, subjected to no loads, is resting on two simple supports separated by a_1.

Introducing the global loads:

$$N_1 = A_{11}\frac{\partial u_1^0}{\partial x_1} + B_{11}\frac{\partial \psi_1}{\partial x_1},$$

$$N_2 = A_{12}\frac{\partial u_1^0}{\partial x_1},$$

$$N_6 = A_{66}\frac{\partial u_2^0}{\partial x_1},$$

$$M_1 = B_{11}\frac{\partial u_1^0}{\partial x_1} + D_{11}\frac{\partial \psi_1}{\partial x_1},$$

$$M_2 = D_{12}\frac{\partial \psi_1}{\partial x_1},$$

$$M_6 = 0,$$

$$N_4 = 0,$$

$$N_5 = KA_{55}\left(\psi_1 + \frac{\partial u_3^0}{\partial x_1}\right),$$

into the global equations of motion:

$$\frac{\partial N_1}{\partial x_1} = I_0\frac{\partial^2 u_1^0}{\partial t^2} + I_1\frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial N_6}{\partial x_1} = I_0\frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial M_1}{\partial x_1} - N_5 = I_1\frac{\partial^2 u_1^0}{\partial t^2} + I_2\frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial M_6}{\partial x_1} - N_4 = I_1\frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial N_5}{\partial x_1} = I_0\frac{\partial^2 u_3^0}{\partial t^2},$$

we obtain the five equations:

$$A_{11}\frac{\partial^2 u_1^0}{\partial x_1^2} + B_{11}\frac{\partial^2 \psi_1}{\partial x_1^2} = I_0\frac{\partial^2 u_1^0}{\partial t^2} + I_1\frac{\partial^2 \psi_1}{\partial t^2},$$

$$A_{66}\frac{\partial^2 u_2^0}{\partial x_1^2} = I_0\frac{\partial^2 u_2^0}{\partial t^2},$$

$$B_{11}\frac{\partial^2 u_1^0}{\partial x_1^2} + D_{11}\frac{\partial^2 \psi_1}{\partial x_1^2} - KA_{55}\left(\psi_1 + \frac{\partial u_3^0}{\partial x_1}\right) = I_1\frac{\partial^2 u_1^0}{\partial t^2} + I_2\frac{\partial^2 \psi_1}{\partial t^2},$$

$$0 = I_1\frac{\partial^2 u_2^0}{\partial t^2},$$

$$KA_{55}\left(\frac{\partial \psi_1}{\partial x_1} + \frac{\partial^2 u_3^0}{\partial x_1^2}\right) = I_0\frac{\partial^2 u_3^0}{\partial t^2}.$$

The boundary conditions at $x_1 = 0$ and $x_1 = a_1$ are:

$$N_1 = A_{11}\frac{\partial u_1^0}{\partial x_1} + B_{11}\frac{\partial \psi_1}{\partial x_1} = 0,$$

$$N_6 = A_{66}\frac{\partial u_2^0}{\partial x_1} = 0,$$

$$M_1 = B_{11}\frac{\partial u_1^0}{\partial x_1} + D_{11}\frac{\partial \psi_1}{\partial x_1} = 0,$$

$$u_3^0 = 0.$$

The displacement field:

$$u_1^0 = U_{m_1}^1 \cos\frac{m_1\pi x_1}{a_1}\sin\left(\omega_{m_1}t + \varphi_{m_1}\right),$$

$$u_2^0 = U_{m_1}^2 \cos\frac{m_1\pi x_1}{a_1}\sin\left(\omega_{m_1}t + \varphi_{m_1}\right),$$

$$u_3^0 = U_{m_1}^3 \sin\frac{m_1\pi x_1}{a_1}\sin\left(\omega_{m_1}t + \varphi_{m_1}\right),$$

$$\psi_1 = \Psi_{m_1}^1 \cos\frac{m_1\pi x_1}{a_1}\sin\left(\omega_{m_1}t + \varphi_{m_1}\right),$$

satisfies the boundary conditions and the global equations of motion. Introducing the displacements into the global equations and after simplification, we obtain:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11} - I_0\omega_{m_1}^2\right]U_{m_1}^1 + \left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{11} - I_1\omega_{m_1}^2\right]\Psi_{m_1}^1 = 0,$$

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{66} - I_0\omega_{m_1}^2\right]U_{m_1}^2 = 0,$$

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{11} - I_1\omega_{m_1}^2\right]U_{m_1}^1 + \frac{m_1\pi}{a_1}KA_{55}U_{m_1}^3 + \left[\left(\frac{m_1\pi}{a_1}\right)^2 D_{11} + KA_{55} - I_2\omega_{m_1}^2\right]\Psi_{m_1}^1 = 0,$$

$$I_1\omega_{m_1}^2 U_{m_1}^2 = 0,$$

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 KA_{55} - I_0\omega_{m_1}^2\right]U_{m_1}^3 + \frac{m_1\pi}{a_1}KA_{55}\Psi_{m_1}^1 = 0.$$

From the fourth equation, we have $U_{m_1}^2 = 0$, from which $u_2^0 = 0$.

The three other equations can be put in the following matrix form:

$$\begin{bmatrix} \left(\frac{m_1\pi}{a_1}\right)^2 A_{11} - I_0\omega_{m_1}^2 & 0 & \left(\frac{m_1\pi}{a_1}\right)^2 B_{11} - I_1\omega_{m_1}^2 \\ 0 & \left(\frac{m_1\pi}{a_1}\right)^2 KA_{55} - I_0\omega_{m_1}^2 & \frac{m_1\pi}{a_1}KA_{55} \\ \left(\frac{m_1\pi}{a_1}\right)^2 B_{11} - I_1\omega_{m_1}^2 & \frac{m_1\pi}{a_1}KA_{55} & \left(\frac{m_1\pi}{a_1}\right)^2 D_{11} + KA_{55} - I_2\omega_{m_1}^2 \end{bmatrix} \begin{bmatrix} U_{m_1}^1 \\ U_{m_1}^3 \\ \Psi_{m_1}^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This set has a solution other than the trivial solution $U_{m_1}^1 = U_{m_1}^3 = 0$, $\Psi_{m_1}^1 = 0$ for the values of ω_{m_1} which cancel its determinant, from which the equation for the natural frequencies in ω_{m_1}:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11} - I_0\omega_{m_1}^2\right]\left[\left(\frac{m_1\pi}{a_1}\right)^2 KA_{55} - I_0\omega_{m_1}^2\right]\left[\left(\frac{m_1\pi}{a_1}\right)^2 D_{11} + KA_{55} - I_2\omega_{m_1}^2\right]\ldots$$

$$\ldots - \left[\left(\frac{m_1\pi}{a_1}\right)^2 KA_{55} - I_0\omega_{m_1}^2\right]\left[\left(\frac{m_1\pi}{a_1}\right)^2 B_{11} - I_1\omega_{m_1}^2\right]^2\ldots$$

$$\ldots - \left(\frac{m_1\pi}{a_1}KA_{55}\right)^2\left[\left(\frac{m_1\pi}{a_1}\right)^2 A_{11} - I_0\omega_{m_1}^2\right] = 0,$$

of the form:

$$-A\omega_{m_1m_2}^6 + B\omega_{m_1m_2}^4 - C\omega_{m_1m_2}^2 + D = 0,$$

with:

$$A = I_0\left(I_0 I_2 - I_1^2\right),$$

$$B = \left[\left(\frac{m_1\pi}{a_1}\right)^2 D_{11} + KA_{55}\right] I_0^2 + \left(\frac{m_1\pi}{a_1}\right)^2 \left[(A_{11} + KA_{55})I_0 I_2 - 2B_{11}I_0 I_1 - KA_{55}I_1^2\right],$$

$$C = \left(\frac{m_1\pi}{a_1}\right)^2 \left\{\left(\frac{m_1\pi}{a_1}\right)^2 (A_{11} + KA_{55})D_{11}I_0 + A_{11}KA_{55}\left[\left(\frac{m_1\pi}{a_1}\right)^2 I_2 + I_0\right] \ldots\right.$$

$$\left.\ldots - \left(\frac{m_1\pi}{a_1}\right)^2 B_{11}(B_{11}I_0 + 2KA_{55}I_1)\right\},$$

$$D = \left(\frac{m_1\pi}{a_1}\right)^6 \left(D_{11}A_{11} - B_{11}^2\right)KA_{55},$$

are derived.

For each value of m_1 we obtain three natural frequencies.

11.7. Buckling

The plate studied, resting on two simple supports at $x_1 = 0$ and $x_1 = a_1$, is only loaded in compression $-N_1^0$, with $N_1^0 > 0$.

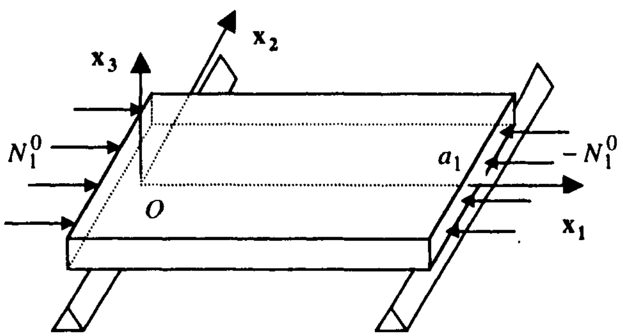

Figure 11.2. *Plate in buckling situation*

By introducing the global loads:

$$N_1 = A_{11}\frac{du_1^0}{dx_1} + B_{11}\frac{d\psi_1}{dx_1},$$

$$N_2 = A_{12}\frac{du_1^0}{dx_1},$$

$$N_6 = A_{66}\frac{du_2^0}{dx_1},$$

$$M_1 = B_{11}\frac{du_1^0}{dx_1} + D_{11}\frac{d\psi_1}{dx_1},$$

$$M_2 = D_{12}\frac{d\psi_1}{dx_1},$$

$$M_6 = 0,$$

$$N_4 = 0,$$

$$N_5 = KA_{55}\left(\psi_1 + \frac{du_3^0}{dx_1}\right),$$

into the equations for global buckling:

$$\frac{dN_1}{dx_1} = 0,$$

$$\frac{dN_6}{dx_1} = 0,$$

$$\frac{dM_1}{dx_1} - N_5 = 0,$$

$$\frac{dM_6}{dx_1} - N_4 = 0,$$

$$\frac{dN_5}{dx_1} - N_1^0\frac{d^2u_3^0}{dx_1^2} = 0,$$

we obtain the four equations:

$$A_{11}\frac{d^2u_1^0}{dx_1^2} + B_{11}\frac{d^2\psi_1}{dx_1^2} = 0,$$

$$A_{66}\frac{d^2u_2^0}{dx_1^2} = 0,$$

$$B_{11}\frac{d^2u_1^0}{dx_1^2} + D_{11}\frac{d^2\psi_1}{dx_1^2} - KA_{55}\left(\psi_1 + \frac{du_3^0}{dx_1}\right) = 0,$$

$$KA_{55}\left(\frac{d\psi_1}{dx_1}+\frac{d^2u_3^0}{dx_1^2}\right)-N_1^0\frac{d^2u_3^0}{dx_1^2}=0.$$

The boundary conditions at $x_1 = 0$ and $x_1 = a_1$ are written:

$$N_1 = A_{11}\frac{du_1^0}{dx_1}+B_{11}\frac{d\psi_1}{dx_1}=0,$$

$$N_6 = A_{66}\frac{du_2^0}{dx_1}=0,$$

$$M_1 = B_{11}\frac{du_1^0}{dx_1}+D_{11}\frac{d\psi_1}{dx_1}=0,$$

$$u_3^0 = 0.$$

The displacement field:

$$u_1^0 = U_{m_1}^1\cos\frac{m_1\pi x_1}{a_1},$$

$$u_2^0 = U_{m_1}^2\cos\frac{m_1\pi x_1}{a_1},$$

$$u_3^0 = U_{m_1}^3\sin\frac{m_1\pi x_1}{a_1},$$

$$\psi_1 = \Psi_{m_1}^1\cos\frac{m_1\pi x_1}{a_1},$$

satisfies the boundary conditions and the global buckling equations. By introducing the values of u_1^0, u_2^0, u_3^0 and ψ_1 into the equations of global buckling, we obtain the set:

$$\left[-\left(\frac{m_1\pi}{a_1}\right)^2 A_{11}U_{m_1}^1-\left(\frac{m_1\pi}{a_1}\right)^2 B_{11}\Psi_{m_1}^1\right]\cos\frac{m_1\pi x_1}{a_1}=0,$$

$$-\left(\frac{m_1\pi}{a_1}\right)^2 A_{66}U_{m_1}^2\cos\frac{m_1\pi x_1}{a_1}=0,$$

$$\left[-\left(\frac{m_1\pi}{a_1}\right)^2 B_{11}U_{m_1}^1-\left(\frac{m_1\pi}{a_1}\right)^2 D_{11}\Psi_{m_1}^1-KA_{55}\left(\Psi_{m_1}^1+\frac{m_1\pi}{a_1}U_{m_1}^3\right)\right]\cos\frac{m_1\pi x_1}{a_1}=0,$$

$$\left\{-KA_{55}\left[\frac{m_1\pi}{a_1}\Psi_{m_1}^1+\left(\frac{m_1\pi}{a_1}\right)^2 U_{m_1}^3\right]+N_1^0\left(\frac{m_1\pi}{a_1}\right)^2 U_{m_1}^3\right\}\sin\frac{m_1\pi x_1}{a_1}=0.$$

The second equation gives $U_{m_1}^2 = 0$, then $u_2^0 = 0$.

After simplification, the three remaining equations provide the following matrix set:

$$\begin{bmatrix} \left(\frac{m_1\pi}{a_1}\right)^2 A_{11} & 0 & \left(\frac{m_1\pi}{a_1}\right)^2 B_{11} \\ 0 & \left(\frac{m_1\pi}{a_1}\right)^2 \left(KA_{55} - N_1^0\right) & \frac{m_1\pi}{a_1} KA_{55} \\ \left(\frac{m_1\pi}{a_1}\right)^2 B_{11} & \frac{m_1\pi}{a_1} KA_{55} & \left(\frac{m_1\pi}{a_1}\right)^2 D_{11} + KA_{55} \end{bmatrix} \begin{bmatrix} U_{m_1}^1 \\ U_{m_1}^3 \\ \Psi_{m_1}^1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

The critical buckling loads are the values of N_1^0 which cancel the determinant of the homogeneous system, hence the equation:

$$\left(\frac{m_1\pi}{a_1}\right)^4 A_{11}\left(KA_{55} - N_1^0\right)\left[\left(\frac{m_1\pi}{a_1}\right)^2 D_{11} + KA_{55}\right] - \left(\frac{m_1\pi}{a_1}\right)^6 B_{11}^2\left(KA_{55} - N_1^0\right)\ldots$$

$$\ldots - \left(\frac{m_1\pi}{a_1}\right)^4 A_{11}\left(KA_{55}\right)^2 = 0,$$

which is written as:

$$\left[\left(\frac{m_1\pi}{a_1}\right)^2 \left(A_{11}D_{11} - B_{11}^2\right) + A_{11}KA_{55}\right]\left(KA_{55} - N_1^0\right) - A_{11}\left(KA_{55}\right)^2 = 0,$$

and for which the solution is:

$$N_1^0 = KA_{55} - \frac{A_{11}\left(KA_{55}\right)^2}{\left(\frac{m_1\pi}{a_1}\right)^2 \left(A_{11}D_{11} - B_{11}^2\right) + A_{11}KA_{55}}.$$

The critical buckling loads are given by:

$$N_1^0 = \frac{\left(\frac{m_1\pi}{a_1}\right)^2 \left(A_{11}D_{11} - B_{11}^2\right) KA_{55}}{\left(\frac{m_1\pi}{a_1}\right)^2 \left(A_{11}D_{11} - B_{11}^2\right) + A_{11}KA_{55}}.$$

PART III
Multi-Layer Beams

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Chapter 12

Symmetrical multi-layer beams in tension-compression

12.1. Introduction

In this and the two following chapters we will study multi-layer symmetric beams, which are composites for which the ratio of width over length is small.

The determination of equivalent stiffnesses allows us to use the formulae and methods currently used in strength of materials studies.

The present chapter is devoted to the longitudinal behaviour of multi-layer symmetric beams in static and vibration loading.

12.2. Strains, stresses, global equation of tension-compression

The global constitutive relation for a symmetrical plate subjected to membrane loads may be written as:

$$N = A\varepsilon^0 \text{ or } \begin{bmatrix} N_1 \\ N_2 \\ N_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{bmatrix},$$

or, after inversion:

$$\varepsilon^0 = A^{-1}N \text{ or } \begin{bmatrix} \varepsilon_1^0 \\ \varepsilon_2^0 \\ \varepsilon_6^0 \end{bmatrix} = \begin{bmatrix} A_{11}^* & A_{12}^* & A_{16}^* \\ A_{12}^* & A_{22}^* & A_{26}^* \\ A_{16}^* & A_{26}^* & A_{66}^* \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_6 \end{bmatrix}.$$

In the case of tension loading in the $\mathbf{x}_1$ direction, we have:

$$N_2 = N_6 = 0,$$

and the global strains are equal to:

$$\varepsilon_j^0 = A_{1j}^* N_1,$$

or:

$$\varepsilon_1^0 = A_{11}^* N_1,$$

$$\varepsilon_2^0 = A_{12}^* N_1,$$
$$\varepsilon_6^0 = A_{16}^* N_1,$$

with:

$$A_{11}^* = \frac{A_{22}A_{66} - A_{26}^2}{A},$$
$$A_{12}^* = -\frac{A_{12}A_{66} - A_{16}A_{26}}{A},$$
$$A_{16}^* = \frac{A_{12}A_{26} - A_{22}A_{16}}{A},$$
$$A = \left(A_{11}A_{22} - A_{12}^2\right)A_{66} + 2A_{12}A_{16}A_{26} - A_{11}A_{26}^2 - A_{22}A_{16}^2,$$

and:

$$\varepsilon_1^0 = \frac{\partial u_1^0}{\partial x_1}, \quad \varepsilon_2^0 = \frac{\partial u_2^0}{\partial x_2}, \quad \varepsilon_6^0 = \frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}.$$

The stresses in layer k are determined from the expressions:

$$\sigma_i^k = Q_{ij}^k \varepsilon_j^0 = Q_{ij}^k A_{1j}^* N_1,$$

or:

$$\sigma_1^k = \left(Q_{11}^k A_{11}^* + Q_{12}^k A_{12}^* + Q_{16}^k A_{16}^*\right)N_1,$$
$$\sigma_2^k = \left(Q_{12}^k A_{11}^* + Q_{22}^k A_{12}^* + Q_{26}^k A_{16}^*\right)N_1,$$
$$\sigma_6^k = \left(Q_{16}^k A_{11}^* + Q_{26}^k A_{12}^* + Q_{66}^k A_{16}^*\right)N_1.$$

From these calculations, the boundary conditions at the free edges of each layer $\sigma_2^k = \sigma_6^k = 0$ are not satisfied. However, globally they are satisfied. The stresses obtained from the theory developed above are not correct near the free edges where the stress state is three-dimensional. To minimize the influence of the free edges, the ratio of width over height of the section should be sufficiently large.

By introducing the expression:

$$\frac{\partial u_1^0}{\partial x_1} = A_{11}^* N_1,$$

or:

$$N_1 = \frac{1}{A_{11}^*}\frac{\partial u_1^0}{\partial x_1},$$

into the global equation:

$$\frac{\partial N_1}{\partial x_1} + p_1 = I_0 \frac{\partial^2 u_1^0}{\partial t^2},$$

we obtain the equation:

$$\frac{1}{A_{11}^*}\frac{\partial^2 u_1^0}{\partial x_1^2} + p_1 = I_0 \frac{\partial^2 u_1^0}{\partial t^2},$$

with:

$$I_0 = \sum_{k=1}^{N} \rho^k \left(z_k - z_{k-1}\right),$$

$$p_1 = \sum_{k=1}^{N} f_1^k \left(z_k - z_{k-1}\right).$$

12.3. Single layer orthotropic beam

In this particular case, the global membrane stiffnesses may be written:

$$A_{ij} = h\overline{Q}_{ij},$$

with:

$$\overline{Q}_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad \overline{Q}_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}},$$

$$\overline{Q}_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}.$$

The global compliances are given by:

$$A_{11}^* = \frac{A_{22}}{A_{11}A_{22} - A_{12}^2} = \frac{1}{h}\frac{\overline{Q}_{22}}{\overline{Q}_{11}\overline{Q}_{22} - \overline{Q}_{12}^2} = \frac{1}{h}\frac{\dfrac{E_2}{1-\nu_{12}\nu_{21}}}{\dfrac{E_1E_2}{\left(1-\nu_{12}\nu_{21}\right)^2} - \dfrac{\nu_{21}E_1\nu_{12}E_2}{\left(1-\nu_{12}\nu_{21}\right)^2}},$$

$$A_{11}^* = \frac{1}{E_1 h},$$

$$A_{12}^* = -\frac{A_{12}}{A_{11}A_{22} - A_{12}^2} = -\frac{1}{h}\frac{\overline{Q}_{12}}{\overline{Q}_{11}\overline{Q}_{22} - \overline{Q}_{12}^2} = -\frac{1}{h}\frac{\dfrac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}}{\dfrac{E_1E_2}{\left(1-\nu_{12}\nu_{21}\right)^2} - \dfrac{\nu_{21}E_1\nu_{12}E_2}{\left(1-\nu_{12}\nu_{21}\right)^2}},$$

$$A_{12}^* = -\frac{\nu_{12}}{E_1 h},$$

$$A_{16}^* = 0.$$

The membrane strains are equal to:

$$\varepsilon_1^0 = A_{11}^* N_1,$$

$$\varepsilon_2^0 = A_{12}^* N_1,$$

$$\varepsilon_6^0 = 0,$$

and the stresses to:

$$\sigma_1 = \left(\overline{Q}_{11}A_{11}^* + \overline{Q}_{12}A_{12}^*\right)N_1 = \frac{N_1}{h},$$

$$\sigma_2 = \left(\overline{Q}_{12}A_{11}^* + \overline{Q}_{22}A_{12}^*\right)N_1 = 0,$$

$$\sigma_6 = 0.$$

Also, we have:

$$I_0 = \rho h.$$

Multiplying by b the two members of the following equations:

$$N_1 = \frac{1}{A_{11}^*}\frac{\partial u_1^0}{\partial x_1} = E_1 h \frac{\partial u_1^0}{\partial x_1},$$

$$\frac{\partial N_1}{\partial x_1} + p_1 = \rho h \frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{1}{A_{11}^*}\frac{\partial^2 u_1^0}{\partial x_1^2} + p_1 = \rho h \frac{\partial^2 u_1^0}{\partial t^2},$$

and putting $N = N_1 b$, $q_0 = p_1 b$ and $S = bh$, we derive the classic equations:

$$N = E_1 S \frac{\partial u_1^0}{\partial x_1},$$

$$\frac{\partial N}{\partial x_1} + q_0 = \rho S \frac{\partial^2 u_1^0}{\partial t^2},$$

$$E_1 S \frac{\partial^2 u_1^0}{\partial x_1^2} + q_0 = \rho S \frac{\partial^2 u_1^0}{\partial t^2}.$$

12.4. General equations for beams in tension-compression

By introducing, for multi-layer symmetric beams, the equivalent characteristics:

$$(ES)_0 = \frac{b}{A_{11}^*},$$

$$(\rho S)_0 = I_0 b,$$

as well as:

$$q_0 = p_1 b,$$

we obtain, in tension-compression, the following global equations:

$$N = (ES)_0 \frac{\partial u_1^0}{\partial x_1},$$

$$\frac{\partial N}{\partial x_1} + q_0 = (\rho S)_0 \frac{\partial^2 u_1^0}{\partial t^2},$$

$$(ES)_0 \frac{\partial^2 u_1^0}{\partial x_1^2} + q_0 = (\rho S)_0 \frac{\partial^2 u_1^0}{\partial t^2}.$$

If we assume, as we do in the classical approach for beam theory in tension-compression, that the displacement u_1 only depends on x_1 and on t, we have:

$$u_1 = u_1^0(x_1|t),$$

$$\varepsilon_1 = \frac{\partial u_1^0}{\partial x_1}.$$

For an orthotropic beam we have:

$$\sigma_1^k = E^k \varepsilon_1 = E^k \frac{\partial u_1^0}{\partial x_1},$$

where E^k is Young's modulus in the direction of orthotropy $\vec{x}_1$ of layer k.

The axial load is then given by:

$$N = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_1 \, dx_3 dx_2 = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} E \frac{\partial u_1^0}{\partial x_1} dx_3 dx_2,$$

$$N = \int_{-\frac{b}{2}}^{\frac{b}{2}} \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} E^k \frac{\partial u_1^0}{\partial x_1} dx_3 dx_2,$$

or:

$$N = (ES)_0 \frac{\partial u_1^0}{\partial x_1},$$

with:

$$(ES)_0 = b \sum_{k=1}^{N} E^k (z_k - z_{k-1}).$$

12.5. Built-in beam under its own weight and subjected to a force

The beam OA of length l, mean plane $(O|\mathbf{x}_1, \mathbf{x}_2)$, built-in at O is subjected at A to the force $F\mathbf{x}_1$ and to the action of its weight, $\mathbf{x}_1$ is descending vertical.

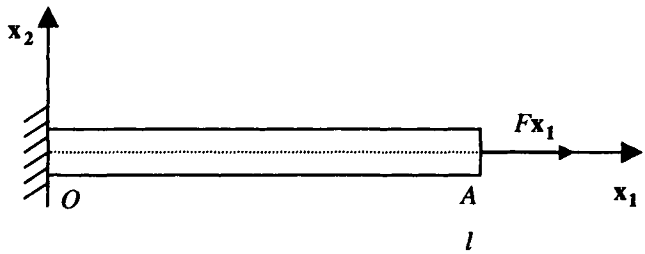

Figure 12.1. *Beam in tension*

The integration of the equation of equilibrium:

$$\frac{d^2u_1^0}{dx_1^2} = -\frac{q_0}{(ES)_0},$$

gives, in the case where q_0 is constant, the displacement u_1^0:

$$\frac{du_1^0}{dx_1} = -\frac{q_0}{(ES)_0}(x_1 + C),$$

$$u_1^0 = -\frac{q_0}{(ES)_0}\left(\frac{x_1^2}{2} + Cx_1 + D\right),$$

and the axial force:

$$N = (ES)_0\frac{du_1^0}{dx_1} = -q_0(x_1 + C).$$

The two integration constants C and D are obtained from the two conditions:

$$u_1^0(0) = 0,$$

$$N(l) = F,$$

from the system:

$$D = 0,$$

$$F = -q_0(l + C),$$

the second equation of which gives:

$$C = -\left(\frac{F}{q_0} + l\right).$$

The displacement u_1^0 and the axial load N are therefore equal to:

$$u_1^0 = \frac{x_1}{(ES)_0}\left[F + q_0\left(l - \frac{x_1}{2}\right)\right],$$

$$N = F + q_0(l - x_1).$$

The elongation of the beam is equal to:

$$u_1^0(l) = \frac{l}{(ES)_0}\left(F + q_0\frac{l}{2}\right).$$

We can find the expression for u_1^0 by integrating the expression:

$$\frac{du_1^0}{dx_1} = \frac{N}{(ES)_0}.$$

The expression for the axial force is:

$$N = F + q_0(l - x_1),$$

so we obtain:

$$u_1^0 = \frac{1}{(ES)_0}\left[Fx_1 + q_0\left(lx_1 - \frac{x_1^2}{2}\right) + C\right].$$

The built-in condition $u_1^0(0) = 0$, gives $C = 0$.

12.6. Vibration of a built-in beam

The beam *OA*, of length *l*, mean plane $(O|\mathbf{x}_1, \mathbf{x}_2)$ and subjected to no external load, is built-in at *O* and at *A*.

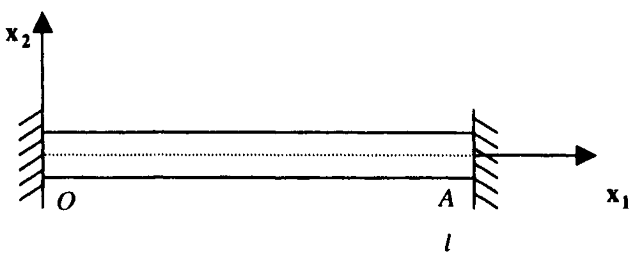

Figure 12.2. *Beam under longitudinal vibrations*

The longitudinal vibrations of the beam are governed by the equation:

$$(ES)_0\frac{\partial^2 u_1^0}{\partial x_1^2} = (\rho S)_0\frac{\partial^2 u_1^0}{\partial t^2}.$$

The solution to this equation, which satisfies the displacement boundary conditions:

$$u_1^0(0|t)=0,$$
$$u_1^0(l|t)=0,$$

is of the form:

$$u_1^0(x_1|t)=U_n^1\sin\frac{n\pi x_1}{l}\sin(\omega_n t+\varphi_n).$$

Introducing this into the equations of motion we obtain the expression:

$$-\left(\frac{n\pi}{l}\right)^2(ES)_0U_n^1\sin\frac{n\pi x_1}{l}\sin(\omega_n t+\varphi_n)=-\omega_n^2(\rho S)_0U_n^1\sin\frac{n\pi x_1}{l}\sin(\omega_n t+\varphi_n),$$

which, after simplification provides the natural frequencies:

$$\omega_n=\frac{n\pi}{l}\sqrt{\frac{(ES)_0}{(\rho S)_0}}.$$

Chapter 13

Symmetrical multi-layer beams in flexure without transverse shear strain

13.1. Introduction

In this chapter we will develop the theory of beams in flexure neglecting transverse shear strains. We will examine flexure, vibration and buckling.

Bernoulli's theory, used in the strength of materials approach, will be applied here.

13.2. Strains, stresses, equations of motion

The global constitutive relation of a symmetric plate subjected to flexural loads is written as:

$$M = D\kappa \text{ or } \begin{bmatrix} M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix},$$

after inversion it is written as:

$$\kappa = D^{-1}M \text{ or } \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} = \begin{bmatrix} D^*_{11} & D^*_{12} & D^*_{16} \\ D^*_{12} & D^*_{22} & D^*_{26} \\ D^*_{16} & D^*_{26} & D^*_{66} \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_6 \end{bmatrix}.$$

In the case of flexure in-plane $(O|\mathbf{x}_1, \mathbf{x}_3)$, we have:

$$M_2 = M_6 = 0,$$

as well as the curvatures:

$$\kappa_j = D^*_{1j} M_1,$$

or:

$$\kappa_1 = D^*_{11} M_1,$$

$$\kappa_2 = D^*_{12} M_1,$$
$$\kappa_6 = D^*_{16} M_1,$$

with:

$$D^*_{11} = \frac{D_{22}D_{66} - D_{26}^2}{D},$$
$$D^*_{12} = -\frac{D_{12}D_{66} - D_{16}D_{26}}{D},$$
$$D^*_{16} = \frac{D_{12}D_{26} - D_{22}D_{16}}{D},$$
$$D = \left(D_{11}D_{22} - D_{12}^2\right)D_{66} + 2D_{12}D_{16}D_{26} - D_{11}D_{26}^2 - D_{22}D_{16}^2,$$

and:

$$\kappa_1 = -\frac{\partial^2 u_3^0}{\partial x_1^2}, \quad \kappa_2 = -\frac{\partial^2 u_3^0}{\partial x_2^2}, \quad \kappa_6 = -2\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2}.$$

The expressions:

$$\frac{\partial^2 u_3^0}{\partial x_1^2} = -D^*_{11} M_1, \quad \frac{\partial^2 u_3^0}{\partial x_2^2} = -D^*_{12} M_1, \quad \frac{\partial^2 u_3^0}{\partial x_1 \partial x_2} = -\frac{1}{2} D^*_{16} M_1,$$

show that u_3^0 depends on the two variables x_1 and x_2.

The term D^*_{12} is at the origin of a curvature of the mean surface of the beam in the transverse plane which is orthogonal to the axis of the beam. The contact of a beam resting on two simple parallel supports is not a line contact.

When D^*_{16} is not zero, bending-twisting appears which is superposed on the previous phenomenon. This bending-twisting is zero for symmetric cross-ply beams.

These two actions mean that rectilinear contact cannot be maintained across the whole width of the beam when it is loaded.

These phenomena can be neglected when the ratio of width over length is small. In this case we can assume that u_3^0 only depends on x_1 and t.

The stresses in layer k are calculated from the expression:

$$\sigma_i^k = x_3 Q_{ij}^k \kappa_j = x_3 Q_{ij}^k D^*_{1j} M_1,$$

or:

$$\sigma_1^k = x_3\left(Q_{11}^k D^*_{11} + Q_{12}^k D^*_{12} + Q_{16}^k D^*_{16}\right)M_1,$$
$$\sigma_2^k = x_3\left(Q_{12}^k D^*_{11} + Q_{22}^k D^*_{12} + Q_{26}^k D^*_{16}\right)M_1,$$
$$\sigma_6^k = x_3\left(Q_{16}^k D^*_{11} + Q_{26}^k D^*_{12} + Q_{66}^k D^*_{16}\right)M_1.$$

These expressions show that the free edge boundary conditions are not satisfied; they are only globally satisfied. To minimize the influence of the free edges the ratio of height over width must be sufficiently small.

The transverse shear stress σ_5^k , in layer k, is obtained by integration of the first equation of local motion.

In the case of flexure, the first equilibrium equation:

$$\frac{\partial\sigma_{11}}{\partial x_1}+\frac{\partial\sigma_{12}}{\partial x_2}+\frac{\partial\sigma_{13}}{\partial x_3}=0\,,$$

is written, if we assume that u_3^0 only depends on x_1:

$$\frac{\partial\sigma_1}{\partial x_1}+\frac{\partial\sigma_5}{\partial x_3}=0\,,$$

for σ_6 is independent of x_2, hence:

$$\sigma_5^k=-\int_{-\frac{h}{2}}^{x_3}\frac{\partial\sigma_1}{\partial x_1}d\zeta\,,\ \text{with}\ z_{k-1}\le x_3\le z_k\,.$$

The layer by layer treatment enables the stress σ_5^k to be written in the form:

$$\sigma_5^k=-\sum_{j=1}^{k-1}\int_{z_{j-1}}^{z_j}\frac{\partial\sigma_1^j}{\partial x_1}d\zeta-\int_{z_{k-1}}^{x_3}\frac{\partial\sigma_1^k}{\partial x_1}d\zeta\,.$$

From the expression:

$$\sigma_1^j=x_3\left(Q_{11}^jD_{11}^*+Q_{12}^jD_{12}^*+Q_{16}^jD_{16}^*\right)M_1\,,$$

and the global equilibrium equation:

$$\frac{dM_1}{dx_1}-N_5=0\,,$$

we obtain:

$$\frac{\partial\sigma_1^j}{\partial x_1}=x_3\left(Q_{11}^jD_{11}^*+Q_{12}^jD_{12}^*+Q_{16}^jD_{16}^*\right)N_5\,.$$

The transverse shear stress σ_5^k can therefore be written as:

$$\sigma_5^k=-N_5\left\{\sum_{j=1}^{k-1}\int_{z_{j-1}}^{z_j}\left(Q_{11}^jD_{11}^*+Q_{12}^jD_{12}^*+Q_{16}^jD_{16}^*\right)\zeta d\zeta\ldots\right.$$

$$\left.\ldots+\int_{z_{k-1}}^{x_3}\left(Q_{11}^kD_{11}^*+Q_{12}^kD_{12}^*+Q_{16}^kD_{16}^*\right)\zeta d\zeta\right\},$$

or:

$$\sigma_5^k = -\frac{N_5}{2}\left\{\sum_{j=1}^{k-1}\left(Q_{11}^j D_{11}^* + Q_{12}^j D_{12}^* + Q_{16}^j D_{16}^*\right)\left(z_j^2 - z_{j-1}^2\right)\ldots\right.$$

$$\left.\ldots+\left(Q_{11}^k D_{11}^* + Q_{12}^k D_{12}^* + Q_{16}^k D_{16}^*\right)\left(x_3^2 - z_{k-1}^2\right)\right\}.$$

The global plate equation:

$$\frac{\partial^2 M_1}{\partial x_1^2} + 2\frac{\partial^2 M_6}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} - N_1^0 \frac{\partial^2 u_3^0}{\partial x_1^2} - 2N_6^0 \frac{\partial^2 u_3^0}{\partial x_1 \partial x_2} - N_2^0 \frac{\partial^2 u_3^0}{\partial x_2^2} + q = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

reduces, for beams, to the following:

$$\frac{\partial^2 M_1}{\partial x_1^2} - N_1^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

with:

$$I_0 = \sum_{k=1}^{N} \rho^k \left(z_k - z_{k-1}\right),$$

$$q = q_3 + \sum_{k=1}^{N} f_3^k \left(z_k - z_{k-1}\right).$$

By introducing the global flexural moment:

$$M_1 = -\frac{1}{D_{11}^*}\frac{\partial^2 u_3^0}{\partial x_1^2},$$

we obtain the equation:

$$-\frac{1}{D_{11}^*}\frac{\partial^4 u_3^0}{\partial x_1^4} - N_1^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

which in the following particular cases becomes:

– flexure:

$$\frac{1}{D_{11}^*}\frac{d^4 u_3^0}{dx_1^4} = q,$$

– vibration:

$$\frac{1}{D_{11}^*}\frac{\partial^4 u_3^0}{\partial x_1^4} + I_0 \frac{\partial^2 u_3^0}{\partial t^2} = 0,$$

– buckling:

$$\frac{1}{D_{11}^*}\frac{d^4 u_3^0}{dx_1^4} + N_1^0 \frac{d^2 u_3^0}{dx_1^2} = 0.$$

13.3. Monolayer orthotropic beam

The global flexural stiffnesses are equal to:

$$D_{ij} = \frac{h^3}{12}\overline{Q}_{ij},$$

with:

$$\overline{Q}_{11} = \frac{E_1}{1-\nu_{12}\nu_{21}}, \quad \overline{Q}_{22} = \frac{E_2}{1-\nu_{12}\nu_{21}},$$

$$\overline{Q}_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}.$$

The corresponding global compliances are equal to:

$$D_{11}^* = \frac{D_{22}}{D_{11}D_{22} - D_{12}^2} = \frac{12}{h^3}\frac{\overline{Q}_{22}}{\overline{Q}_{11}\overline{Q}_{22} - \overline{Q}_{12}^2} = \frac{12}{E_1 h^3},$$

$$D_{12}^* = -\frac{D_{12}}{D_{11}D_{22} - D_{12}^2} = -\frac{12}{h^3}\frac{\overline{Q}_{12}}{\overline{Q}_{11}\overline{Q}_{22} - \overline{Q}_{12}^2} = -\frac{12\nu_{12}}{E_1 h^3},$$

$$D_{16}^* = 0.$$

The strains are given by:

$$\varepsilon_1 = x_3 D_{11}^* M_1,$$

$$\varepsilon_2 = x_3 D_{12}^* M_1,$$

$$\varepsilon_6 = 0,$$

and the stresses by:

$$\sigma_1 = x_3\left(\overline{Q}_{11}D_{11}^* + \overline{Q}_{12}D_{12}^*\right)M_1 = \frac{12x_3M_1}{h^3},$$

$$\sigma_2 = x_3\left(\overline{Q}_{12}D_{11}^* + \overline{Q}_{22}D_{12}^*\right)M_1 = 0,$$

$$\sigma_6 = 0,$$

$$\sigma_5 = \frac{N_5}{2}\left(\overline{Q}_{11}D_{11}^* + \overline{Q}_{12}D_{12}^*\right)\left(\frac{h^2}{4} - x_3^2\right) = \frac{6N_5}{h^3}\left(\frac{h^2}{4} - x_3^2\right)\ldots$$

$$\ldots = \frac{3}{2h}\left[1 - 4\left(\frac{x_3}{h}\right)^2\right]N_5.$$

Multiplying by b the two parts of the following equations:

$$\frac{\partial M_1}{\partial x_1} - N_5 = 0,$$

$$M_1 = -\frac{1}{D_{11}^*}\frac{\partial^2 u_3^0}{\partial x_1^2} = -\frac{E_1 h^3}{12}\frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$\frac{\partial^2 M_1}{\partial x_1^2} - N_1^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

$$-\frac{E_1 h^3}{12} \frac{\partial^4 u_3^0}{\partial x_1^4} - N_1^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q = \rho h \frac{\partial^2 u_3^0}{\partial t^2},$$

and putting:

$$M_{f_2} = M_1 b, \quad T_3 = N_5 b, \quad N^0 = N_1^0 b, \quad q_0 = qb, \quad S = bh, \quad I_{22} = \frac{bh^3}{12},$$

we obtain the following well-known expressions:

$$T_3 = \frac{\partial M_{f_2}}{\partial x_1},$$

$$M_{f_2} = -E_1 I_{22} \frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$\frac{\partial^2 M_{f_2}}{\partial x_1^2} - N^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q_0 = \rho S \frac{\partial^2 u_3^0}{\partial t^2},$$

$$-E_1 I_{22} \frac{\partial^4 u_3^0}{\partial x_1^4} - N^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q_0 = \rho S \frac{\partial^2 u_3^0}{\partial t^2}.$$

13.4. General beam equations

In this section, we will write the global equations for beams in flexure with the notations used for isotropic beams. We will detail these equations in the case of flexure in the plane $(O|\mathbf{x}_1, \mathbf{x}_3)$ and in the plane $(O|\mathbf{x}_1, \mathbf{x}_2)$. It will be assumed, as for the classic flexural theory of beams, that the transverse displacement only depends on x_1 and t.

13.4.1. *Flexure in the plane* $(O|\mathbf{x}_1, \mathbf{x}_3)$

Introducing, for symmetrical monolayer beams, the equivalent characteristics:

$$(EI)_0 = \frac{b}{D_{11}^*},$$

$$(\rho S)_0 = I_0 b,$$

as well as:

$$q_0 = qb,$$

we obtain, in flexure, the global equations:

$$T_3 = -(EI)_0 \frac{\partial^3 u_3^0}{\partial x_1^3},$$

$$M_{f_2} = -(EI)_0 \frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$\frac{\partial^2 M_{f_2}}{\partial x_1^2} - N^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q_0 = (\rho S)_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

$$-(EI)_0 \frac{\partial^4 u_3^0}{\partial x_1^4} - N^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q_0 = (\rho S)_0 \frac{\partial^2 u_3^0}{\partial t^2}.$$

This case corresponds to that we used in the general composite theory.

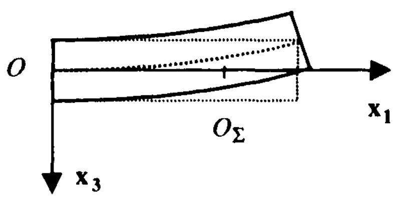

Figure 13.1. *Beam element deformed in the plane* $(O|\mathbf{x}_1, \mathbf{x}_3)$

In addition, starting from the displacement field defined by:

$$\mathbf{U}(M|t) = \mathbf{U}(O_\Sigma|t) + \mathbf{\Omega}(t) \times \mathbf{O_\Sigma M},$$

with:

$$\mathbf{U}(O_\Sigma|t) = u_3^0 \mathbf{x}_3, \ \mathbf{\Omega}(t) = \alpha_2 \mathbf{x}_2 \ \text{ and } \ \mathbf{O_\Sigma M} = x_2 \mathbf{x}_2 + x_3 \mathbf{x}_3,$$

we obtain the displacements:

$$u_1 = x_3 \alpha_2 (x_1|t),$$

$$u_3 = u_3^0 (x_1|t).$$

The strain tensor therefore has the components:

$$\varepsilon_1 = x_3 \frac{\partial \alpha_2}{\partial x_1},$$

$$\varepsilon_5 = 2\varepsilon_{13} = \frac{\partial u_3^0}{\partial x_1} + \alpha_2.$$

The hypothesis that $\varepsilon_5 = 0$ gives $\alpha_2 = -\frac{\partial u_3^0}{\partial x_1}$, from which:

$$u_1 = -x_3 \frac{\partial u_3^0}{\partial x_1},$$

$$\varepsilon_1 = -x_3 \frac{\partial^2 u_3^0}{\partial x_1^2}.$$

The axial stress in layer k of an orthotropic beam:

$$\sigma_1^k = -E^k x_3 \frac{\partial^2 u_3^0}{\partial x_1^2},$$

where E^k is Young's modulus in the direction of orthotropy $\mathbf{x}_1$ of layer k.

By definition, the flexural moment is equal to:

$$M_{f_2} = \mathbf{x}_2 \cdot \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} (x_2 \mathbf{x}_2 + x_3 \mathbf{x}_3) \times \sigma_1 \mathbf{x}_1 dx_3 dx_2,$$

or:

$$M_{f_2} = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_1 x_3 dx_3 dx_2,$$

from which:

$$M_{f_2} = -(EI)_0 \frac{\partial^2 u_3^0}{\partial x_1^2},$$

with:

$$(EI)_0 = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} E^k x_3^2 dx_3 dx_2 = \frac{b}{3} \sum_{k=1}^{N} E^k \left(z_k^3 - z_{k-1}^3 \right).$$

A positive flexural moment results in a negative curvature.

The global beam equations are written:

$$\frac{\partial M_{f_2}}{\partial x_1} - T_3 = 0,$$

$$\frac{\partial^2 M_{f_2}}{\partial x_1^2} - N^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q_0 = (\rho S)_0 \frac{\partial^2 u_3^0}{\partial t^2}.$$

The shear force is given by:

$$T_3 = \frac{\partial M_{f_2}}{\partial x_1} = -(EI)_0 \frac{\partial^3 u_3^0}{\partial x_1^3},$$

and the flexural moment by:

$$M_{f_2} = -(EI)_0 \frac{\partial^2 u_3^0}{\partial x_1^2}.$$

The global beam equation can be presented in the form:

$$-(EI)_0 \frac{\partial^4 u_3^0}{\partial x_1^4} - N^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q_0 = (\rho S)_0 \frac{\partial^2 u_3^0}{\partial t^2}.$$

13.4.2. *Flexure in the plane* $(O|\mathbf{x}_1, \mathbf{x}_2)$

This is the case generally used to present beam theory in strength of materials studies.

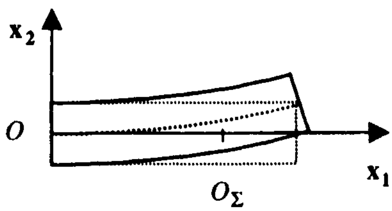

Figure 13.2. *Beam element deformed in the plane* $(O|\mathbf{x}_1, \mathbf{x}_2)$

The displacement field is expressed:

$$\mathbf{U}(M|t) = \mathbf{U}(O_\Sigma|t) + \mathbf{\Omega}(t) \times \mathbf{O_\Sigma M},$$

where:

$$\mathbf{U}(O_\Sigma|t) = u_2^0 \mathbf{x}_2, \quad \mathbf{\Omega}(t) = \alpha_3 \mathbf{x}_3 \quad \text{and} \quad \mathbf{O_\Sigma M} = x_2 \mathbf{x}_2 + x_3 \mathbf{x}_3.$$

The displacement vector has the components:

$$u_1 = -x_2 \alpha_3(x_1|t),$$

$$u_2 = u_2^0(x_1|t),$$

and the corresponding strains are equal to:

$$\varepsilon_1 = -x_2 \frac{\partial \alpha_3}{\partial x_1},$$

$$\varepsilon_6 = 2\varepsilon_{12} = \frac{\partial u_2^0}{\partial x_1} - \alpha_3.$$

The assumption $\varepsilon_6 = 0$ leads to:

$$\alpha_3 = \frac{\partial u_2^0}{\partial x_1}, \text{ as well as:}$$

$$u_1 = -x_3 \frac{\partial u_2^0}{\partial x_1},$$

$$\varepsilon_1 = -x_2 \frac{\partial^2 u_2^0}{\partial x_1^2}.$$

The axial stress in layer k, of an orthotropic beam is equal to:

$$\sigma_1^k = -E^k x_2 \frac{\partial^2 u_2^0}{\partial x_1^2},$$

and the flexural moment to:

$$M_{f_3} = \mathbf{x}_3 \cdot \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} (x_2 \mathbf{x}_2 + x_3 \mathbf{x}_3) \times \sigma_1 \mathbf{x}_1 dx_2 dx_3,$$

$$M_{f_3} = -\int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_1 x_2 dx_2 dx_3,$$

or:

$$M_{f_3} = (EI)_0 \frac{\partial^2 u_2^0}{\partial x_1^2},$$

with:

$$(EI)_0 = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} E^k x_2^2 dx_2 dx_3 = \frac{b}{3} \sum_{k=1}^{N} E^k \left(y_k^3 - y_{k-1}^3 \right).$$

Contrary to the previous case, here the flexural moment and the curvature have the same signs.

Since:

$$\int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_1 x_2 dx_2 dx_3 = -M_{f_3},$$

the global beam equations are given by the following expressions:

$$-\frac{\partial M_{f_3}}{\partial x_1} - T_2 = 0,$$

$$-\frac{\partial^2 M_{f_3}}{\partial x_1^2} - N^0 \frac{\partial^2 u_2^0}{\partial x_1^2} + q_0 = (\rho S)_0 \frac{\partial^2 u_2^0}{\partial t^2},$$

where the flexural moment and the shear force are given by the expressions:

$$M_{f_3} = (EI)_0 \frac{\partial^2 u_2^0}{\partial x_1^2},$$

$$T_2 = -\frac{\partial M_{f_3}}{\partial x_1} = -(EI)_0 \frac{\partial^3 u_2^0}{\partial x_1^3}.$$

The global equation is written therefore:

$$-(EI)_0 \frac{\partial^4 u_2^0}{\partial x_1^4} - N^0 \frac{\partial^2 u_2^0}{\partial x_1^2} + q_0 = (\rho S)_0 \frac{\partial^2 u_2^0}{\partial t^2}.$$

13.5. Simply supported beam subjected to sinusoidal loads

The beam OA of length l rests at O and A on two simple supports. The beam, of mean plane $(O|\mathbf{x}_1, \mathbf{x}_2)$ is subjected to a sinusoidal load:

$$q_0 = q_n \sin \frac{n\pi x_1}{l}.$$

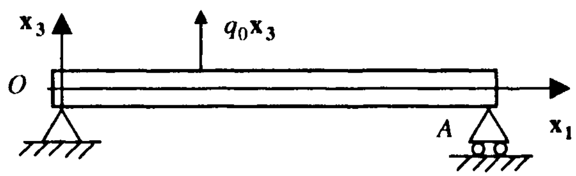

Figure 13.3. *Beam subjected to a sinusoidal load*

The global equilibrium equation is written:

$$-(EI)_0 \frac{d^4 u_3^0}{dx_1^4} + q_0 = 0,$$

and the boundary conditions are:

$$u_3^0(0) = u_3^0(l) = 0,$$
$$M_{f_2}(0) = M_{f_2}(l) = 0,$$

with:

$$M_{f_2} = -(EI)_0 \frac{d^2 u_3^0}{dx_1^2}.$$

The transverse displacement:

$$u_3^0(x) = U_n^3 \sin \frac{n\pi x_1}{l},$$

satisfies the boundary conditions and the global equilibrium equation. By introducing it into the latter we obtain the expression:

$$\left[-\left(\frac{n\pi}{l}\right)^4 (EI)_0 U_n^3 + q_n\right]\sin\frac{n\pi x_1}{l} = 0,$$

which has as solution:

$$U_n^3 = \left(\frac{l}{n\pi}\right)^4 \frac{q_n}{(EI)_0},$$

hence:

$$u_3^0(x) = \left(\frac{l}{n\pi}\right)^4 \frac{q_n}{(EI)_0}\sin\frac{n\pi x_1}{l}.$$

13.6. Vibrations of a simply supported beam

The beam OA, of length l and mean plane $(O|\mathbf{x}_1, \mathbf{x}_2)$, rests at O and A on two simple supports.

The transverse vibrations of the beam are governed by the equation:

$$(EI)_0 \frac{\partial^4 u_3^0}{\partial x_1^4} + (\rho S)_0 \frac{\partial^2 u_3^0}{\partial t^2} = 0.$$

The boundary conditions are:

$$u_3^0(0|t) = u_3^0(l|t) = 0,$$
$$M_{f_2}(0|t) = M_{f_2}(l|t) = 0,$$

with:

$$M_{f_2} = -(EI)_0 \frac{\partial^2 u_3^0}{\partial x_1^2}.$$

The boundary conditions and the global equation of vibration are satisfied by:

$$u_3^0(x|t) = U_n^3 \sin\frac{n\pi x_1}{l}\sin(\omega_n t + \varphi_n).$$

The equation of motion which becomes:

$$\left[\left(\frac{n\pi}{l}\right)^4 (EI)_0 - (\rho S)_0 \omega_n^2\right] U_n^3 \sin\frac{n\pi x_1}{l}\sin(\omega_n t + \varphi_n) = 0,$$

provides the natural frequencies:

$$\omega_n = \left(\frac{n\pi}{l}\right)^2 \sqrt{\frac{(EI)_0}{(\rho S)_0}}, \text{ with : } n = 1, 2, 3, \ldots.$$

13.7. Buckling of a simply supported beam

The beam OA of length l, has the mean plane $(O|\mathbf{x}_1,\mathbf{x}_2)$. It is articulated at O and rests at A on a simple support without friction. The beam is only subjected at A to the compression force $-F\mathbf{x}_1$, with $F>0$.

Figure 13.4. *Beam buckling*

The differential equation of global buckling is:

$$(EI)_0 \frac{d^4 u_3^0}{dx_1^4} + F\frac{d^2 u_3^0}{dx_1^2} = 0,$$

and the boundary conditions are written as:

$$u_3^0(0) = u_3^0(l) = 0,$$
$$M_{f_2}(0) = M_{f_2}(l) = 0,$$

with:

$$M_{f_2} = -(EI)_0 \frac{d^2 u_3^0}{dx_1^2}.$$

The transverse displacement:

$$u_3^0(x) = U_n^3 \sin\frac{n\pi x_1}{l},$$

satisfies the differential equation of global buckling and the boundary conditions.

The buckling equation which becomes:

$$\left[\left(\frac{n\pi}{l}\right)^4 (EI)_0 - \left(\frac{n\pi}{l}\right)^2 F\right] U_n^3 \sin\frac{n\pi x_1}{l} = 0,$$

leads to the critical buckling forces:

$$F_n = \left(\frac{n\pi}{l}\right)^2 (EI)_0.$$

For the first buckling mode the critical force is:

$$F_1 = \frac{\pi^2 (EI)_0}{l^2}.$$

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Chapter 14

Symmetrical multi-layer beams in flexure with transverse shear strain

14.1. Introduction

Having considered beams neglecting the transverse shear strains, we will now examine the influence of these strains on the transverse displacements, the natural frequencies and the critical buckling load.

The theory presented in this chapter corresponds in strength of materials to the Timoshenko beam theory for flexure.

14.2. Strains, stresses, global equations

The global constitutive relation for a symmetric plate in flexure is written:

$$\begin{bmatrix} M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix}, \quad \begin{bmatrix} N_4 \\ N_5 \end{bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{bmatrix},$$

and after inversion:

$$\begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} = \begin{bmatrix} D_{11}^* & D_{12}^* & D_{16}^* \\ D_{12}^* & D_{22}^* & D_{26}^* \\ D_{16}^* & D_{26}^* & D_{66}^* \end{bmatrix} \begin{bmatrix} M_1 \\ M_2 \\ M_6 \end{bmatrix}, \quad \begin{bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{bmatrix} = \frac{1}{K} \begin{bmatrix} A_{44}^* & A_{45}^* \\ A_{45}^* & A_{55}^* \end{bmatrix} \begin{bmatrix} N_4 \\ N_5 \end{bmatrix},$$

with:

$$\kappa_1 = \frac{\partial \psi_1}{\partial x_1}, \quad \kappa_2 = \frac{\partial \psi_2}{\partial x_2}, \quad \kappa_6 = \frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1},$$

and:

$$\varepsilon_4 = \frac{\partial u_3^0}{\partial x_2} + \psi_2, \quad \varepsilon_5 = \frac{\partial u_3^0}{\partial x_1} + \psi_1.$$

In the case of flexure in the $\mathbf{x_2}$ direction, we have:

$$M_2 = M_6 = 0 \text{ and } N_4 = 0,$$

hence:

$$\frac{\partial \psi_1}{\partial x_1} = D_{11}^* M_1,$$

$$\frac{\partial \psi_2}{\partial x_2} = D_{12}^* M_1,$$

$$\frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1} = D_{16}^* M_1,$$

and:

$$\frac{\partial u_3^0}{\partial x_2} + \psi_2 = \frac{A_{45}^*}{K} N_5,$$

$$\frac{\partial u_3^0}{\partial x_1} + \psi_1 = \frac{A_{55}^*}{K} N_5,$$

with:

$$D_{11}^* = \frac{D_{22}D_{66} - D_{26}^2}{D},$$

$$D_{12}^* = -\frac{D_{12}D_{66} - D_{16}D_{26}}{D},$$

$$D_{16}^* = \frac{D_{12}D_{26} - D_{22}D_{16}}{D},$$

$$D = \left(D_{11}D_{22} - D_{12}^2\right)D_{66} + 2D_{12}D_{16}D_{26} - D_{11}D_{26}^2 - D_{22}D_{16}^2,$$

and:

$$A_{55}^* = \frac{A_{44}}{A_{44}A_{55} - A_{45}^2},$$

$$A_{45}^* = -\frac{A_{45}}{A_{44}A_{55} - A_{45}^2}.$$

From these expressions ψ_1, ψ_2 and u_3^0 depend on x_1 and x_2.

As in the previous chapter, when the ratio of length over width is large, we can assume that ψ_1 and u_3^0 only depend on x_1 and on t.

The stresses in layer k, determined with the help of:

$$\sigma_i^k = x_3 Q_{ij}^k \kappa_j = x_3 Q_{ij}^k D_{1j}^* M_1 \qquad (i, j = 1, 2, 6),$$

$$\sigma_i^k = Q_{ij}^k \varepsilon_j = Q_{ij}^k \frac{A_{j5}}{K} N_5 \qquad (i, j = 4, 5),$$

are written:

$$\sigma_1^k = x_3\left(Q_{11}^k D_{11}^* + Q_{12}^k D_{12}^* + Q_{16}^k D_{16}^*\right)M_1,$$
$$\sigma_2^k = x_3\left(Q_{12}^k D_{11}^* + Q_{22}^k D_{12}^* + Q_{26}^k D_{16}^*\right)M_1,$$
$$\sigma_6^k = x_3\left(Q_{16}^k D_{11}^* + Q_{26}^k D_{12}^* + Q_{66}^k D_{16}^*\right)M_1,$$

and:

$$\sigma_4^k = \frac{1}{K}\left(Q_{44}^k A_{45}^* + Q_{45}^k A_{55}^*\right)N_5,$$
$$\sigma_5^k = \frac{1}{K}\left(Q_{45}^k A_{45}^* + Q_{55}^k A_{55}^*\right)N_5.$$

The boundary conditions for stress at the free edges of the beam are not satisfied. Their influence is minimized when the ratio of height to width is sufficiently small.

The global plate equations in flexure reduce to two equations:

$$\frac{\partial M_1}{\partial x_1} - N_5 = I_2 \frac{\partial^2 \psi_1}{\partial t^2},$$
$$\frac{\partial N_5}{\partial x_1} - N_1^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

which, after the introduction of:

$$M_1 = \frac{1}{D_{11}^*}\frac{\partial \psi_1}{\partial x_1},$$
$$N_5 = \frac{K}{A_{55}^*}\left(\frac{\partial u_3^0}{\partial x_1} + \psi_1\right),$$

are written

$$\frac{1}{D_{11}^*}\frac{\partial^2 \psi_1}{\partial x_1^2} - \frac{K}{A_{55}^*}\left(\frac{\partial u_3^0}{\partial x_1} + \psi_1\right) = I_2 \frac{\partial^2 \psi_1}{\partial t^2},$$
$$\frac{K}{A_{55}^*}\left(\frac{\partial^2 u_3^0}{\partial x_1^2} + \frac{\partial \psi_1}{\partial x_1}\right) - N_1^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q = I_0 \frac{\partial^2 u_3^0}{\partial t^2}.$$

In the following particular cases we have:

– flexure:

$$\frac{1}{D_{11}^*}\frac{d^2 \psi_1}{dx_1^2} - \frac{K}{A_{55}^*}\left(\frac{du_3^0}{dx_1} + \psi_1\right) = 0,$$

$$\frac{K}{A_{55}^*}\left(\frac{d^2u_3^0}{dx_1^2}+\frac{d\psi_1}{dx_1}\right)+q=0,$$

– vibration:

$$\frac{1}{D_{11}^*}\frac{\partial^2\psi_1}{\partial x_1^2}-\frac{K}{A_{55}^*}\left(\frac{\partial u_3^0}{\partial x_1}+\psi_1\right)=I_2\frac{\partial^2\psi_1}{\partial t^2},$$

$$\frac{K}{A_{55}^*}\left(\frac{\partial^2u_3^0}{\partial x_1^2}+\frac{\partial\psi_1}{\partial x_1}\right)=I_0\frac{\partial^2u_3^0}{\partial t^2},$$

– buckling:

$$\frac{1}{D_{11}^*}\frac{d^2\psi_1}{dx_1^2}-\frac{K}{A_{55}^*}\left(\frac{du_3^0}{dx_1}+\psi_1\right)=0,$$

$$\frac{K}{A_{55}^*}\left(\frac{d^2u_3^0}{dx_1^2}+\frac{d\psi_1}{dx_1}\right)-N_1^0\frac{d^2u_3^0}{dx_1^2}=0.$$

14.3. Monolayer orthotropic beam

In addition to the global stiffnesses:

$$D_{11}^*=\frac{12}{E_1h^3},\ D_{12}^*=-\frac{12\nu_{12}}{E_1h^3},$$

we have:

$$A_{55}^*=\frac{1}{A_{55}}=\frac{1}{G_{13}h}.$$

The strains are equal to:

$$\varepsilon_1=x_3D_{11}^*M_1,$$

$$\varepsilon_2=x_3D_{12}^*M_1,$$

$$\varepsilon_5=\frac{A_{55}^*}{K}N_5,$$

$$\varepsilon_3=\varepsilon_4=\varepsilon_6=0,$$

and the stresses to:

$$\sigma_1=x_3\left(\overline{Q}_{11}D_{11}^*+\overline{Q}_{12}D_{12}^*\right)M_1=\frac{12x_3}{h^3}M_1,$$

$$\sigma_2=x_3\left(\overline{Q}_{12}D_{11}^*+\overline{Q}_{22}D_{12}^*\right)M_1=0,$$

$$\sigma_5 = \overline{Q}_{55} \frac{A_{55}^*}{K} N_5,$$
$$\sigma_3 = \sigma_4 = \sigma_6 = 0.$$

Multiplying by b the two members of:

$$M_1 = \frac{1}{D_{11}^*} \frac{\partial \psi_1}{\partial x_1} = \frac{E_1 h^3}{12} \frac{\partial \psi_1}{\partial x_1},$$

$$N_5 = \frac{K}{A_{55}^*} \left(\frac{\partial u_3^0}{\partial x_1} + \psi_1 \right) = KG_{13} h \left(\frac{\partial u_3^0}{\partial x_1} + \psi_1 \right),$$

$$\frac{\partial M_1}{\partial x_1} - N_5 = I_2 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial N_5}{\partial x_1} - N_1^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

$$\frac{E_1 h^3}{12} \frac{\partial^2 \psi_1}{\partial x_1^2} - KG_{13} h \left(\frac{\partial u_3^0}{\partial x_1} + \psi_1 \right) = \frac{\rho h^3}{12} \frac{\partial^2 \psi_1}{\partial t^2},$$

$$KG_{13} h \left(\frac{\partial^2 u_3^0}{\partial x_1^2} + \frac{\partial \psi_1}{\partial x_1} \right) - N_1^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q = \rho h \frac{\partial^2 u_3^0}{\partial t^2},$$

and introducing:

$$M_{f_2} = M_1 b, \quad T = N_5 b, \quad N^0 = N_1^0 b, \quad q_0 = qb, \quad S = bh, \quad I_{22} = \frac{bh^3}{12},$$

we obtain the following expressions:

$$M_{f_2} = E_1 I_{22} \frac{\partial \psi_1}{\partial x_1},$$

$$T = KG_{13} S \left(\frac{\partial u_3^0}{\partial x_1} + \psi_1 \right),$$

$$\frac{\partial M_{f_2}}{\partial x_1} - T = I_{22} \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial T}{\partial x_1} - N^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q_0 = \rho S \frac{\partial^2 u_3^0}{\partial t^2},$$

$$E_1 I_{22} \frac{\partial^2 \psi_1}{\partial x_1^2} - KG_{13} S \left(\frac{\partial u_3^0}{\partial x_1} + \psi_1 \right) = I_{22} \frac{\partial^2 \psi_1}{\partial t^2},$$

$$KG_{13} S \left(\frac{\partial^2 u_3^0}{\partial x_1^2} + \frac{\partial \psi_1}{\partial x_1} \right) - N^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q_0 = \rho S \frac{\partial^2 u_3^0}{\partial t^2}.$$

14.4. General beam equations

As in the previous chapter, we will detail the equations for flexure in the planes $(O|\mathbf{x}_1, \mathbf{x}_3)$ and $(O|\mathbf{x}_1, \mathbf{x}_2)$.

14.4.1. *Flexure in the plane* $(O|\mathbf{x}_1, \mathbf{x}_3)$

Introducing, for multi-layer, symmetric beams, the equivalent characteristics:

$$(EI)_0 = \frac{b}{D_{11}^*}, \ (\rho S)_0 = I_0 b, \ (KGS)_0 = \frac{Kb}{A_{55}^*},$$

as well as:

$$q_0 = qb,$$

with:

$$A_{55}^* = \frac{A_{44}}{A_{44}A_{55} - A_{45}^2},$$

$$A_{ij} = \sum_{k=1}^{N} Q_{ij}^k \left(z_k - z_{k-1}\right),$$

and:

$$(\rho I)_0 = I_2 b,$$

we obtain, in flexure, the following global equations:

$$M_{f_2} = (EI)_0 \frac{\partial \psi_1}{\partial x_1},$$

$$T_3 = (KGS)_0 \left(\frac{\partial u_3^0}{\partial x_1} + \psi_1 \right),$$

$$\frac{\partial M_{f_2}}{\partial x_1} - T_3 = (\rho I)_0 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial T_3}{\partial x_1} - N^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q_0 = (\rho S)_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

$$(EI)_0 \frac{\partial^2 \psi_1}{\partial x_1^2} - (KGS)_0 \left(\frac{\partial u_3^0}{\partial x_1} + \psi_1 \right) = (\rho I)_0 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$(KGS)_0 \left(\frac{\partial^2 u_3^0}{\partial x_1^2} + \frac{\partial \psi_1}{\partial x_1} \right) - N^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q_0 = (\rho S)_0 \frac{\partial^2 u_3^0}{\partial t^2}.$$

With the following expressions for the displacements:

$$u_1 = x_3\alpha_2(x_1|t),$$
$$u_3 = u_3(x_1|t),$$

we obtain:

– the strains:

$$\varepsilon_1 = x_3\frac{\partial\alpha_2}{\partial x_1},$$
$$\varepsilon_5 = \frac{\partial u_3}{\partial x_1} + \alpha_2,$$

– the stresses in layer k of an orthotropic beam:

$$\sigma_1^k = E_1^k x_3\frac{\partial\alpha_2}{\partial x_1},$$
$$\sigma_5^k = G_{13}^k\left(\frac{\partial u_3}{\partial x_1} + \alpha_2\right),$$

– the flexural moment:

$$M_{f_2} = \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_1 x_3 dx_3 dx_2,$$
$$M_{f_2} = (EI)_0\frac{\partial\alpha_2}{\partial x_1}, \text{ with } (EI)_0 = \frac{b}{3}\sum_{k=1}^{N} E_1^k\left(z_k^3 - z_{k-1}^3\right),$$

– the shear force:

$$T_3 = (KGS)_0\left(\frac{\partial u_3}{\partial x_1} + \alpha_2\right),$$

with :

$$(KGS)_0 = \int_{-\frac{b}{2}}^{\frac{b}{2}}\int_{-\frac{h}{2}}^{\frac{h}{2}} KG_{13} dx_3 dx_2 = Kb\sum_{k=1}^{N} G_{13}^k(z_k - z_{k-1}).$$

The global beam equations:

$$\frac{\partial M_{f_2}}{\partial x_1} - T_3 = (\rho I)_0\frac{\partial^2\alpha_2}{\partial t^2},$$
$$\frac{\partial T_3}{\partial x_1} - N^0\frac{\partial^2 u_3^0}{\partial x_1^2} + q_0 = (\rho S)_0\frac{\partial^2 u_3^0}{\partial t^2},$$

are written, after introduction of the expressions for M_{f_2} and T_3, in the following form:

$$(EI)_0 \frac{\partial^2 \alpha_2}{\partial x_1^2} - (KGS)_0 \left(\frac{\partial u_3^0}{\partial x_1} + \alpha_2 \right) = (\rho I)_0 \frac{\partial^2 \alpha_2}{\partial t^2},$$

$$(KGS)_0 \left(\frac{\partial^2 u_3^0}{\partial x_1^2} + \frac{\partial \alpha_2}{\partial x_1} \right) - N^0 \frac{\partial^2 u_3^0}{\partial x_1^2} + q_0 = (\rho S)_0 \frac{\partial^2 u_3^0}{\partial t^2}.$$

14.4.2. *Flexure in the plane* $(O|\mathbf{x}_1, \mathbf{x}_2)$

From the expressions:

$$u_1 = -x_2 \alpha_3 (x_1 | t),$$

$$u_2 = u_2^0 (x_1 | t),$$

we obtain:

$$\varepsilon_1 = -x_2 \frac{\partial \alpha_3}{\partial x_1},$$

$$\varepsilon_6 = \frac{\partial u_2^0}{\partial x_1} - \alpha_3,$$

as well as:

$$\sigma_1^k = -E_1^k x_2 \frac{\partial \alpha_3}{\partial x_1},$$

$$\sigma_6^k = G_{12}^k \left(\frac{\partial u_2^0}{\partial x_1} - \alpha_3 \right).$$

The flexural moment is equal to :

$$M_{f_3} = -\int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_1 x_2 dx_2 dx_3,$$

or:

$$M_{f_3} = (EI)_0 \frac{\partial \alpha_3}{\partial x_1}, \text{ with } (EI)_0 = \frac{b}{3} \sum_{k=1}^{N} E_1^k \left(y_k^3 - y_{k-1}^3 \right),$$

and the shear force to:

$$T_2 = \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_6 dx_2 dx_3,$$

or:

$$T_2 = (KGS)_0 \left(\frac{\partial u_2}{\partial x_1} - \alpha_3 \right), \text{ with } (KGS)_0 = Kb \sum_{k=1}^{N} G_{12}^k (y_k - y_{k-1}).$$

The global beam equations can be written:

$$\frac{\partial M_{f_3}}{\partial x_1} + T_2 = (\rho I)_0 \frac{\partial^2 \alpha_3}{\partial t^2},$$

$$\frac{\partial T_2}{\partial x_1} - N^0 \frac{\partial^2 u_2^0}{\partial x_1^2} + q_0 = (\rho S)_0 \frac{\partial^2 u_2^0}{\partial t^2},$$

or:

$$(EI)_0 \frac{\partial^2 \alpha_3}{\partial x_1^2} + (KGS)_0 \left(\frac{\partial u_2^0}{\partial x_1} - \alpha_3 \right) = (\rho I)_0 \frac{\partial^2 \alpha_3}{\partial t^2},$$

$$(KGS)_0 \left(\frac{\partial^2 u_2^0}{\partial x_1^2} - \frac{\partial \alpha_3}{\partial x_1} \right) - N^0 \frac{\partial^2 u_2^0}{\partial x_1^2} + q_0 = (\rho S)_0 \frac{\partial^2 u_2^0}{\partial t^2}.$$

14.5. Simply supported beam subjected to a sinusoidal load

The multi-layer beam OA, with mean plane $(O|\mathbf{x}_1, \mathbf{x}_2)$ and of length l, rests at O and A on two simple supports. It is subjected to a sinusoidal load:

$$q_0 = q_n \sin \frac{n\pi x_1}{l}.$$

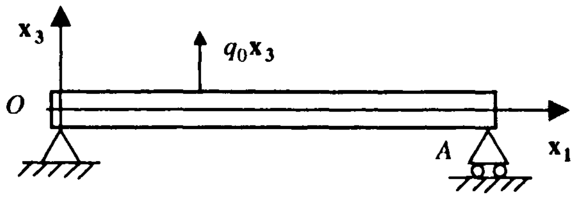

Figure 14.1. *Beam subjected to a sinusoidal load*

The global equilibrium equations are written:

$$(EI)_0 \frac{d^2 \psi_1}{dx_1^2} - (KGS)_0 \left(\frac{du_3^0}{dx_1} + \psi_1 \right) = 0,$$

$$(KGS)_0 \left(\frac{d^2 u_3^0}{dx_1^2} + \frac{d\psi_1}{dx_1} \right) + q_0 = 0,$$

and the boundary conditions are:

$$u_3^0(0) = u_3^0(l) = 0,$$
$$M_{f_2}(0) = M_{f_2}(l) = 0,$$

with:

$$M_{f_2} = (EI)_0 \frac{d\psi_1}{dx_1}.$$

The displacements field defined by:

$$u_3^0 = U_n^3 \sin\frac{n\pi x_1}{l},$$

$$\psi_1 = \Psi_n^1 \cos\frac{n\pi x_1}{l},$$

satisfies the equilibrium equations and the boundary conditions. The two equilibrium equations provide the system:

$$\left[\left(\frac{n\pi}{l}\right)^2 (EI)_0 \Psi_n^1 + (KGS)_0\left(\frac{n\pi}{l}U_n^3 + \Psi_n^1\right)\right]\cos\frac{n\pi x_1}{l} = 0,$$

$$\left\{(KGS)_0\left[\left(\frac{n\pi}{l}\right)^2 U_n^3 + \frac{n\pi}{l}\Psi_n^1\right] - q_n\right\}\sin\frac{n\pi x_1}{l} = 0,$$

which can be presented in the form:

$$\begin{bmatrix} \left(\frac{n\pi}{l}\right)^2 (EI)_0 + (KGS)_0 & \frac{n\pi}{l}(KGS)_0 \\ \frac{n\pi}{l}(KGS)_0 & \left(\frac{n\pi}{l}\right)^2 (KGS)_0 \end{bmatrix} \begin{bmatrix} \Psi_n^1 \\ U_n^3 \end{bmatrix} = \begin{bmatrix} 0 \\ q_n \end{bmatrix}.$$

Given the values of the determinant of the system:

$$\Delta = \left(\frac{n\pi}{l}\right)^4 (EI)_0 (KGS)_0,$$

and of the determinants relative to the two unknowns Ψ_n^1 and U_n^3:

$$\Delta_1 = -\frac{n\pi}{l}(KGS)_0 q_n, \quad \Delta_2 = \left[\left(\frac{n\pi}{l}\right)^2 (EI)_0 + (KGS)_0\right] q_n,$$

the solution to the system is:

$$\Psi_n^1 = -\left(\frac{l}{n\pi}\right)^3 \frac{q_n}{(EI)_0},$$

$$U_n^3 = \frac{\left(\frac{n\pi}{l}\right)^2 (EI)_0 + (KGS)_0}{\left(\frac{n\pi}{l}\right)^4 (EI)_0 (KGS)_0} q_n = \left(\frac{l}{n\pi}\right)^4 \left[1 + \left(\frac{n\pi}{l}\right)^2 \frac{(EI)_0}{(KGS)_0}\right] \frac{q_n}{(EI)_0}.$$

Finally, the displacement field is defined by:

$$\psi_1 = -\left(\frac{l}{n\pi}\right)^3 \frac{q_n}{(EI)_0}\cos\frac{n\pi x_1}{l},$$

$$u_3^0 = \left(\frac{l}{n\pi}\right)^4\left[1+\left(\frac{n\pi}{l}\right)^2\frac{(EI)_0}{(KGS)_0}\right]\frac{q_n}{(EI)_0}\sin\frac{n\pi x_1}{l}.$$

The maximum deflection, obtained at $x_1 = \frac{l}{2}$, is equal to:

$$U_n^3 = \left(\frac{l}{n\pi}\right)^4\left[1+\left(\frac{n\pi}{l}\right)^2\frac{(EI)_0}{(KGS)_0}\right]\frac{q_n}{(EI)_0}.$$

If we ignore the transverse shear strains, it is equal to:

$$\overline{U}_n^3 = \left(\frac{l}{n\pi}\right)^4\frac{q_n}{(EI)_0},$$

and the relative variation of the deflection is:

$$\frac{U_n^3 - \overline{U}_n^3}{\overline{U}_n^3} = \left(\frac{n\pi}{l}\right)^2\frac{(EI)_0}{(KGS)_0}.$$

14.6. Vibration of a simply supported beam

The beam OA of length l and mean plane $(O|\mathbf{x}_1,\mathbf{x}_2)$, rests on two simple supports at O and at A. It is not subjected to any given load.

The transverse vibrations of the beam are governed by the two equations:

$$(EI)_0\frac{\partial^2\psi_1}{\partial x_1^2} - (KGS)_0\left(\frac{\partial u_3^0}{\partial x_1}+\psi_1\right) = (\rho I)_0\frac{\partial^2\psi_1}{\partial t^2},$$

$$(KGS)_0\left(\frac{\partial^2 u_3^0}{\partial x_1^2}+\frac{\partial\psi_1}{\partial x_1}\right) = (\rho S)_0\frac{\partial^2 u_3^0}{\partial t^2}.$$

The boundary conditions are written:

$$u_3^0(0|t) = u_3^0(l|t) = 0,$$

$$M_{f_2}(0|t) = M_{f_2}(l|t) = 0,$$

with:

$$M_{f_2} = (EI)_0\frac{\partial\psi_1}{\partial x_1}.$$

The following expressions for u_3^0 and ψ_1:

$$u_3^0(x_1|t) = U_n^3 \sin\frac{n\pi x_1}{l}\sin(\omega_n t + \varphi_n),$$

$$\psi_1(x_1|t) = \Psi_n^1 \cos\frac{n\pi x_1}{l}\sin(\omega_n t + \varphi_n),$$

satisfy the boundary conditions and the global equations of motion which become:

$$\left[\left(\frac{n\pi}{l}\right)^2 (EI)_0 \Psi_n^1 + (KGS)_0\left(\frac{n\pi}{l}U_n^3 + \Psi_n^1\right)\ldots\right.$$

$$\left.\ldots - (\rho I)_0\omega_n^2\Psi_n^1 \right]\cos\frac{n\pi x_1}{l}\sin(\omega_n t + \varphi_n) = 0,$$

$$\left\{(KGS)_0\left[\left(\frac{n\pi}{l}\right)^2 U_n^3 + \frac{n\pi}{l}\Psi_n^1\right] - (\rho S)_0\omega_n^2 U_n^3\right\}\sin\frac{n\pi x_1}{l}\sin(\omega_n t + \varphi_n) = 0.$$

After simplification, we obtain the system:

$$\begin{bmatrix} \left(\frac{n\pi}{l}\right)^2 (EI)_0 + (KGS)_0 - (\rho I)_0\omega_n^2 & \frac{n\pi}{l}(KGS)_0 \\ \frac{n\pi}{l}(KGS)_0 & \left(\frac{n\pi}{l}\right)^2 (KGS)_0 - (\rho S)_0\omega_n^2 \end{bmatrix} \begin{bmatrix} \Psi_n^1 \\ U_n^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which has a solution other than the trivial solution $\Psi_n^1 = 0$ and $U_n^3 = 0$ for the values of ω_n which cancels the determinant of the system. The equation for the natural frequencies:

$$\left[\left(\frac{n\pi}{l}\right)^2 (EI)_0 + (KGS)_0 - (\rho I)_0\omega_n^2\right]\left[\left(\frac{n\pi}{l}\right)^2 (KGS)_0 - (\rho S)_0\omega_n^2\right]\ldots$$

$$\ldots - \left[\frac{n\pi}{l}(KGS)_0\right]^2 = 0,$$

is written:

$$(\rho I)_0(\rho S)_0\omega_n^4 - \left\{\left[\left(\frac{n\pi}{l}\right)^2 (EI)_0 + (KGS)_0\right](\rho S)_0 + \left(\frac{n\pi}{l}\right)^2 (KGS)_0(\rho I)_0\right\}\omega_n^2\ldots$$

$$\ldots + \left(\frac{n\pi}{l}\right)^4 (EI)_0(KGS)_0 = 0,$$

or:

$$\omega_n^4 - \left\{ \left(\frac{n\pi}{l} \right)^2 \left[\frac{(EI)_0}{(\rho I)_0} + \frac{(KGS)_0}{(\rho S)_0} \right] + \frac{(KGS)_0}{(\rho I)_0} \right\} \omega_n^2 + \left(\frac{n\pi}{l} \right)^4 \frac{(EI)_0 (KGS)_0}{(\rho I)_0 (\rho S)_0} = 0 .$$

The discriminant of this equation:

$$\Delta = \left(\frac{n\pi}{l} \right)^4 \left[\frac{(EI)_0}{(\rho I)_0} + \frac{(KGS)_0}{(\rho S)_0} \right]^2 + \left[\frac{(KGS)_0}{(\rho I)_0} \right]^2 \ldots$$

$$\ldots + 2 \left(\frac{n\pi}{l} \right)^2 \frac{(KGS)_0}{(\rho I)_0} \left[\frac{(EI)_0}{(\rho I)_0} + \frac{(KGS)_0}{(\rho S)_0} \right] - 4 \left(\frac{n\pi}{l} \right)^4 \frac{(EI)_0 (KGS)_0}{(\rho I)_0 (\rho S)_0},$$

or:

$$\Delta = \left(\frac{n\pi}{l} \right)^4 \left[\frac{(EI)_0}{(\rho I)_0} - \frac{(KGS)_0}{(\rho S)_0} \right]^2 + \left[\frac{(KGS)_0}{(\rho I)_0} \right]^2 \ldots$$

$$\ldots + 2 \left(\frac{n\pi}{l} \right)^2 \frac{(KGS)_0}{(\rho I)_0} \left[\frac{(EI)_0}{(\rho I)_0} + \frac{(KGS)_0}{(\rho S)_0} \right],$$

is positive.

The natural frequencies are equal to:

$$\omega_{n,1}^2 = \frac{1}{2} \left\{ \left(\frac{n\pi}{l} \right)^2 \left[\frac{(EI)_0}{(\rho I)_0} + \frac{(KGS)_0}{(\rho S)_0} \right] + \frac{(KGS)_0}{(\rho I)_0} \ldots \right.$$

$$\left. \ldots - \sqrt{ \left\{ \left(\frac{n\pi}{l} \right)^2 \left[\frac{(EI)_0}{(\rho I)_0} + \frac{(KGS)_0}{(\rho S)_0} \right] + \frac{(KGS)_0}{(\rho I)_0} \right\}^2 - 4 \left(\frac{n\pi}{l} \right)^4 \frac{(EI)_0 (KGS)_0}{(\rho I)_0 (\rho S)_0} } \right\},$$

$$\omega_{n,2}^2 = \frac{1}{2} \left\{ \left(\frac{n\pi}{l} \right)^2 \left[\frac{(EI)_0}{(\rho I)_0} + \frac{(KGS)_0}{(\rho S)_0} \right] + \frac{(KGS)_0}{(\rho I)_0} \ldots \right.$$

$$\left. \ldots + \sqrt{ \left\{ \left(\frac{n\pi}{l} \right)^2 \left[\frac{(EI)_0}{(\rho I)_0} + \frac{(KGS)_0}{(\rho S)_0} \right] + \frac{(KGS)_0}{(\rho I)_0} \right\}^2 - 4 \left(\frac{n\pi}{l} \right)^4 \frac{(EI)_0 (KGS)_0}{(\rho I)_0 (\rho S)_0} } \right\}.$$

For each value of n, there are two natural frequencies.

If we neglect the inertia of rotation, the two global vibration equations may be written:

$$(EI)_0 \frac{\partial^2 \psi_1}{\partial x_1^2} - (KGS)_0 \left(\frac{\partial u_3^0}{\partial x_1} + \psi_1 \right) = 0,$$

$$(KGS)_0\left(\frac{\partial^2 u_3^0}{\partial x_1^2}+\frac{\partial \psi_1}{\partial x_1}\right)=(\rho S)_0\frac{\partial^2 u_3^0}{\partial t^2}.$$

With:

$$u_3^0(x_1|t)=U_n^3\sin\frac{n\pi x_1}{l}\sin(\omega_n t+\varphi_n),$$

$$\psi_1(x_1|t)=\Psi_n^1\cos\frac{n\pi x_1}{l}\sin(\omega_n t+\varphi_n),$$

we obtain the system:

$$\begin{bmatrix}\left(\frac{n\pi}{l}\right)^2(EI)_0+(KGS)_0 & \frac{n\pi}{l}(KGS)_0\\ \frac{n\pi}{l}(KGS)_0 & \left(\frac{n\pi}{l}\right)^2(KGS)_0-(\rho S)_0\omega_n^2\end{bmatrix}\begin{bmatrix}\Psi_n^1\\ U_n^3\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix},$$

and the equation for the natural frequencies:

$$\left(\frac{n\pi}{l}\right)^4(EI)_0(KGS)_0-\left[\left(\frac{n\pi}{l}\right)^2(EI)_0+(KGS)_0\right](\rho S)_0\omega_n^2=0,$$

for which the solution is:

$$\omega_n^2=\frac{\left(\frac{n\pi}{l}\right)^4\frac{(EI)_0}{(\rho S)_0}}{1+\left(\frac{n\pi}{l}\right)^2\frac{(EI)_0}{(KGS)_0}}.$$

The values of natural frequencies thus obtained are lower than those calculated neglecting the transverse shear strains:

$$\overline{\omega}_n^2=\left(\frac{n\pi}{l}\right)^4\frac{(EI)_0}{(\rho S)_0},$$

the corresponding relative error is equal to:

$$\frac{\omega_n^2-\overline{\omega}_n^2}{\overline{\omega}_n^2}=\frac{1}{1+\left(\frac{n\pi}{l}\right)^2\frac{(EI)_0}{(KGS)_0}}-1.$$

14.7. Buckling of a simply supported beam

The beam OA, of length l and mean plane $(O|\mathbf{x}_1,\mathbf{x}_2)$, is subjected at A, as shown in figure 14.2, to the force $-F\mathbf{x}_1$, with $F>0$.

Figure 14.2. *Beam buckling*

The differential equations for global buckling are written:

$$(EI)_0 \frac{d^2\psi_1}{dx_1^2} - (KGS)_0\left(\frac{du_3^0}{dx_1} + \psi_1\right) = 0,$$

$$(KGS)_0\left(\frac{d^2u_3^0}{dx_1^2} + \frac{d\psi_1}{dx_1}\right) - F\frac{d^2u_3^0}{dx_1^2} = 0,$$

and the boundary conditions:

$$u_3^0(0) = u_3^0(l) = 0,$$

$$M_{f_2}(0) = M_{f_2}(l) = 0,$$

with:

$$M_{f_2} = (EI)_0 \frac{d\psi_1}{dx_1}.$$

The buckling equations and the boundary conditions are satisfied by:

$$u_3^0 = U_n^3 \sin\frac{n\pi x_1}{l},$$

$$\psi_1 = \Psi_n^1 \cos\frac{n\pi x_1}{l}.$$

Introduction into the buckling equations leads to the two expressions:

$$\left[\left(\frac{n\pi}{l}\right)^2 (EI)_0 \Psi_n^1 + (KGS)_0\left(\frac{n\pi}{l}U_n^3 + \Psi_n^1\right)\right]\cos\frac{n\pi x_1}{l} = 0,$$

$$\left\{(KGS)_0\left[\left(\frac{n\pi}{l}\right)^2 U_n^3 + \frac{n\pi}{l}\Psi_n^1\right] - \left(\frac{n\pi}{l}\right)^2 FU_n^3\right\}\sin\frac{n\pi x_1}{l} = 0,$$

and the system:

$$\begin{bmatrix} \left(\frac{n\pi}{l}\right)^2 (EI)_0 + (KGS)_0 & \frac{n\pi}{l}(KGS)_0 \\ \frac{n\pi}{l}(KGS)_0 & \left(\frac{n\pi}{l}\right)^2 (KGS)_0 - \left(\frac{n\pi}{l}\right)^2 F \end{bmatrix} \begin{bmatrix} \Psi_n^1 \\ U_n^3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which has a solution other than the trivial solution for the values of F which cancel its determinant, that is for the F solution of the equation:

$$\left(\frac{n\pi}{l}\right)^4 (EI)_0 (KGS)_0 - \left(\frac{n\pi}{l}\right)^2 \left[\left(\frac{n\pi}{l}\right)^2 (EI)_0 + (KGS)_0\right] F = 0 .$$

The critical force of the buckling mode n is equal to:

$$F_n = \frac{\left(\frac{n\pi}{l}\right)^2 (EI)_0}{1 + \left(\frac{n\pi}{l}\right)^2 \frac{(EI)_0}{(KGS)_0}} .$$

These values are lower than those obtained when we neglect the transverse shear strains:

$$\overline{F}_n = \left(\frac{n\pi}{l}\right)^2 (EI)_0 .$$

The corresponding relative error is:

$$\frac{F_n - \overline{F}_n}{\overline{F}_n} = \frac{1}{1 + \left(\frac{n\pi}{l}\right)^2 \frac{(EI)_0}{(KGS)_0}} - 1 .$$

The first buckling mode is characterized by the critical force:

$$F_1 = \frac{\frac{\pi^2 (EI)_0}{l^2}}{1 + \frac{\pi^2 (EI)_0}{l^2 (KGS)_0}} .$$

APPENDICES

Global Plate Equations

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Chapter 15

Global plate equations neglecting large transverse displacements

15.1. Introduction

In this appendix, we present the plate analysis equations, integrating the local equations when the three absolute values of displacements are low in comparison with the plate thickness, and when the absolute values of the components of displacement vector gradient are much lower than one.

We will limit this study to the resultant forces and moments and the inertia forces in the Reissner-Mindlin and Kirchhoff-Love type analyses.

15.2. Hypothesis relating to plates

A plate is a continuum limited by two parallel planes corresponding to the lower and top surfaces of the plate, and by a cylindrical surface (edge of the plate) orthogonal to the faces of the plate.

The middle plane of the plate is equidistant from the lower and upper surfaces and these are separated by a distance h.

The middle plane of the plate is located in the plane $(O|\mathbf{x}_1,\mathbf{x}_2)$ of the reference axes $(O|\mathbf{x}_1,\mathbf{x}_2,\mathbf{x}_3)$.

The rectangular plate as shown below has the middle surface represented by a rectangle, the dimensions of which are a_1 and a_2 respectively.

The plate is specified as "thin" if the thickness is small in comparison with the middle surface dimensions, which is the case when the thickness to characteristic dimension of the middle surface ratio is lower than 1/20. The term "moderately thick" is used when the ratio is between 1/5 and 1/20.

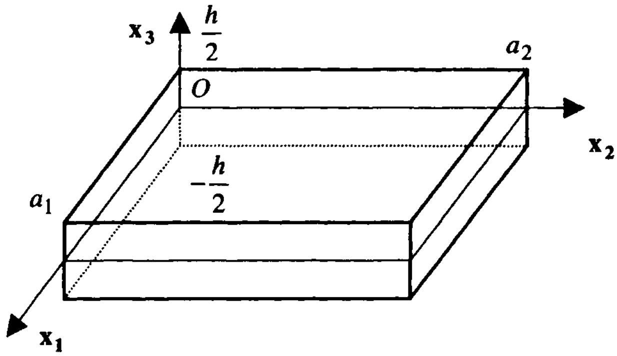

Figure 15.1. *Thin plate*

In this appendix, we assume that the absolute values of displacements u_i are small in comparison with h and that the absolute values of the partially derived functions $\frac{\partial u_i}{\partial x_j}$ are less than 1.

15.3. Reissner-Mindlin and Kirchhoff-Love plate theories

In these two plate analyses, we are assuming that u_1 and u_2 depend on x_1, x_2, x_3, and t, and u_3 depends on x_1, x_2 and t only. The chosen displacement field is written as:

$$u_1 = u_1^0(x_1, x_2|t) + x_3\psi_1(x_1, x_2|t),$$
$$u_2 = u_2^0(x_1, x_2|t) + x_3\psi_2(x_1, x_2|t),$$
$$u_3 = u_3^0(x_1, x_2|t),$$

The strain field is then defined by:

$$\varepsilon_{11} = \frac{\partial u_1^0}{\partial x_1} + x_3\frac{\partial \psi_1}{\partial x_1}, \quad 2\varepsilon_{12} = \frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1} + x_3\left(\frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1}\right),$$

$$\varepsilon_{22} = \frac{\partial u_2^0}{\partial x_2} + x_3\frac{\partial \psi_2}{\partial x_2}, \quad 2\varepsilon_{13} = \psi_1 + \frac{\partial u_3^0}{\partial x_1},$$

$$\varepsilon_{33} = 0, \quad 2\varepsilon_{23} = \psi_2 + \frac{\partial u_3^0}{\partial x_2}.$$

We consider the set of particles M placed on a line segment AB, which is orthogonal to the middle surface of the plate and goes through particle M^0 placed in the centre.

With the chosen displacement field, the particles placed on the line segment AB are placed on the line segment $A'B'$ after deformation of the plate as shown below:

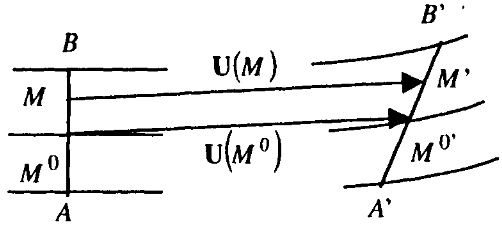

Figure 15.2. *Displacement field*

The particles placed on the line segment AB are defined by:

$$\mathbf{M^0M} = x_3\mathbf{x_3}, \text{ with } -\frac{h}{2} \leq x_3 \leq \frac{h}{2} \text{ and } \mathbf{OM^0} = x_1\mathbf{x_1} + x_2\mathbf{x_2}.$$

Starting from the relation:

$$\mathbf{M^{0'}M'} = \mathbf{M^{0'}M^0} + \mathbf{M^0M} + \mathbf{MM'},$$

and introducing the displacement vectors of particles M^0 and M, we obtain:

$$\mathbf{M^{0'}M'} = \mathbf{M^0M} + \mathbf{U}(M|t) - \mathbf{U}(M^0|t),$$

so:

$$\mathbf{M^{0'}M'} = x_3\mathbf{x_3} + \left(u_1^0 + x_3\psi_1\right)\mathbf{x_1} + \left(u_2^0 + x_3\psi_2\right)\mathbf{x_2} + u_3^0\mathbf{x_3}\ldots$$
$$\ldots - \left(u_1^0\mathbf{x_1} + u_2^0\mathbf{x_2} + u_3^0\mathbf{x_3}\right),$$
$$\mathbf{M^{0'}M'} = x_3\left(\psi_1\mathbf{x_1} + \psi_2\mathbf{x_2} + \mathbf{x_3}\right).$$

This last equation shows that the particles placed on a line segment AB remain aligned after deformation of the plate and are placed on the line segment $A'B'$.

This displacement field chosen corresponds to the moment field:

$$\mathbf{U}(M|t) = \mathbf{U}(M^0|t) + \mathbf{\Omega} \times \mathbf{M^0M},$$

with:

$$\mathbf{U}(M^0|t) = u_1^0\mathbf{x_1} + u_2^0\mathbf{x_2} + u_3^0\mathbf{x_3},\ \mathbf{\Omega} = \Omega_1\mathbf{x_1} + \Omega_2\mathbf{x_2},\ \mathbf{M^0M} = x_3\mathbf{x_3},$$

we obtain the relation:

$$\mathbf{U}(M|t) = \left(u_1^0 + x_3\Omega_2\right)\mathbf{x_1} + \left(u_2^0 - x_3\Omega_1\right)\mathbf{x_2} + u_3^0\mathbf{x_3},$$

which, identifying the chosen displacement field, gives the two relations:

$$\Omega_1 = -\psi_2 \, , \ \Omega_2 = \psi_1 ,$$

which represent the two infinitesimal rotations of the line segment AB measured respectively on $\vec{x}_1$ and $\vec{x}_2$. These two rotations are shown in the figure below:

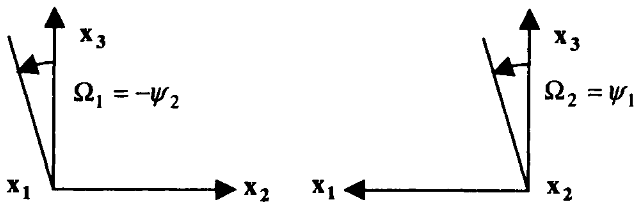

Figure 15.3. *Infinitesimal rotations*

The middle surface of the plate is defined by the vectorial equation:

$$\mathbf{f}(x_1, x_2|t) = \mathbf{OM}^{0'} = \mathbf{OM}^0 + \mathbf{U}(M^0|t),$$

$$\mathbf{f}(x_1, x_2|t) = x_1\mathbf{x}_1 + x_2\mathbf{x}_2 + u_1^0\mathbf{x}_1 + u_2^0\mathbf{x}_2 + u_3^0\mathbf{x}_3,$$

$$\mathbf{f}(x_1, x_2|t) = (x_1 + u_1^0)\mathbf{x}_1 + (x_2 + u_2^0)\mathbf{x}_2 + u_3^0\mathbf{x}_3 .$$

The two partially derived functions of $\mathbf{f}$ with respect to x_1 and x_2:

$$\frac{\partial \mathbf{f}}{\partial x_1} = \left(1 + \frac{\partial u_1^0}{\partial x_1}\right)\mathbf{x}_1 + \frac{\partial u_2^0}{\partial x_1}\mathbf{x}_2 + \frac{\partial u_3^0}{\partial x_1}\mathbf{x}_3 ,$$

$$\frac{\partial \mathbf{f}}{\partial x_2} = \frac{\partial u_1^0}{\partial x_2}\mathbf{x}_1 + \left(1 + \frac{\partial u_2^0}{\partial x_2}\right)\mathbf{x}_2 + \frac{\partial u_3^0}{\partial x_2}\mathbf{x}_3 ,$$

correspond to the tangential plane of the middle surface of the plate which goes through $M^{0'}$.

The principal parts, i.e. the first order terms, of the scalar products of the two previous vectors with the vector $\mathbf{M}^{0'}\mathbf{M}'$ are respectively equal to:

$$\mathbf{M}^{0'}\mathbf{M}' \cdot \frac{\partial \mathbf{f}}{\partial x_1} = x_3\left(\psi_1 + \frac{\partial u_3^0}{\partial x_1}\right),$$

$$\mathbf{M}^{0'}\mathbf{M}' \cdot \frac{\partial \mathbf{f}}{\partial x_2} = x_3\left(\psi_2 + \frac{\partial u_3^0}{\partial x_2}\right).$$

The hypothesis in the Kirchhoff-Love analysis is to say that the transverse shear strains ε_{13} and ε_{23} are zero, so the infinitesimal rotations are expressed in terms of u_3^0 with the two following relations:

$$\psi_1 = -\frac{\partial u_3^0}{\partial x_1},$$

$$\psi_2 = -\frac{\partial u_3^0}{\partial x_2}.$$

Putting these values into the two previous scalar products, we obtain the following equations:

$$\mathbf{M}^{0'}\mathbf{M}' \cdot \frac{\partial \mathbf{f}}{\partial x_1} = \mathbf{M}^{0'}\mathbf{M}' \cdot \frac{\partial \mathbf{f}}{\partial x_2} = 0,$$

which show that the line $A'B'$ is orthogonal to the middle surface of the plate.

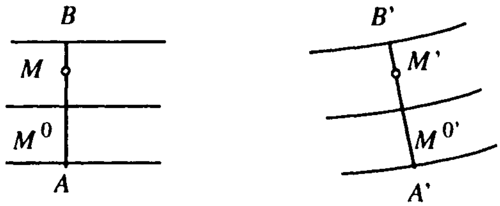

Figure 15.4. *Displacements in Kirchhoff-Love type analysis*

In Reissner-Mindlin type analysis, we have the displacement field:

$$u_1 = u_1^0 + x_3\psi_1,$$

$$u_2 = u_2^0 + x_3\psi_2,$$

$$u_3 = u_3^0,$$

separated into the membrane displacement field:

$$u_1^m = u_1^0,$$

$$u_2^m = u_2^0,$$

$$u_3^m = 0,$$

and the bending displacement field:

$$u_1^f = x_3\psi_1,$$

$$u_2^f = x_3\psi_2,$$

$$u_3^f = u_3^0.$$

In a similar manner, the strain field:

$$\varepsilon_{11}=\frac{\partial u_1^0}{\partial x_1}+x_3\frac{\partial\psi_1}{\partial x_1},\quad 2\varepsilon_{12}=\frac{\partial u_1^0}{\partial x_2}+\frac{\partial u_2^0}{\partial x_1}+x_3\left(\frac{\partial\psi_1}{\partial x_2}+\frac{\partial\psi_2}{\partial x_1}\right),$$

$$\varepsilon_{22}=\frac{\partial u_2^0}{\partial x_2}+x_3\frac{\partial\psi_2}{\partial x_2},\quad 2\varepsilon_{13}=\psi_1+\frac{\partial u_3^0}{\partial x_1},$$

$$\varepsilon_{33}=0,\qquad 2\varepsilon_{23}=\psi_2+\frac{\partial u_3^0}{\partial x_2},$$

is separated into a membrane displacement field:

$$\varepsilon_{11}^m=\frac{\partial u_1^0}{\partial x_1},\quad 2\varepsilon_{12}^m=\frac{\partial u_1^0}{\partial x_2}+\frac{\partial u_2^0}{\partial x_1},$$

$$\varepsilon_{22}^m=\frac{\partial u_2^0}{\partial x_2},\quad 2\varepsilon_{13}^m=0,$$

$$\varepsilon_{33}^m=0,\qquad 2\varepsilon_{23}^m=0,$$

and bending:

$$\varepsilon_{11}^f=x_3\frac{\partial\psi_1}{\partial x_1},\quad 2\varepsilon_{12}^f=x_3\left(\frac{\partial\psi_1}{\partial x_2}+\frac{\partial\psi_2}{\partial x_1}\right),$$

$$\varepsilon_{22}^f=x_3\frac{\partial\psi_2}{\partial x_2},\quad 2\varepsilon_{13}^f=\psi_1+\frac{\partial u_3^0}{\partial x_1},$$

$$\varepsilon_{33}^f=0,\qquad 2\varepsilon_{23}^f=\psi_2+\frac{\partial u_3^0}{\partial x_2}.$$

In Reissner-Mindlin type analysis, the membrane displacements and strains depend only on the two functions $u_1^0(x_1,x_2|t)$ and $u_2^0(x_1,x_2|t)$, while the bending displacements and strains depend only on the three functions $\psi_1(x_1,x_2|t)$, $\psi_2(x_1,x_2|t)$ and $u_3^0(x_1,x_2|t)$.

But in Kirchhoff-Love type analysis, we have the displacement field:

$$u_1=u_1^0-x_3\frac{\partial u_3^0}{\partial x_1},$$

$$u_2=u_2^0-x_3\frac{\partial u_3^0}{\partial x_2},$$

$$u_3=u_3^0,$$

separated into a membrane displacement field:

$$u_1^m=u_1^0,$$

$$u_2^m=u_2^0,$$

$$u_3^m=0,$$

and bending:

$$u_1^f = -x_3 \frac{\partial u_3^0}{\partial x_1},$$

$$u_2^f = -x_3 \frac{\partial u_3^0}{\partial x_2},$$

$$u_3^f = u_3^0.$$

Similarly, the non-zero components of the strain tensor:

$$\varepsilon_{11} = \frac{\partial u_1^0}{\partial x_1} - x_3 \frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$\varepsilon_{22} = \frac{\partial u_2^0}{\partial x_2} - x_3 \frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$2\varepsilon_{12} = \frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1} - 2x_3 \frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

are separated into the membrane strains:

$$\varepsilon_{11}^m = \frac{\partial u_1^0}{\partial x_1},$$

$$\varepsilon_{22}^m = \frac{\partial u_2^0}{\partial x_2},$$

$$2\varepsilon_{12}^m = \frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1},$$

and bending:

$$\varepsilon_{11}^f = -x_3 \frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$\varepsilon_{22}^f = -x_3 \frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$2\varepsilon_{12}^f = -2x_3 \frac{\partial^2 u_3^0}{\partial x_1 \partial x_2}.$$

These last expressions present the curvature of the middle surface:

$$\kappa_{11} = -\frac{\partial^2 u_3^0}{\partial x_1^2},$$

$$\kappa_{22} = -\frac{\partial^2 u_3^0}{\partial x_2^2},$$

$$2\kappa_{12} = -2\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2}.$$

In Kirchhoff-Love type analysis, the membrane displacements and strains depend only on $u_1^0(x_1, x_2|t)$ and $u_2^0(x_1, x_2|t)$, but the bending displacements and strains depend only on $u_3^0(x_1, x_2|t)$.

15.4. Global plate equations

15.4.1. *Force equations*

The resultant forces are defined as:

$$N_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij} dx_3,$$

and they satisfy the symmetry condition $N_{ij} = N_{ji}$.

Multiplying the two members of the motion equations:

$$\frac{\partial \sigma_{ij}}{\partial x_j} + f_i = \rho \frac{\partial^2 u_i}{\partial t^2},$$

by dx_3, and integrating through the thickness from $-\frac{h}{2}$ to $\frac{h}{2}$, the three force equations are obtained:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{ij}}{\partial x_j} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_i dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_i}{\partial t^2} dx_3.$$

– For $i = 1$, the equation is:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{11}}{\partial x_1} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{12}}{\partial x_2} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{13}}{\partial x_3} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_1 dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} dx_3.$$

The first term of this equation is equal to:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{11}}{\partial x_1} dx_3 = \frac{\partial}{\partial x_1} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{11} dx_3 = \frac{\partial N_{11}}{\partial x_1},$$

in a similar manner, the second term is equal to:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{12}}{\partial x_2} dx_3 = \frac{\partial}{\partial x_2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{12} dx_3 = \frac{\partial N_{12}}{\partial x_2}.$$

The third term becomes:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{13}}{\partial x_3} dx_3 = [\sigma_{13}]_{-\frac{h}{2}}^{\frac{h}{2}} = \sigma_{13}\left(x_1, x_2, \frac{h}{2}\middle| t\right) - \sigma_{13}\left(x_1, x_2, -\frac{h}{2}\middle| t\right).$$

The boundary conditions applied on the upper and lower surfaces give the two relations:

$$\sigma_{13}\left(x_1, x_2, \frac{h}{2}\middle| t\right) = \tau_{13}^{+}(x_1, x_2 | t),$$

$$\sigma_{13}\left(x_1, x_2, -\frac{h}{2}\middle| t\right) = \tau_{13}^{-}(x_1, x_2 | t),$$

in which τ_{13}^{+} and τ_{13}^{-} are the surface forces applying on the two faces respectively. From this:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{13}}{\partial x_3} dx_3 = \tau_{13}^{+} - \tau_{13}^{-}.$$

For the body forces:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} f_1 dx_3 = p_1(x_1, x_2 | t),$$

the following equation is obtained:

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + \tau_{13}^{+} - \tau_{13}^{-} + p_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} dx_3.$$

– For $i = 2$, the following equation:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{21}}{\partial x_1} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{22}}{\partial x_2} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{23}}{\partial x_3} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_2 dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} dx_3,$$

which with the previously introduced notation will be written as:

$$\frac{\partial N_{21}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + \tau_{23}^{+} - \tau_{23}^{-} + p_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} dx_3.$$

– For $i = 3$, in a similar manner:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{31}}{\partial x_1} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{32}}{\partial x_2} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{33}}{\partial x_3} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_3 dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_3}{\partial t^2} dx_3,$$

and:

$$\frac{\partial N_{31}}{\partial x_1} + \frac{\partial N_{32}}{\partial x_2} + \tau_{33}^{+} - \tau_{33}^{-} + p_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_3}{\partial t^2} dx_3.$$

15.4.2. *Moment equations*

The resultant moments are equal to:

$$M_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij} x_3 dx_3,$$

and satisfy the condition $M_{ij} = M_{ji}$.

Multiplying the two members of the motion equations by $x_3 dx_3$ and integrating through the thickness of the plate, we obtain the moment equation:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{ij}}{\partial x_j} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_i x_3 dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_i}{\partial t^2} x_3 dx_3.$$

– For $i = 1$, the equation is written as:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{11}}{\partial x_1} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{12}}{\partial x_2} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{13}}{\partial x_3} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_1 x_3 dx_3 \ldots$$

$$\ldots = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} x_3 dx_3,$$

from the first term:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{11}}{\partial x_1} x_3 dx_3 = \frac{\partial}{\partial x_1} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{11} x_3 dx_3 = \frac{\partial M_{11}}{\partial x_1},$$

and the second term:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{12}}{\partial x_2} x_3 dx_3 = \frac{\partial}{\partial x_2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{12} x_3 dx_3 = \frac{\partial M_{12}}{\partial x_2}.$$

The third term is equal to:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{13}}{\partial x_3} x_3 dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial}{\partial x_3} (\sigma_{13} x_3) dx_3 - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{13} dx_3,$$

so, using the previous notation:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{13}}{\partial x_3} x_3 dx_3 = \left[\sigma_{13} x_3\right]_{-\frac{h}{2}}^{\frac{h}{2}} - N_{13} = \left(\tau_{13}^+ + \tau_{13}^-\right)\frac{h}{2} - N_{13}.$$

Let, for the relative integral of the body forces:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} f_1 x_3 dx_3 = \mu_1(x_1, x_2|t),$$

we then obtain the equation:

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} + \left(\tau_{13}^+ + \tau_{13}^-\right)\frac{h}{2} + \mu_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} x_3 dx_3 .$$

– For $i = 2$, in a similar way we obtain the equation:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{21}}{\partial x_1} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{22}}{\partial x_2} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{23}}{\partial x_3} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_2 x_3 dx_3 \ldots$$

$$\ldots = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} x_3 dx_3,$$

which can be written as:

$$\frac{\partial M_{21}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - N_{23} + \left(\tau_{23}^+ + \tau_{23}^-\right)\frac{h}{2} + \mu_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} x_3 dx_3 ,$$

using the previous notation.

The equation written with $i = 3$ has no physical application.

15.5. Plate equations in Reissner-Mindlin analysis

15.5.1. *Calculation of second members*

The Reissner-Mindlin displacement field is:

$$u_1 = u_1^0(x_1, x_2|t) + x_3 \psi_1(x_1, x_2|t),$$
$$u_2 = u_2^0(x_1, x_2|t) + x_3 \psi_2(x_1, x_2|t),$$
$$u_3 = u_3^0(x_1, x_2|t),$$

By introducing:

$$I_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho dx_3 ,$$

and:

$$I_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho x_3 dx_3 ,$$

the second members of the three force equations are equal to:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left(\frac{\partial^2 u_1^0}{\partial t^2} + x_3 \frac{\partial^2 \psi_1}{\partial t^2} \right) dx_3 = I_0 \frac{\partial^2 u_1^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left(\frac{\partial^2 u_2^0}{\partial t^2} + x_3 \frac{\partial^2 \psi_2}{\partial t^2} \right) dx_3 = I_0 \frac{\partial^2 u_2^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_2}{\partial t^2},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_3}{\partial t^2} dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_3^0}{\partial t^2} dx_3 = I_0 \frac{\partial^2 u_3^0}{\partial t^2}.$$

By introducing:

$$I_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho x_3^2 dx_3,$$

the second members of the two moment equations are equal to:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} x_3 dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left(\frac{\partial^2 u_1^0}{\partial t^2} + x_3 \frac{\partial^2 \psi_1}{\partial t^2} \right) x_3 dx_3 = I_1 \frac{\partial^2 u_1^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} x_3 dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left(\frac{\partial^2 u_2^0}{\partial t^2} + x_3 \frac{\partial^2 \psi_2}{\partial t^2} \right) x_3 dx_3 = I_1 \frac{\partial^2 u_2^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_2}{\partial t^2}.$$

15.5.2. *Global plate equations*

In the case where the μ_i are equal to 0, and also τ_{i3}^+ and τ_{i3}^- are zero except $\tau_{33}^+ = q_3$, the plate equations are written as:

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 = I_0 \frac{\partial^2 u_1^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 = I_0 \frac{\partial^2 u_2^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_2}{\partial t^2},$$

$$\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + q_3 + p_3 = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} = I_1 \frac{\partial^2 u_1^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - N_{23} = I_1 \frac{\partial^2 u_2^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_2}{\partial t^2}.$$

15.5.3. *Boundary conditions*

n and s are respectively the local normal and tangential co-ordinates on the plate edge, and the conditions around the perimeter of the middle surface of the plate, for an edge in the following case:

– simply supported:

$$N_n = N_s = 0, \quad M_n = 0, \quad \psi_s = 0, \quad u_3^0 = 0,$$

– hinged free in the normal direction:

$$N_n = 0, \quad M_n = 0, \quad \psi_s = 0, \quad u_s^0 = u_3^0 = 0,$$

– hinged free in the tangential direction:

$$N_s = 0, \quad M_n = 0, \quad \psi_s = 0, \quad u_n^0 = u_3^0 = 0,$$

– clamped:

$$u_s^0 = u_n^0 = u_3^0 = 0, \quad \psi_s = \psi_n = 0,$$

– free:

$$N_n = N_s = N_3 = 0, \quad M_n = M_s = 0.$$

15.6. Plate equations in Kirchhoff-Love analysis

15.6.1. *Calculation of second members*

With the Kirchhoff-Love displacement field:

$$u_1 = u_1^0(x_1, x_2|t) - x_3 \frac{\partial u_3^0}{\partial x_1},$$

$$u_2 = u_2^0(x_1, x_2|t) - x_3 \frac{\partial u_3^0}{\partial x_2},$$

$$u_3 = u_3^0(x_1, x_2|t),$$

the second members are obtained:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left(\frac{\partial^2 u_1^0}{\partial t^2} - x_3 \frac{\partial^3 u_3^0}{\partial x_1 \partial t^2} \right) dx_3 = I_0 \frac{\partial^2 u_1^0}{\partial t^2} - I_1 \frac{\partial^3 u_3^0}{\partial x_1 \partial t^2},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left(\frac{\partial^2 u_2^0}{\partial t^2} - x_3 \frac{\partial^3 u_3^0}{\partial x_2 \partial t^2} \right) dx_3 = I_0 \frac{\partial^2 u_2^0}{\partial t^2} - I_1 \frac{\partial^3 u_3^0}{\partial x_2 \partial t^2},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_3}{\partial t^2} dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_3^0}{\partial t^2} dx_3 = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

and:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} x_3 dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left(\frac{\partial^2 u_1^0}{\partial t^2} - x_3 \frac{\partial^3 u_3^0}{\partial x_1 \partial t^2} \right) x_3 dx_3 = I_1 \frac{\partial^2 u_1^0}{\partial t^2} - I_2 \frac{\partial^3 u_3^0}{\partial x_1 \partial t^2},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} x_3 dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left(\frac{\partial^2 u_2^0}{\partial t^2} - x_3 \frac{\partial^3 u_3^0}{\partial x_2 \partial t^2} \right) x_3 dx_3 = I_1 \frac{\partial^2 u_2^0}{\partial t^2} - I_2 \frac{\partial^3 u_3^0}{\partial x_2 \partial t^2}.$$

As in Kirchhoff-Love type analysis, the rotation inertia where I_1 and I_2 appear are neglected and the second members are written as:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} dx_3 = I_0 \frac{\partial^2 u_1^0}{\partial t^2}, \qquad \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} dx_3 = I_0 \frac{\partial^2 u_2^0}{\partial t^2},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_3}{\partial t^2} dx_3 = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

and:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} x_3 dx_3 = 0, \qquad \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} x_3 dx_3 = 0.$$

15.6.2. *Global plate equations*

In the particular case where $\mu_i = 0$, $\tau_{i3}^- = 0$, $\tau_{i3}^+ = 0$ except for $\tau_{33}^+ = q_3$, we have:

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 = I_0 \frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 = I_0 \frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + q_3 + p_3 = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} = 0\,,$$

$$\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - N_{23} = 0\,.$$

Differentiating the two members of the last two equations respectively with respect to x_1 and x_2, we obtain the two equations:

$$\frac{\partial^2 M_{11}}{\partial x_1^2} + \frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} - \frac{\partial N_{13}}{\partial x_1} = 0\,,$$

$$\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 M_{22}}{\partial x_2^2} - \frac{\partial N_{23}}{\partial x_2} = 0\,,$$

leading to the following relation:

$$\frac{\partial^2 M_{11}}{\partial x_1^2} + 2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 M_{22}}{\partial x_2^2} + q_3 + p_3 = I_0 \frac{\partial^2 u_3^0}{\partial t^2}\,.$$

15.6.3. *Boundary edge conditions*

In the particular case previously considered, we have for an edge:

– simply supported:

$$N_n = N_s = 0, \quad M_n = 0, \quad u_3^0 = 0,$$

– hinged free in the normal direction:

$$N_n = 0, \quad M_n = 0, \quad u_s^0 = u_3^0 = 0,$$

– hinged free in the tangential direction:

$$N_s = 0, \quad M_n = 0, \quad u_n^0 = u_3^0 = 0,$$

– clamped:

$$u_n^0 = u_s^0 = u_3^0 = 0, \quad \frac{\partial u_3^0}{\partial n} = 0,$$

– free:

$$N_n = N_s = 0, \quad \frac{\partial M_s}{\partial s} + N_3 = 0, \quad M_n = 0.$$

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Chapter 16

Global plate equations for large transverse displacements

16.1. Introduction

We will now derive the global plate equations by integrating the non-linear equations of motion that the Kirchhoff stress tensor satisfies, for the case where the absolute value of u_3 is not small in comparison with the plate thickness.

This configuration allows the buckling of plates to be studied. In addition, the absolute values of u_1 and u_2 are small in comparison with the plate thickness, and the absolute values of partially derived functions of u_1 and u_2 with respect to x_1, x_2 and x_3 are less than 1.

We will clarify these relations in both Reissner-Mindlin and Kirchhoff-Love type analyses.

16.2. Local plate equations

The components σ_{ij}^K of the Kirchhoff stress tensor are written in this appendix σ_{ij} and satisfy the local relations:

$$\frac{\partial}{\partial x_j}\left[\left(\delta_{ik}+\frac{\partial u_i}{\partial x_k}\right)\sigma_{kj}\right]+f_i=\rho\frac{\partial^2 u_i}{\partial t^2}.$$

In the present plate analysis, we consider the displacement field to be written as:

$$u_1=u_1(x_1,x_2,x_3|t),$$
$$u_2=u_2(x_1,x_2,x_3|t),$$
$$u_3=u_3(x_1,x_2|t),$$

and in the local plate equations, we only conserve as non-linear terms those containing the partially derived functions of u_3 with respect to x_1 and x_2.

Under these conditions, the local equations of movement are written as:

– for $i = 1$:

$$\frac{\partial}{\partial x_j}\left[\left(\delta_{1k} + \frac{\partial u_1}{\partial x_k}\right)\sigma_{kj}\right] + f_1 = \rho\frac{\partial^2 u_1}{\partial t^2},$$

$$\frac{\partial \sigma_{1j}}{\partial x_j} + f_1 = \rho\frac{\partial^2 u_1}{\partial t^2},$$

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = \rho\frac{\partial^2 u_1}{\partial t^2},$$

– for $i = 2$:

$$\frac{\partial}{\partial x_j}\left[\left(\delta_{2k} + \frac{\partial u_2}{\partial x_k}\right)\sigma_{kj}\right] + f_2 = \rho\frac{\partial^2 u_2}{\partial t^2},$$

$$\frac{\partial \sigma_{2j}}{\partial x_j} + f_2 = \rho\frac{\partial^2 u_2}{\partial t^2},$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + f_2 = \rho\frac{\partial^2 u_2}{\partial t^2},$$

– for $i = 3$:

$$\frac{\partial}{\partial x_j}\left[\left(\delta_{3k} + \frac{\partial u_3}{\partial x_k}\right)\sigma_{kj}\right] + f_3 = \rho\frac{\partial^2 u_3}{\partial t^2},$$

$$\frac{\partial}{\partial x_j}\left(\sigma_{3j} + \sigma_{kj}\frac{\partial u_3}{\partial x_k}\right) + f_3 = \rho\frac{\partial^2 u_3}{\partial t^2},$$

$$\frac{\partial}{\partial x_1}\left(\sigma_{13} + \sigma_{11}\frac{\partial u_3}{\partial x_1} + \sigma_{12}\frac{\partial u_3}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(\sigma_{23} + \sigma_{12}\frac{\partial u_3}{\partial x_1} + \sigma_{22}\frac{\partial u_3}{\partial x_2}\right)\ldots$$

$$\ldots + \frac{\partial}{\partial x_3}\left(\sigma_{33} + \sigma_{13}\frac{\partial u_3}{\partial x_1} + \sigma_{23}\frac{\partial u_3}{\partial x_2}\right) + f_3 = \rho\frac{\partial^2 u_3}{\partial t^2}.$$

16.3. Global plate equations

16.3.1. *Global plate summation equations*

Multiplying the two members of the local equation of movement $i = 1$ by dx_3 and integrating through the thickness of the plate from $-\frac{h}{2}$ to $\frac{h}{2}$, we obtain the relation:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{11}}{\partial x_1} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{12}}{\partial x_2} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{13}}{\partial x_3} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_1 dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} dx_3,$$

including the following integrals:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{11}}{\partial x_1} dx_3 = \frac{\partial}{\partial x_1} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{11} dx_3 = \frac{\partial N_{11}}{\partial x_1},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{12}}{\partial x_2} dx_3 = \frac{\partial}{\partial x_2} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{12} dx_3 = \frac{\partial N_{12}}{\partial x_2},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{13}}{\partial x_3} dx_3 = [\sigma_{13}]_{-\frac{h}{2}}^{\frac{h}{2}} = \sigma_{13}\left(x_1, x_2, \frac{h}{2} \middle| t \right) - \sigma_{13}\left(x_1, x_2, -\frac{h}{2} \middle| t \right).$$

From the edge boundary conditions on the top and lower surfaces, we have:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{13}}{\partial x_3} dx_3 = \tau_{13}^{+} - \tau_{13}^{-},$$

let:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} f_1 dx_3 = p_1(x_1, x_2 | t),$$

we now obtain the equation:

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + \tau_{13}^{+} - \tau_{13}^{-} + p_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} dx_3.$$

An analogous transformation of the local equation $i = 2$ gives:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{12}}{\partial x_1} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{22}}{\partial x_2} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{23}}{\partial x_3} dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_2 dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} dx_3,$$

then:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{12}}{\partial x_1} dx_3 = \frac{\partial N_{12}}{\partial x_1}, \qquad \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{22}}{\partial x_2} dx_3 = \frac{\partial N_{22}}{\partial x_2},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{23}}{\partial x_3} dx_3 = \tau_{23}^{+} - \tau_{23}^{-}, \qquad \int_{-\frac{h}{2}}^{\frac{h}{2}} f_2 dx_3 = p_2,$$

from which the second equation:

$$\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}+\tau_{23}^{+}-\tau_{23}^{-}+p_2=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\frac{\partial^2 u_2}{\partial t^2}dx_3.$$

From the local equation $i=3$, we obtain by integrating through the thickness of the plate the following relation:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{\partial}{\partial x_1}\left(\sigma_{13}+\sigma_{11}\frac{\partial u_3}{\partial x_1}+\sigma_{12}\frac{\partial u_3}{\partial x_2}\right)dx_3+\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{\partial}{\partial x_2}\left(\sigma_{23}+\sigma_{12}\frac{\partial u_3}{\partial x_1}+\sigma_{22}\frac{\partial u_3}{\partial x_2}\right)dx_3\ldots$$

$$\ldots+\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{\partial}{\partial x_3}\left(\sigma_{33}+\sigma_{13}\frac{\partial u_3}{\partial x_1}+\sigma_{23}\frac{\partial u_3}{\partial x_2}\right)dx_3+\int_{-\frac{h}{2}}^{\frac{h}{2}}f_3dx_3=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\frac{\partial^2 u_3}{\partial t^2}dx_3,$$

in which the following terms appear:

$$J_1=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{\partial}{\partial x_1}\left(\sigma_{13}+\sigma_{11}\frac{\partial u_3}{\partial x_1}+\sigma_{12}\frac{\partial u_3}{\partial x_2}\right)dx_3,$$

$$J_1=\frac{\partial}{\partial x_1}\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(\sigma_{13}+\sigma_{11}\frac{\partial u_3}{\partial x_1}+\sigma_{12}\frac{\partial u_3}{\partial x_2}\right)dx_3,$$

$$J_1=\frac{\partial}{\partial x_1}\left(N_{13}+N_{11}\frac{\partial u_3}{\partial x_1}+N_{12}\frac{\partial u_3}{\partial x_2}\right),\text{ because } u_3 \text{ does not depend on } x_3,$$

$$J_1=\frac{\partial N_{13}}{\partial x_1}+\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3}{\partial x_1}+N_{12}\frac{\partial u_3}{\partial x_2}\right),$$

$$J_2=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{\partial}{\partial x_2}\left(\sigma_{23}+\sigma_{12}\frac{\partial u_3}{\partial x_1}+\sigma_{22}\frac{\partial u_3}{\partial x_2}\right)dx_3,$$

or:

$$J_2=\frac{\partial N_{23}}{\partial x_2}+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3}{\partial x_1}+N_{22}\frac{\partial u_3}{\partial x_2}\right),$$

$$J_3=\int_{-\frac{h}{2}}^{\frac{h}{2}}\frac{\partial}{\partial x_3}\left(\sigma_{33}+\sigma_{13}\frac{\partial u_3}{\partial x_1}+\sigma_{23}\frac{\partial u_3}{\partial x_2}\right)dx_3,$$

$$J_3=\left[\sigma_{33}+\sigma_{13}\frac{\partial u_3}{\partial x_1}+\sigma_{23}\frac{\partial u_3}{\partial x_2}\right]_{-\frac{h}{2}}^{\frac{h}{2}},$$

or:

$$J_3 = \tau_{33}^{+} - \tau_{33}^{-} + \tau_{13}^{+}\frac{\partial u_3}{\partial x_1}\left(x_1, x_2, \frac{h}{2}\middle| t\right) - \tau_{13}^{-}\frac{\partial u_3}{\partial x_1}\left(x_1, x_2, -\frac{h}{2}\middle| t\right)\ldots$$

$$\ldots + \tau_{23}^{+}\frac{\partial u_3}{\partial x_2}\left(x_1, x_2, \frac{h}{2}\middle| t\right) - \tau_{23}^{-}\frac{\partial u_3}{\partial x_2}\left(x_1, x_2, -\frac{h}{2}\middle| t\right),$$

let:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} f_3 dx_3 = p_3,$$

we thus obtain the third summation equation:

$$\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + \frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3}{\partial x_1} + N_{12}\frac{\partial u_3}{\partial x_2}\right) + \frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3}{\partial x_1} + N_{22}\frac{\partial u_3}{\partial x_2}\right)\ldots$$

$$\ldots + \tau_{33}^{+} - \tau_{33}^{-} + \left(\tau_{13}^{+} + \tau_{23}^{+}\right)\frac{\partial u_3}{\partial x_1}\left(x_1, x_2, \frac{h}{2}\middle| t\right)\ldots$$

$$\ldots - \left(\tau_{13}^{-} + \tau_{23}^{-}\right)\frac{\partial u_3}{\partial x_1}\left(x_1, x_2, -\frac{h}{2}\middle| t\right) + p_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_3}{\partial t^2} dx_3.$$

16.3.2. *Global plate moment equations*

Multiplying the two members of the local equation of movement $i = 1$ by $x_3 dx_3$ and integrating through the thickness of the plate from $-\frac{h}{2}$ to $\frac{h}{2}$, we obtain the relation:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{11}}{\partial x_1} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{12}}{\partial x_2} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{13}}{\partial x_3} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_1 x_3 dx_3 \ldots$$

$$\ldots = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} x_3 dx_3,$$

in which the following terms appear:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{11}}{\partial x_1} x_3 dx_3 = \frac{\partial}{\partial x_1}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{11} x_3 dx_3 = \frac{\partial M_{11}}{\partial x_1},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{12}}{\partial x_2} x_3 dx_3 = \frac{\partial}{\partial x_2}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{12} x_3 dx_3 = \frac{\partial M_{12}}{\partial x_2},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{13}}{\partial x_3} x_3 dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial}{\partial x_3}(\sigma_{13} x_3) dx_3 - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{13} dx_3 \ldots$$

$$\ldots = \left[\sigma_{13} x_3\right]_{-\frac{h}{2}}^{\frac{h}{2}} - N_{13} = \left(\tau_{13}^{+} + \tau_{13}^{-}\right)\frac{h}{2} - N_{13}.$$

In other respects, we let:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} f_1 x_3 dx_3 = \mu_1\left(x_1, x_2 \middle| t\right).$$

The first moment equation is written as:

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} + \left(\tau_{13}^{+} + \tau_{13}^{-}\right)\frac{h}{2} + \mu_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} x_3 dx_3.$$

The second moment equation is obtained in the same manner by multiplying the two members of the local equation $i = 2$ by $x_3 dx_3$ and by integrating through the thickness of the plate, which gives:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{12}}{\partial x_1} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{22}}{\partial x_2} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{23}}{\partial x_3} x_3 dx_3 + \int_{-\frac{h}{2}}^{\frac{h}{2}} f_2 x_3 dx_3 \ldots$$

$$\ldots = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} x_3 dx_3,$$

with:

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{12}}{\partial x_1} x_3 dx_3 = \frac{\partial M_{12}}{\partial x_1},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{22}}{\partial x_2} x_3 dx_3 = \frac{\partial M_{22}}{\partial x_2},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{23}}{\partial x_3} x_3 dx_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial}{\partial x_3}(\sigma_{23} x_3) dx_3 - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{23} dx_3 = \left(\tau_{23}^{+} + \tau_{23}^{-}\right)\frac{h}{2} - N_{23},$$

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} f_2 x_3 dx_3 = \mu_2\left(x_1, x_2 \middle| t\right),$$

from which the second moment equation:

$$\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - N_{23} + \left(\tau_{23}^{+} + \tau_{23}^{-}\right)\frac{h}{2} + \mu_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} x_3 dx_3.$$

16.4. Global plate equations for static, vibration and buckling cases

16.4.1. *Global plate equations*

In the particular case where:

$$\tau_{13}^{+} = \tau_{13}^{-} = \tau_{23}^{+} = \tau_{23}^{-} = \tau_{33}^{-} = 0,\ \tau_{33}^{+} = q_3,$$

$$\mu_1 = \mu_2 = 0,$$

the five equations of the plate analysis involve:

– the three summation expressions:

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} dx_3,$$

$$\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} dx_3,$$

$$\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + \frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3}{\partial x_1} + N_{12}\frac{\partial u_3}{\partial x_2} \right) + \frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3}{\partial x_1} + N_{22}\frac{\partial u_3}{\partial x_2} \right) \dots$$

$$\dots + q_3 + p_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_3}{\partial t^2} dx_3,$$

– the two moment equations:

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_1}{\partial t^2} x_3 dx_3,$$

$$\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - N_{23} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_2}{\partial t^2} x_3 dx_3 .$$

16.4.2. *Global plate equilibrium equations*

The non-linear terms are neglected and the five global equations of equilibrium are written as:

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 = 0,$$

$$\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 = 0,$$

$$\frac{\partial N_{13}}{\partial x_1}+\frac{\partial N_{23}}{\partial x_2}+q_3+p_3=0,$$

$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}-N_{13}=0,$$

$$\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}-N_{23}=0.$$

The elimination of N_{13} and N_{23} in the last three equations gives:

$$\frac{\partial^2 M_{11}}{\partial x_1^2}+2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2}+\frac{\partial^2 M_{22}}{\partial x_2^2}+q_3+p_3=0.$$

16.4.3. *Global plate vibration equations*

Neglecting the non-linear terms, we thus obtain the global plate vibration equations:

$$\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\frac{\partial^2 u_1}{\partial t^2}dx_3,$$

$$\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\frac{\partial^2 u_2}{\partial t^2}dx_3,$$

$$\frac{\partial N_{13}}{\partial x_1}+\frac{\partial N_{23}}{\partial x_2}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\frac{\partial^2 u_3}{\partial t^2}dx_3,$$

$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}-N_{13}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\frac{\partial^2 u_1}{\partial t^2}x_3dx_3,$$

$$\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}-N_{23}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\frac{\partial^2 u_2}{\partial t^2}x_3dx_3.$$

Eliminating N_{13} and N_{23} in the three last equations, we have:

$$\frac{\partial^2 M_{11}}{\partial x_1^2}+2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2}+\frac{\partial^2 M_{22}}{\partial x_2^2}=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\frac{\partial^2 u_3}{\partial t^2}dx_3+\frac{\partial}{\partial x_1}\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\frac{\partial^2 u_1}{\partial t^2}x_3dx_3\ldots$$

$$\ldots+\frac{\partial}{\partial x_2}\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\frac{\partial^2 u_2}{\partial t^2}x_3dx_3.$$

16.4.4. *Global plate buckling equations*

The global plate buckling equations which involve the non-linear terms of the equations of plate analysis, are written as:

$$\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}=0,$$

$$\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}=0,$$

$$\frac{\partial N_{13}}{\partial x_1}+\frac{\partial N_{23}}{\partial x_2}+N_{11}\frac{\partial^2 u_3}{\partial x_1^2}+2N_{12}\frac{\partial^2 u_3}{\partial x_1 \partial x_2}+N_{22}\frac{\partial^2 u_3}{\partial x_2^2}=0,$$

$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}-N_{13}=0,$$

$$\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}-N_{23}=0,$$

where the first two equations have been used to write the third one. The elimination of N_{13} and N_{23} in the last equations gives the following relation:

$$\frac{\partial^2 M_{11}}{\partial x_1^2}+2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2}+\frac{\partial^2 M_{22}}{\partial x_2^2}+N_{11}\frac{\partial^2 u_3}{\partial x_1^2}+2N_{12}\frac{\partial^2 u_3}{\partial x_1 \partial x_2}+N_{22}\frac{\partial^2 u_3}{\partial x_2^2}=0.$$

16.5. Reissner-Mindlin global plate equations

Taking account of the calculations of the second member developed in the previous chapter, the global plate equations are written as:

$$\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}+p_1=I_0\frac{\partial^2 u_1^0}{\partial t^2}+I_1\frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}+p_2=I_0\frac{\partial^2 u_2^0}{\partial t^2}+I_1\frac{\partial^2 \psi_2}{\partial t^2},$$

$$\frac{\partial N_{13}}{\partial x_1}+\frac{\partial N_{23}}{\partial x_2}+\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1}+N_{12}\frac{\partial u_3^0}{\partial x_2}\right)\ldots$$

$$\ldots+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1}+N_{22}\frac{\partial u_3^0}{\partial x_2}\right)+q_3+p_3=I_0\frac{\partial^2 u_3^0}{\partial t^2},$$

$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}-N_{13}=I_1\frac{\partial^2 u_1^0}{\partial t^2}+I_2\frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}-N_{23}=I_1\frac{\partial^2 u_2^0}{\partial t^2}+I_2\frac{\partial^2 \psi_2}{\partial t^2}.$$

16.6. Kirchhoff-Love global plate equations

The simplifying hypothesis of Kirchhoff-Love type analysis and the previous calculations enable us to write the global equations of the plate analysis in the following form:

$$\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}+p_1=I_0\frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}+p_2=I_0\frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial N_{13}}{\partial x_1}+\frac{\partial N_{23}}{\partial x_2}+\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1}+N_{12}\frac{\partial u_3^0}{\partial x_2}\right)\ldots$$

$$\ldots+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1}+N_{22}\frac{\partial u_3^0}{\partial x_2}\right)+q_3+p_3=I_0\frac{\partial^2 u_3^0}{\partial t^2},$$

$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}-N_{13}=0,$$

$$\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}-N_{23}=0.$$

Alternatively, we have the equation:

$$\frac{\partial^2 M_{11}}{\partial x_1^2}+2\frac{\partial^2 M_{12}}{\partial x_1\partial x_2}+\frac{\partial^2 M_{22}}{\partial x_2^2}+\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1}+N_{12}\frac{\partial u_3^0}{\partial x_2}\right)\ldots$$

$$\ldots+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1}+N_{22}\frac{\partial u_3^0}{\partial x_2}\right)+q_3+p_3=I_0\frac{\partial^2 u_3^0}{\partial t^2}.$$

Chapter 17

Global plate equations: Kirchhoff-Love theory variational formulation

17.1. Introduction

The object of this appendix is to present the Kirchhoff-Love plate theory, in which the transverse shear is not taken into account.

From the Kirchhoff-Love displacements u_1, u_2 and u_3, we calculate the Von Karman strains used when the transverse displacement u_3 is higher than u_1 and u_2.

The plate equations are obtained from variational formulation. From these equations, we write the relations for bending, vibration and buckling of plates.

17.2. Von Karman strains

The components of the Green-Lagrange strain tensor:

$$\varepsilon_{ij}^{L} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} + \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right),$$

provide a linear part:

$$\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and a non-linear part:

$$\frac{1}{2}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_i}\frac{\partial u_1}{\partial x_j} + \frac{\partial u_2}{\partial x_i}\frac{\partial u_2}{\partial x_j} + \frac{\partial u_3}{\partial x_i}\frac{\partial u_3}{\partial x_j}\right).$$

When the transverse displacement u_3 is higher than u_1 and u_2, the non-linear term:

$$\frac{1}{2}\left(\frac{\partial u_1}{\partial x_i}\frac{\partial u_1}{\partial x_j}+\frac{\partial u_2}{\partial x_i}\frac{\partial u_2}{\partial x_j}\right),$$

can be neglected with respect to:

$$\frac{1}{2}\frac{\partial u_3}{\partial x_i}\frac{\partial u_3}{\partial x_j}.$$

Under these conditions, the Green-Lagrange strains are reduced to Von Karman strains:

$$\varepsilon_{ij}^{VK}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\frac{1}{2}\frac{\partial u_3}{\partial x_i}\frac{\partial u_3}{\partial x_j}.$$

The displacement field of Kirchhoff-Love theory:

$$u_1=u_1^0(x_1,x_2|t)-x_3\frac{\partial u_3^0}{\partial x_1}(x_1,x_2|t),$$

$$u_2=u_2^0(x_1,x_2|t)-x_3\frac{\partial u_3^0}{\partial x_2}(x_1,x_2|t),$$

$$u_3=u_3^0(x_1,x_2|t),$$

leads to the Von Karman strain field:

$$\varepsilon_{11}^{VK}=\frac{\partial u_1^0}{\partial x_1}-x_3\frac{\partial^2 u_3^0}{\partial x_1^2}+\frac{1}{2}\left(\frac{\partial u_3^0}{\partial x_1}\right)^2,$$

$$\varepsilon_{22}^{VK}=\frac{\partial u_2^0}{\partial x_2}-x_3\frac{\partial^2 u_3^0}{\partial x_2^2}+\frac{1}{2}\left(\frac{\partial u_3^0}{\partial x_2}\right)^2,$$

$$\varepsilon_{12}^{VK}=\frac{1}{2}\left(\frac{\partial u_1^0}{\partial x_2}+\frac{\partial u_2^0}{\partial x_1}\right)-x_3\frac{\partial^2 u_3^0}{\partial x_1\partial x_2}+\frac{1}{2}\frac{\partial u_3^0}{\partial x_1}\frac{\partial u_3^0}{\partial x_2},$$

$$\varepsilon_{13}^{VK}=\varepsilon_{23}^{VK}=\varepsilon_{33}^{VK}=0.$$

We can introduce in the strain relations:

$$\varepsilon_{ij}^{L}=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)+\frac{1}{2}\frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j},$$

the expression:

$$\frac{\partial u_k}{\partial x_l}=\varepsilon_{kl}+\omega_{kl},$$

which leads to:

$$\varepsilon_{ij}^{L}=\varepsilon_{ij}+\frac{1}{2}(\varepsilon_{ki}+\omega_{ki})(\varepsilon_{kj}+\omega_{kj}),$$

so:

$$\varepsilon_{ij}^{L} = \varepsilon_{ij} + \frac{1}{2}\left(\varepsilon_{ki}\varepsilon_{kj} + \varepsilon_{ki}\omega_{kj} + \varepsilon_{kj}\omega_{ki} + \omega_{ki}\omega_{kj}\right).$$

For the case:

$$0 < \left|\varepsilon_{ij}\right| < \left|\omega_{ij}\right| < 1,$$

we have:

$$0 < \left|\varepsilon_{ij}\varepsilon_{kl}\right| < \left|\omega_{ij}\varepsilon_{kl}\right| < \left|\varepsilon_{kl}\right|,$$

and:

$$0 < \left|\varepsilon_{ij}\omega_{kl}\right| < \left|\omega_{ij}\omega_{kl}\right| < \left|\omega_{kl}\right|.$$

Taking into account these inequality relations, we can neglect $\left|\varepsilon_{ki}\varepsilon_{kj}\right|$, $\left|\varepsilon_{ki}\omega_{kj}\right|$ and $\left|\omega_{ki}\varepsilon_{kj}\right|$ compared to $\left|\omega_{ki}\omega_{kj}\right|$.

The Kirchhoff-Love displacement field gives the partial derivatives

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_1^0}{\partial x_1} - x_3\frac{\partial^2 u_3^0}{\partial x_1^2}, \quad \frac{\partial u_1}{\partial x_2} = \frac{\partial u_1^0}{\partial x_2} - x_3\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2}, \quad \frac{\partial u_1}{\partial x_3} = -\frac{\partial u_3^0}{\partial x_1},$$

$$\frac{\partial u_2}{\partial x_1} = \frac{\partial u_2^0}{\partial x_1} - x_3\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2}, \quad \frac{\partial u_2}{\partial x_2} = \frac{\partial u_2^0}{\partial x_2} - x_3\frac{\partial^2 u_3^0}{\partial x_2^2}, \quad \frac{\partial u_2}{\partial x_3} = -\frac{\partial u_3^0}{\partial x_2},$$

$$\frac{\partial u_3}{\partial x_1} = \frac{\partial u_3^0}{\partial x_1}, \quad \frac{\partial u_3}{\partial x_2} = \frac{\partial u_3^0}{\partial x_2}, \quad \frac{\partial u_3}{\partial x_3} = 0,$$

from which we can calculate:

$$\varepsilon_{11} = \frac{\partial u_1^0}{\partial x_1} - x_3\frac{\partial^2 u_3^0}{\partial x_1^2}, \quad \varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - x_3\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2},$$

$$\varepsilon_{22} = \frac{\partial u_2^0}{\partial x_2} - x_3\frac{\partial^2 u_3^0}{\partial x_2^2}, \quad \varepsilon_{23} = 0,$$

$$\varepsilon_{33} = 0, \quad \varepsilon_{31} = 0,$$

and:

$$\omega_{12} = \frac{1}{2}\left(\frac{\partial u_1^0}{\partial x_2} - \frac{\partial u_2^0}{\partial x_1}\right),$$

$$\omega_{23} = -\frac{\partial u_3^0}{\partial x_2},$$

$$\omega_{31} = \frac{\partial u_3^0}{\partial x_1}.$$

If the transverse displacements are significant, the ω_{12} component of infinitesimal rotation is neglected compared to ω_{23} and ω_{31}.

Thus the Von Karman strains are obtained:

$$\varepsilon_{11}^{VK} = \varepsilon_{11} + \frac{1}{2}\omega_{31}^2 = \frac{\partial u_1^0}{\partial x_1} - x_3\frac{\partial^2 u_3^0}{\partial x_1^2} + \frac{1}{2}\left(\frac{\partial u_3^0}{\partial x_1}\right)^2,$$

$$\varepsilon_{22}^{VK} = \varepsilon_{22} + \frac{1}{2}\omega_{32}^2 = \frac{\partial u_2^0}{\partial x_2} - x_3\frac{\partial^2 u_3^0}{\partial x_2^2} + \frac{1}{2}\left(\frac{\partial u_3^0}{\partial x_2}\right)^2,$$

$$\varepsilon_{12}^{VK} = \varepsilon_{12} + \frac{1}{2}\omega_{31}\omega_{32} = \frac{1}{2}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - x_3\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2} + \frac{1}{2}\frac{\partial u_3^0}{\partial x_1}\frac{\partial u_3^0}{\partial x_2},$$

$$\varepsilon_{13}^{VK} = \varepsilon_{23}^{VK} = \varepsilon_{33}^{VK} = 0.$$

The Von Karman strains ε_{ij} are as follows:

$$\varepsilon_{11} = \frac{\partial u_1^0}{\partial x_1} - x_3\frac{\partial^2 u_3^0}{\partial x_1^2} + \frac{1}{2}\left(\frac{\partial u_3^0}{\partial x_1}\right)^2,$$

$$\varepsilon_{22} = \frac{\partial u_2^0}{\partial x_2} - x_3\frac{\partial^2 u_3^0}{\partial x_2^2} + \frac{1}{2}\left(\frac{\partial u_3^0}{\partial x_2}\right)^2,$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1}\right) - x_3\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2} + \frac{1}{2}\frac{\partial u_3^0}{\partial x_1}\frac{\partial u_3^0}{\partial x_2},$$

$$\varepsilon_{13} = \varepsilon_{23} = \varepsilon_{33} = 0.$$

The Kirchhoff-Love virtual displacement field:

$$\delta u_1 = \delta u_1^0(x_1, x_2|t) - x_3\frac{\partial \delta u_3^0}{\partial x_1}(x_1, x_2|t),$$

$$\delta u_2 = \delta u_2^0(x_1, x_2|t) - x_3\frac{\partial \delta u_3^0}{\partial x_2}(x_1, x_2|t),$$

$$\delta u_3 = \delta u_3^0(x_1, x_2|t).$$

is associated with the Von Karman virtual strains:

$$\delta\varepsilon_{11} = \frac{\partial \delta u_1^0}{\partial x_1} - x_3\frac{\partial^2 \delta u_3^0}{\partial x_1^2} + \frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_1},$$

$$\delta\varepsilon_{22} = \frac{\partial \delta u_2^0}{\partial x_2} - x_3\frac{\partial^2 \delta u_3^0}{\partial x_2^2} + \frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_2},$$

$$\delta\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial \delta u_1^0}{\partial x_2} + \frac{\partial \delta u_2^0}{\partial x_1}\right) - x_3\frac{\partial^2 \delta u_3^0}{\partial x_1 \partial x_2} + \frac{1}{2}\left(\frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_1} + \frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_2}\right),$$

$$\delta\varepsilon_{13} = \delta\varepsilon_{23} = \delta\varepsilon_{33} = 0.$$

17.3. Variational formulation

The variational formulation of a problem with fixed surface efforts is written:

$$\left| \begin{array}{l} \int_\Omega \sigma_{ij}\delta\varepsilon_{ij}d\Omega - \int_{\Sigma_F} F_i\delta u_i dS - \int_\Omega f_i \delta u_i d\Omega + \int_\Omega \rho \frac{\partial^2 u_i}{\partial t^2}\delta u_i d\Omega = 0, \\ \text{where } u_i \text{ satisfies the relation } \forall \delta u_i. \end{array} \right.$$

17.3.1. *Virtual work of internal forces*

The virtual work of the internal forces δW_i is given by:

$$-\delta W_i = \int_\Omega \sigma_{ij}\delta\varepsilon_{ij}d\Omega .$$

Decomposing the volume integral into a simple integral and a double integral, we obtain the following relation:

$$-\delta W_i = \int_S \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(\sigma_{11}\delta\varepsilon_{11} + \sigma_{22}\delta\varepsilon_{22} + 2\sigma_{12}\delta\varepsilon_{12}\right)dx_3 \right] dS .$$

Replacing the virtual strains with this expression, we obtain:

$$-\delta W_i = \int_S \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\sigma_{11}\left(\frac{\partial \delta u_1^0}{\partial x_1} - x_3 \frac{\partial^2 \delta u_3^0}{\partial x_1^2} + \frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_1} \right) \right.\right. \dots$$

$$\dots + \sigma_{22}\left(\frac{\partial \delta u_2^0}{\partial x_2} - x_3 \frac{\partial^2 \delta u_3^0}{\partial x_2^2} + \frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_2} \right) \dots$$

$$\left.\left. \dots + \sigma_{12}\left(\frac{\partial \delta u_1^0}{\partial x_2} + \frac{\partial \delta u_2^0}{\partial x_1} - 2x_3 \frac{\partial^2 \delta u_3^0}{\partial x_1 \partial x_2} + \frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_1} + \frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_2} \right) \right] dx_3 \right\} dS.$$

By introducing the resultant forces and moments:

$$N_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij}dx_3 \quad \text{and} \quad M_{ij} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{ij}x_3 dx_3 \ ,$$

we have:

$$-\delta W_i = \int_S \left[N_{11}\frac{\partial \delta u_1^0}{\partial x_1} - M_{11}\frac{\partial^2 \delta u_3^0}{\partial x_1^2} + N_{11}\frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_1} + N_{22}\frac{\partial \delta u_2^0}{\partial x_2} \right. \dots$$

$$\dots - M_{22}\frac{\partial^2 \delta u_3^0}{\partial x_2^2} + N_{22}\frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_2} + N_{12}\left(\frac{\partial \delta u_1^0}{\partial x_2} + \frac{\partial \delta u_2^0}{\partial x_1} \right) \dots$$

$$\ldots - 2M_{12}\frac{\partial^2 \delta u_3^0}{\partial x_1 \partial x_2} + N_{12}\left(\frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_1} + \frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_2}\right)\Bigg] dS.$$

Using the derivative formula $uv' = (uv)' - u'v$, we obtain the following relations:

$$N_{11}\frac{\partial \delta u_1^0}{\partial x_1} = \frac{\partial}{\partial x_1}\left(N_{11}\delta u_1^0\right) - \frac{\partial N_{11}}{\partial x_1}\delta u_1^0,$$

$$M_{11}\frac{\partial^2 \delta u_3^0}{\partial x_1^2} = \frac{\partial}{\partial x_1}\left(M_{11}\frac{\partial \delta u_3^0}{\partial x_1}\right) - \frac{\partial M_{11}}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_1} = \frac{\partial}{\partial x_1}\left(M_{11}\frac{\partial \delta u_3^0}{\partial x_1}\right)\ldots$$

$$\ldots - \frac{\partial}{\partial x_1}\left(\frac{\partial M_{11}}{\partial x_1}\delta u_3^0\right) + \frac{\partial^2 M_{11}}{\partial x_1^2}\delta u_3^0,$$

$$N_{11}\frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_1} = \frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1}\delta u_3^0\right) - \frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1}\right)\delta u_3^0,$$

$$N_{22}\frac{\partial \delta u_2^0}{\partial x_2} = \frac{\partial}{\partial x_2}\left(N_{22}\delta u_2^0\right) - \frac{\partial N_{22}}{\partial x_2}\delta u_2^0,$$

$$M_{22}\frac{\partial^2 \delta u_3^0}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(M_{22}\frac{\partial \delta u_3^0}{\partial x_2}\right) - \frac{\partial M_{22}}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_2} = \frac{\partial}{\partial x_2}\left(M_{22}\frac{\partial \delta u_3^0}{\partial x_2}\right)\ldots$$

$$\ldots - \frac{\partial}{\partial x_2}\left(\frac{\partial M_{22}}{\partial x_2}\delta u_3^0\right) + \frac{\partial^2 M_{22}}{\partial x_2^2}\delta u_3^0,$$

$$N_{22}\frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_2} = \frac{\partial}{\partial x_2}\left(N_{22}\frac{\partial u_3^0}{\partial x_2}\delta u_3^0\right) - \frac{\partial}{\partial x_2}\left(N_{22}\frac{\partial u_3^0}{\partial x_2}\right)\delta u_3^0,$$

$$N_{12}\left(\frac{\partial \delta u_1^0}{\partial x_2} + \frac{\partial \delta u_2^0}{\partial x_1}\right) = \frac{\partial}{\partial x_2}\left(N_{12}\delta u_1^0\right) - \frac{\partial N_{12}}{\partial x_2}\delta u_1^0 + \frac{\partial}{\partial x_1}\left(N_{12}\delta u_2^0\right) - \frac{\partial N_{12}}{\partial x_1}\delta u_2^0,$$

$$2M_{12}\frac{\partial^2 \delta u_3^0}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(M_{12}\frac{\partial \delta u_3^0}{\partial x_2}\right) - \frac{\partial M_{12}}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_2} + \frac{\partial}{\partial x_2}\left(M_{12}\frac{\partial \delta u_3^0}{\partial x_1}\right)\ldots$$

$$\ldots - \frac{\partial M_{12}}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_1},$$

$$2M_{12}\frac{\partial^2 \delta u_3^0}{\partial x_1 \partial x_2} = \frac{\partial}{\partial x_1}\left(M_{12}\frac{\partial \delta u_3^0}{\partial x_2}\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial M_{12}}{\partial x_1}\delta u_3^0\right) + \frac{\partial^2 M_{12}}{\partial x_1 \partial x_2}\delta u_3^0\ldots$$

$$\ldots + \frac{\partial}{\partial x_2}\left(M_{12}\frac{\partial \delta u_3^0}{\partial x_1}\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial M_{12}}{\partial x_2}\delta u_3^0\right) + \frac{\partial^2 M_{12}}{\partial x_1 \partial x_2}\delta u_3^0,$$

$$N_{12}\left(\frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_1}+\frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_2}\right)=\frac{\partial}{\partial x_1}\left(N_{12}\frac{\partial u_3^0}{\partial x_2}\delta u_3^0\right)-\frac{\partial}{\partial x_1}\left(N_{12}\frac{\partial u_3^0}{\partial x_2}\right)\delta u_3^0\ldots$$

$$\ldots+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1}\delta u_3^0\right)-\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1}\right)\delta u_3^0,$$

allowing us to write the virtual work of body forces as below:

$$-\delta W_i=\int_S\left\{\frac{\partial}{\partial x_1}\left[N_{11}\delta u_1^0+N_{12}\delta u_2^0+\left(N_{11}\frac{\partial u_3^0}{\partial x_1}+N_{12}\frac{\partial u_3^0}{\partial x_2}+\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}\right)\delta u_3^0\ldots\right.\right.$$

$$\left.\ldots-M_{11}\frac{\partial \delta u_3^0}{\partial x_1}-M_{12}\frac{\partial \delta u_3^0}{\partial x_2}\right]+\frac{\partial}{\partial x_2}\left[N_{12}\delta u_1^0+N_{22}\delta u_2^0+\left(N_{12}\frac{\partial u_3^0}{\partial x_1}+N_{22}\frac{\partial u_3^0}{\partial x_2}\ldots\right.\right.$$

$$\left.\left.\left.\ldots+\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}\right)\delta u_3^0-M_{12}\frac{\partial \delta u_3^0}{\partial x_1}-M_{22}\frac{\partial \delta u_3^0}{\partial x_2}\right]\right\}dS-\int_S\left\{\left(\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}\right)\delta u_1^0\ldots\right.$$

$$\ldots+\left(\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}\right)\delta u_2^0+\left[\frac{\partial^2 M_{11}}{\partial x_1^2}+2\frac{\partial^2 M_{12}}{\partial x_1\partial x_2}+\frac{\partial^2 M_{22}}{\partial x_2^2}+\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1}\ldots\right.\right.$$

$$\left.\left.\left.\ldots+N_{12}\frac{\partial u_3^0}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1}+N_{22}\frac{\partial u_3^0}{\partial x_2}\right)\right]\delta u_3^0\right\}dS.$$

Taking into account the formula:

$$\int_S g_{j,j}dS=\int_\Gamma g_j n_j ds,$$

the virtual work of the internal forces is also written:

$$-\delta W_i=\int_\Gamma\left\{(N_{11}n_1+N_{12}n_2)\delta u_1^0+(N_{12}n_1+N_{22}n_2)\delta u_2^0+\left[(N_{11}n_1+N_{12}n_2)\frac{\partial u_3^0}{\partial x_1}\ldots\right.\right.$$

$$\left.\ldots+(N_{12}n_1+N_{22}n_2)\frac{\partial u_3^0}{\partial x_2}+\left(\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}\right)n_1+\left(\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}\right)n_2\right]\delta u_3^0\ldots$$

$$\left.\ldots-(M_{11}n_1+M_{12}n_2)\frac{\partial \delta u_3^0}{\partial x_1}-(M_{12}n_1+M_{22}n_2)\frac{\partial \delta u_3^0}{\partial x_2}\right\}ds\ldots$$

$$\ldots-\int_S\left\{\left(\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}\right)\delta u_1^0+\left(\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}\right)\delta u_2^0+\left[\frac{\partial^2 M_{11}}{\partial x_1^2}+2\frac{\partial^2 M_{12}}{\partial x_1\partial x_2}\ldots\right.\right.$$

$$\left.\left.\ldots+\frac{\partial^2 M_{22}}{\partial x_2^2}+\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1}+N_{12}\frac{\partial u_3^0}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1}+N_{22}\frac{\partial u_3^0}{\partial x_2}\right)\right]\delta u_3^0\right\}dS.$$

17.3.2. *Virtual work of transverse surface forces*

The virtual work developed by the transverse forces $q_3(x_1,x_2|t)x_3$ applied on the top surface of the plate is equal to:

$$\delta W_{F_t} = \int_{S_t} q_i \delta u_i dS ,$$

or:

$$\delta W_{F_t} = \int_S q_3 \delta u_3 dS ,$$

in fact, the integration applied on the top surface of the plate is reduced to an integration applied to the middle plane of the plate.

17.3.3. *Virtual work of external lateral surface forces*

The virtual work of the external lateral surface forces applied to the perimeter of the plate is given by the following relation:

$$\delta W_{F_c} = \int_{\Sigma_F} F_i \delta u_i dS .$$

Decomposing the surface integral to a simple integral applied on the thickness of the plate and to a simple integral applied to the outline of the middle surface of the plate, the previous relation is written:

$$\delta W_{F_c} = \int_\Gamma \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} F_i \delta u_i dx_3 \right) ds .$$

On introducing the virtual displacement δu_i, this relation becomes:

$$\delta W_{F_c} = \int_\Gamma \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[F_1 \left(\delta u_1^0 - x_3 \frac{\partial \delta u_3^0}{\partial x_1} \right) + F_2 \left(\delta u_2^0 - x_3 \frac{\partial \delta u_3^0}{\partial x_2} \right) + F_3 \delta u_3^0 \right] dx_3 \right\} ds .$$

Let:

$$Q_i(x_1,x_2|t) = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_i(x_1,x_2,x_3|t) dx_3 \quad (i = 1,2,3),$$

$$C_i(x_1,x_2|t) = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_i(x_1,x_2,x_3|t) x_3 dx_3 \quad (i = 1,2),$$

the virtual work developed by the surface forces applied on the outline of the plate is written:

$$\delta W_{F_c} = \int_\Gamma \left(Q_1 \delta u_1^0 + Q_2 \delta u_2^0 + Q_3 \delta u_3^0 - C_1 \frac{\partial \delta u_3^0}{\partial x_1} - C_2 \frac{\partial \delta u_3^0}{\partial x_2} \right) ds .$$

17.3.4. *Virtual work of body forces*

The virtual work developed by the body forces is obtained using the relation:

$$\delta W_f = \int_\Omega f_i \delta u_i d\Omega ,$$

which, decomposing the triple integral to a simple integral applied to the thickness of the plate and to a double integral applied to the middle surface, is written:

$$\delta W_f = \int_S \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} f_i \delta u_i dx_3 \right) dS .$$

The introduction of virtual displacements leads to:

$$\delta W_f = \int_S \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[f_1 \left(\delta u_1^0 - x_3 \frac{\partial \delta u_3^0}{\partial x_1} \right) + f_2 \left(\delta u_2^0 - x_3 \frac{\partial \delta u_3^0}{\partial x_2} \right) + f_3 \delta u_3^0 \right] dx_3 \right\} dS .$$

Let:

$$p_i(x_1, x_2 | t) = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_i(x_1, x_2, x_3 | t) dx_3 \quad (i = 1,2,3),$$

$$m_i(x_1, x_2 | t) = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_i(x_1, x_2, x_3 | t) x_3 dx_3 \quad (i = 1,2),$$

the virtual work developed by the body forces is written:

$$\delta W_f = \int_S \left(p_1 \delta u_1^0 + p_2 \delta u_2^0 + p_3 \delta u_3^0 - m_1 \frac{\partial \delta u_3^0}{\partial x_1} - m_2 \frac{\partial \delta u_3^0}{\partial x_2} \right) dS .$$

The differentiation formula $uv' = (uv)' - u'v$ allows us to write:

$$\delta W_f = \int_S \left[p_1 \delta u_1^0 + p_2 \delta u_2^0 + p_3 \delta u_3^0 - \frac{\partial}{\partial x_1}\left(m_1 \delta u_3^0\right) + \frac{\partial m_1}{\partial x_1} \delta u_3^0 \ldots \right.$$

$$\left. \ldots - \frac{\partial}{\partial x_2}\left(m_2 \delta u_3^0\right) + \frac{\partial m_2}{\partial x_2} \delta u_3^0 \right] dS.$$

With:

$$\int_S g_{j,j} dS = \int_\Gamma g_j n_j ds ,$$

the virtual work developed by the body forces is as written below:

$$W_f = \int_S \left[p_1 \delta u_1^0 + p_2 \delta u_2^0 + \left(p_3 + \frac{\partial m_1}{\partial x_1} + \frac{\partial m_2}{\partial x_2} \right) \delta u_3^0 \right] dS \ldots$$

$$\ldots - \int_\Gamma (m_1 n_1 + m_2 n_2) \delta u_3^0 ds.$$

17.3.5. *Virtual work done by inertial forces*

The virtual work done by inertial forces is written as:

$$\delta W_a = \int_\Omega \rho \frac{\partial^2 u_i}{\partial t^2} \delta u_i d\Omega ,$$

which, on decomposing the volume integral to a simple integral and a double integral, becomes:

$$\delta W_a = \int_S \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_i}{\partial t^2} \delta u_i dx_3 \right) dS .$$

On introducing the virtual displacement, we obtain:

$$\delta W_a = \int_S \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left[\left(\frac{\partial^2 u_1^0}{\partial t^2} - x_3 \frac{\partial^3 u_3^0}{\partial x_1 \partial t^2} \right) \left(\delta u_1^0 - x_3 \frac{\partial \delta u_3^0}{\partial x_1} \right) \ldots \right.\right.$$

$$\left.\left. \ldots + \left(\frac{\partial^2 u_2^0}{\partial t^2} - x_3 \frac{\partial^3 u_3^0}{\partial x_2 \partial t^2} \right) \left(\delta u_2^0 - x_3 \frac{\partial \delta u_3^0}{\partial x_2} \right) + \frac{\partial^2 u_3^0}{\partial t^2} \delta u_3^0 \right] dx_3 \right\} dS .$$

Let:

$$I_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho dx_3, \quad I_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho x_3 dx_3, \quad I_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho x_3^2 dx_3,$$

thus we obtain:

$$\delta W_a = \int_S \left[\left(I_0 \frac{\partial^2 u_1^0}{\partial t^2} - I_1 \frac{\partial^3 u_3^0}{\partial x_1 \partial t^2} \right) \delta u_1^0 + \left(I_0 \frac{\partial^2 u_2^0}{\partial t^2} - I_1 \frac{\partial^3 u_3^0}{\partial x_2 \partial t^2} \right) \delta u_2^0 \ldots \right.$$

$$\ldots + I_0 \frac{\partial^2 u_3^0}{\partial t^2} \delta u_3^0 - \left(I_1 \frac{\partial^2 u_1^0}{\partial t^2} - I_2 \frac{\partial^3 u_3^0}{\partial x_1 \partial t^2} \right) \frac{\partial \delta u_3^0}{\partial x_1} \ldots$$

$$\left. \ldots - \left(I_1 \frac{\partial^2 u_2^0}{\partial t^2} - I_2 \frac{\partial^3 u_3^0}{\partial x_2 \partial t^2} \right) \frac{\partial \delta u_3^0}{\partial x_2} \right] dS.$$

Kirchhoff-Love analysis neglects the terms I_1 and I_2 compared with I_0. The virtual work developed by the inertia forces is reduced to the following relation:

$$\delta W_a = \int_S I_0 \left(\frac{\partial^2 u_1^0}{\partial t^2} \delta u_1^0 + \frac{\partial^2 u_2^0}{\partial t^2} \delta u_2^0 + \frac{\partial^2 u_3^0}{\partial t^2} \delta u_3^0 \right) dS .$$

17.3.6. *Variational formulation*

Taking into account the previous relations, the variational formulation of plates in Kirchhoff-Love type analysis is written as:

$$\int_\Gamma \left\{ (N_{11}n_1 + N_{12}n_2)\delta u_1^0 + (N_{12}n_1 + N_{22}n_2)\delta u_2^0 + \left[(N_{11}n_1 + N_{12}n_2)\frac{\partial u_3^0}{\partial x_1} \right.\right. \ldots$$

$$\ldots + (N_{12}n_1 + N_{22}n_2)\frac{\partial u_3^0}{\partial x_2} + \left(\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2}\right)n_1 + \left.\left(\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2}\right)n_2 \right]\delta u_3^0 \ldots$$

$$\ldots - (M_{11}n_1 + M_{12}n_2)\frac{\partial \delta u_3^0}{\partial x_1} - \left.(M_{12}n_1 + M_{22}n_2)\frac{\partial \delta u_3^0}{\partial x_2}\right\} ds \ldots$$

$$\ldots - \int_S \left\{ \left(\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2}\right)\delta u_1^0 + \left(\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2}\right)\delta u_2^0 + \left[\frac{\partial^2 M_{11}}{\partial x_1^2} + 2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} \right.\right. \ldots$$

$$\ldots + \frac{\partial^2 M_{22}}{\partial x_2^2} + \frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1} + N_{12}\frac{\partial u_3^0}{\partial x_2}\right) + \left.\left.\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1} + N_{22}\frac{\partial u_3^0}{\partial x_2}\right)\right]\delta u_3^0 \right\} dS \ldots$$

$$\ldots - \int_S q_3 \delta u_3 dS - \int_\Gamma \left(Q_1 \delta u_1^0 + Q_2 \delta u_2^0 + Q_3 \delta u_3^0 - C_1 \frac{\partial \delta u_3^0}{\partial x_1} - C_2 \frac{\partial \delta u_3^0}{\partial x_2}\right) ds \ldots$$

$$\ldots - \int_S \left[p_1 \delta u_1^0 + p_2 \delta u_2^0 + \left(p_3 + \frac{\partial m_1}{\partial x_1} + \frac{\partial m_2}{\partial x_2}\right)\delta u_3^0 \right] dS + \int_\Gamma (m_1 n_1 + m_2 n_2)\delta u_3^0 ds \ldots$$

$$\ldots + \int_S I_0 \left(\frac{\partial^2 u_1^0}{\partial t^2}\delta u_1^0 + \frac{\partial^2 u_2^0}{\partial t^2}\delta u_2^0 + \frac{\partial^2 u_3^0}{\partial t^2}\delta u_3^0 \right) dS = 0,$$

where u_1^0, u_2^0 and u_3^0 satisfy the relation $\forall \delta u_1^0, \forall \delta u_2^0, \forall \delta u_3^0, \forall \frac{\partial \delta u_3^0}{\partial x_1}$ and $\forall \frac{\partial \delta u_3^0}{\partial x_2}$.

Or:

$$-\int_S \left\{ \left(\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 - I_0 \frac{\partial^2 u_1^0}{\partial t^2}\right)\delta u_1^0 + \left(\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 - I_0 \frac{\partial^2 u_2^0}{\partial t^2}\right)\delta u_2^0 \right. \ldots$$

$$\ldots + \left[\frac{\partial^2 M_{11}}{\partial x_1^2} + 2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} + \frac{\partial^2 M_{22}}{\partial x_2^2} + \frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1} + N_{12}\frac{\partial u_3^0}{\partial x_2}\right) \right. \ldots$$

$$\ldots + \frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1} + N_{22}\frac{\partial u_3^0}{\partial x_2}\right) + q_3 + p_3 + \frac{\partial m_1}{\partial x_1} + \frac{\partial m_2}{\partial x_2} - \left.\left. I_0 \frac{\partial^2 u_3^0}{\partial t^2}\right]\delta u_3^0 \right\} dS \ldots$$

$$\ldots+\int_{\Gamma}\Big\{(N_{11}n_1+N_{12}n_2-Q_1)\delta u_1^0+(N_{12}n_1+N_{22}n_2-Q_2)\delta u_2^0\ldots$$

$$\ldots+\left[(N_{11}n_1+N_{12}n_2)\frac{\partial u_3^0}{\partial x_1}+(N_{12}n_1+N_{22}n_2)\frac{\partial u_3^0}{\partial x_2}+\left(\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}\right)n_1\ldots\right.$$

$$\left.\ldots+\left(\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}\right)n_2-Q_3+m_1n_1+m_2n_2\right]\delta u_3^0-(M_{11}n_1+M_{12}n_2-C_1)\frac{\partial \delta u_3^0}{\partial x_1}\ldots$$

$$\ldots-(M_{12}n_1+M_{22}n_2-C_2)\frac{\partial \delta u_3^0}{\partial x_2}\Big\}ds=0,$$

where u_1^0, u_2^0 and u_3^0 satisfy the relation $\forall\delta u_1^0,\forall\delta u_2^0,\forall\delta u_3^0,\forall\frac{\partial \delta u_3^0}{\partial x_1}$ and $\forall\frac{\partial \delta u_3^0}{\partial x_2}$.

17.4. Global plate equations, boundary edge conditions

17.4.1. *Global plate equations*

The relation just obtained is satisfied, whatever the values of $\delta u_1^0,\delta u_2^0,\delta u_3^0$, $\frac{\partial \delta u_3^0}{\partial x_1}$ and $\frac{\partial \delta u_3^0}{\partial x_2}$; it is still true if these values are imposed as zero on the perimeter Γ of the middle surface S of the plate.

From the variational formulation:

$$\int_S\left\{\left(\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}+p_1-I_0\frac{\partial^2 u_1^0}{\partial t^2}\right)\delta u_1^0+\left(\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}+p_2-I_0\frac{\partial^2 u_2^0}{\partial t^2}\right)\delta u_2^0\ldots\right.$$

$$\ldots+\left[\frac{\partial^2 M_{11}}{\partial x_1^2}+2\frac{\partial^2 M_{12}}{\partial x_1\partial x_2}+\frac{\partial^2 M_{22}}{\partial x_2^2}+\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1}+N_{12}\frac{\partial u_3^0}{\partial x_2}\right)\ldots\right.$$

$$\left.\left.\ldots+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1}+N_{22}\frac{\partial u_3^0}{\partial x_2}\right)+q_3+p_3+\frac{\partial m_1}{\partial x_1}+\frac{\partial m_2}{\partial x_2}-I_0\frac{\partial^2 u_3^0}{\partial t^2}\right]\delta u_3^0\right\}dS=0,$$

where u_1^0,u_2^0 and u_3^0, satisfy the relation $\forall\delta u_1^0,\forall\delta u_2^0$ and $\forall\delta u_3^0\ldots$

$$\ldots/\,\delta u_1^0=\delta u_2^0=\delta u_3^0=0 \text{ on } \Gamma,$$

we obtain, for all the middle surface, the following equations:

$$\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}+p_1=I_0\frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}+p_2=I_0\frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial^2 M_{11}}{\partial x_1^2}+2\frac{\partial^2 M_{12}}{\partial x_1\partial x_2}+\frac{\partial^2 M_{22}}{\partial x_2^2}+\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1}+N_{12}\frac{\partial u_3^0}{\partial x_2}\right)\ldots$$

$$\ldots+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1}+N_{22}\frac{\partial u_3^0}{\partial x_2}\right)+q_3+p_3+\frac{\partial m_1}{\partial x_1}+\frac{\partial m_2}{\partial x_2}=I_0\frac{\partial^2 u_3^0}{\partial t^2}.$$

17.4.2. *Boundary edge conditions*

Taking the three equations in the variational formulation, we obtain the following formulation:

$$\int_\Gamma\left\{(N_{11}n_1+N_{12}n_2-Q_1)\delta u_1^0+(N_{12}n_1+N_{22}n_2-Q_2)\delta u_2^0\ldots\right.$$

$$\ldots+\left[(N_{11}n_1+N_{12}n_2)\frac{\partial u_3^0}{\partial x_1}+(N_{12}n_1+N_{22}n_2)\frac{\partial u_3^0}{\partial x_2}+\left(\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}\right)n_1\ldots\right.$$

$$\left.\ldots+\left(\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}\right)n_2-Q_3+m_1n_1+m_2n_2\right]\delta u_3^0-(M_{11}n_1+M_{12}n_2-C_1)\frac{\partial \delta u_3^0}{\partial x_1}\ldots$$

$$\left.\ldots-(M_{12}n_1+M_{22}n_2-C_2)\frac{\partial \delta u_3^0}{\partial x_2}\right\}ds=0,$$

where u_1^0, u_2^0 and u_3^0 satisfy the relation $\forall\delta u_1^0, \forall\delta u_2^0, \forall\delta u_3^0, \forall\frac{\partial \delta u_3^0}{\partial x_1}$ and $\forall\frac{\partial \delta u_3^0}{\partial x_2}$,

from which we deduce the edge boundary conditions:

$$Q_1=N_{11}n_1+N_{12}n_2,$$

$$Q_2=N_{12}n_1+N_{22}n_2,$$

$$Q_3=(N_{11}n_1+N_{12}n_2)\frac{\partial u_3^0}{\partial x_1}+(N_{12}n_1+N_{22}n_2)\frac{\partial u_3^0}{\partial x_2}+\left(\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}\right)n_1\ldots$$

$$\ldots+\left(\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}\right)n_{23}+m_1n_1+m_2n_2,$$

$$C_1=M_{11}n_1+M_{12}n_2,$$

$$C_2=M_{12}n_1+M_{22}n_2.$$

Subsequently in this appendix, the particular case is applied where:

$$m_1 = m_2 = 0 .$$

On introducing for all M points of the perimeter Γ the direct local co-ordinates:

$$(b) = (\mathbf{n}, \boldsymbol{\tau}, \mathbf{x}_3),$$

where $\mathbf{n}$ is orthogonal and external to the edge, and where $\boldsymbol{\tau}$ is tangential to it and:

$$\mathbf{n} = n_1\mathbf{x}_1 + n_2\mathbf{x}_2,$$

$$\boldsymbol{\tau} = \mathbf{x}_3 \times \mathbf{n} = -n_2\mathbf{x}_1 + n_1\mathbf{x}_2.$$

The components of the virtual displacement vector:

$$\delta\mathbf{U} = \delta u_1^0\mathbf{x}_1 + \delta u_2^0\mathbf{x}_2 + \delta u_3^0\mathbf{x}_3 = \delta u_n^0\mathbf{n} + \delta u_s^0\boldsymbol{\tau} + \delta u_3^0\mathbf{x}_3,$$

are linked by:

$$\delta u_1^0 = \delta u_n^0 n_1 - \delta u_s^0 n_2,$$

$$\delta u_2^0 = \delta u_n^0 n_2 + \delta u_s^0 n_1.$$

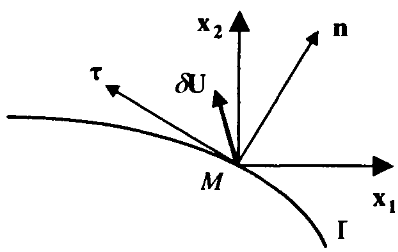

Figure 17.1. *Local axes associated with the edge*

The partial differential functions of δu_3^0, with respect to x_1 and x_2, and in relation to n and s, are linked by:

$$\frac{\partial \delta u_3^0}{\partial x_1} = \frac{\partial \delta u_3^0}{\partial n} n_1 - \frac{\partial \delta u_3^0}{\partial s} n_2,$$

$$\frac{\partial \delta u_3^0}{\partial x_2} = \frac{\partial \delta u_3^0}{\partial n} n_2 + \frac{\partial \delta u_3^0}{\partial s} n_1.$$

The simple integral in the virtual work of internal forces:

$$-\delta W_{i_\Gamma} = \int_\Gamma \Bigg\{ (N_{11}n_1 + N_{12}n_2)\delta u_1^0 + (N_{12}n_1 + N_{22}n_2)\delta u_2^0 + \Bigg[(N_{11}n_1 + N_{12}n_2)\frac{\partial u_3^0}{\partial x_1} \ldots$$

$$\ldots + (N_{12}n_1 + N_{22}n_2)\frac{\partial u_3^0}{\partial x_2} + \left(\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} \right) n_1 + \left(\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} \right) n_2 \Bigg] \delta u_3^0 \ldots$$

$$\ldots-\left(M_{11}n_1+M_{12}n_2\right)\frac{\partial\delta u_3^0}{\partial x_1}-\left(M_{12}n_1+M_{22}n_2\right)\frac{\partial\delta u_3^0}{\partial x_2}\Bigg\}ds,$$

is written as:

$$-\delta W_{i_\Gamma}=\int_\Gamma\Bigg(\Big[N_{11}n_1^2+N_{22}n_2^2+2N_{12}n_1n_2\Big]\delta u_n^0\ +\big[-\left(N_{11}-N_{22}\right)n_1n_2\ldots$$

$$\ldots+N_{12}\left(n_1^2-n_2^2\right)\big]\delta u_s^0+\Big\{\Big[N_{11}n_1^2+N_{22}n_2^2+2N_{12}n_1n_2\Big]\frac{\partial u_3^0}{\partial n}+\big[-\left(N_{11}-N_{22}\right)n_1n_2\ldots$$

$$\ldots+N_{12}\left(n_1^2-n_2^2\right)\big]\frac{\partial u_3^0}{\partial s}+\left(\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}\right)n_1+\left(\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}\right)n_2\Big\}\delta u_3^0\ldots$$

$$\ldots-\Big[M_{11}n_1^2+M_{22}n_2^2+2M_{12}n_1n_2\Big]\frac{\partial\delta u_3^0}{\partial n}-\big[-\left(M_{11}-M_{22}\right)n_1n_2\ldots$$

$$\ldots+M_{12}\left(n_1^2-n_2^2\right)\big]\frac{\partial\delta u_3^0}{\partial s}\Bigg)ds.$$

The resultant forces and moments do not have the following components in the local axes (b):

$$N_n=N_{11}n_1^2+N_{22}n_2^2+2N_{12}n_1n_2,$$
$$N_s=-\left(N_{11}-N_{22}\right)n_1n_2+N_{12}\left(n_1^2-n_2^2\right),$$
$$M_n=M_{11}n_1^2+M_{22}n_2^2+2M_{12}n_1n_2,$$
$$M_s=-\left(M_{11}-M_{22}\right)n_1n_2+M_{12}\left(n_1^2-n_2^2\right).$$

By introducing:

$$N_3=N_n\frac{\partial u_3^0}{\partial n}+N_s\frac{\partial u_3^0}{\partial s}+\left(\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}\right)n_1+\left(\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}\right)n_2\ ,$$

The virtual work of internal forces is as written below:

$$-\delta W_i=\int_\Gamma\left(N_n\delta u_n^0+N_s\delta u_s^0+N_3\delta u_3^0-M_n\frac{\partial\delta u_3^0}{\partial n}-M_s\frac{\partial\delta u_3^0}{\partial s}\right)ds\ldots$$

$$\ldots-\int_S\Bigg\{\left(\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}\right)\delta u_1^0+\left(\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}\right)\delta u_2^0+\Bigg[\frac{\partial^2M_{11}}{\partial x_1^2}+2\frac{\partial^2M_{12}}{\partial x_1\partial x_2}\ldots$$

$$\ldots+\frac{\partial^2M_{22}}{\partial x_2^2}+\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1}+N_{12}\frac{\partial u_3^0}{\partial x_2}\right)+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1}+N_{22}\frac{\partial u_3^0}{\partial x_2}\right)\Bigg]\delta u_3^0\Bigg\}dS.$$

The virtual work developed by the surface forces applied on the edge of the plate is given by:

$$\delta W_{F_c} = \int_{\Gamma}\left[\int_{-\frac{h}{2}}^{\frac{h}{2}}\left(F_n\delta u_n + F_s\delta u_s + F_3\delta u_3^0\right)\right]ds\,.$$

By introducing the virtual displacements:

$$\delta u_n = \delta u_n^0 - x_3\frac{\partial \delta u_3^0}{\partial n},$$

$$\delta u_s = \delta u_s^0 - x_3\frac{\partial \delta u_3^0}{\partial s},$$

$$\delta u_3 = \delta u_3^0,$$

we obtain:

$$\delta W_{F_c} = \int_{\Gamma}\left\{\int_{-\frac{h}{2}}^{\frac{h}{2}}\left[F_n\left(\delta u_n^0 - x_3\frac{\partial \delta u_3^0}{\partial n}\right) + F_s\left(\delta u_s^0 - x_3\frac{\partial \delta u_3^0}{\partial x_s}\right)\ldots\right.\right.$$

$$\left.\left.\ldots + F_3\delta u_3^0\right]dx_3\right\}ds.$$

Let:

$$Q_n = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_n dx_3, \qquad Q_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_s dx_3, \qquad Q_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_3 dx_3,$$

$$C_n = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_n x_3 dx_3, \quad C_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_s x_3 dx_3,$$

thus the virtual work developed by the surface forces applied on the outline of the plate is written as:

$$\delta W_{F_c} = \int_{\Gamma}\left(Q_n\delta u_n^0 + Q_s\delta u_s^0 + Q_3\delta u_3^0 - C_n\frac{\partial \delta u_3^0}{\partial n} - C_s\frac{\partial \delta u_3^0}{\partial s}\right)ds\,.$$

The boundary edge conditions are written as:

$$Q_n = N_n,$$

$$Q_s = N_s,$$

$$Q_3 = N_3,$$

$$C_n = M_n,$$

$$C_s = M_s.$$

In Kirchoff-Love type analysis, the third and fifth conditions are combined into a single condition that includes the transverse shear effort that we obtain when we transform the following term:

$$\int_{\Gamma} - M_s\frac{\partial \delta u_3^0}{\partial s}ds\,,$$

which is present in the virtual work of the internal forces. Using the differentiation formula for a product, we obtain:

$$\int_\Gamma -M_s \frac{\partial \delta u_3^0}{\partial s} ds = \int_\Gamma -\frac{\partial}{\partial s}\left(M_s \delta u_3^0\right) ds + \int_\Gamma \frac{\partial M_s}{\partial s} \delta u_3^0 ds,$$

$$\int_\Gamma -M_s \frac{\partial \delta u_3^0}{\partial s} ds = \int_\Gamma \frac{\partial M_s}{\partial s} \delta u_3^0 ds - \left[M_s \delta u_3^0\right]_\Gamma.$$

The last term between the brackets is zero when the edge Γ has no angular points, otherwise there is a transverse force acting at the point being considered.

The virtual work of internal forces is written as:

$$-\delta W_i = \int_\Gamma \left(N_n \delta u_n^0 + N_s \delta u_s^0 + \left(N_3 + \frac{\partial M_s}{\partial s} \right) \delta u_3^0 - M_n \frac{\partial \delta u_3^0}{\partial n} \right) ds - \left[M_s \delta u_3^0\right]_\Gamma \ldots$$

$$\ldots - \int_S \left\{ \left(\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} \right) \delta u_1^0 + \left(\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} \right) \delta u_2^0 + \left[\frac{\partial^2 M_{11}}{\partial x_1^2} + 2 \frac{\partial^2 M_{12}}{\partial x_1 \partial x_2} \ldots \right.\right.$$

$$\left.\left. \ldots + \frac{\partial^2 M_{22}}{\partial x_2^2} + \frac{\partial}{\partial x_1}\left(N_{11} \frac{\partial u_3^0}{\partial x_1} + N_{12} \frac{\partial u_3^0}{\partial x_2} \right) + \frac{\partial}{\partial x_2}\left(N_{12} \frac{\partial u_3^0}{\partial x_1} + N_{22} \frac{\partial u_3^0}{\partial x_2} \right) \right] \delta u_3^0 \right\} dS,$$

and the virtual work of effort applied at the edge is written as below:

$$\delta W_F = \int_\Gamma \left(Q_n \delta u_n^0 + Q_s \delta u_s^0 + Q_3^{eff} \delta u_3^0 - C_n \frac{\partial \delta u_3^0}{\partial n} \right) ds + \sum_i F_i \delta u_{3i}^0.$$

The edge boundary conditions are written as:

$$Q_n = N_n,$$

$$Q_s = N_s,$$

$$Q_3^{eff} = N_3 + \frac{\partial M_s}{\partial s},$$

$$C_n = M_n,$$

$$F_i = -\left[M_{si}\right] = M_{si}^- - M_{si}^+.$$

The last condition only applies for the rough points.

17.5. Global plate equations in static, vibration and buckling cases

The global plate equations:

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 = I_0 \frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 = I_0 \frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial^2 M_{11}}{\partial x_1^2}+2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2}+\frac{\partial^2 M_{22}}{\partial x_2^2}+\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1}+N_{12}\frac{\partial u_3^0}{\partial x_2}\right)\dots$$

$$\dots+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1}+N_{22}\frac{\partial u_3^0}{\partial x_2}\right)+q_3+p_3=I_0\frac{\partial^2 u_3^0}{\partial t^2},$$

are written as:

– static:

$$\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}+p_1=0,$$

$$\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}+p_2=0,$$

$$\frac{\partial^2 M_{11}}{\partial x_1^2}+2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2}+\frac{\partial^2 M_{22}}{\partial x_2^2}+q_3+p_3=0,$$

– vibration:

$$\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}=I_0\frac{\partial^2 u_1^0}{\partial t^2},$$

$$\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}=I_0\frac{\partial^2 u_2^0}{\partial t^2},$$

$$\frac{\partial^2 M_{11}}{\partial x_1^2}+2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2}+\frac{\partial^2 M_{22}}{\partial x_2^2}=I_0\frac{\partial^2 u_3^0}{\partial t^2},$$

– buckling:

$$\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}=0,$$

$$\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}=0,$$

$$\frac{\partial^2 M_{11}}{\partial x_1^2}+2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2}+\frac{\partial^2 M_{22}}{\partial x_2^2}+N_{11}\frac{\partial^2 u_3^0}{\partial x_1^2}+2N_{12}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2}\dots$$

$$\dots+N_{22}\frac{\partial^2 u_3^0}{\partial x_2^2}+\left(\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}\right)\frac{\partial u_3^0}{\partial x_1}+\left(\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}\right)\frac{\partial u_3^0}{\partial x_2}=0,$$

taking into account the first two relations, the third equation is reduced to:

$$\frac{\partial^2 M_{11}}{\partial x_1^2}+2\frac{\partial^2 M_{12}}{\partial x_1 \partial x_2}+\frac{\partial^2 M_{22}}{\partial x_2^2}+N_{11}\frac{\partial^2 u_3^0}{\partial x_1^2}+2N_{12}\frac{\partial^2 u_3^0}{\partial x_1 \partial x_2}+N_{22}\frac{\partial^2 u_3^0}{\partial x_2^2}=0.$$

Chapter 18

Global plate equations: Reissner-Mindlin theory variational formulation

18.1. Introduction

This appendix, which refers to different subjects to the previous one, presents the Reissner-Mindlin theory of plates taking into account transverse shear.

18.2. Von Karman strains

Including the Reissner-Mindlin displacement field:

$$u_1 = u_1^0(x_1, x_2|t) + x_3\psi_1(x_1, x_2|t),$$
$$u_2 = u_2^0(x_1, x_2|t) + x_3\psi_2(x_1, x_2|t),$$
$$u_3 = u_3^0(x_1, x_2|t),$$

in Von Karman strains:

$$\varepsilon_{ij}^{VK} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\frac{\partial u_3}{\partial x_i}\frac{\partial u_3}{\partial x_j},$$

leads to:

$$\varepsilon_{11} = \frac{\partial u_1^0}{\partial x_1} + x_3\frac{\partial \psi_1}{\partial x_1} + \frac{1}{2}\left(\frac{\partial u_3^0}{\partial x_1}\right)^2,$$

$$\varepsilon_{22} = \frac{\partial u_2^0}{\partial x_2} + x_3\frac{\partial \psi_2}{\partial x_2} + \frac{1}{2}\left(\frac{\partial u_3^0}{\partial x_2}\right)^2,$$

$$\varepsilon_{33} = 0,$$

$$\varepsilon_{12} = \frac{1}{2}\left[\frac{\partial u_1^0}{\partial x_2} + \frac{\partial u_2^0}{\partial x_1} + x_3\left(\frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1}\right)\right] + \frac{1}{2}\frac{\partial u_3^0}{\partial x_1}\frac{\partial u_3^0}{\partial x_2},$$

$$\varepsilon_{13} = \frac{1}{2}\left(\psi_1 + \frac{\partial u_3^0}{\partial x_1}\right),$$

$$\varepsilon_{23} = \frac{1}{2}\left(\psi_2 + \frac{\partial u_3^0}{\partial x_2}\right).$$

With the virtual displacements of Reissner-Mindlin plate theory:

$$\delta u_1 = \delta u_1^0(x_1, x_2|t) + x_3 \delta\psi_1(x_1, x_2|t),$$

$$\delta u_2 = \delta u_2^0(x_1, x_2|t) + x_3 \delta\psi_2(x_1, x_2|t),$$

$$\delta u_3 = \delta u_3^0(x_1, x_2|t),$$

are associated the virtual strains:

$$\delta\varepsilon_{11} = \frac{\partial \delta u_1^0}{\partial x_1} + x_3 \frac{\partial \delta\psi_1}{\partial x_1} + \frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_1},$$

$$\delta\varepsilon_{22} = \frac{\partial \delta u_2^0}{\partial x_2} + x_3 \frac{\partial \delta\psi_2}{\partial x_2} + \frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_2},$$

$$\delta\varepsilon_{33} = 0,$$

$$\delta\varepsilon_{12} = \frac{1}{2}\left[\frac{\partial \delta u_1^0}{\partial x_2} + \frac{\partial \delta u_2^0}{\partial x_1} + x_3\left(\frac{\partial \delta\psi_1}{\partial x_2} + \frac{\partial \delta\psi_2}{\partial x_1}\right)\right]\ldots$$

$$\ldots + \frac{1}{2}\left(\frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_1} + \frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_2}\right),$$

$$\delta\varepsilon_{13} = \frac{1}{2}\left(\delta\psi_1 + \frac{\partial \delta u_3^0}{\partial x_1}\right),$$

$$\delta\varepsilon_{23} = \frac{1}{2}\left(\delta\psi_2 + \frac{\partial \delta u_3^0}{\partial x_2}\right).$$

18.3. Variational formulation

The variational formulation of the elasto-dynamic problem with surface forces applied is written as:

$$\left|\begin{array}{l}\int_\Omega \sigma_{ij}\delta\varepsilon_{ij}d\Omega - \int_{\Sigma_F} F_i \delta u_i dS - \int_\Omega f_i \delta u_i d\Omega + \int_\Omega \rho \frac{\partial^2 u_i}{\partial t^2}\delta u_i d\Omega = 0, \\ \text{where } u_i \text{ satisfies the relation } \forall \delta u_i.\end{array}\right.$$

18.3.1. *Virtual work of internal forces*

The virtual work of internal forces δW_i is presented by the relationship:

$$-\delta W_i = \int_\Omega \sigma_{ij} \delta\varepsilon_{ij} d\Omega ,$$

which can be written as:

$$-\delta W_i = \int_S \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} (\sigma_{11}\delta\varepsilon_{11} + \sigma_{22}\delta\varepsilon_{22} + 2\sigma_{12}\delta\varepsilon_{12} + 2\sigma_{13}\delta\varepsilon_{13} \ldots \right.$$

$$\left. \ldots + 2\sigma_{23}\delta\varepsilon_{23})dx_3 \right] dS.$$

Including the virtual strains within the virtual work of the internal forces gives:

$$-\delta W_i = \int_S \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[\left(\sigma_{11} \frac{\partial \delta u_1^0}{\partial x_1} + x_3 \frac{\partial \delta\psi_1}{\partial x_1} + \frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_1} \right) \ldots \right.\right.$$

$$\ldots + \sigma_{22}\left(\frac{\partial \delta u_2^0}{\partial x_2} + x_3 \frac{\partial \delta\psi_2}{\partial x_2} + \frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_2} \right) \ldots$$

$$\ldots + \sigma_{12}\left[\frac{\partial \delta u_1^0}{\partial x_2} + \frac{\partial \delta u_2^0}{\partial x_1} + x_3\left(\frac{\partial \delta\psi_1}{\partial x_2} + \frac{\partial \delta\psi_2}{\partial x_1} \right) + \frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_1} + \frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_2} \right] \ldots$$

$$\left.\left. \ldots + \sigma_{13}\left(\delta\psi_1 + \frac{\partial \delta u_3^0}{\partial x_1} \right) + \sigma_{23}\left(\delta\psi_2 + \frac{\partial \delta u_3^0}{\partial x_2} \right) \right] dx_3 \right\} dS,$$

and the introduction of the resultant forces and moments allows the following expression to be written:

$$-\delta W_i = \int_S \left[N_{11} \frac{\partial \delta u_1^0}{\partial x_1} + M_{11} \frac{\partial \delta\psi_1}{\partial x_1} + N_{11} \frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_1} \ldots \right.$$

$$\ldots + N_{22} \frac{\partial \delta u_2^0}{\partial x_2} + M_{22} \frac{\partial \delta\psi_2}{\partial x_2} + N_{22} \frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_2} \ldots$$

$$\ldots + N_{12}\left(\frac{\partial \delta u_1^0}{\partial x_2} + \frac{\partial \delta u_2^0}{\partial x_1} \right) + M_{12}\left(\frac{\partial \delta\psi_1}{\partial x_2} + \frac{\partial \delta\psi_2}{\partial x_1} \right) \ldots$$

$$\ldots + N_{12}\left(\frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_1} + \frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_2} \right) + N_{13}\left(\delta\psi_1 + \frac{\partial \delta u_3^0}{\partial x_1} \right) \ldots$$

$$\left. \ldots + N_{23}\left(\delta\psi_2 + \frac{\partial \delta u_3^0}{\partial x_2} \right) \right] dS.$$

With the following relationships:

$$N_{11}\frac{\partial \delta u_1^0}{\partial x_1}=\frac{\partial}{\partial x_1}\left(N_{11}\delta u_1^0\right)-\frac{\partial N_{11}}{\partial x_1}\delta u_1^0,$$

$$M_{11}\frac{\partial \delta \psi_1}{\partial x_1}=\frac{\partial}{\partial x_1}\left(M_{11}\delta \psi_1\right)-\frac{\partial M_{11}}{\partial x_1}\delta \psi_1,$$

$$N_{11}\frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_1}=\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1}\delta u_3^0\right)-\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1}\right)\delta u_3^0,$$

$$N_{22}\frac{\partial \delta u_1^0}{\partial x_2}=\frac{\partial}{\partial x_2}\left(N_{22}\delta u_2^0\right)-\frac{\partial N_{22}}{\partial x_2}\delta u_2^0,$$

$$M_{22}\frac{\partial \delta \psi_2}{\partial x_2}=\frac{\partial}{\partial x_2}\left(M_{22}\delta \psi_2\right)-\frac{\partial M_{22}}{\partial x_2}\delta \psi_2,$$

$$N_{22}\frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_2}=\frac{\partial}{\partial x_2}\left(N_{22}\frac{\partial u_3^0}{\partial x_2}\delta u_3^0\right)-\frac{\partial}{\partial x_2}\left(N_{22}\frac{\partial u_3^0}{\partial x_2}\right)\delta u_3^0,$$

$$N_{12}\left(\frac{\partial \delta u_1^0}{\partial x_2}+\frac{\partial \delta u_2^0}{\partial x_1}\right)=\frac{\partial}{\partial x_2}\left(N_{12}\delta u_1^0\right)-\frac{\partial N_{12}}{\partial x_2}\delta u_1^0+\frac{\partial}{\partial x_1}\left(N_{12}\delta u_2^0\right)-\frac{\partial N_{12}}{\partial x_1}\delta u_2^0,$$

$$M_{12}\left(\frac{\partial \delta \psi_1}{\partial x_2}+\frac{\partial \delta \psi_2}{\partial x_1}\right)=\frac{\partial}{\partial x_2}\left(M_{12}\delta \psi_1\right)-\frac{\partial M_{12}}{\partial x_2}\delta \psi_1+\frac{\partial}{\partial x_1}\left(M_{12}\delta \psi_2\right)\ldots$$

$$\ldots-\frac{\partial M_{12}}{\partial x_1}\delta \psi_2,$$

$$N_{12}\left(\frac{\partial u_3^0}{\partial x_2}\frac{\partial \delta u_3^0}{\partial x_1}+\frac{\partial u_3^0}{\partial x_1}\frac{\partial \delta u_3^0}{\partial x_2}\right)=\frac{\partial}{\partial x_1}\left(N_{12}\frac{\partial u_3^0}{\partial x_2}\delta u_3^0\right)-\frac{\partial}{\partial x_1}\left(N_{12}\frac{\partial u_3^0}{\partial x_2}\right)\delta u_3^0\ldots$$

$$\ldots+\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1}\delta u_3^0\right)-\frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1}\right)\delta u_3^0,$$

$$N_{13}\left(\delta \psi_1+\frac{\partial \delta u_3^0}{\partial x_1}\right)=N_{13}\delta \psi_1+\frac{\partial}{\partial x_1}\left(N_{13}\delta u_3^0\right)-\frac{\partial N_{13}}{\partial x_1}\delta u_3^0,$$

$$N_{23}\left(\delta \psi_2+\frac{\partial \delta u_3^0}{\partial x_2}\right)=N_{23}\delta \psi_2+\frac{\partial}{\partial x_2}\left(N_{23}\delta u_3^0\right)-\frac{\partial N_{23}}{\partial x_2}\delta u_3^0,$$

we obtain:

$$-\delta W_i = \int_S \left\{ \frac{\partial}{\partial x_1}\left[N_{11}\delta u_1^0 + N_{12}\delta u_2^0 + M_{11}\delta\psi_1 + M_{12}\delta\psi_2 + \left(N_{11}\frac{\partial u_3^0}{\partial x_1} + N_{12}\frac{\partial u_3^0}{\partial x_2} \dots \right.\right.\right.$$

$$\left.\left. \dots + N_{13} \right)\delta u_3^0 \right] + \frac{\partial}{\partial x_2}\left[N_{12}\delta u_1^0 + N_{22}\delta u_2^0 + M_{12}\delta\psi_1 + M_{22}\delta\psi_2 + \left(N_{12}\frac{\partial u_3^0}{\partial x_1} \dots \right.\right.$$

$$\left.\left.\left. \dots + N_{22}\frac{\partial u_3^0}{\partial x_2} + N_{23} \right)\delta u_3^0 \right] \right\} dS \dots$$

$$\dots - \int_S \left\{ \left(\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} \right)\delta u_1^0 + \left(\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} \right)\delta u_2^0 + \left(\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} \right)\delta\psi_1 \dots \right.$$

$$\dots + \left(\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - N_{23} \right)\delta\psi_2 + \left[\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1} + N_{12}\frac{\partial u_3^0}{\partial x_2} + N_{13} \right) \dots \right.$$

$$\left.\left. \dots + \frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1} + N_{22}\frac{\partial u_3^0}{\partial x_2} + N_{23} \right) \right]\delta u_3^0 \right\} dS.$$

The relation:

$$\int_S g_{j,j}\, dS = \int_\Gamma g_j n_j\, ds\,,$$

allows the virtual work of the internal forces to be written as:

$$-\delta W_i = \int_\Gamma \left\{ (N_{11}n_1 + N_{12}n_2)\delta u_1^0 + (N_{12}n_1 + N_{22}n_2)\delta u_2^0 + (M_{11}n_1 + M_{12}n_2)\delta\psi_1 \dots \right.$$

$$\dots + (M_{12}n_1 + M_{22}n_2)\delta\psi_2 + \left[(N_{11}n_1 + N_{12}n_2)\frac{\partial u_3^0}{\partial x_1} + (N_{12}n_1 + N_{22}n_2)\frac{\partial u_3^0}{\partial x_2} \dots \right.$$

$$\left.\left. \dots + (N_{13}n_1 + N_{23}n_2) \right]\delta u_3^0 \right\} ds \dots$$

$$\dots - \int_S \left\{ \left(\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} \right)\delta u_1^0 + \left(\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} \right)\delta u_2^0 + \left(\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} \right)\delta\psi_1 \dots \right.$$

$$\dots + \left(\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - N_{23} \right)\delta\psi_2 + \left[\frac{\partial}{\partial x_1}\left(N_{11}\frac{\partial u_3^0}{\partial x_1} + N_{12}\frac{\partial u_3^0}{\partial x_2} + N_{13} \right) \dots \right.$$

$$\left.\left. \dots + \frac{\partial}{\partial x_2}\left(N_{12}\frac{\partial u_3^0}{\partial x_1} + N_{22}\frac{\partial u_3^0}{\partial x_2} + N_{23} \right) \right]\delta u_3^0 \right\} dS.$$

18.3.2. *Virtual work of transverse surface forces*

The virtual work developed by the surface forces $q_3(x_1, x_2|t)\vec{x}_3$ operating on the top surface of the plate is equal to:

$$\delta W_{F_t} = \int_{S_t} q_i \delta u_i dS ,$$

or:

$$\delta W_{F_t} = \int_S q_3 \delta u_3^0 dS .$$

18.3.3. *Virtual work of external lateral surface forces*

This virtual work is given by the expression:

$$\delta W_{F_c} = \int_{\Sigma_F} F_i \delta u_i dS ,$$

or:

$$\delta W_{F_c} = \int_\Gamma \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} F_i \delta u_i dx_3 \right) ds ,$$

or:

$$\delta W_{F_c} = \int_\Gamma \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[F_1 \left(\delta u_1^0 + x_3 \delta \psi_1 \right) + F_2 \left(\delta u_2^0 + x_3 \delta \psi_2 \right) + F_3 \delta u_3^0 \right] dx_3 \right\} ds .$$

With:

$$Q_i(x_1, x_2|t) = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_i(x_1, x_2, x_3|t) dx_3 \quad (i = 1,2,3),$$

$$C_i(x_1, x_2|t) = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_i(x_1, x_2, x_3|t) x_3 dx_3 \quad (i = 1,2),$$

we obtain the relation:

$$\delta W_{F_c} = \int_\Gamma \left(Q_1 \delta u_1^0 + Q_2 \delta u_2^0 + Q_3 \delta u_3^0 + C_1 \delta \psi_1 + C_2 \delta \psi_2 \right) ds .$$

18.3.4. *Virtual work of body forces*

The virtual work is obtained from:

$$\delta W_f = \int_\Omega f_i \delta u_i d\Omega ,$$

or:

$$\delta W_f = \int_S \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} f_i \delta u_i dx_3 \right) dS ,$$

or:

$$\delta W_f = \int_S \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[f_1 \left(\delta u_1^0 + x_3 \delta \psi_1 \right) + f_2 \left(\delta u_2^0 + x_3 \delta \psi_2 \right) + f_3 \delta u_3^0 \right] dx_3 \right\} dS .$$

Let:

$$p_i \left(x_1, x_2 | t \right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_i \left(x_1, x_2, x_3 | t \right) dx_3 \quad (i = 1,2,3),$$

$$m_i \left(x_1, x_2 | t \right) = \int_{-\frac{h}{2}}^{\frac{h}{2}} f_i \left(x_1, x_2, x_3 | t \right) x_3 dx_3 \quad (i = 1,2),$$

thus we obtain:

$$\delta W_f = \int_S \left(p_1 \delta u_1^0 + p_2 \delta u_2^0 + p_3 \delta u_3^0 + m_1 \delta \psi_1 + m_2 \delta \psi_2 \right) dS .$$

18.3.5. *Virtual work done by inertial forces*

The virtual work done by inertial forces is:

$$\delta W_a = \int_\Omega \rho \frac{\partial^2 u_i}{\partial t^2} \delta u_i d\Omega ,$$

or:

$$\delta W_a = \int_S \left(\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \frac{\partial^2 u_i}{\partial t^2} \delta u_i dx_3 \right) dS ,$$

$$\delta W_a = \int_S \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho \left[\left(\frac{\partial^2 u_1^0}{\partial t^2} + x_3 \frac{\partial^2 \psi_1}{\partial t^2} \right) \left(\delta u_1^0 + x_3 \delta \psi_1 \right) \ldots \right. \right.$$

$$\left. \left. \ldots + \left(\frac{\partial^2 u_2^0}{\partial t^2} + x_3 \frac{\partial^2 \psi_2}{\partial t^2} \right) \left(\delta u_2^0 + x_3 \delta \psi_2 \right) + \frac{\partial^2 u_3^0}{\partial t^2} \delta u_3^0 \right] dx_3 \right\} dS.$$

With:

$$I_0 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho dx_3, \quad I_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho x_3 dx_3, \quad I_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} \rho x_3^2 dx_3,$$

we obtain:

$$\delta W_a = \int_S \left[\left(I_0 \frac{\partial^2 u_1^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_1}{\partial t^2} \right) \delta u_1^0 + \left(I_0 \frac{\partial^2 u_2^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_2}{\partial t^2} \right) \delta u_2^0 \ldots \right.$$

$$\ldots + \left(I_1 \frac{\partial^2 u_1^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_1}{\partial t^2} \right) \delta \psi_1 + \left(I_1 \frac{\partial^2 u_2^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_2}{\partial t^2} \right) \delta \psi_2 \ldots$$

$$\left. \ldots + I_0 \frac{\partial^2 u_3^0}{\partial t^2} \delta u_3^0 \right] dS.$$

18.3.6. *Variational formulation*

The variational formulation of the Reissner-Mindlin theory of plates is given by:

$$\int_\Gamma \left\{ (N_{11}n_1 + N_{12}n_2)\delta u_1^0 + (N_{12}n_1 + N_{22}n_2)\delta u_2^0 + (M_{11}n_1 + M_{12}n_2)\delta \psi_1 \ldots \right.$$

$$\ldots + (M_{12}n_1 + M_{22}n_2)\delta \psi_2 + \left[(N_{11}n_1 + N_{12}n_2)\frac{\partial u_3^0}{\partial x_1} + (N_{12}n_1 + N_{22}n_2)\frac{\partial u_3^0}{\partial x_2} \ldots \right.$$

$$\left. \left. \ldots + (N_{13}n_1 + N_{23}n_2) \right] \delta u_3^0 \right\} ds \ldots$$

$$\ldots - \int_S \left\{ \left(\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} \right) \delta u_1^0 + \left(\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} \right) \delta u_2^0 + \left(\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} \right) \delta \psi_1 \ldots \right.$$

$$\ldots + \left(\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - N_{23} \right) \delta \psi_2 + \left[\frac{\partial}{\partial x_1} \left(N_{11} \frac{\partial u_3^0}{\partial x_1} + N_{12} \frac{\partial u_3^0}{\partial x_2} + N_{13} \right) \ldots \right.$$

$$\left. \left. \ldots + \frac{\partial}{\partial x_2} \left(N_{12} \frac{\partial u_3^0}{\partial x_1} + N_{22} \frac{\partial u_3^0}{\partial x_2} + N_{23} \right) \right] \delta u_3^0 \right\} dS \ldots$$

$$\ldots - \int_S q_3 \delta u_3^0 dS - \int_\Gamma \left(Q_1 \delta u_1^0 + Q_2 \delta u_2^0 + Q_3 \delta u_3^0 + C_1 \delta \psi_1 + C_2 \delta \psi_2 \right) ds \ldots$$

$$\ldots - \int_S \left(p_1 \delta u_1^0 + p_2 \delta u_2^0 + p_3 \delta u_3^0 + m_1 \delta \psi_1 + m_2 \delta \psi_2 \right) dS \ldots$$

$$\ldots + \int_S \left[\left(I_0 \frac{\partial^2 u_1^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_1}{\partial t^2} \right) \delta u_1^0 + \left(I_0 \frac{\partial^2 u_2^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_2}{\partial t^2} \right) \delta u_2^0 \ldots \right.$$

$$\left. \ldots + \left(I_1 \frac{\partial^2 u_1^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_1}{\partial t^2} \right) \delta \psi_1 + \left(I_1 \frac{\partial^2 u_2^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_2}{\partial t^2} \right) \delta \psi_2 + I_0 \frac{\partial^2 u_3^0}{\partial t^2} \delta u_3^0 \right] dS = 0,$$

where $u_1^0, u_2^0, u_3^0, \psi_1$ and ψ_2 satisfy the relation $\forall \delta u_1^0, \forall \delta u_2^0, \forall \delta u_3^0, \forall \delta\psi_1$ and $\forall \delta\psi_2$.

Therefore:

$$-\int_S \left\{ \left(\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 - I_0 \frac{\partial^2 u_1^0}{\partial t^2} - I_1 \frac{\partial^2 \psi_1}{\partial t^2} \right) \delta u_1^0 \ldots \right.$$

$$\ldots + \left(\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 - I_0 \frac{\partial^2 u_2^0}{\partial t^2} - I_1 \frac{\partial^2 \psi_2}{\partial t^2} \right) \delta u_2^0 \ldots$$

$$\ldots + \left(\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} + m_1 - I_1 \frac{\partial^2 u_1^0}{\partial t^2} - I_2 \frac{\partial^2 \psi_1}{\partial t^2} \right) \delta\psi_1 \ldots$$

$$\ldots + \left(\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - N_{23} + m_2 - I_1 \frac{\partial^2 u_2^0}{\partial t^2} - I_2 \frac{\partial^2 \psi_2}{\partial t^2} \right) \delta\psi_2 \ldots$$

$$\ldots + \left[\frac{\partial}{\partial x_1} \left(N_{11} \frac{\partial u_3^0}{\partial x_1} + N_{12} \frac{\partial u_3^0}{\partial x_2} + N_{13} \right) + \frac{\partial}{\partial x_2} \left(N_{12} \frac{\partial u_3^0}{\partial x_1} + N_{22} \frac{\partial u_3^0}{\partial x_2} + N_{23} \right) \ldots \right.$$

$$\left. \left. \ldots + q_3 + p_3 - I_0 \frac{\partial^2 u_3^0}{\partial t^2} \right] \delta u_3^0 \right\} dS \ldots$$

$$\ldots + \int_\Gamma \left\{ (N_{11} n_1 + N_{12} n_2 - Q_1) \delta u_1^0 + (N_{12} n_1 + N_{22} n_2 - Q_2) \delta u_2^0 \ldots \right.$$

$$\ldots + (M_{11} n_1 + M_{12} n_2 - C_1) \delta\psi_1 + (M_{12} n_1 + M_{22} n_2 - C_2) \delta\psi_2 \ldots$$

$$\ldots + \left[(N_{11} n_1 + N_{12} n_2) \frac{\partial u_3^0}{\partial x_1} + (N_{12} n_1 + N_{22} n_2) \frac{\partial u_3^0}{\partial x_2} \ldots \right.$$

$$\left. \left. \ldots + N_{13} n_1 + N_{23} n_2 - Q_3 \right] \delta u_3^0 \right\} ds = 0,$$

where $u_1^0, u_2^0, u_3^0, \psi_1$ and ψ_2 satisfy the relation $\forall \delta u_1^0, \forall \delta u_2^0, \forall \delta u_3^0, \forall \delta\psi_1$ and $\forall \delta\psi_2$.

18.4. Global equations, boundary edge conditions

18.4.1. *Global plate equations*

Taking the variational formulation previously presented:

$\delta u_i^0 = 0$ and $\delta\psi_i = 0$ on Γ,

we obtain:

$$\int_S \left\{ \left(\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 - I_0 \frac{\partial^2 u_1^0}{\partial t^2} - I_1 \frac{\partial^2 \psi_1}{\partial t^2} \right) \delta u_1^0 \ldots \right.$$

$$\ldots + \left(\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 - I_0 \frac{\partial^2 u_2^0}{\partial t^2} - I_1 \frac{\partial^2 \psi_2}{\partial t^2} \right) \delta u_2^0 \ldots$$

$$\ldots + \left(\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} + m_1 - I_1 \frac{\partial^2 u_1^0}{\partial t^2} - I_2 \frac{\partial^2 \psi_1}{\partial t^2} \right) \delta \psi_1 \ldots$$

$$\ldots + \left(\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - N_{23} + m_2 - I_1 \frac{\partial^2 u_2^0}{\partial t^2} - I_2 \frac{\partial^2 \psi_2}{\partial t^2} \right) \delta \psi_2 \ldots$$

$$\ldots + \left[\frac{\partial}{\partial x_1} \left(N_{11} \frac{\partial u_3^0}{\partial x_1} + N_{12} \frac{\partial u_3^0}{\partial x_2} + N_{13} \right) + \frac{\partial}{\partial x_2} \left(N_{12} \frac{\partial u_3^0}{\partial x_1} + N_{22} \frac{\partial u_3^0}{\partial x_2} + N_{23} \right) \ldots \right.$$

$$\left. \left. \ldots + q_3 + p_3 - I_0 \frac{\partial^2 u_3^0}{\partial t^2} \right] \delta u_3^0 \right\} dS = 0,$$

where u_i^0 and ψ_i satisfy the relation $\forall \delta u_i^0$ and $\forall \delta \psi_i \,/\, \delta u_i^0 = 0$ and $\delta \psi_i = 0$ on Γ.

Thus, the global equations of plates can be written:

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 = I_0 \frac{\partial^2 u_1^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 = I_0 \frac{\partial^2 u_2^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_2}{\partial t^2},$$

$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} + m_1 = I_1 \frac{\partial^2 u_1^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_1}{\partial t^2},$$

$$\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - N_{23} + m_2 = I_1 \frac{\partial^2 u_2^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_2}{\partial t^2},$$

$$\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + \frac{\partial}{\partial x_1} \left(N_{11} \frac{\partial u_3^0}{\partial x_1} + N_{12} \frac{\partial u_3^0}{\partial x_2} \right) \ldots$$

$$\ldots + \frac{\partial}{\partial x_2} \left(N_{12} \frac{\partial u_3^0}{\partial x_1} + N_{22} \frac{\partial u_3^0}{\partial x_2} \right) + q_3 + p_3 = I_0 \frac{\partial^2 u_3^0}{\partial t^2}.$$

In the following part of this appendix, we take $m_1 = m_2 = 0$.

18.4.2. *Boundary edge conditions*

Substituting the five global equations of plate analysis in the variational formulation, we obtain:

$$\int_\Gamma \Big\{ (N_{11}n_1 + N_{12}n_2 - Q_1)\delta u_1^0 + (N_{12}n_1 + N_{22}n_2 - Q_2)\delta u_2^0 \ldots$$

$$\ldots + (M_{11}n_1 + M_{12}n_2 - C_1)\delta\psi_1 + (M_{12}n_1 + M_{22}n_2 - C_2)\delta\psi_2 \ldots$$

$$\ldots + \Bigg[(N_{11}n_1 + N_{12}n_2)\frac{\partial u_3^0}{\partial x_1} + (N_{12}n_1 + N_{22}n_2)\frac{\partial u_3^0}{\partial x_2} \ldots$$

$$\ldots + N_{13}n_1 + N_{23}n_2 - Q_3 \Bigg] \delta u_3^0 \Big\} ds = 0,$$

From which we get the boundary edge conditions:

$$Q_1 = N_{11}n_1 + N_{12}n_2,$$

$$Q_2 = N_{12}n_1 + N_{22}n_2,$$

$$Q_3 = N_{13}n_1 + N_{23}n_2 + (N_{11}n_1 + N_{12}n_2)\frac{\partial u_3^0}{\partial x_1} + (N_{12}n_1 + N_{22}n_2)\frac{\partial u_3^0}{\partial x_2},$$

$$C_1 = M_{11}n_1 + M_{12}n_2,$$

$$C_2 = M_{12}n_1 + M_{22}n_2.$$

Introducing the local axes $(b) = (\mathbf{n}, \boldsymbol{\tau}, \mathbf{x}_3)$, where $\mathbf{n}$ and $\boldsymbol{\tau}$ are respectively orthogonal and tangential to Γ, enables us to write:

$$\delta u_1^0 = \delta u_n^0 n_1 - \delta u_s^0 n_2,$$

$$\delta u_2^0 = \delta u_n^0 n_2 + \delta u_s^0 n_1,$$

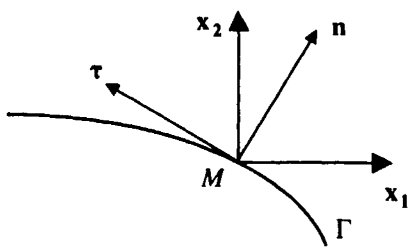

Figure 18.1. *Local axes attached to an edge*

In addition:

$$\frac{\partial \delta u_3^0}{\partial x_1} = \frac{\partial \delta u_3^0}{\partial n} n_1 - \frac{\partial \delta u_3^0}{\partial s} n_2,$$

$$\frac{\partial \delta u_3^0}{\partial x_2} = \frac{\partial \delta u_3^0}{\partial n} n_2 + \frac{\partial \delta u_3^0}{\partial s} n_1,$$

and:

$$\delta\psi_1 = \delta\psi_n n_1 - \delta\psi_s n_2,$$
$$\delta\psi_1 = \delta\psi_n n_2 + \delta\psi_s n_1.$$

The following integral in the virtual work of the internal forces:

$$-\delta W_{i_\Gamma} = \int_\Gamma \left\{ (N_{11}n_1 + N_{12}n_2)\delta u_1^0 + (N_{12}n_1 + N_{22}n_2)\delta u_2^0 + (M_{11}n_1 + M_{12}n_2)\delta\psi_1 \ldots\right.$$
$$\ldots + (M_{12}n_1 + M_{22}n_2)\delta\psi_2 + \left[(N_{11}n_1 + N_{12}n_2)\frac{\partial u_3^0}{\partial x_1} + (N_{12}n_1 + N_{22}n_2)\frac{\partial u_3^0}{\partial x_2} \ldots\right.$$
$$\left.\left. \ldots + N_{13}n_1 + N_{23}n_2 \right] \delta u_3^0 \right\} ds,$$

becomes:

$$-\delta W_{i_\Gamma} = \int_\Gamma \left(\left[N_{11}n_1^2 + N_{22}n_2^2 + 2N_{12}n_1n_2 \right] \delta u_n^0 \quad + \left[-(N_{11} - N_{22})n_1n_2 \ldots\right.\right.$$
$$\left.\ldots + N_{12}\left(n_1^2 - n_2^2\right)\right] \delta u_s^0 + \left[M_{11}n_1^2 + M_{22}n_2^2 + 2M_{12}n_1n_2 \right] \delta\psi_n + \left[-(M_{11} - M_{22})n_1n_2 \ldots\right.$$
$$\left.\ldots + M_{12}\left(n_1^2 - n_2^2\right)\right] \delta\psi_s + \left\{ \left[N_{11}n_1^2 + N_{22}n_2^2 + 2N_{12}n_1n_2 \right] \frac{\partial u_3^0}{\partial n} + \left[-(N_{11} - N_{22})n_1n_2 \ldots\right.\right.$$
$$\left.\left.\left. \ldots + N_{12}\left(n_1^2 - n_2^2\right)\right] \frac{\partial u_3^0}{\partial s} + N_{13}n_1 + N_{23}n_2 \right\} \quad \delta u_3^0 \right) ds.$$

The introduction of:

$$N_n = N_{11}n_1^2 + N_{22}n_2^2 + 2N_{12}n_1n_2,$$
$$N_s = -(N_{11} - N_{22}) + N_{12}\left(n_1^2 - n_2^2\right),$$
$$M_n = M_{11}n_1^2 + M_{22}n_2^2 + 2M_{12}n_1n_2,$$
$$M_s = -(M_{11} - M_{22})n_1n_2 + M_{12}\left(n_1^2 - n_2^2\right),$$

allows us to write:

$$-\delta W_i = \int_\Gamma \left[N_n \delta u_n^0 + N_s \delta u_s^0 + M_n \delta\psi_n + M_s \delta\psi_s \ldots \right.$$

$$\left. \ldots + \left(N_n \frac{\partial u_3^0}{\partial n} + N_s \frac{\partial u_3^0}{\partial s} + N_{13} n_1 + N_{23} n_2 \right) \delta u_3^0 \right] ds.$$

The virtual work developed by the surface forces, which are applied to the plate edge, is:

$$\delta W_{F_c} = \int_\Gamma \left[\int_{-\frac{h}{2}}^{\frac{h}{2}} \left(F_n \delta u_n + F_s \delta u_s + F_3 \delta u_3^0 \right) \right] ds,$$

with:

$$\delta u_n = \delta u_n^0 + x_3 \delta\psi_n ,$$

$$\delta u_s = \delta u_s^0 + x_3 \delta\psi_s ,$$

$$\delta u_3 = \delta u_3^0 ,$$

from which:

$$\delta W_{F_c} = \int_\Gamma \left\{ \int_{-\frac{h}{2}}^{\frac{h}{2}} \left[F_n \left(\delta u_n^0 + x_3 \delta\psi_n \right) + F_s \left(\delta u_s^0 + x_3 \delta\psi_s \right) \ldots \right. \right.$$

$$\left. \left. \ldots + F_3 \delta u_3^0 \right] d\, x_3 \right\} ds.$$

Let:

$$Q_n = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_n dx_3, \qquad Q_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_s dx_3, \qquad Q_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_3 dx_3,$$

$$C_n = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_n x_3 dx_3, \quad C_s = \int_{-\frac{h}{2}}^{\frac{h}{2}} F_s x_3 dx_3,$$

thus we obtain:

$$\delta W_F = \int_\Gamma \left(Q_n \delta u_n^0 + Q_s \delta u_s^0 + Q_3 \delta u_3^0 + C_n \delta\psi_n + C_s \delta\psi_s \right) ds .$$

The boundary conditions are written as:

$$Q_n = N_n,$$
$$Q_s = N_s,$$
$$Q_3 = N_3 + N_n \frac{\partial u_3^0}{\partial n} + N_s \frac{\partial u_3^0}{\partial s},$$
$$C_n = M_n,$$
$$C_s = M_s,$$

with:

$$N_3 = N_{13}n_1 + N_{23}n_2 .$$

18.5. Global static, vibration and buckling equations

The global plate theory equations:

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 = I_0 \frac{\partial^2 u_1^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_1}{\partial t^2},$$
$$\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 = I_0 \frac{\partial^2 u_2^0}{\partial t^2} + I_1 \frac{\partial^2 \psi_2}{\partial t^2},$$
$$\frac{\partial M_{11}}{\partial x_1} + \frac{\partial M_{12}}{\partial x_2} - N_{13} = I_1 \frac{\partial^2 u_1^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_1}{\partial t^2},$$
$$\frac{\partial M_{12}}{\partial x_1} + \frac{\partial M_{22}}{\partial x_2} - N_{23} = I_1 \frac{\partial^2 u_2^0}{\partial t^2} + I_2 \frac{\partial^2 \psi_2}{\partial t^2},$$
$$\frac{\partial N_{13}}{\partial x_1} + \frac{\partial N_{23}}{\partial x_2} + \frac{\partial}{\partial x_1}\left(N_{11} \frac{\partial u_3^0}{\partial x_1} + N_{12} \frac{\partial u_3^0}{\partial x_2} \right) \ldots$$
$$\ldots + \frac{\partial}{\partial x_2}\left(N_{12} \frac{\partial u_3^0}{\partial x_1} + N_{22} \frac{\partial u_3^0}{\partial x_2} \right) + q_3 + p_3 = I_0 \frac{\partial^2 u_3^0}{\partial t^2},$$

are written as:

– static:

$$\frac{\partial N_{11}}{\partial x_1} + \frac{\partial N_{12}}{\partial x_2} + p_1 = 0,$$
$$\frac{\partial N_{12}}{\partial x_1} + \frac{\partial N_{22}}{\partial x_2} + p_2 = 0,$$

$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}-N_{13}=0,$$
$$\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}-N_{23}=0,$$
$$\frac{\partial N_{13}}{\partial x_1}+\frac{\partial N_{23}}{\partial x_2}+q_3+p_3=0,$$

– vibration:

$$\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}=I_0\frac{\partial^2 u_1^0}{\partial t^2}+I_1\frac{\partial^2 \psi_1}{\partial t^2},$$
$$\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}=I_0\frac{\partial^2 u_2^0}{\partial t^2}+I_1\frac{\partial^2 \psi_2}{\partial t^2},$$
$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}-N_{13}=I_1\frac{\partial^2 u_1^0}{\partial t^2}+I_2\frac{\partial^2 \psi_1}{\partial t^2},$$
$$\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}-N_{23}=I_1\frac{\partial^2 u_2^0}{\partial t^2}+I_2\frac{\partial^2 \psi_2}{\partial t^2},$$
$$\frac{\partial N_{13}}{\partial x_1}+\frac{\partial N_{23}}{\partial x_2}=I_0\frac{\partial^2 u_3^0}{\partial t^2},$$

– buckling:

$$\frac{\partial N_{11}}{\partial x_1}+\frac{\partial N_{12}}{\partial x_2}=0,$$
$$\frac{\partial N_{12}}{\partial x_1}+\frac{\partial N_{22}}{\partial x_2}=0,$$
$$\frac{\partial M_{11}}{\partial x_1}+\frac{\partial M_{12}}{\partial x_2}-N_{13}=0,$$
$$\frac{\partial M_{12}}{\partial x_1}+\frac{\partial M_{22}}{\partial x_2}-N_{23}=0,$$
$$\frac{\partial N_{13}}{\partial x_1}+\frac{\partial N_{23}}{\partial x_2}+N_{11}\frac{\partial^2 u_3^0}{\partial x_1^2}+2N_{12}\frac{\partial^2 u_3^0}{\partial x_1\partial x_2}+N_{22}\frac{\partial^2 u_3^0}{\partial x_2^2}=0.$$

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Index