

Fundamental Graph Algorithms

Part Three

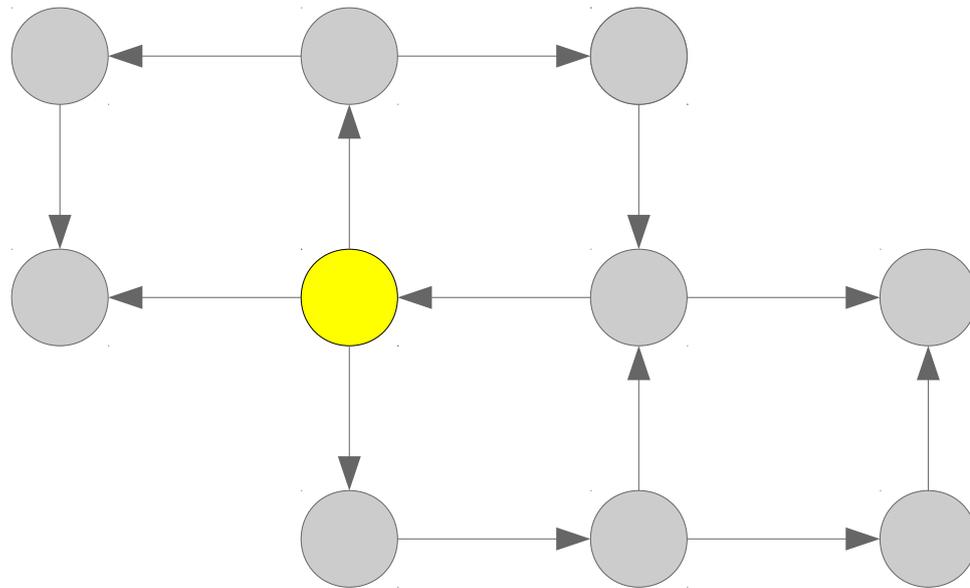
Outline for Today

- **Topological Sorting, Part II**
 - How can we quickly compute a topological ordering on a DAG?
- **Connected Components**
 - How do we find the “pieces” of an undirected graph?
- **Strongly Connected Components**
 - What are the “pieces” of a *directed* graph?
- **Kosaraju's Algorithm, Part I**
 - How do we find strongly connected components?

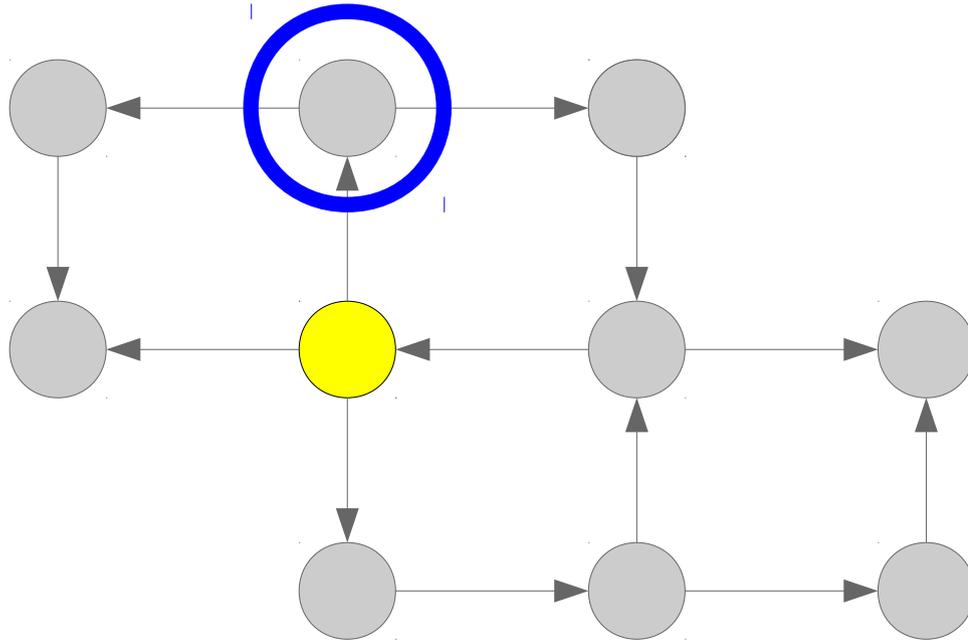
A Correction from Last Time

Theorem: When DFS(s) returns, the set of nodes colored green by that call is precisely the set of nodes reachable from s along a path consisting purely of gray nodes (we'll call this a *gray path*).

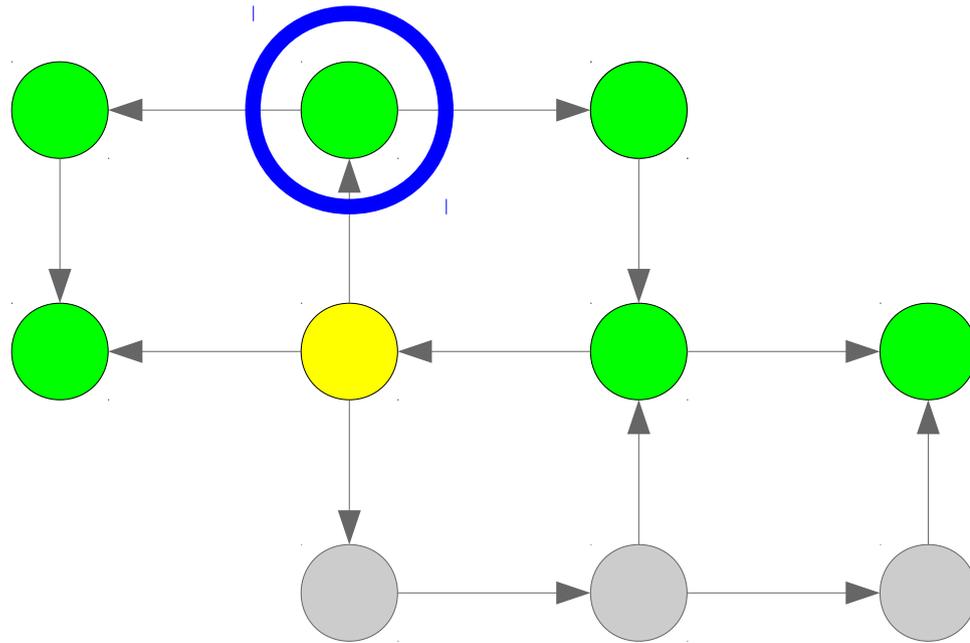
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***Lemma:* Suppose that when $\text{DFS}(s)$ is called, v is gray and there is a path from s to v consisting solely of gray nodes. Then v is green when $\text{DFS}(s)$ returns.**

Proof: $\text{DFS}(s)$ returns only after all recursive DFS calls have returned. $\text{DFS}(v)$ always colors v green, so if v was gray when $\text{DFS}(s)$ was invoked, the only way v wouldn't be green when $\text{DFS}(s)$ ends is if $\text{DFS}(v)$ is never called.

Suppose there is a node v where a gray s - v path P exists when $\text{DFS}(s)$ is called where $\text{DFS}(v)$ is never invoked. Let x be the first node on path P where $\text{DFS}(x)$ wasn't invoked. x can't be s , since $\text{DFS}(s)$ is explicitly invoked, so x is preceded by some node y in P . Consider these cases:

Case 1: $\text{DFS}(y)$ is never invoked. But then x is not the first node on path P where DFS was not called.

Case 2: $\text{DFS}(y)$ was invoked. At this time, x would have been gray, so $\text{DFS}(y)$ would have called $\text{DFS}(x)$.

In both cases, we reach a contradiction, so our assumption must have been wrong. Thus all gray nodes reachable by a gray path must be green when $\text{DFS}(s)$ returns. ■

***Lemma:* Suppose that v was gray when $\text{DFS}(s)$ was called and is green when $\text{DFS}(s)$ ends. Then there is a path from s to v consisting purely of gray nodes.**

Proof: Since v is green when $\text{DFS}(s)$ returns and was gray when $\text{DFS}(s)$ was called, there must have been a path $P = s, x_1, x_2, \dots, x_n, v$ from s to v formed by the recursive calls to DFS . This means that x_1, x_2, \dots, x_n must have been gray when $\text{DFS}(s)$ was called, since otherwise these calls would not have been made. Consequently, P is a path consisting purely of gray nodes from s to v , as required. ■

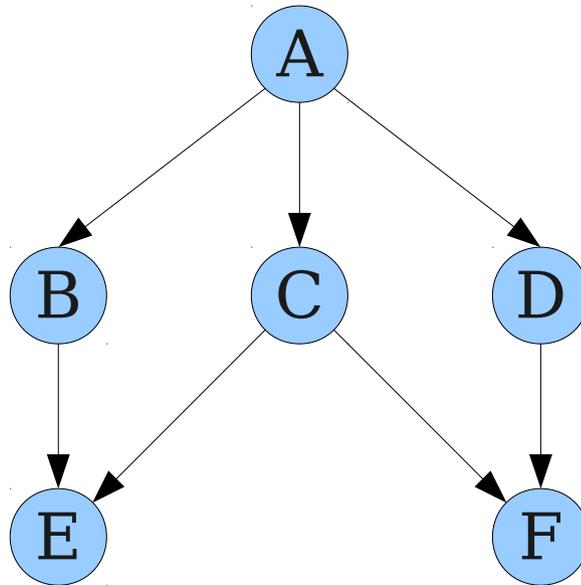
Back to Topological Sorting...

Topological Sorting

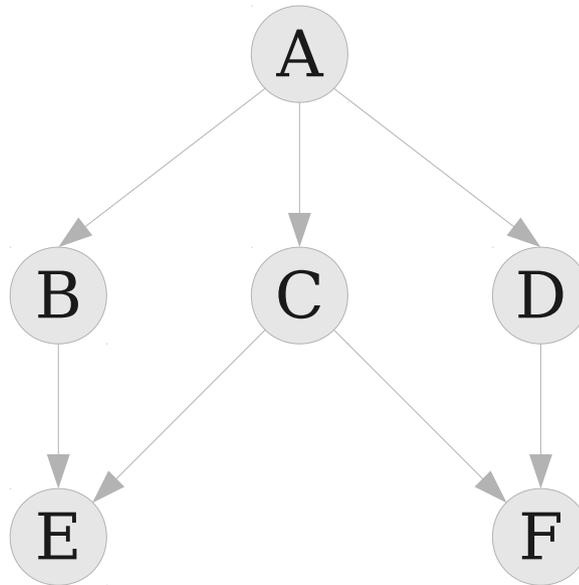
- **Goal:** Order the nodes of a DAG G such that if (u, v) is an edge in G , then u appears before v .
- One simple algorithm is as follows:
repeatedly find a node with no incoming edges, remove it, and add it to the result.
- As mentioned in Kleinberg and Tardos, can be made to run in $\Theta(m + n)$ time.

A Completely Different Algorithm

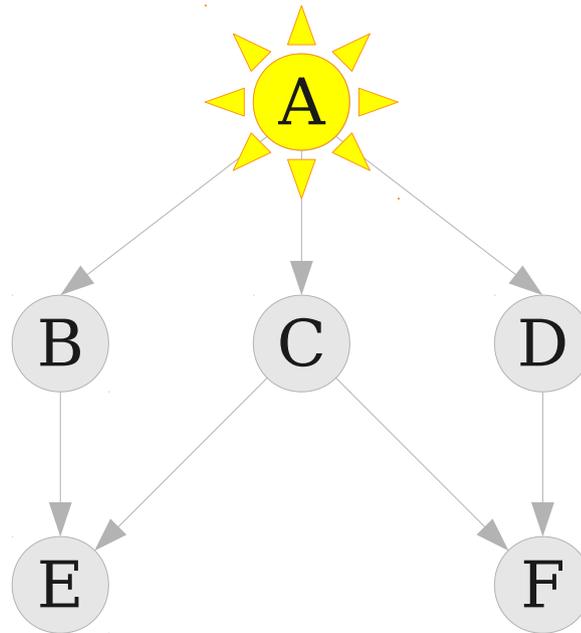
DFS on a DAG



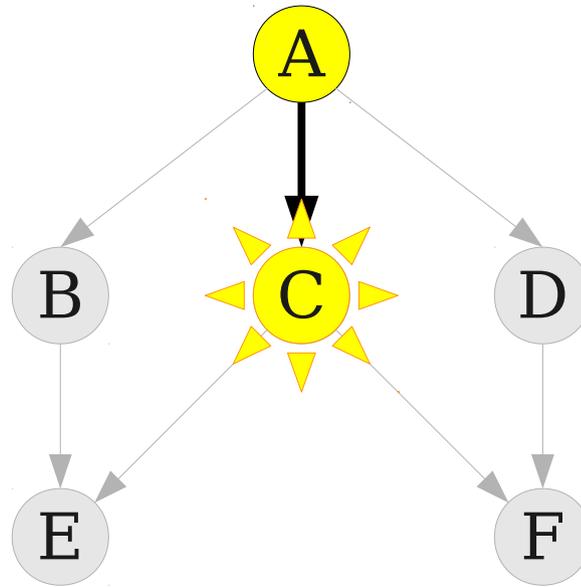
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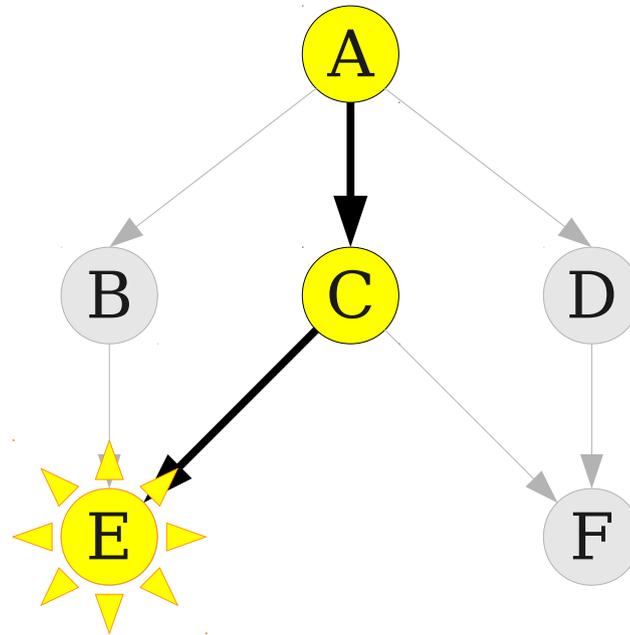
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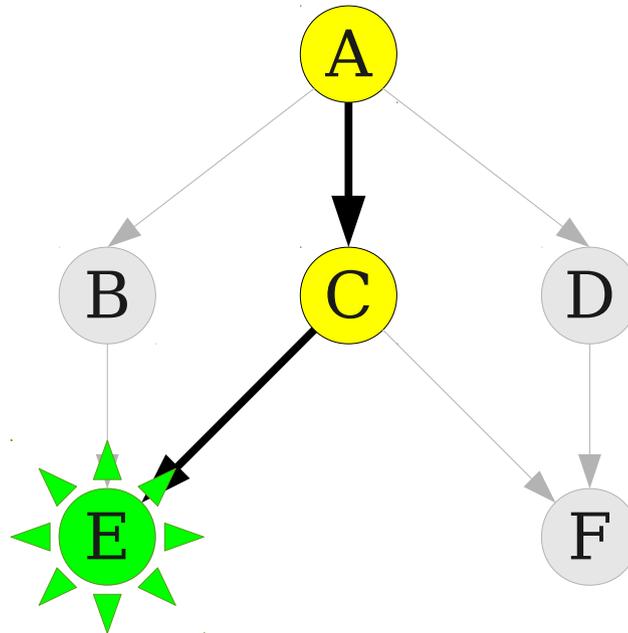
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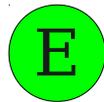
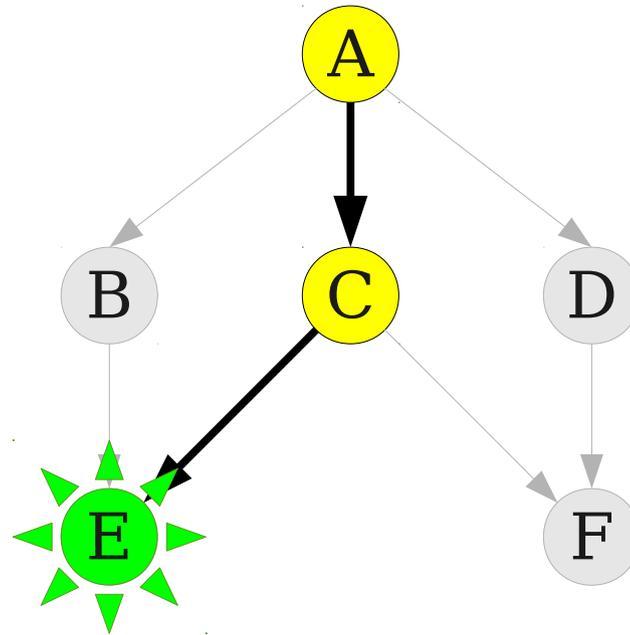
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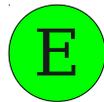
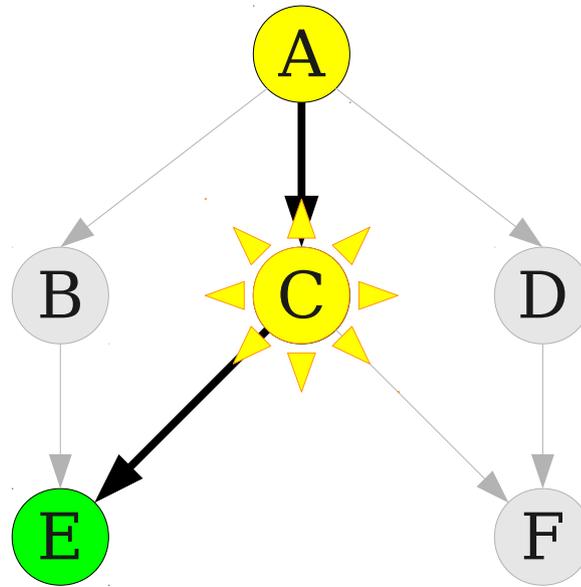
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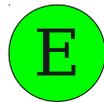
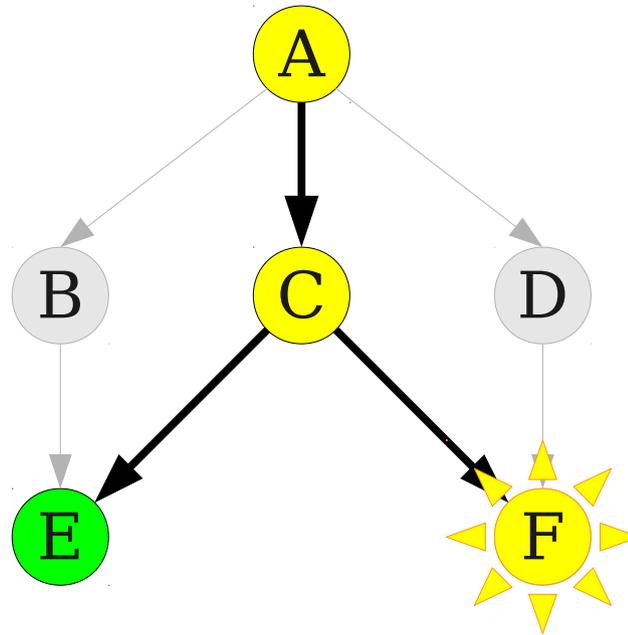
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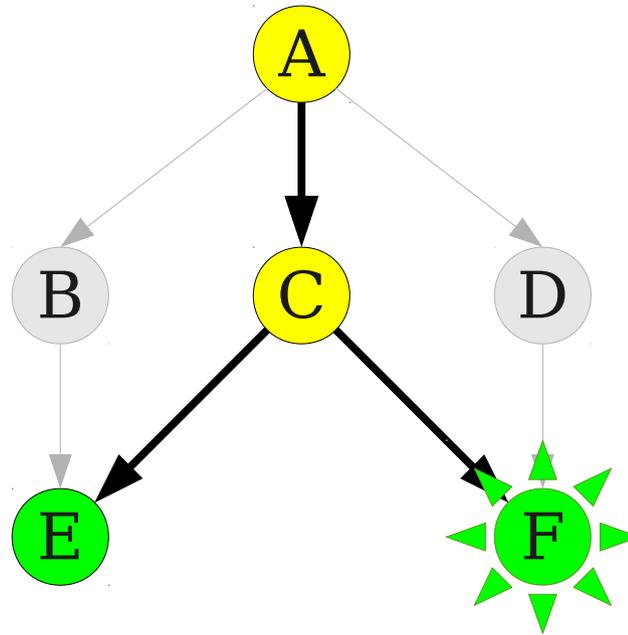
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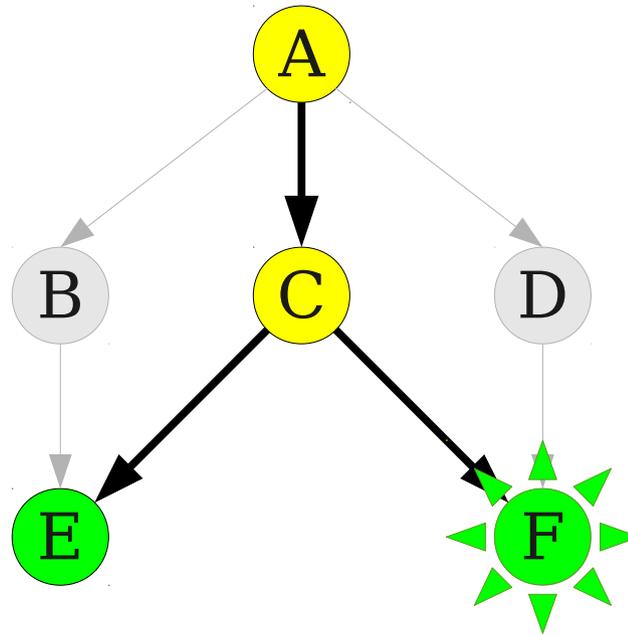
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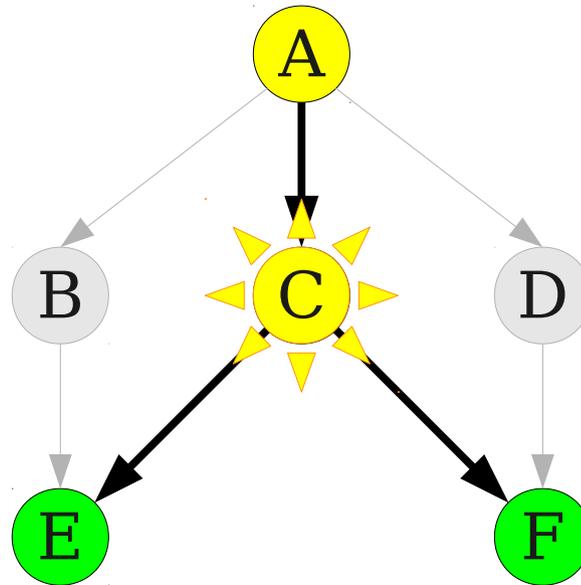
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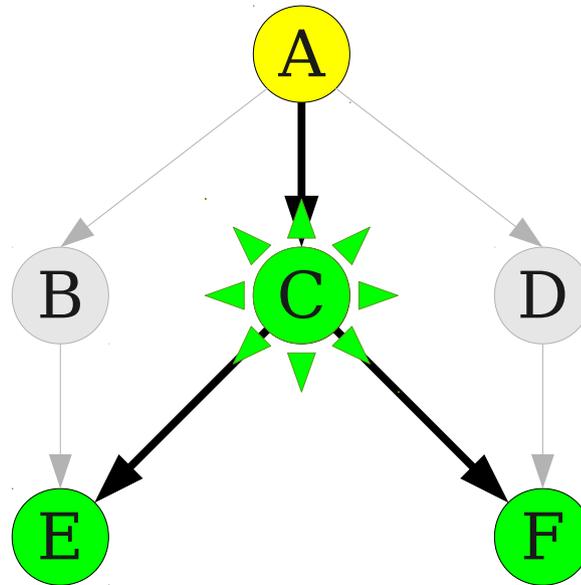
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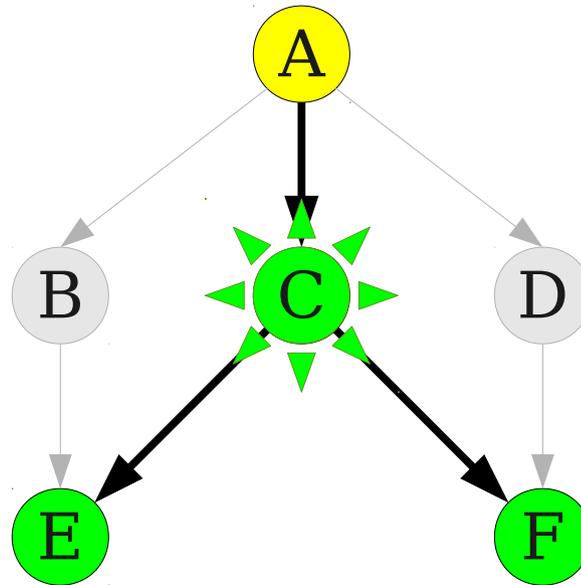
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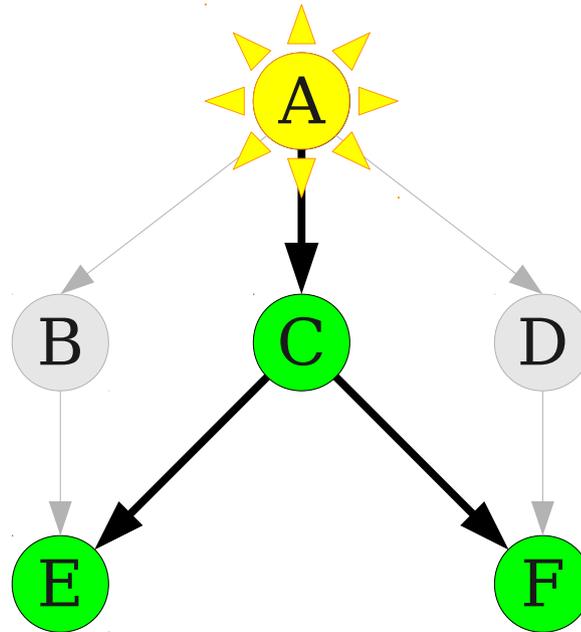
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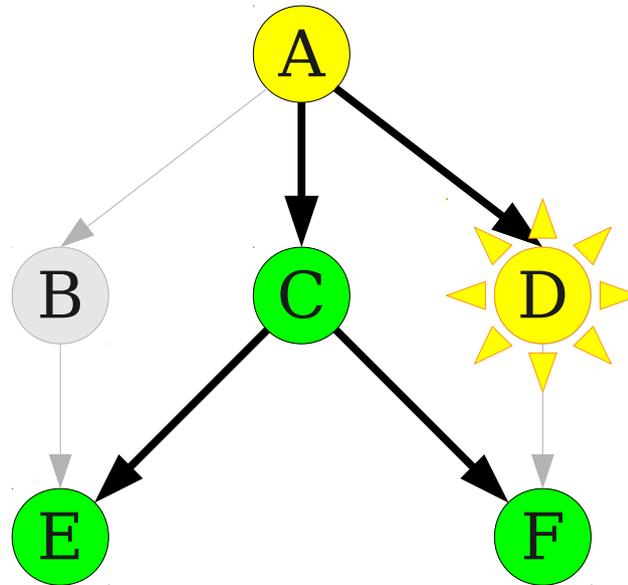
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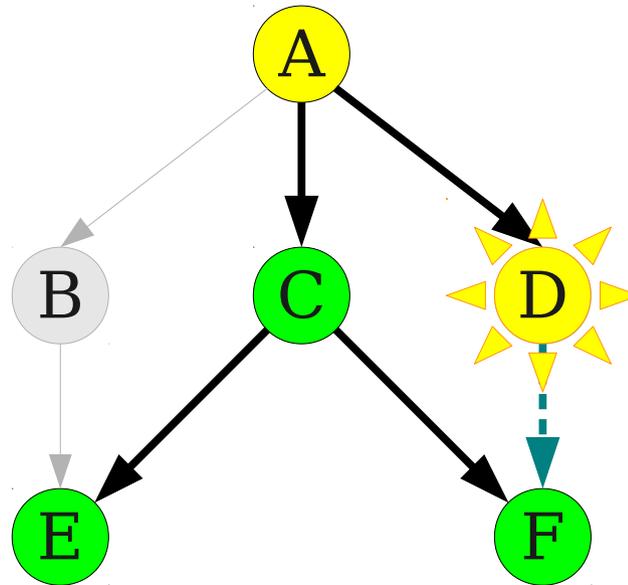
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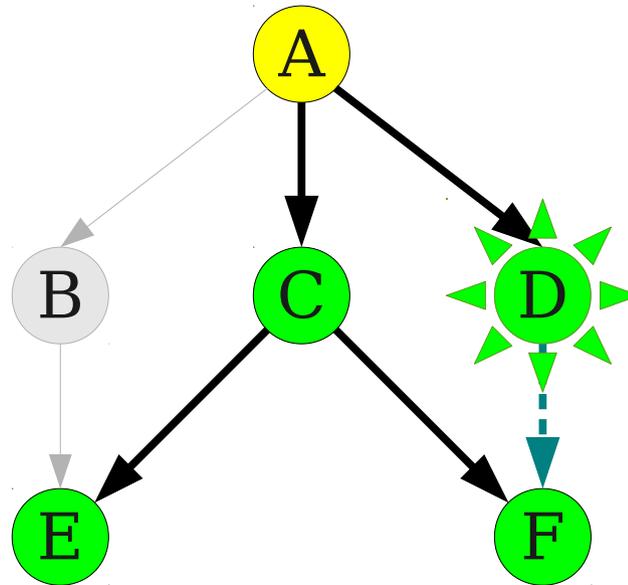
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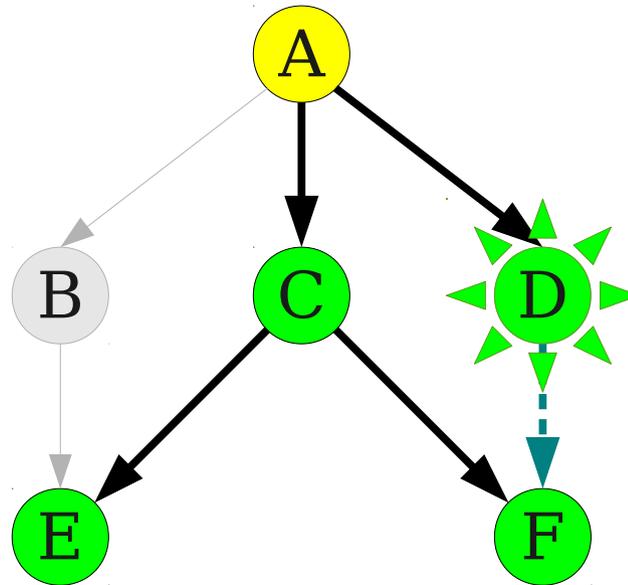
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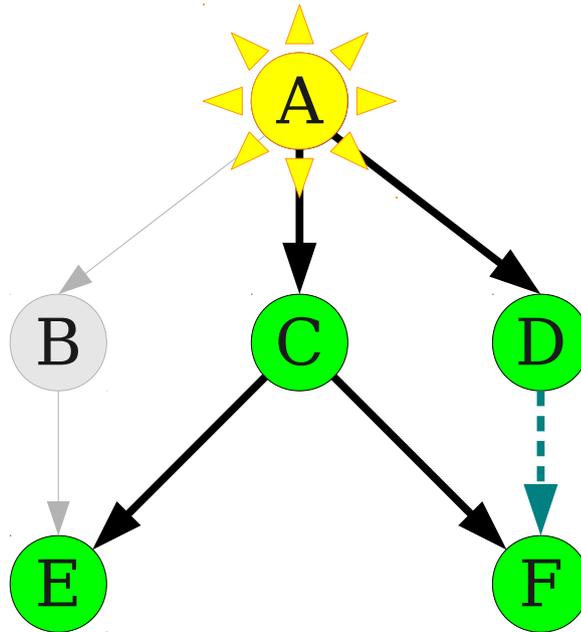
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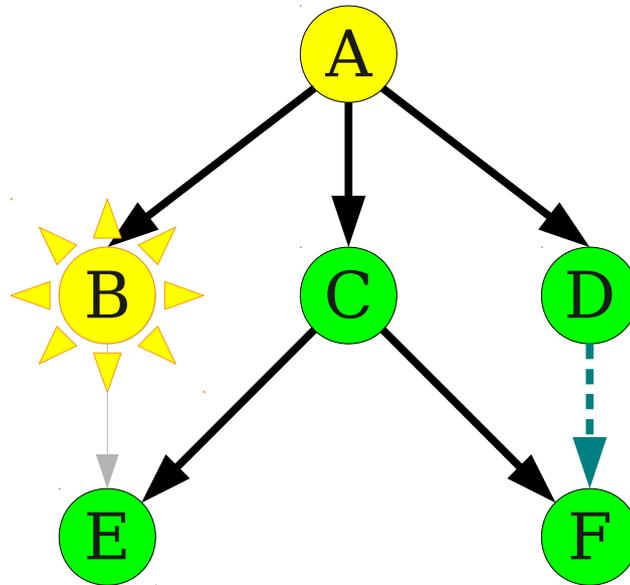
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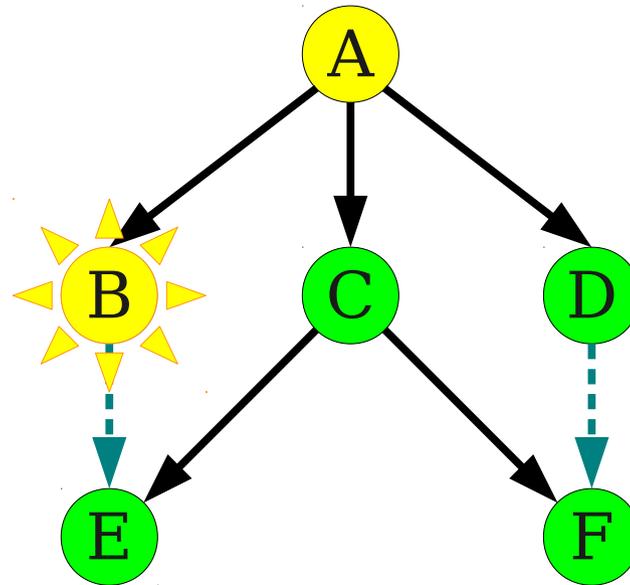
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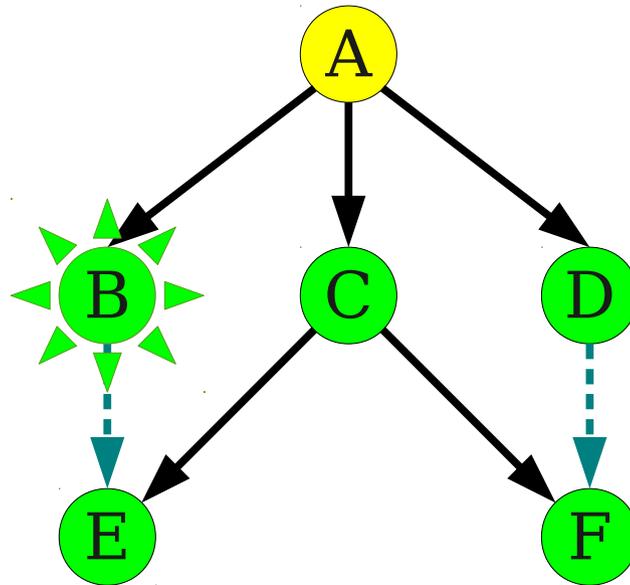
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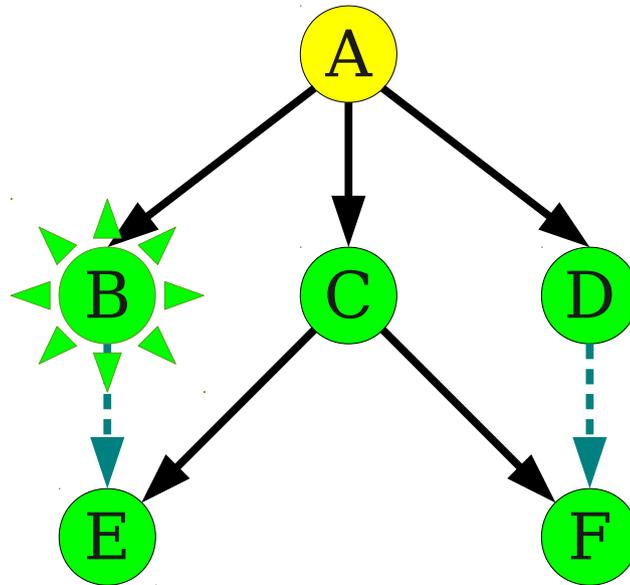
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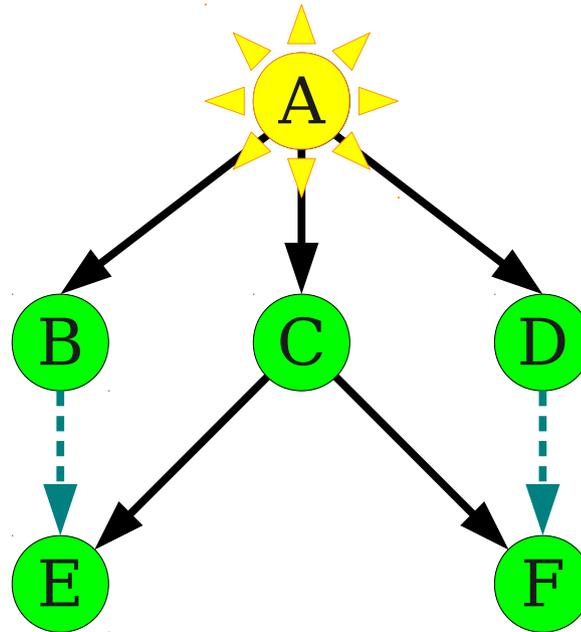
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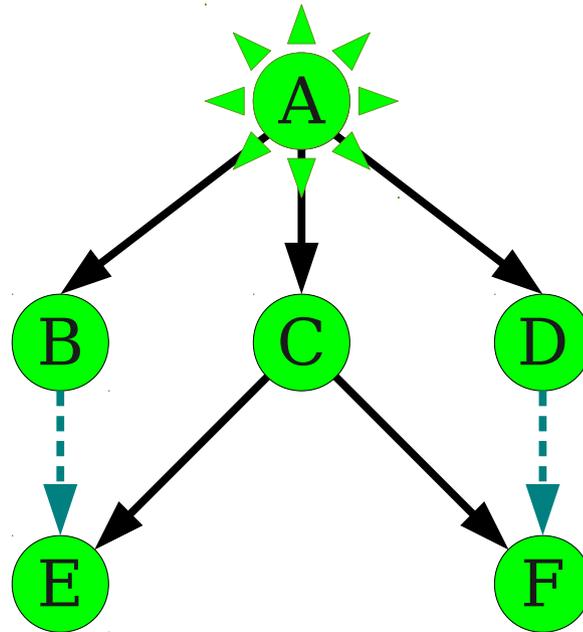
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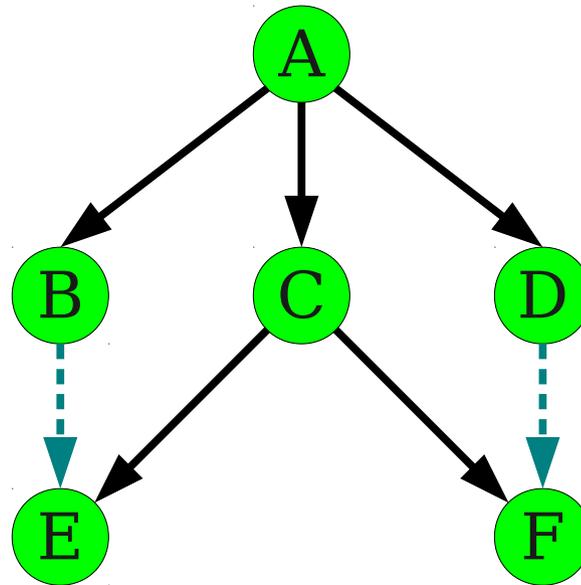
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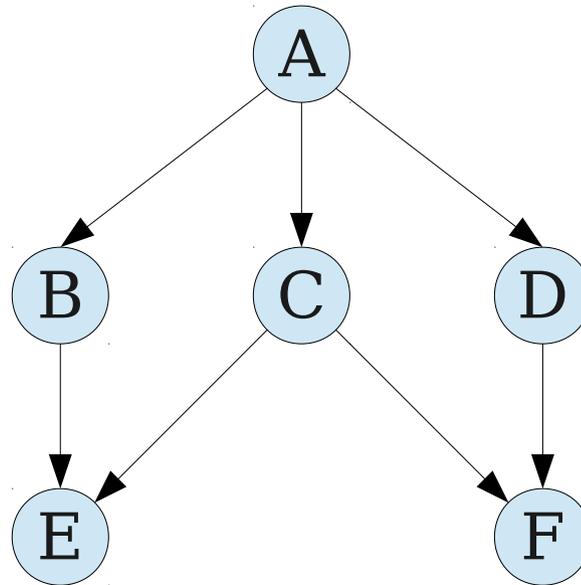
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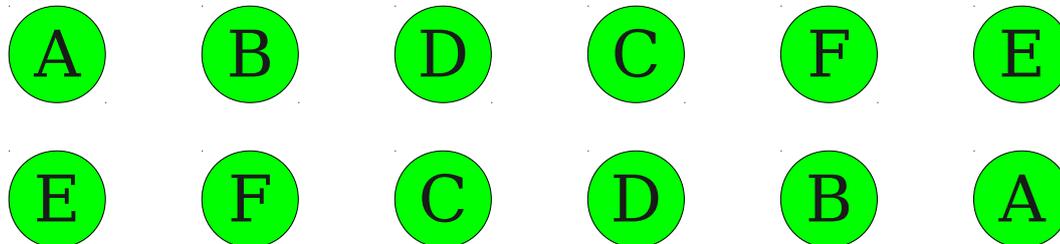
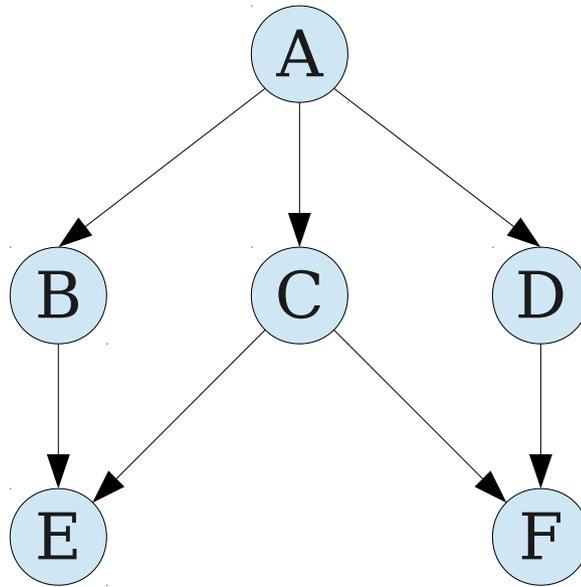
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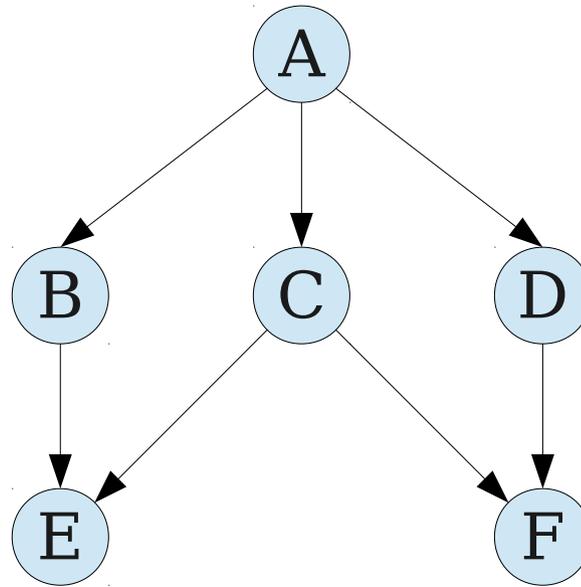
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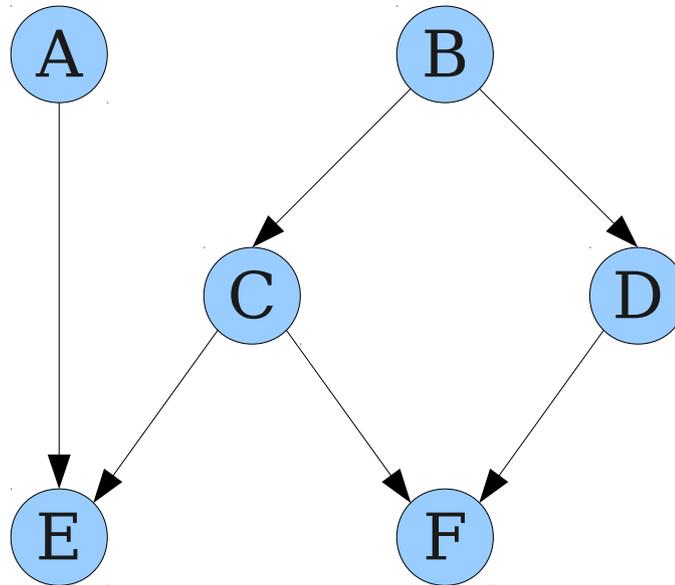
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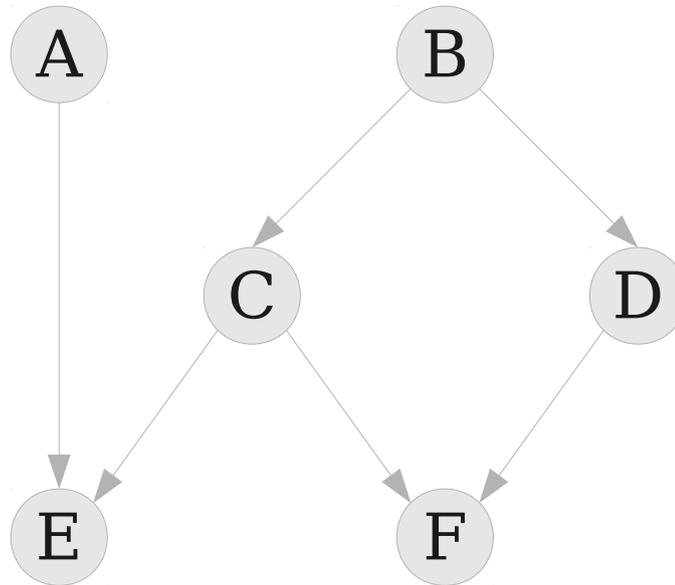
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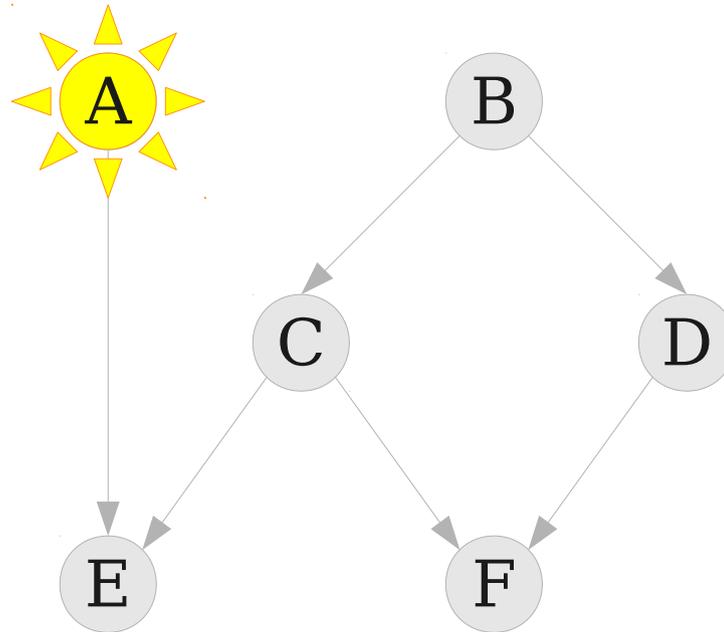
DFS on a DAG, Take II



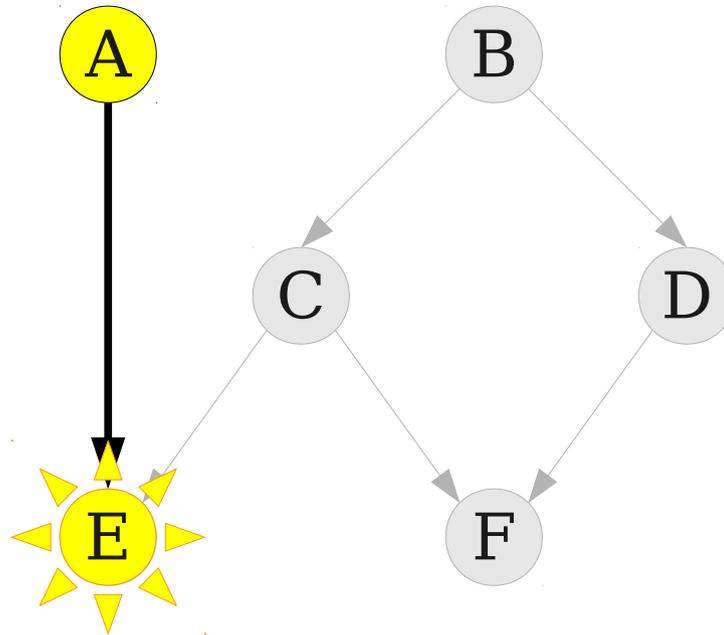
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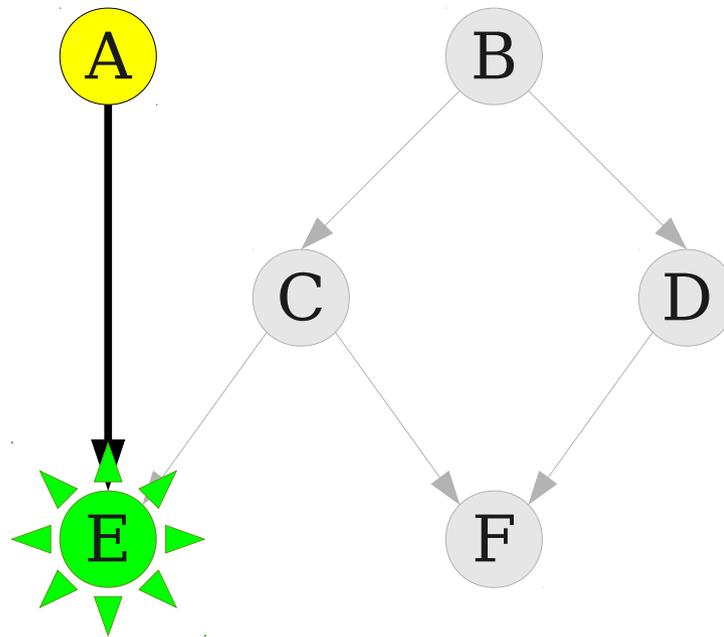
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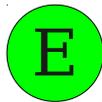
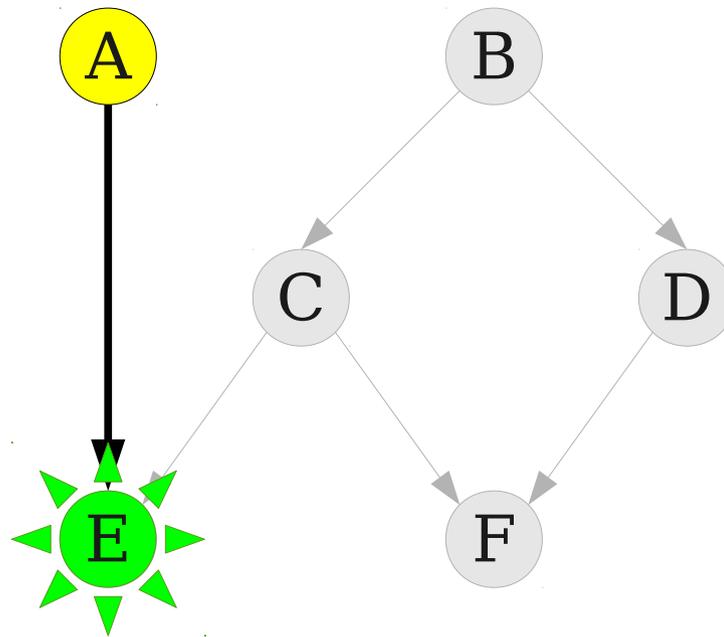
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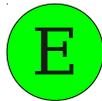
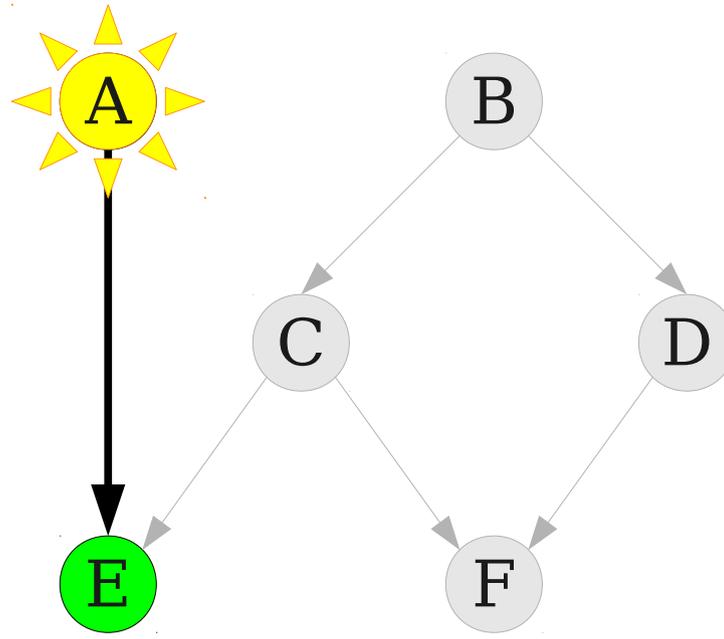
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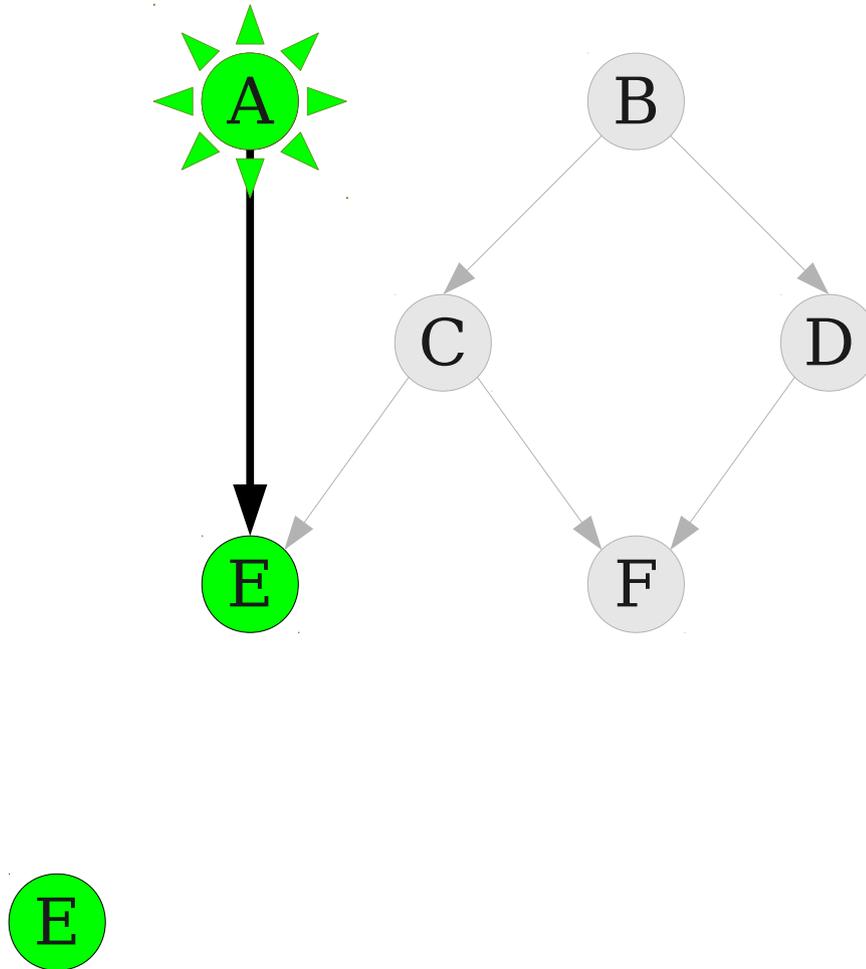
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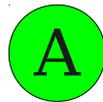
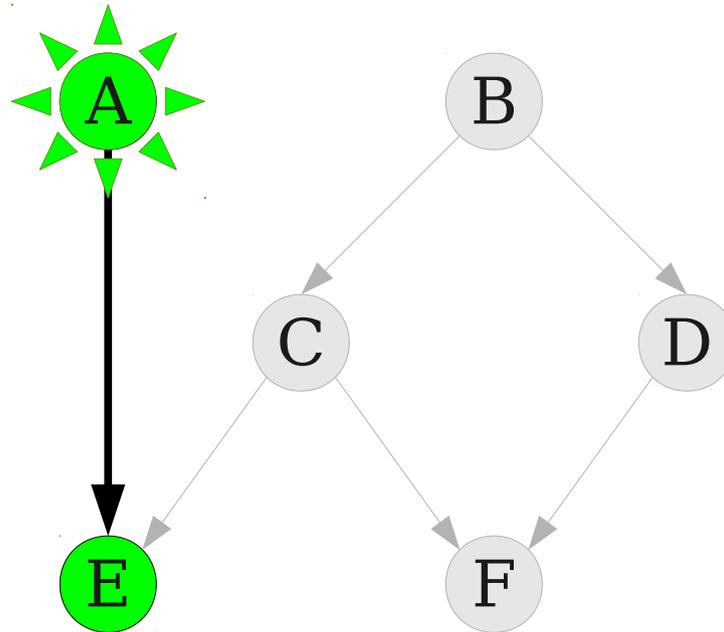
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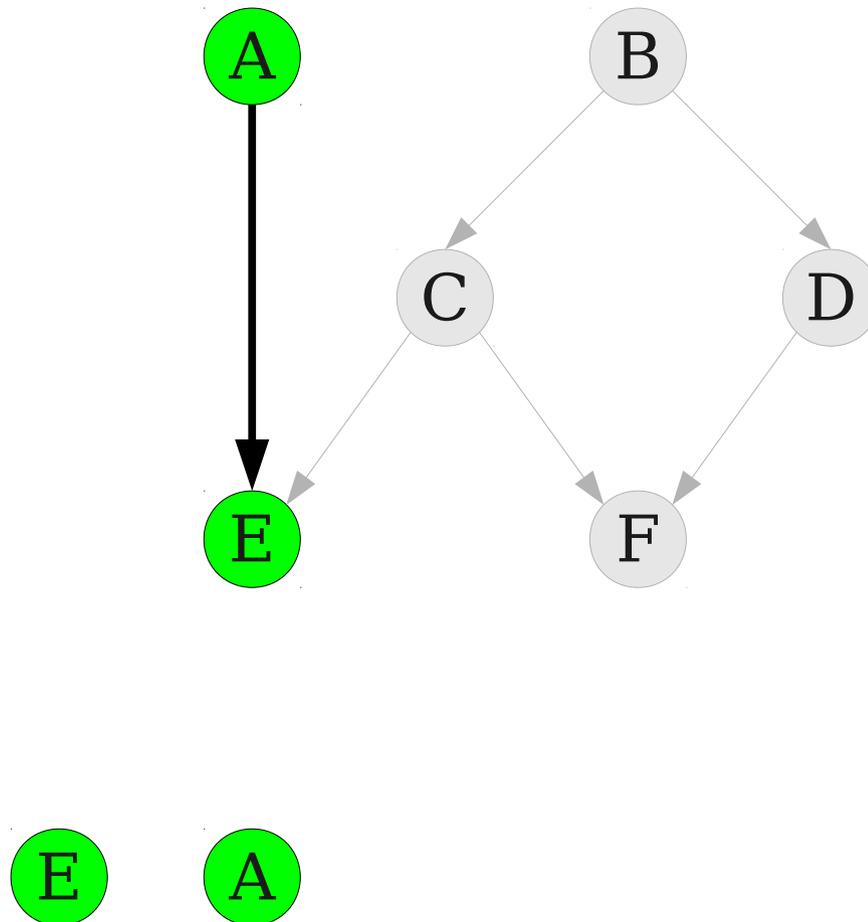
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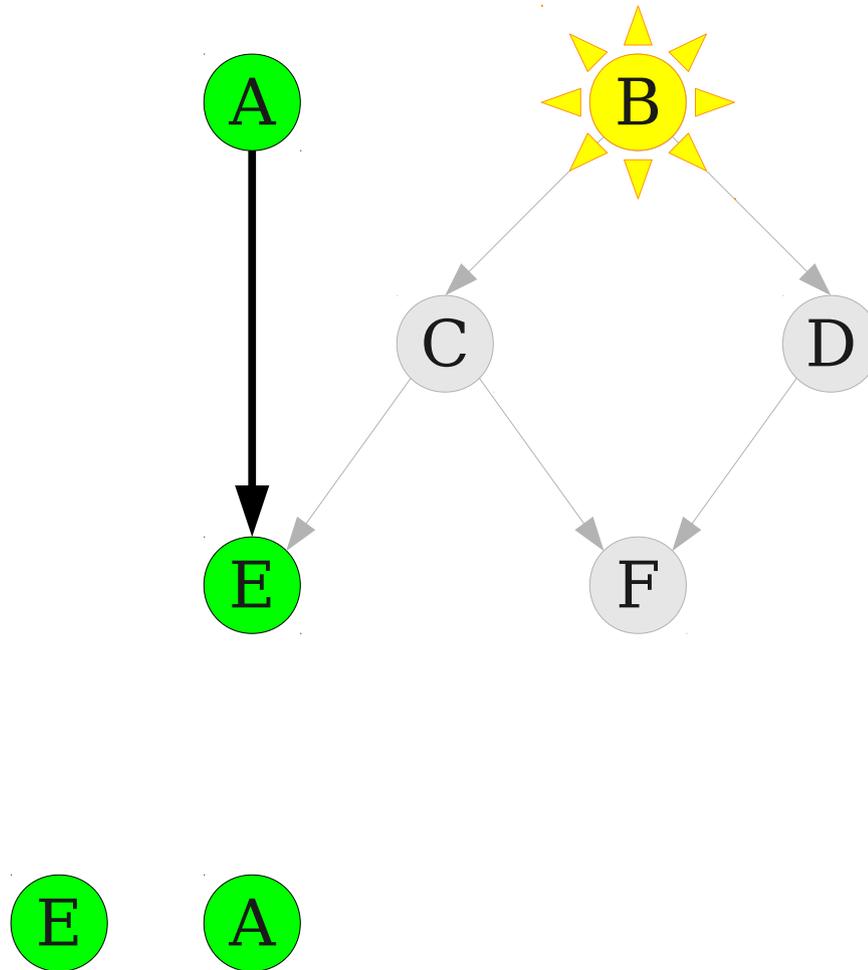
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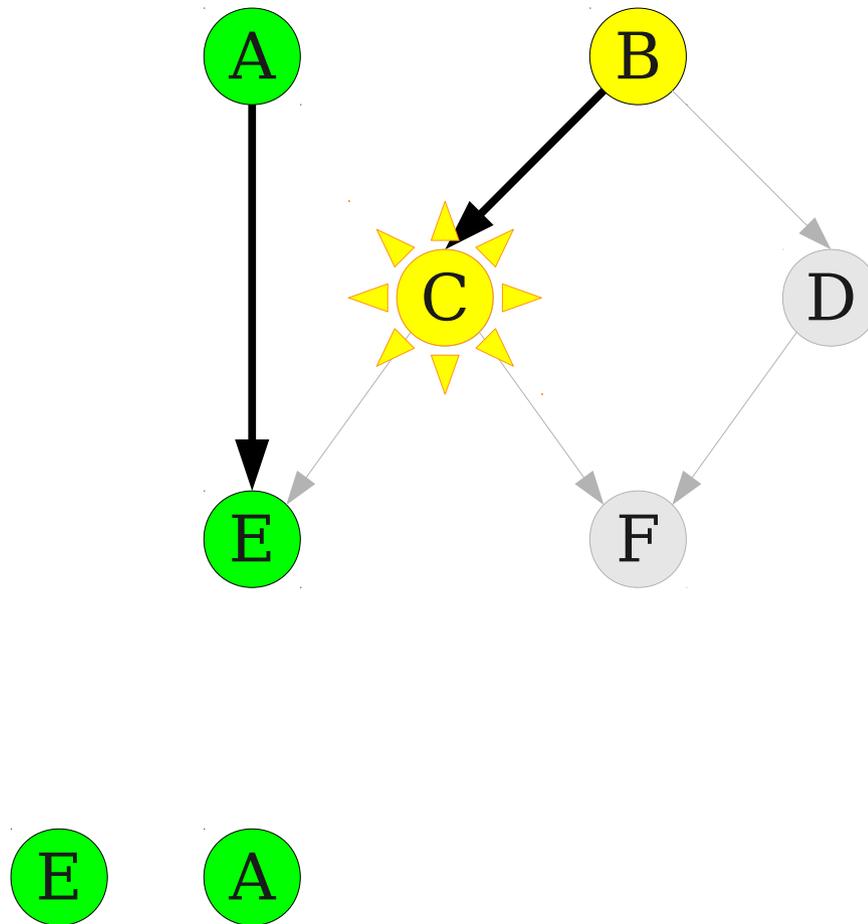
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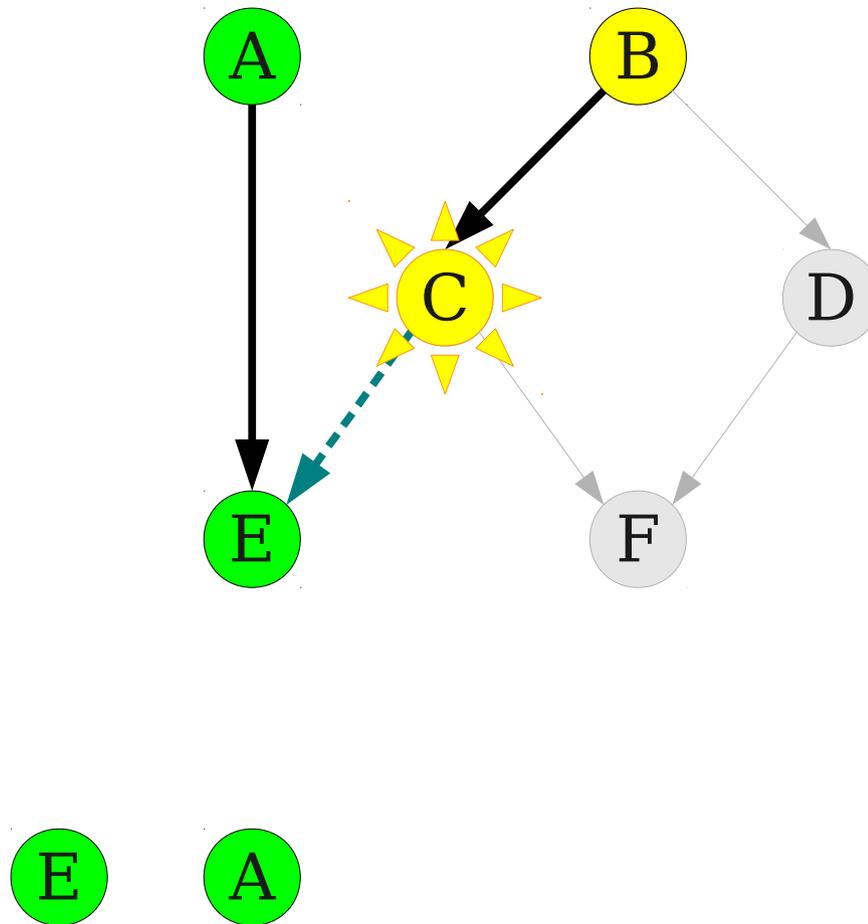
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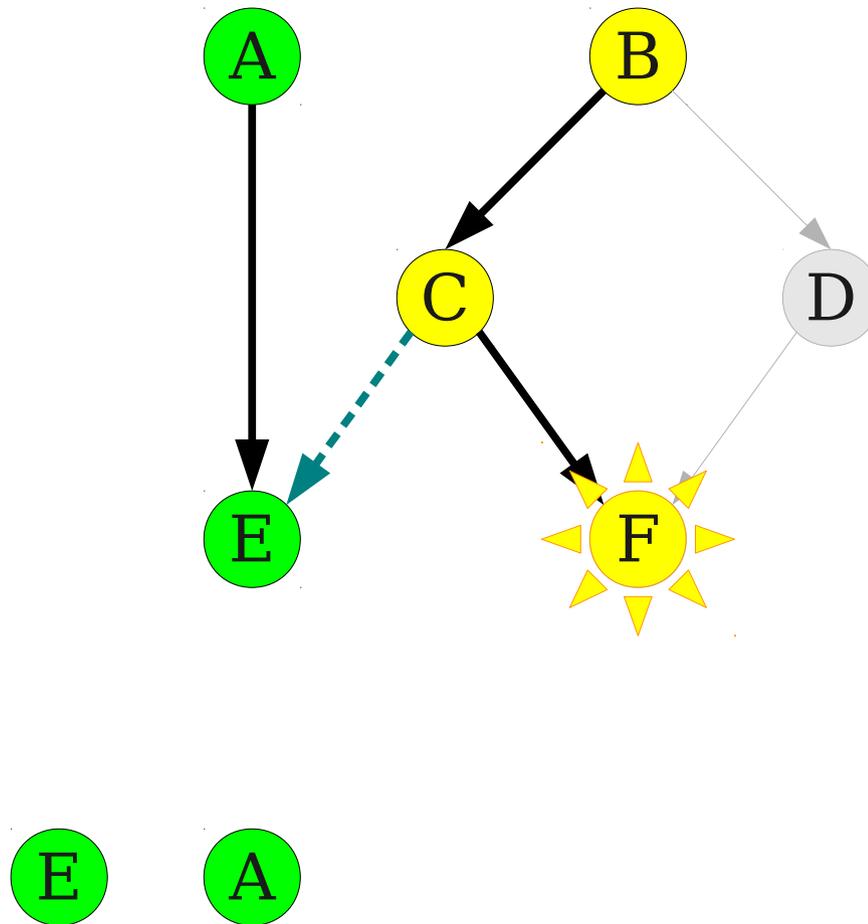
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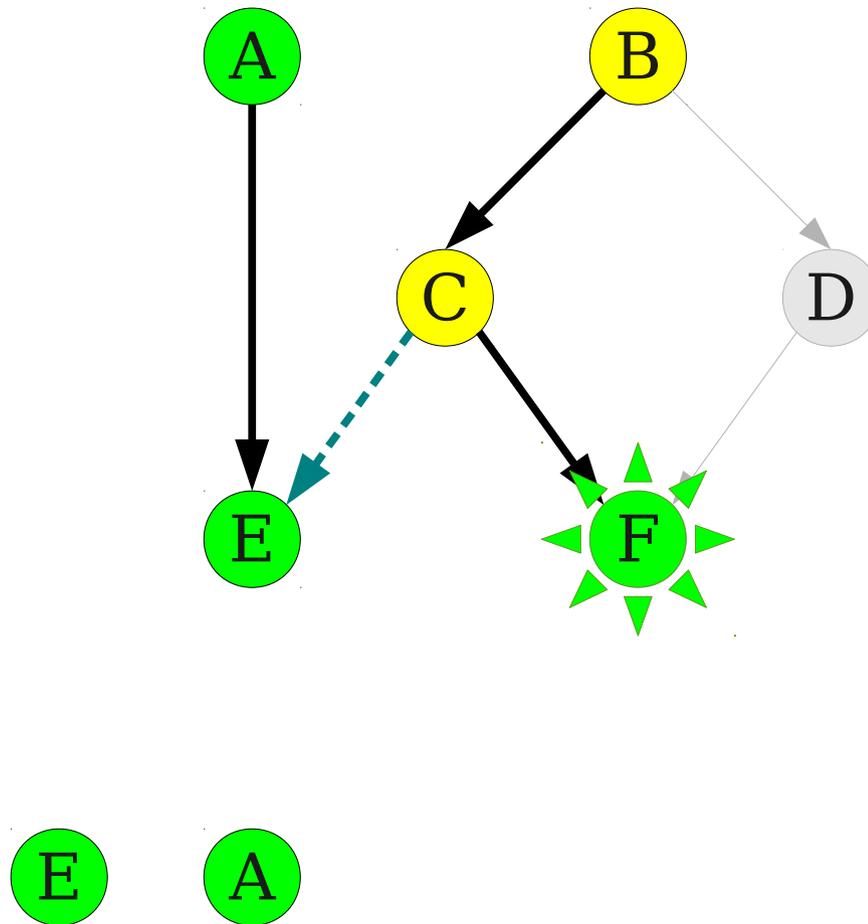
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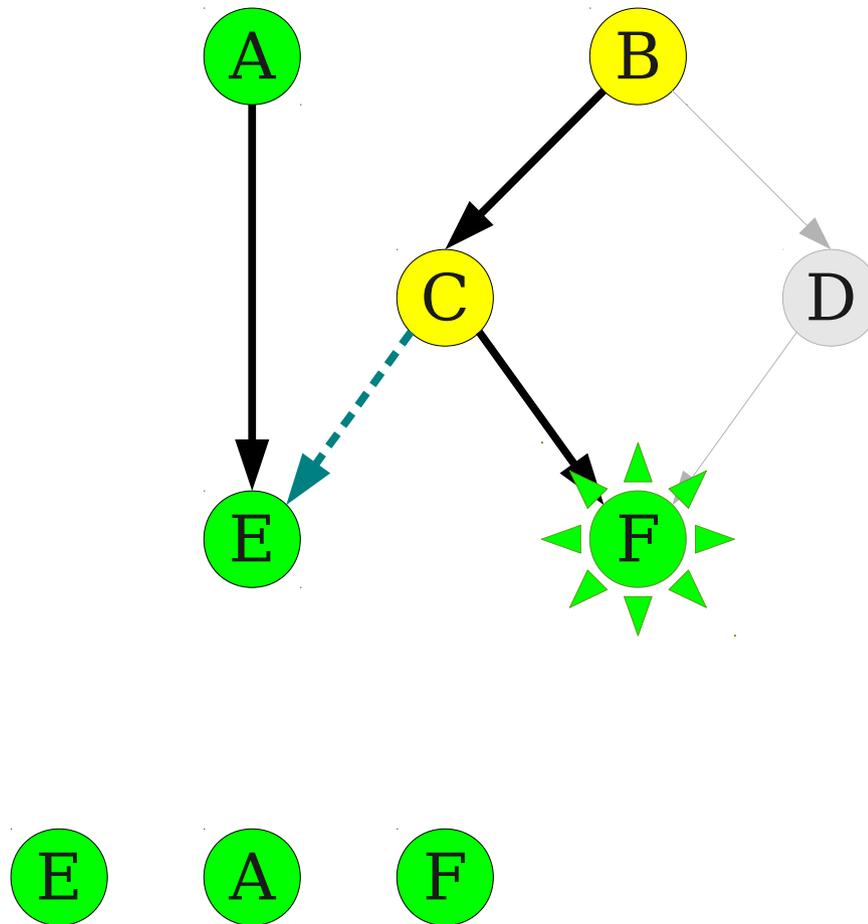
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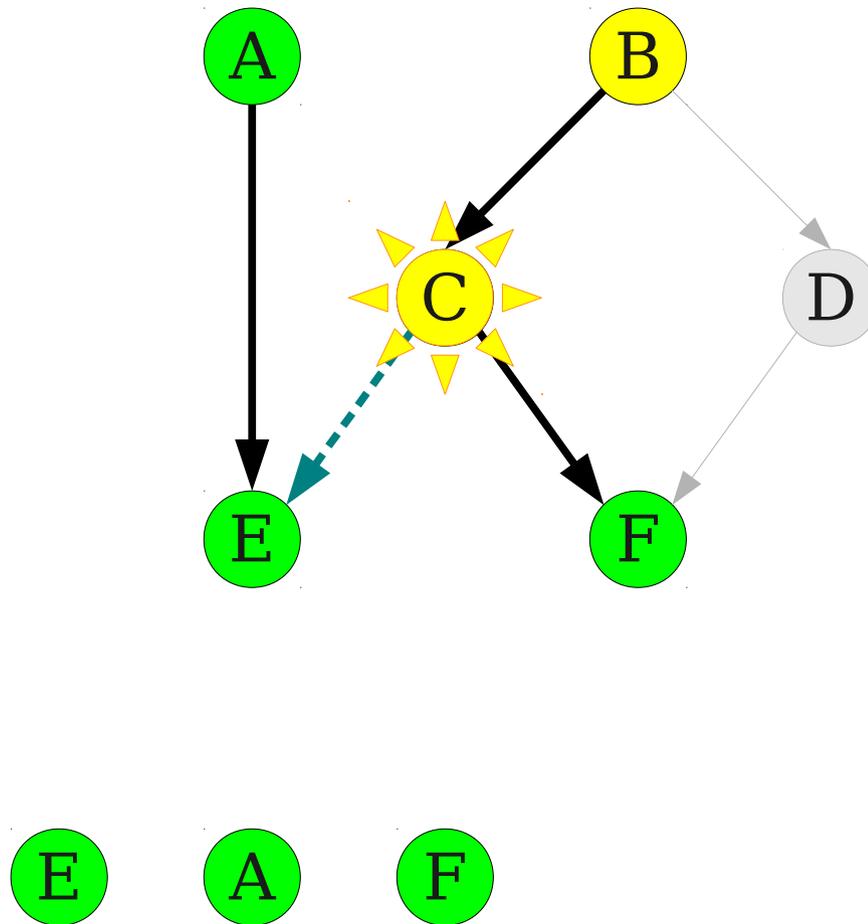
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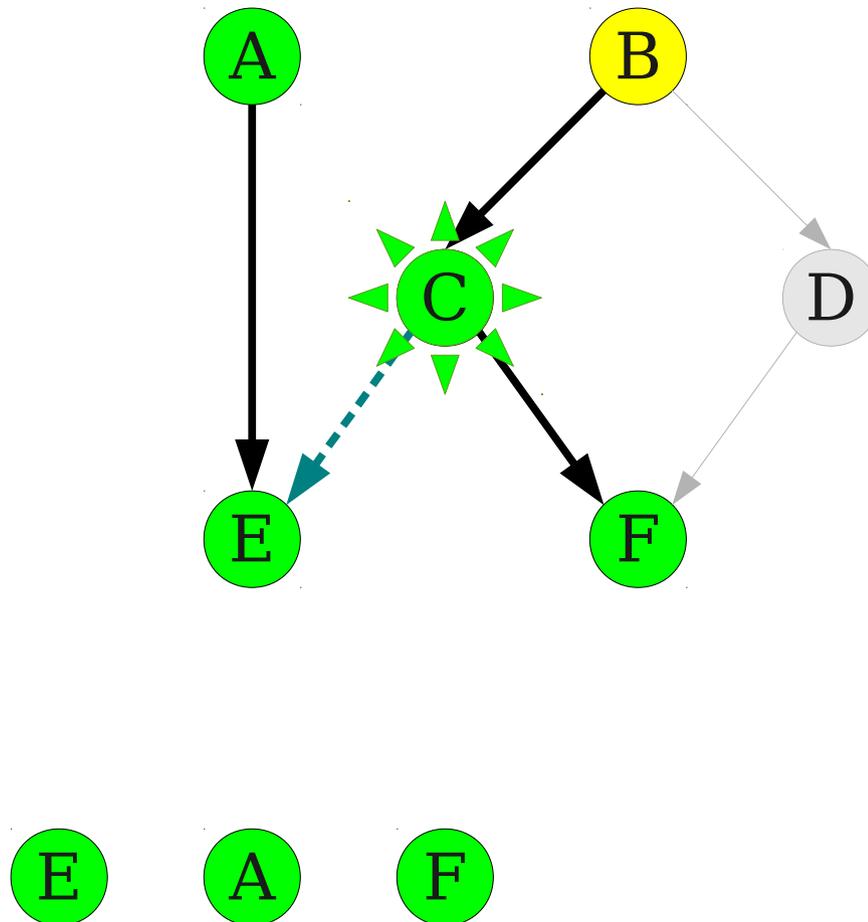
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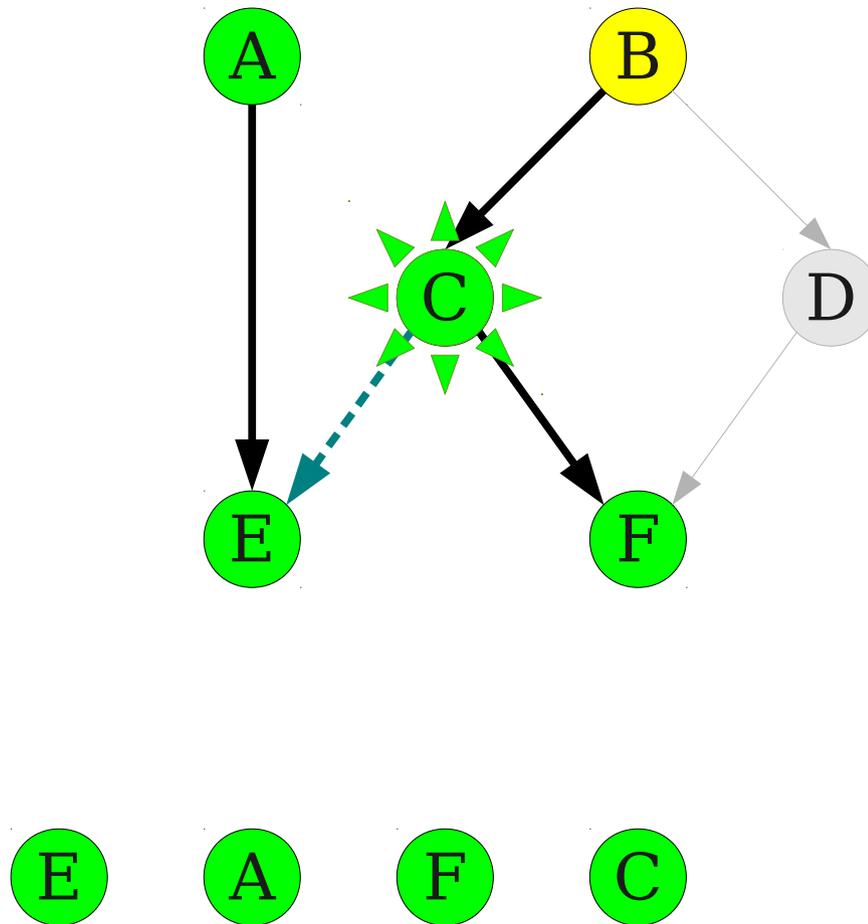
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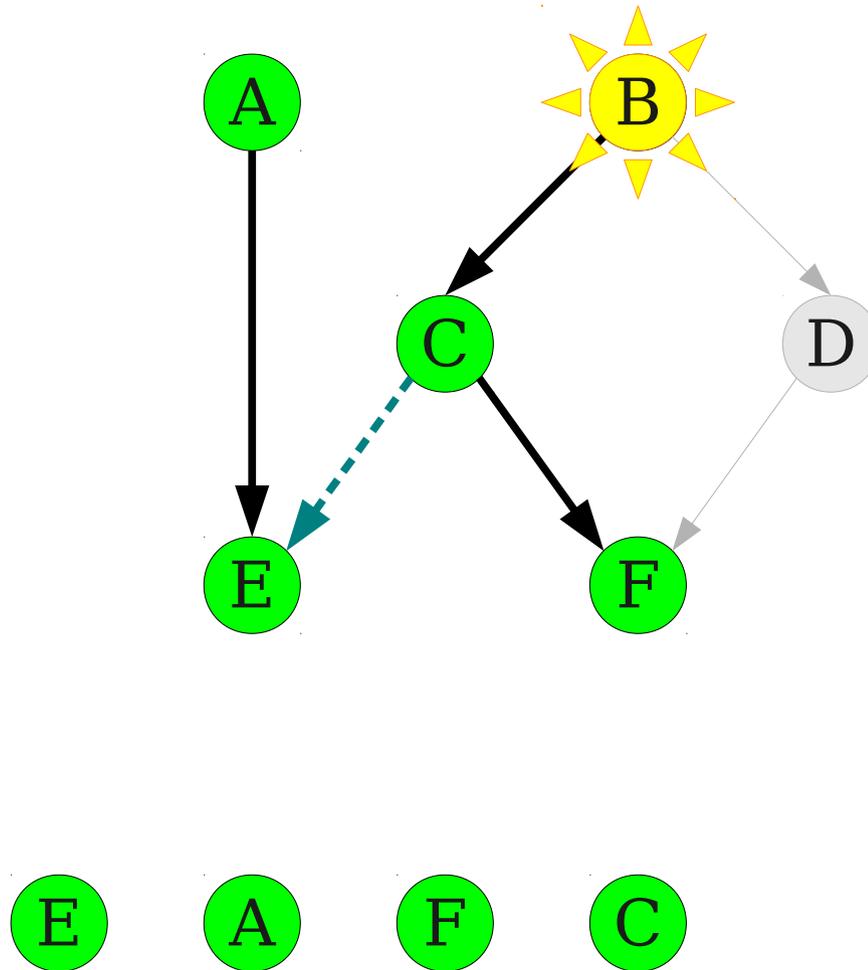
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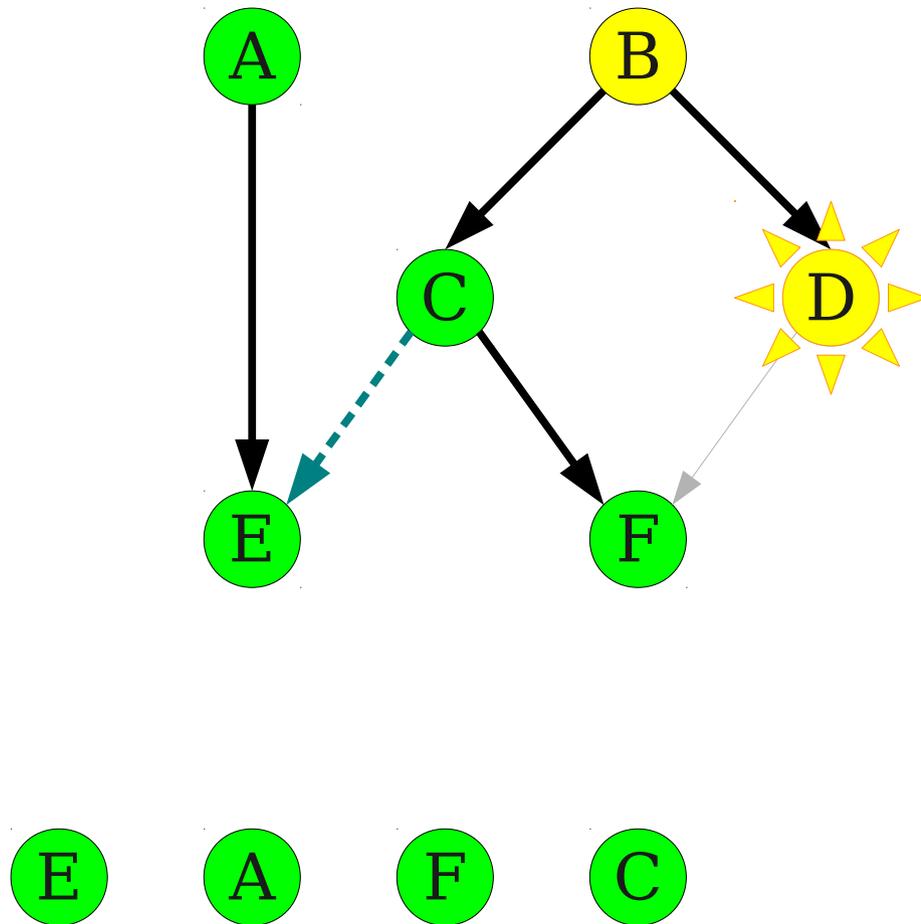
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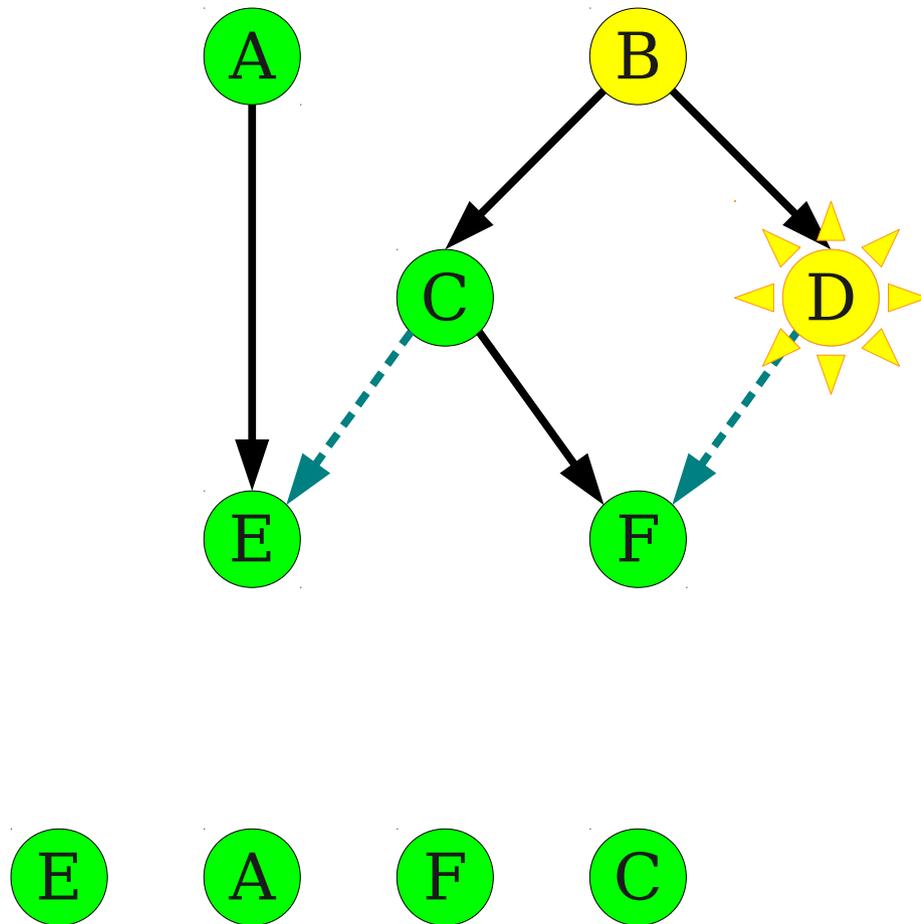
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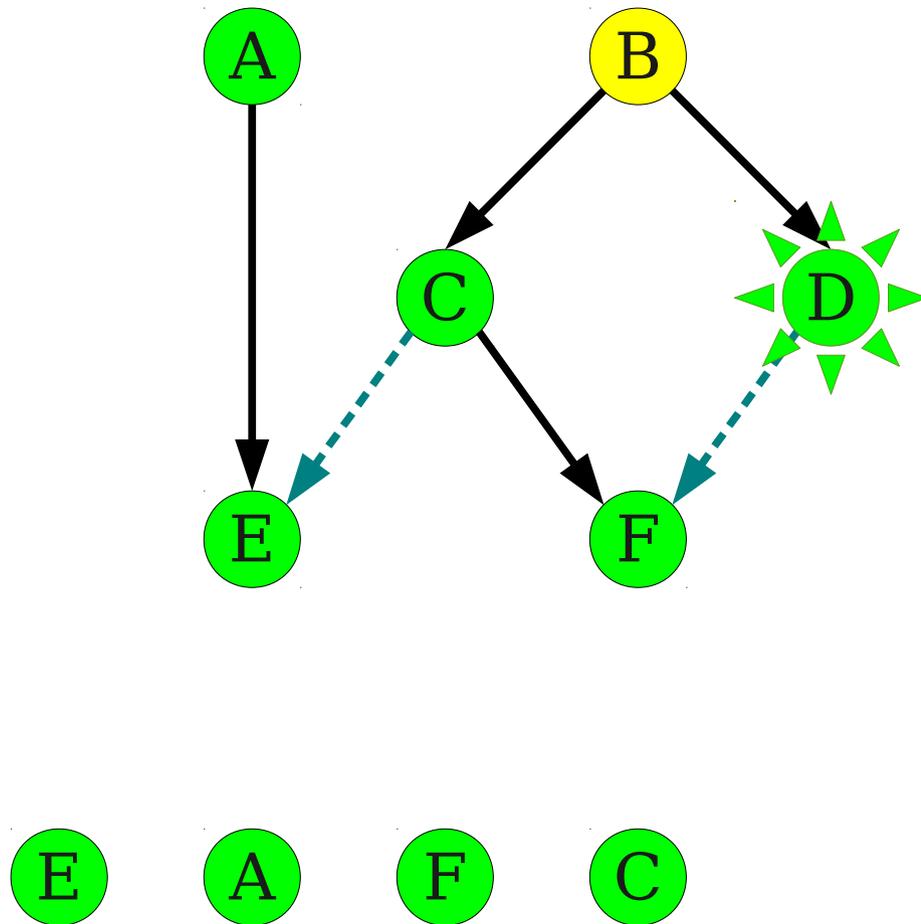
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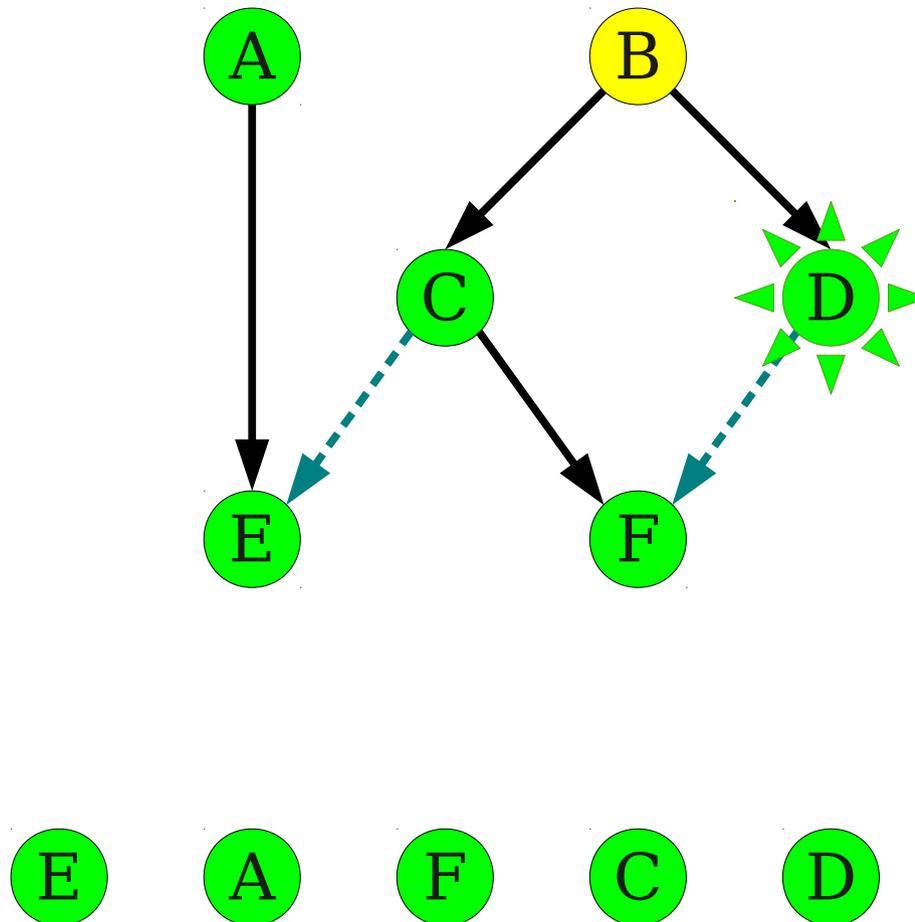
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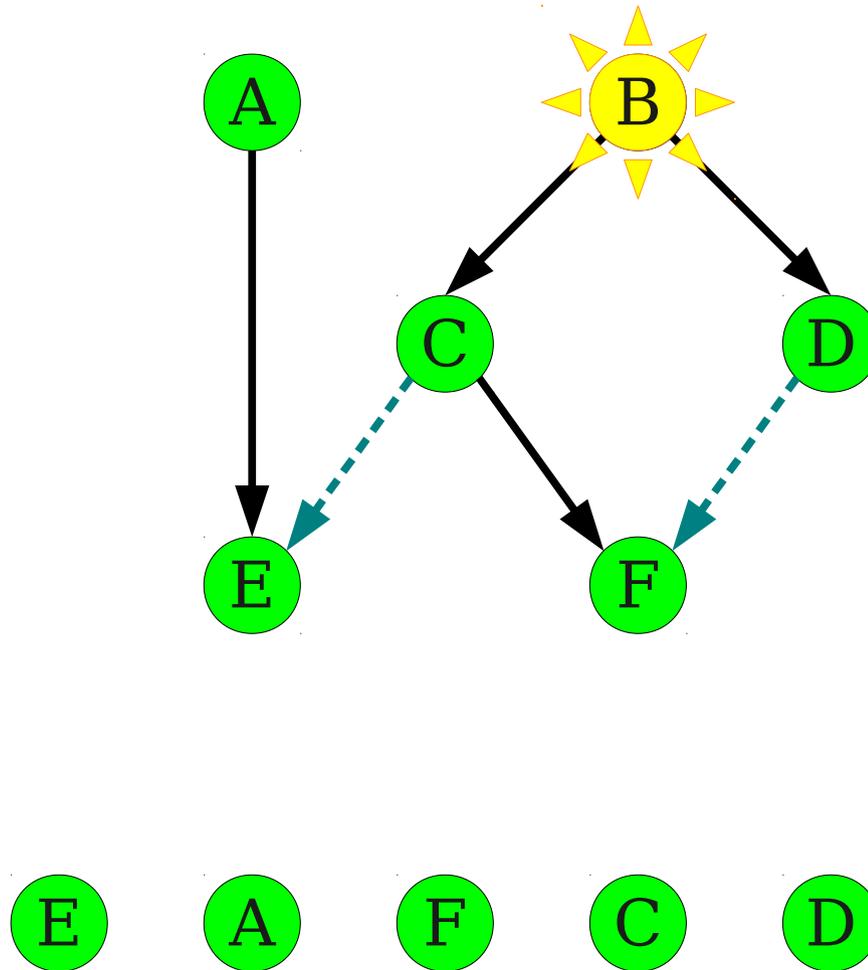
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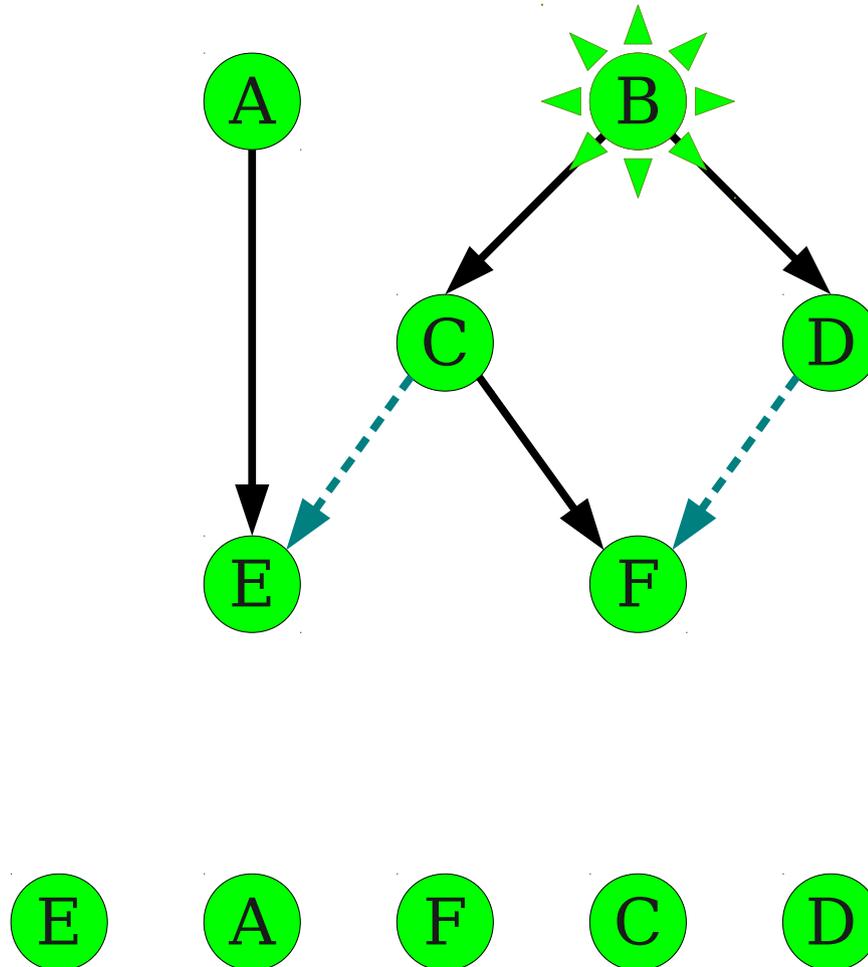
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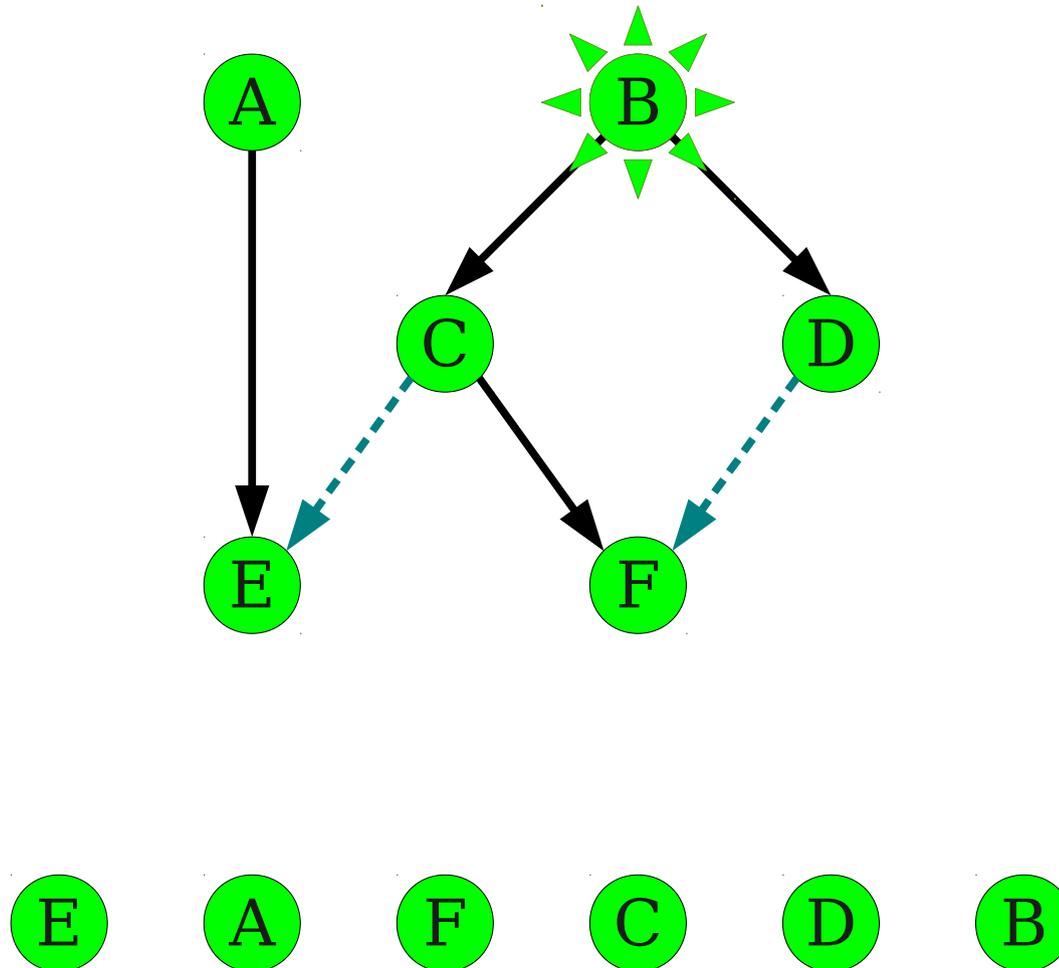
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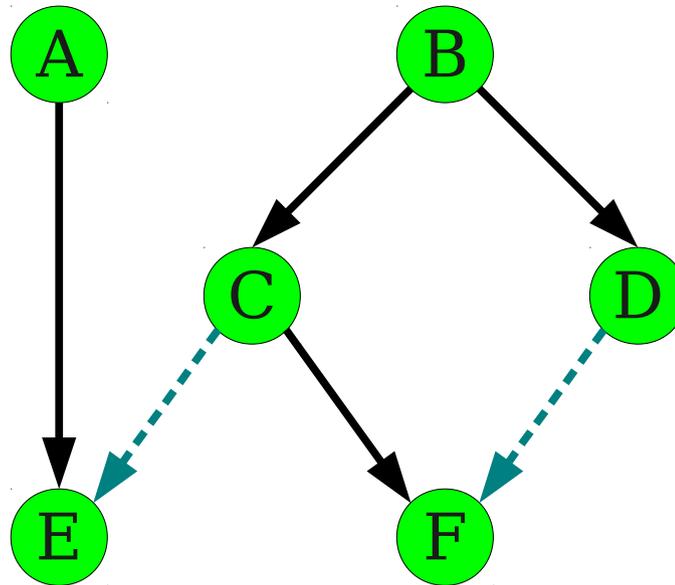
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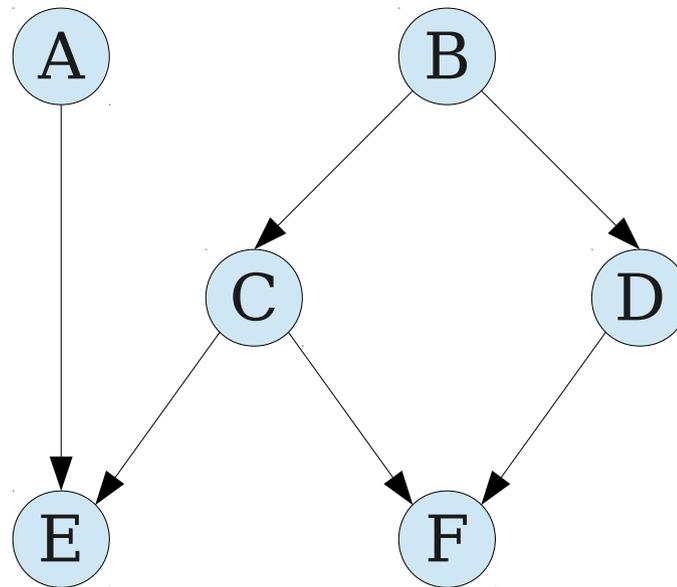
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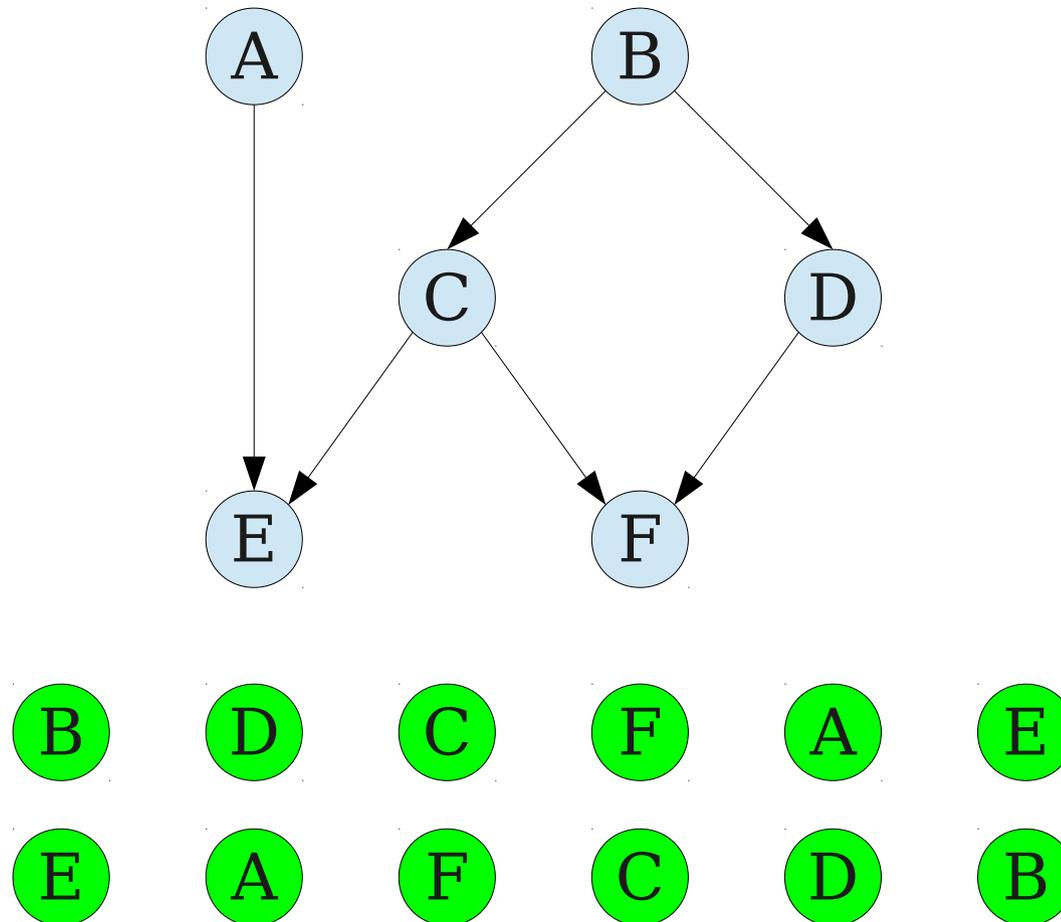
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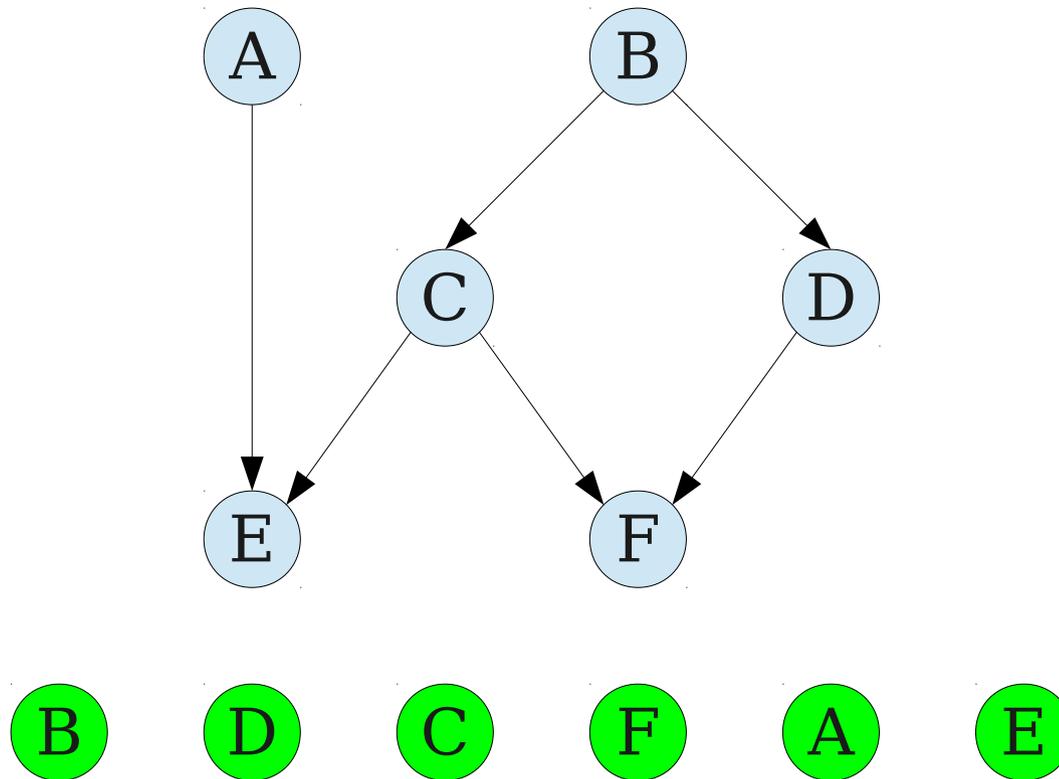
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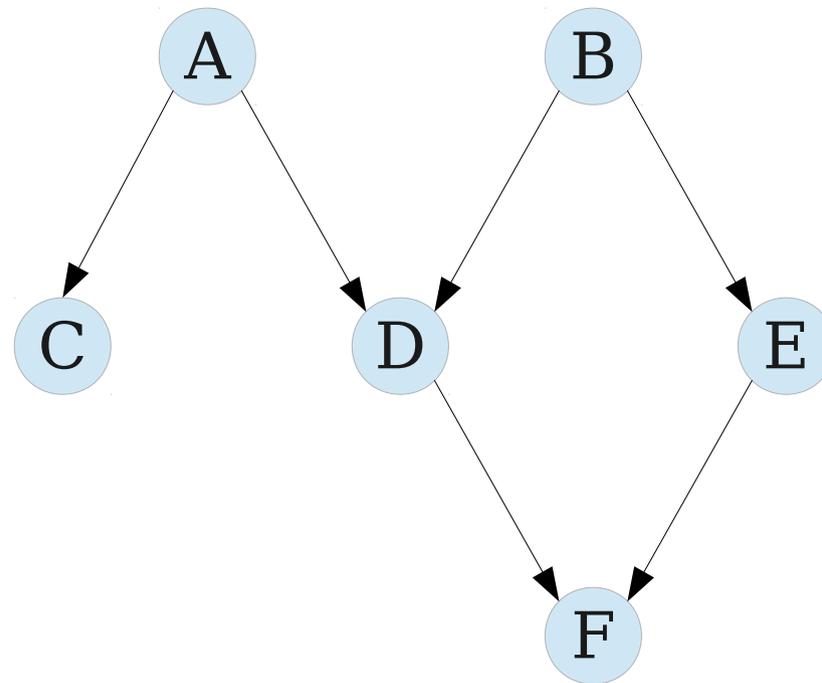
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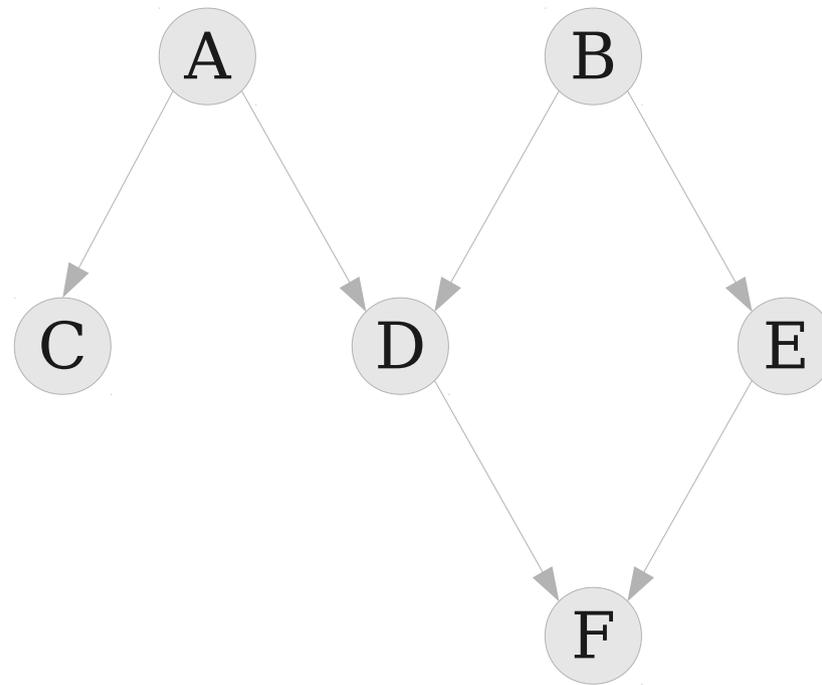
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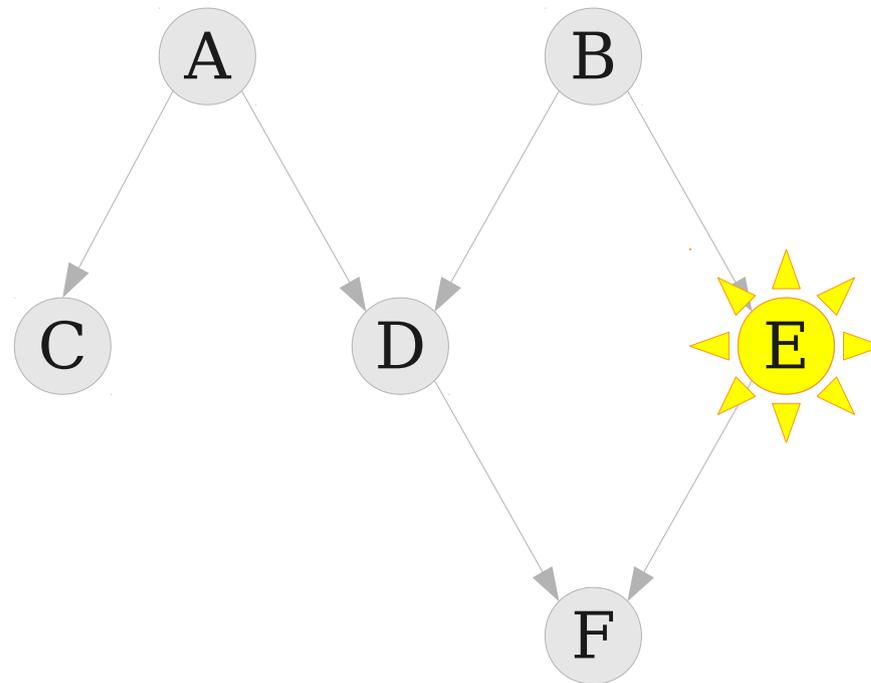
DFS on a DAG, Take III



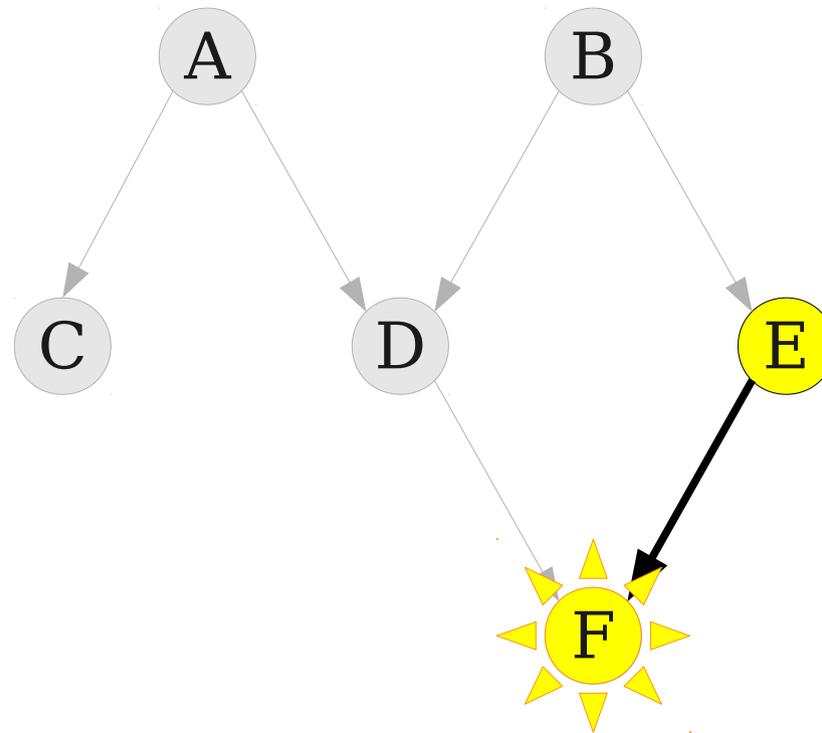
DFS on a DAG, Take III



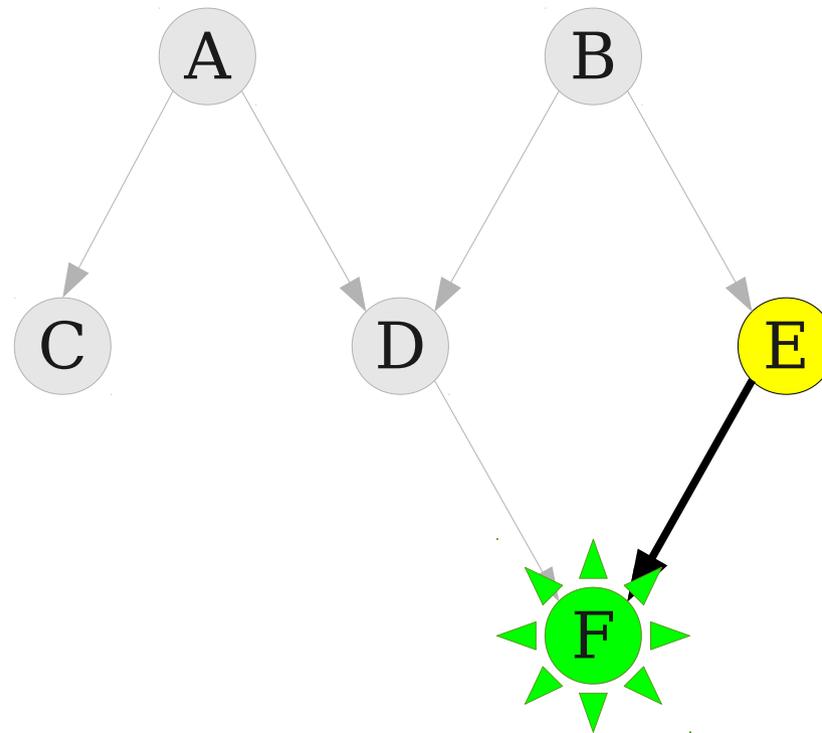
DFS on a DAG, Take III



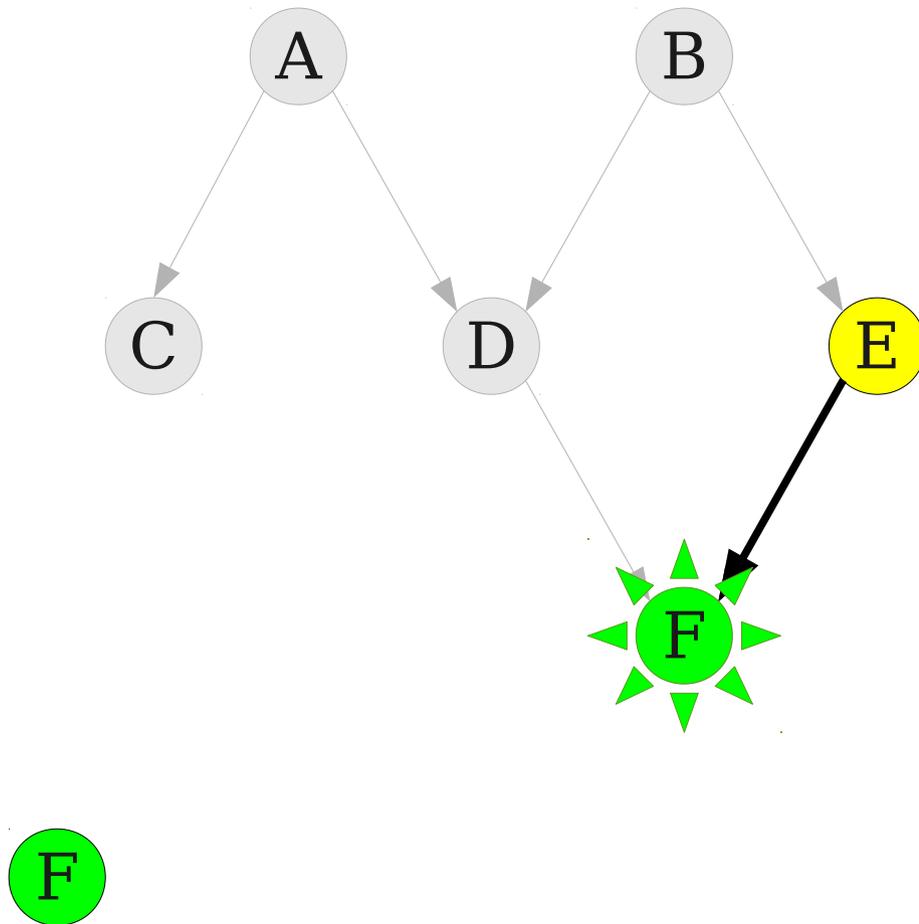
DFS on a DAG, Take III



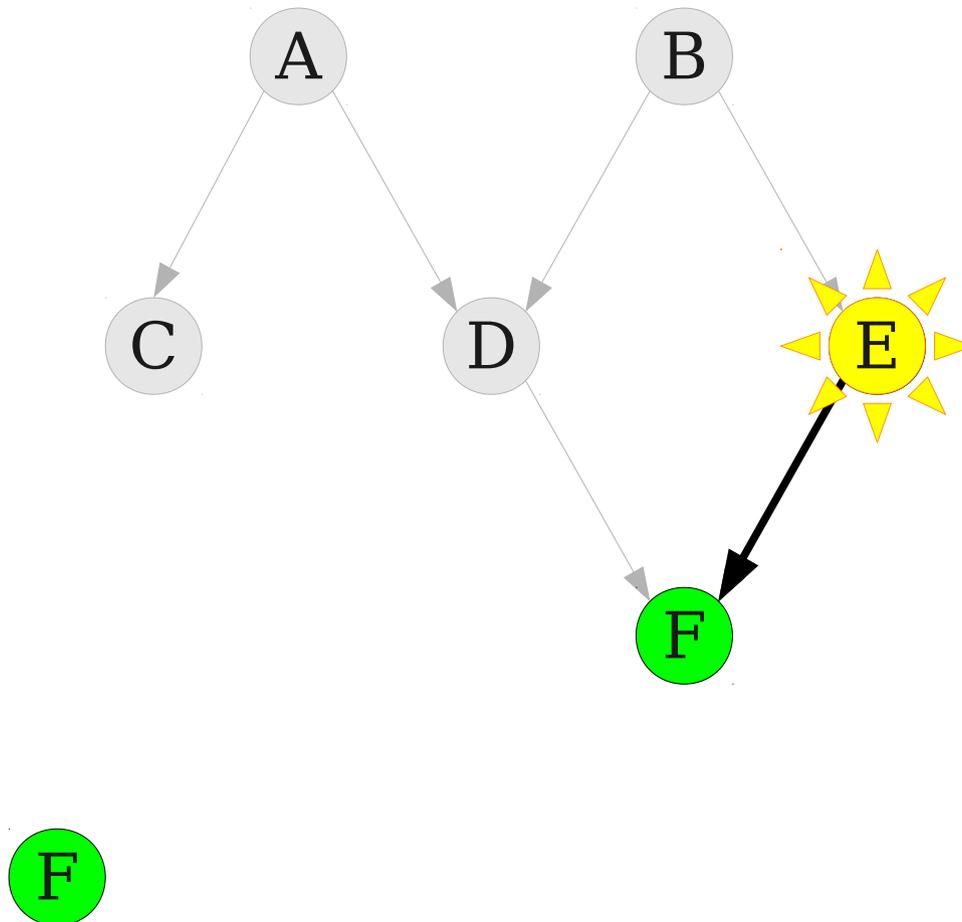
DFS on a DAG, Take III



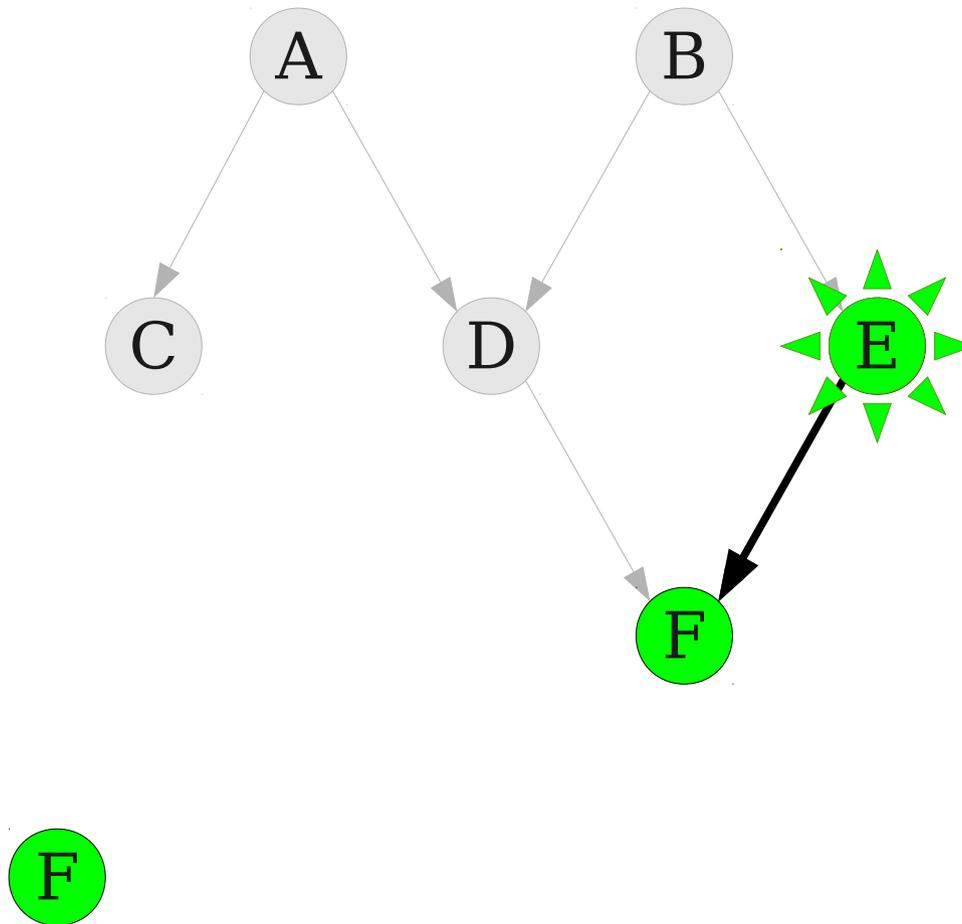
DFS on a DAG, Take III



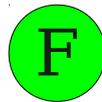
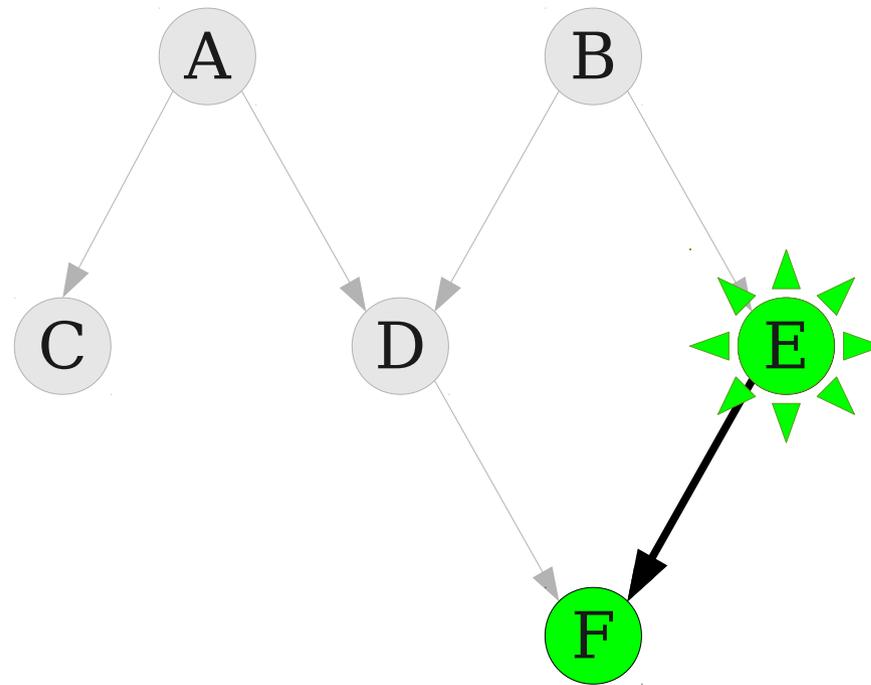
DFS on a DAG, Take III



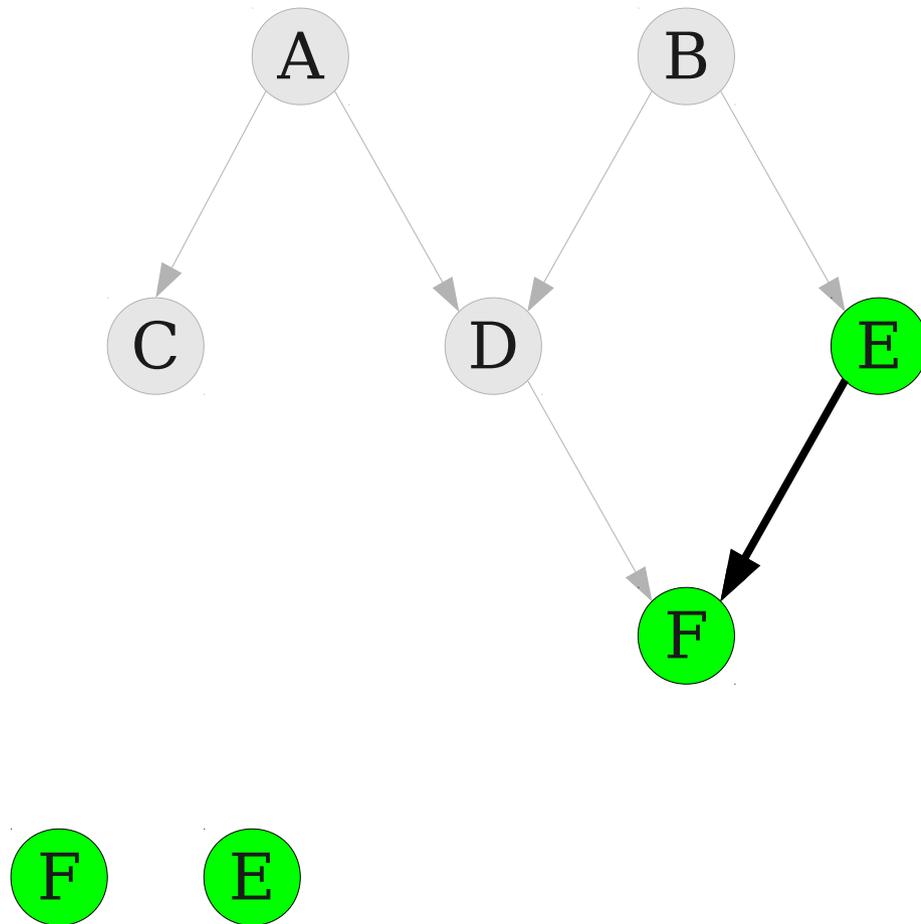
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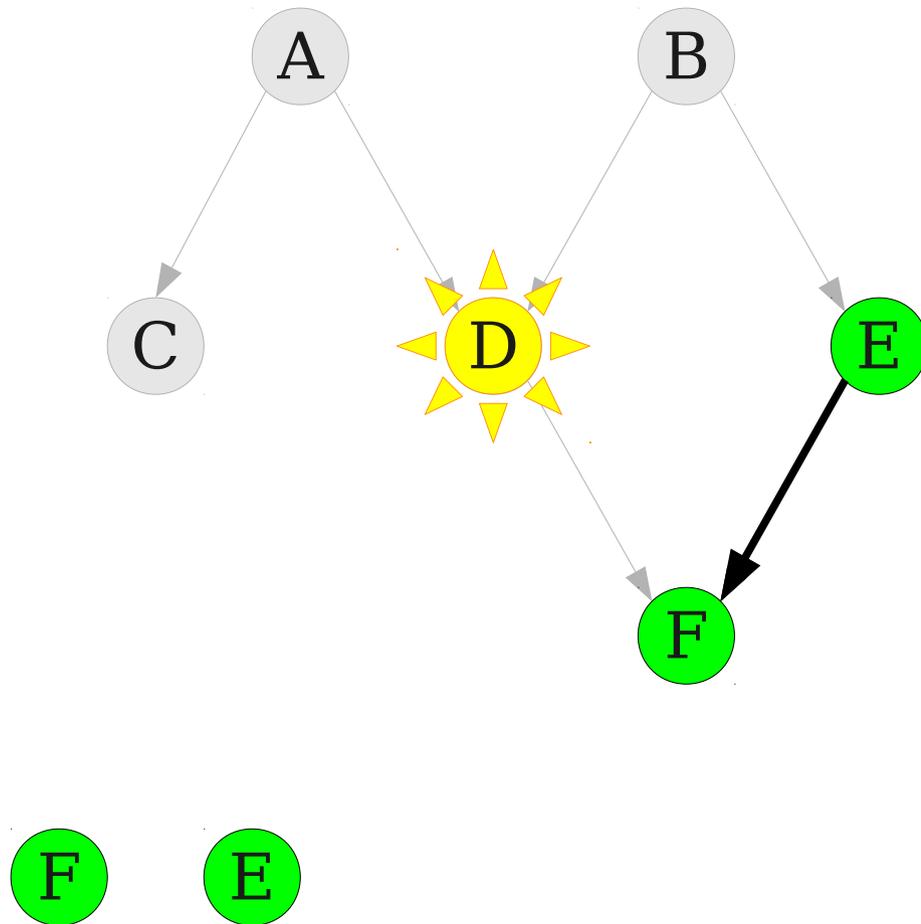
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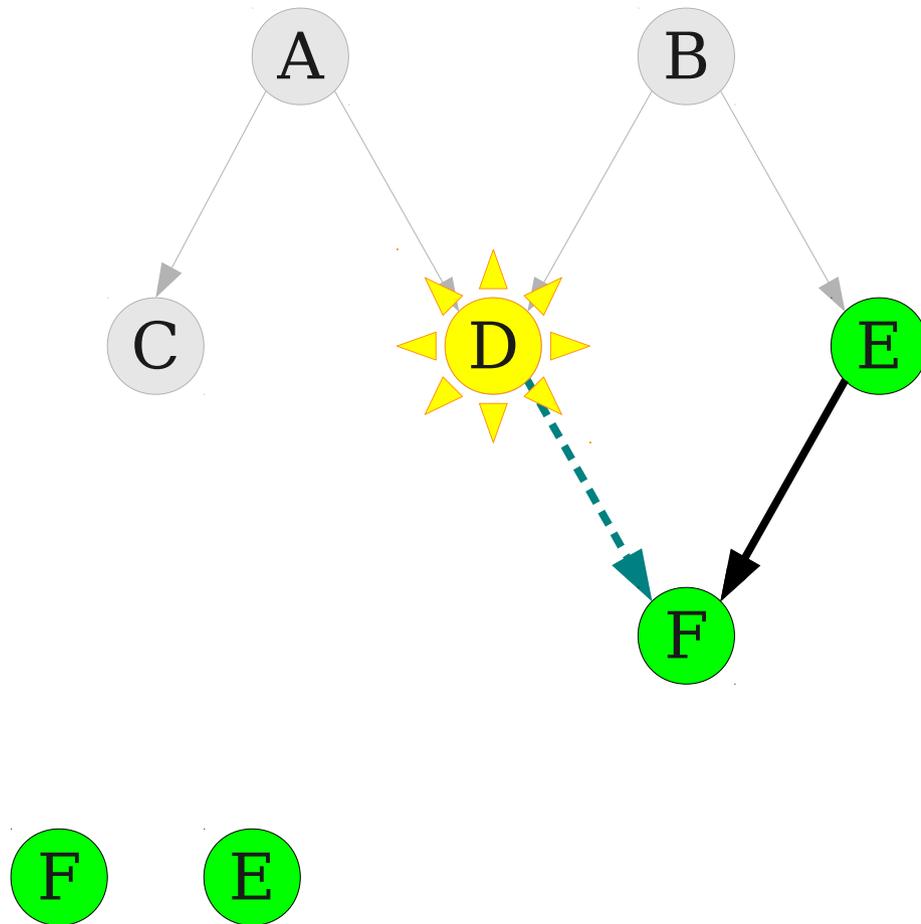
DFS on a DAG, Take III



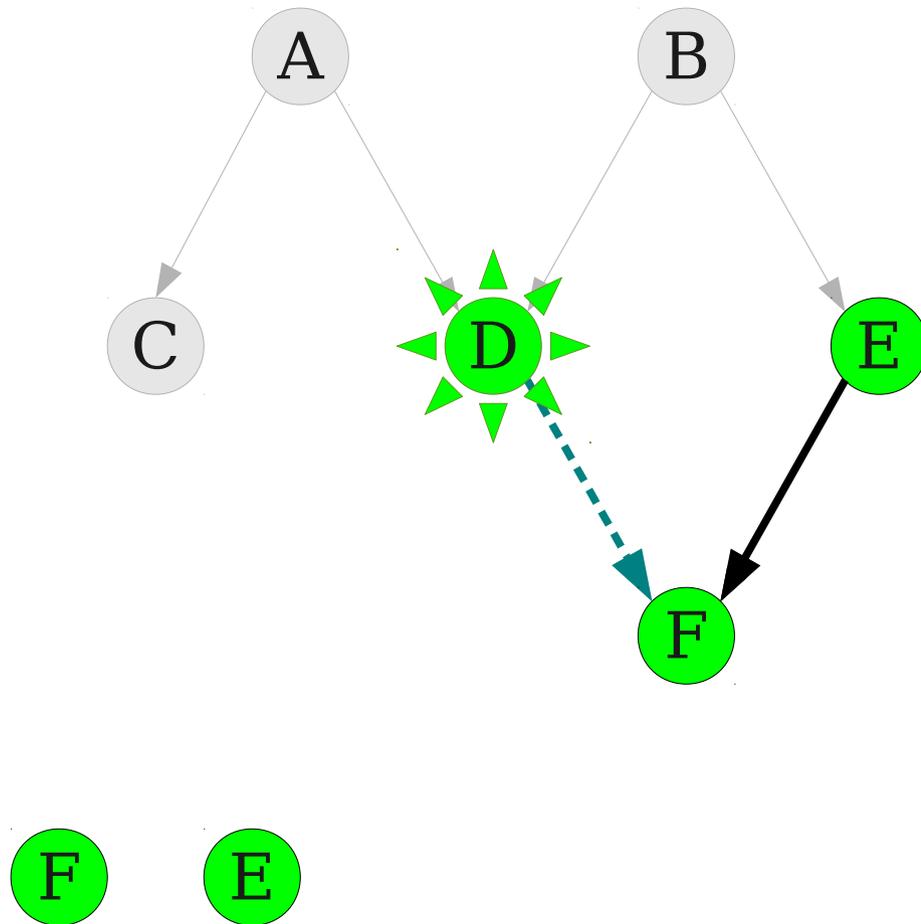
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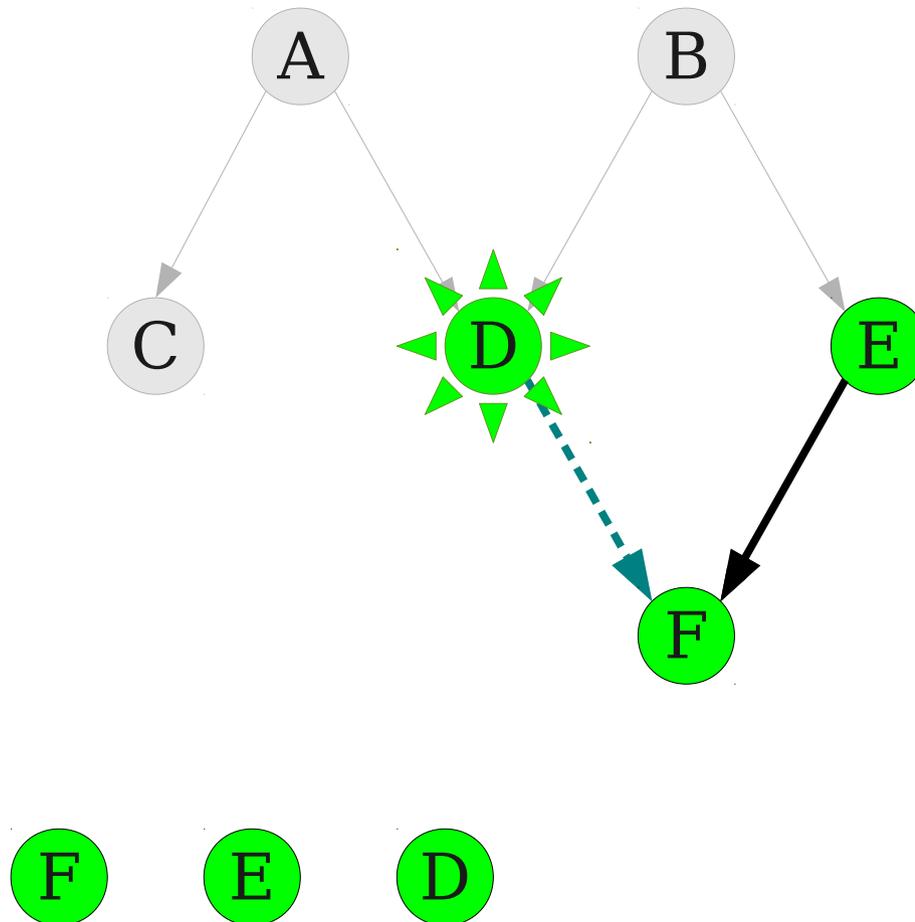
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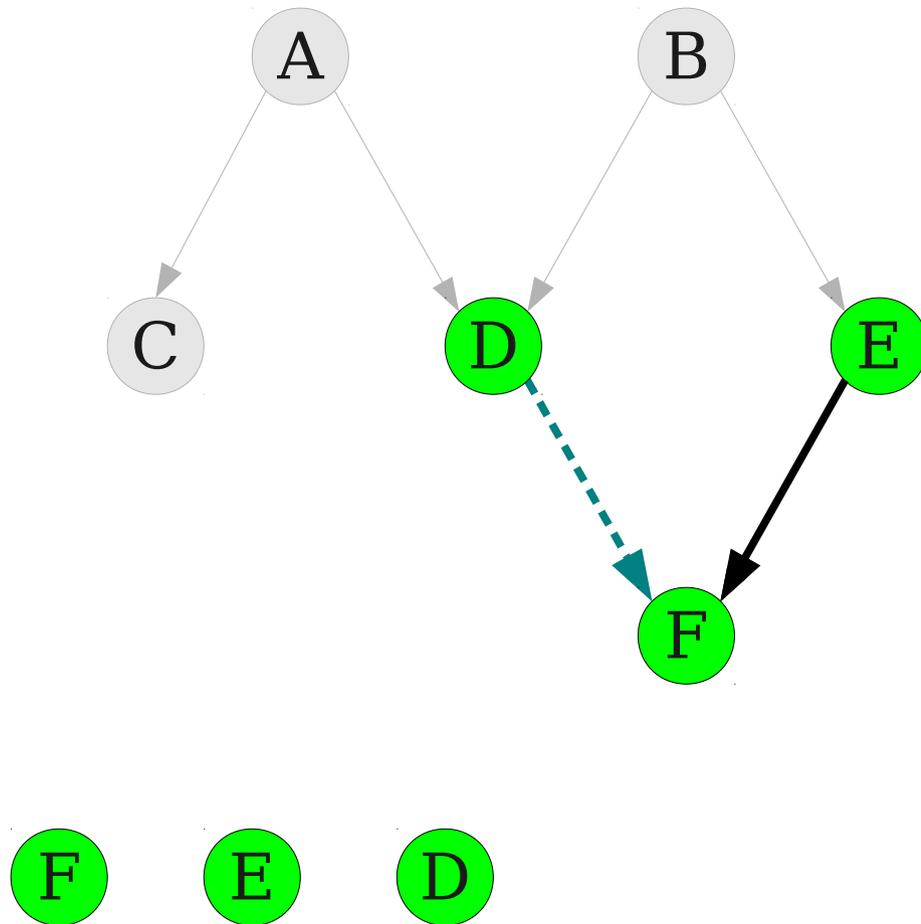
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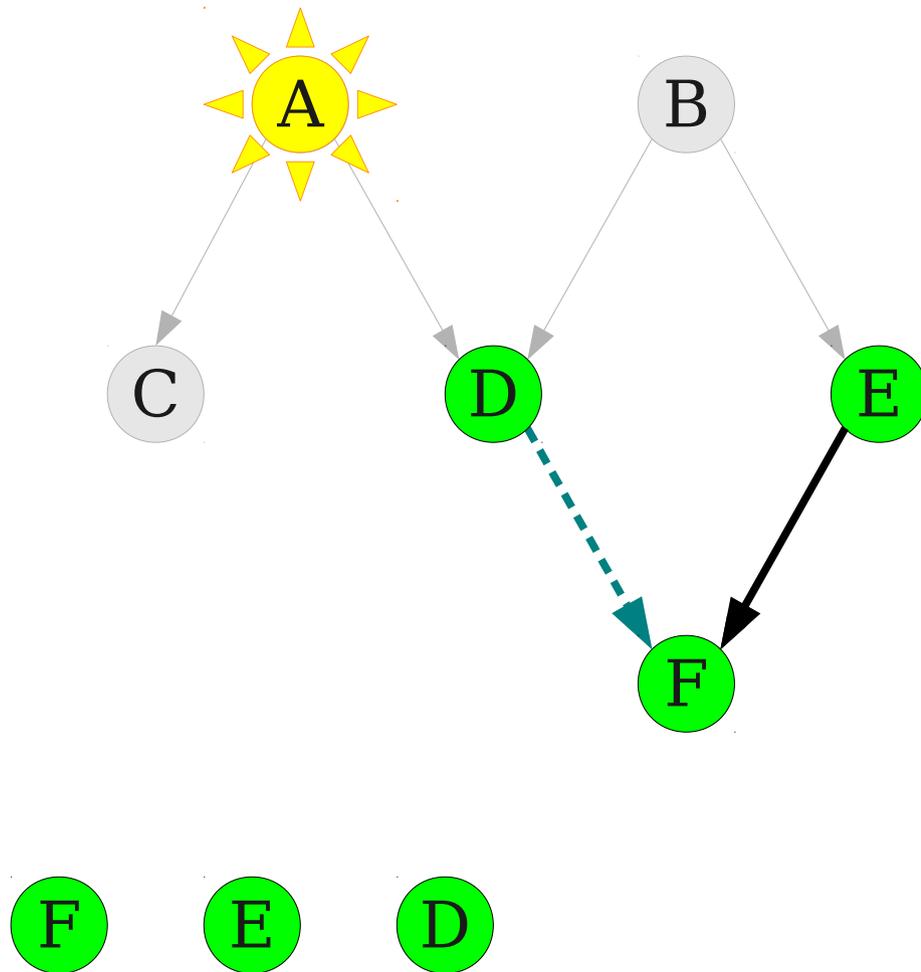
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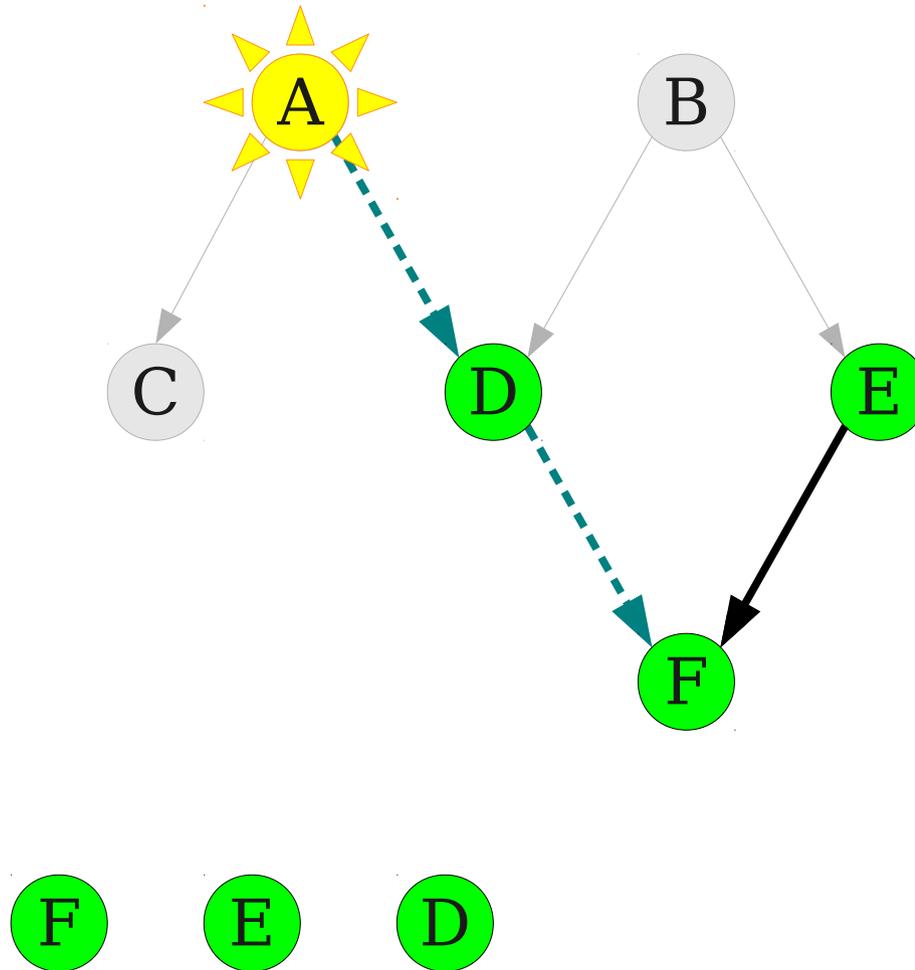
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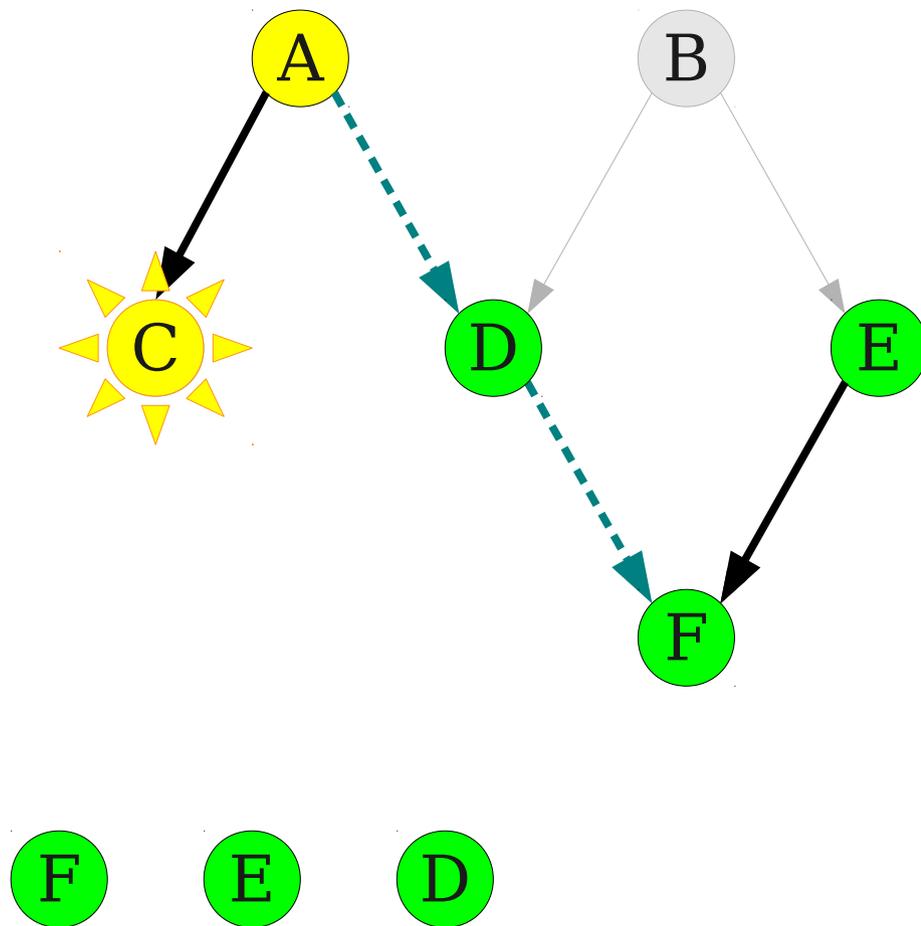
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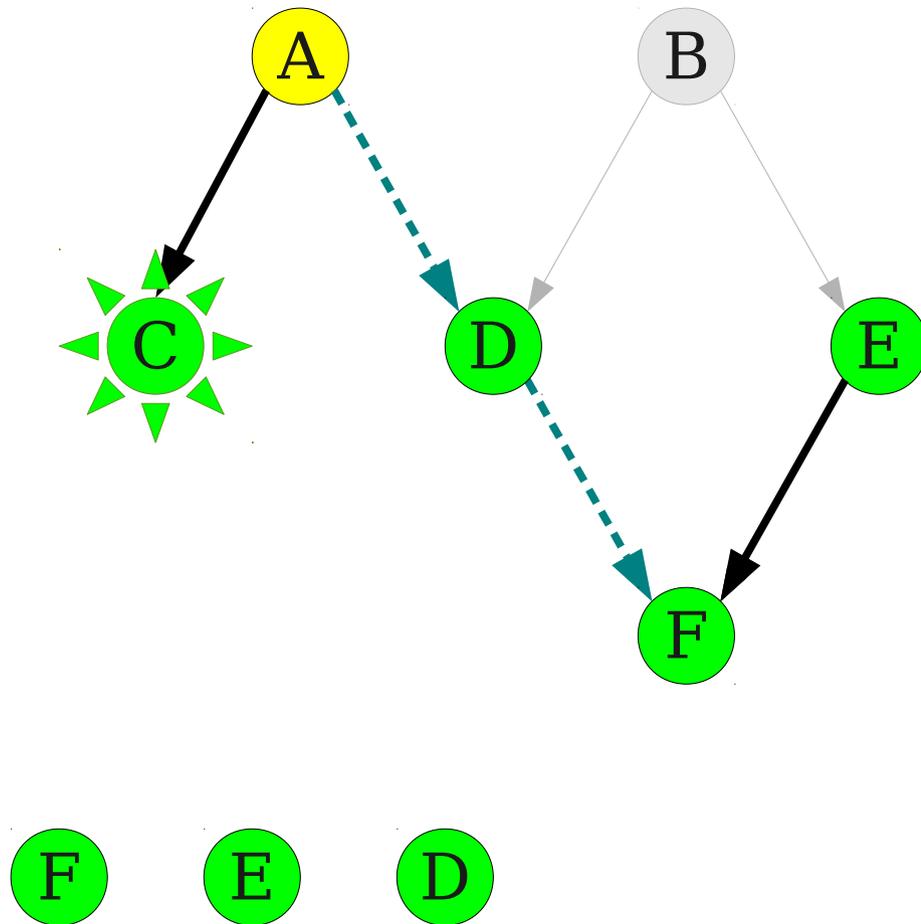
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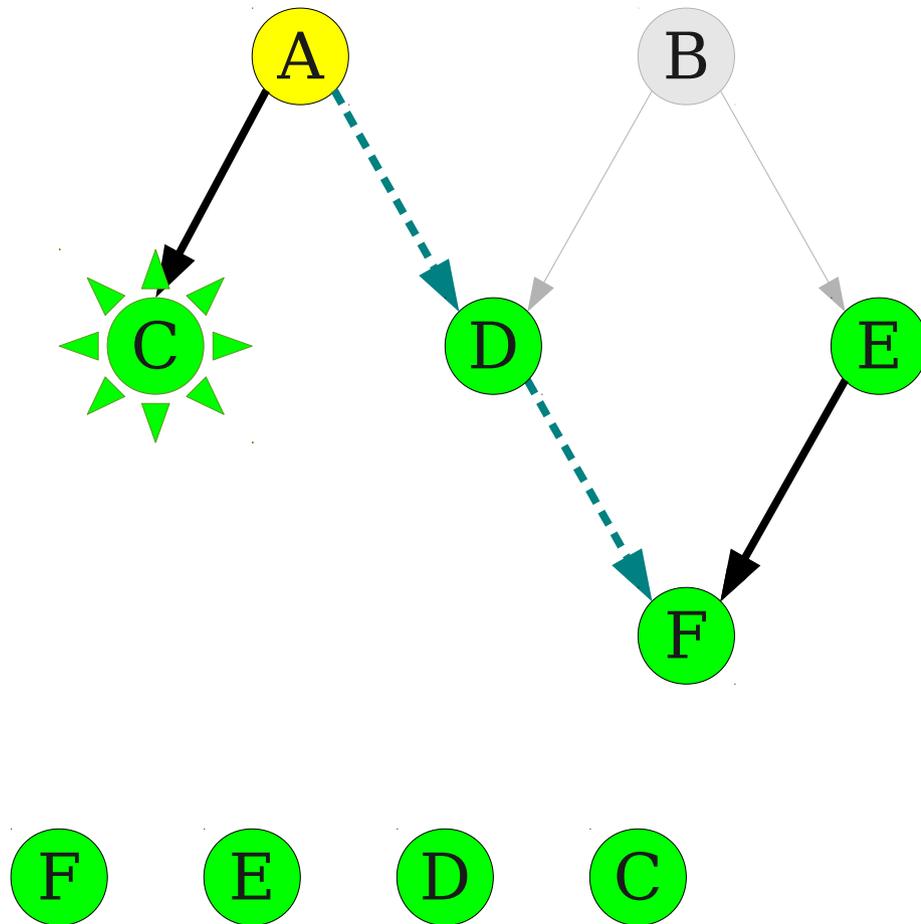
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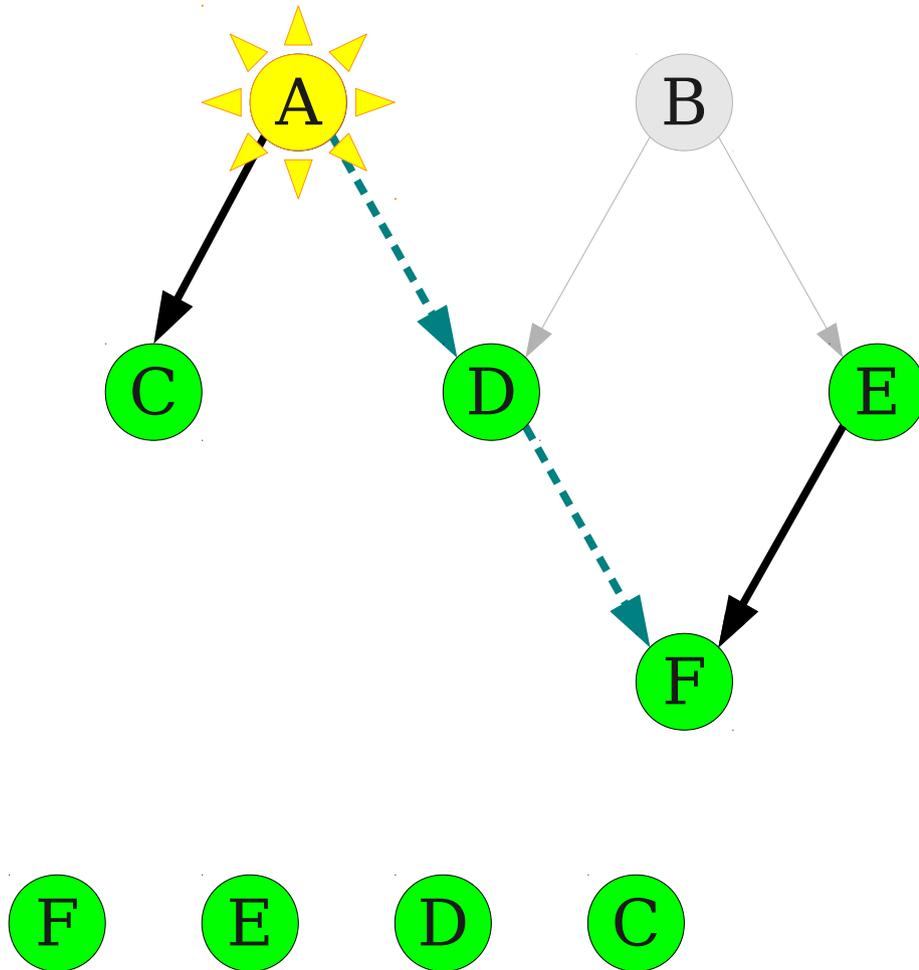
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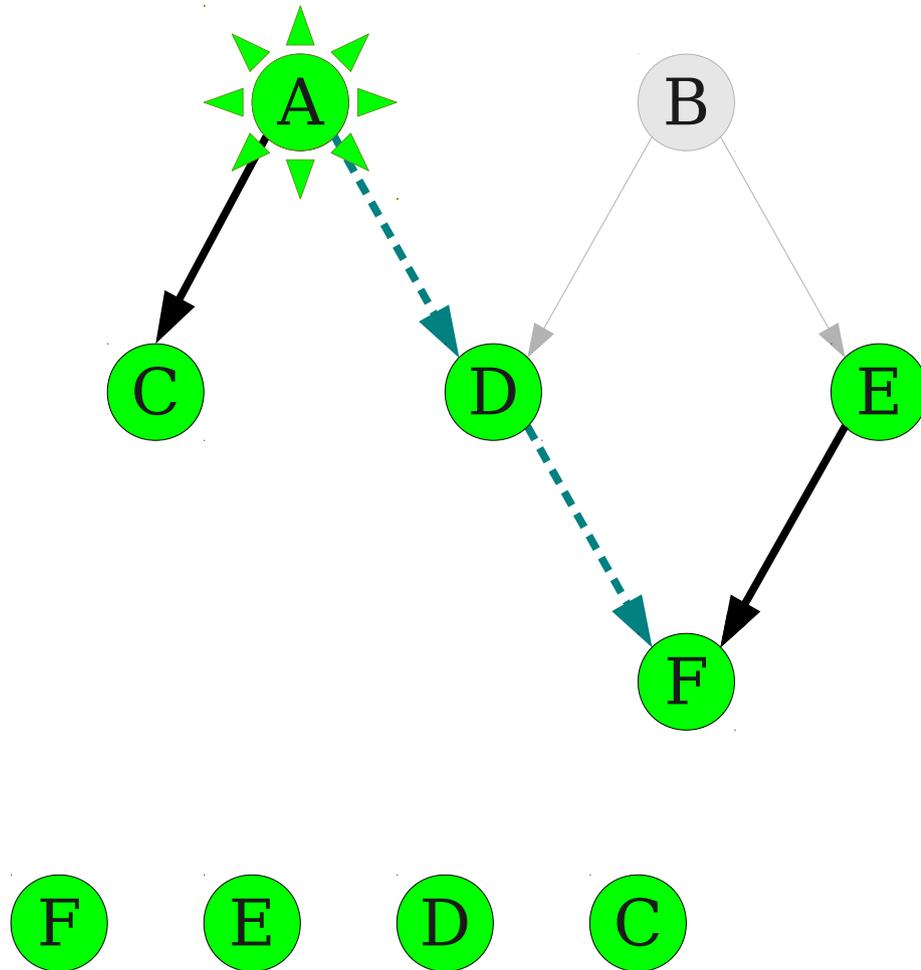
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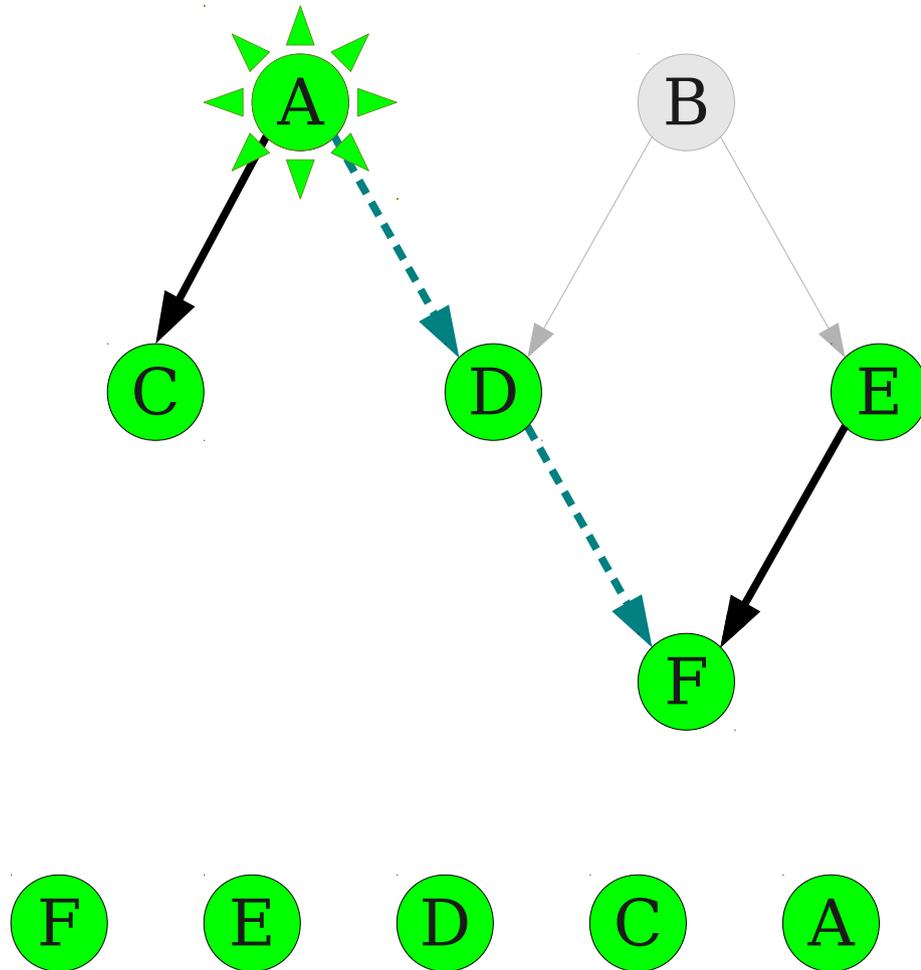
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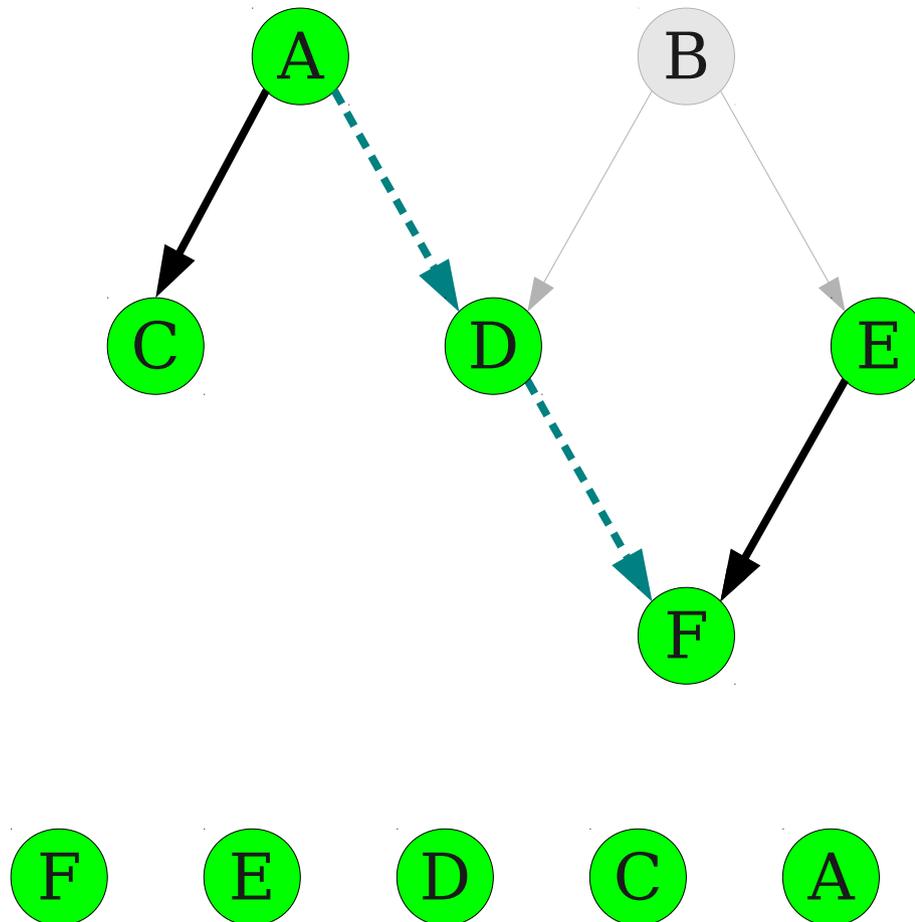
DFS on a DAG, Take III



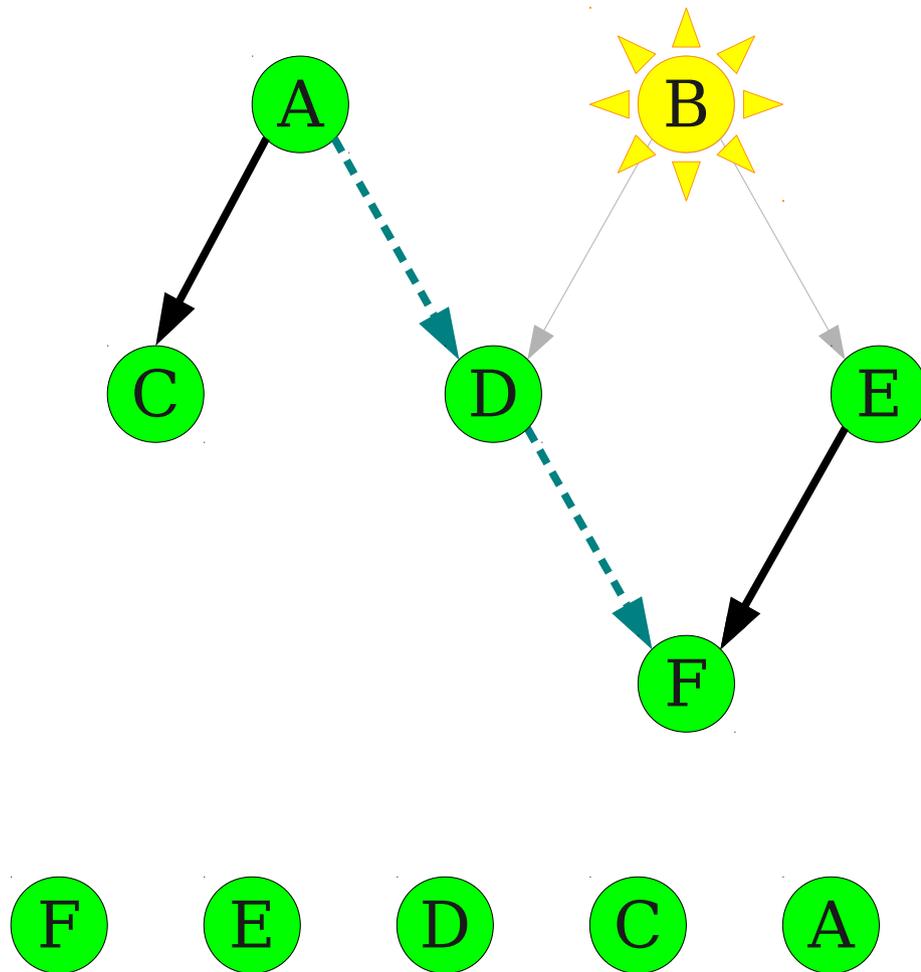
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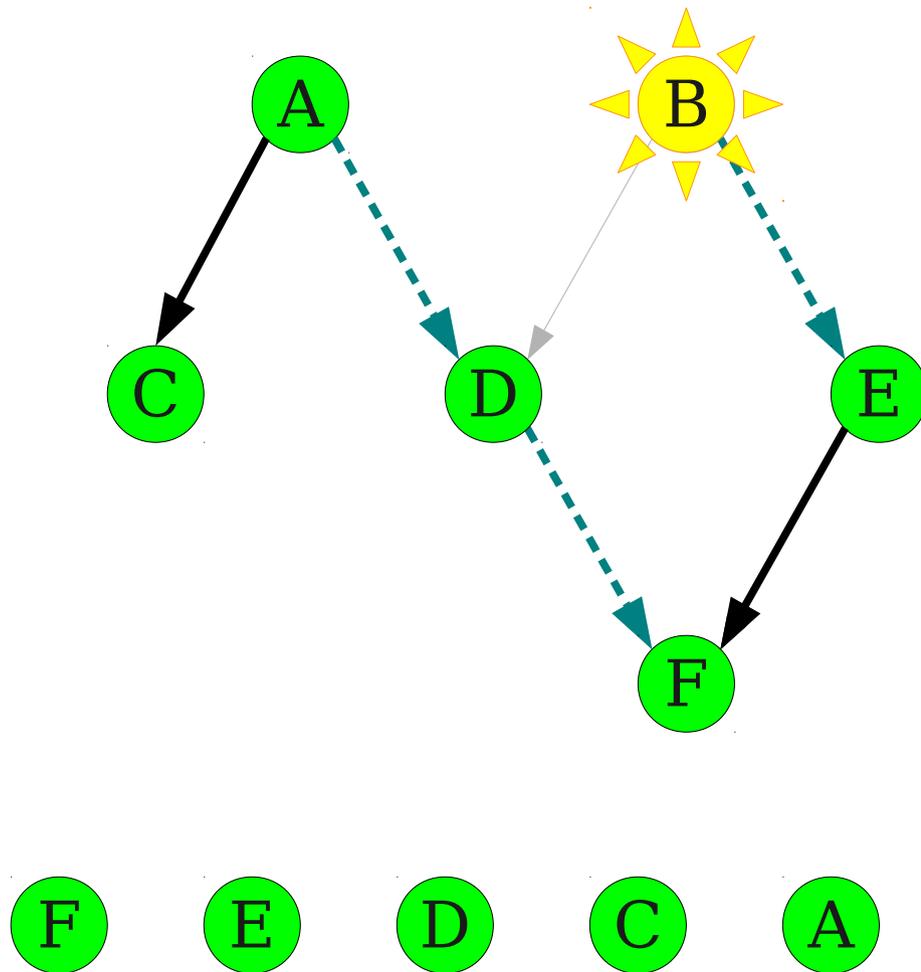
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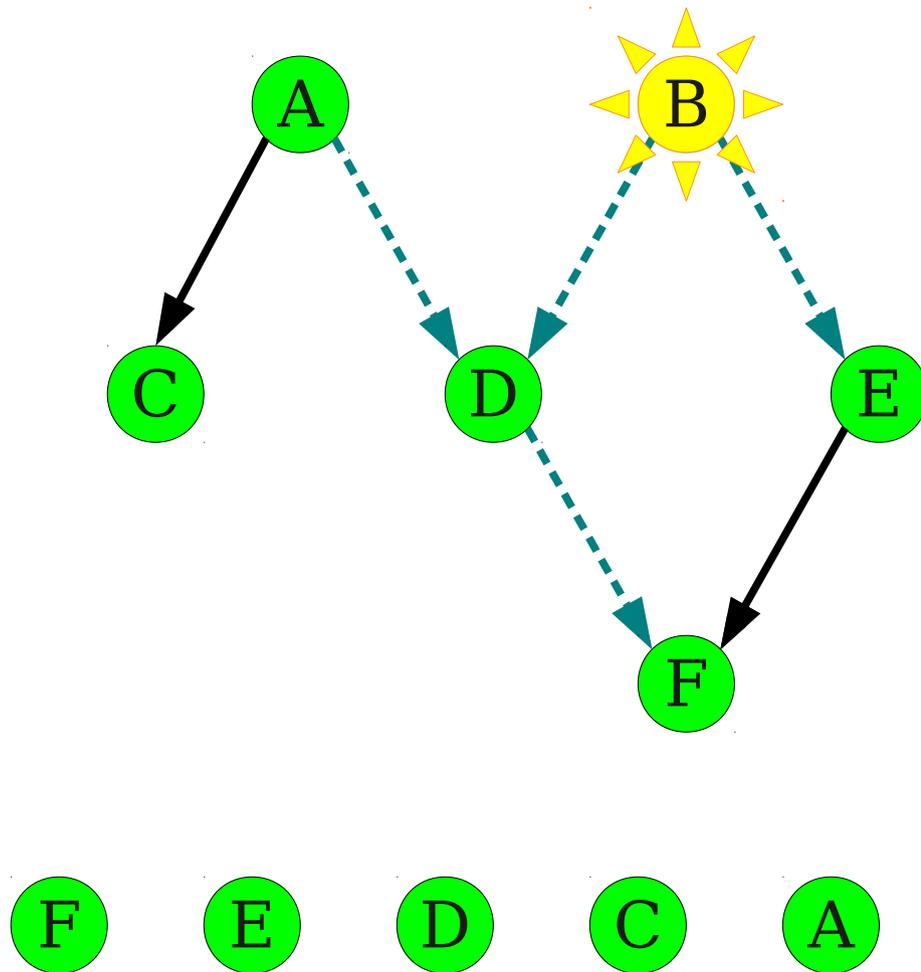
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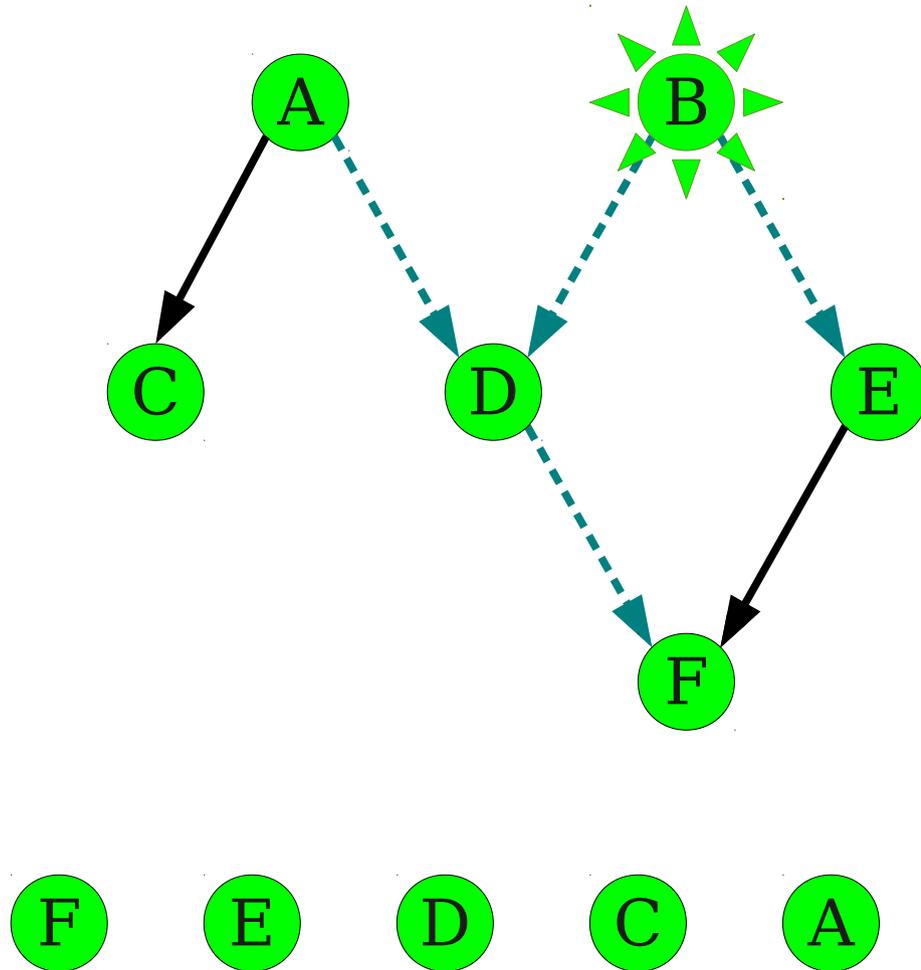
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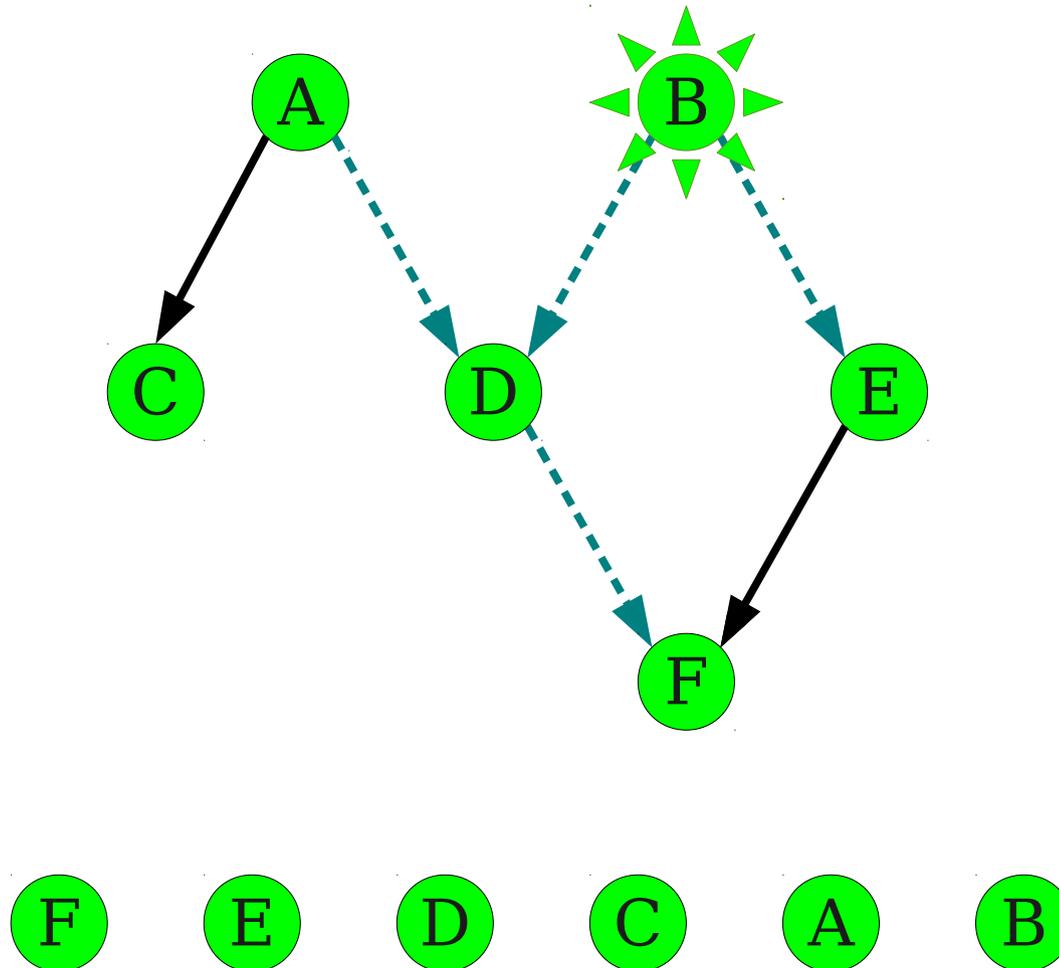
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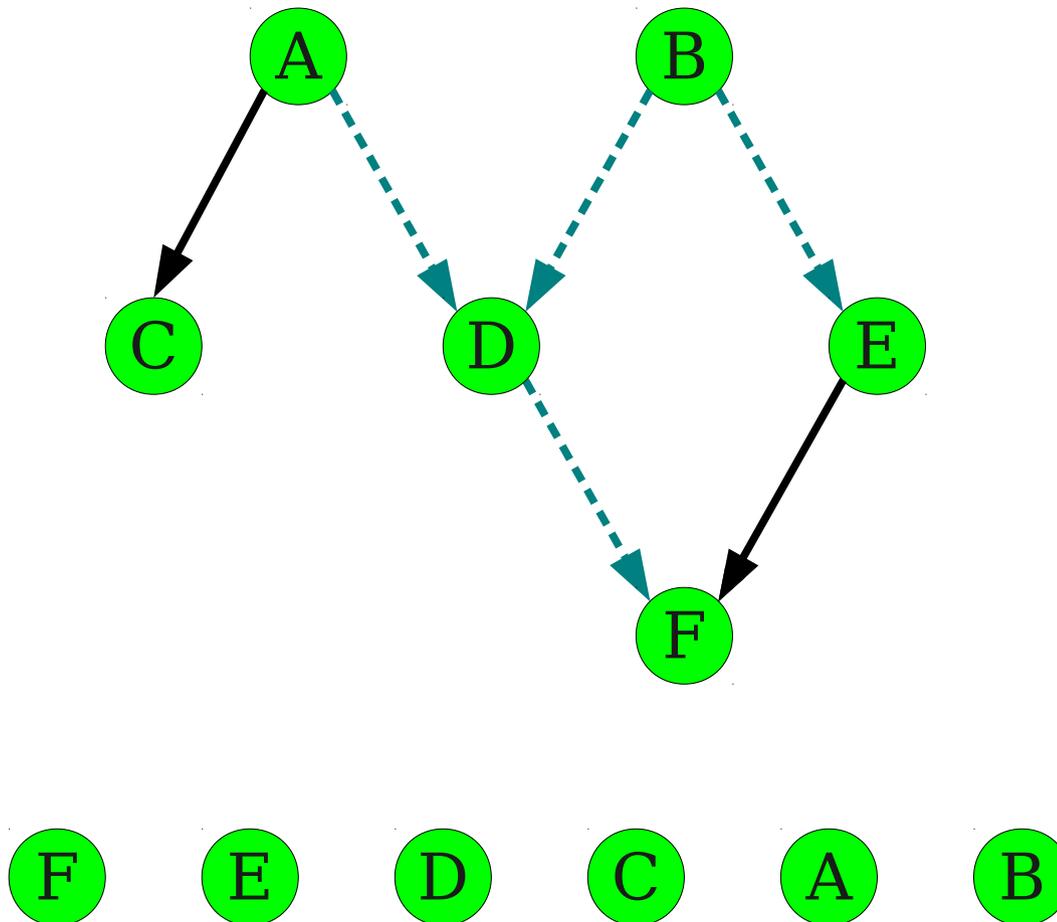
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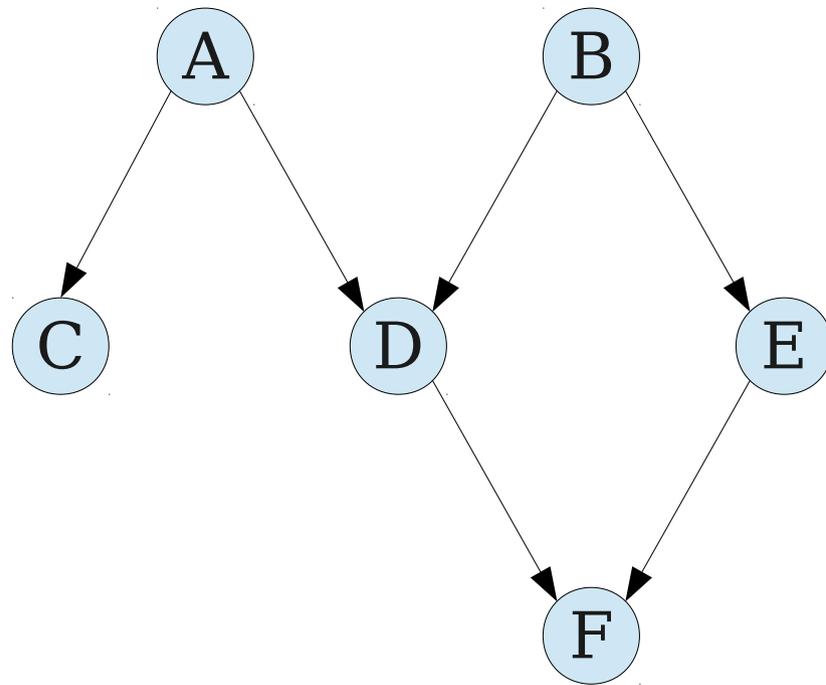
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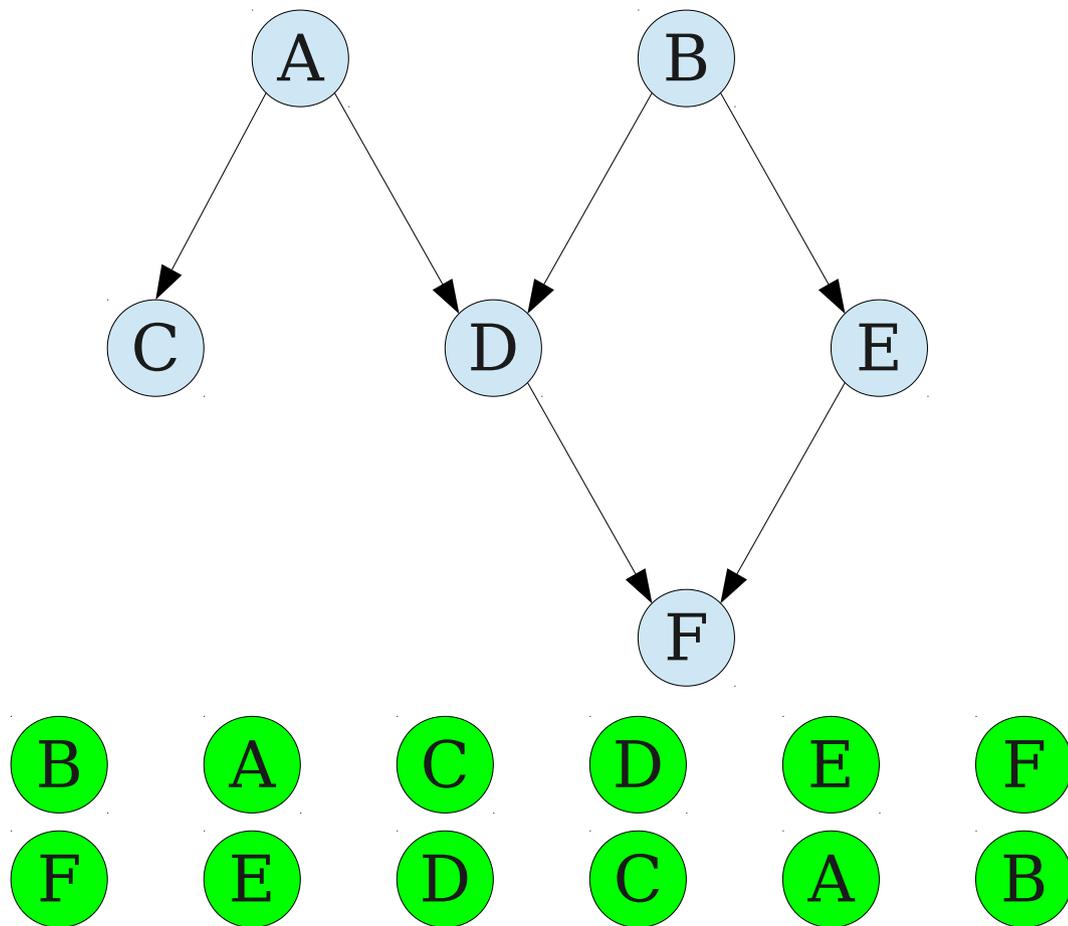
DFS on a DAG, Take III



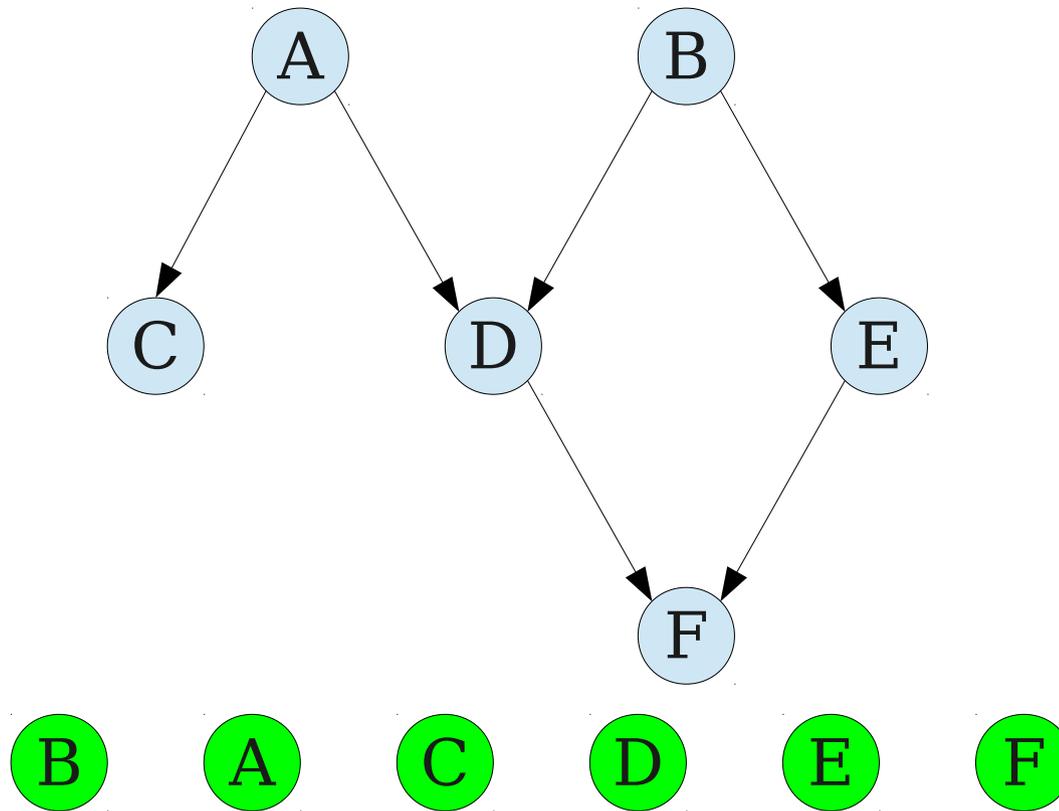
DFS on a DAG, Take III



DFS on a DAG, Take III



DFS on a DAG, Take III



```
procedure dfsTopoSort(DAG G):  
  for each node v in G:  
    color v gray  
  
  let result be an empty list.  
  for each node v in G:  
    if v is gray:  
      run DFS starting from v,  
      appending each node to result as  
      soon as it is colored green.  
  
  return reverse of result
```

Question 1: How do we know this actually produces a topological sort?

Question 2: How *efficiently* does this produce a topological sort?

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Question 2: How *efficiently* does this produce a topological sort?

Observation I

Lemma: Every node appears in the generated list exactly once.

Proof: Nodes are added to the generated list only when they turn green, which can happen at most once. Moreover, every node has DFS called on it at least once, either by a recursive call or when the top-level loop calls it. ■

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Since DFS is called on each node in G , either DFS(u) is called before DFS(v) or vice-versa.

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Otherwise, DFS(u) is called before DFS(v). When DFS(u) terminates, u and v are gray at this time, there is a path from u to v through gray nodes. Thus when DFS(u) terminates, u becomes green. Since the last step of DFS(u) is to color u green. Since the last step of DFS(u) means that v became green before u . ■

Take note of how this proof works. We'll see another one just like it later on.

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Question 1: How do we know this actually produces a topological sort?

Question 2: How *efficiently* does this produce a topological sort?

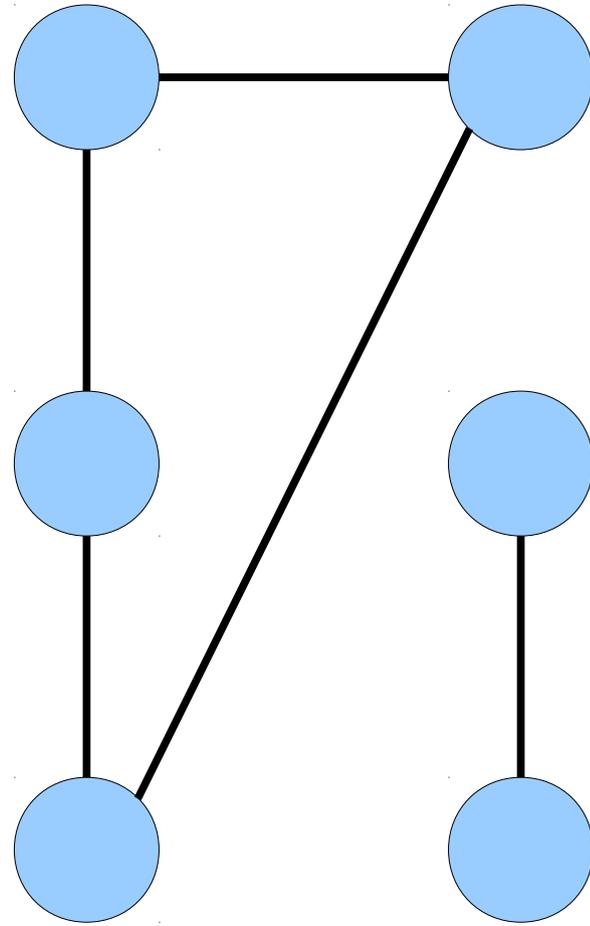
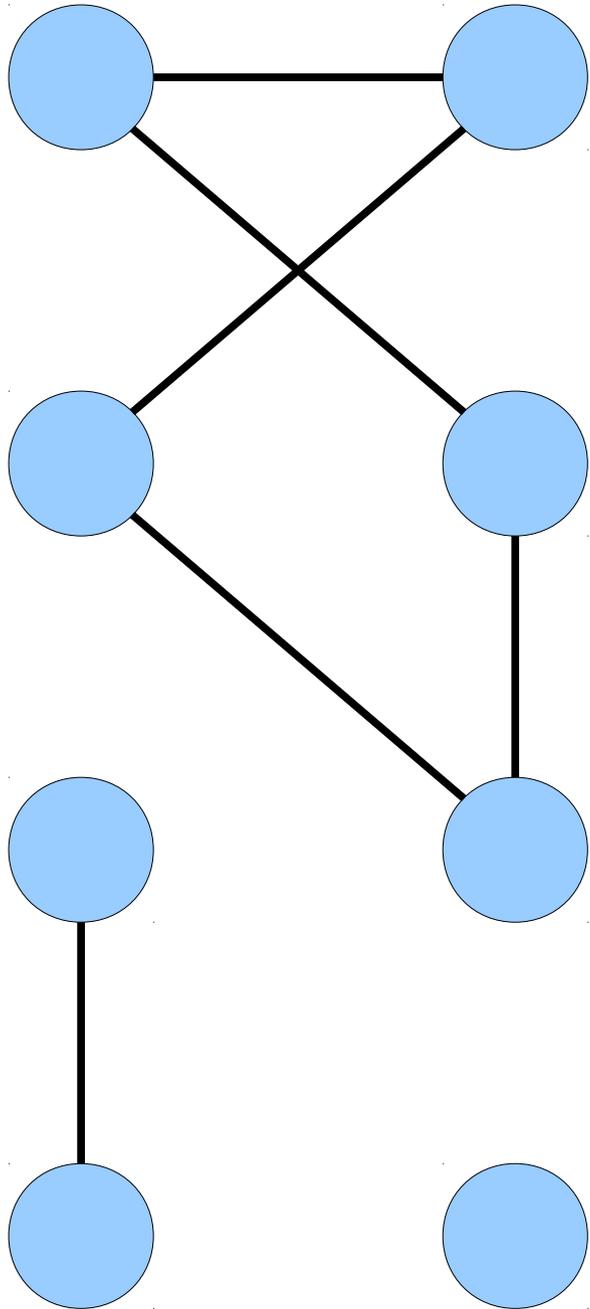
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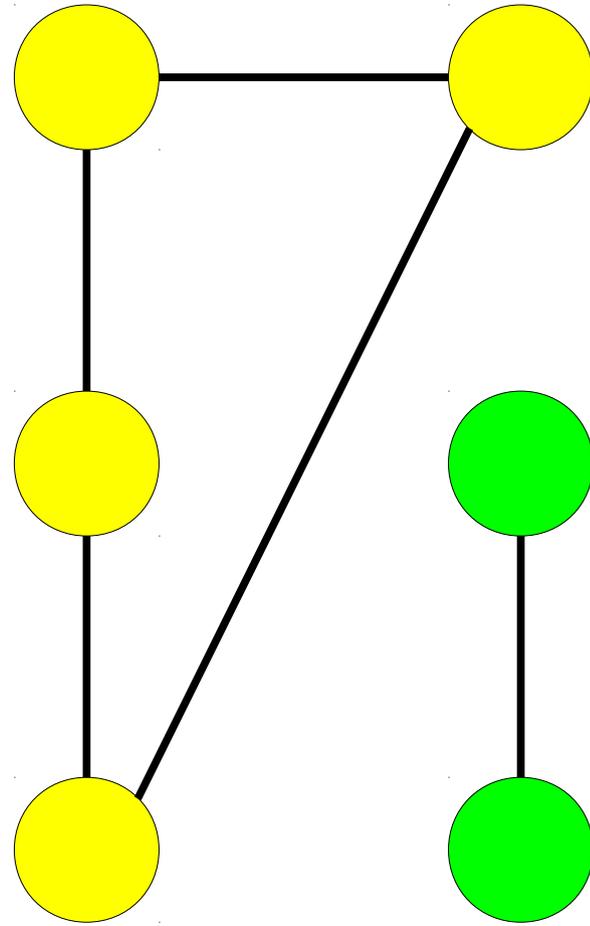
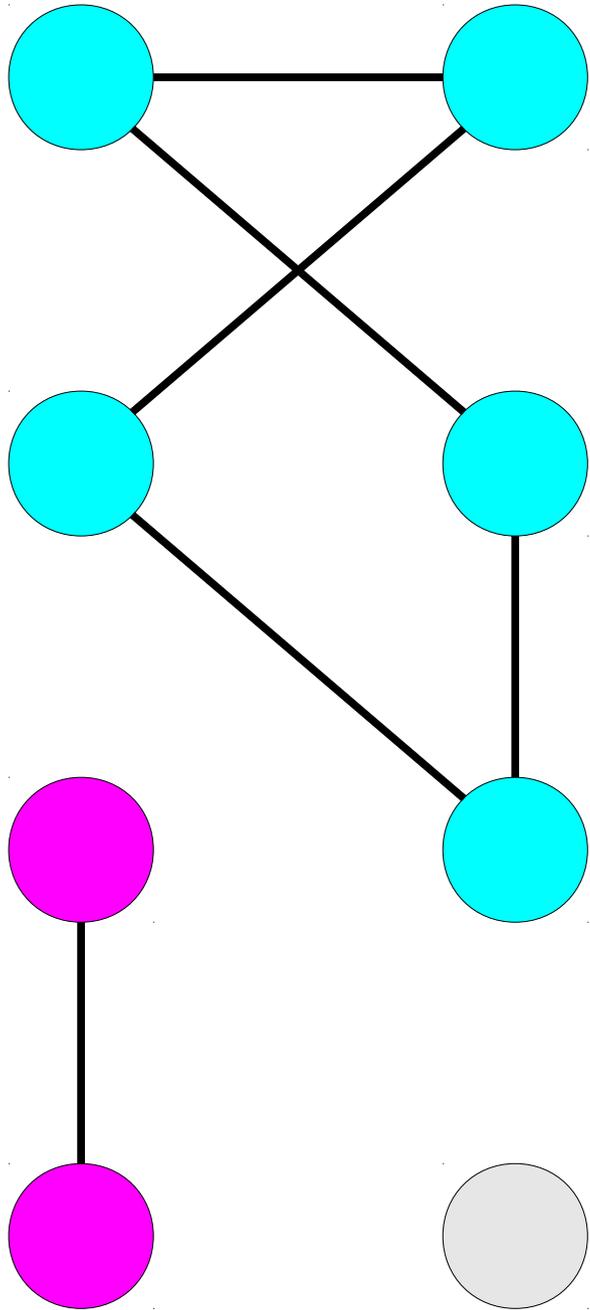
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DFS Topological Sort

- The time complexity of this algorithm is as follows:
 - Coloring all nodes gray can be done in $\Theta(n)$ time.
 - DFS will be invoked exactly once on each node, either by a top-level call in the loop or by a recursive call. This means each node and edge will be visited at most once by DFS. This step takes $\Theta(m + n)$ time.
 - The top-level loop visiting nodes requires $\Theta(n)$ work.
 - Reversing a list of n elements requires $\Theta(n)$ work.
 - Total work required: **$\Theta(m + n)$**
- Asymptotically the same as our previous algorithm, but a lot easier to code up!

Connected Components





Connected Components

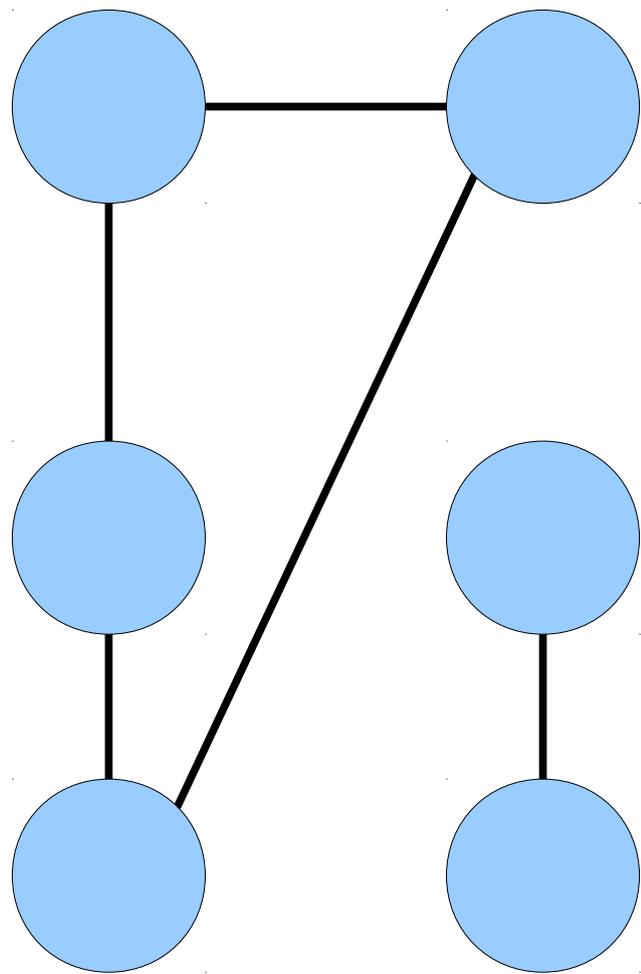
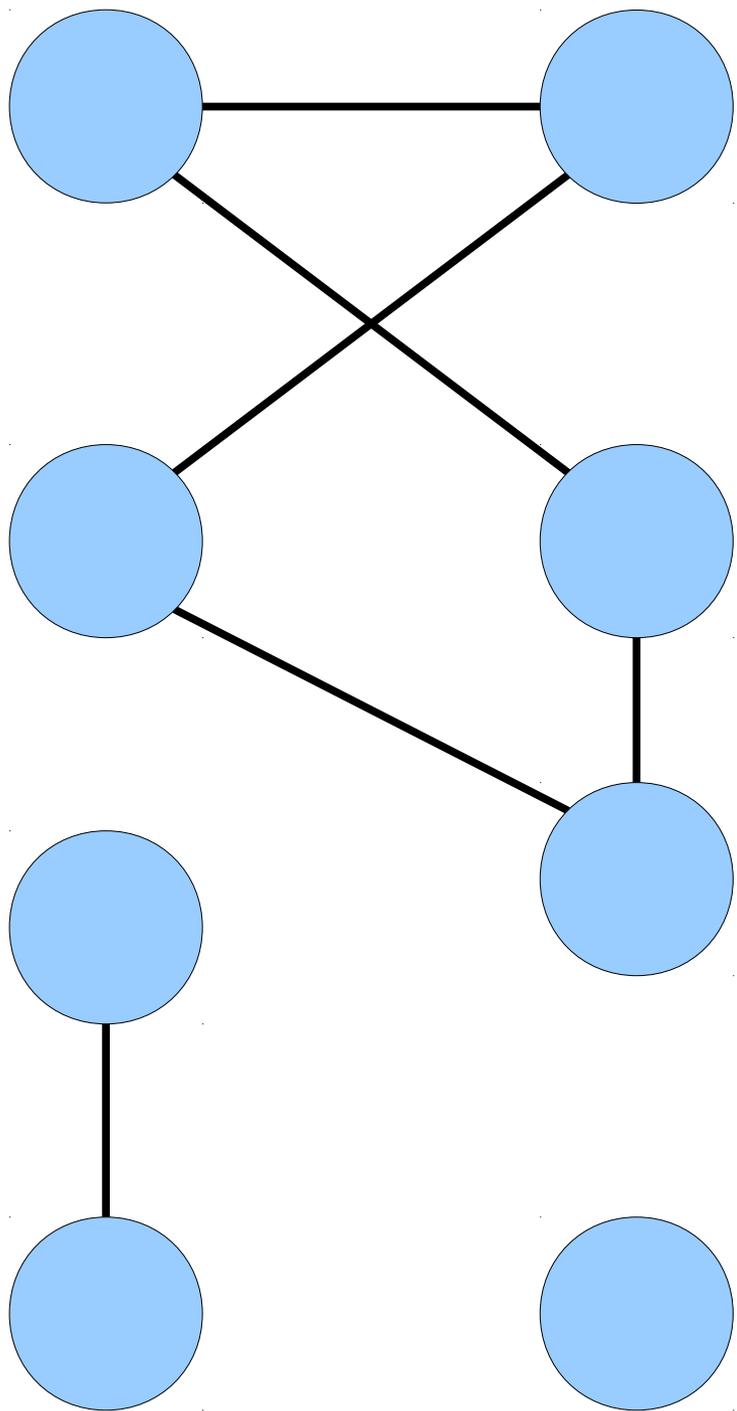
- Let $G = (V, E)$ be an undirected graph.
- Two nodes $u, v \in V$ are called **connected** iff there is a path from u to v .
- A **connected component** of G is a set $C \subseteq V$ with the following properties:
 - C is nonempty.
 - For any $u, v \in C$: u and v are connected.
 - For any $u \in C, v \in V - C$: u and v are not connected.

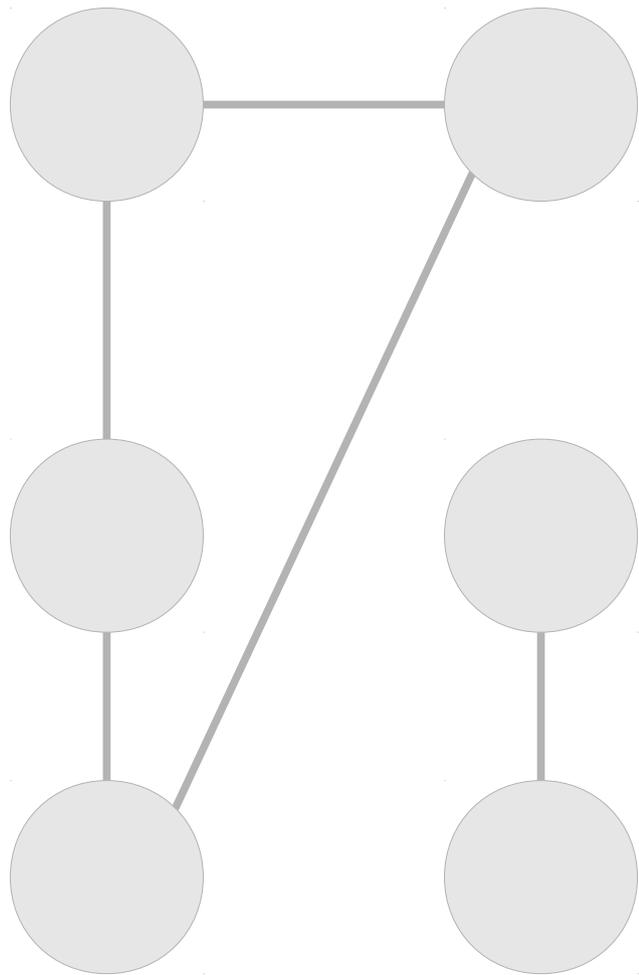
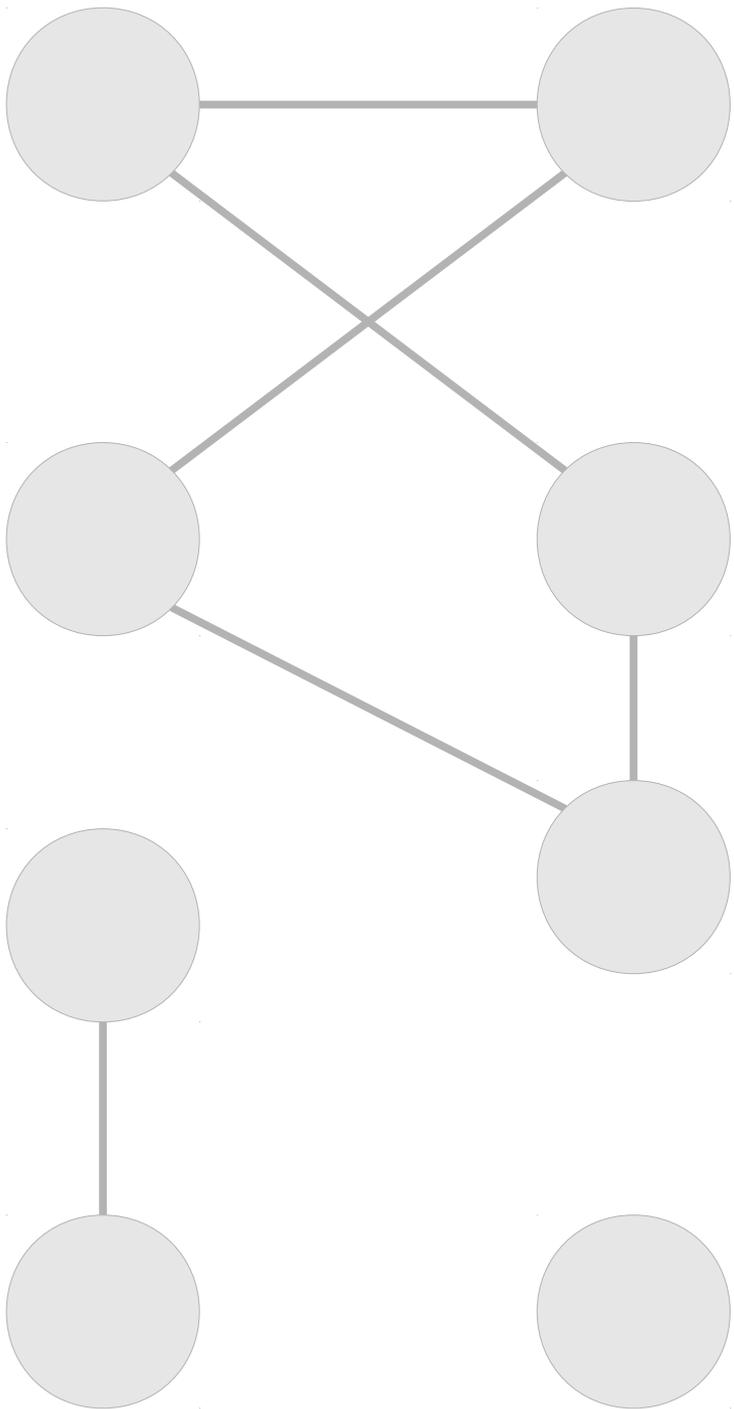
Properties of Connected Components

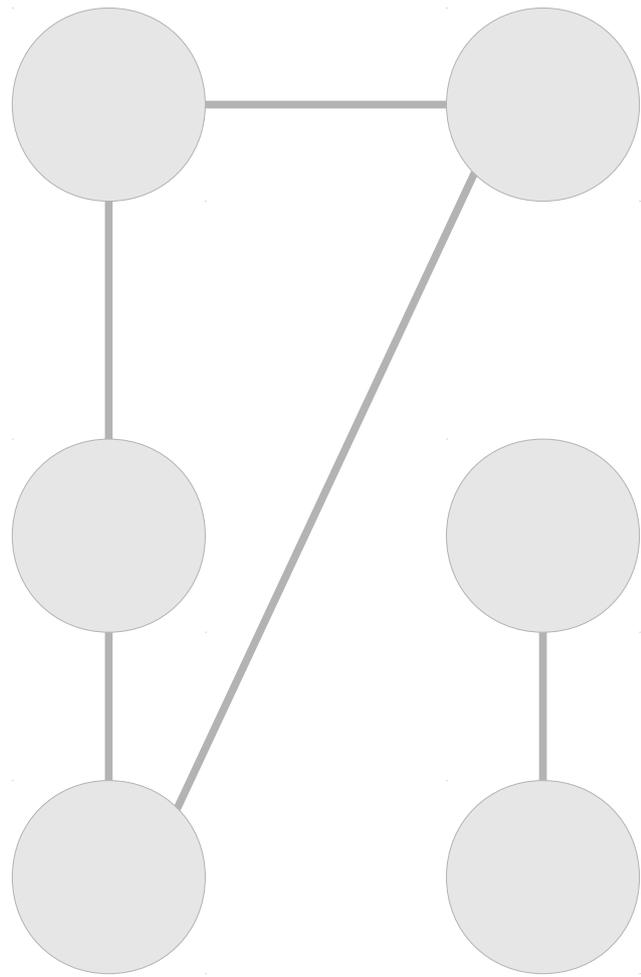
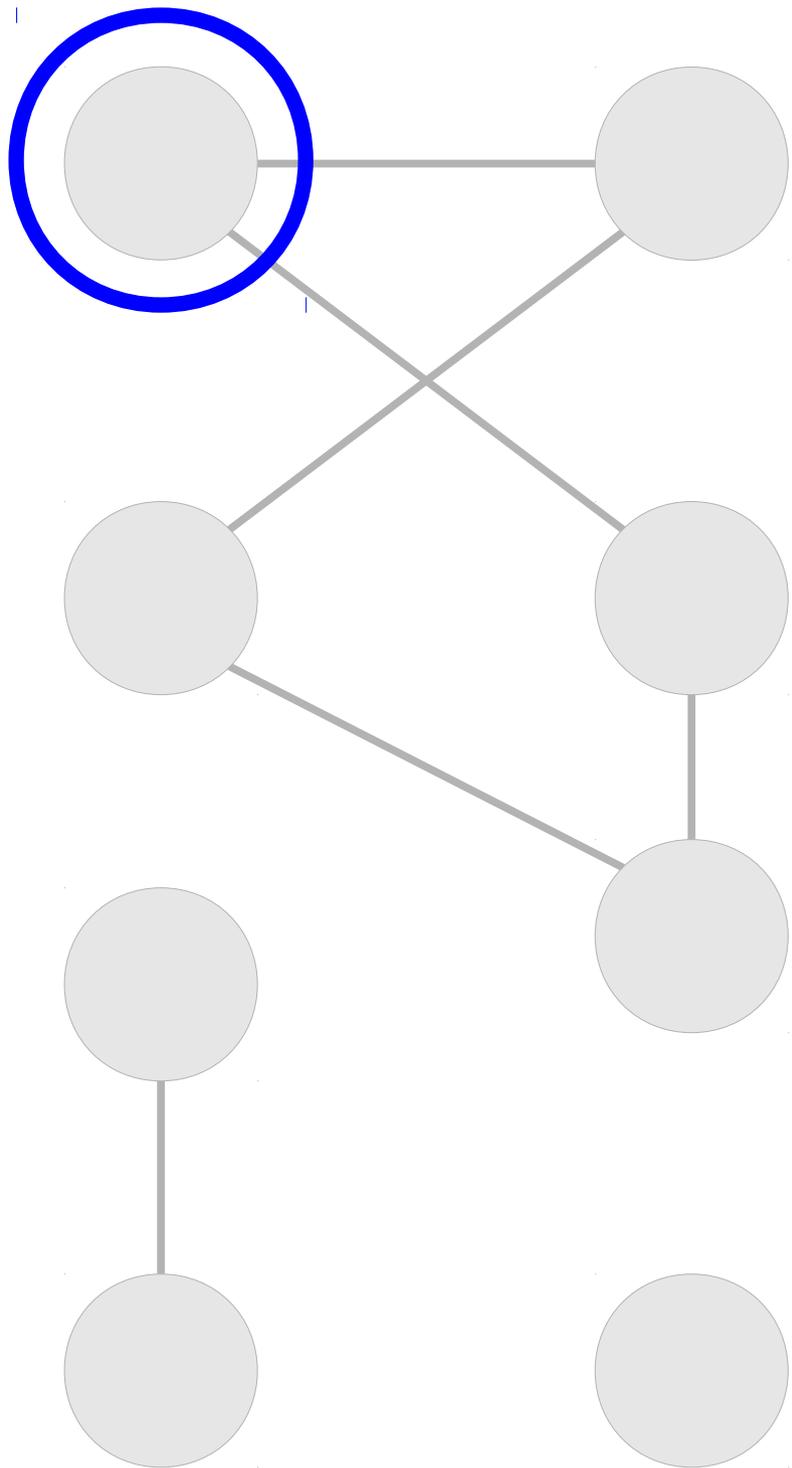
- All of the following are true; it's an interesting exercise to prove them:
 - Any two connected components C_1 and C_2 are either equal or disjoint.
 - Every node in a graph belongs to exactly one connected component.
 - The connected components of a graph form a partition of the nodes of the graph.

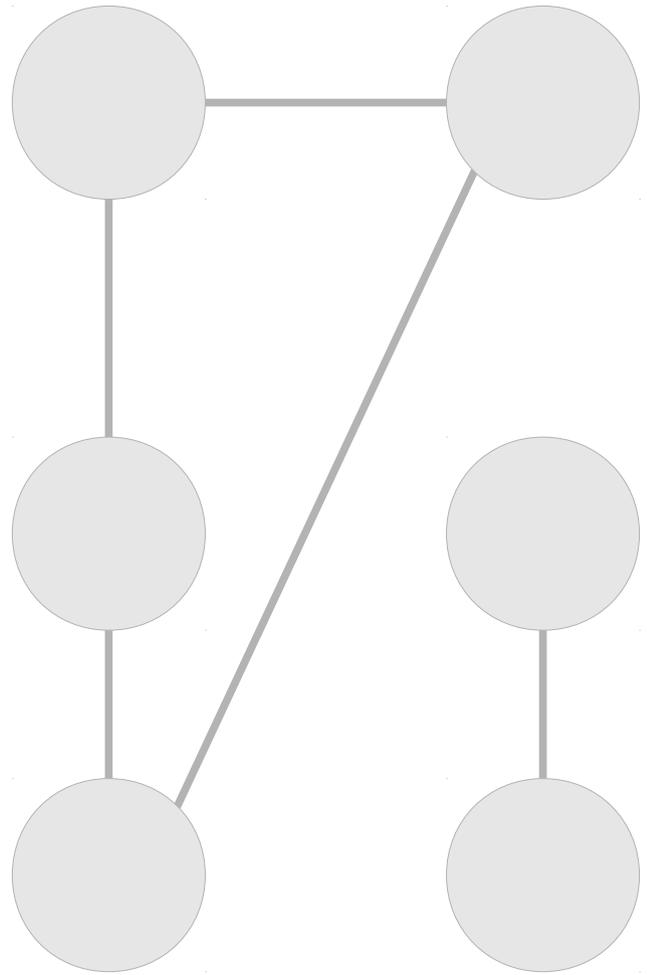
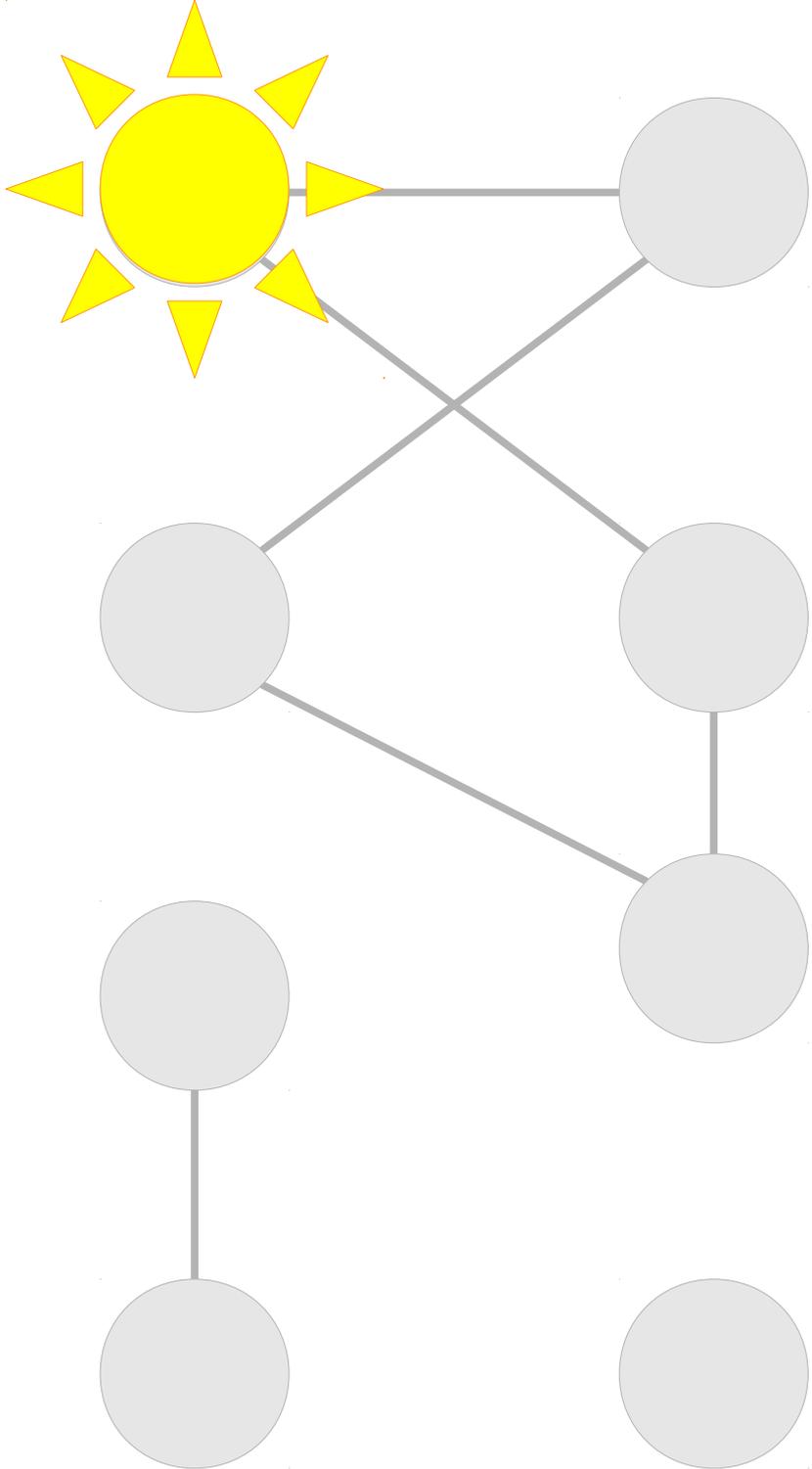
Finding Connected Components

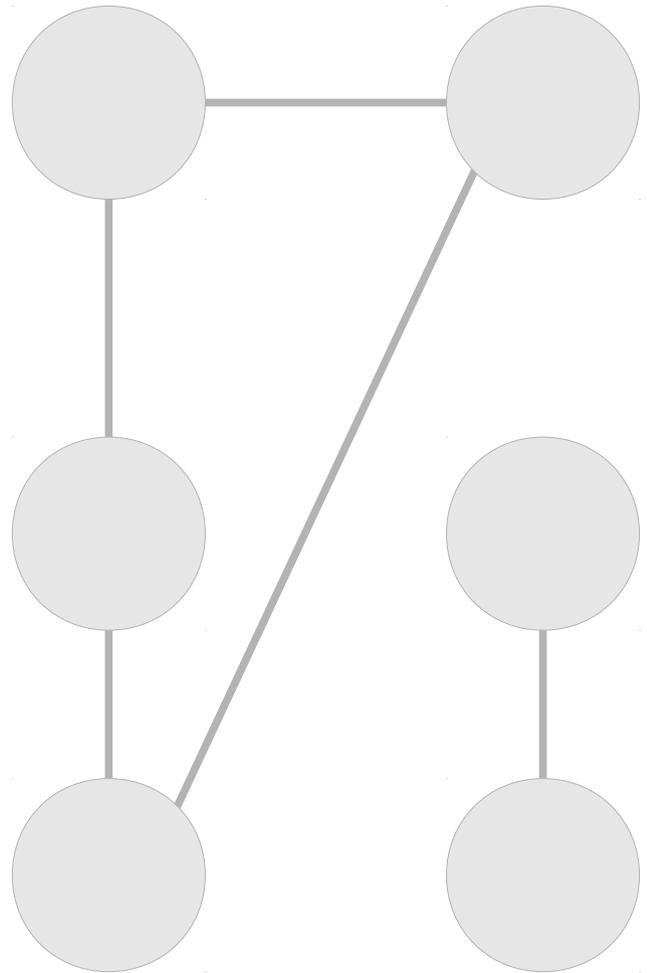
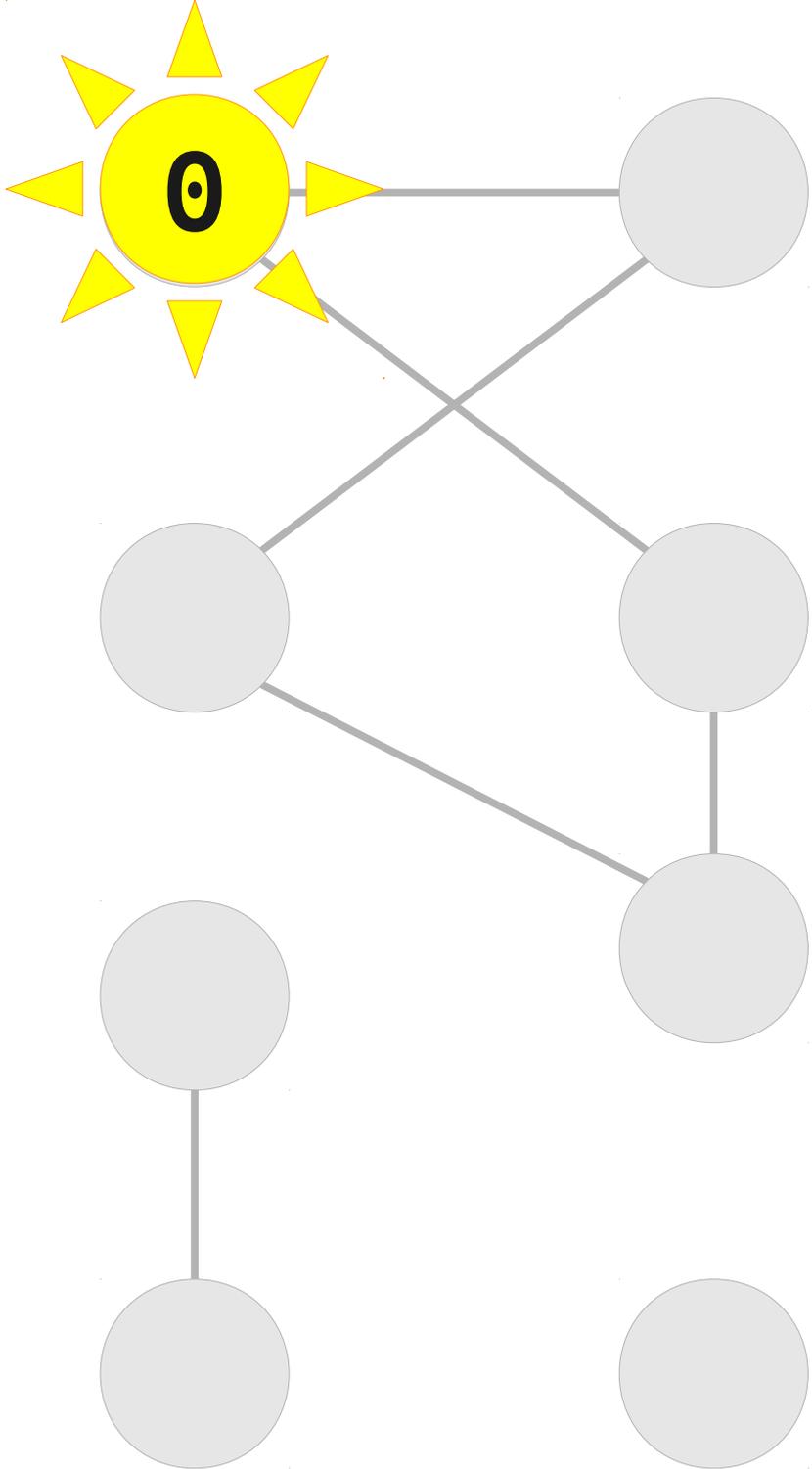
- Recall: When $\text{DFS}(u)$ terminates, u and all gray nodes reachable from u by gray paths will have turned green and no other nodes will have been colored green.
- Suppose that we call DFS in a connected component where we have previously not called DFS before.
- All nodes in the connected component are reachable from one another, and all nodes are gray.
- Therefore, DFS terminates having colored all nodes in that connected component green and coloring no other nodes green.

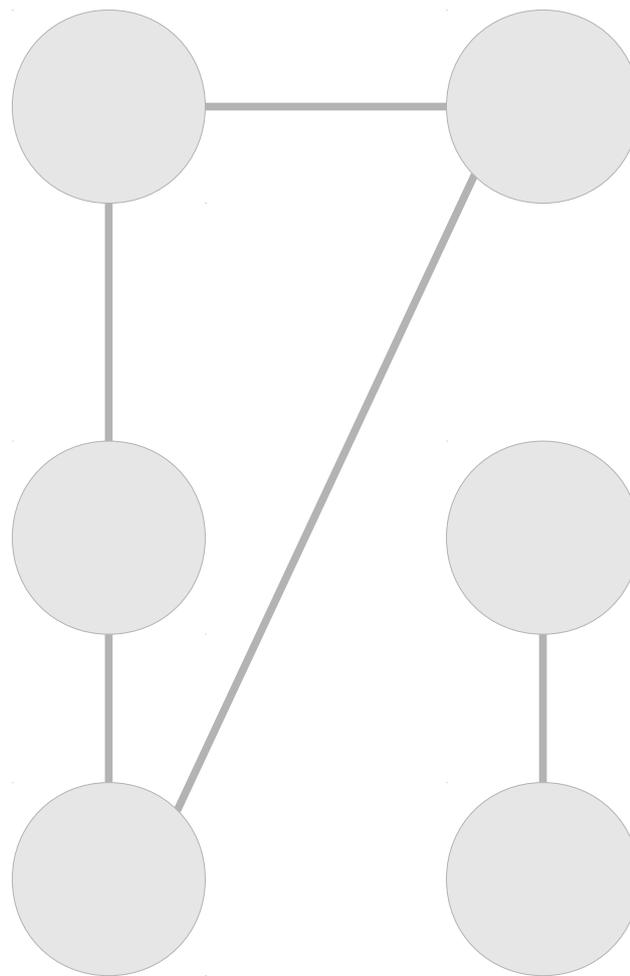
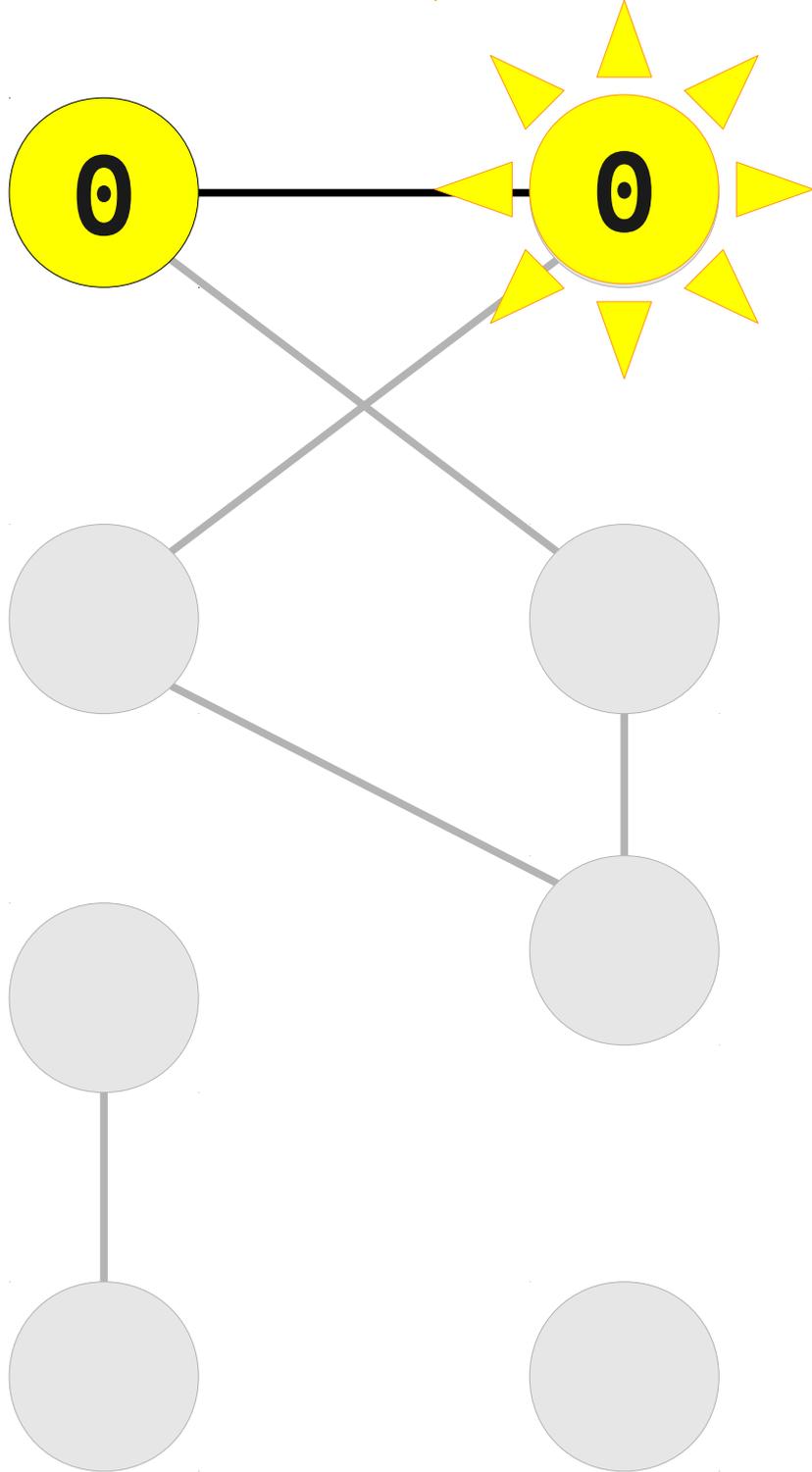


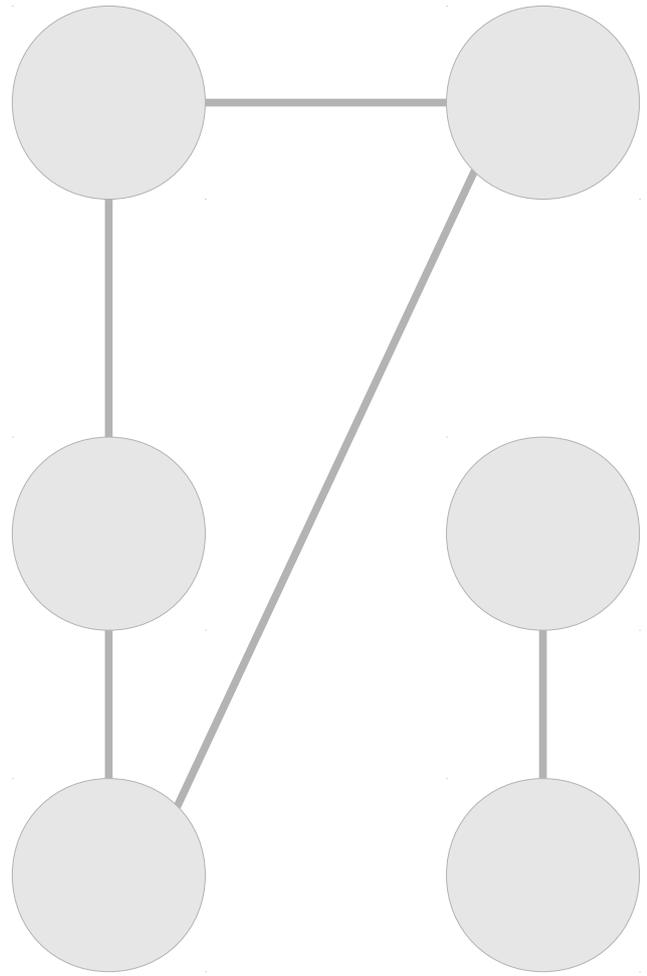
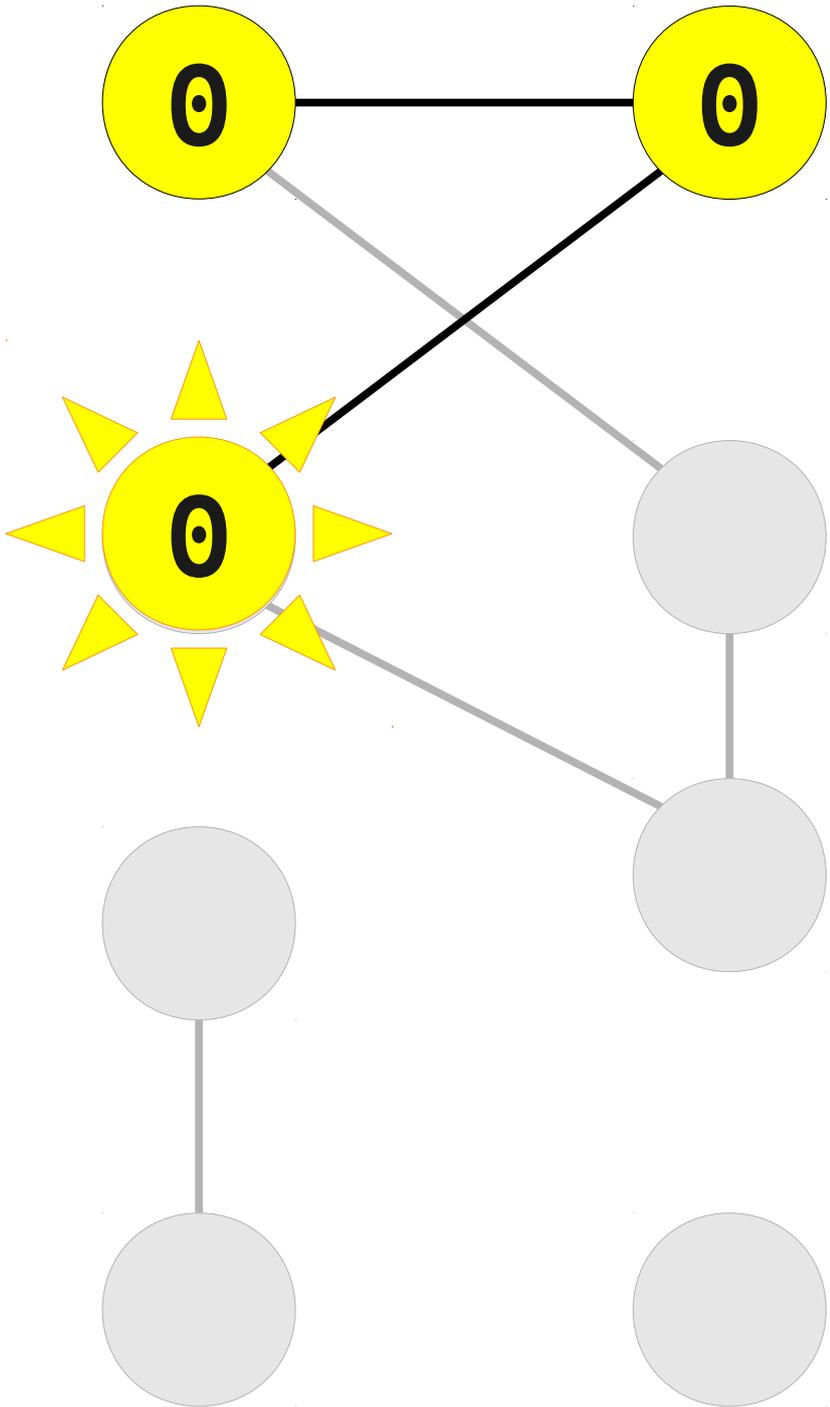


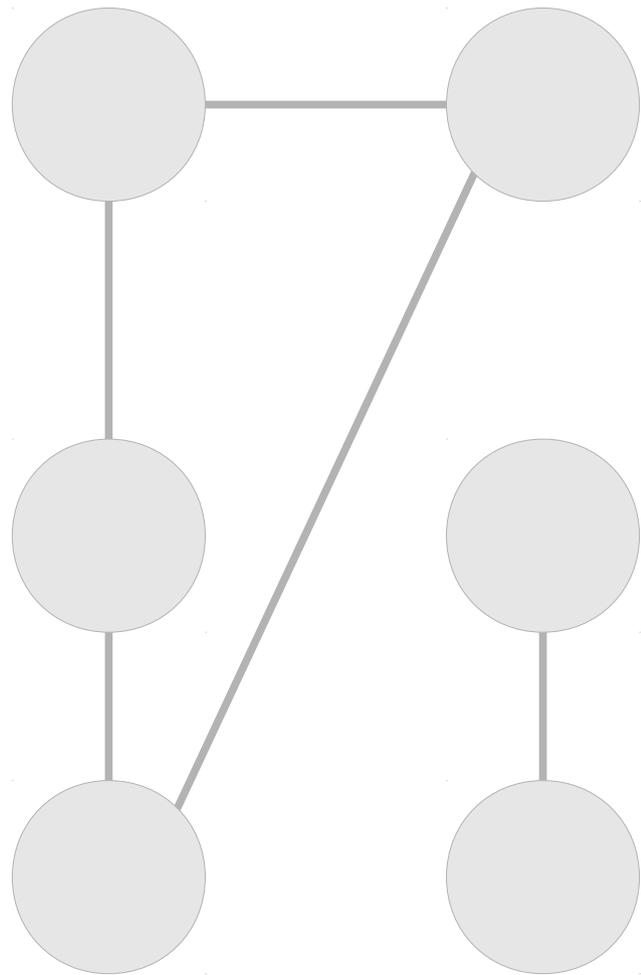
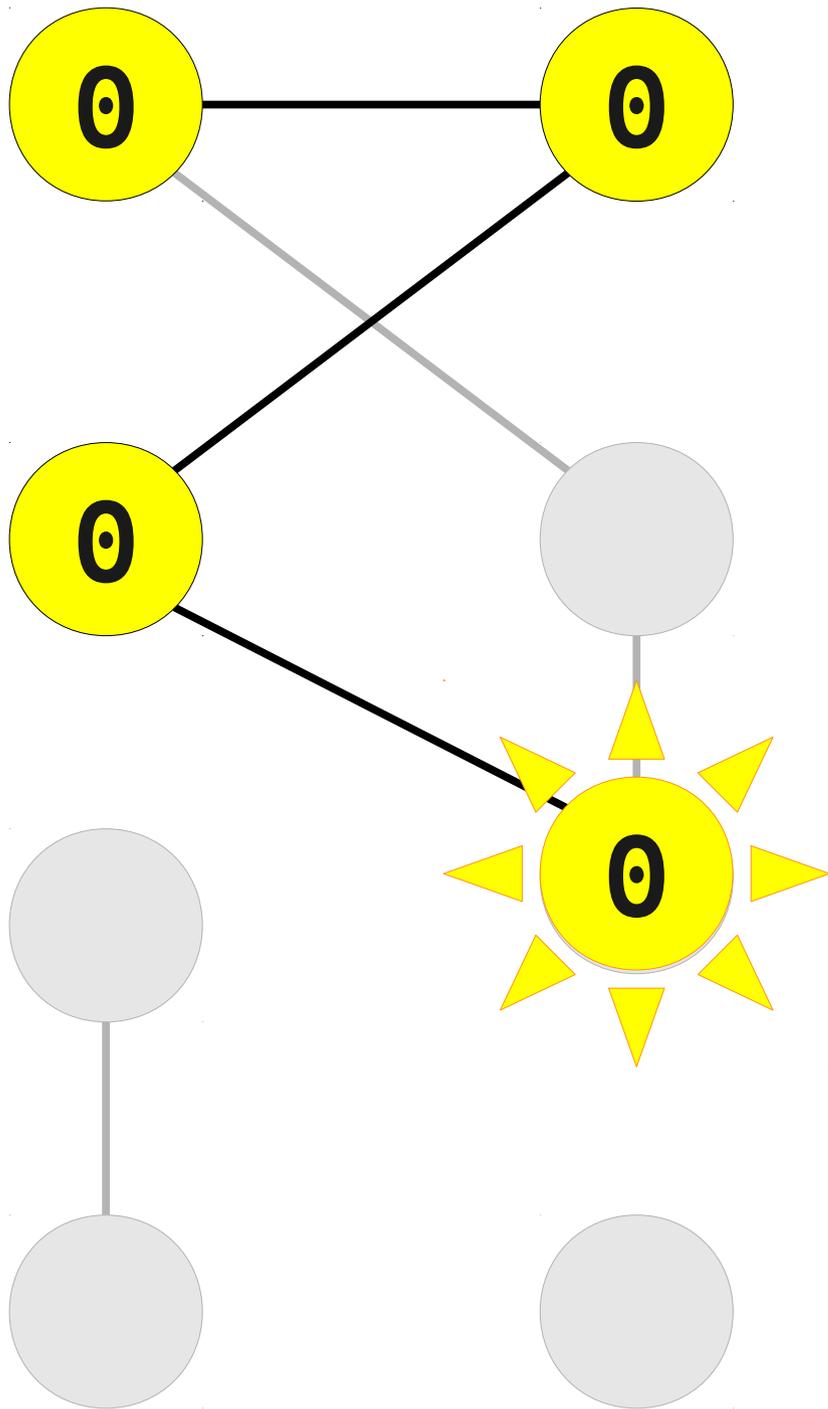


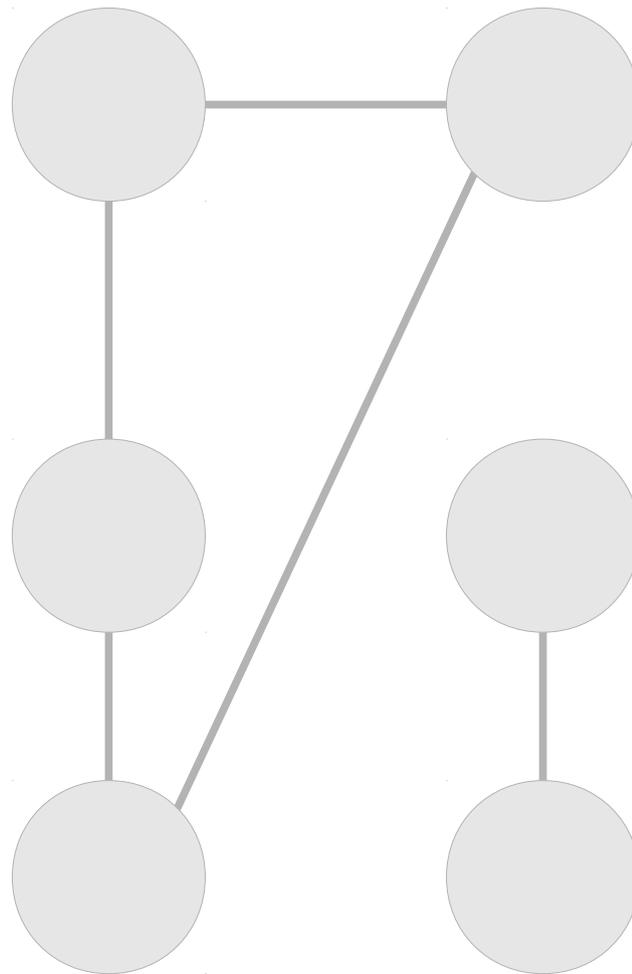
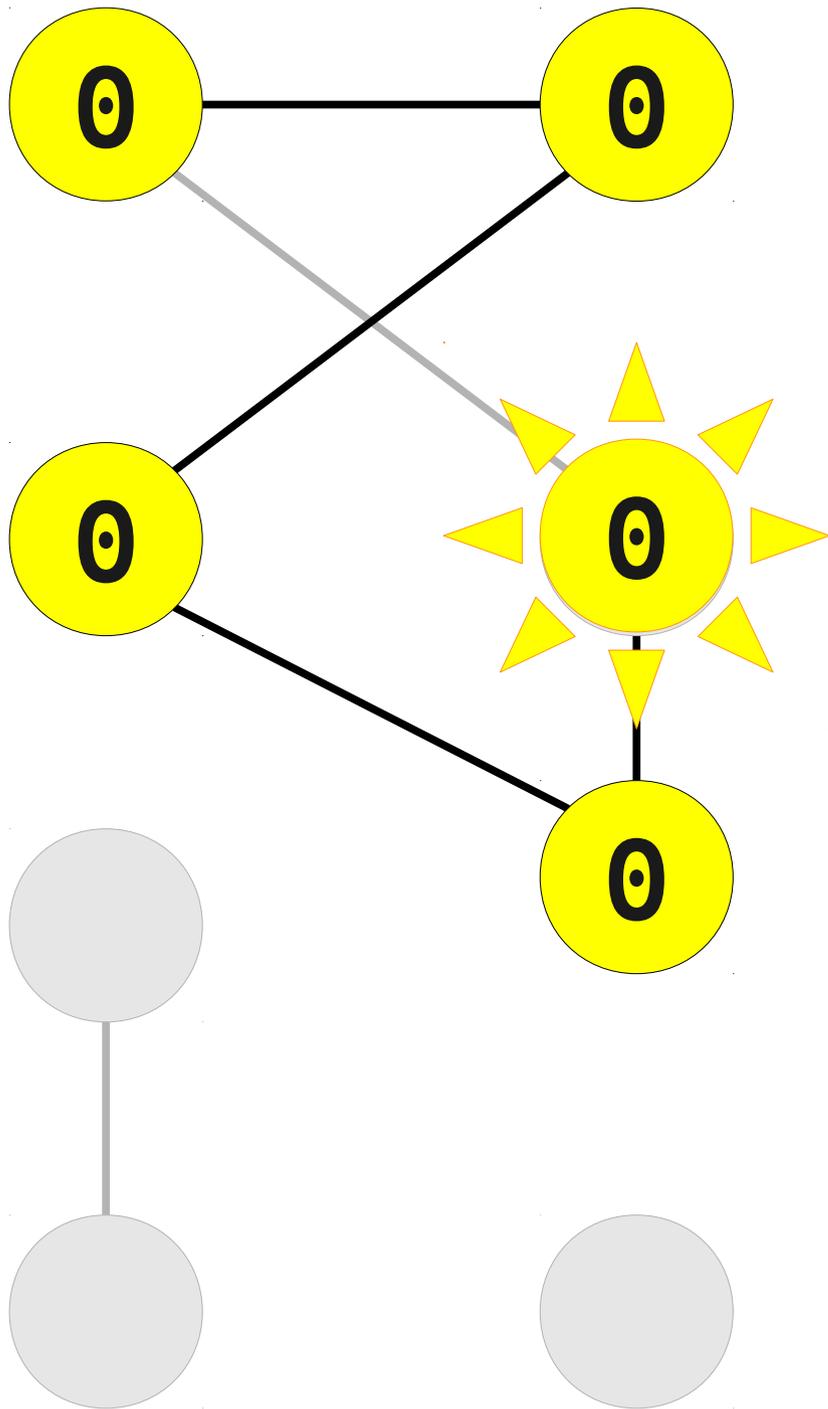


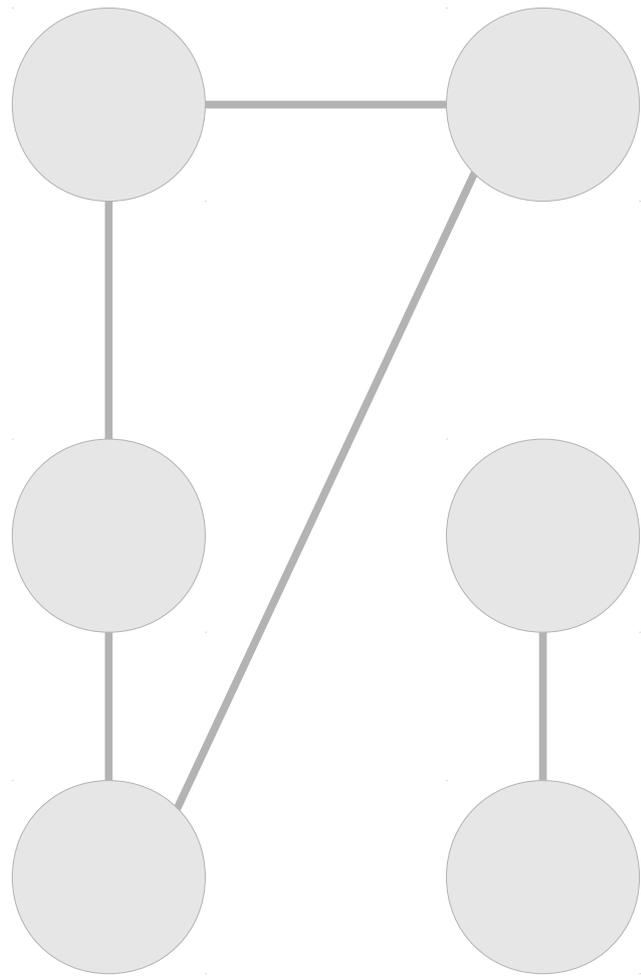
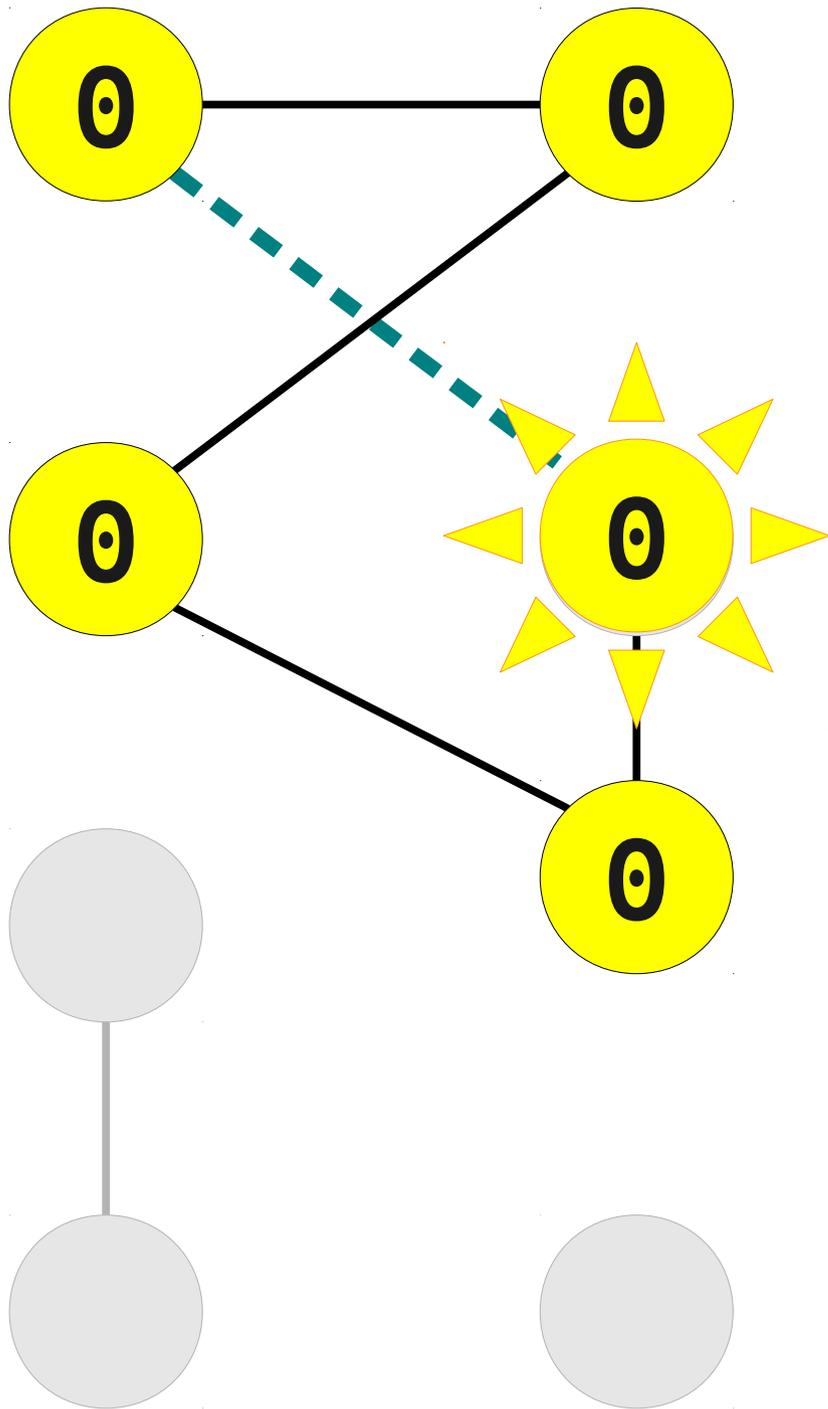


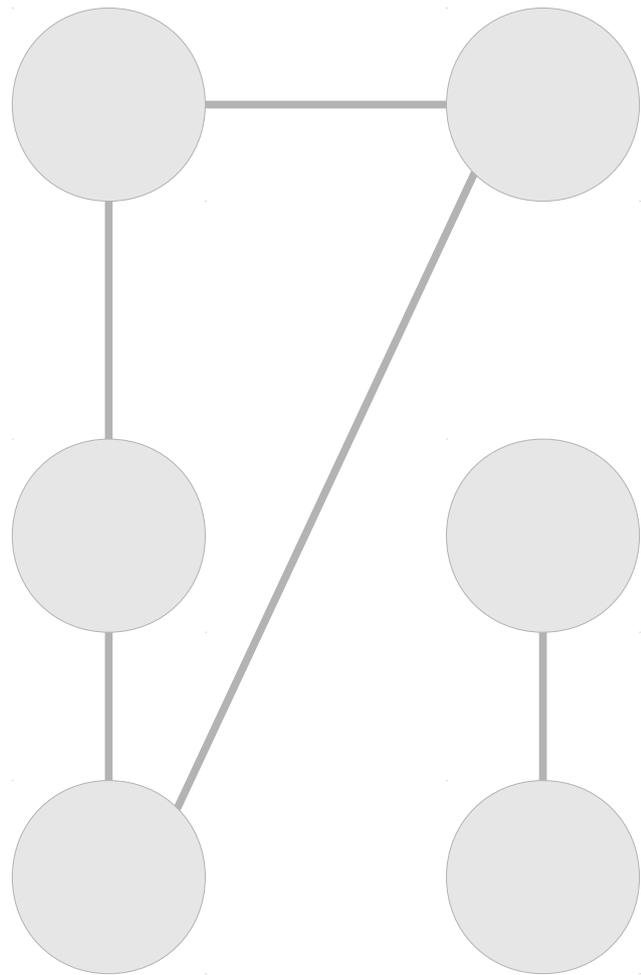
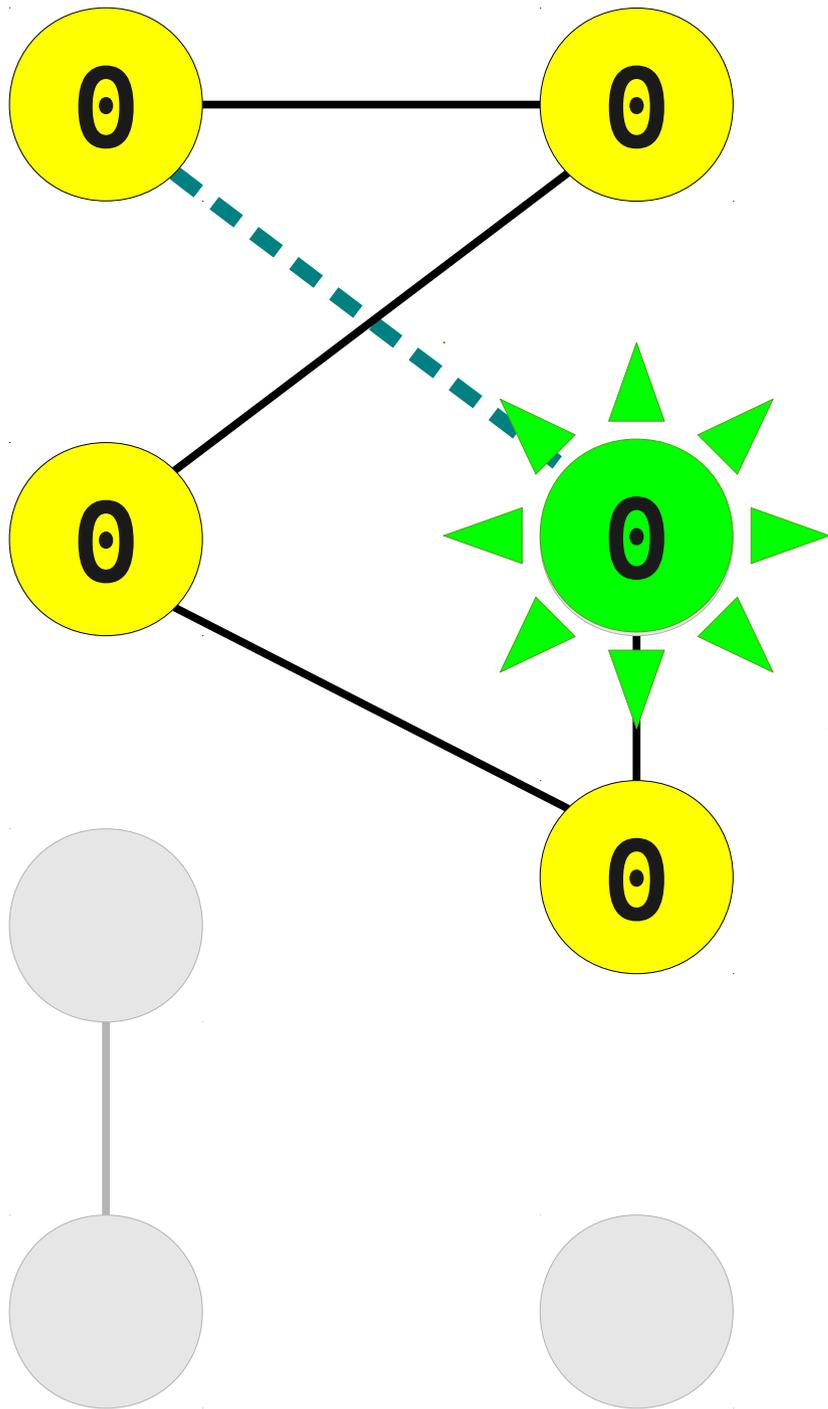


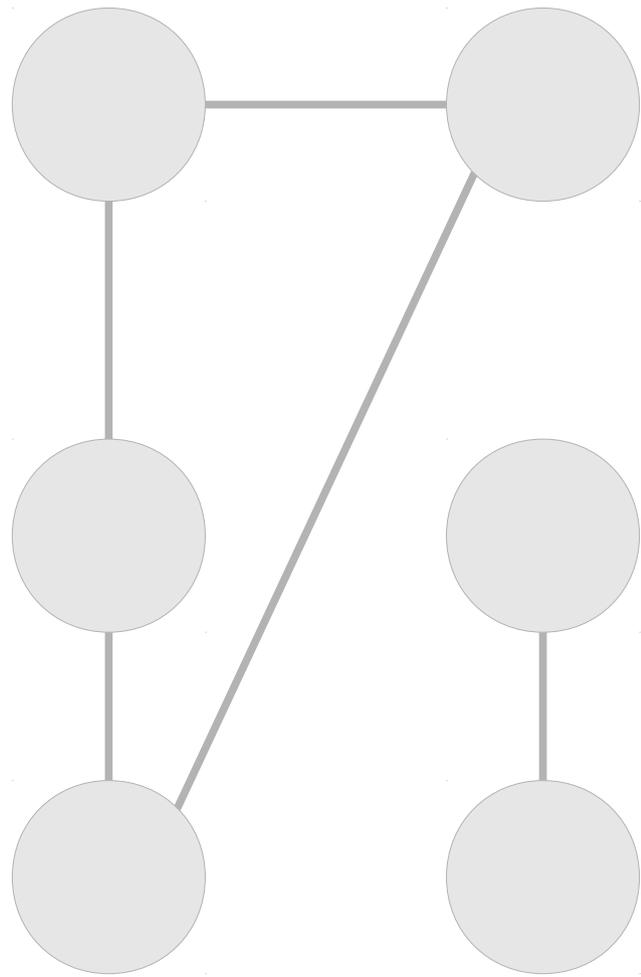
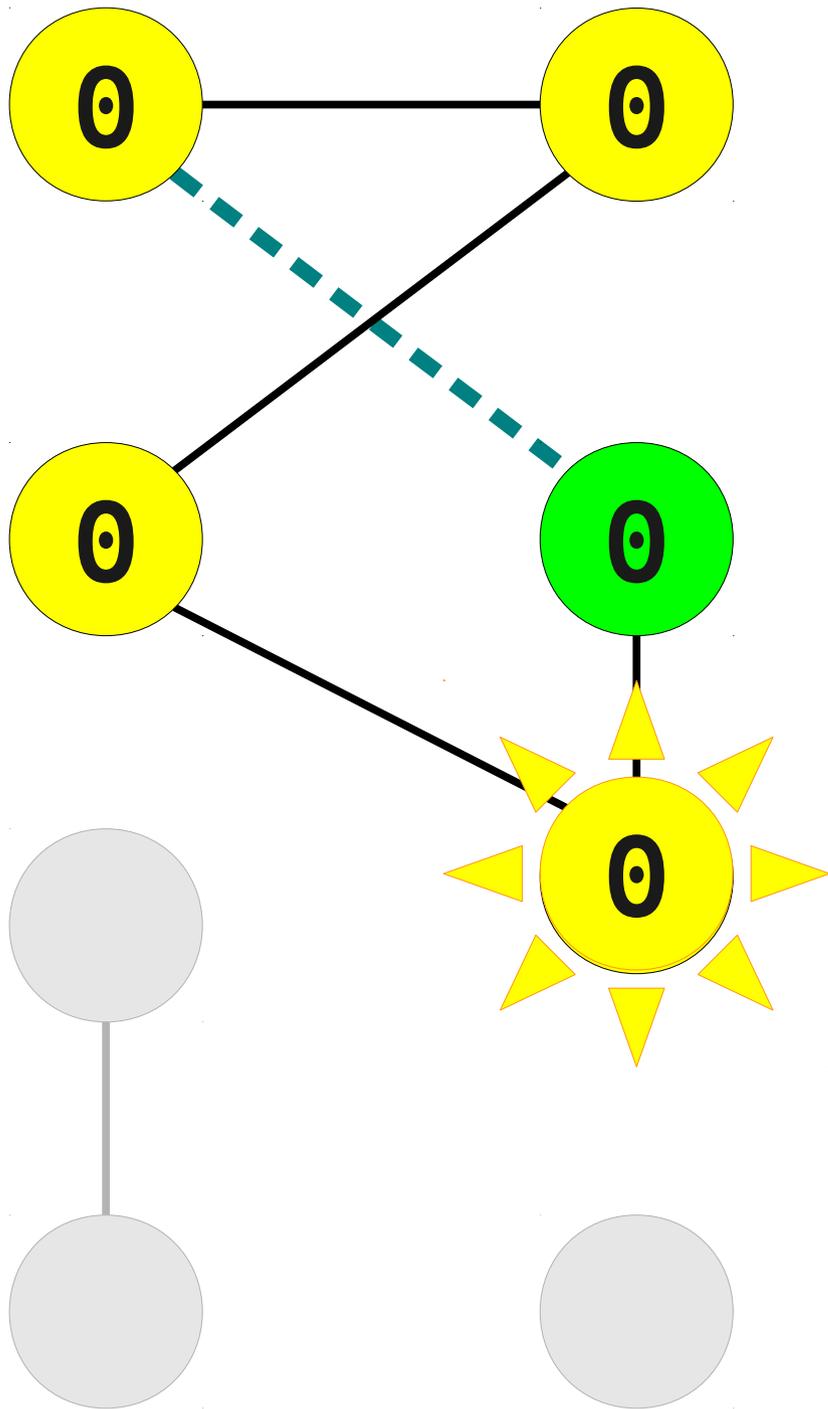


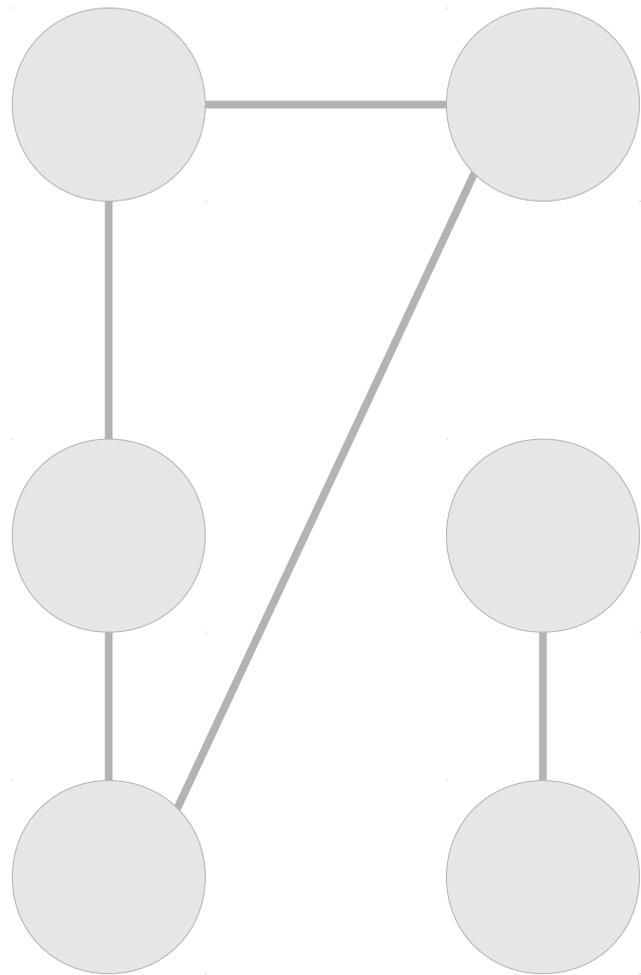
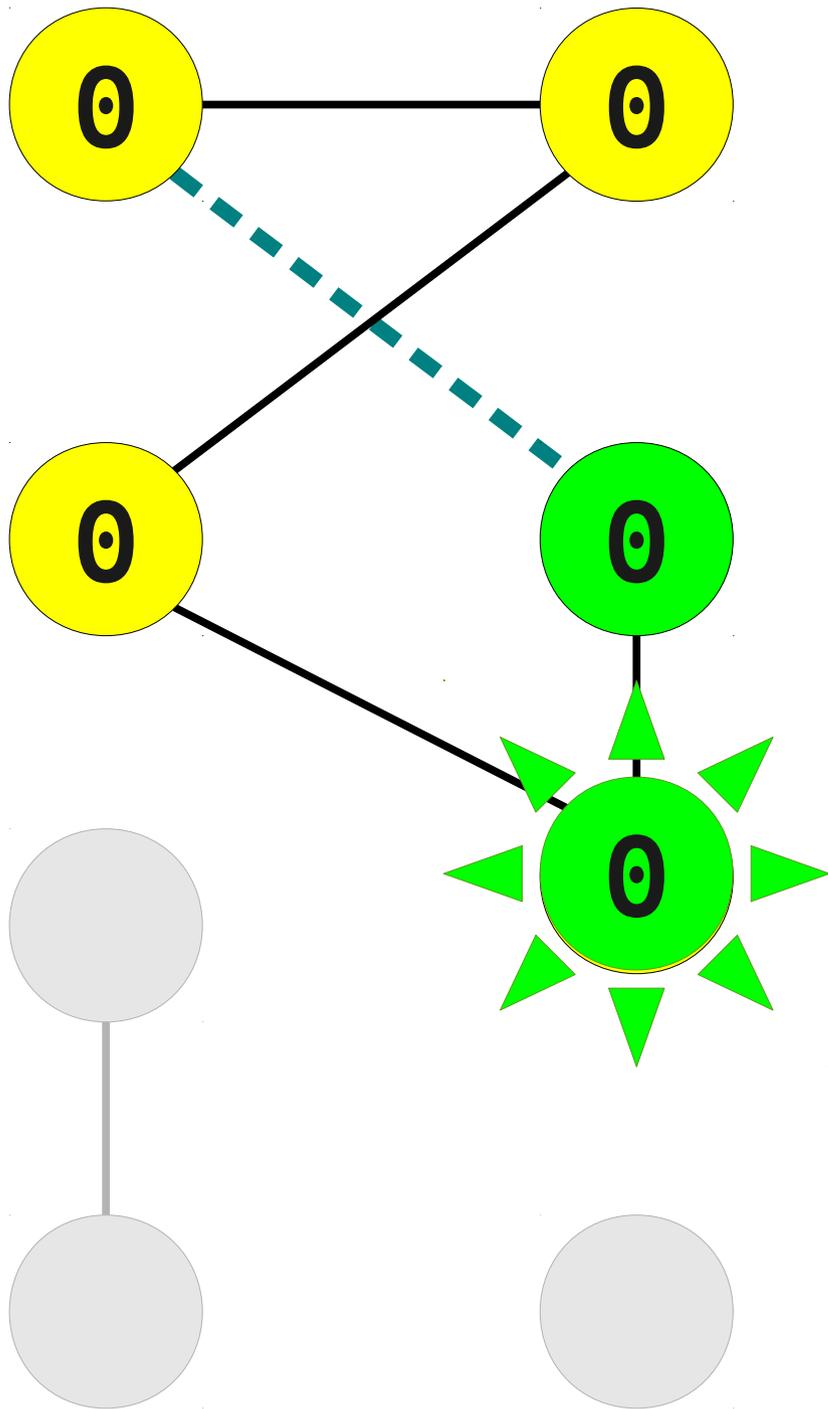


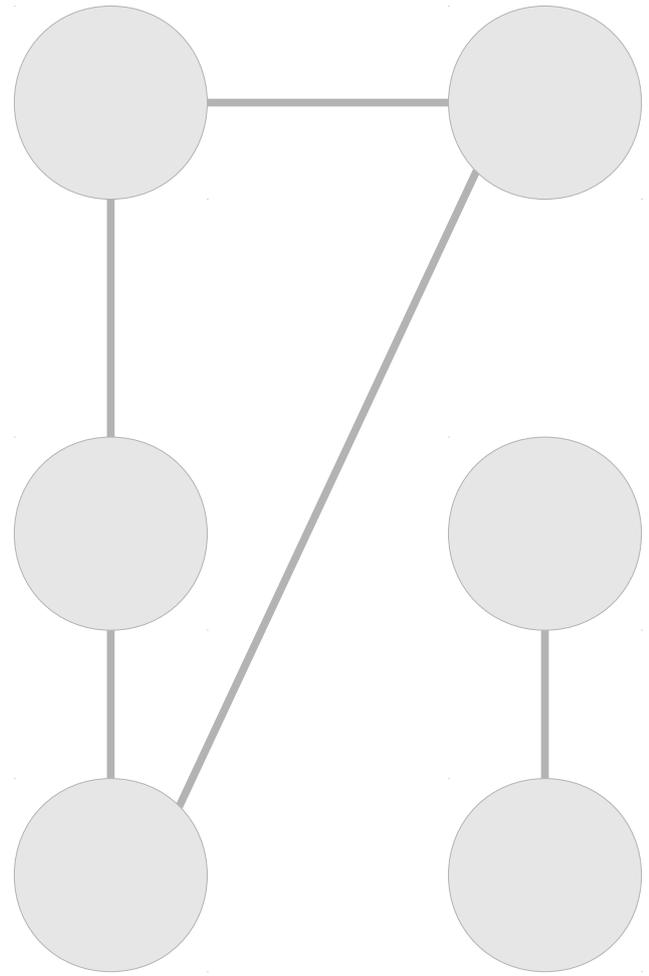
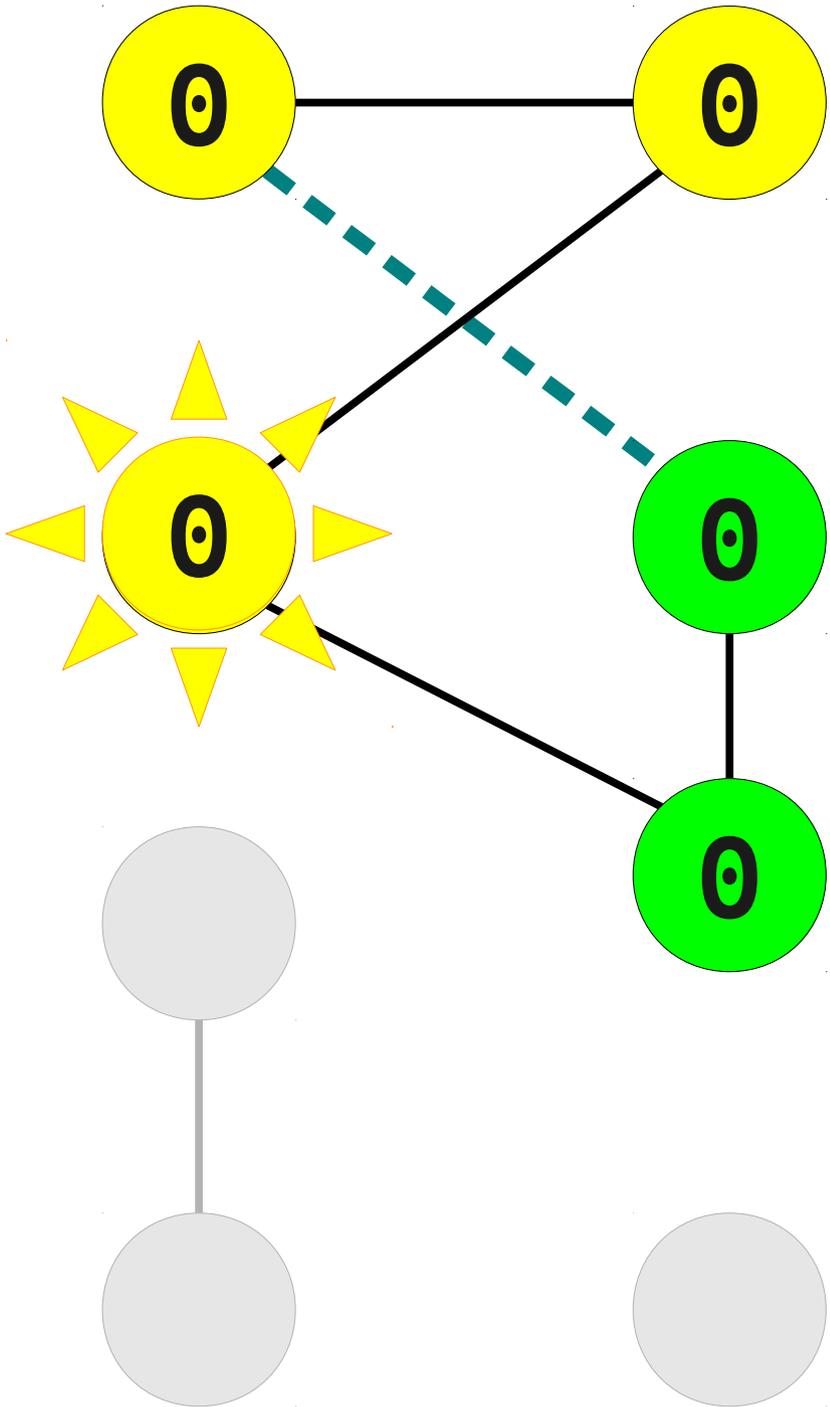


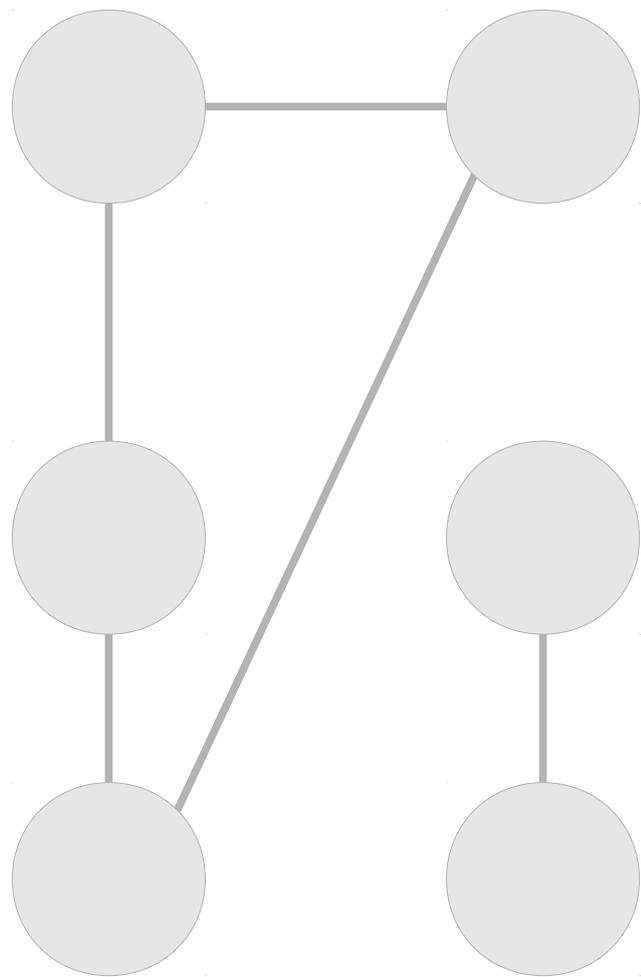
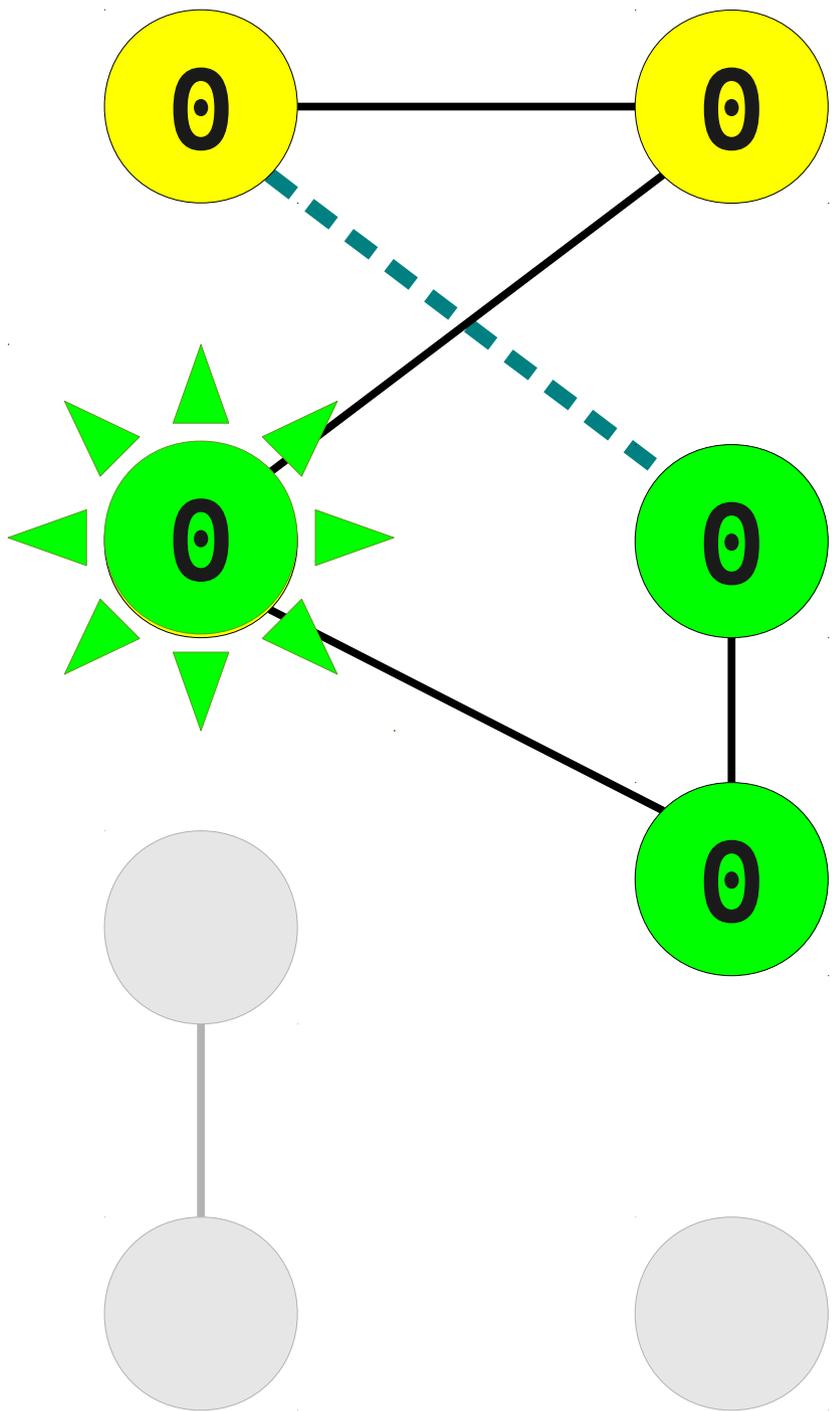


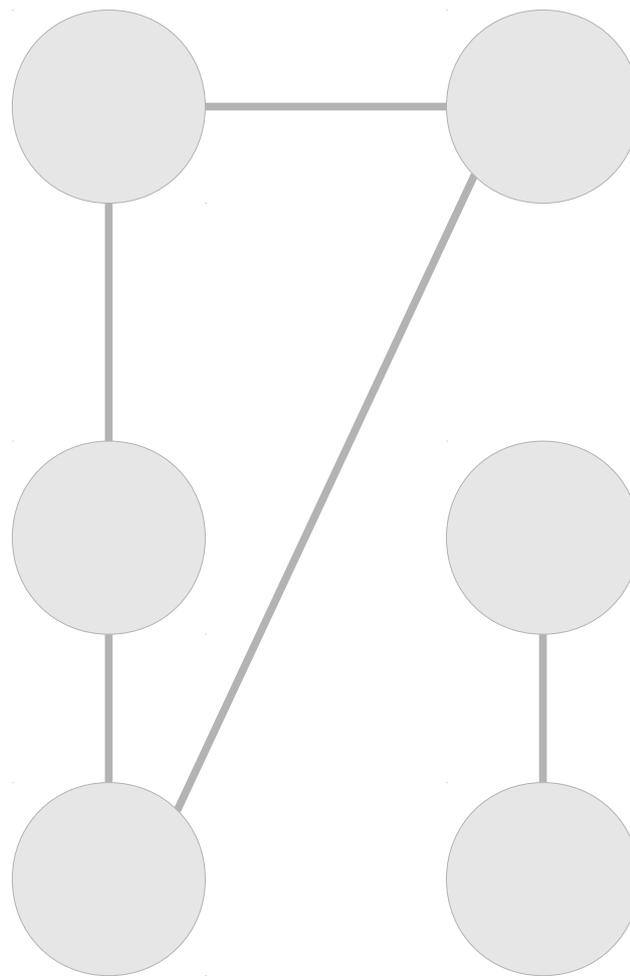
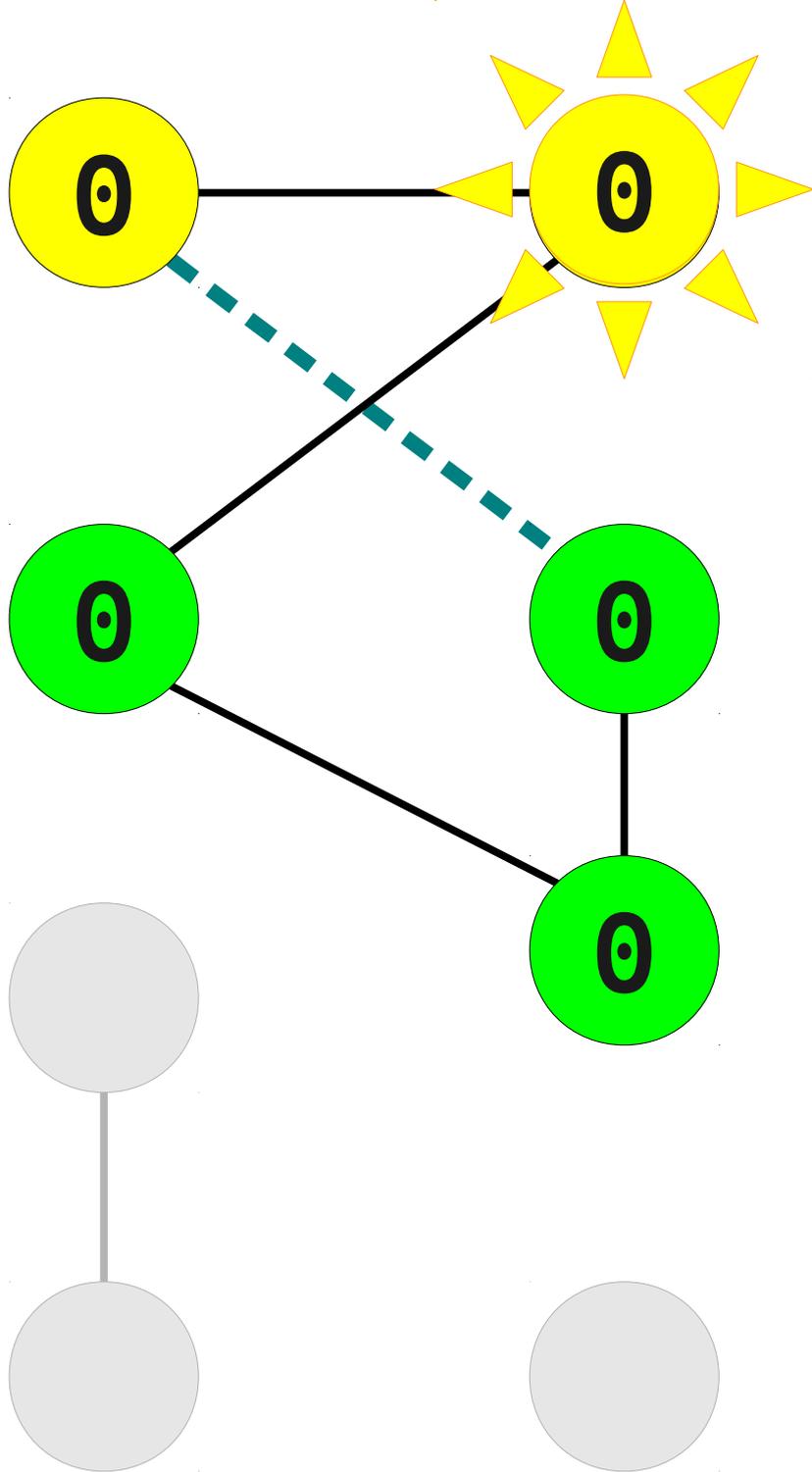


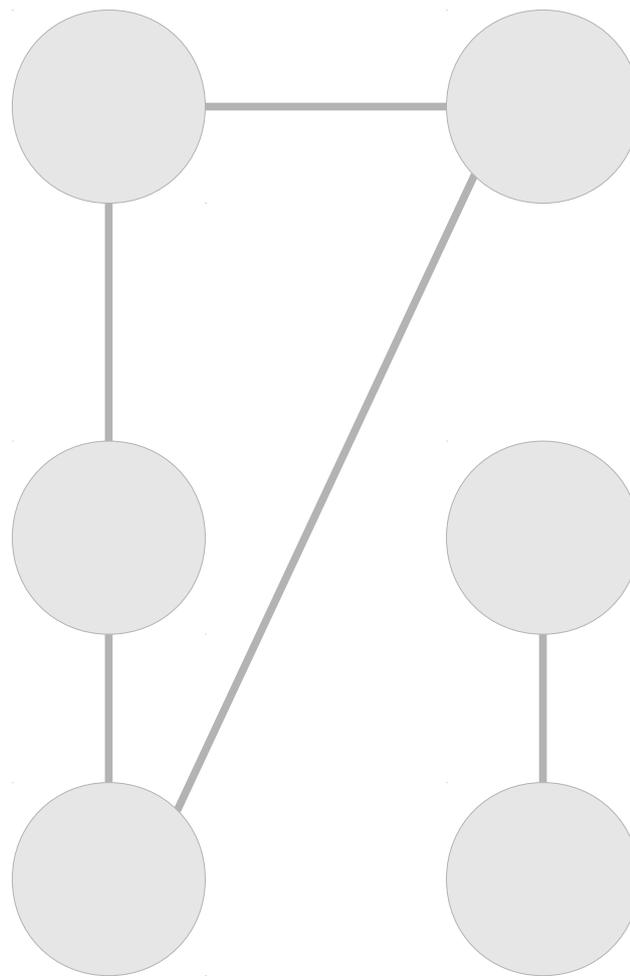
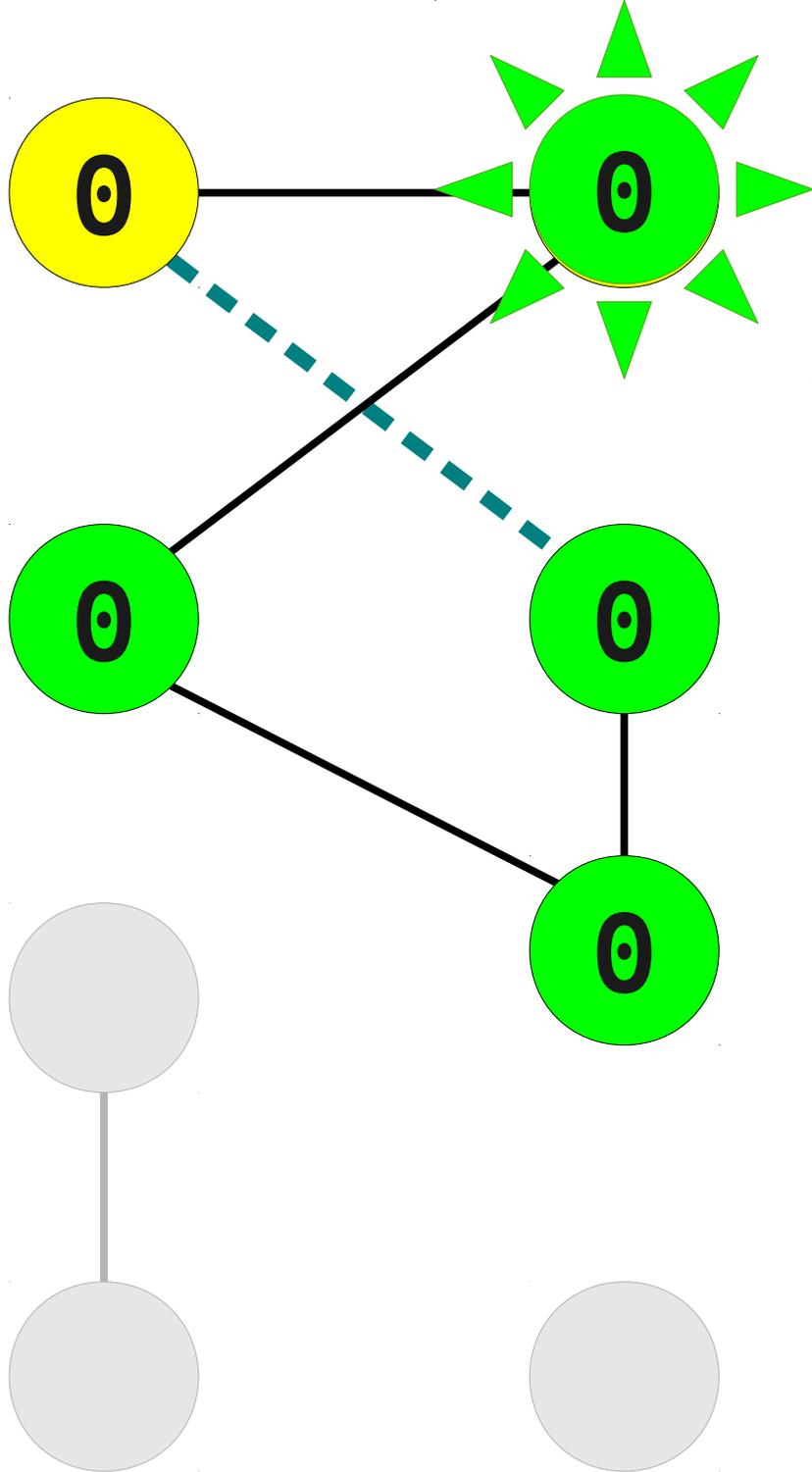


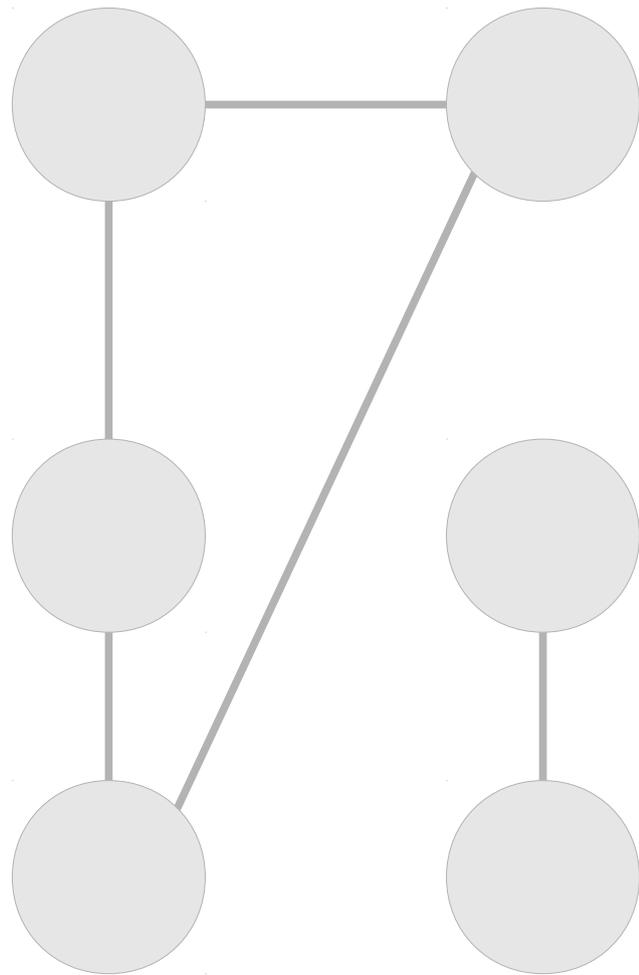
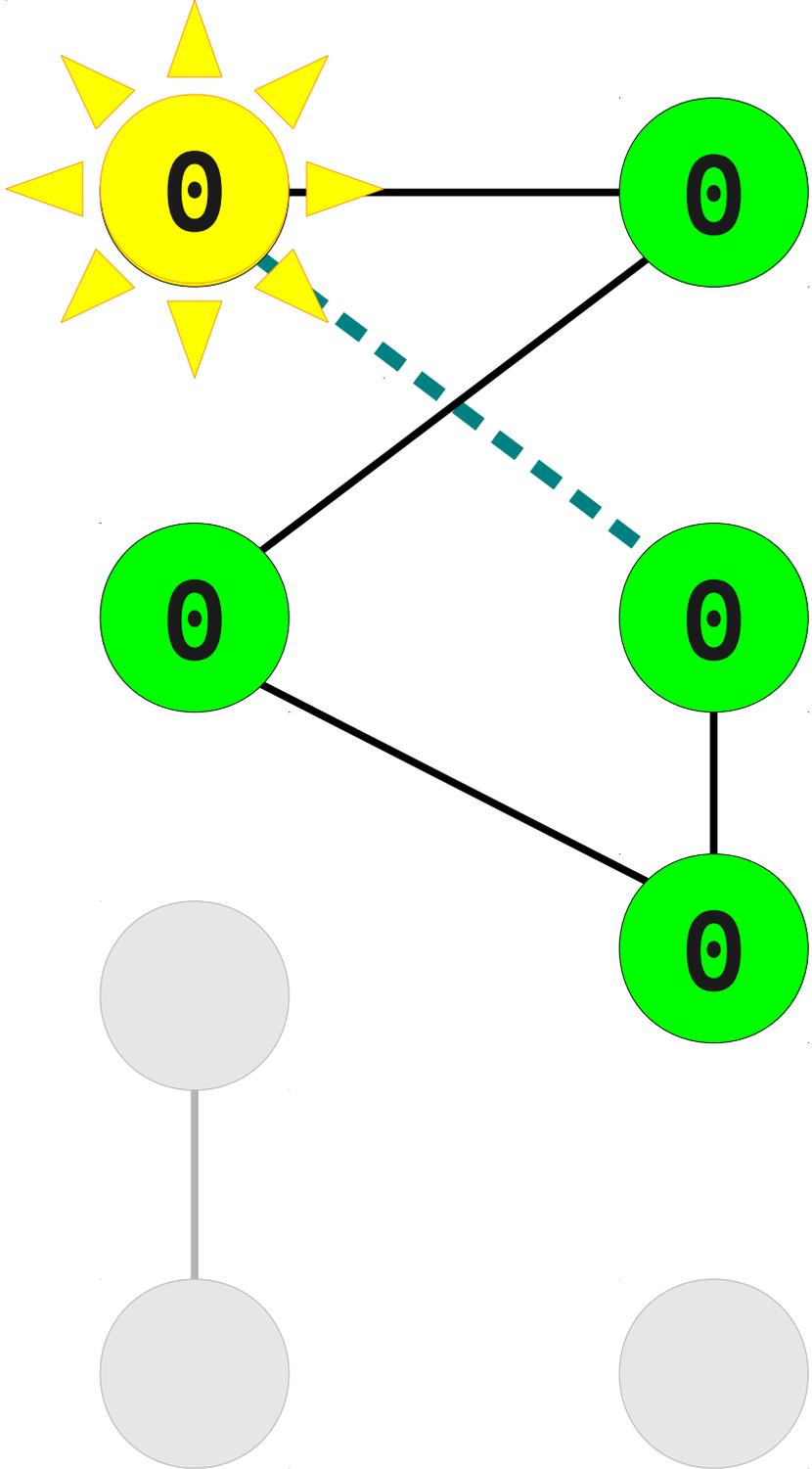


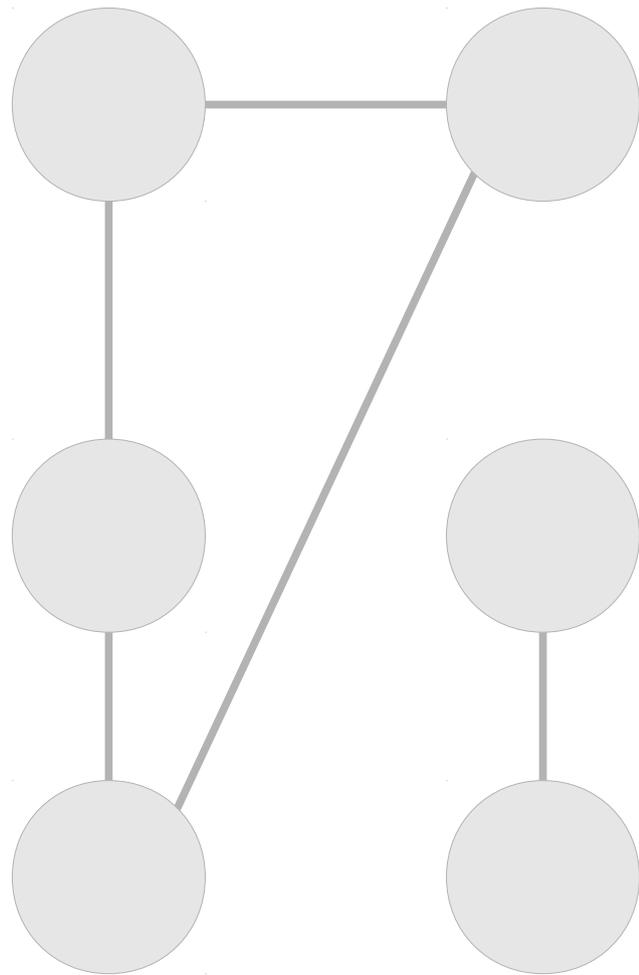
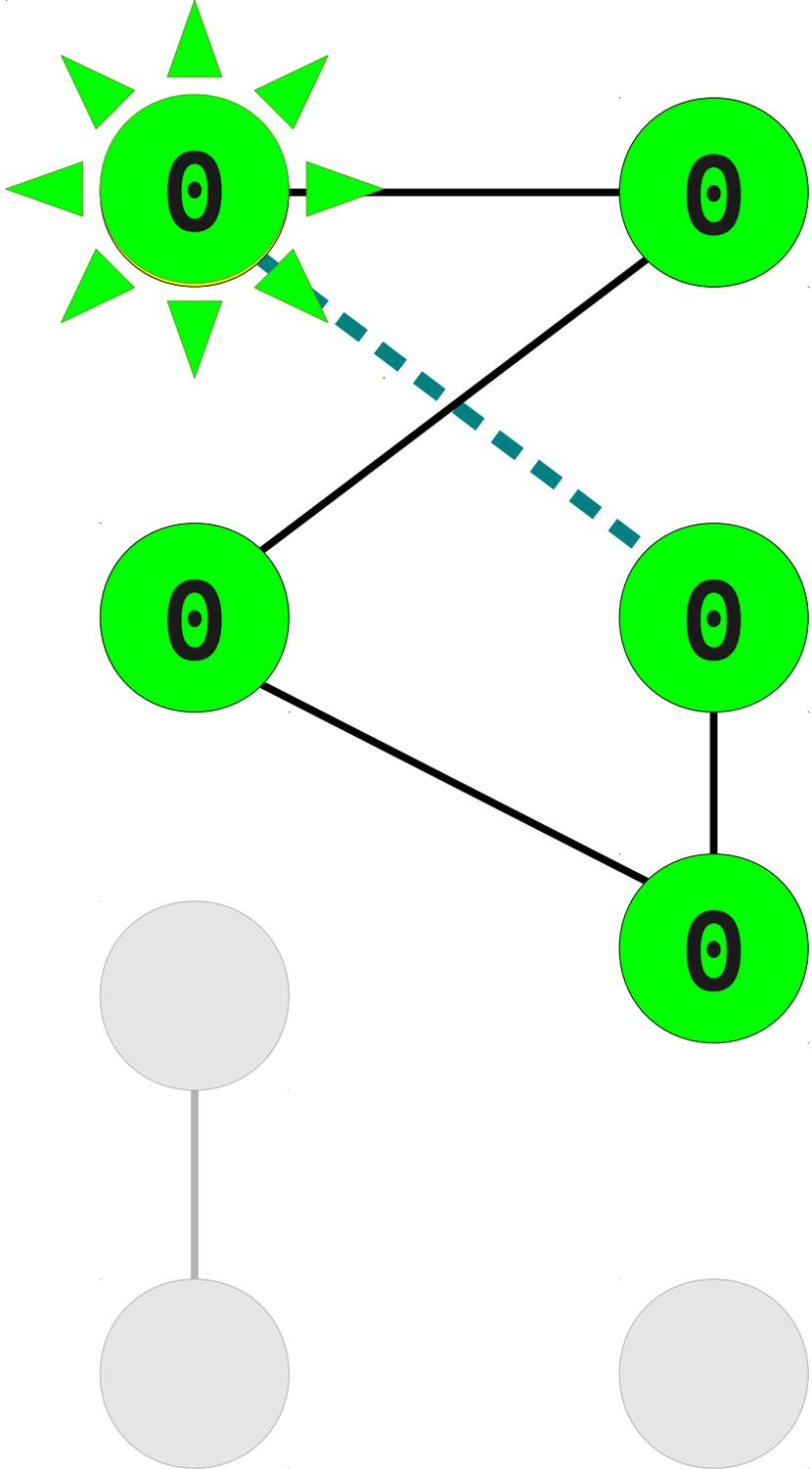


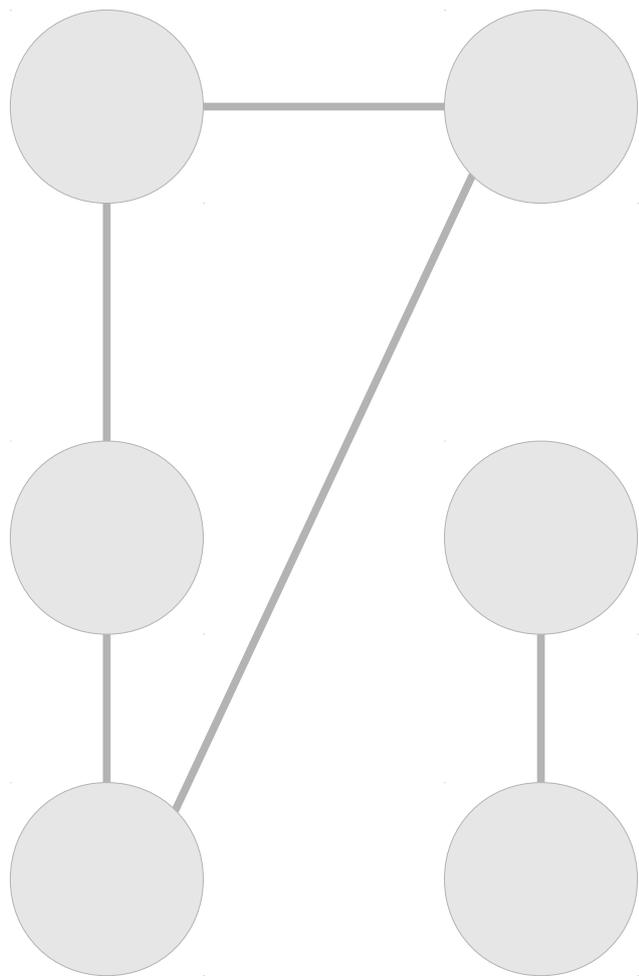
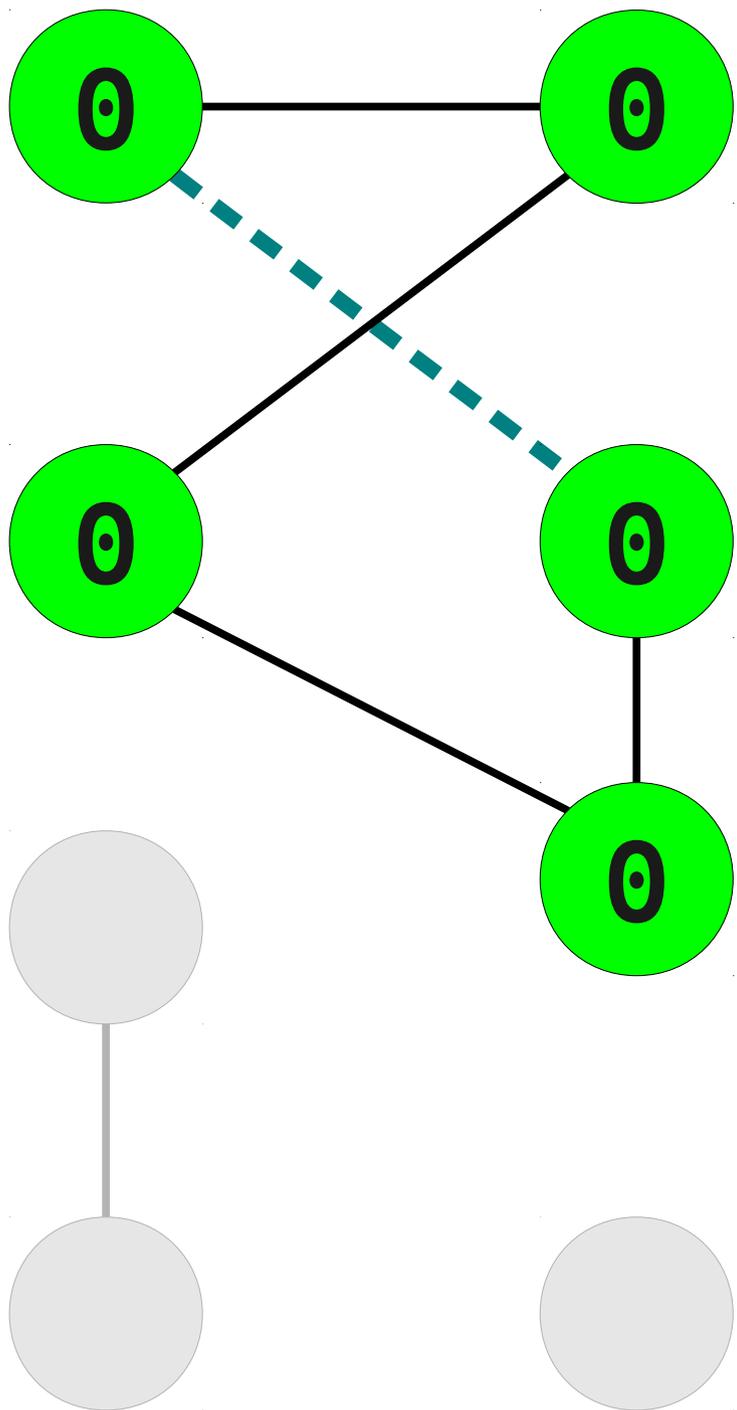


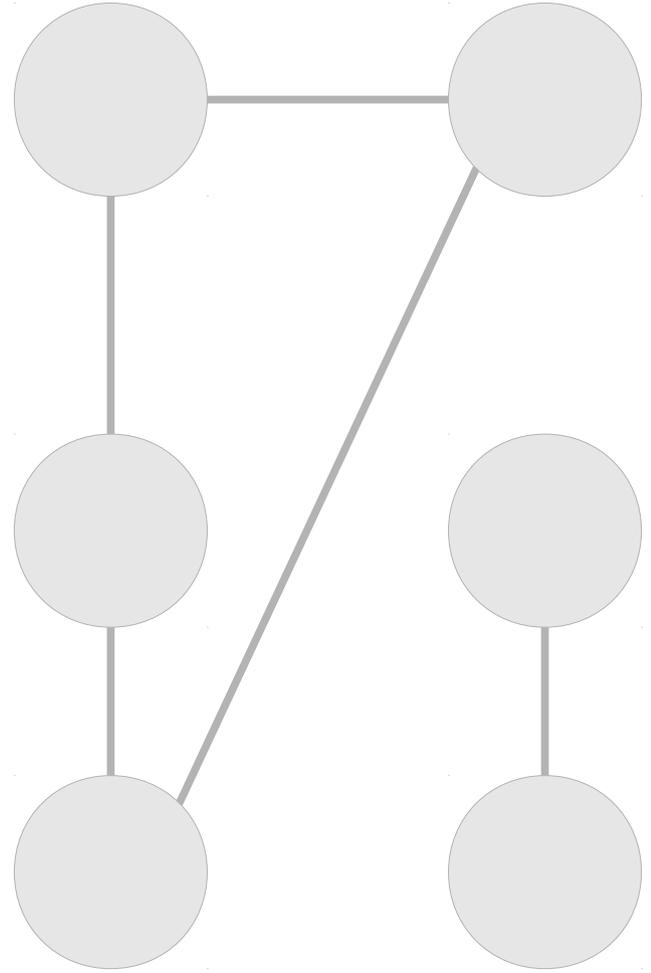
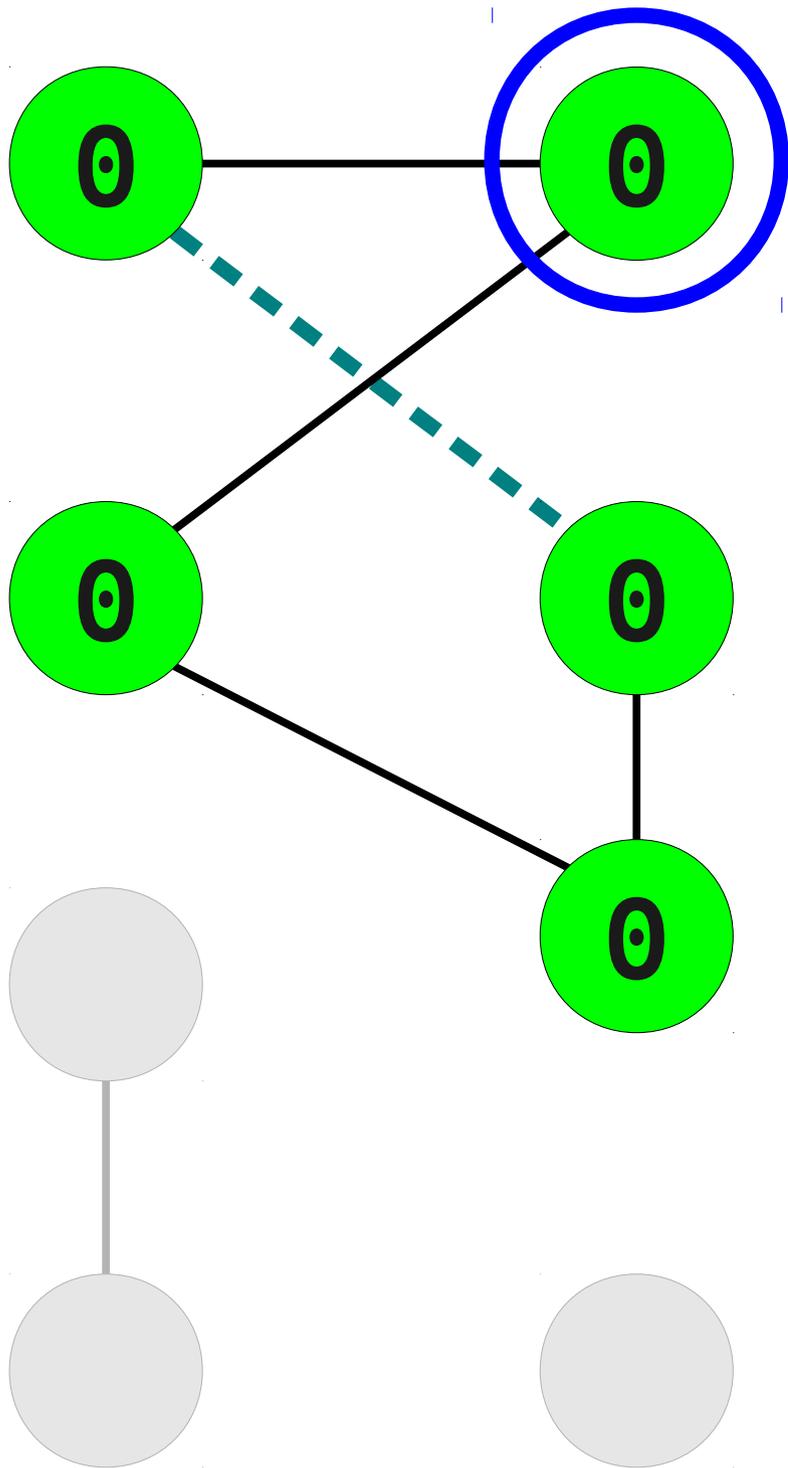


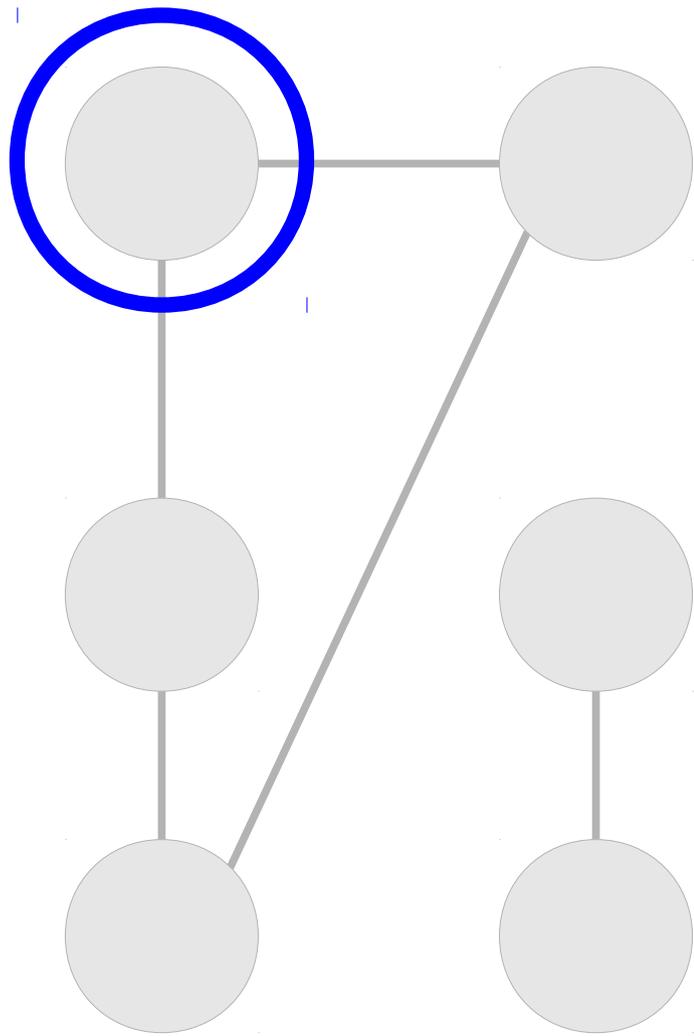
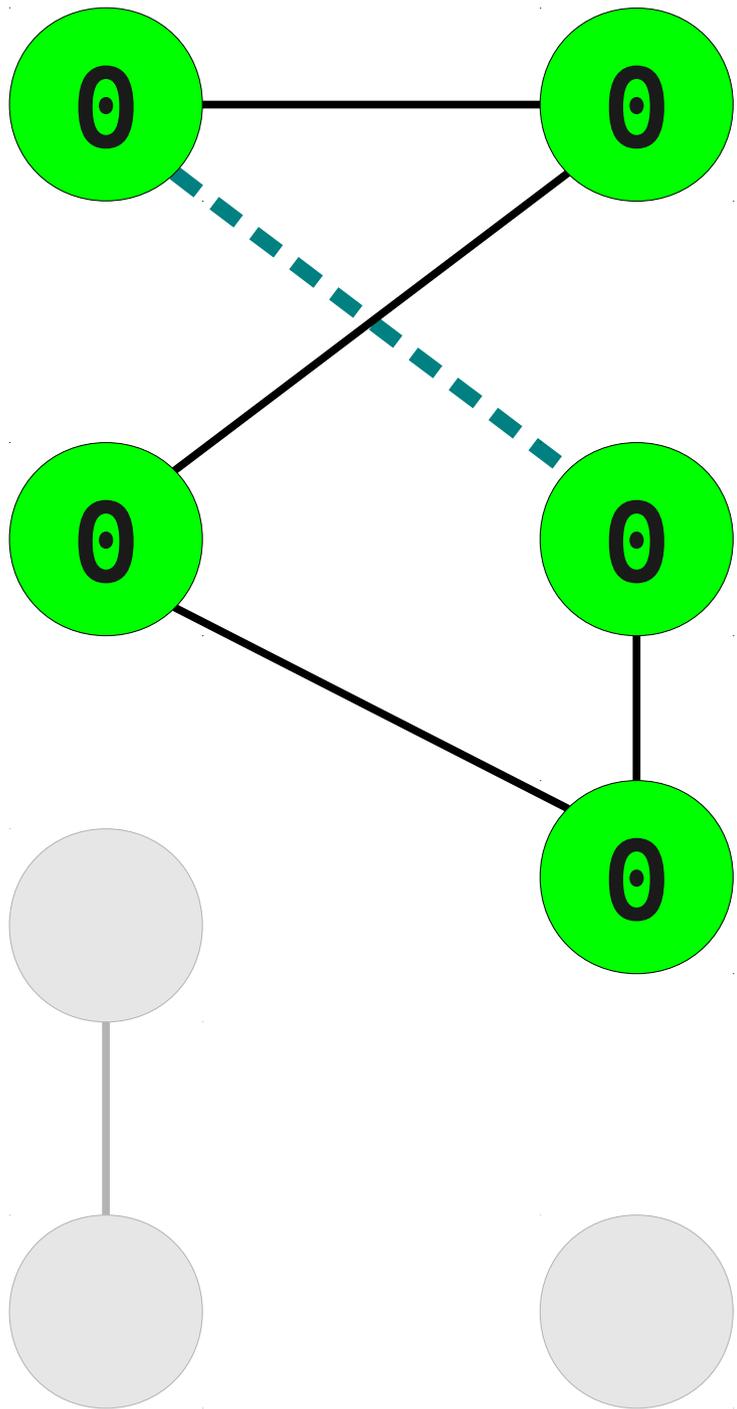


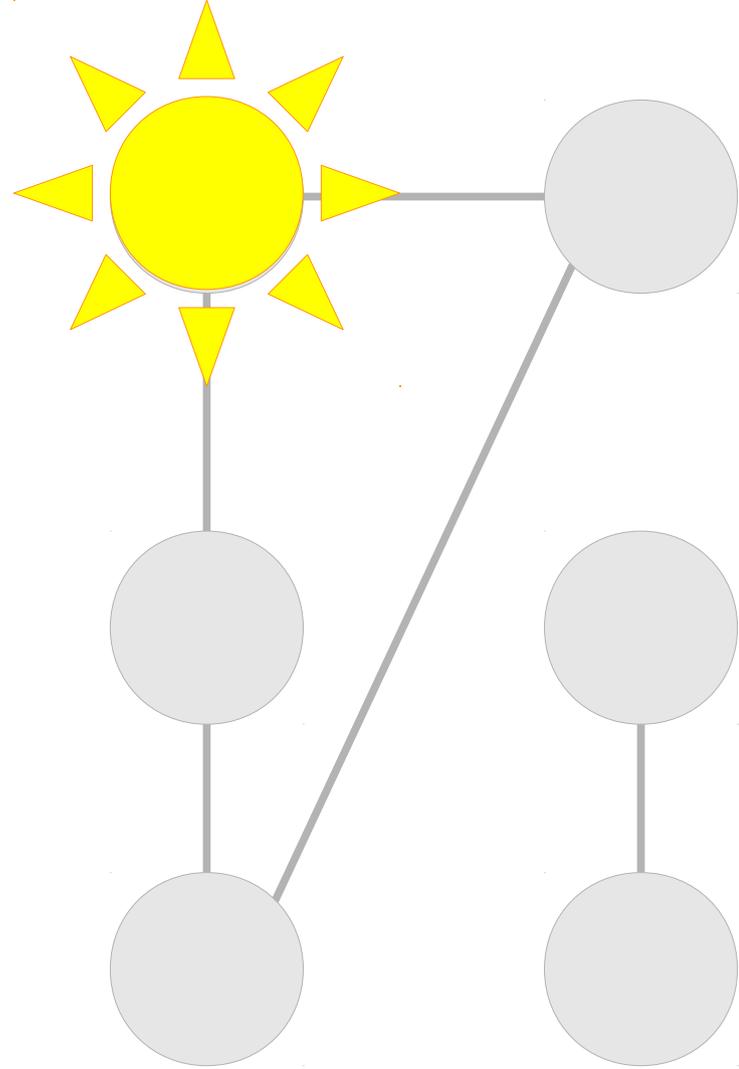
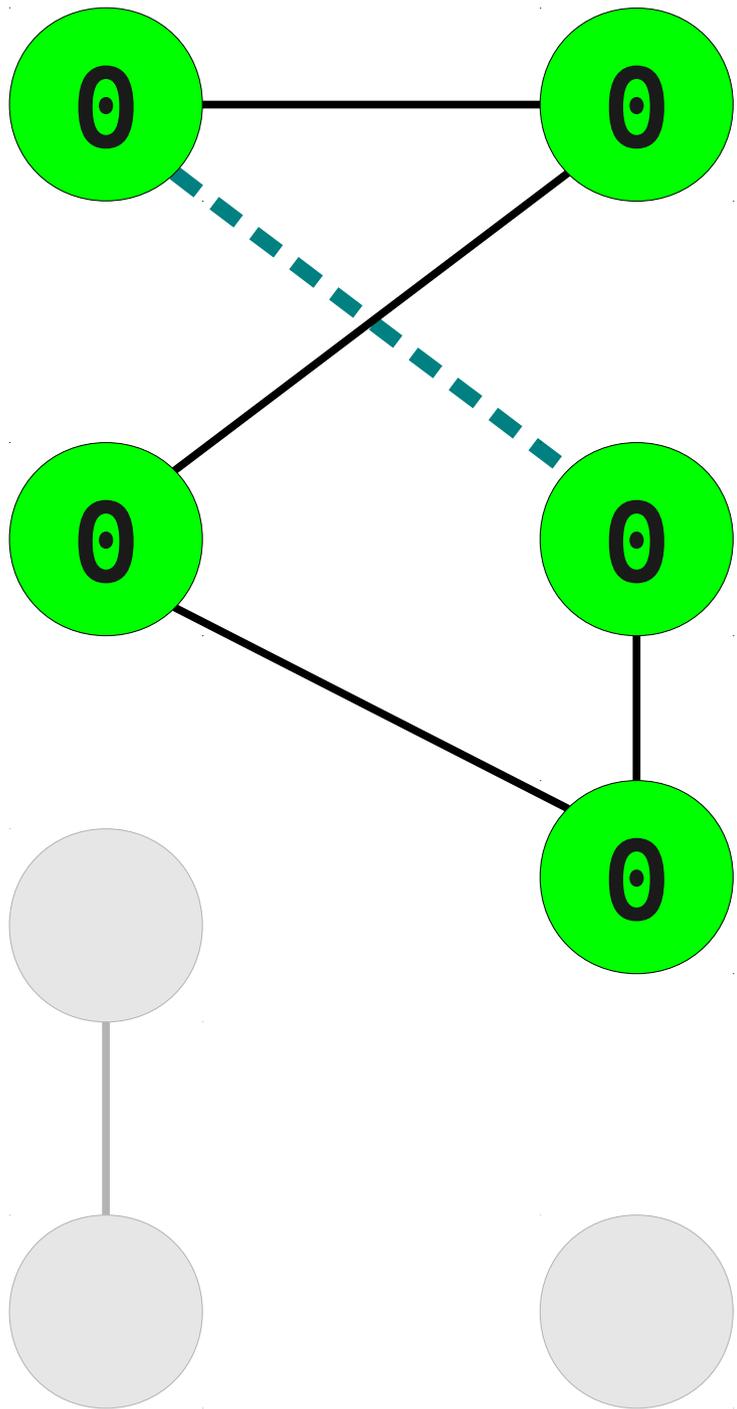


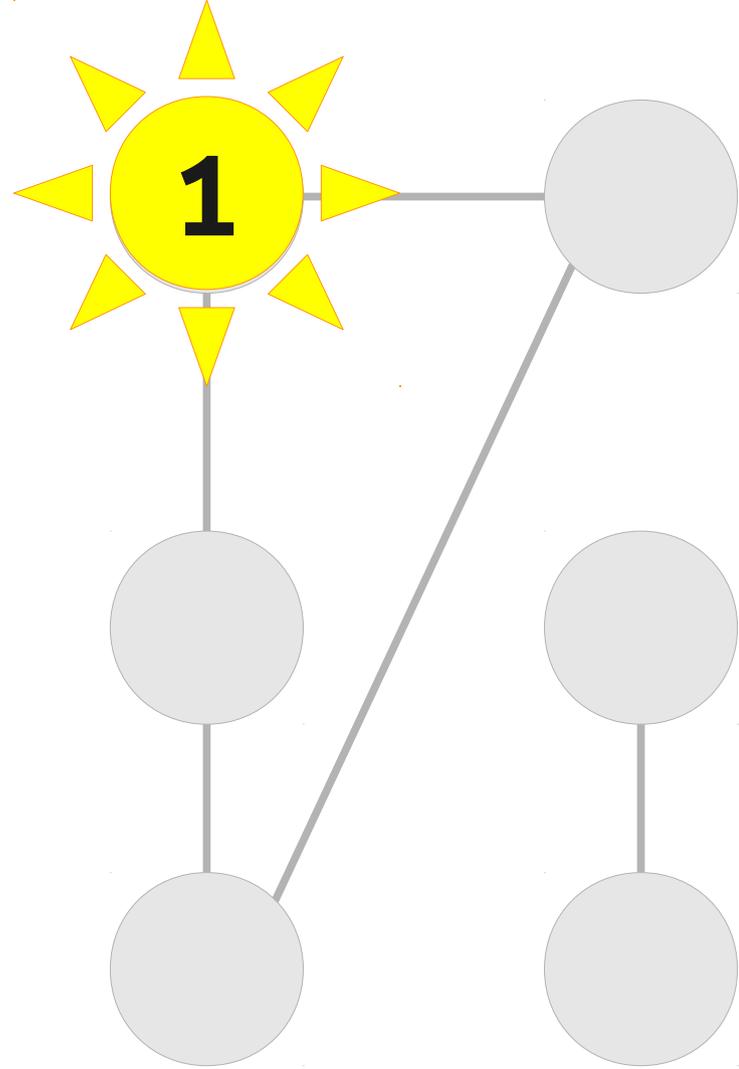
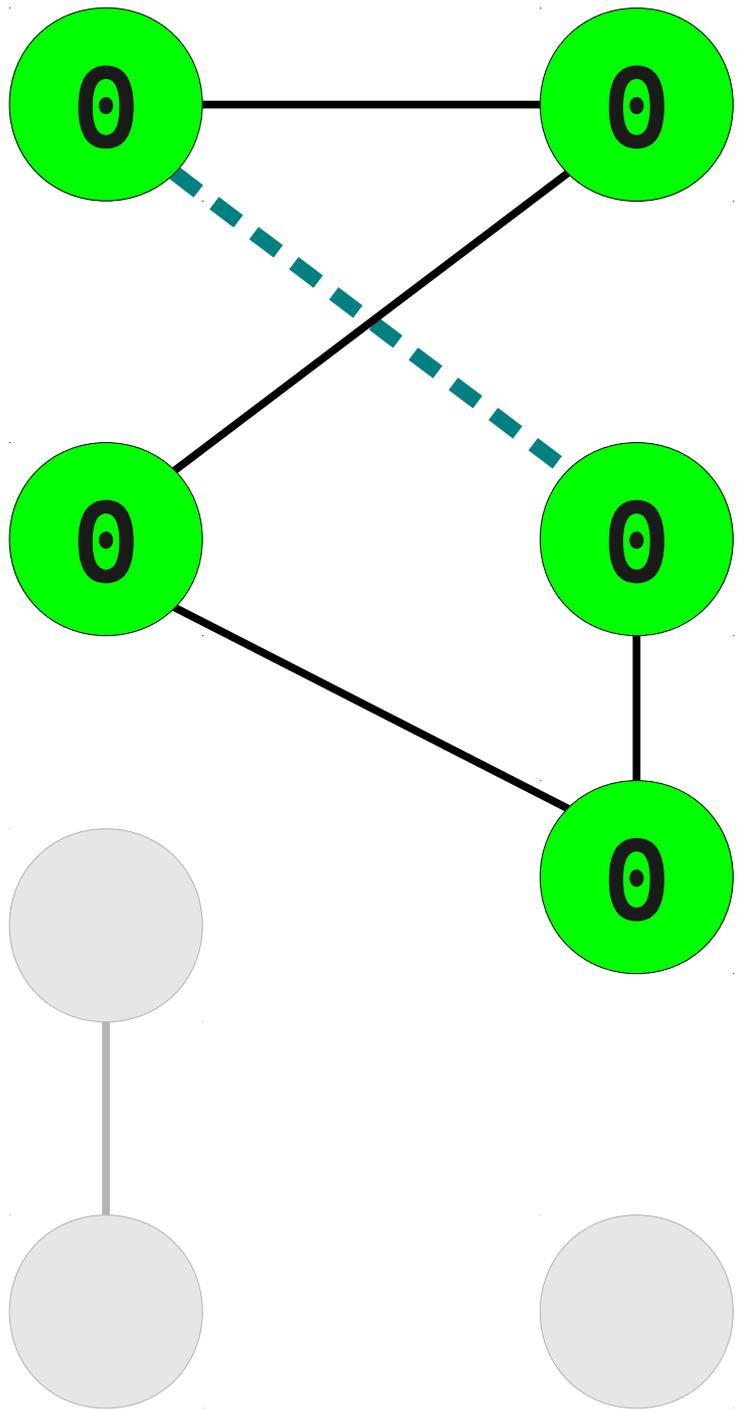


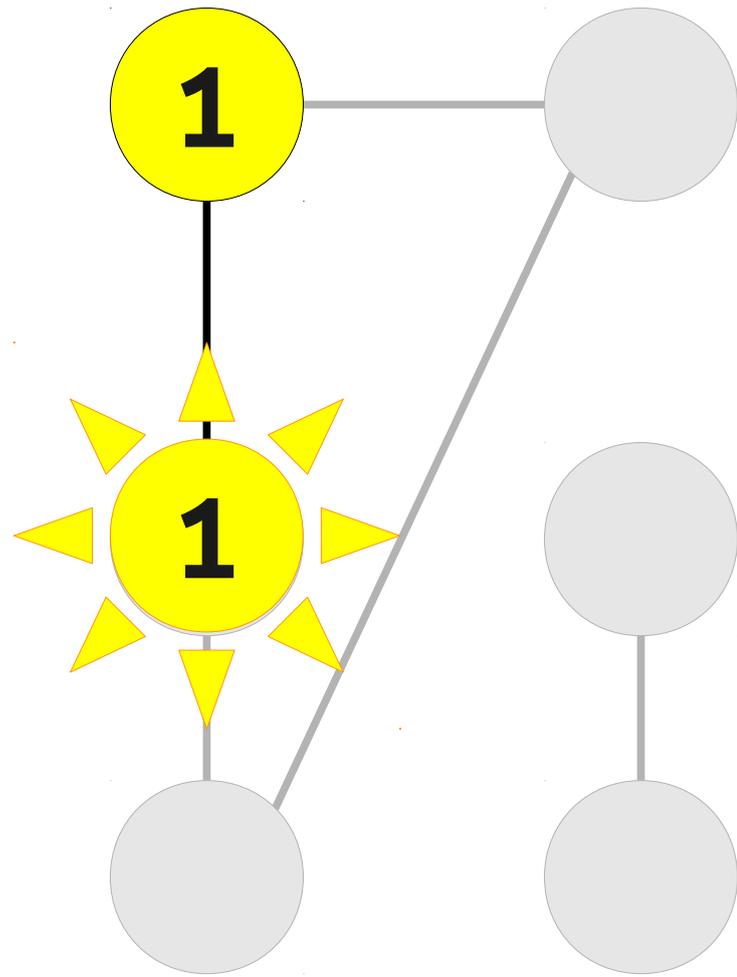
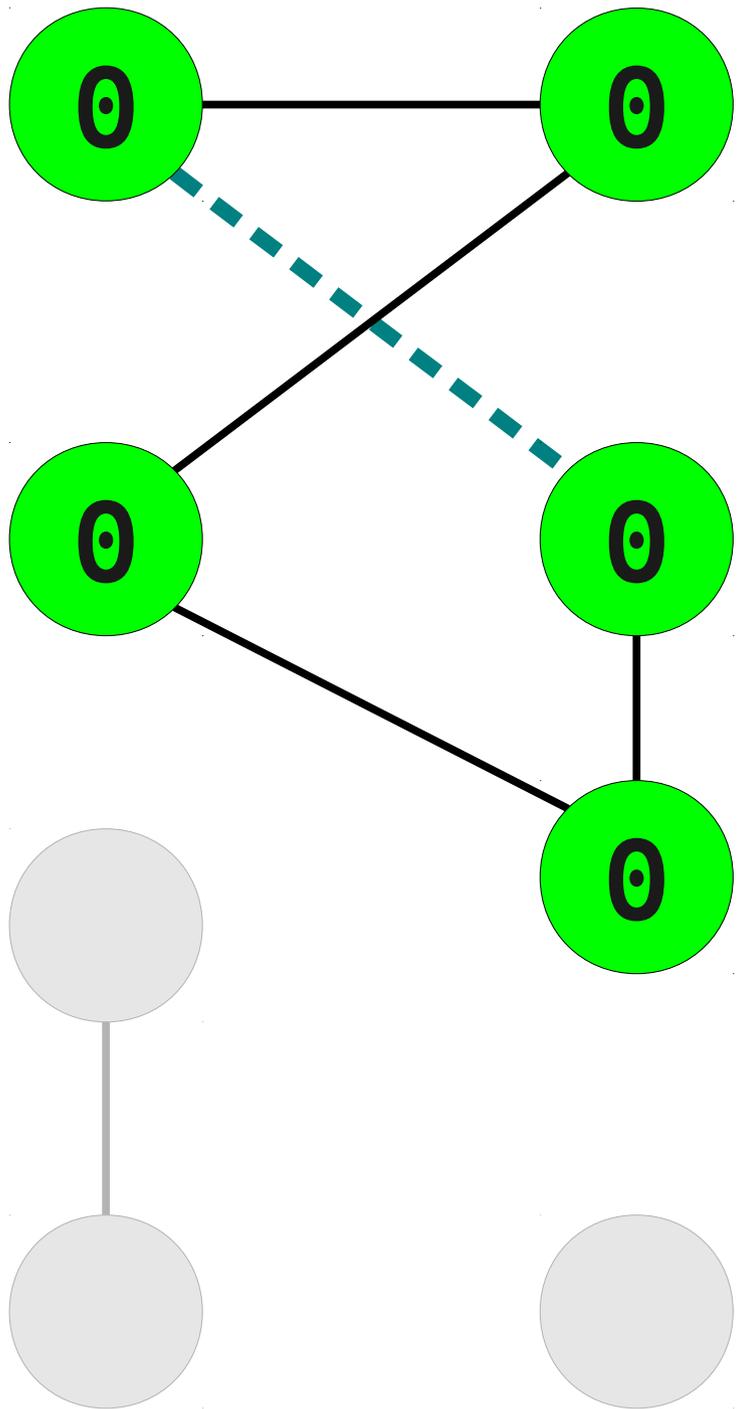


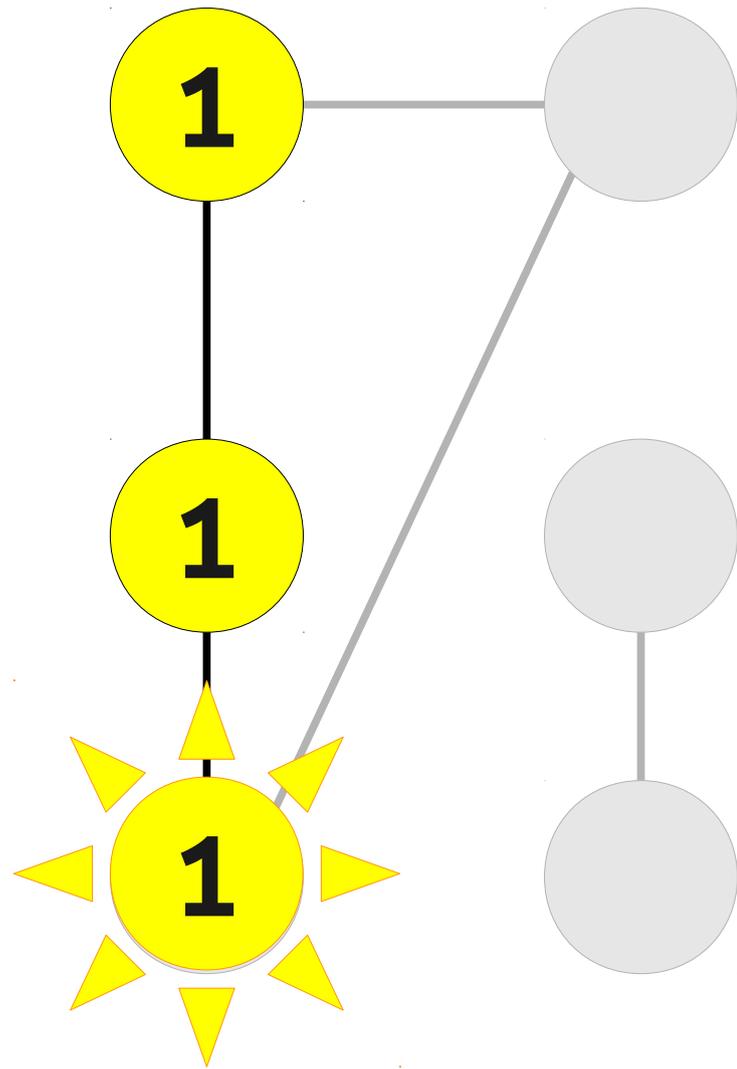
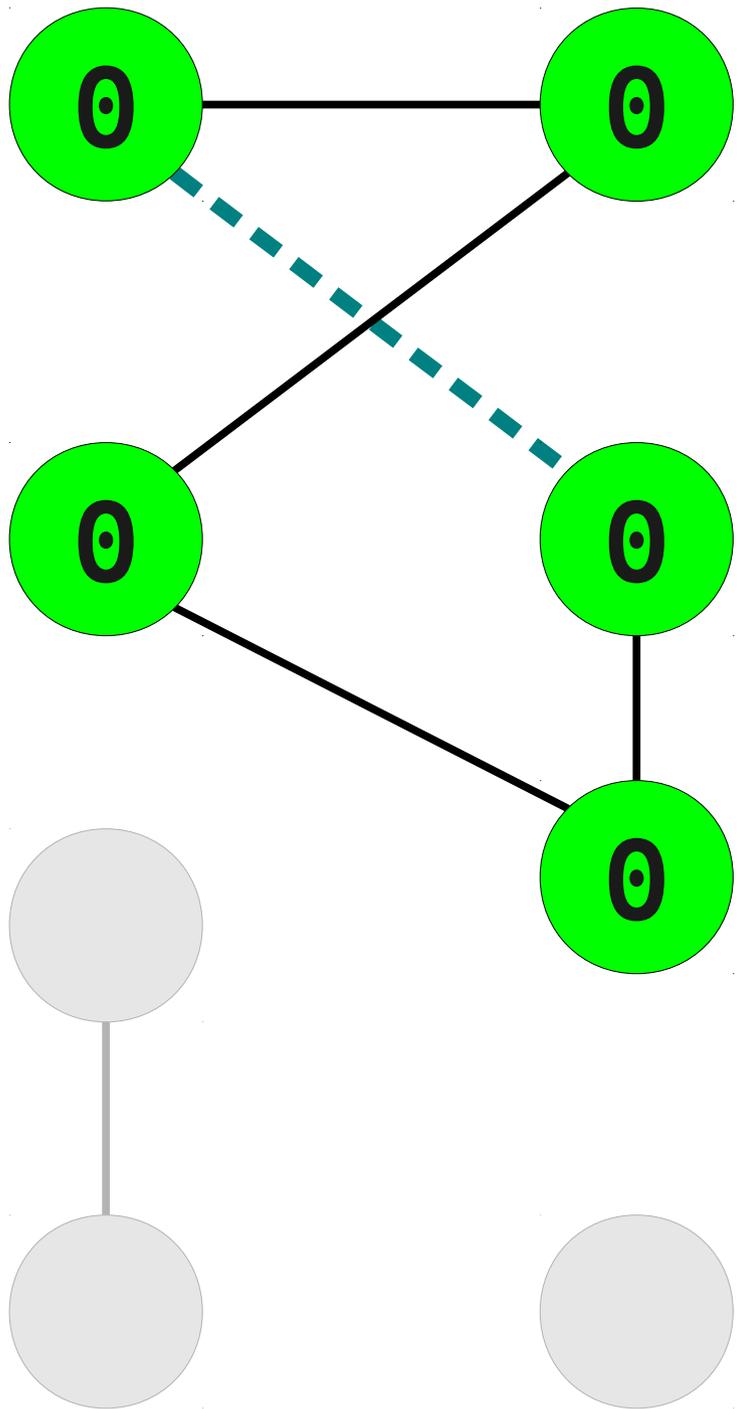


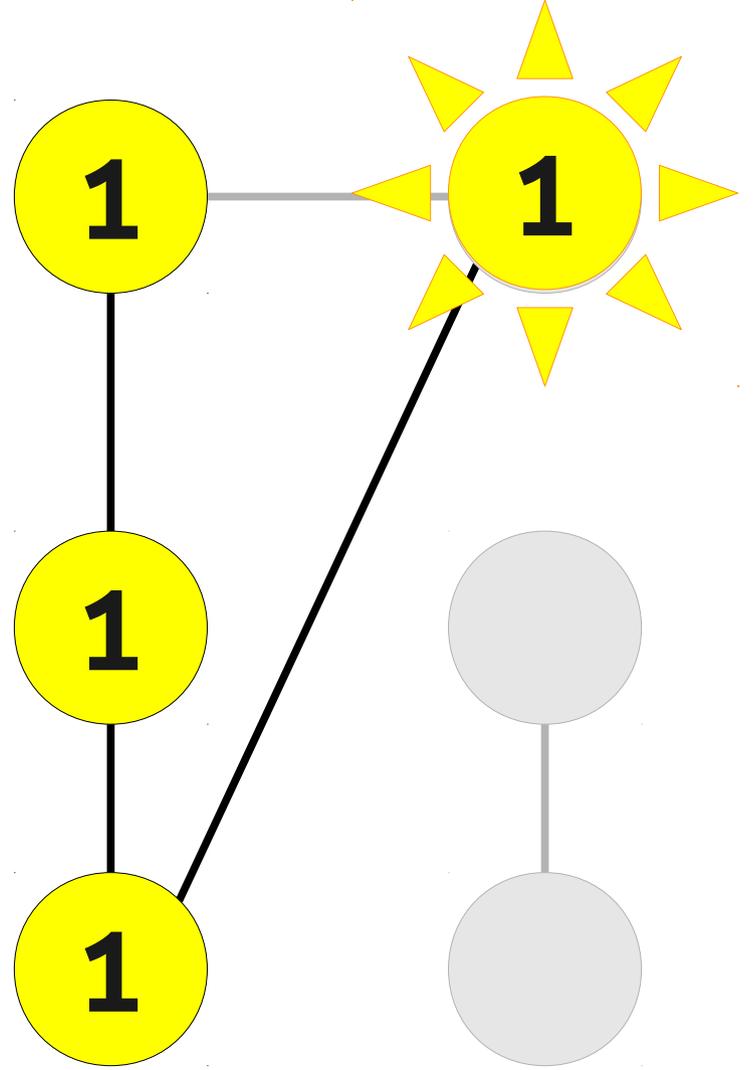
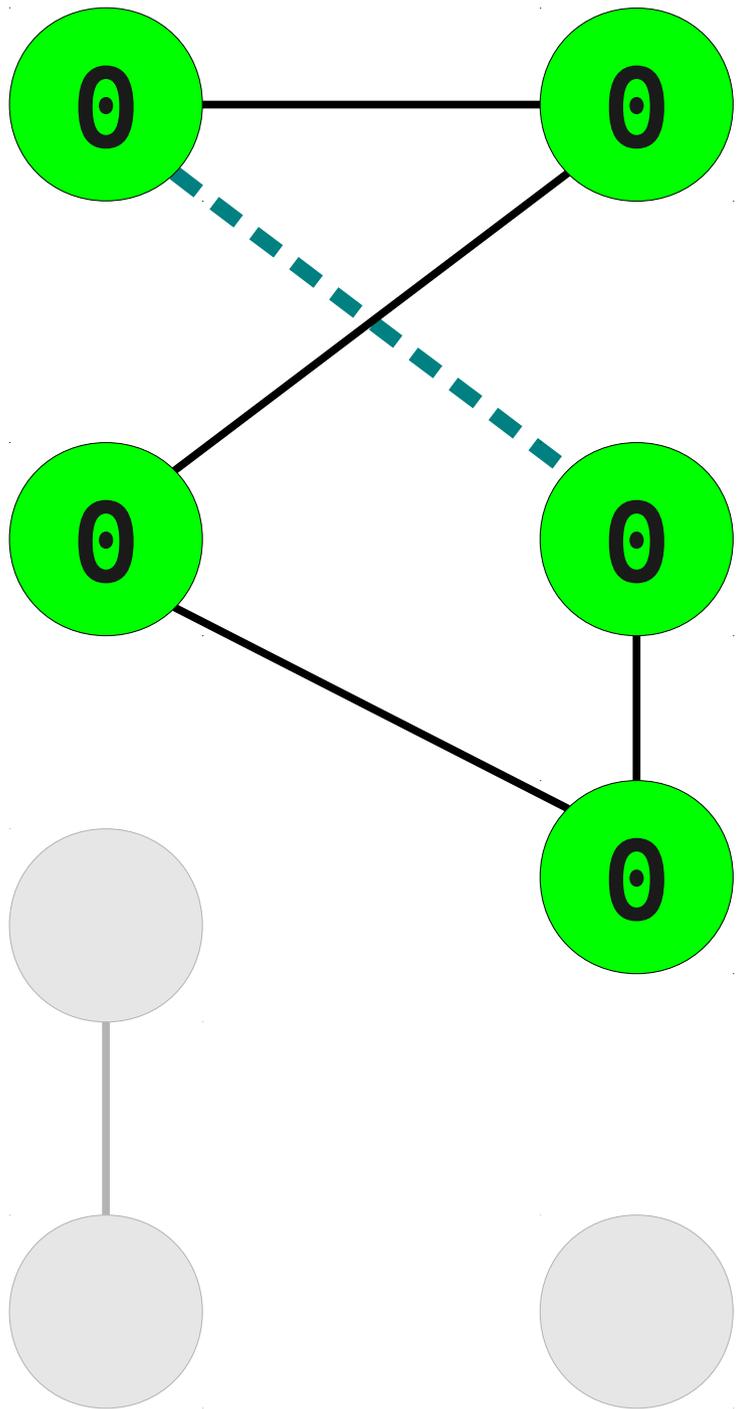


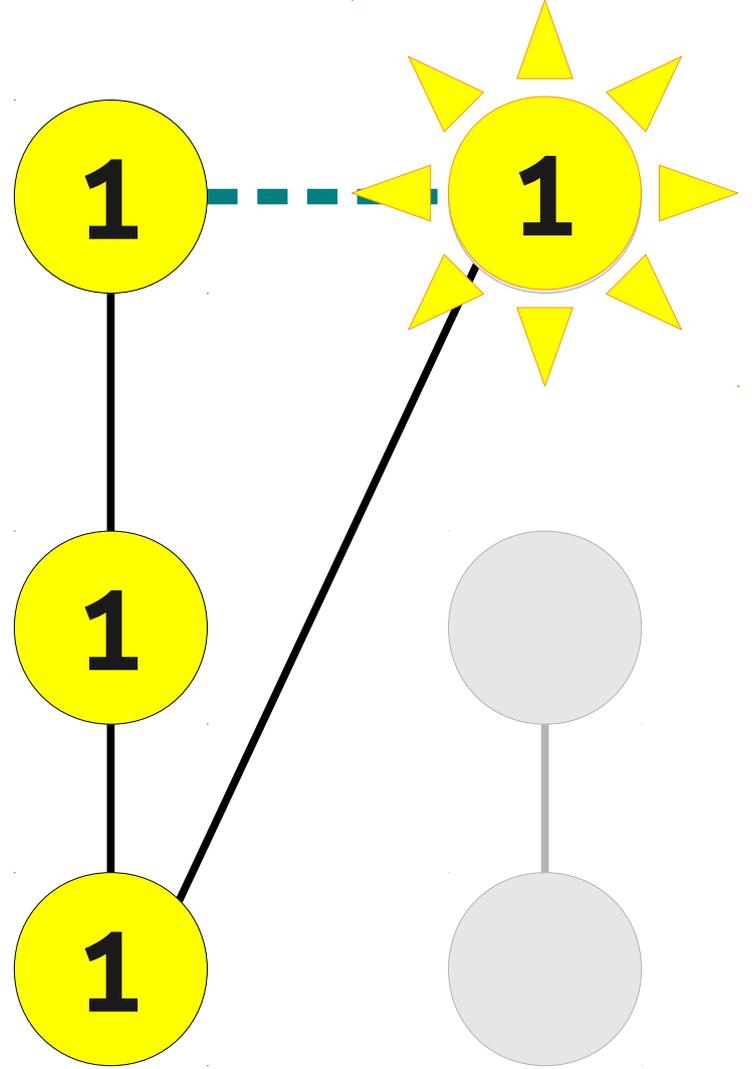
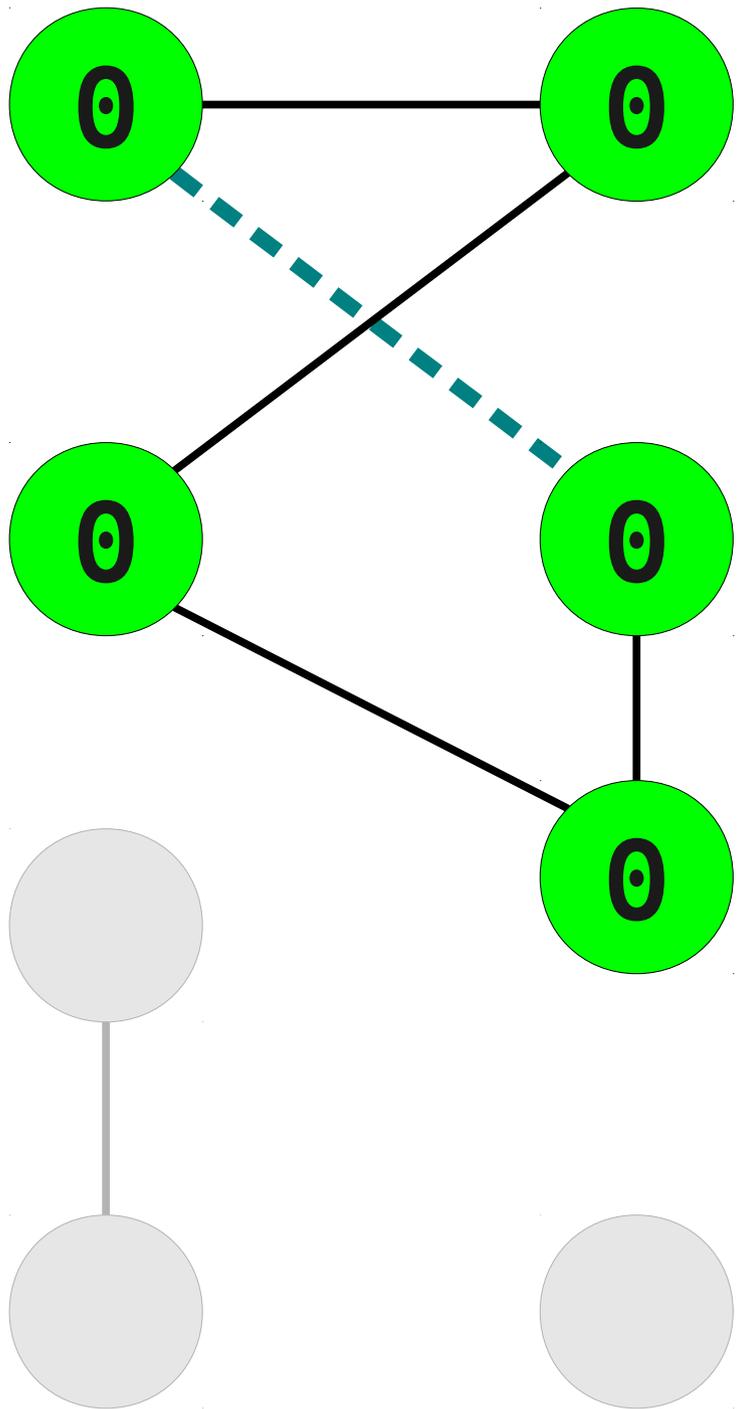


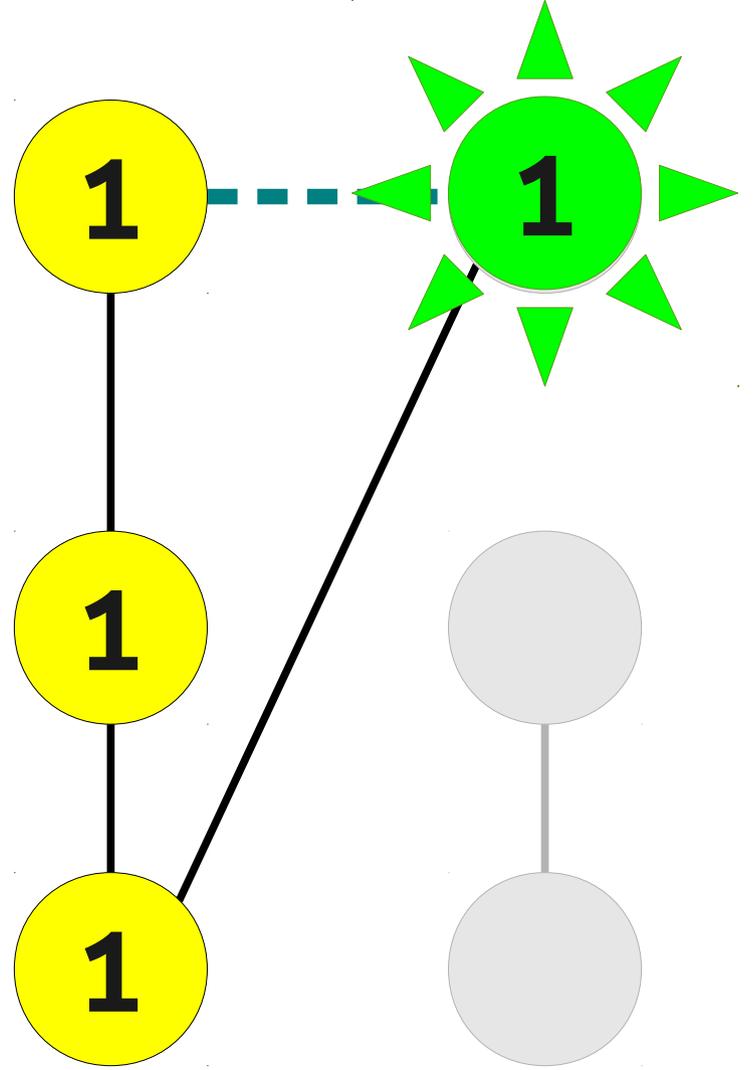
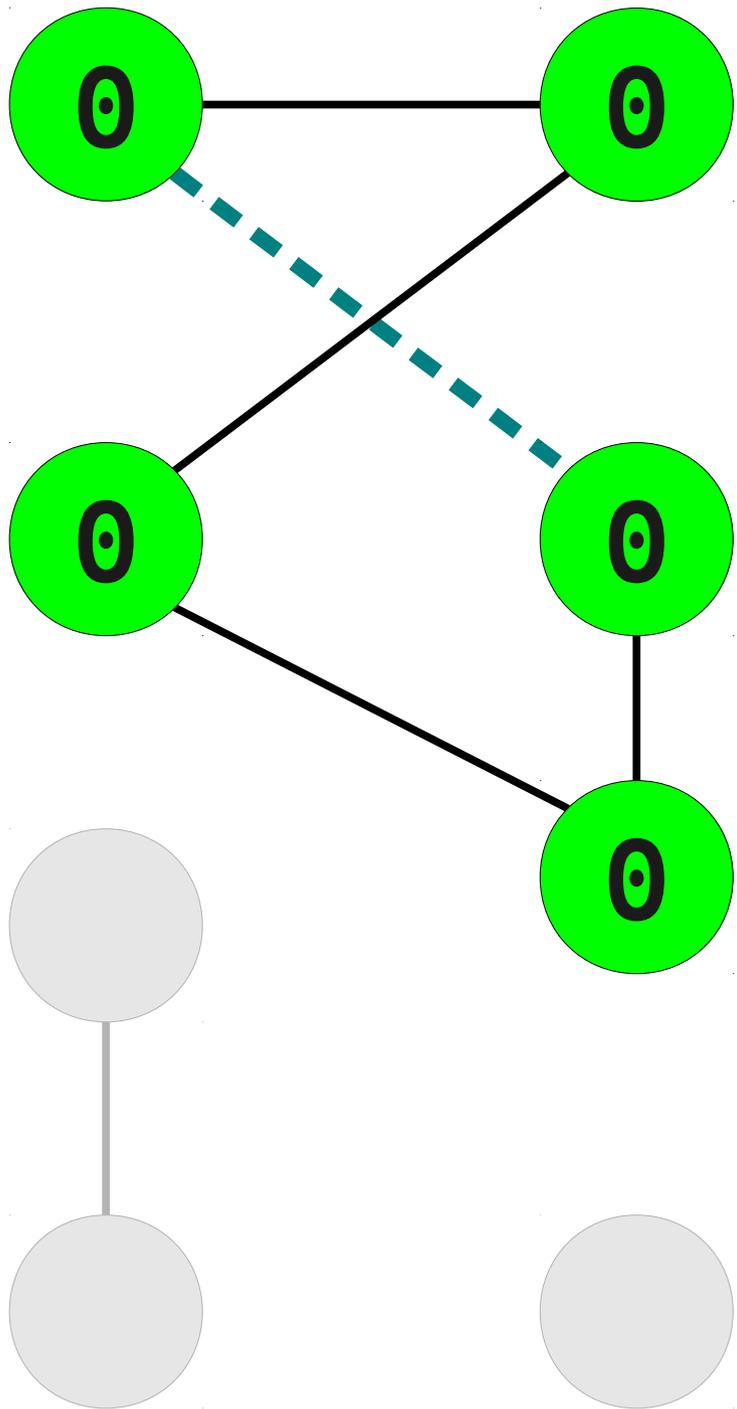


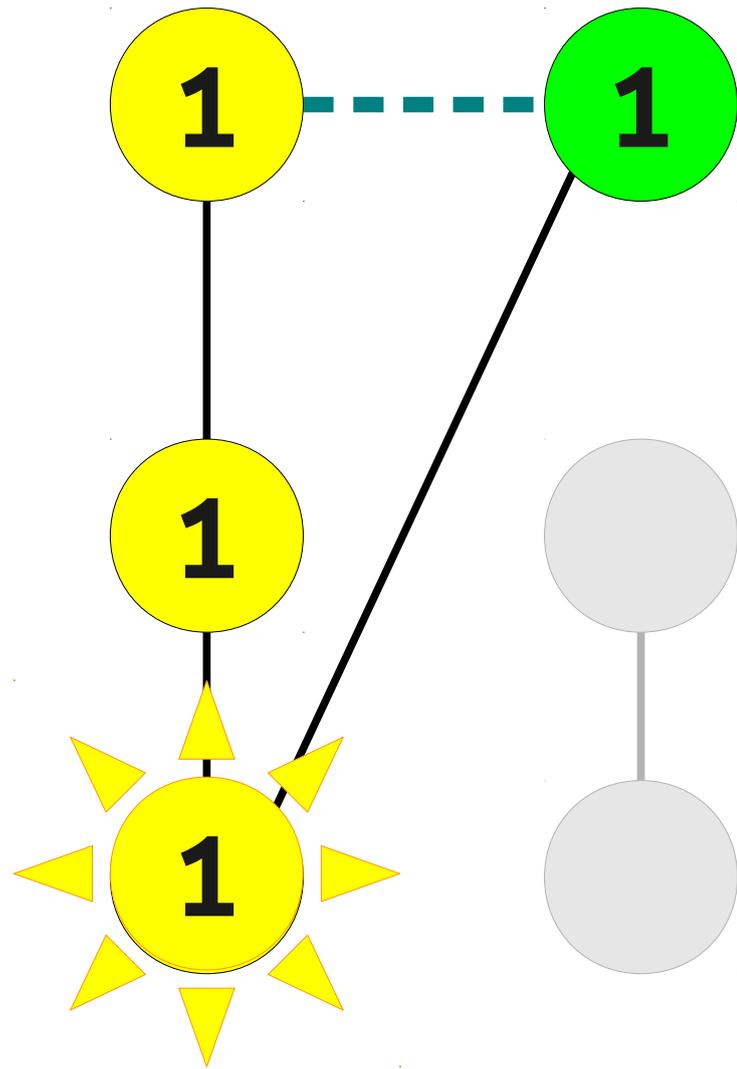
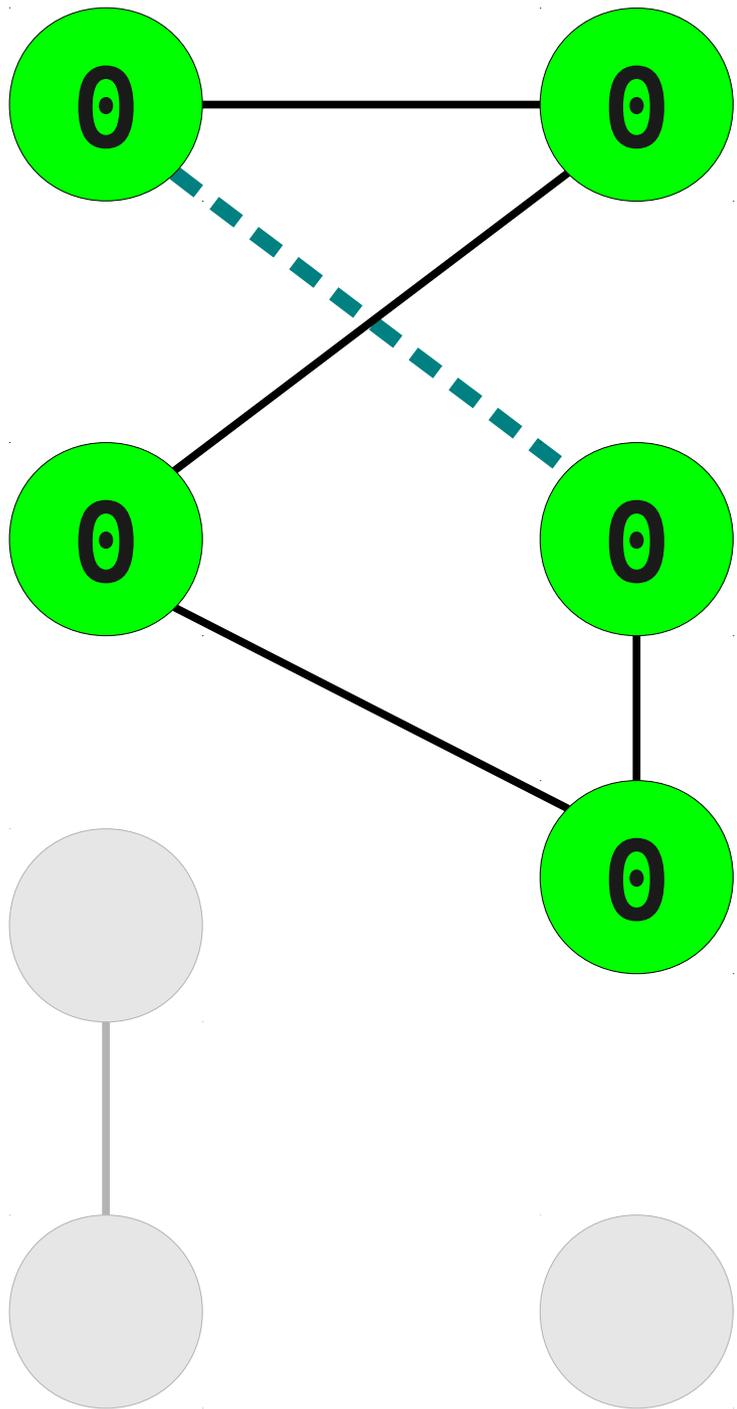


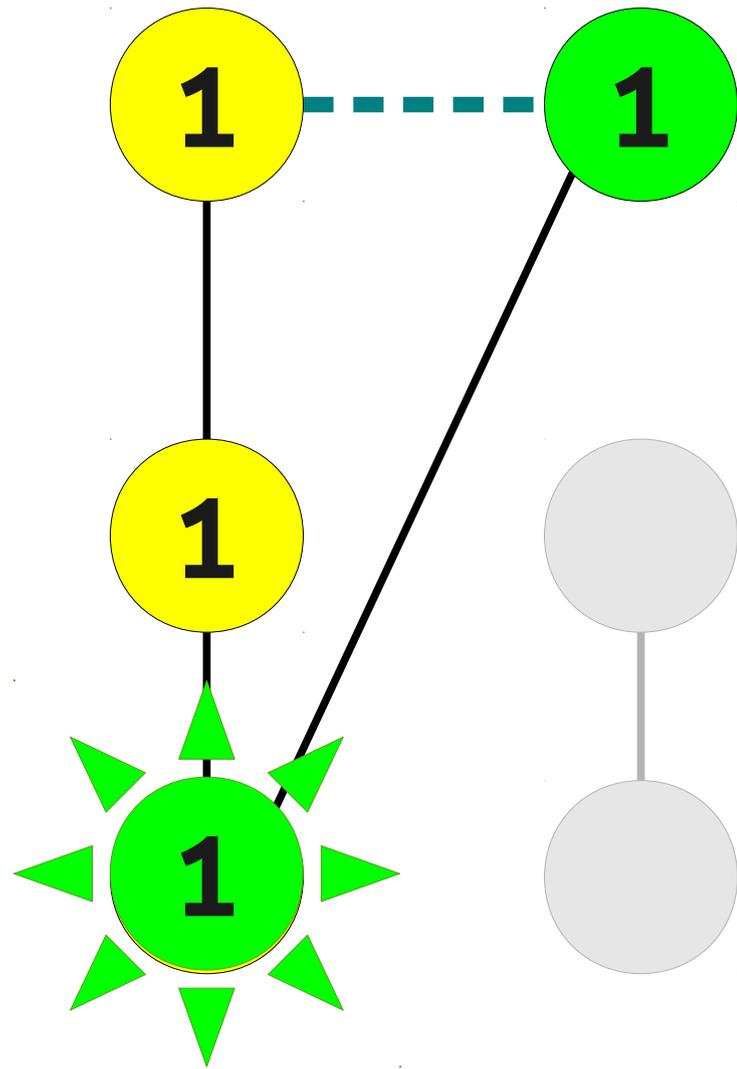
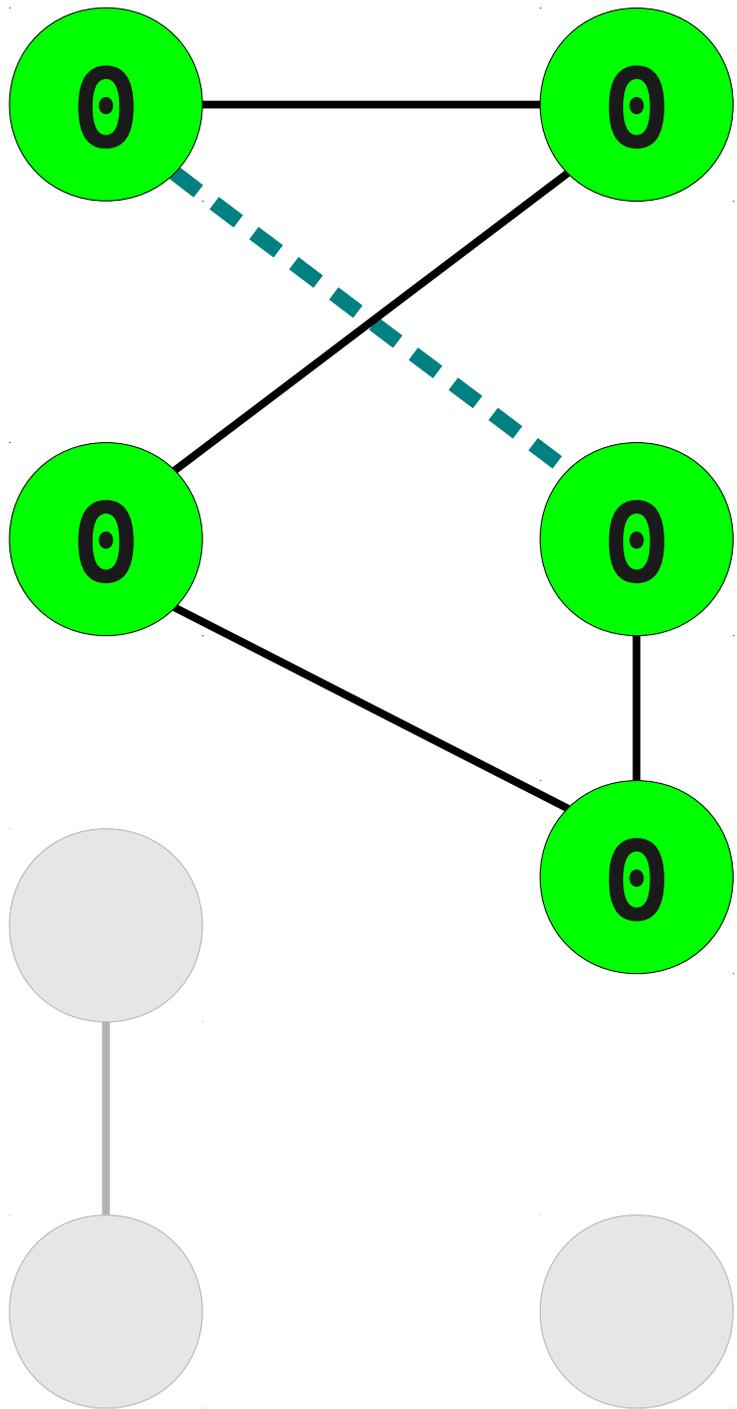


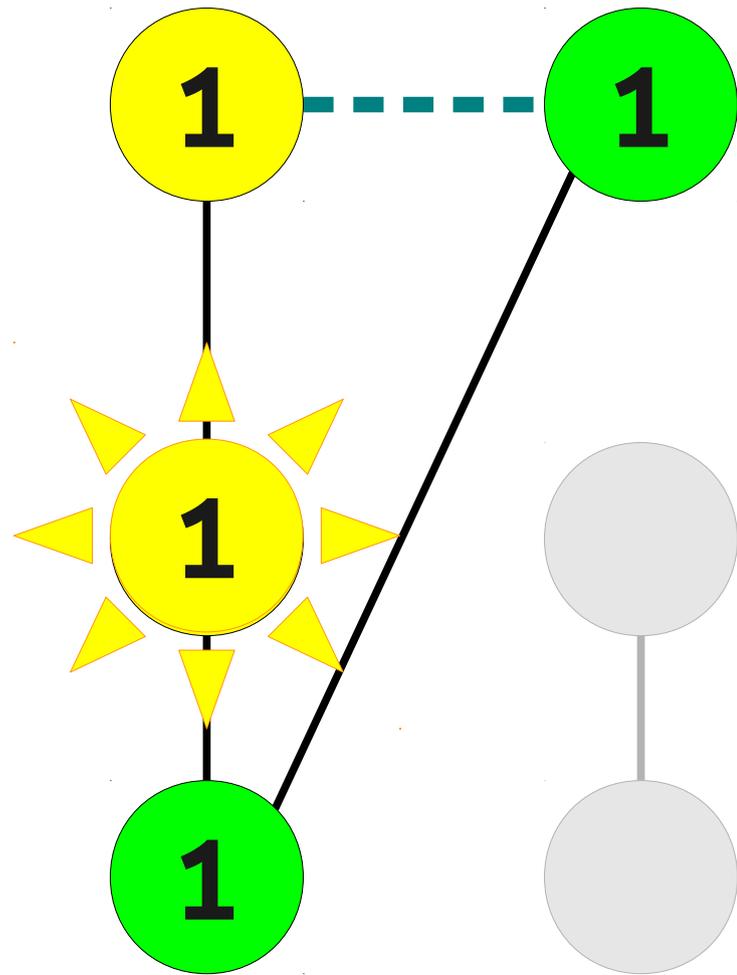
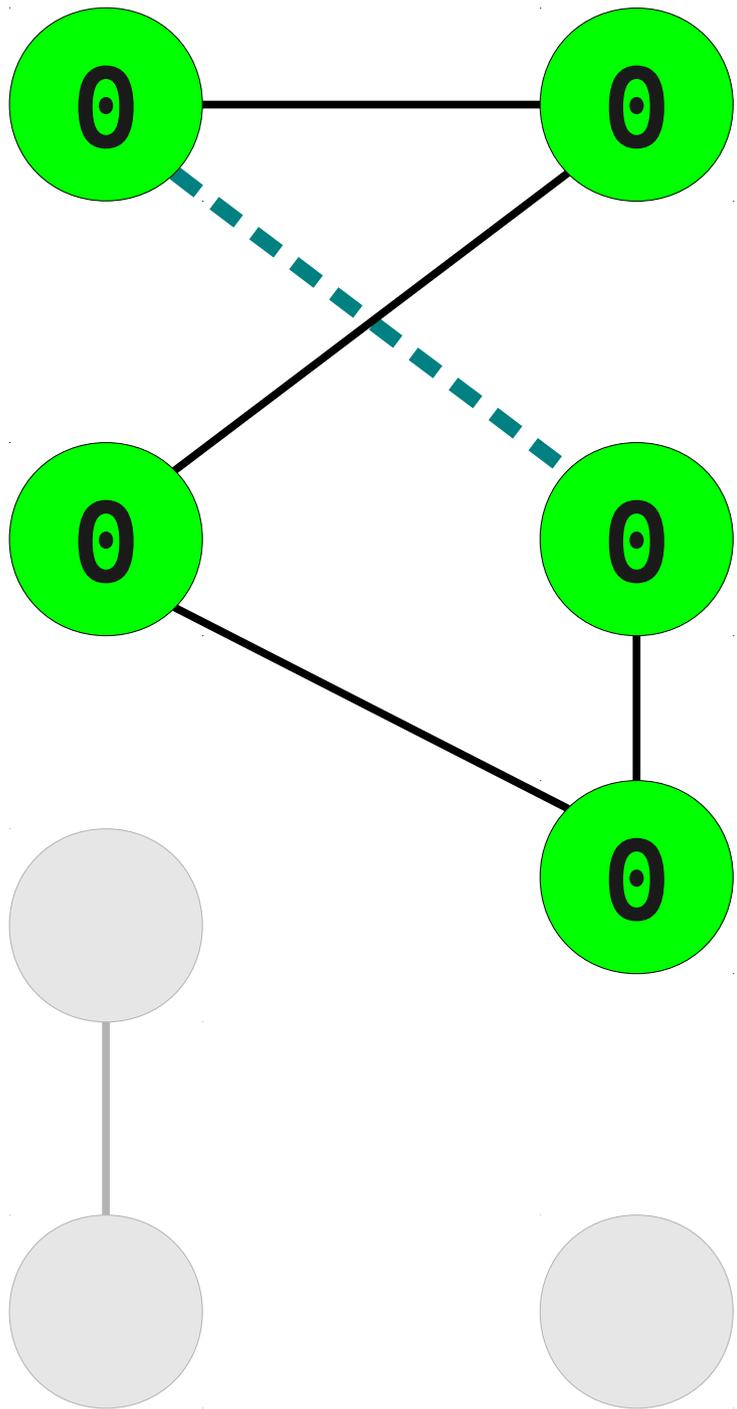


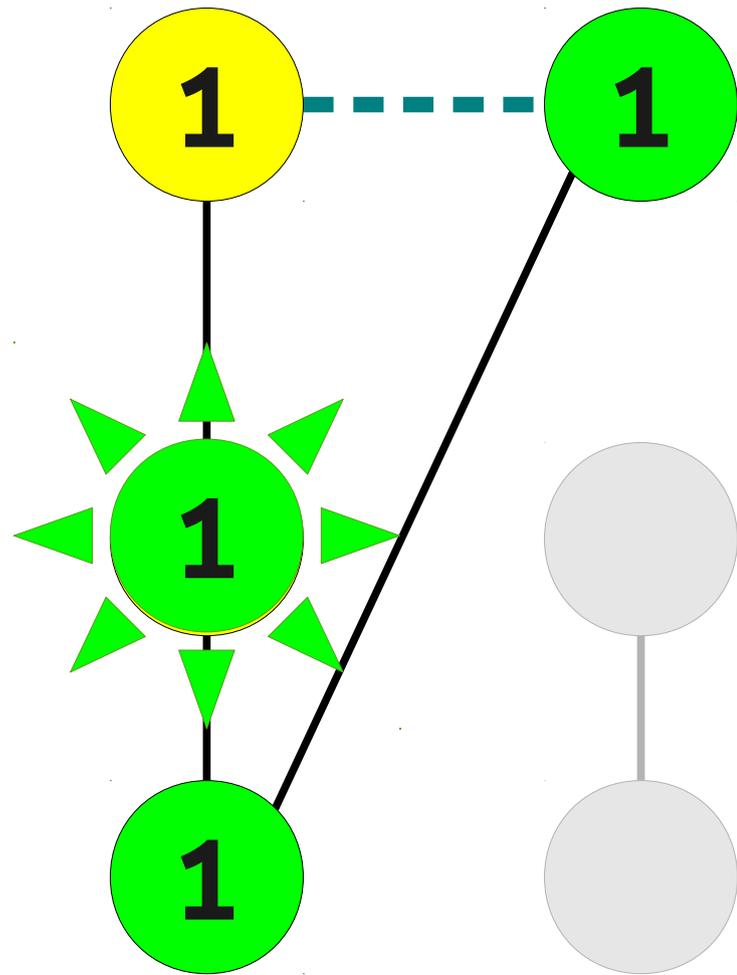
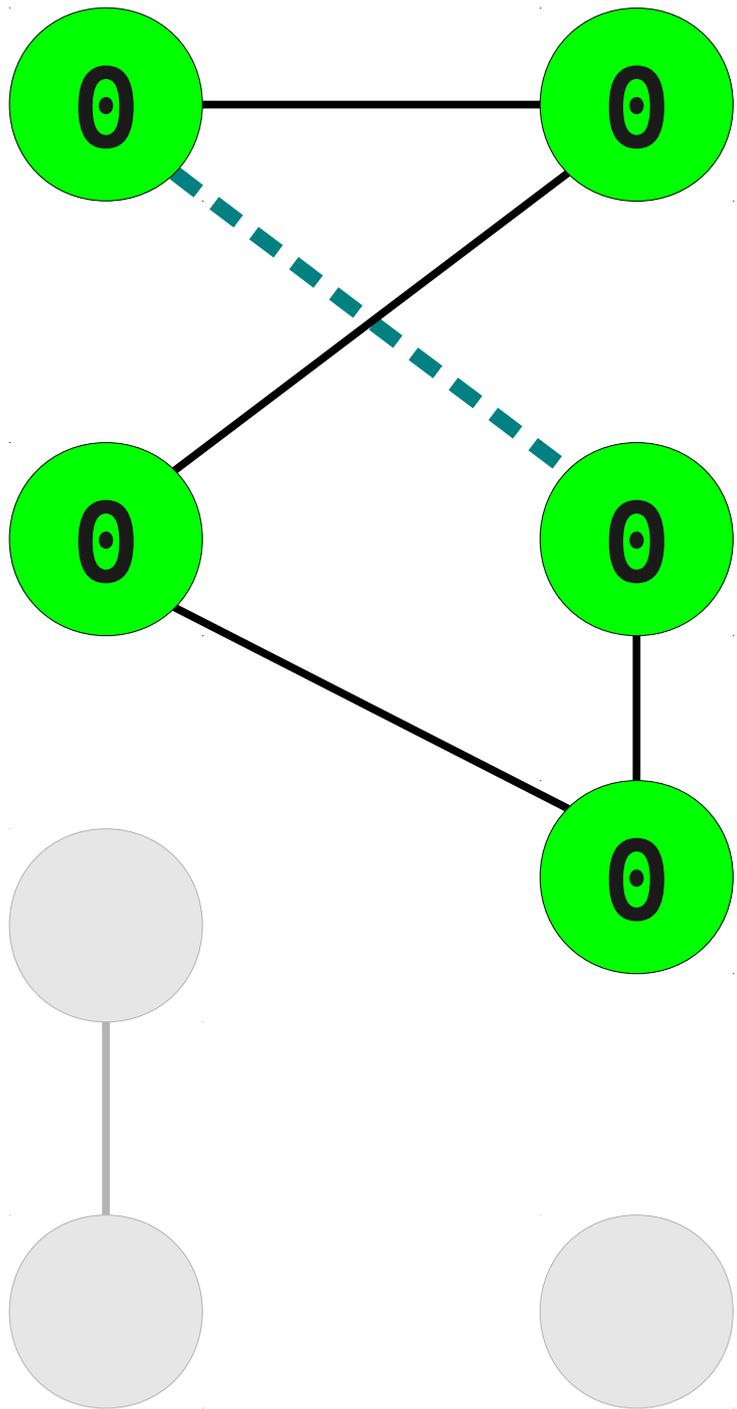


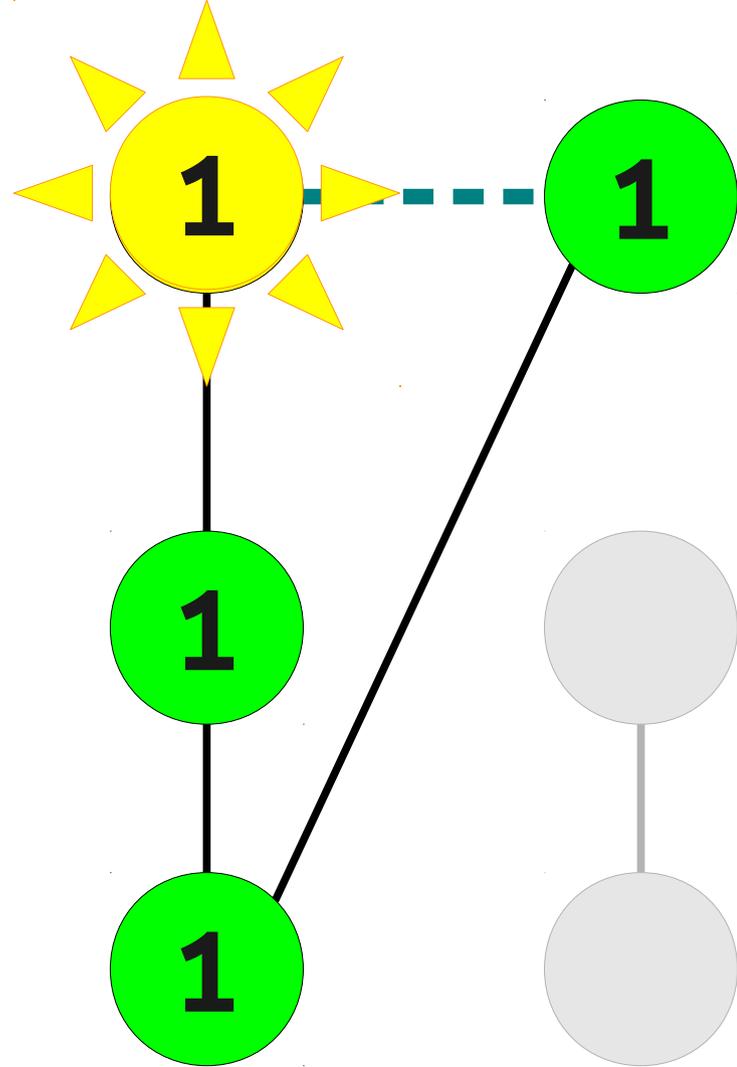
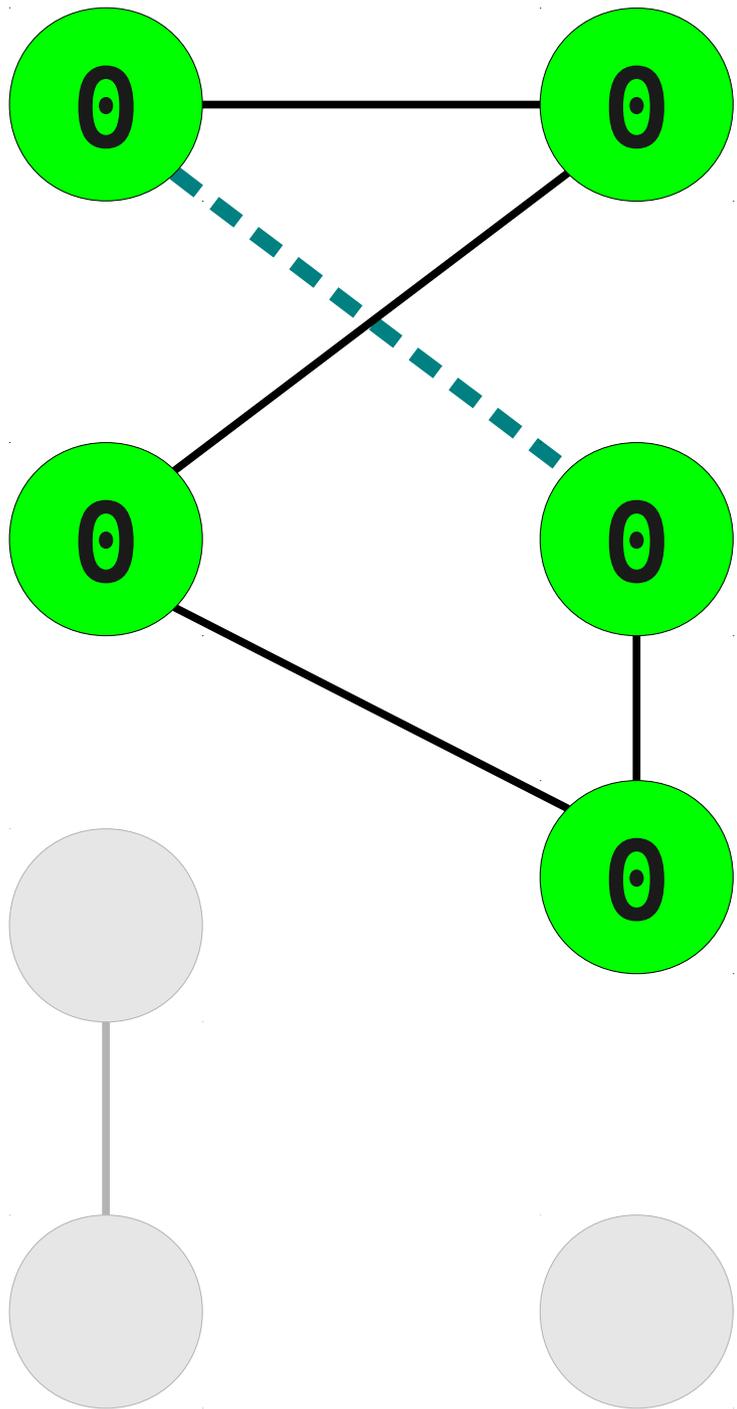


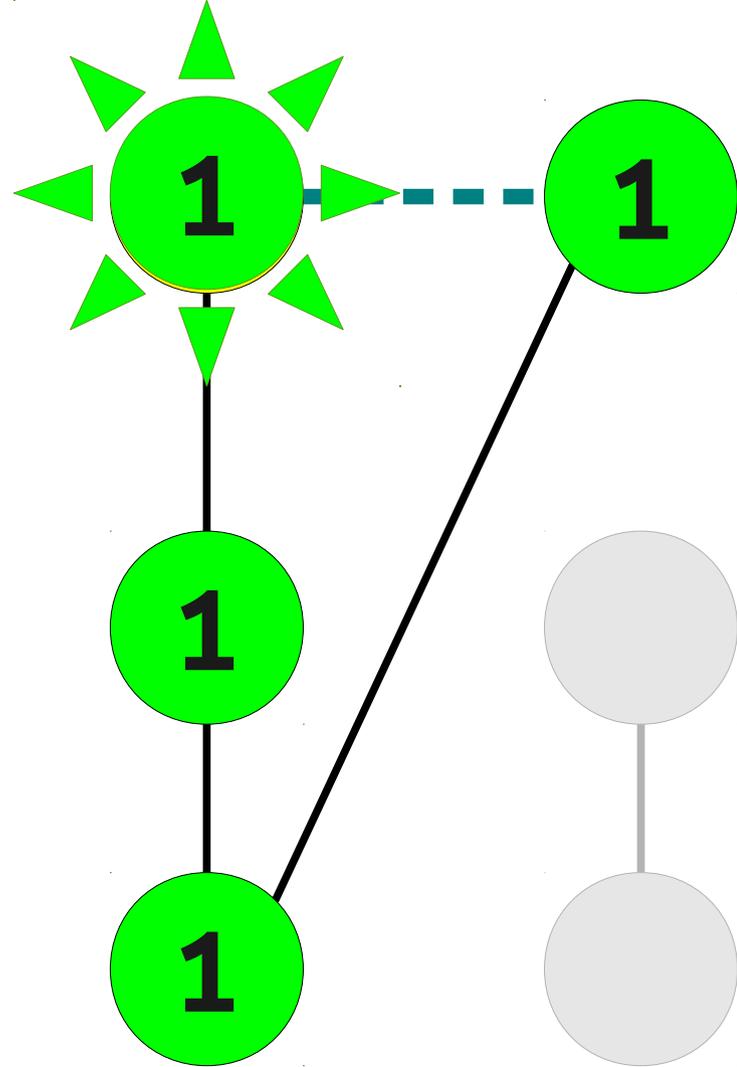
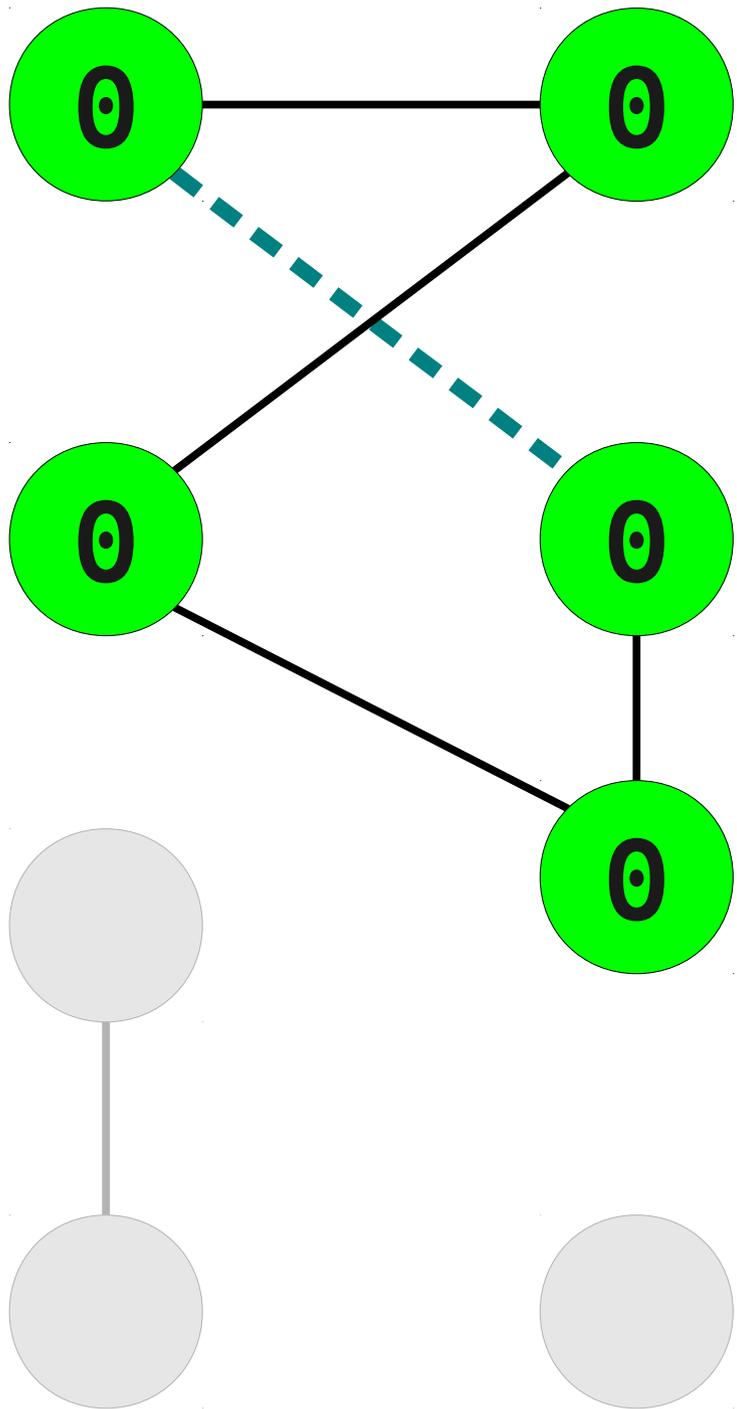


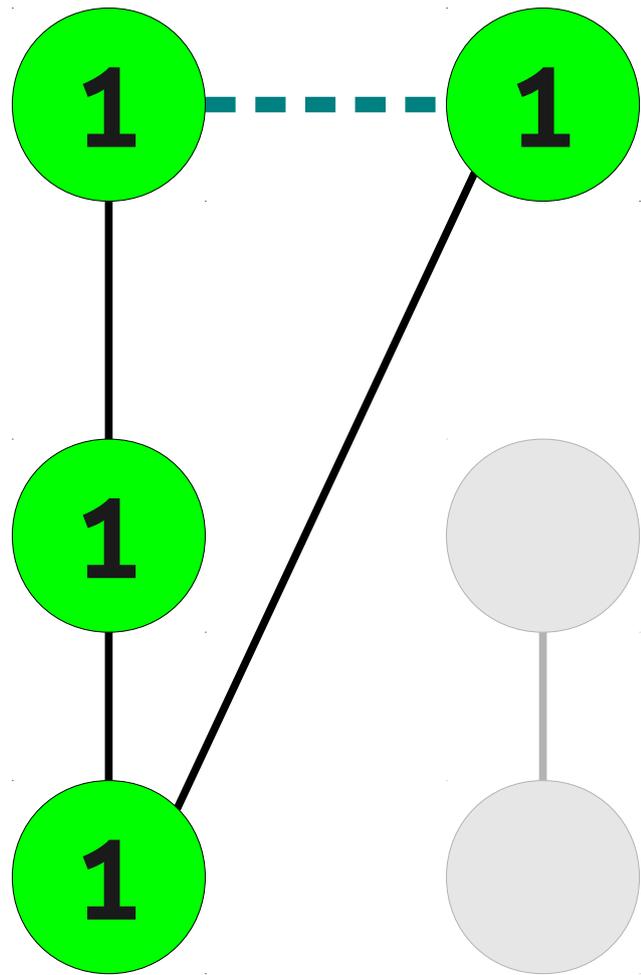
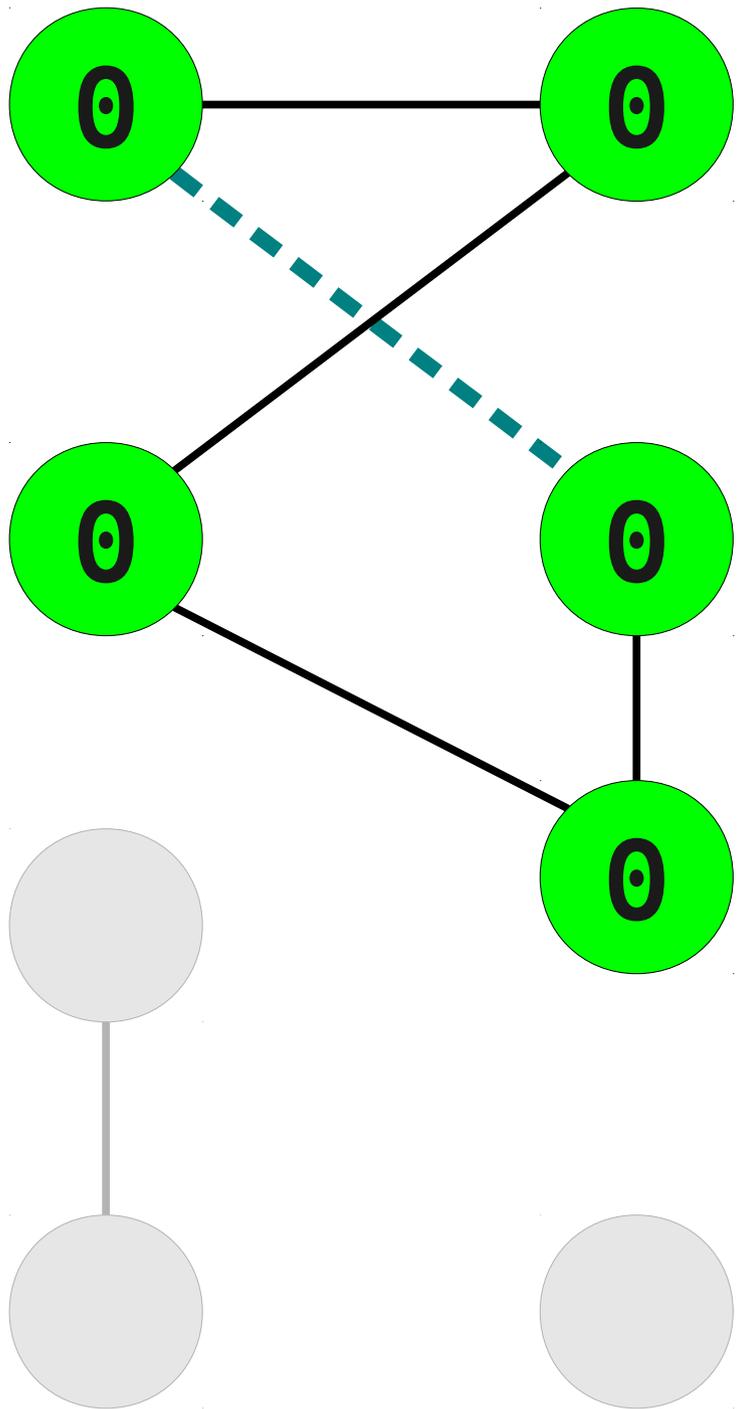


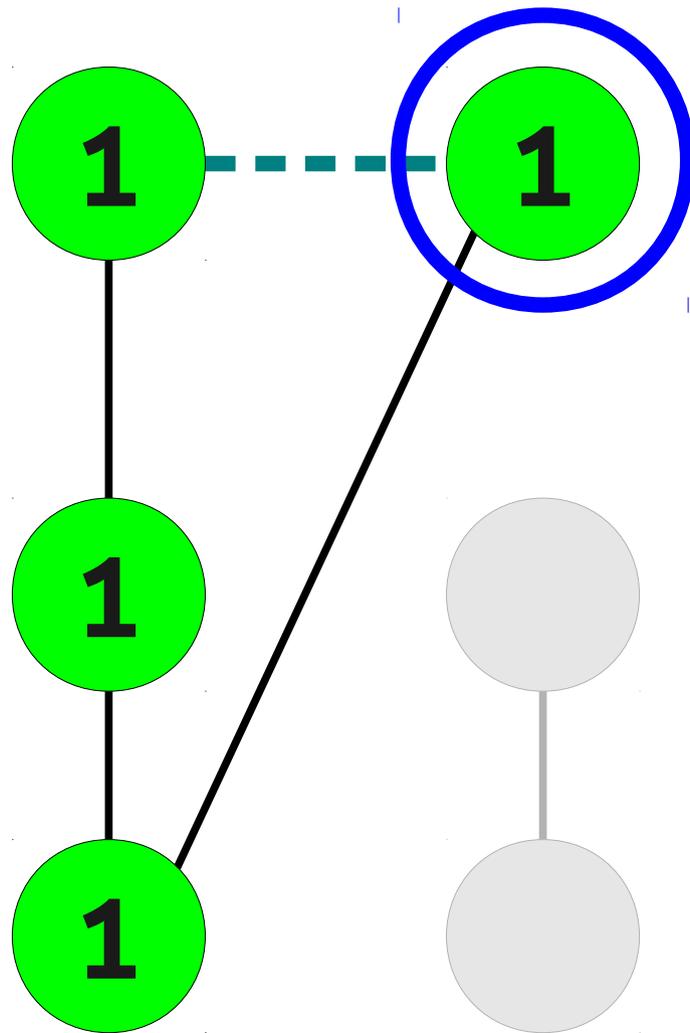
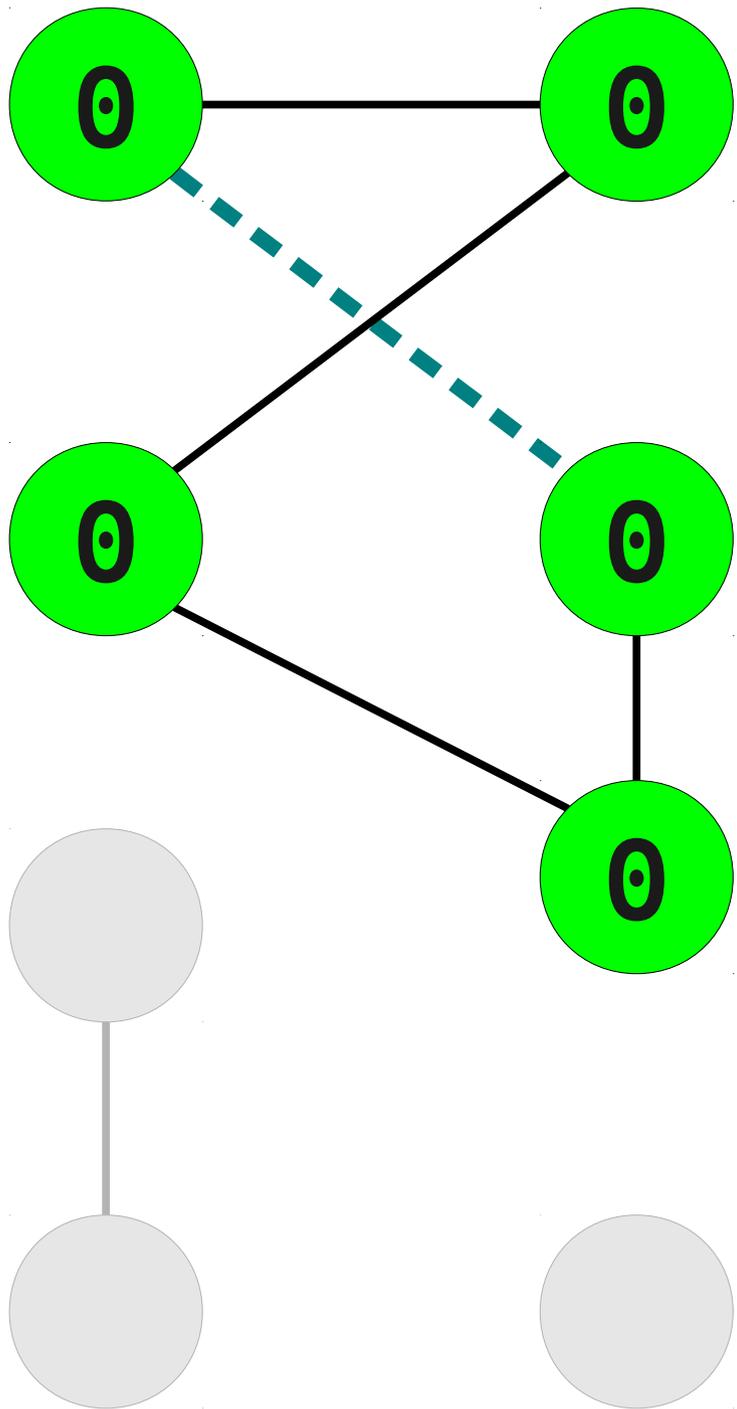


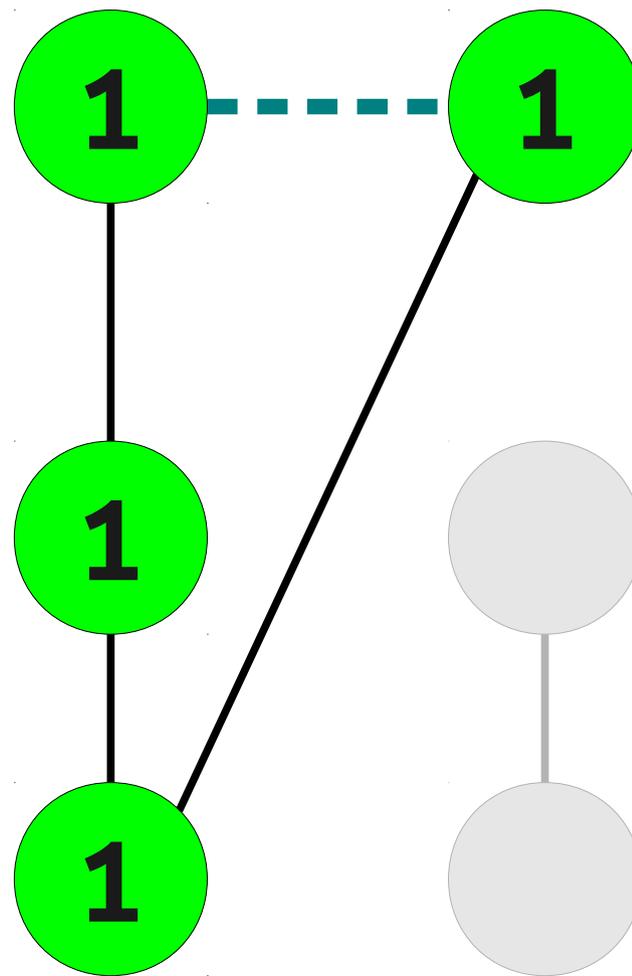
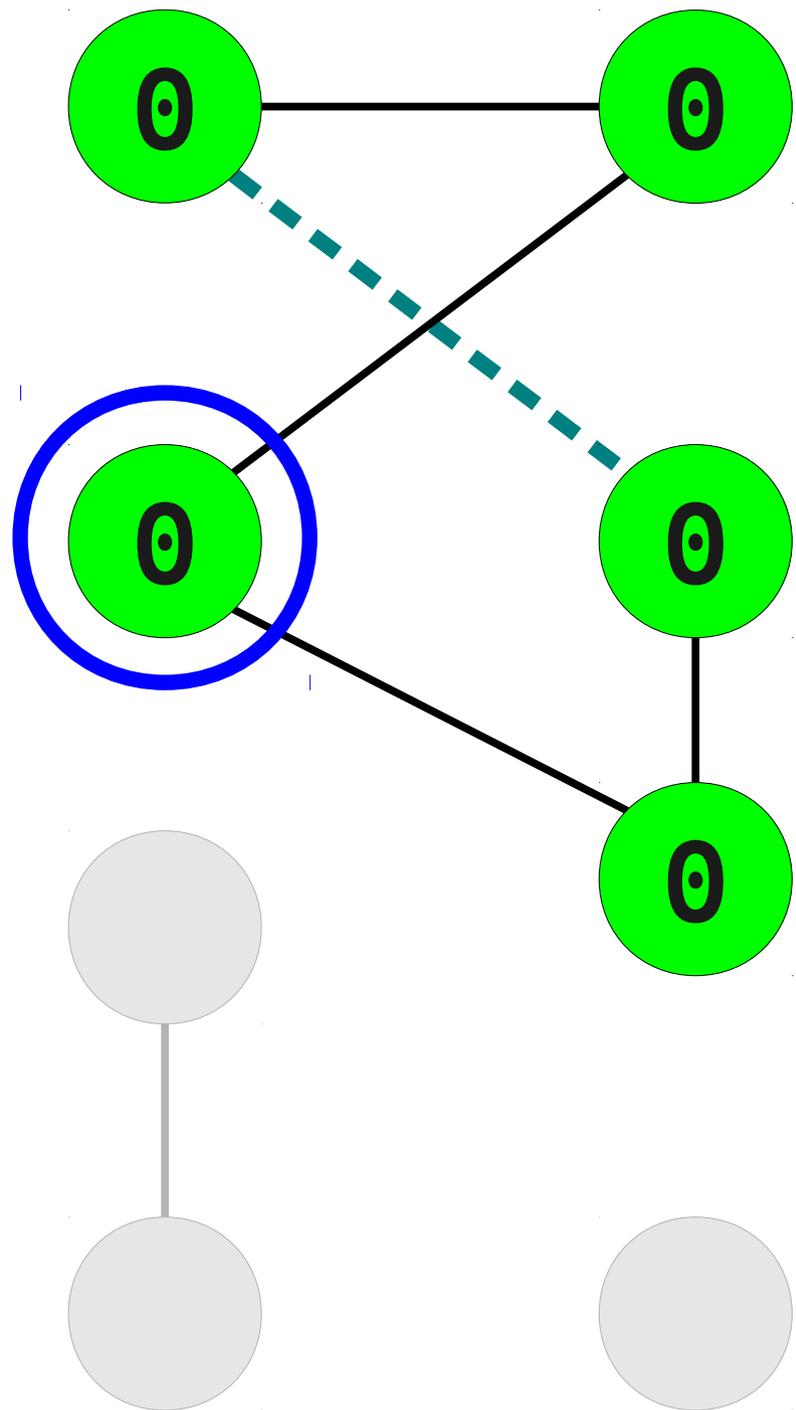


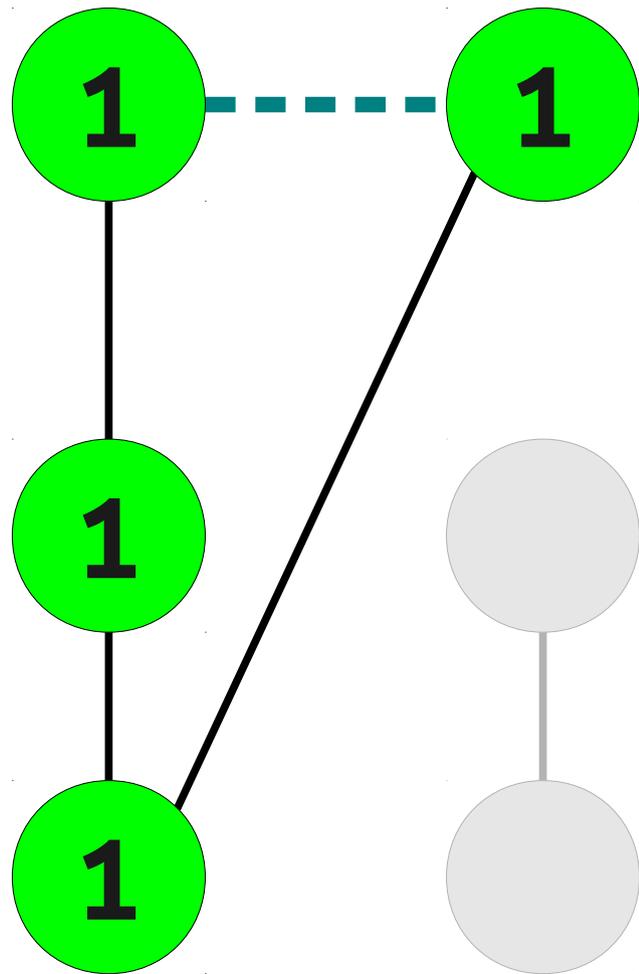
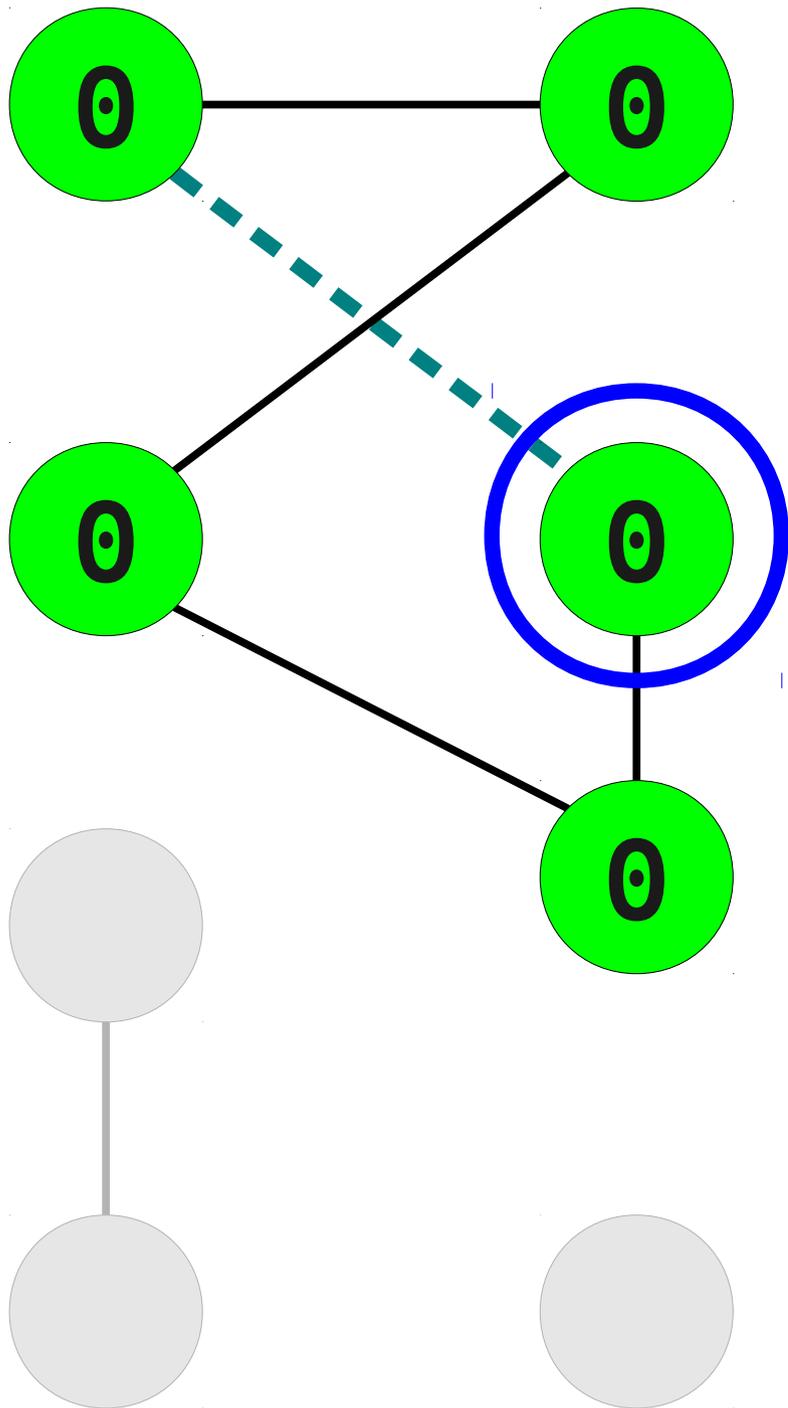


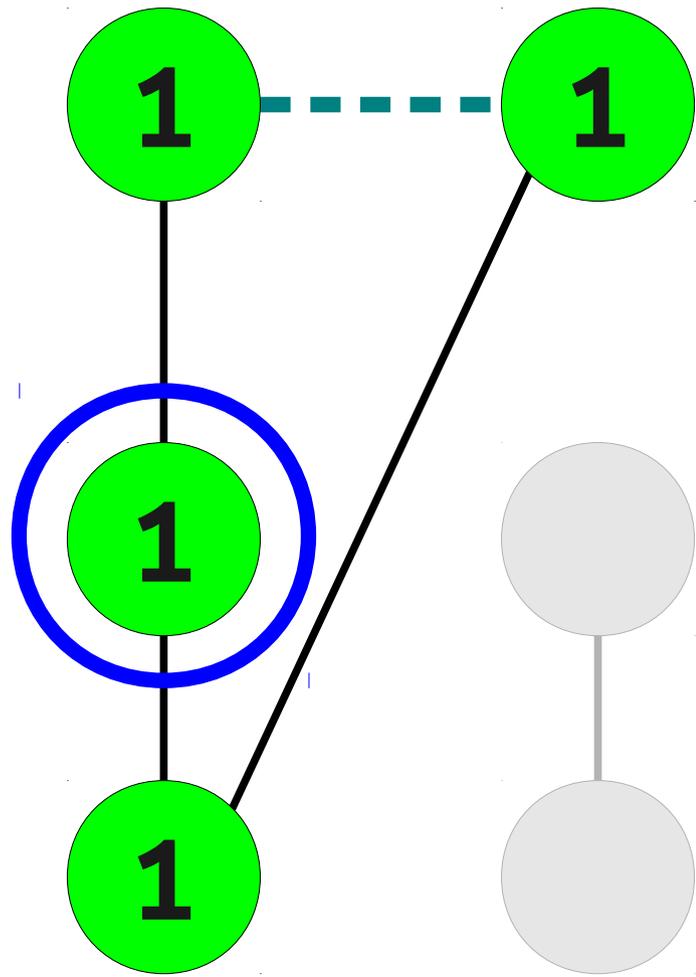
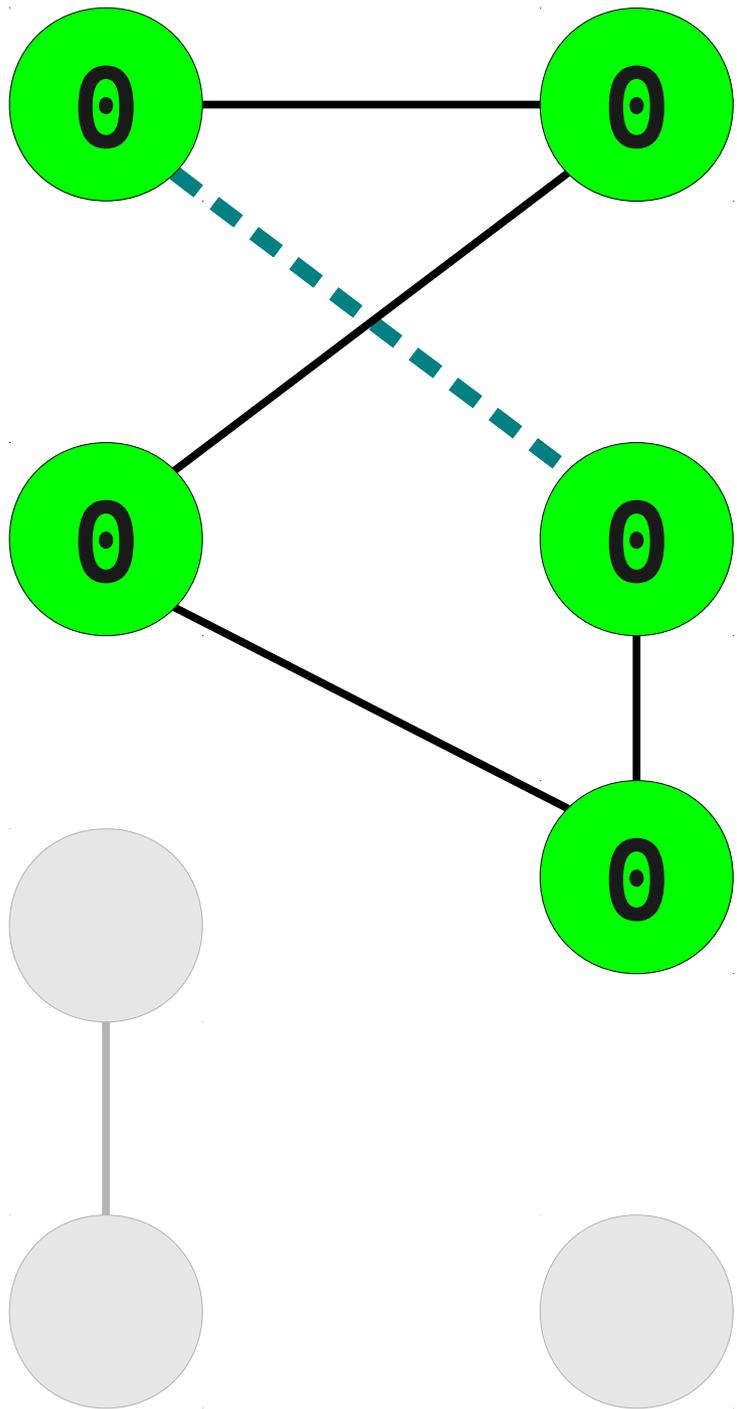


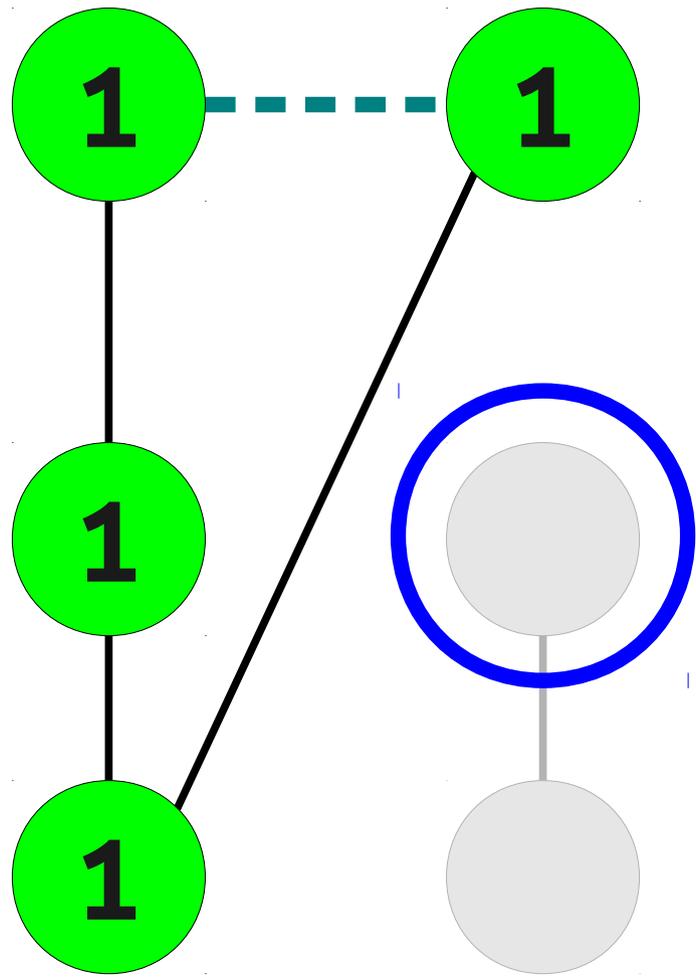
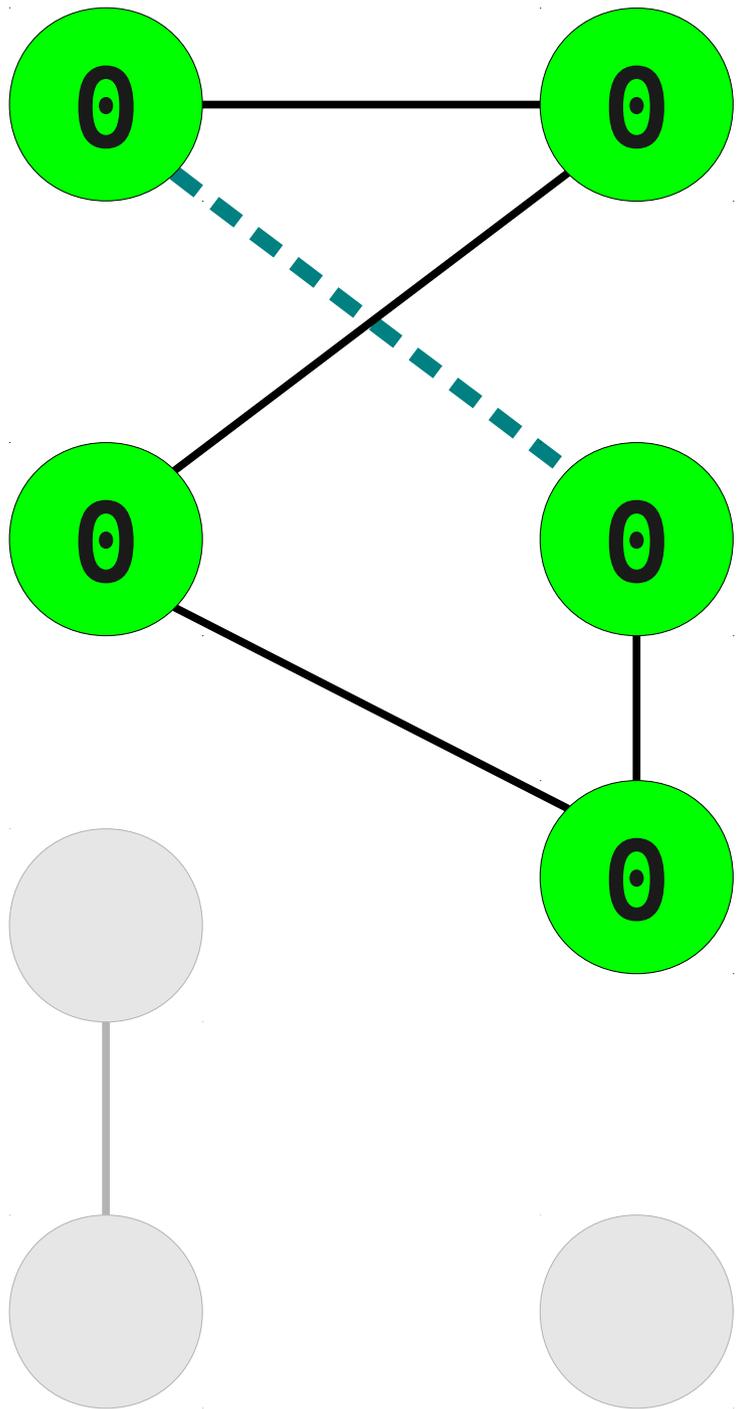


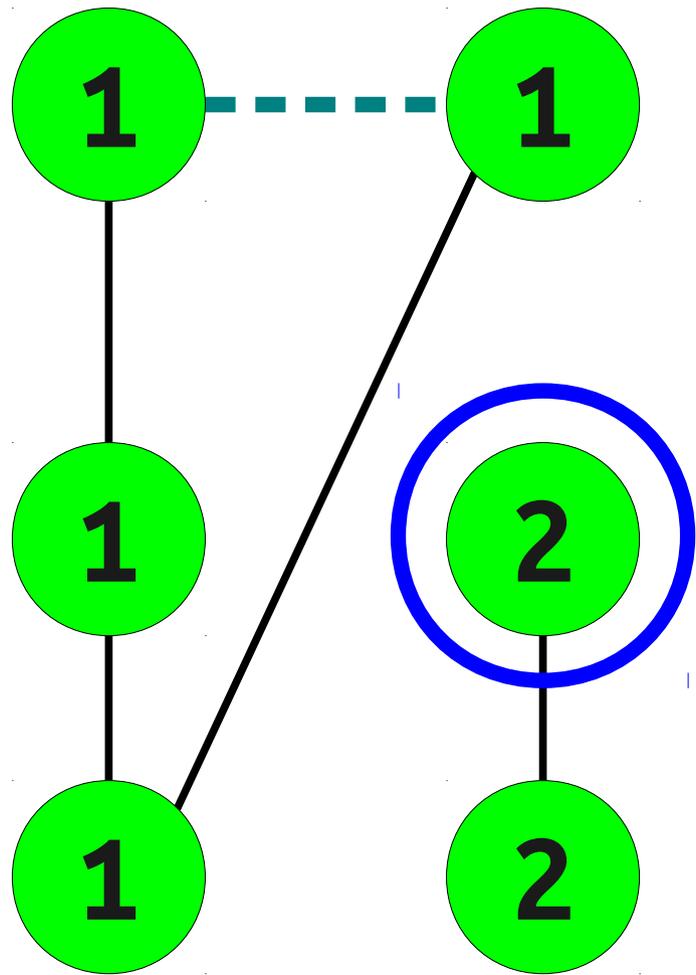
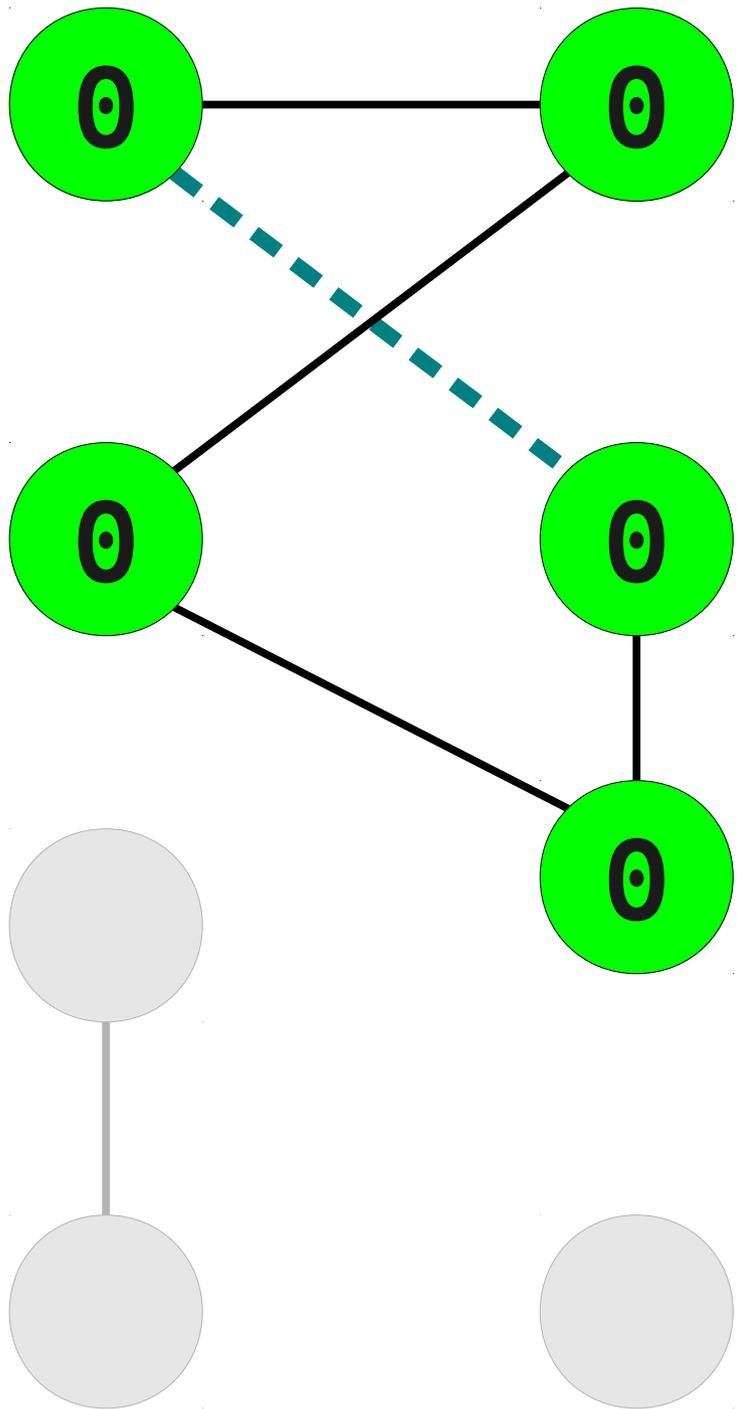












```
procedure findCCs(graph G):  
  for each node v:  
    color v gray  
  
  let cc be an array of size n  
  
  let index = 0  
  for each node in v:  
    if v is gray:  
      run DFS(v), setting cc[u] = index  
      whenever a node u is colored green  
    index = index + 1  
  
  return cc
```

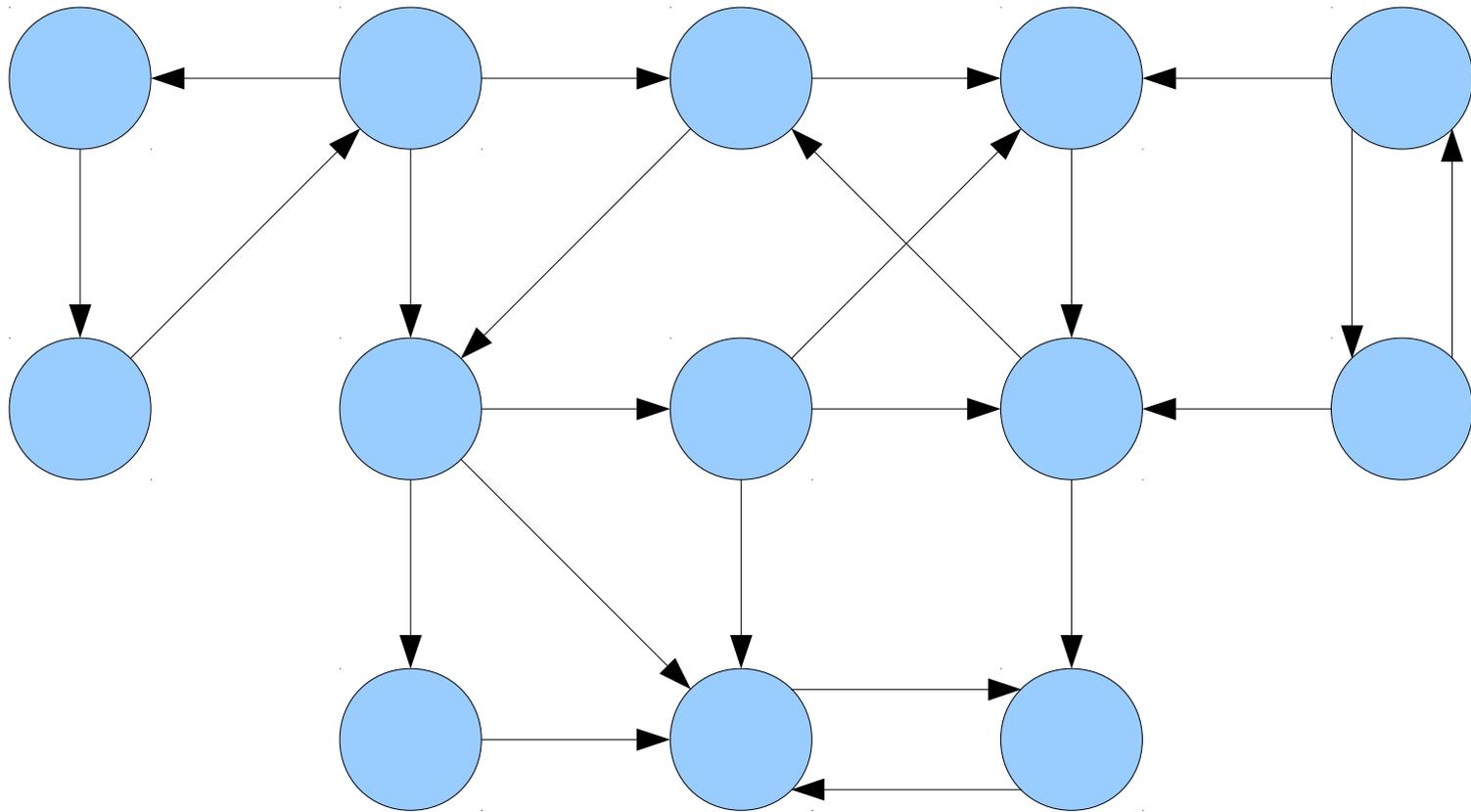
Analyzing the Runtime

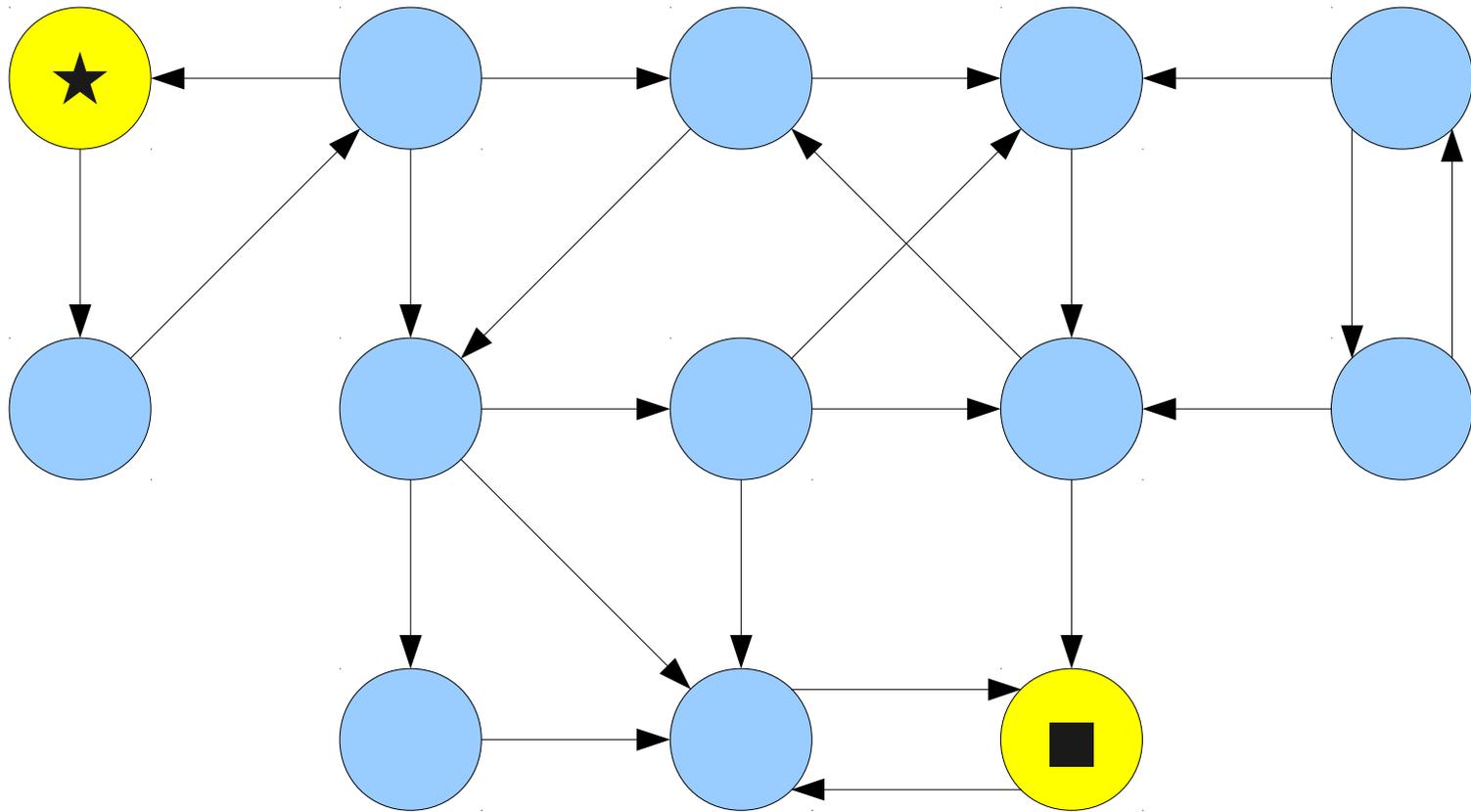
- We do $\Theta(n)$ work initially coloring each node gray.
- Across all iterations of DFS, each node is visited exactly once and each edge is visited exactly once. This takes $\Theta(m + n)$ time.
- Consequently, total work is **$\Theta(m + n)$** .
- Could we also use BFS here? If so, what would the runtime be?

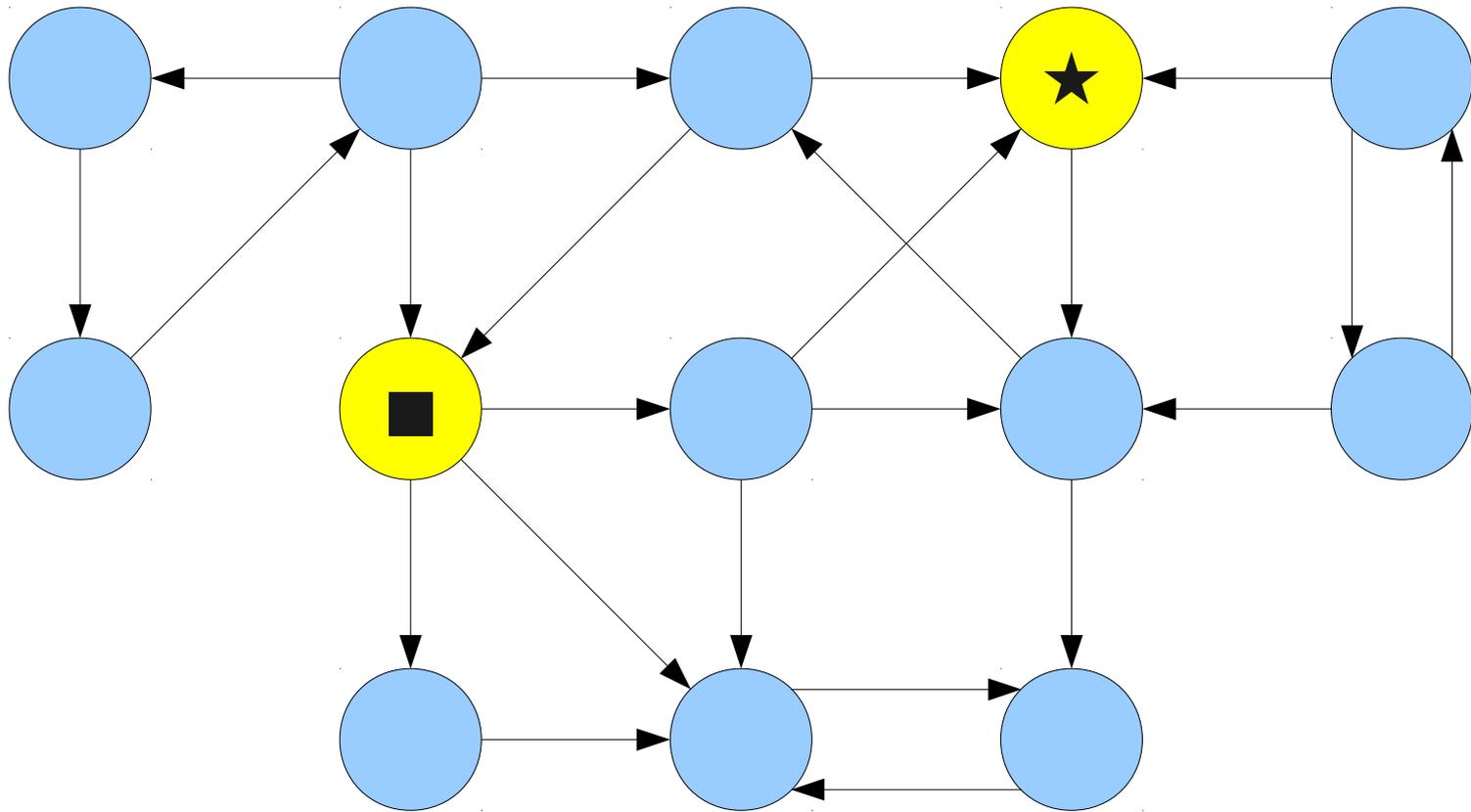
Strongly Connected Components

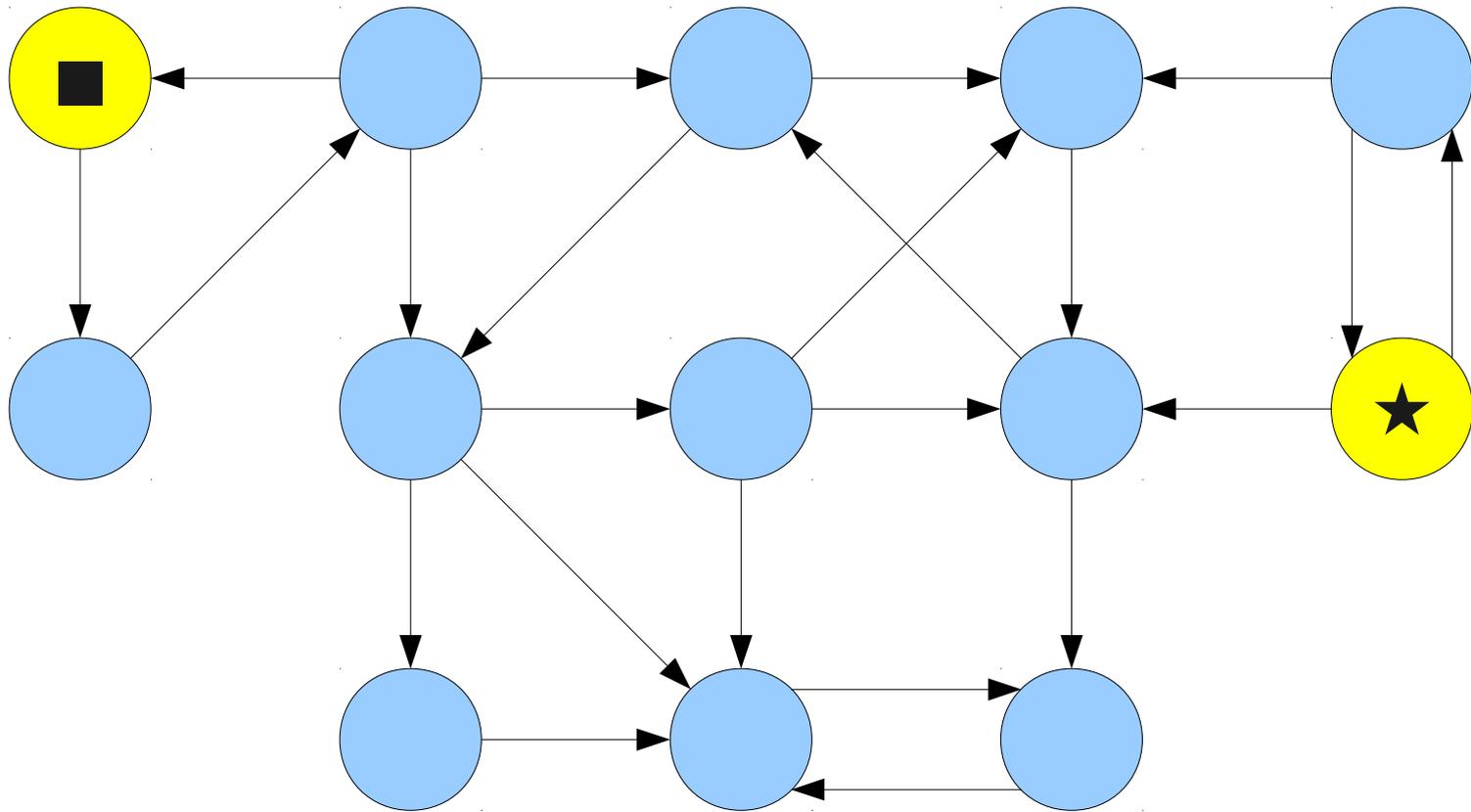
Directed Connectivity

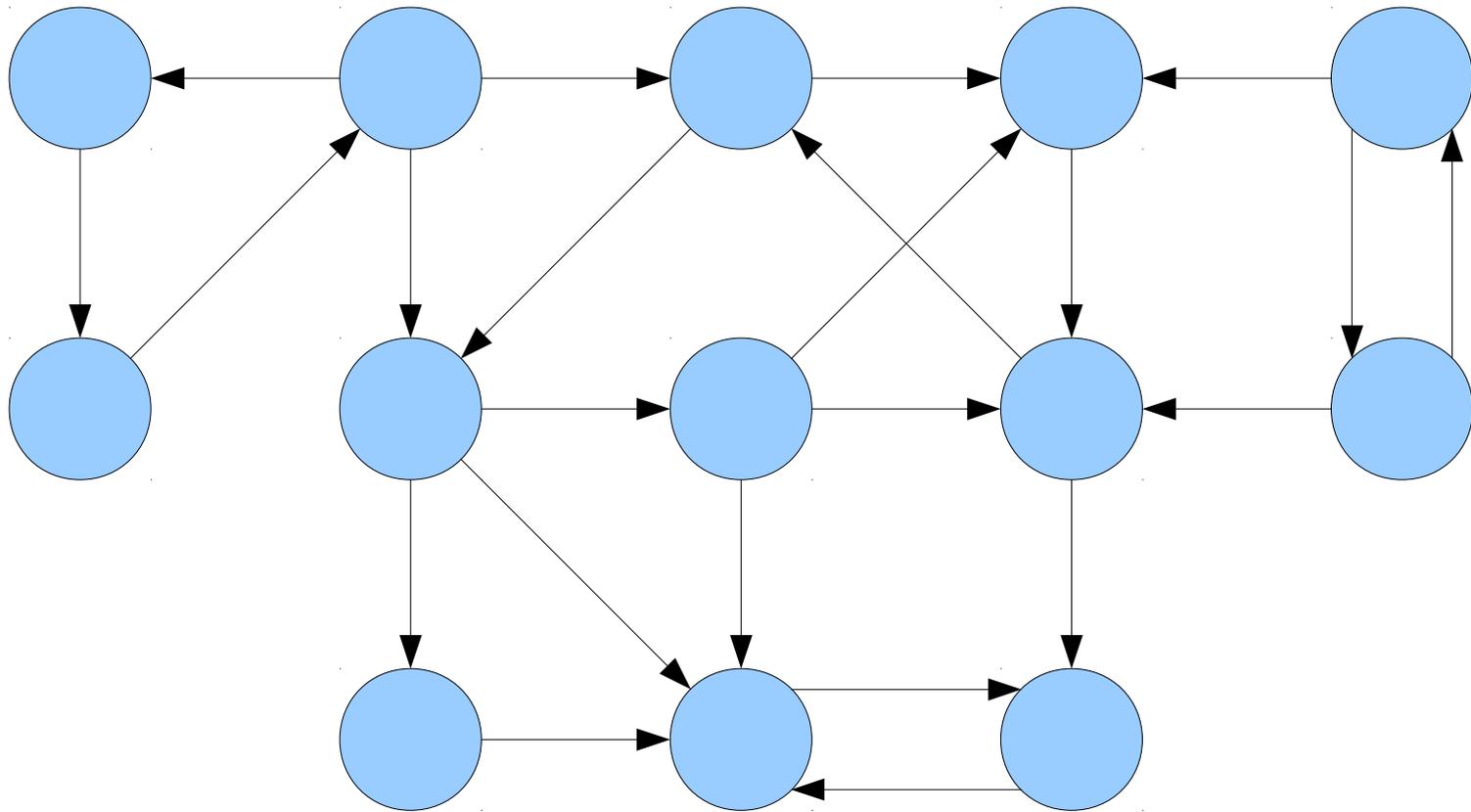
- In a directed graph G , we say v is **reachable** from u iff there is a path from u to v .
- In an undirected graph, if there is a path from u to v , there is also a path from v to u .
- In a directed graph, it is possible for there v to be reachable from u , but for u not to be reachable from v .
- How would we generalize the idea of a connected component to a directed graph?

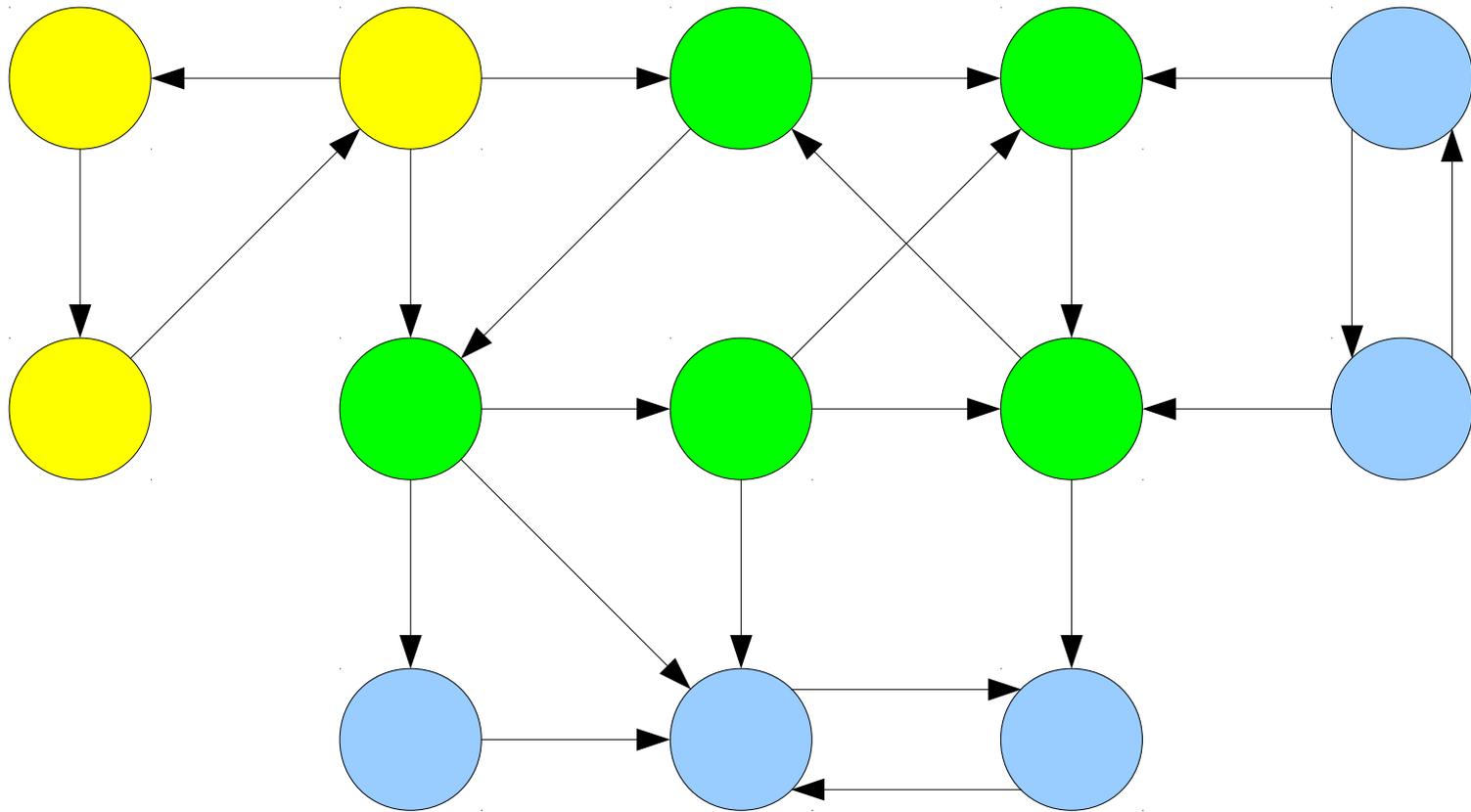


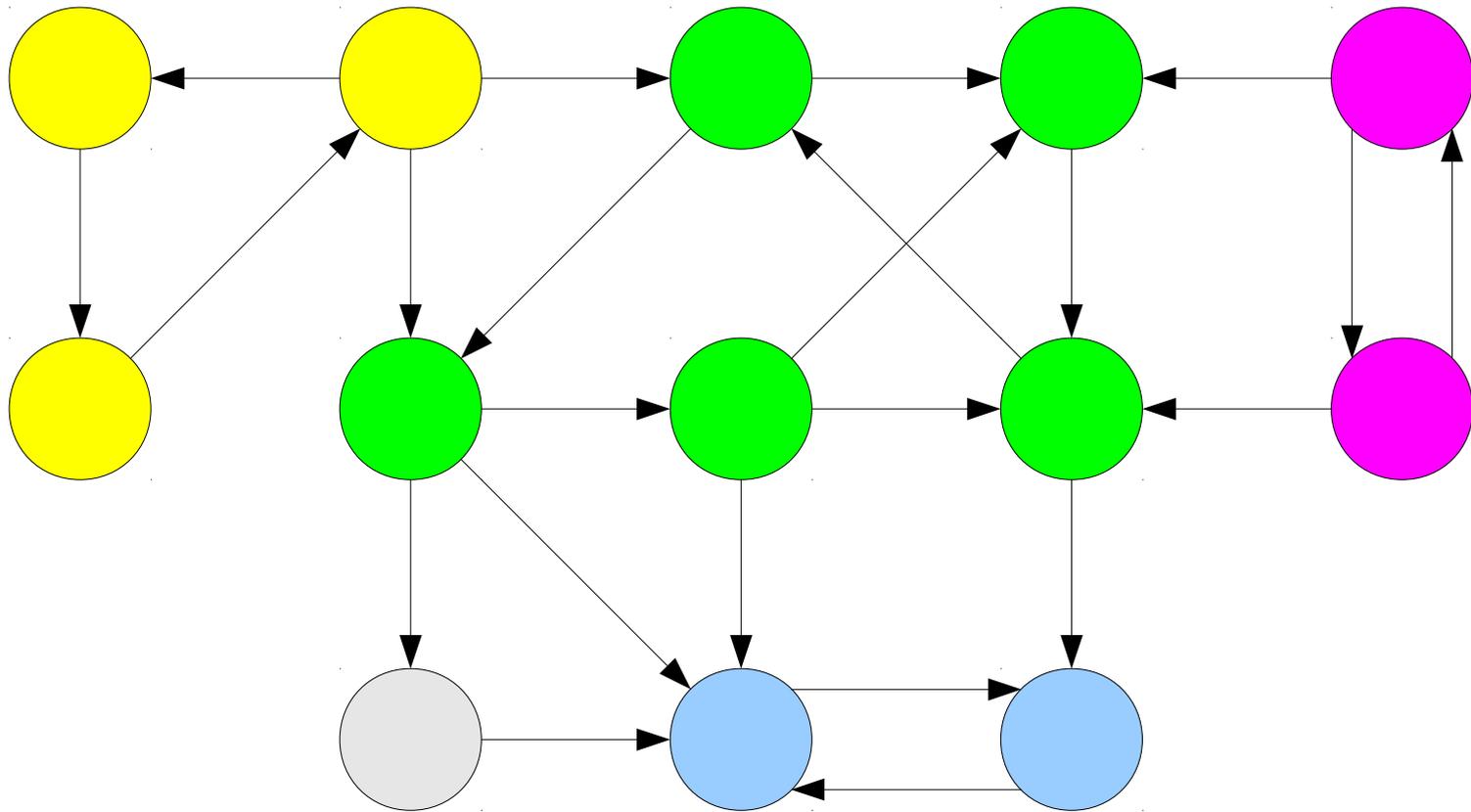


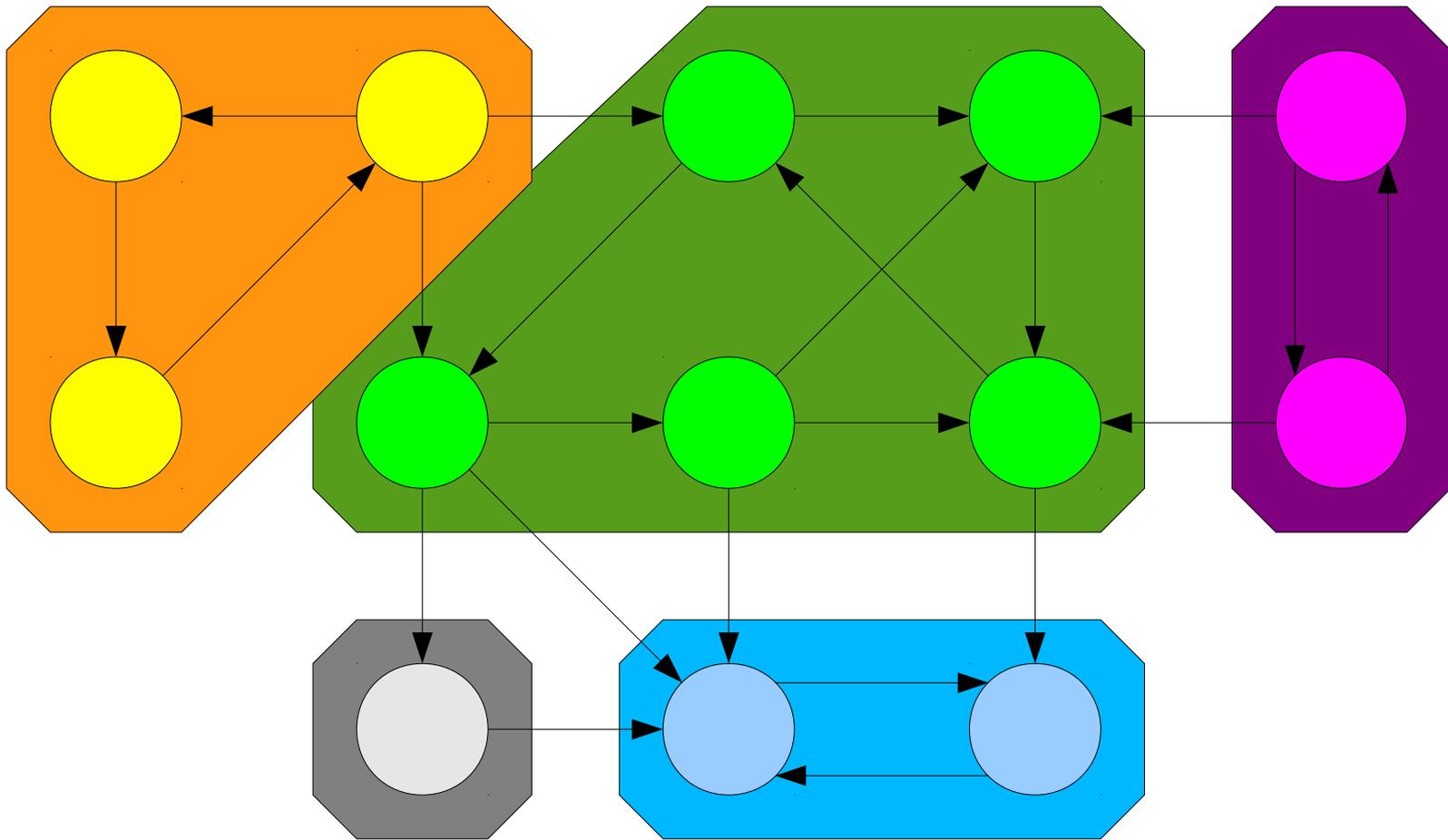






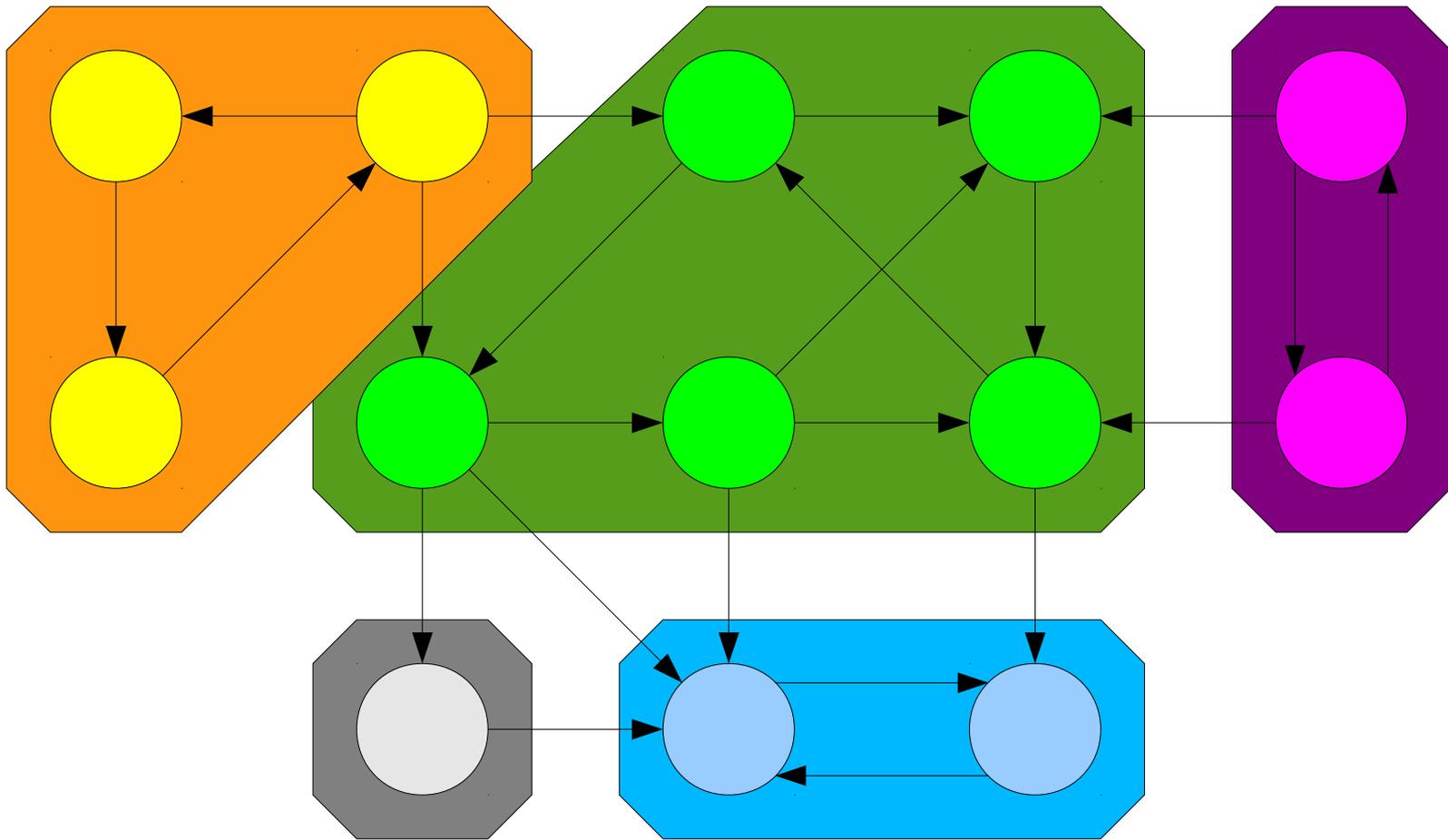


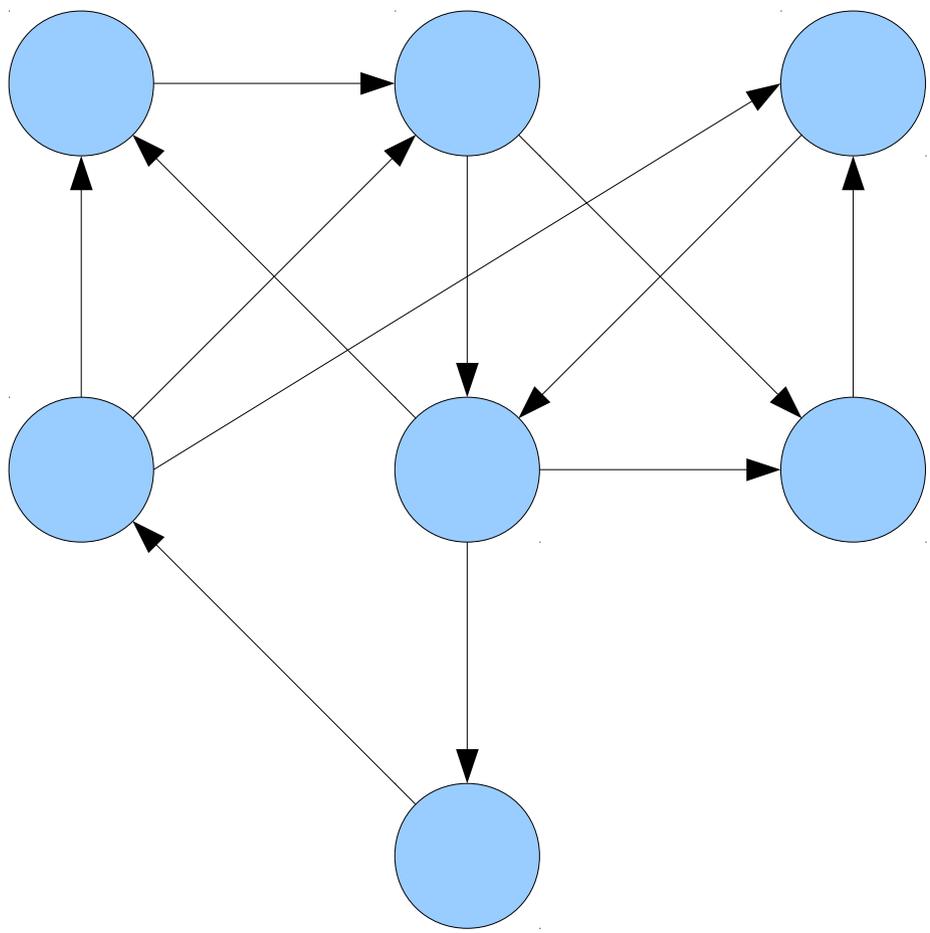


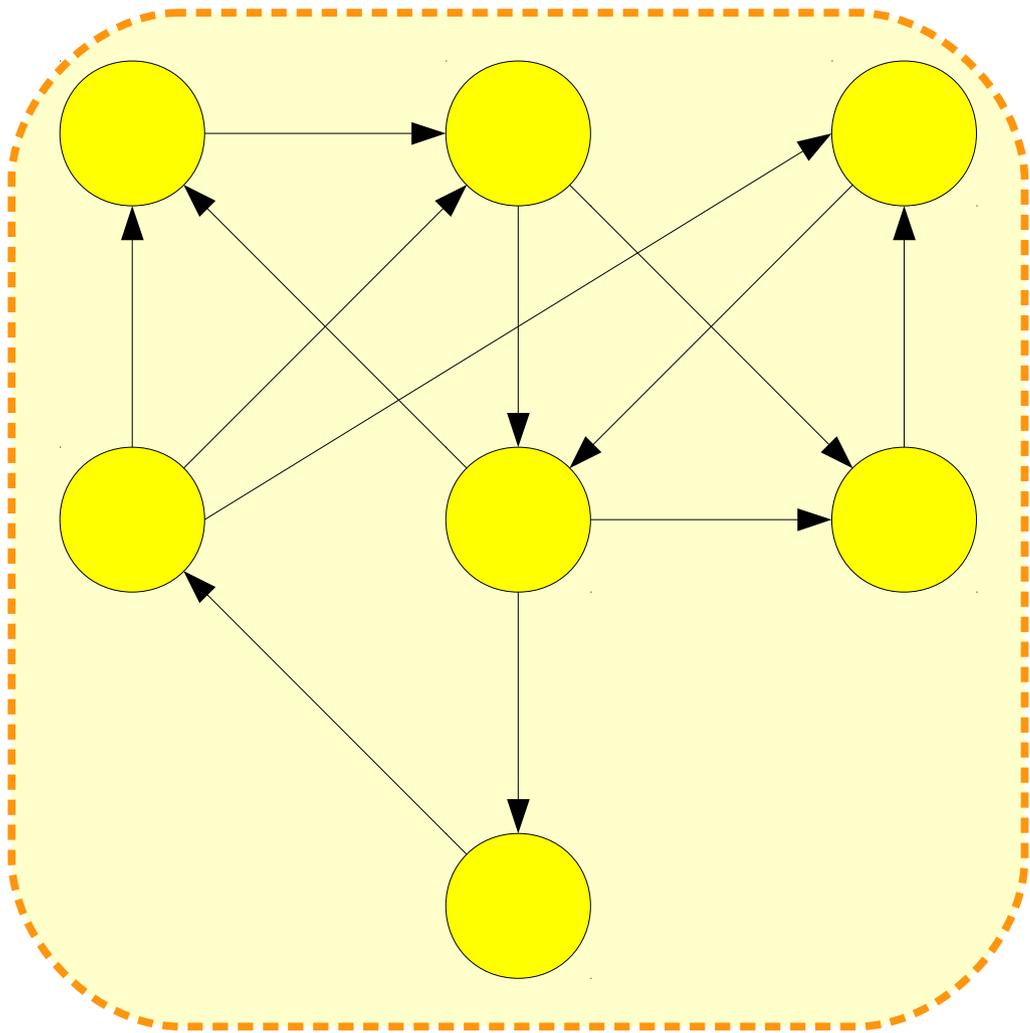


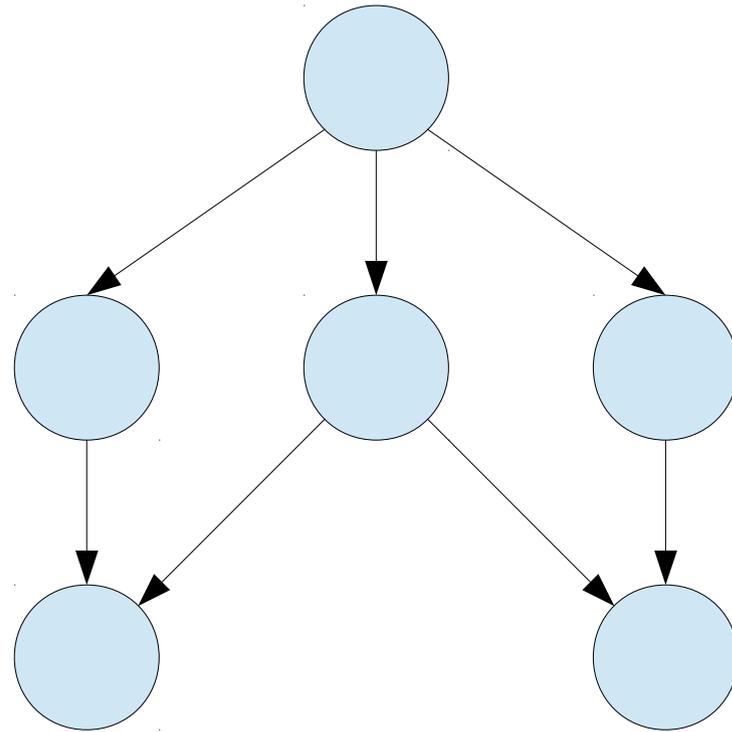
Strongly Connected Components

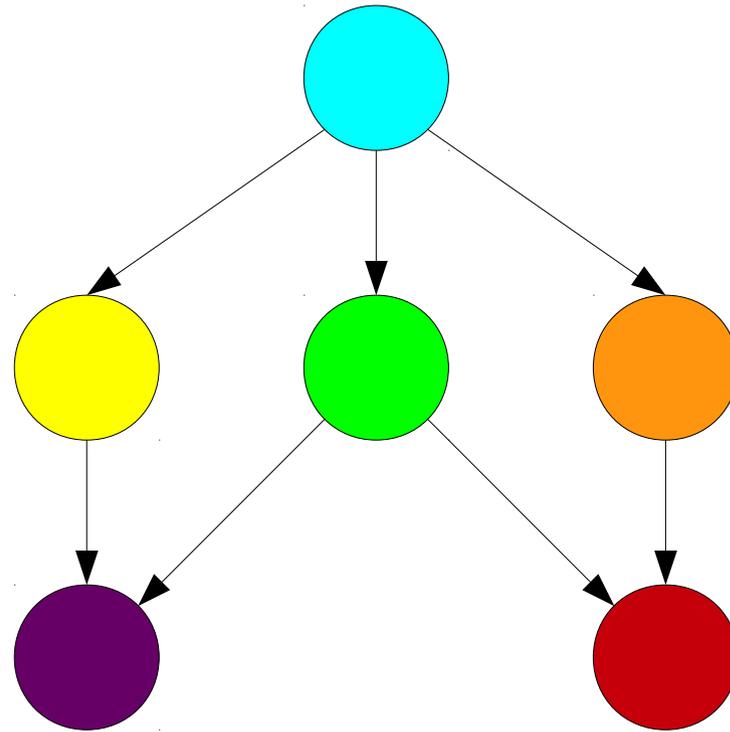
- Let $G = (V, E)$ be a directed graph.
- Two nodes $u, v \in V$ are called **strongly connected** iff v is reachable from u and u is reachable from v .
- A **strongly connected component** (or **SCC**) of G is a set $C \subseteq V$ such that
 - C is not empty.
 - For any $u, v \in C$: u and v are strongly connected.
 - For any $u \in C$ and $v \in V - C$: u and v are not strongly connected.











Properties of SCCs

- The following properties of SCCs are true; it's a good exercise to prove them.
 - Two SCCs C_1 and C_2 are either equal or disjoint.
 - Every node belongs to exactly one SCC.
 - The SCCs of a graph form a partition of the nodes of the graph.

Finding SCCs

- Every graph must have a collection of SCCs.
- In the *undirected* case, it was easy to find all the connected components of a graph by using DFS or BFS.
- Will this find all SCCs in a directed graph?
- **Question:** How can we determine all of the strongly connected components of a directed graph G ?

A Beautiful Observation

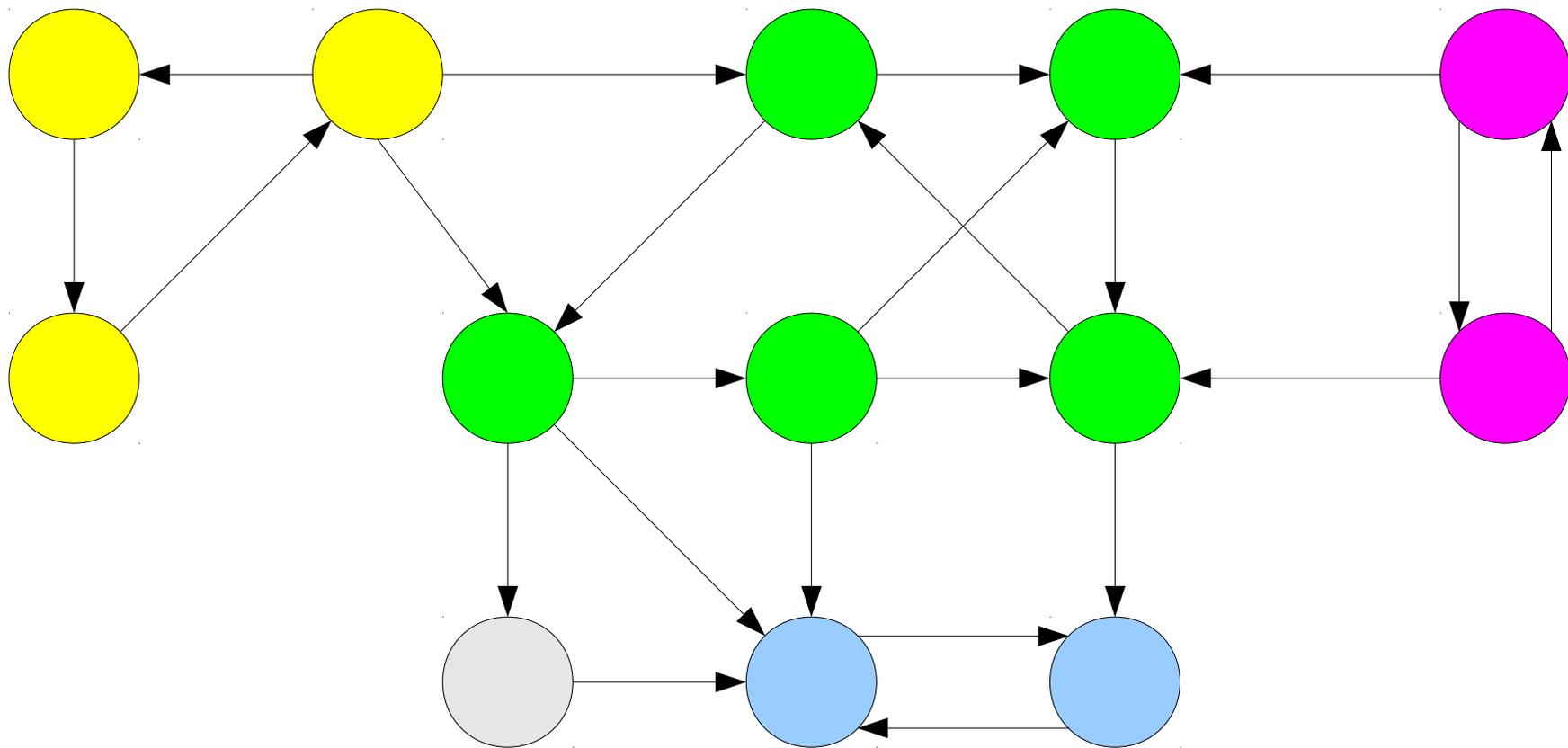
Condensation Graphs

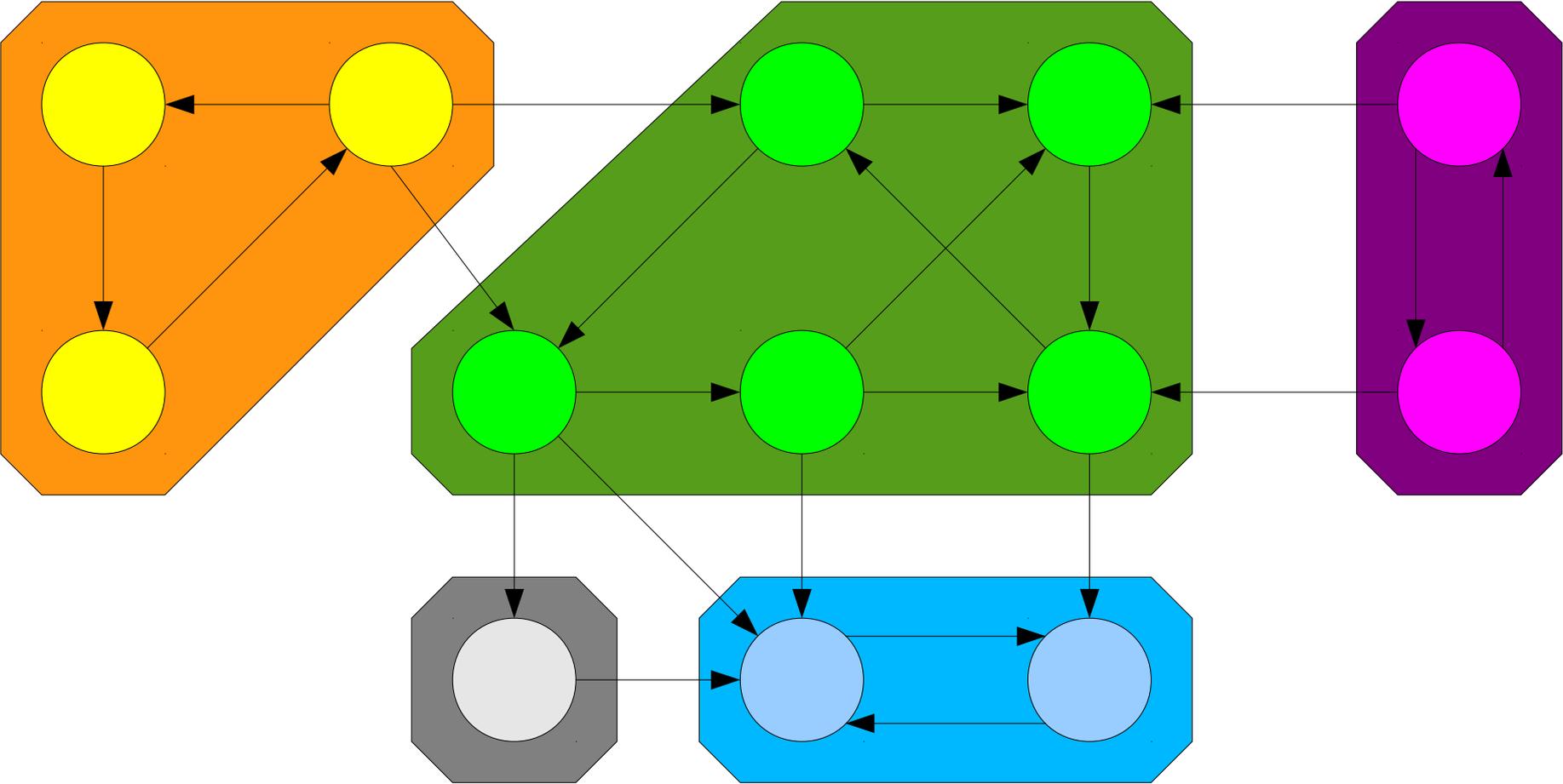
- The **condensation** of a directed graph G is the directed graph G^{SCC} whose nodes are the SCCs of G and whose edges are defined as follows:

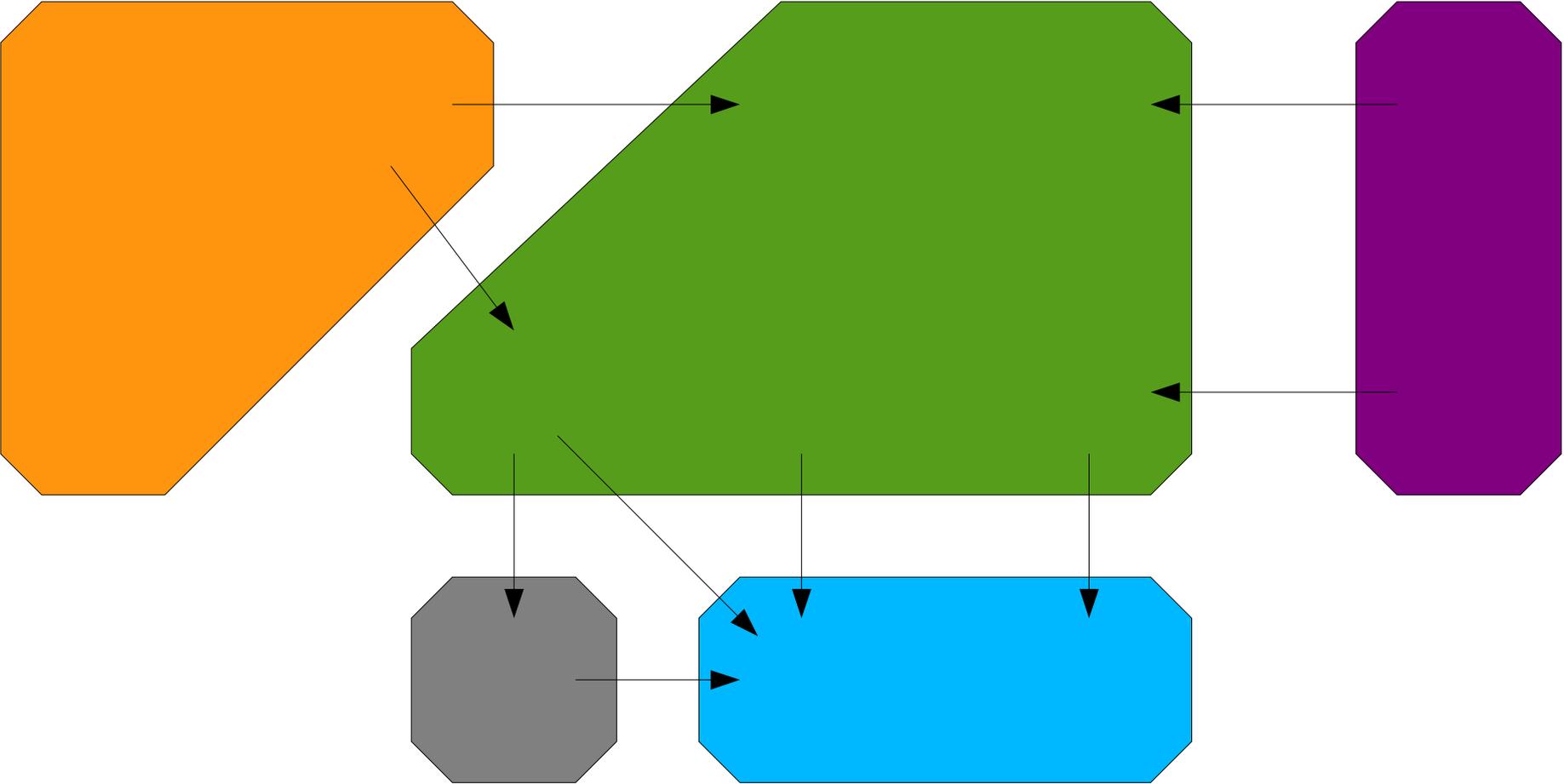
(C_1, C_2) is an edge in G^{SCC} iff

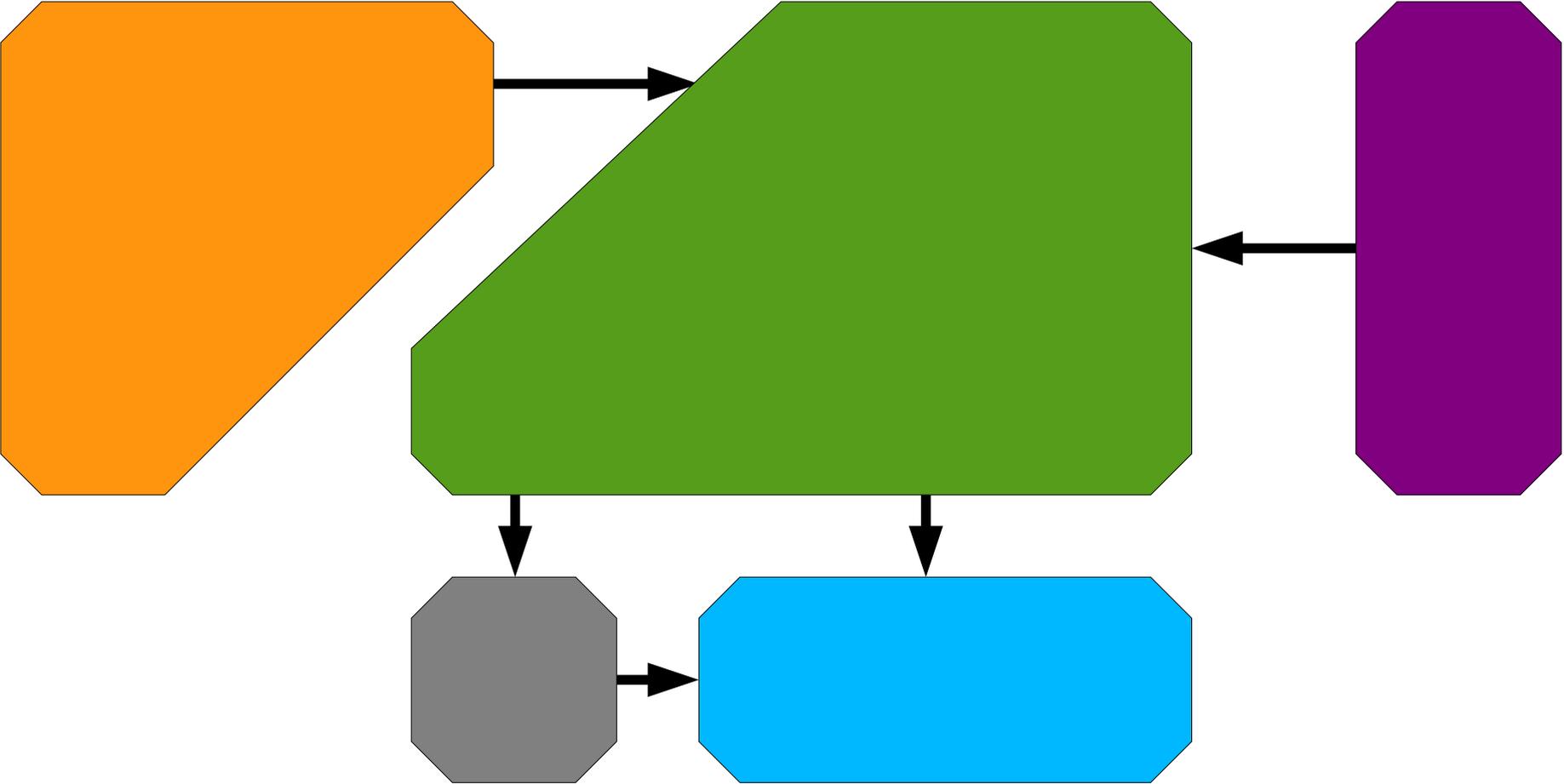
$\exists u \in C_1, v \in C_2. (u, v)$ is an edge in G .

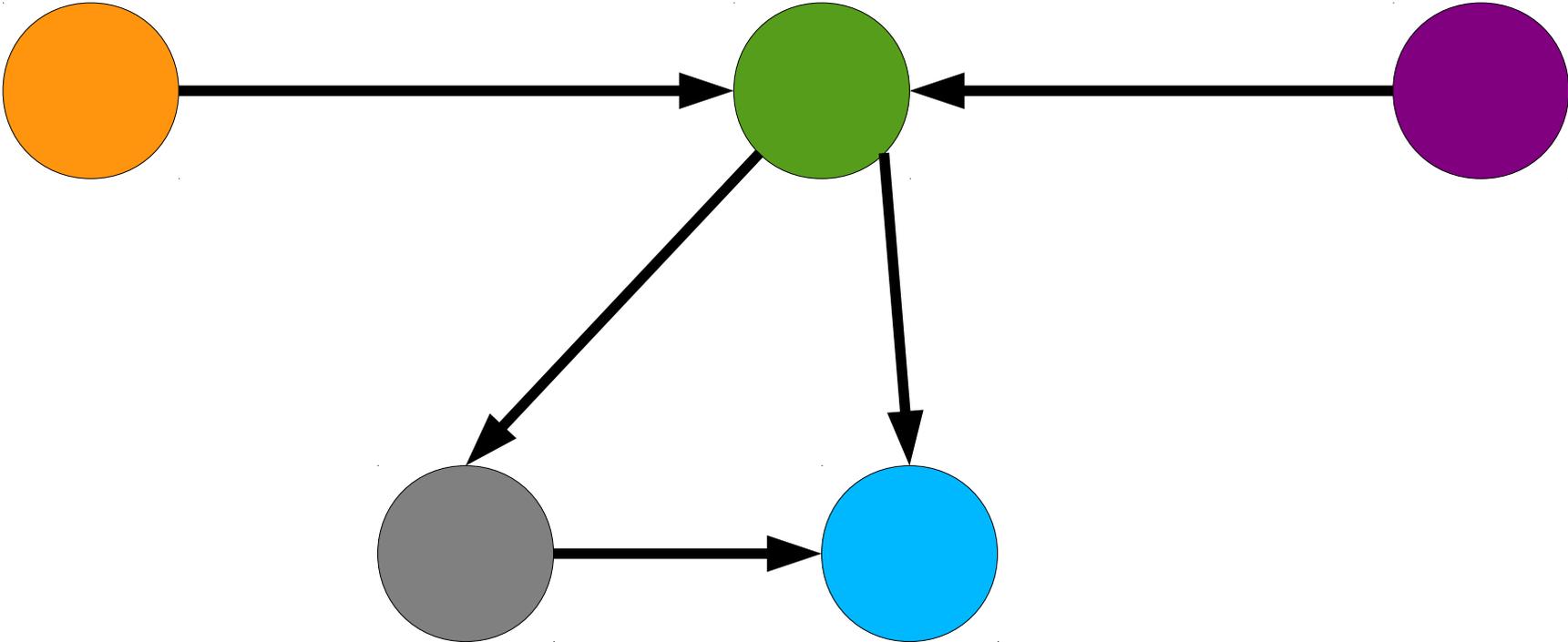
- In other words, if there is an edge in G from *any* node in C_1 to *any* node in C_2 , there is an edge in G^{SCC} from C_1 to C_2 .

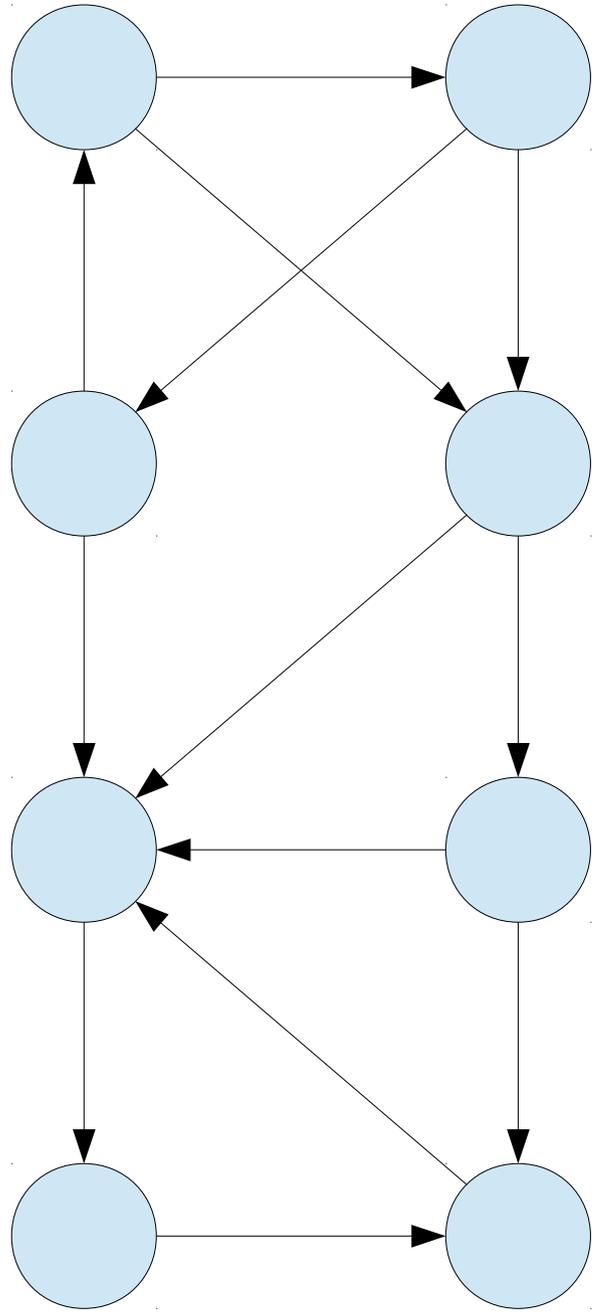


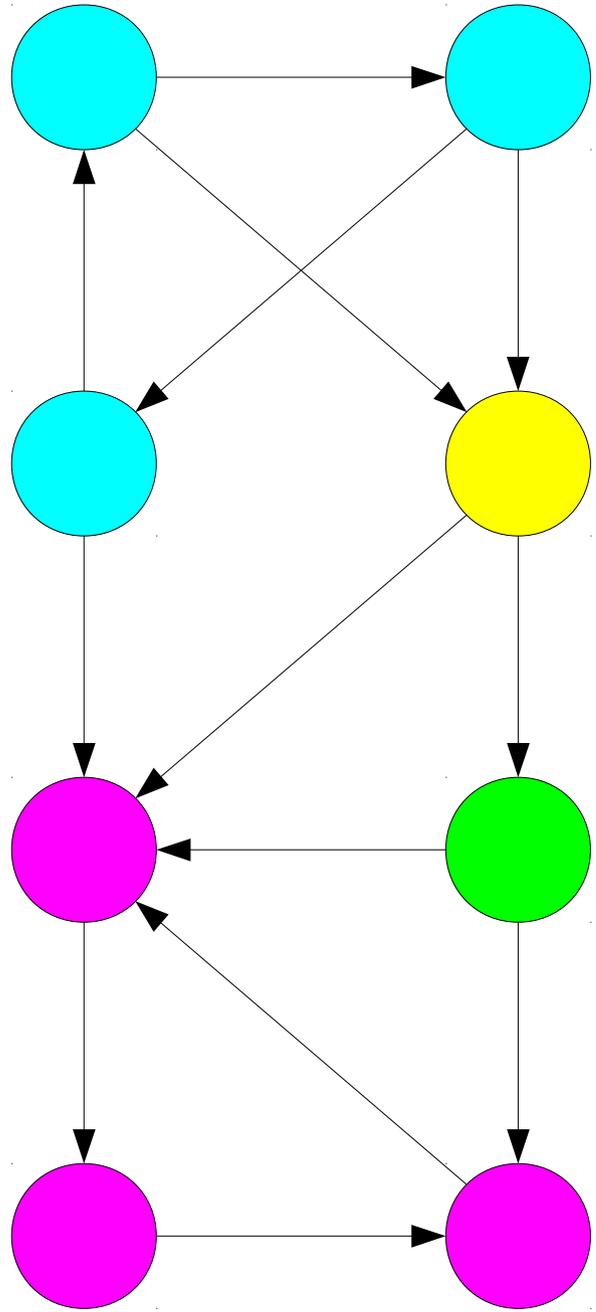


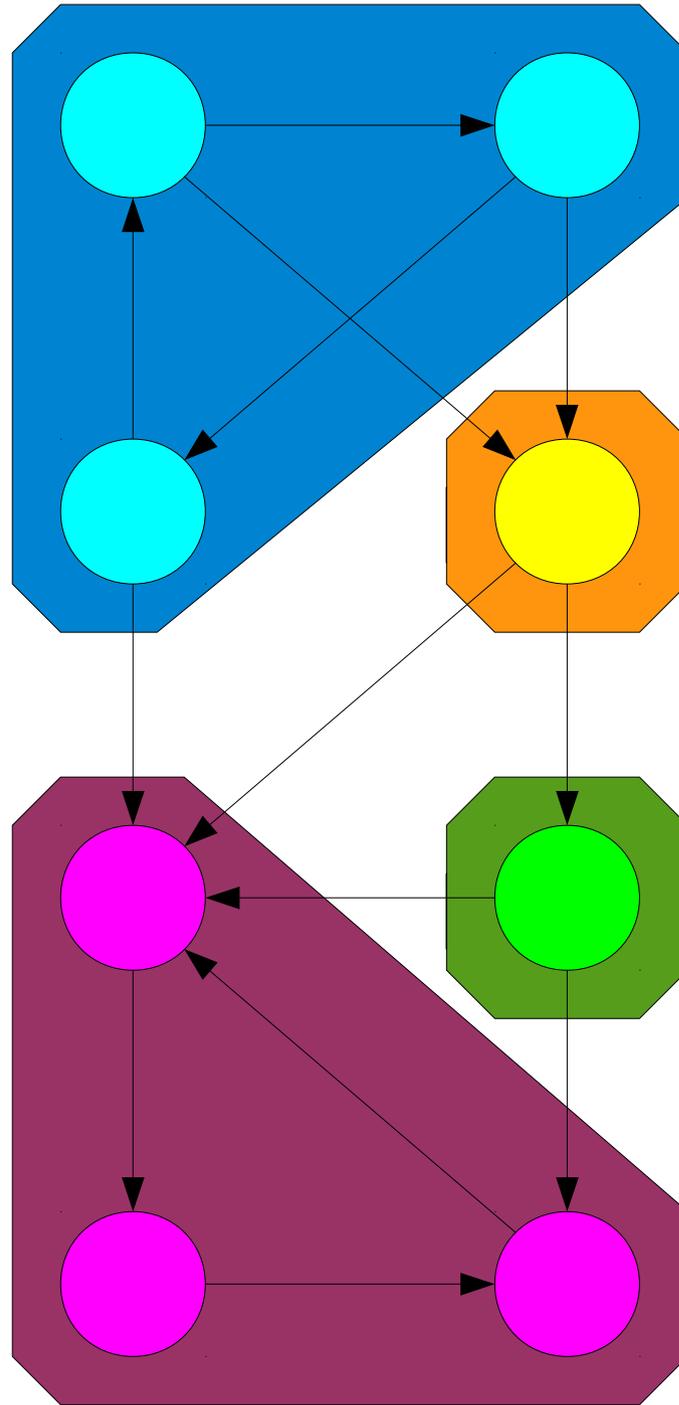


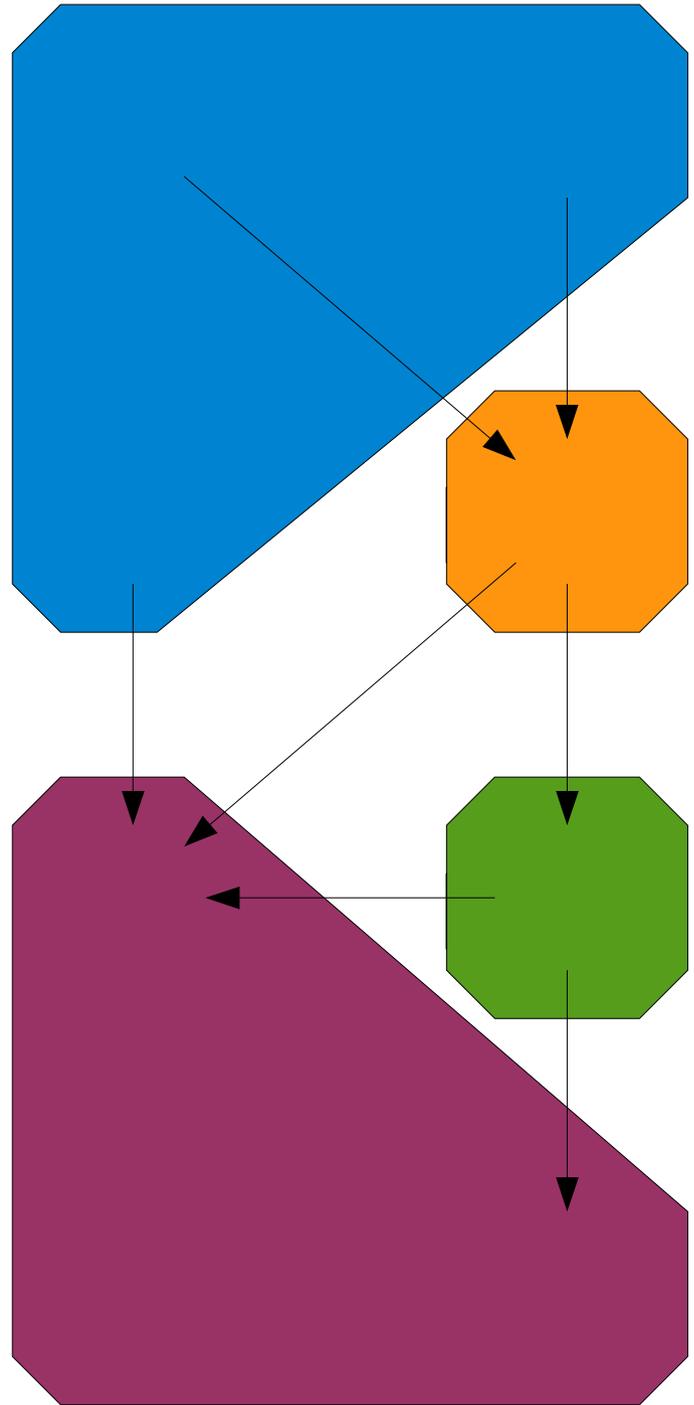


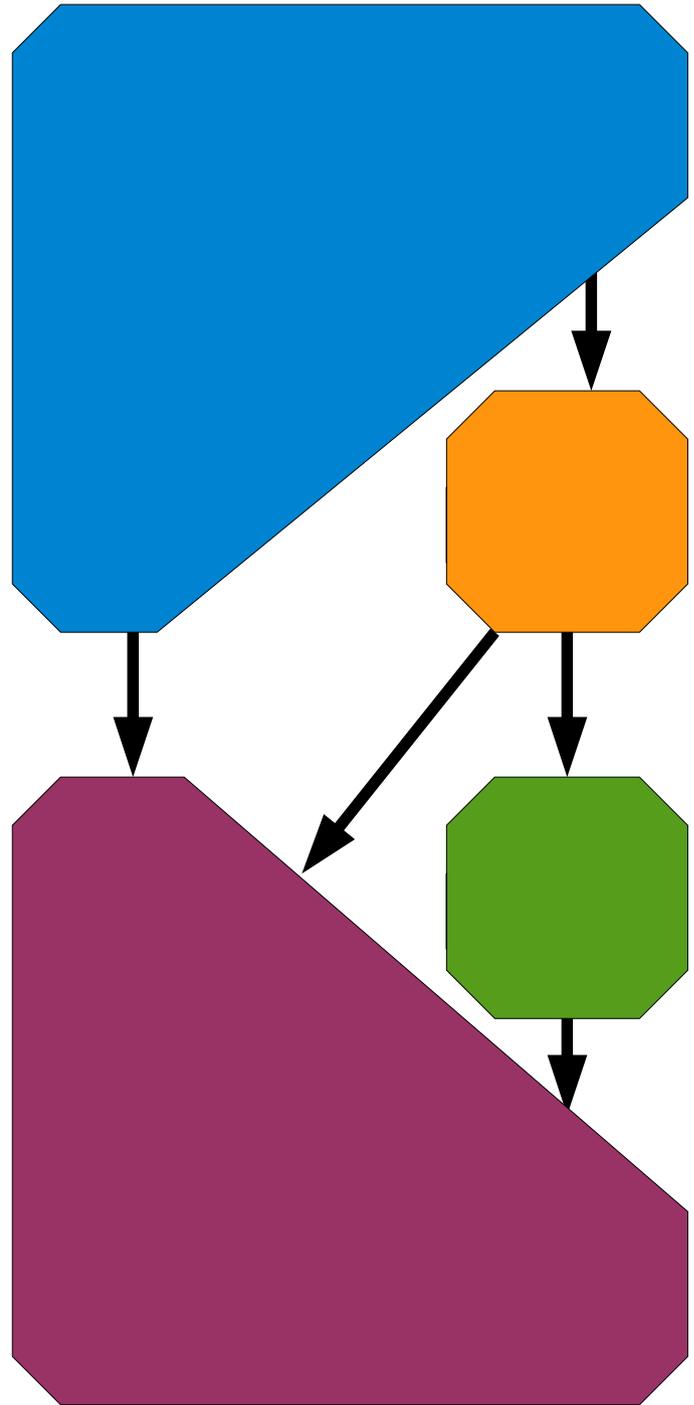


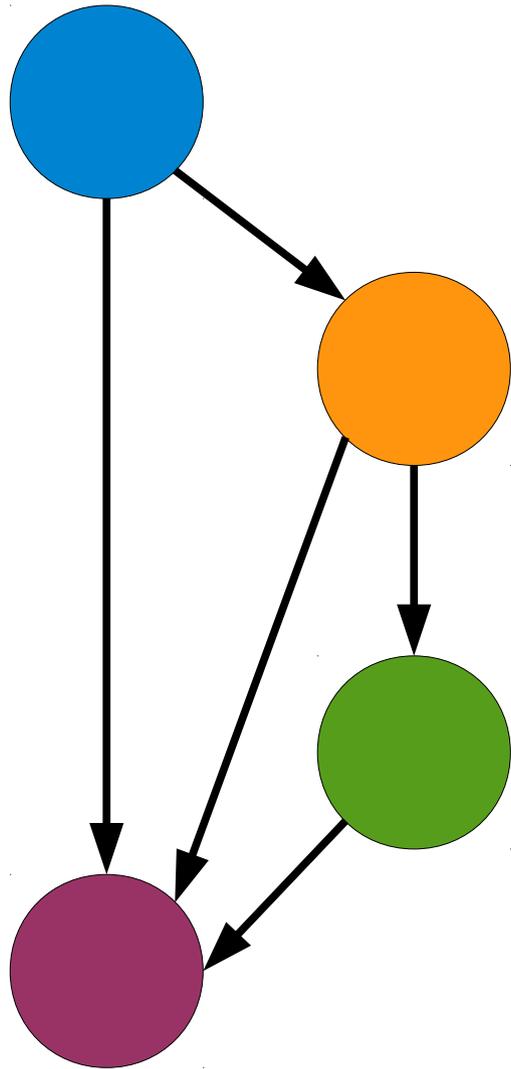


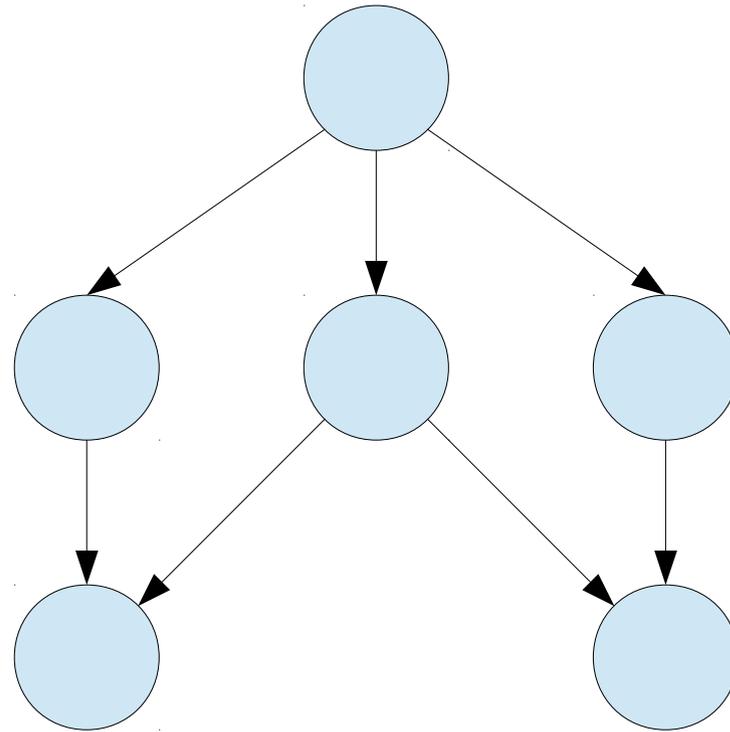


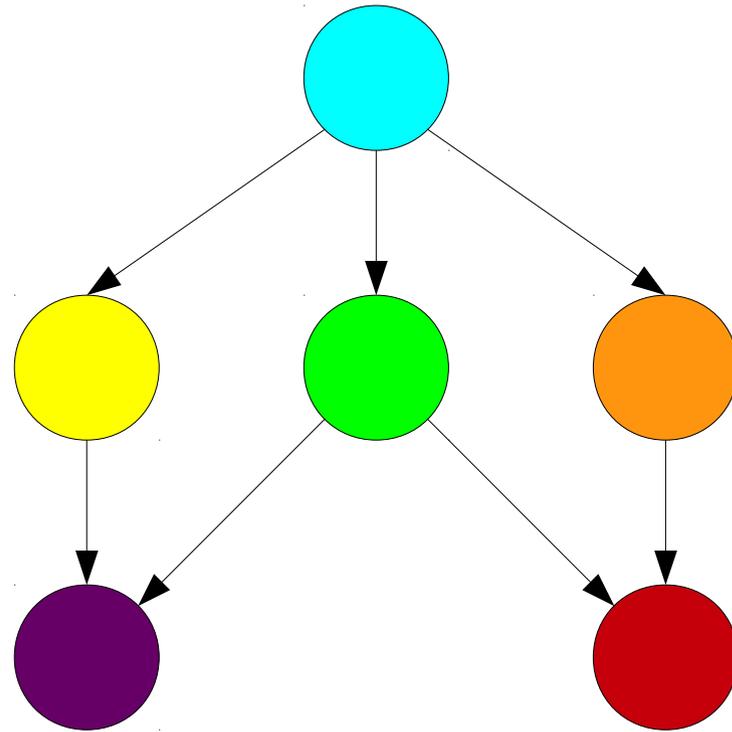










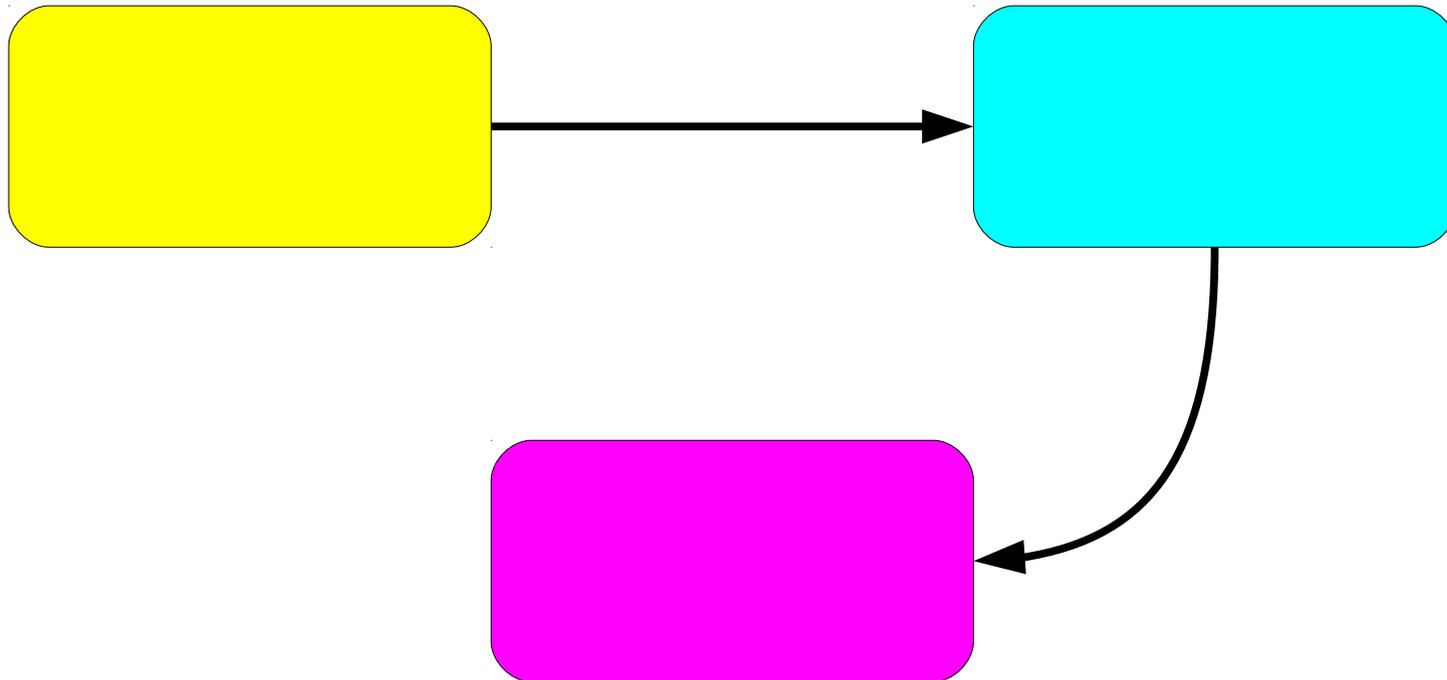


An Amazing Result

- ***Theorem:*** For any directed graph G , the condensation G^{SCC} of G is a DAG.
- *Proof Sketch:*

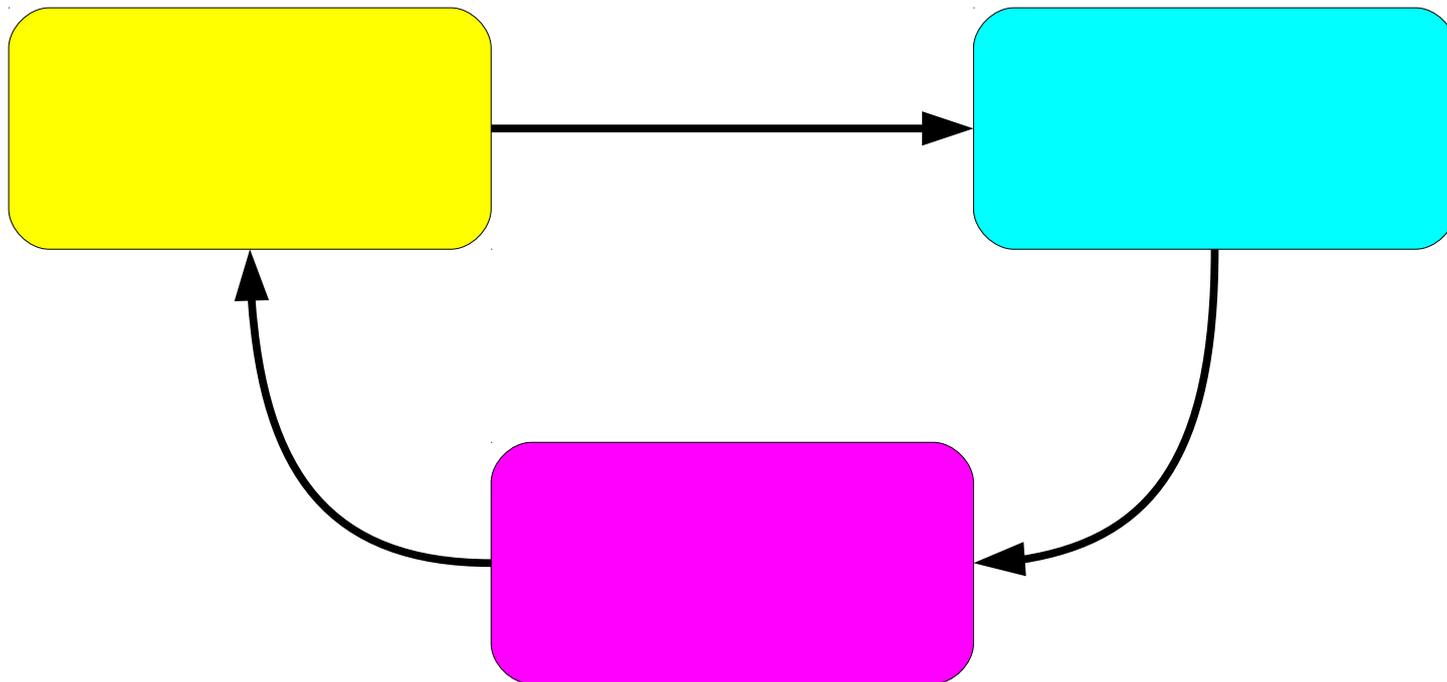
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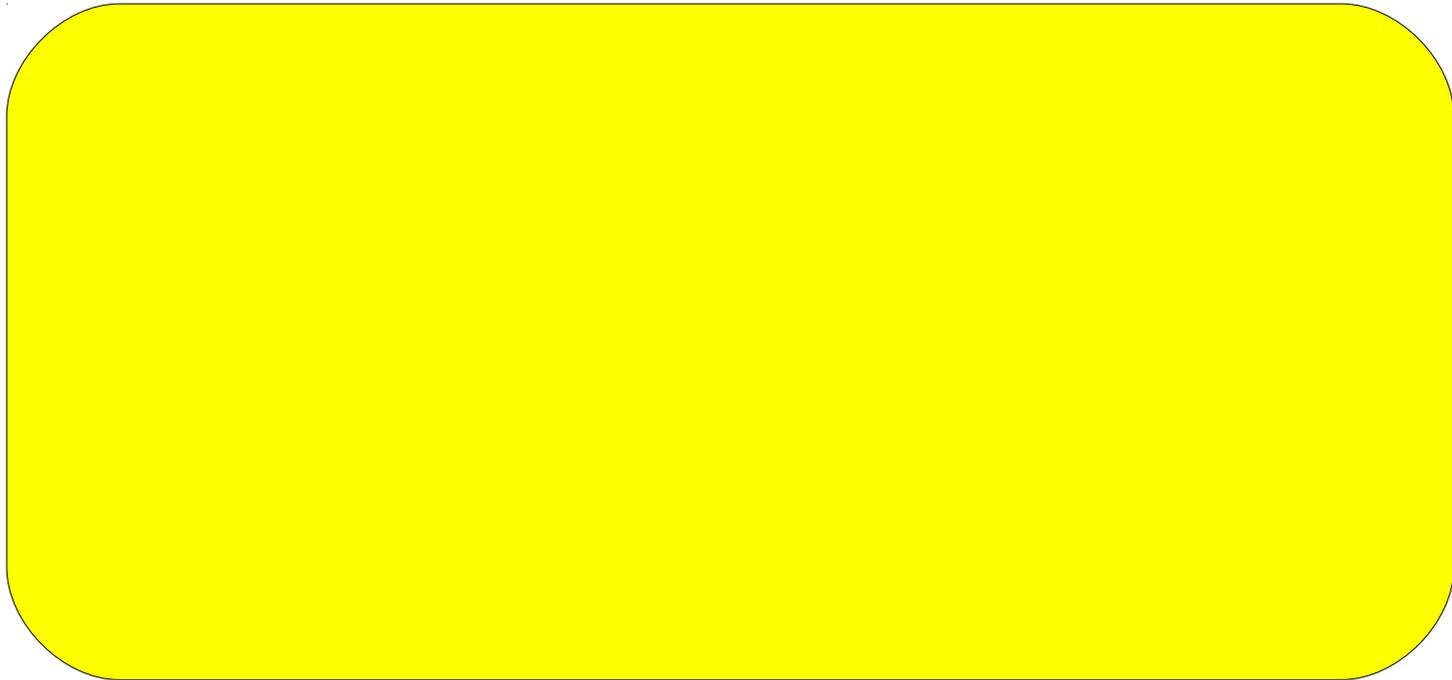
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An Amazing Result

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SCCs and DAGs

- We now see that there is a close connection between SCCs and DAGs: **the SCCs of a graph form a DAG.**
- Intuitively, you can think of a graph as a two-layer structure:
 - At a high level, a graph is a DAG of SCCs showing the top-level connections between clusters of nodes.
 - At a lower level, you can see the connections between nodes in the same SCC.

SCCs and DAGs

- Now that we have found a connection between SCCs and DAGs, can we adapt any of our algorithms on DAGs to find SCCs?
- Right now, our main operation on DAGs is topological sort, and we have two algorithms we can use:

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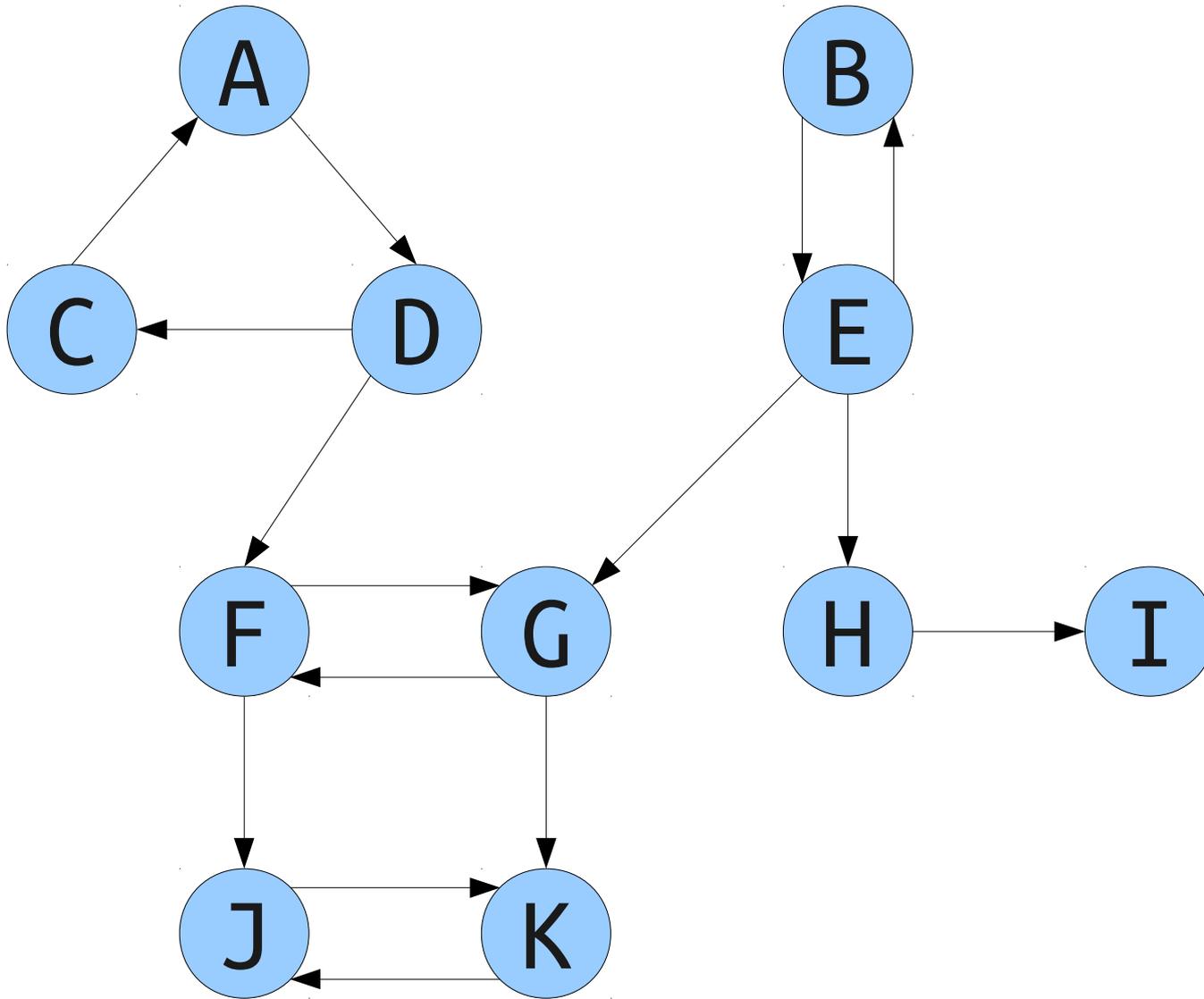
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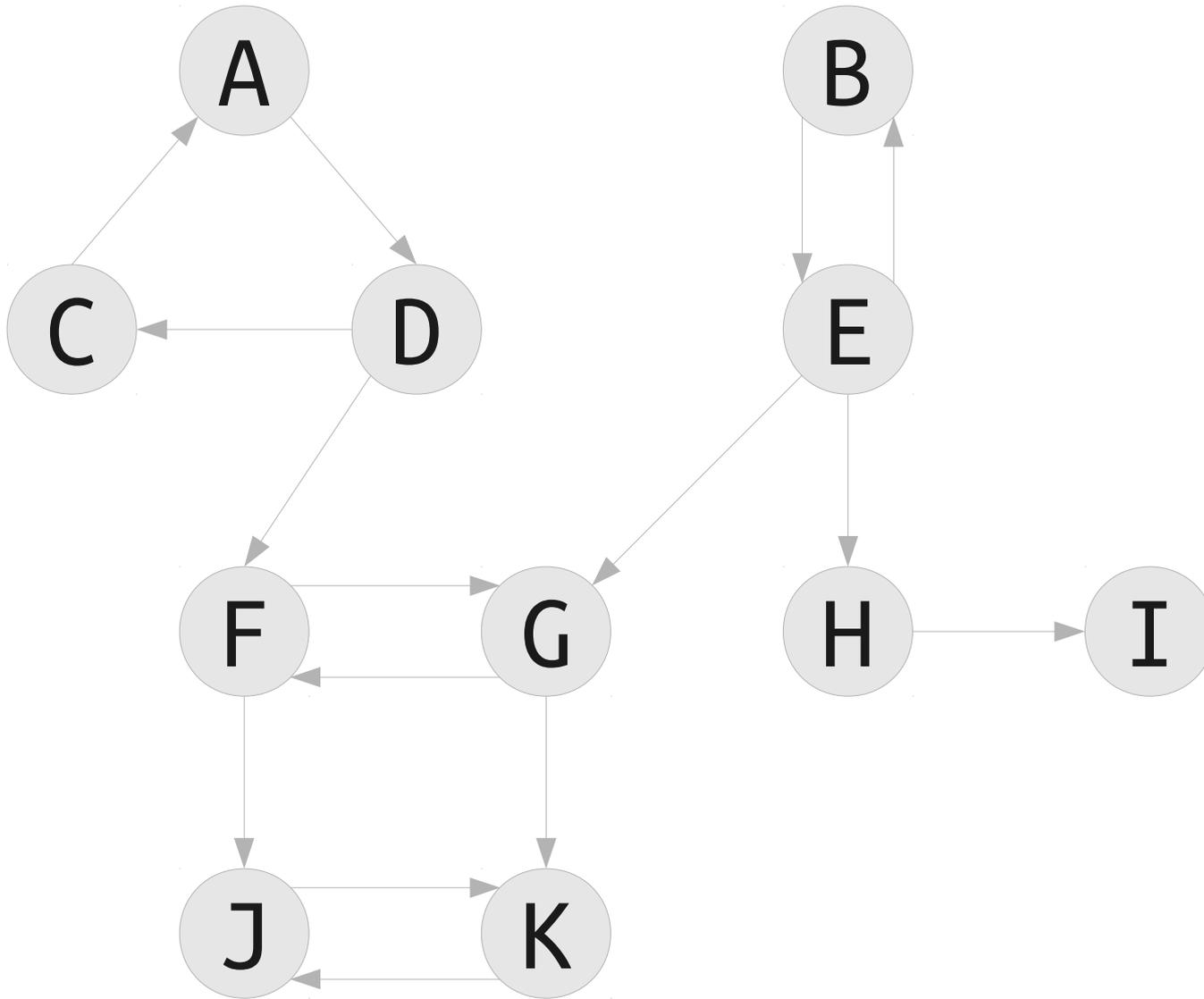
SCCs and DAGs

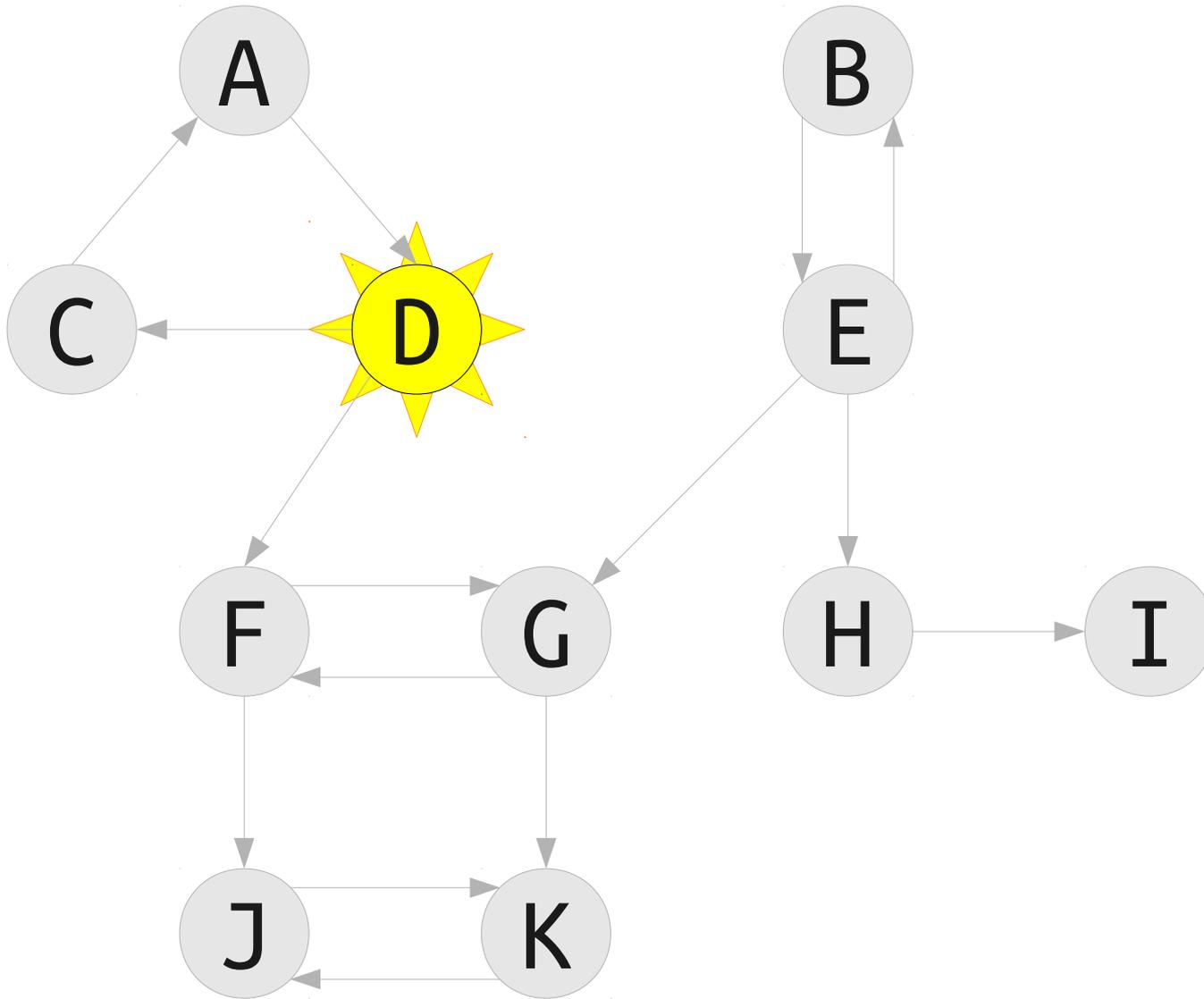
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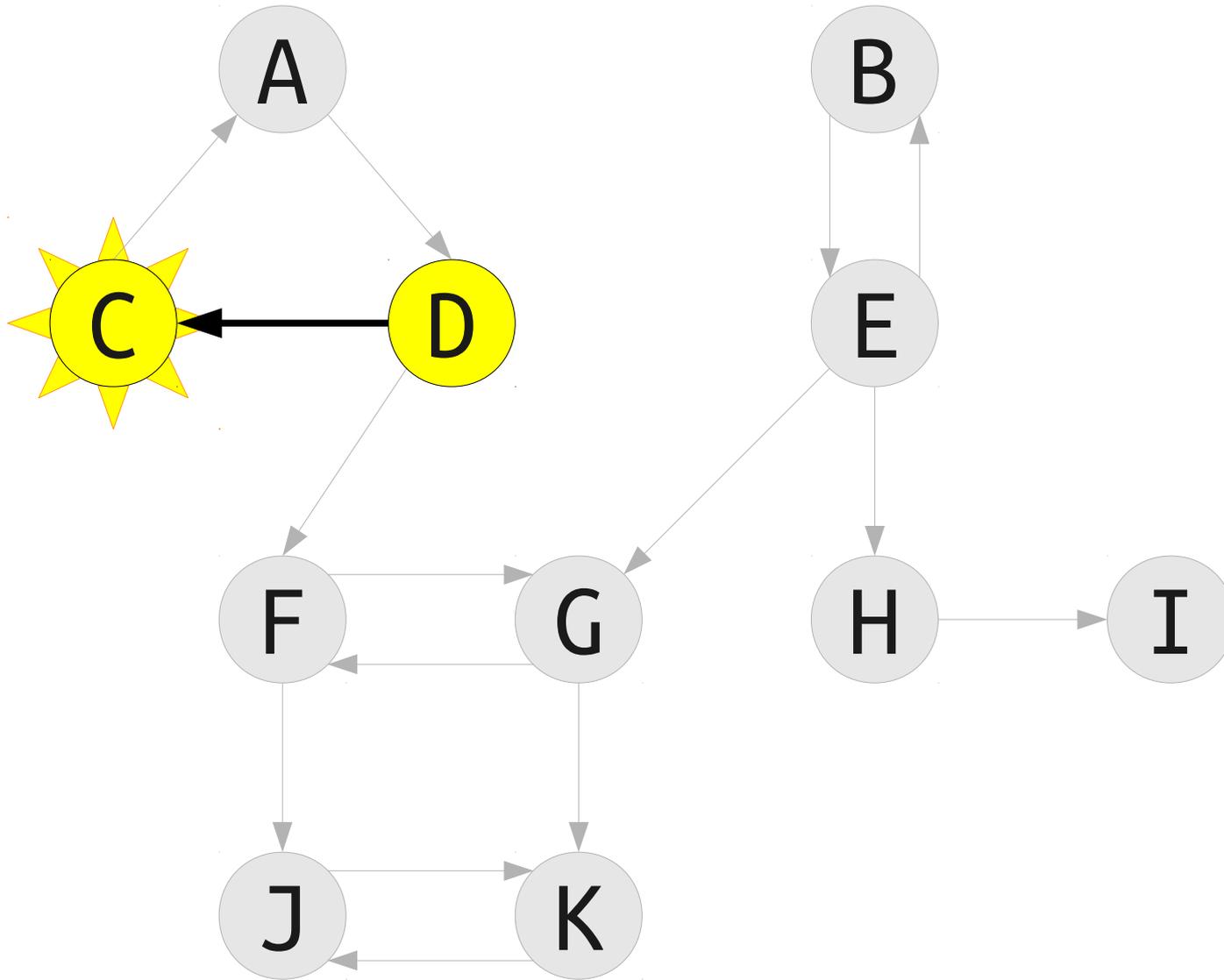
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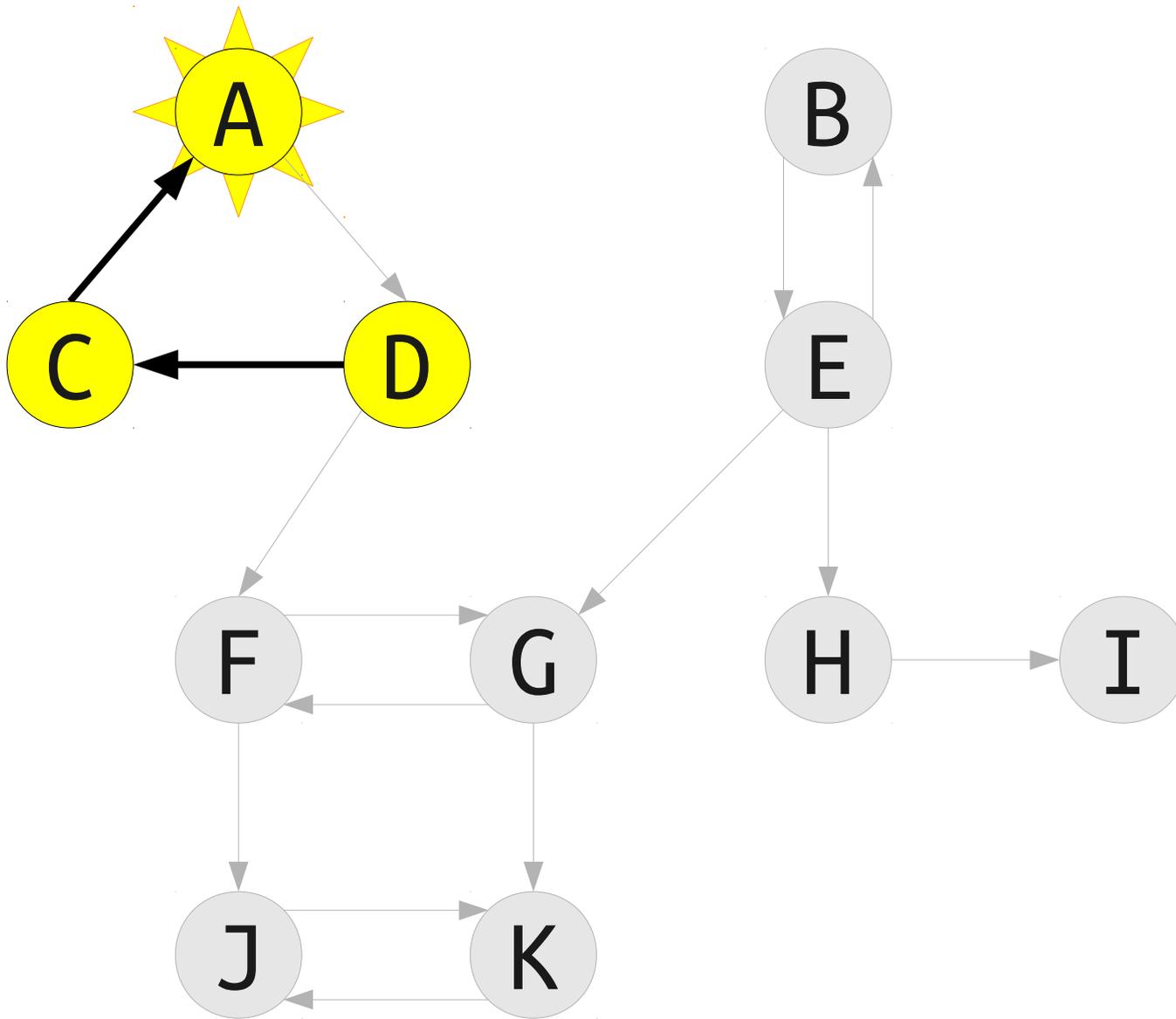
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 - *Repeatedly removing a source node.* That won't help us here, since we can't easily tell if a node is in a source SCC.
 - *Running DFS and reversing the result.* So what happens if we try that out?

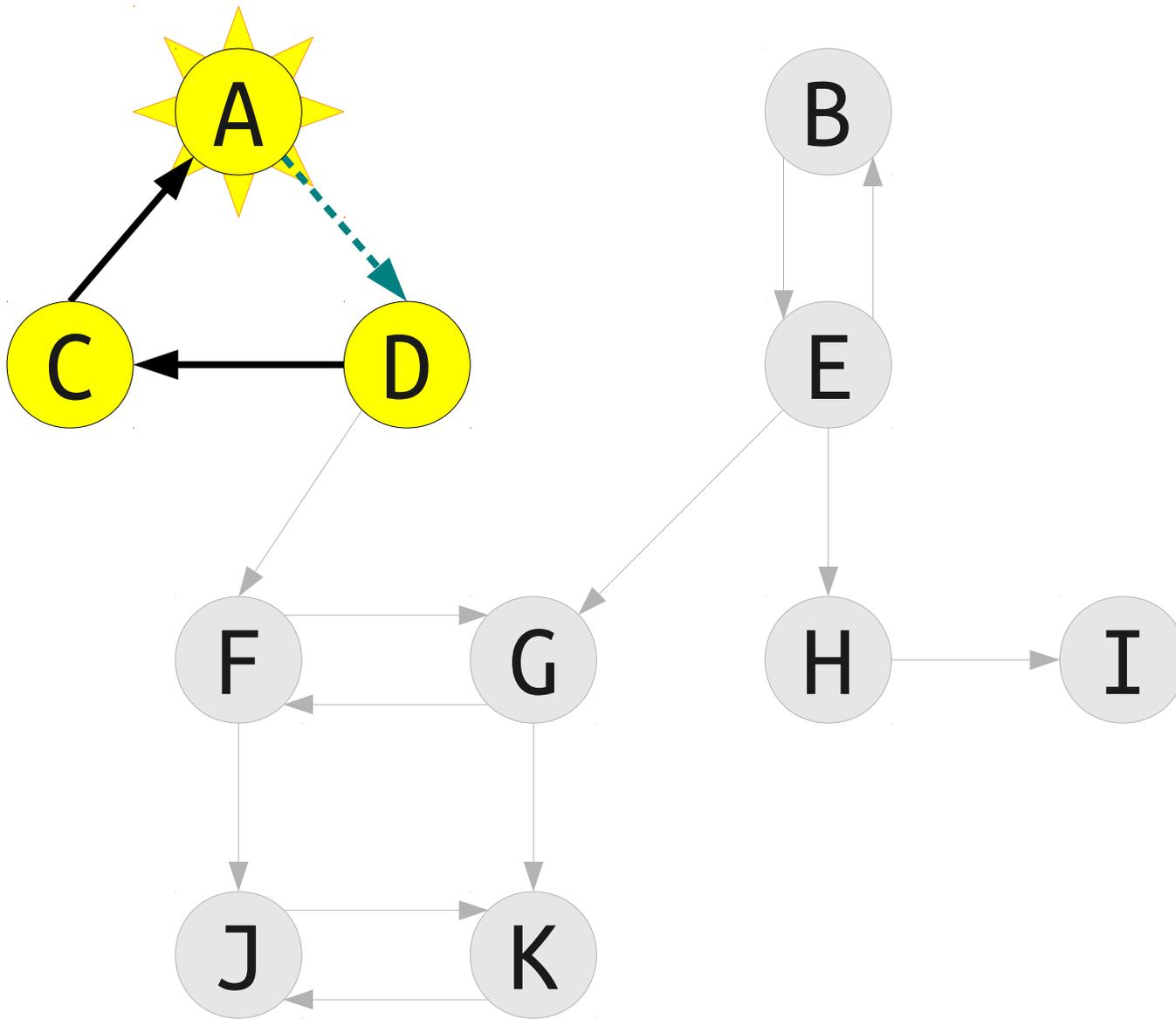


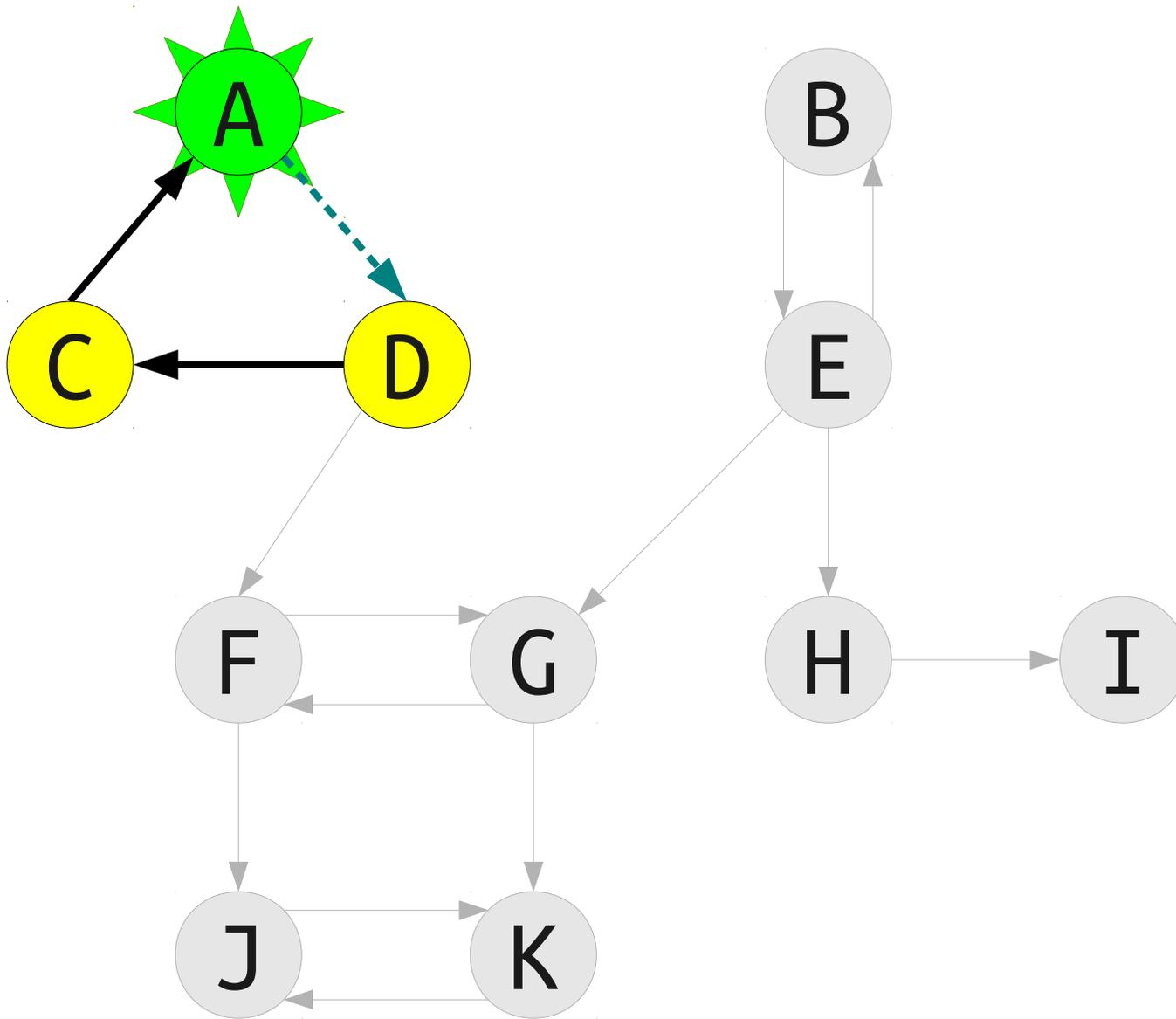


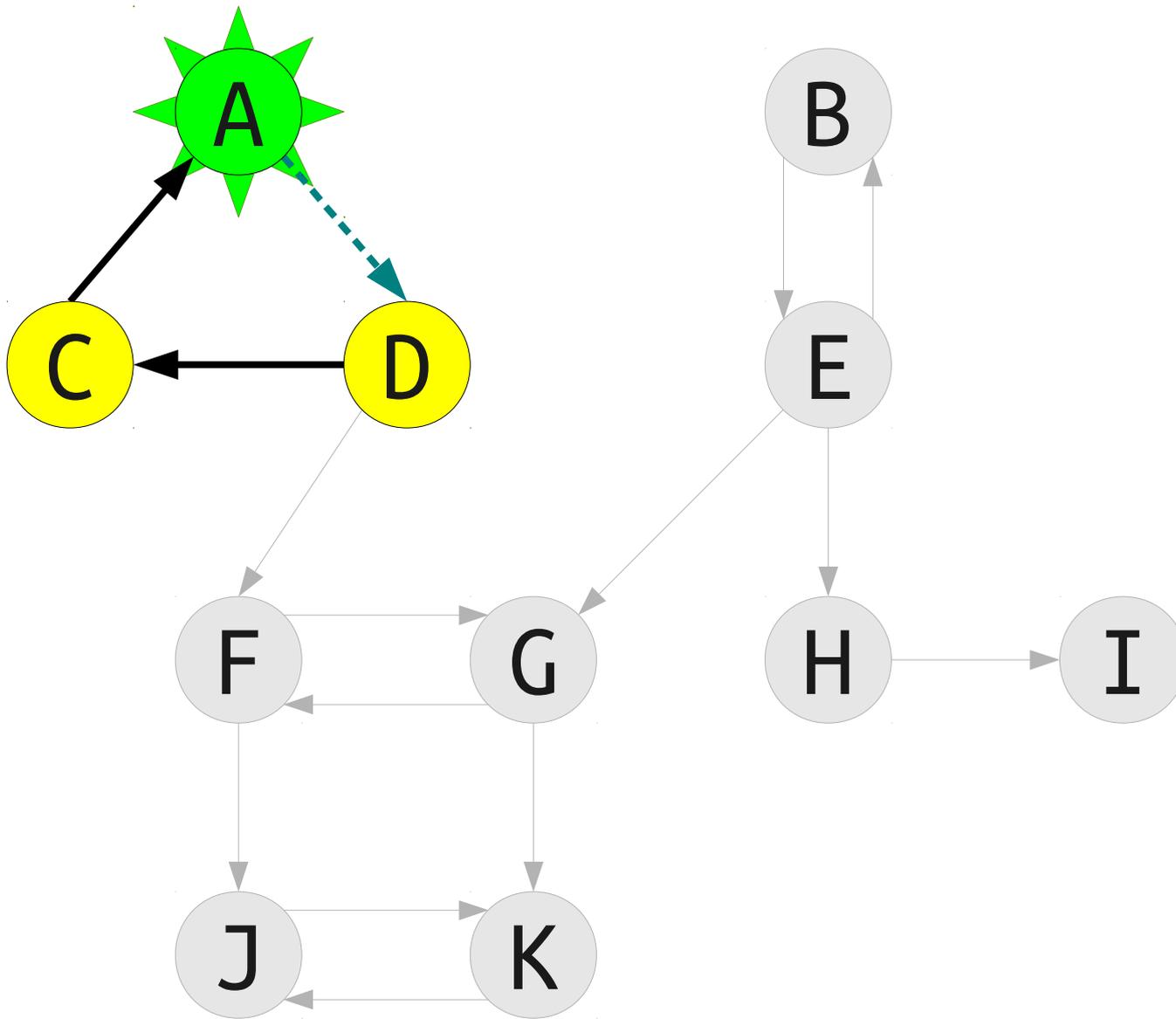


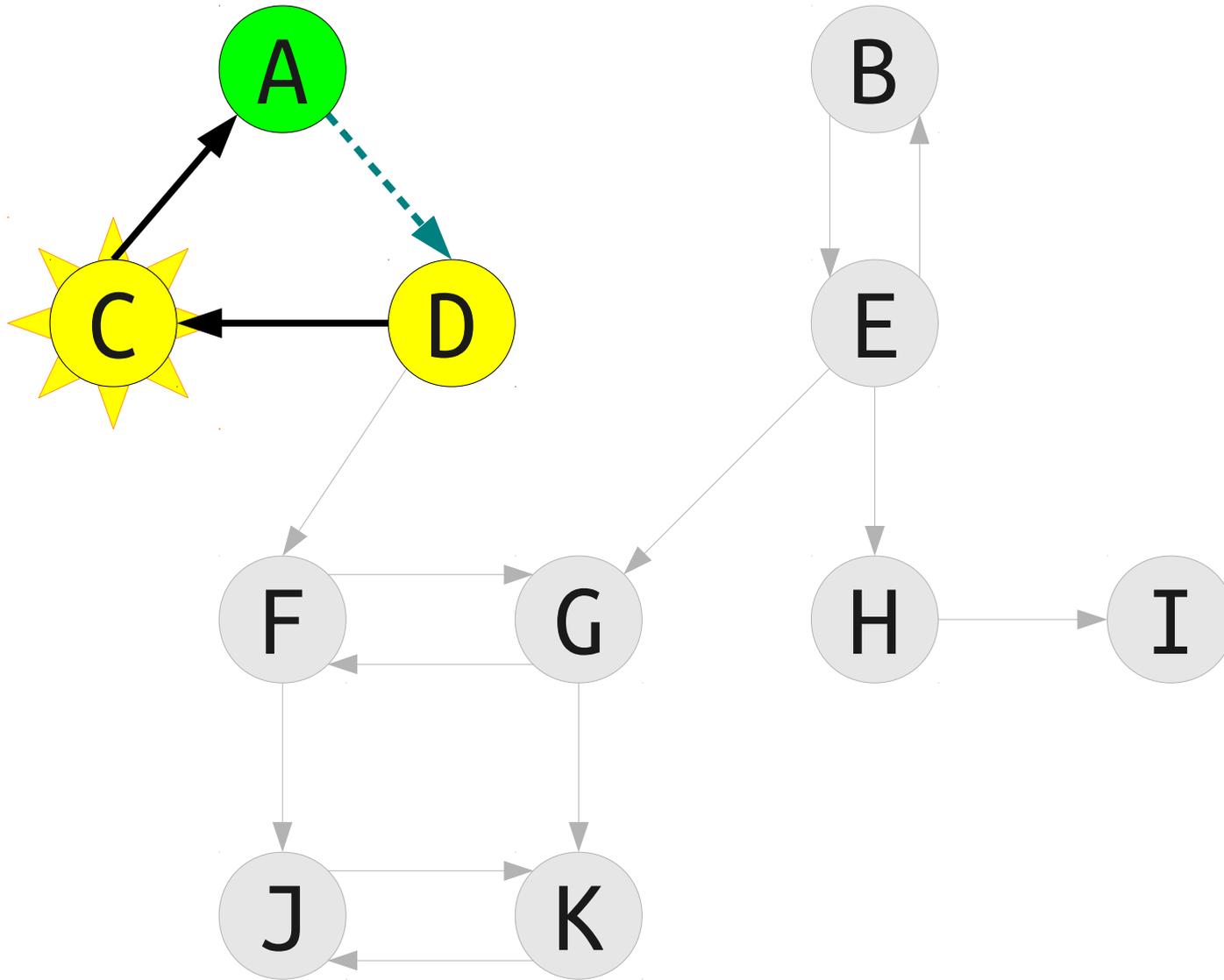


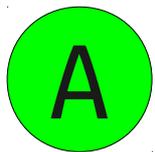
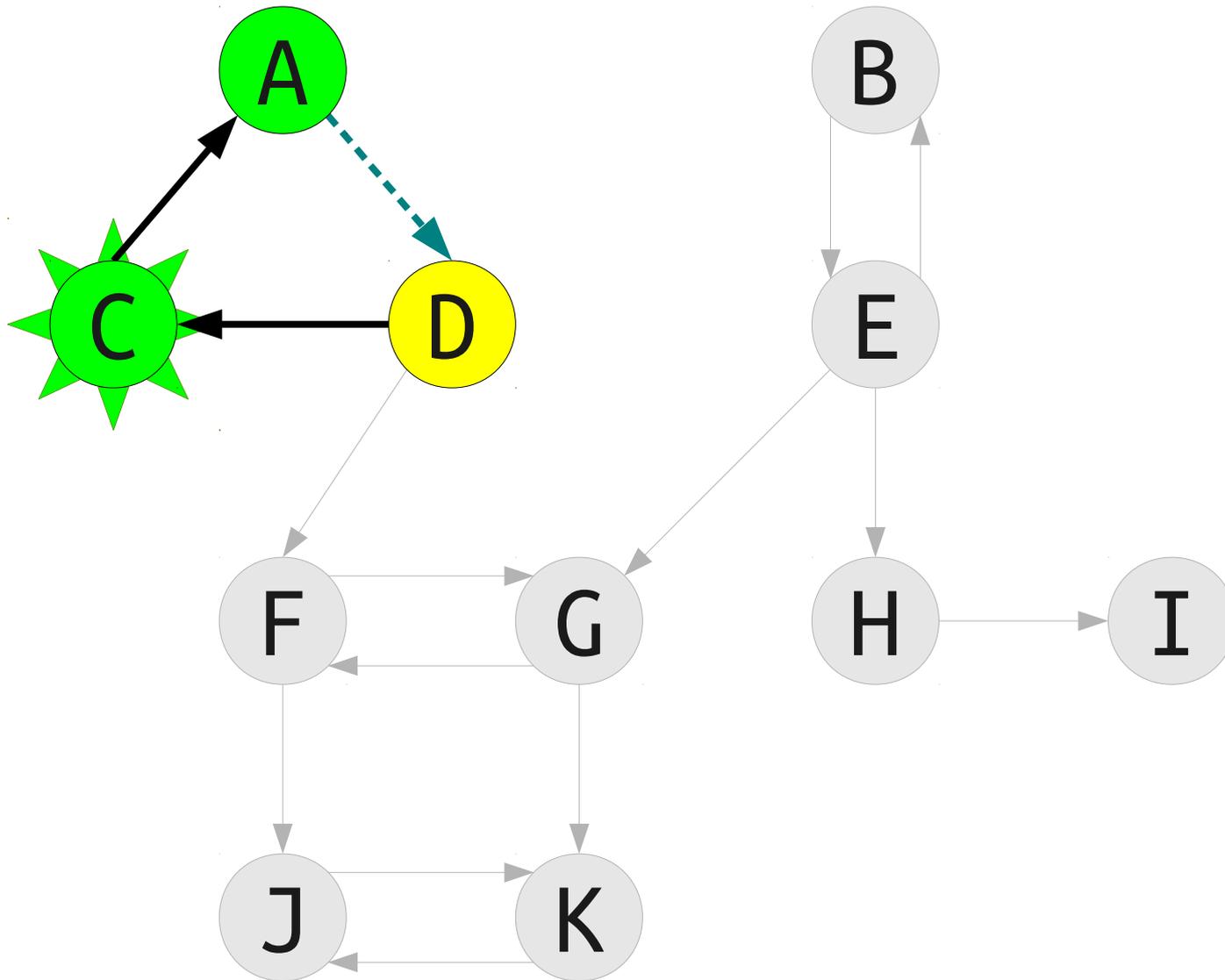


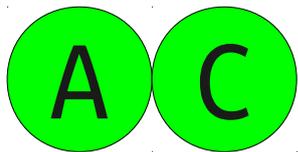
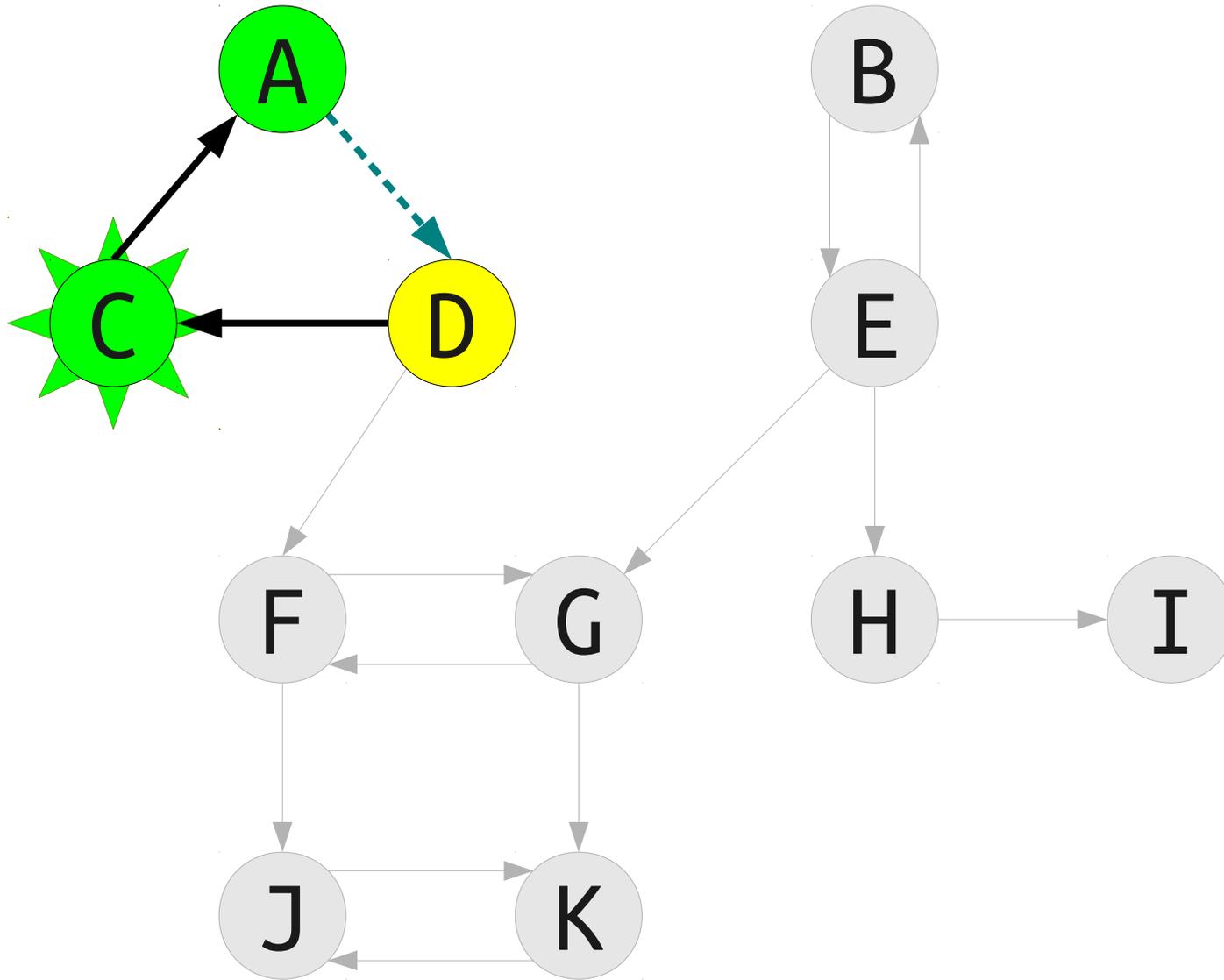


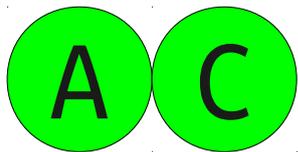
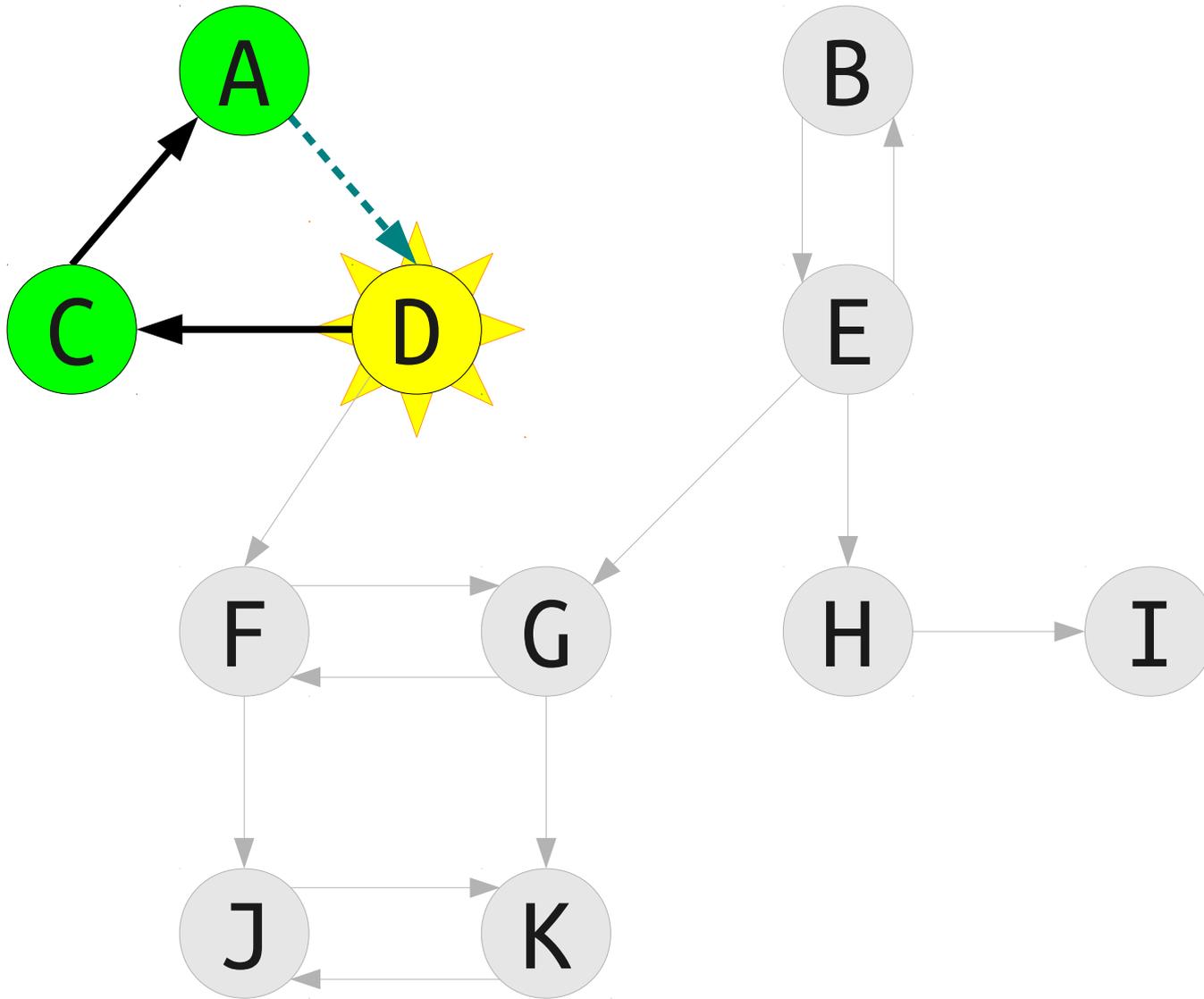


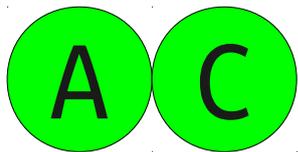
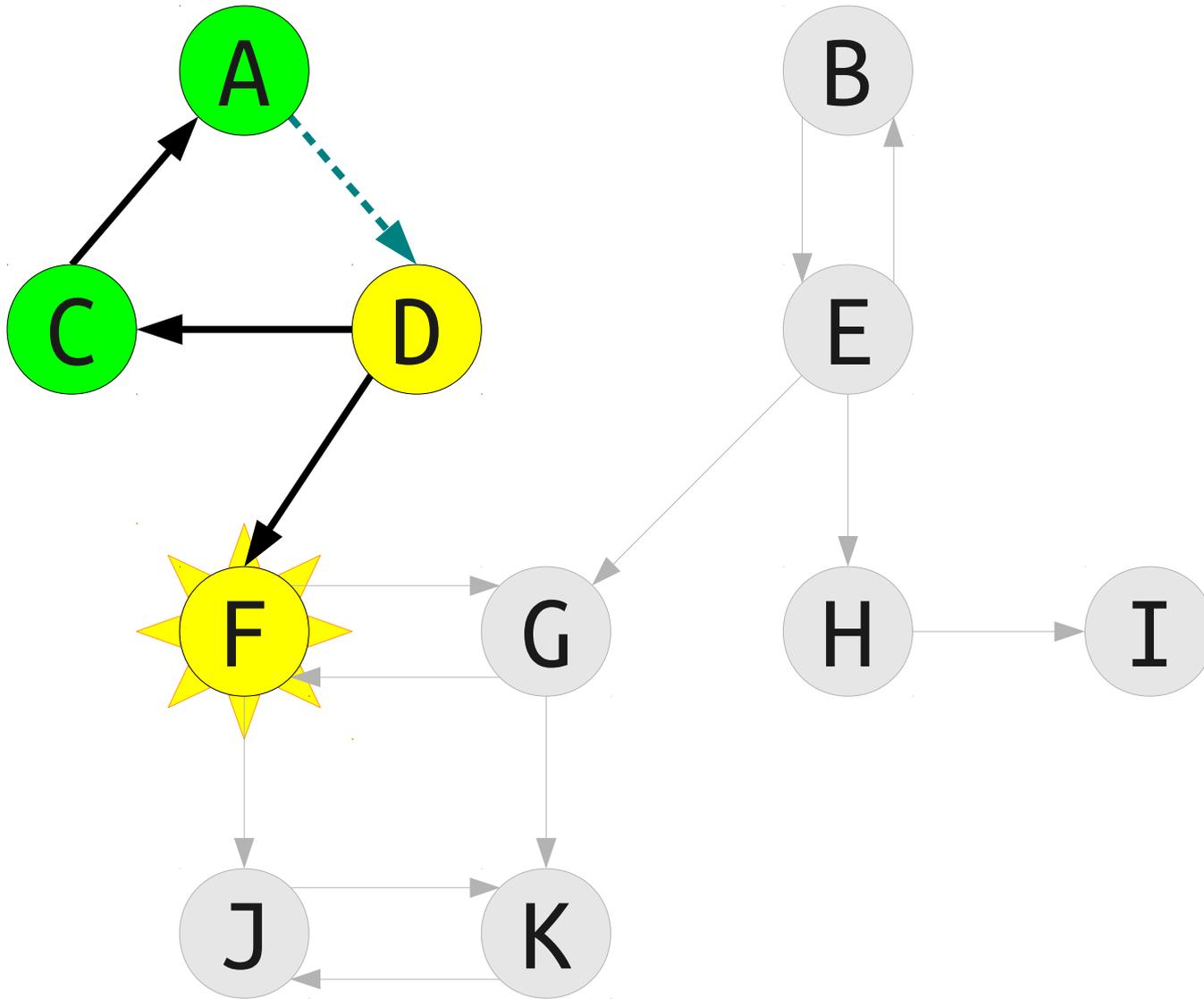


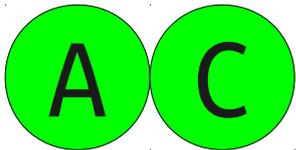
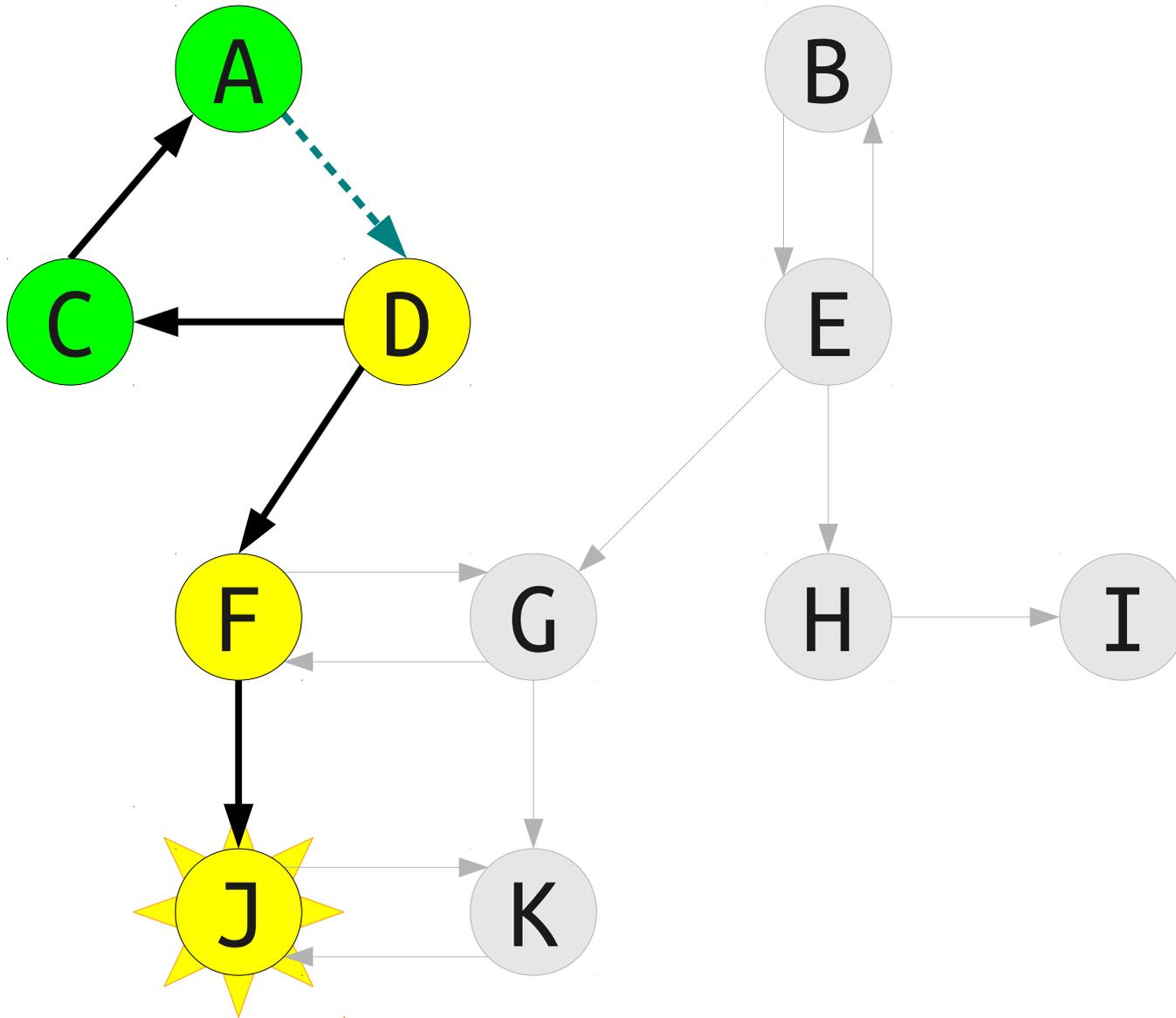


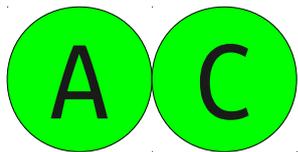
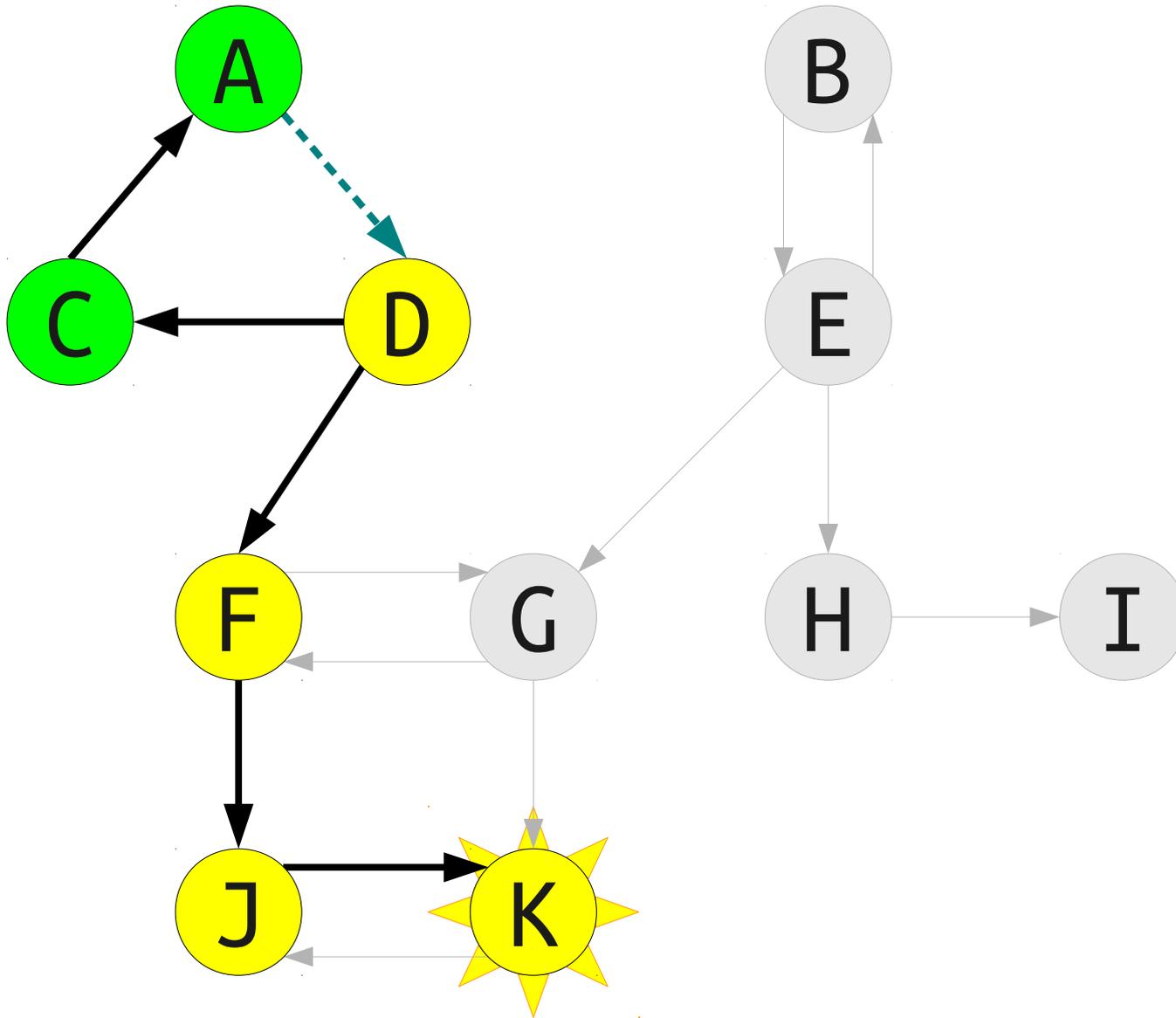


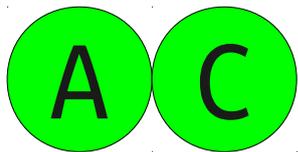
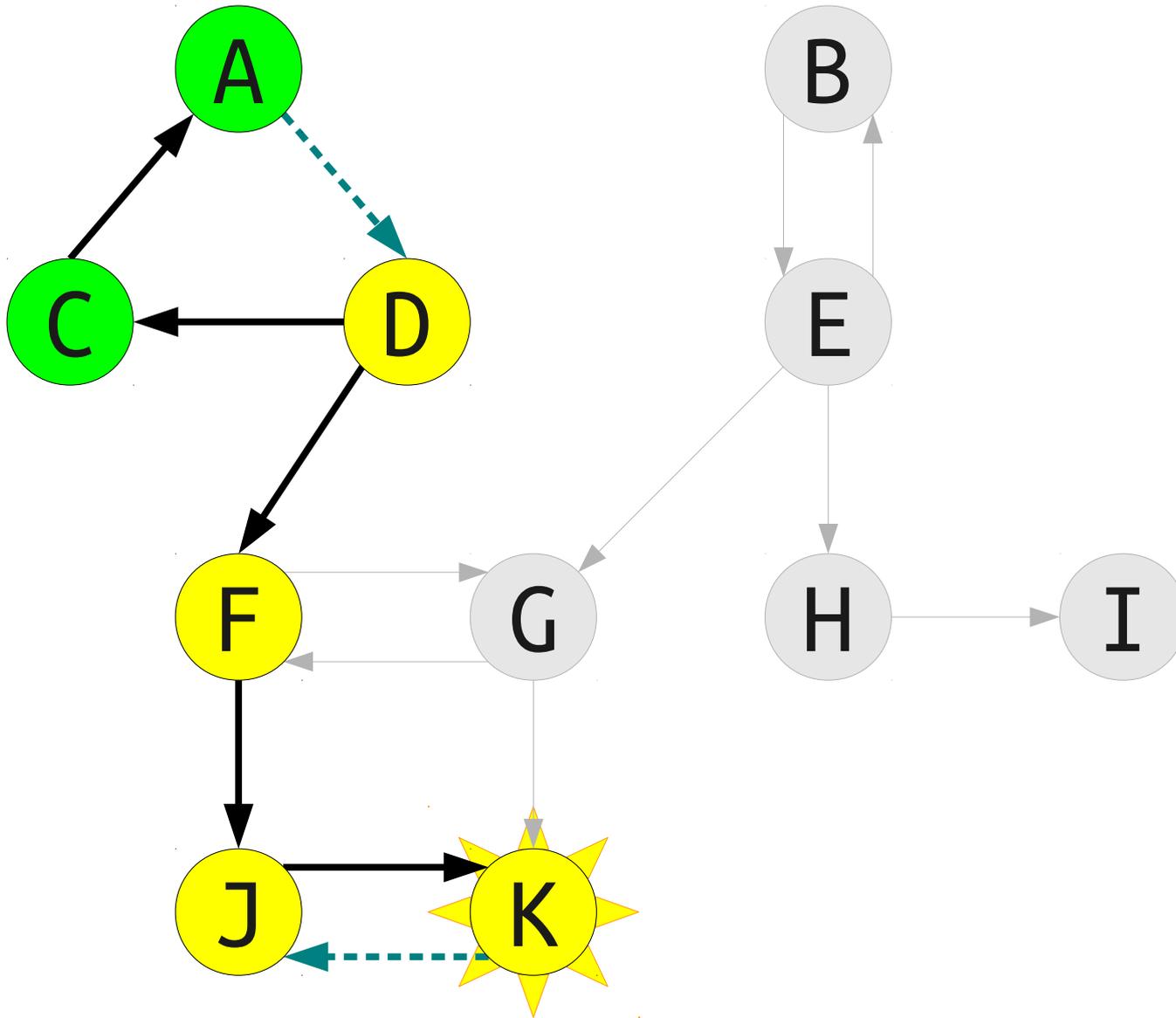


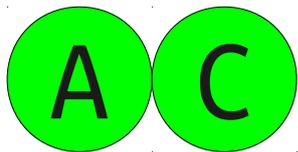
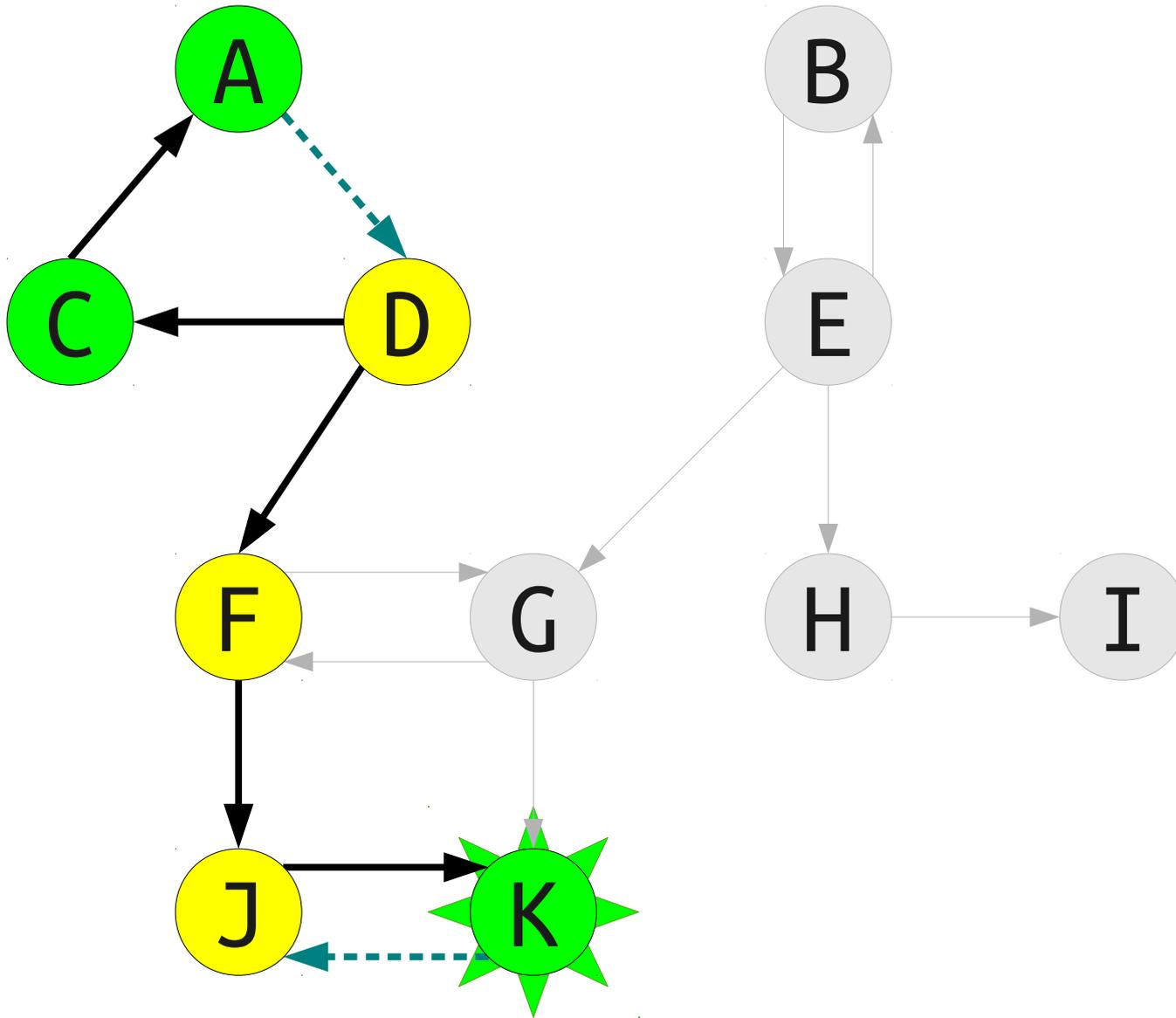


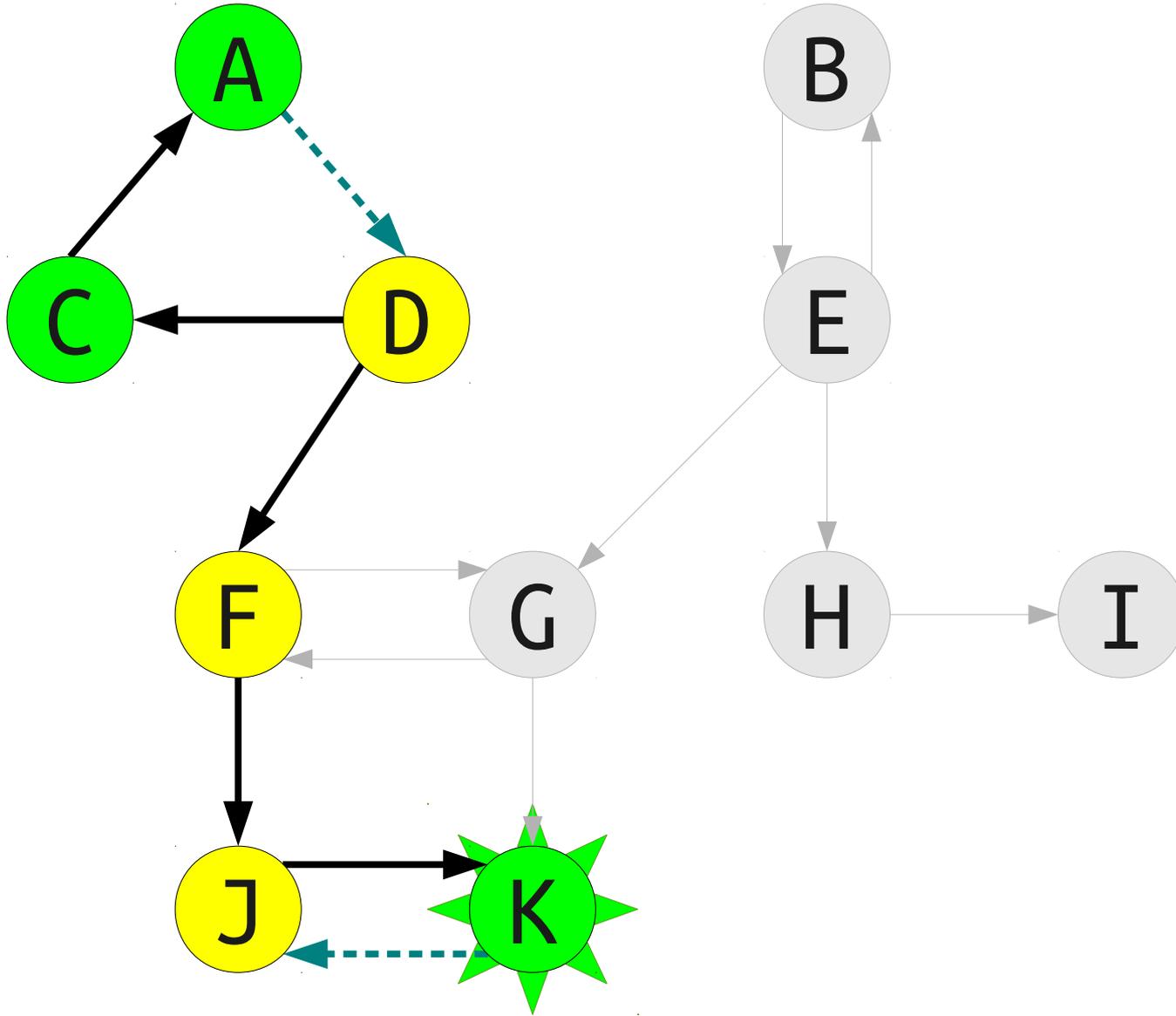




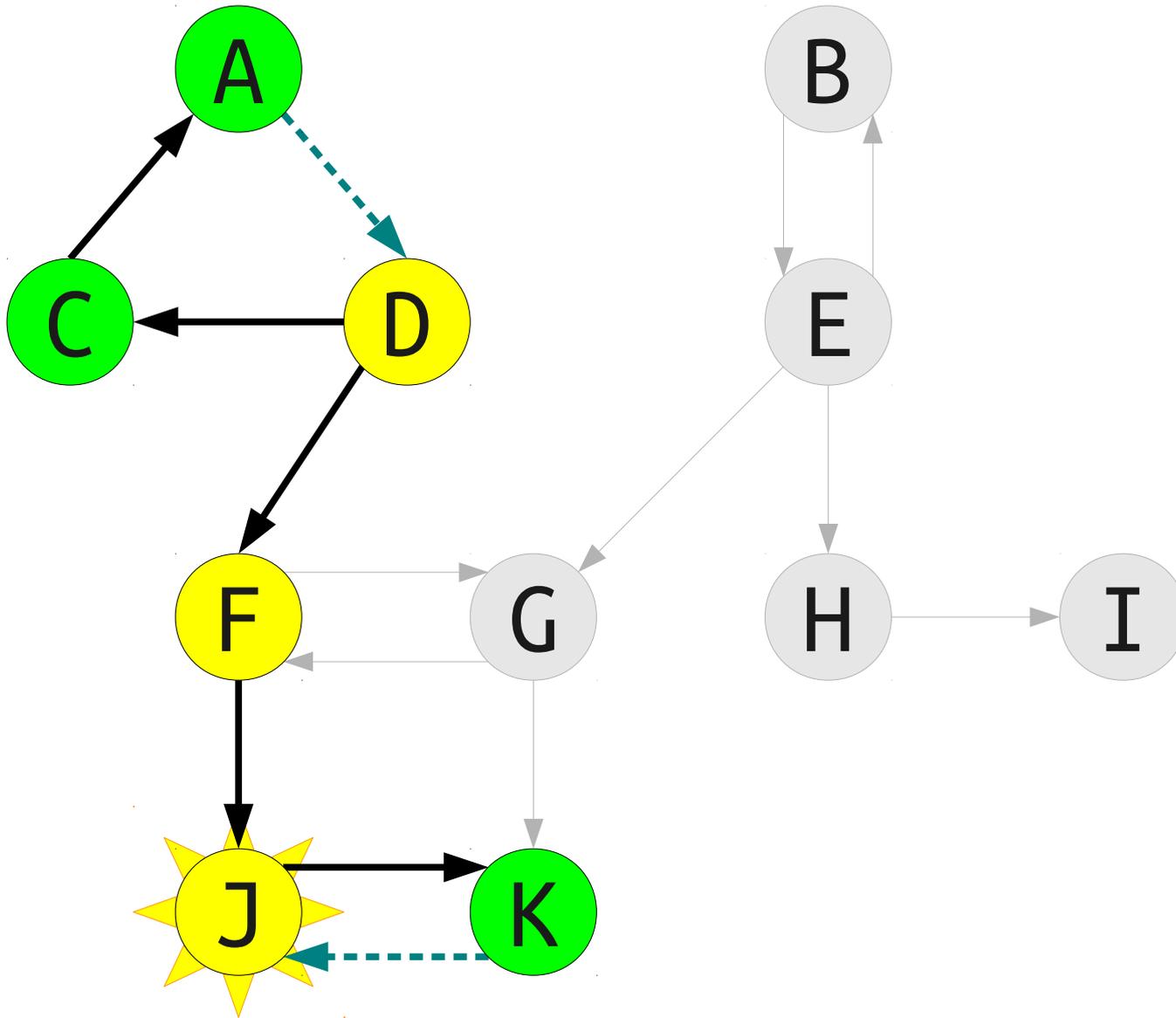




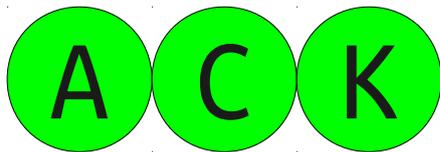
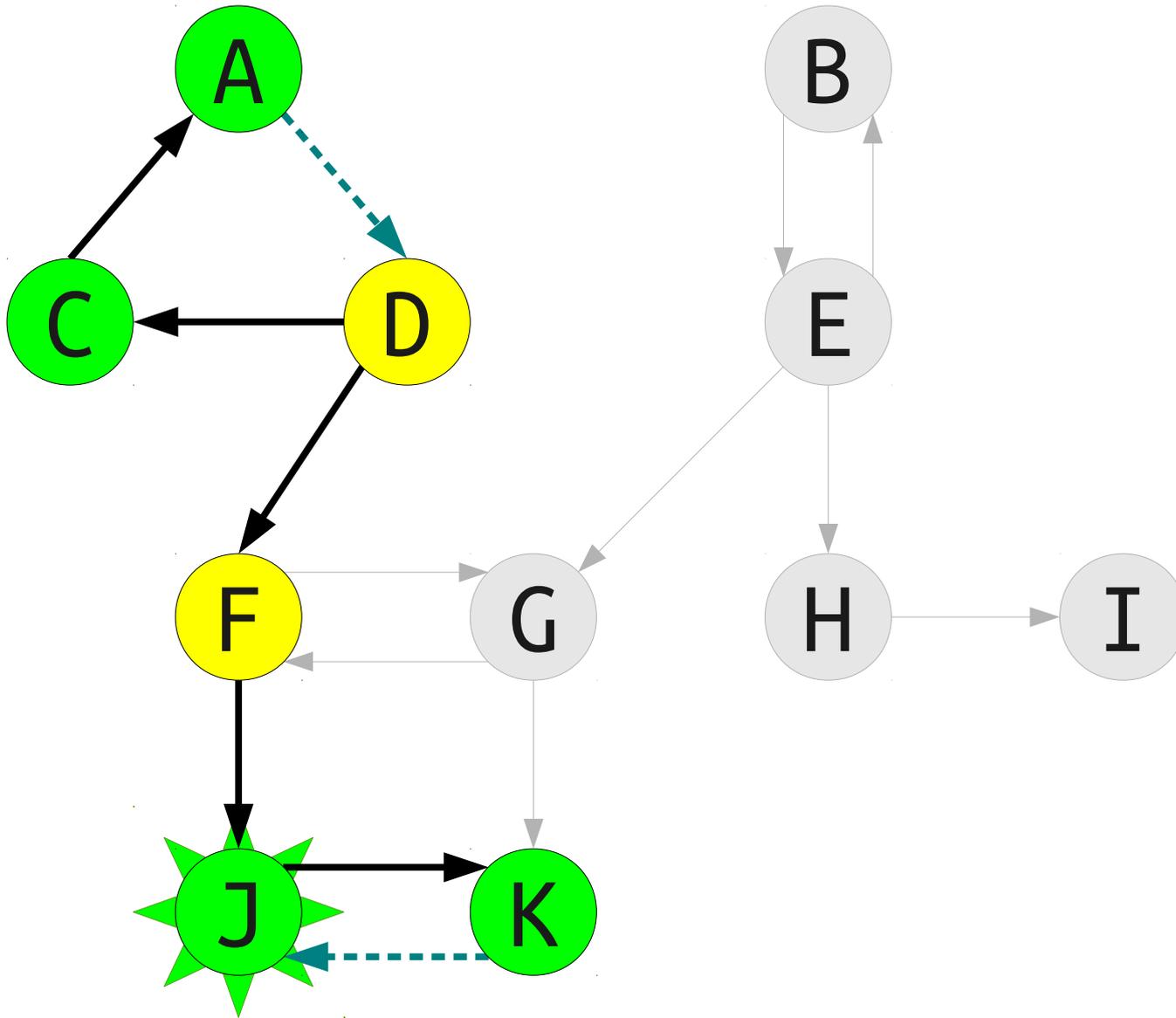


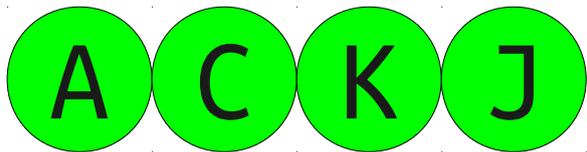
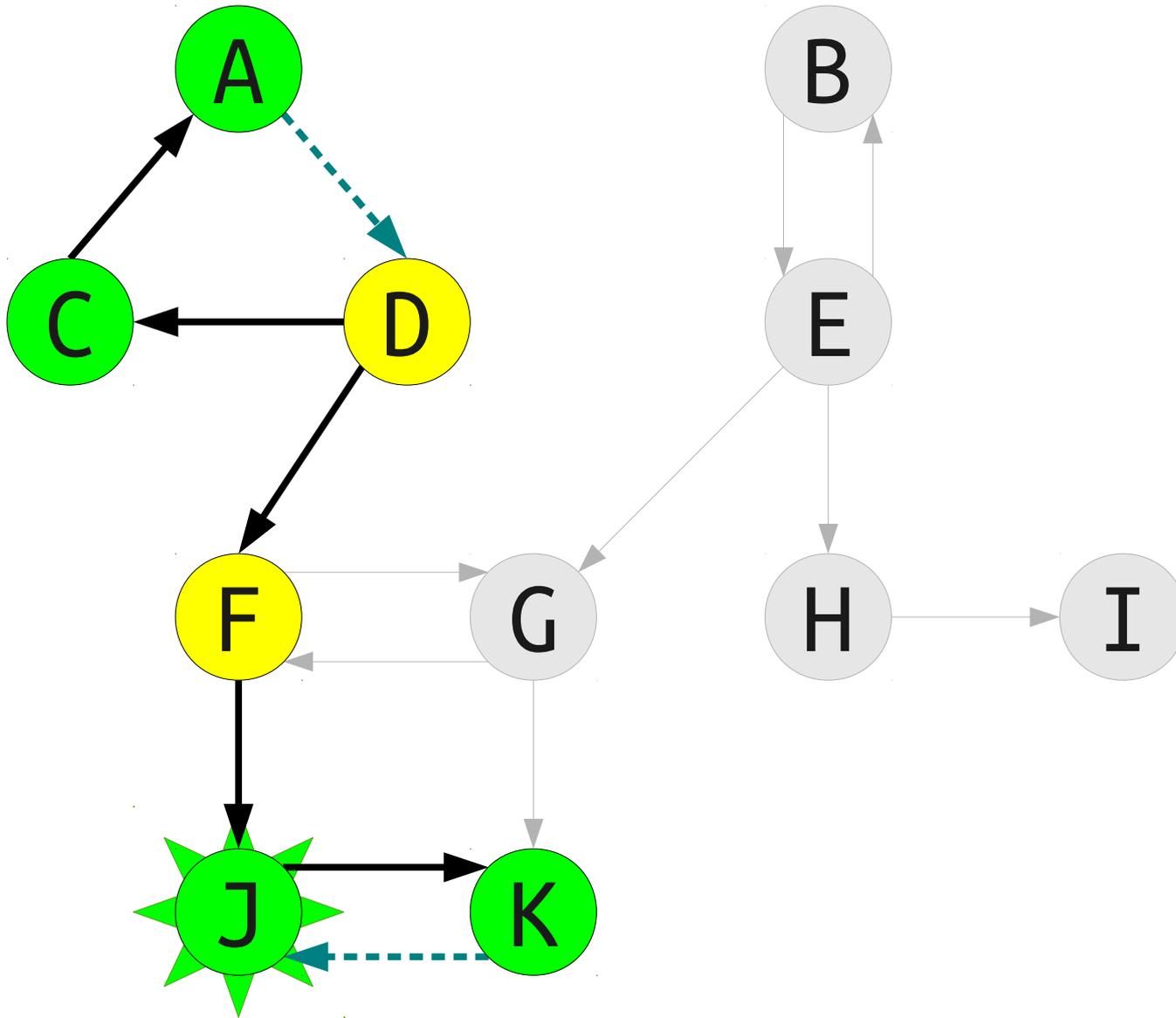


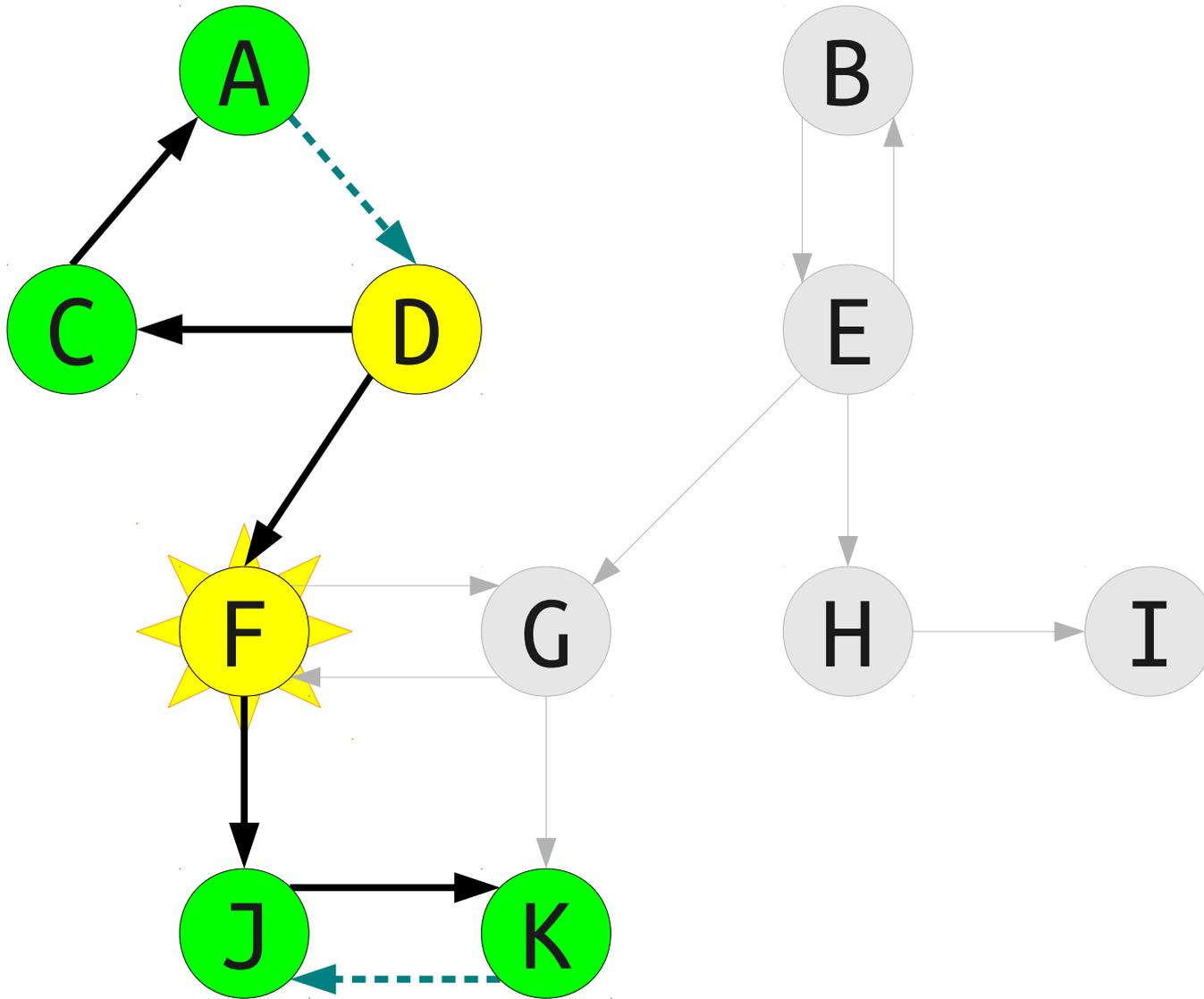
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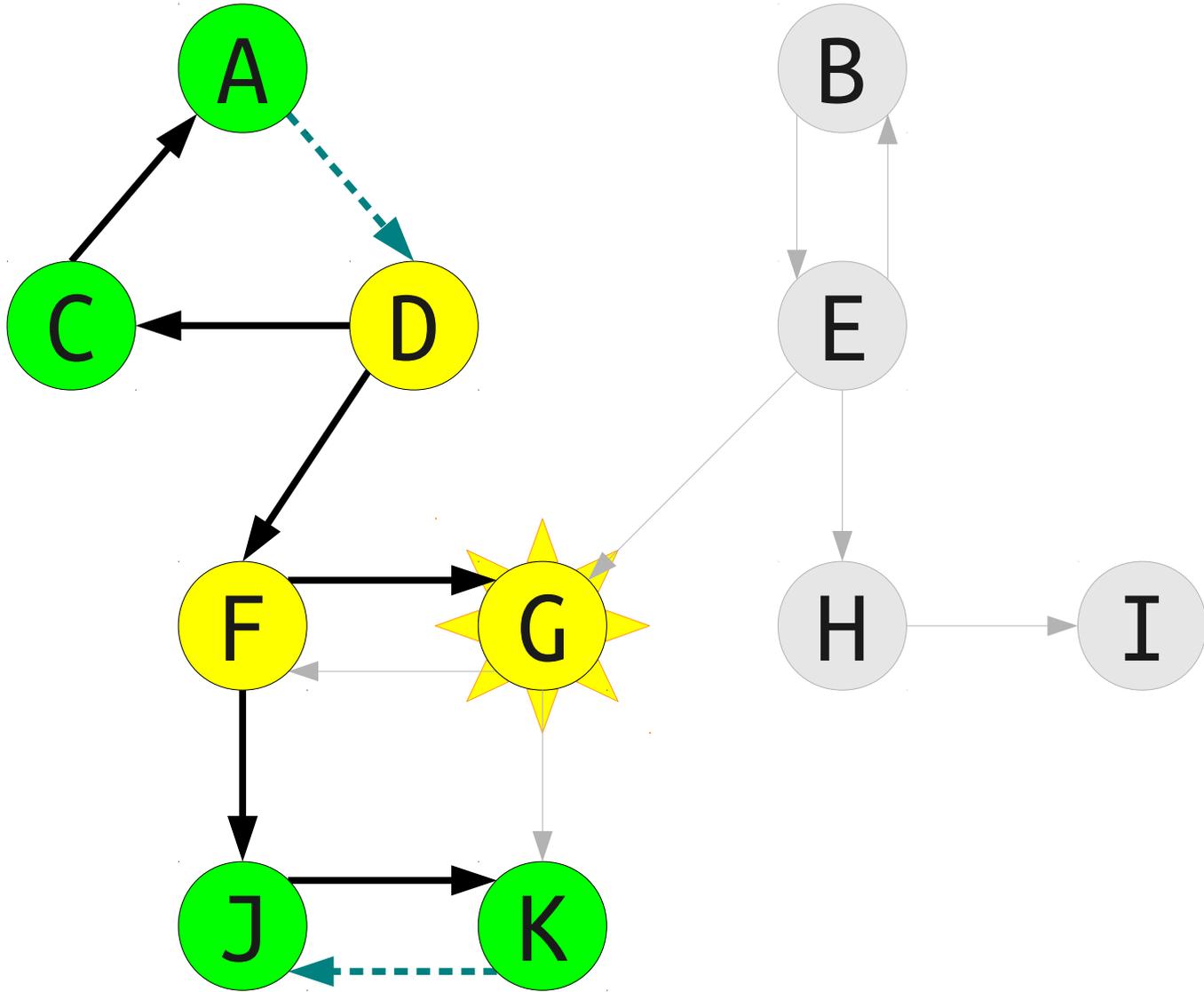
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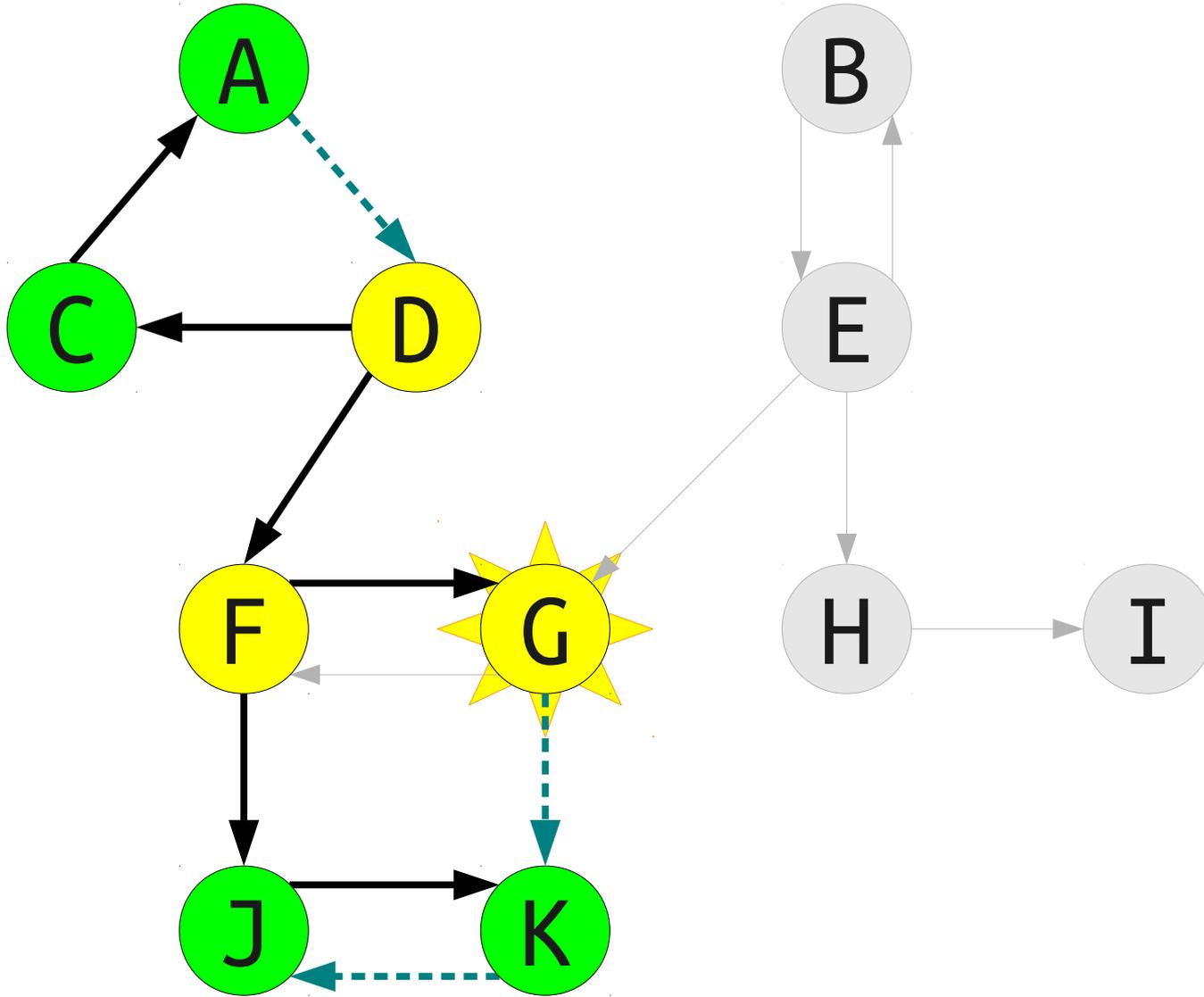




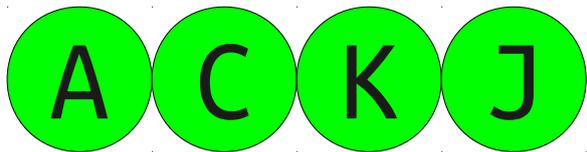
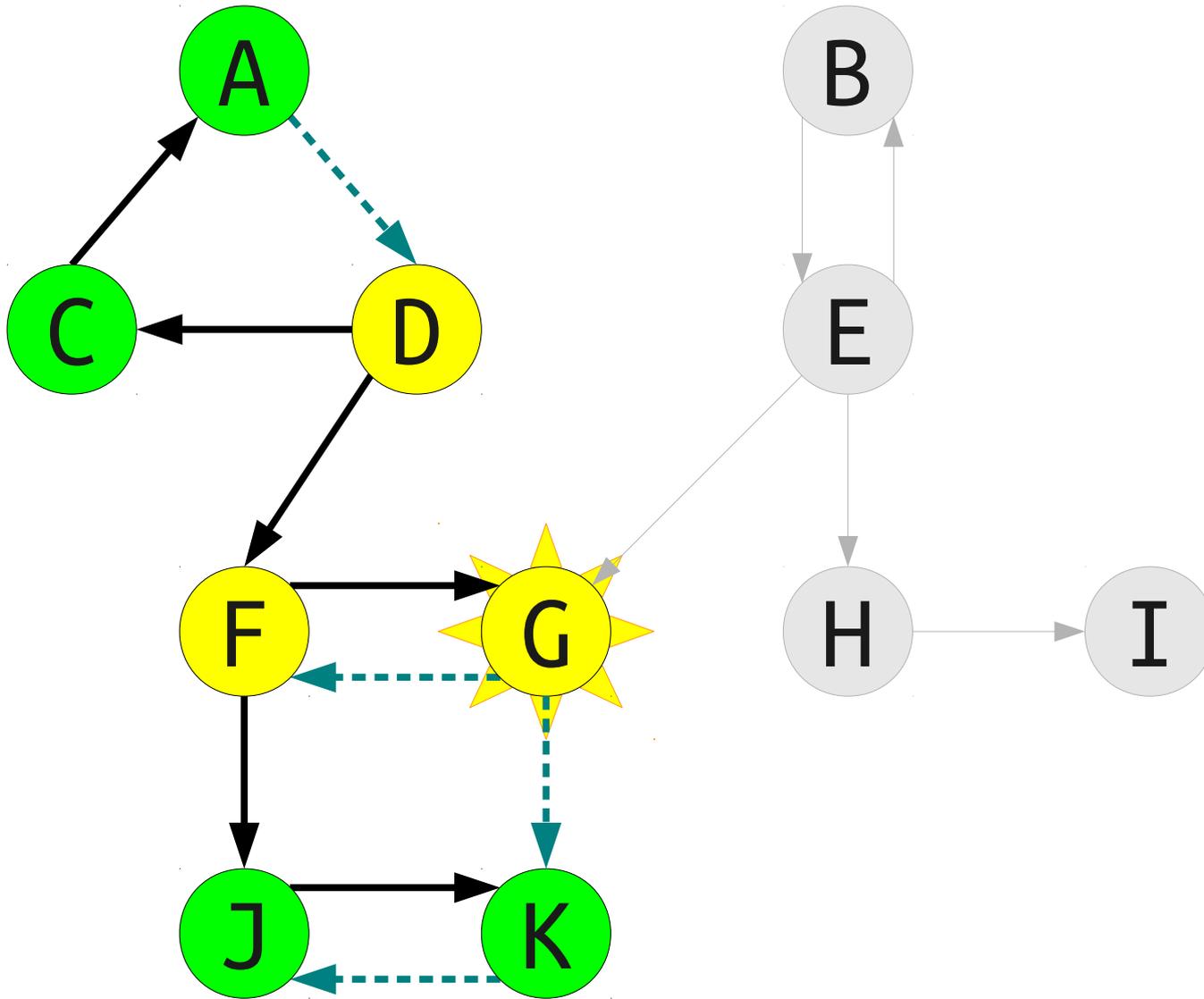
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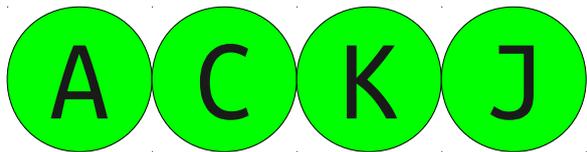
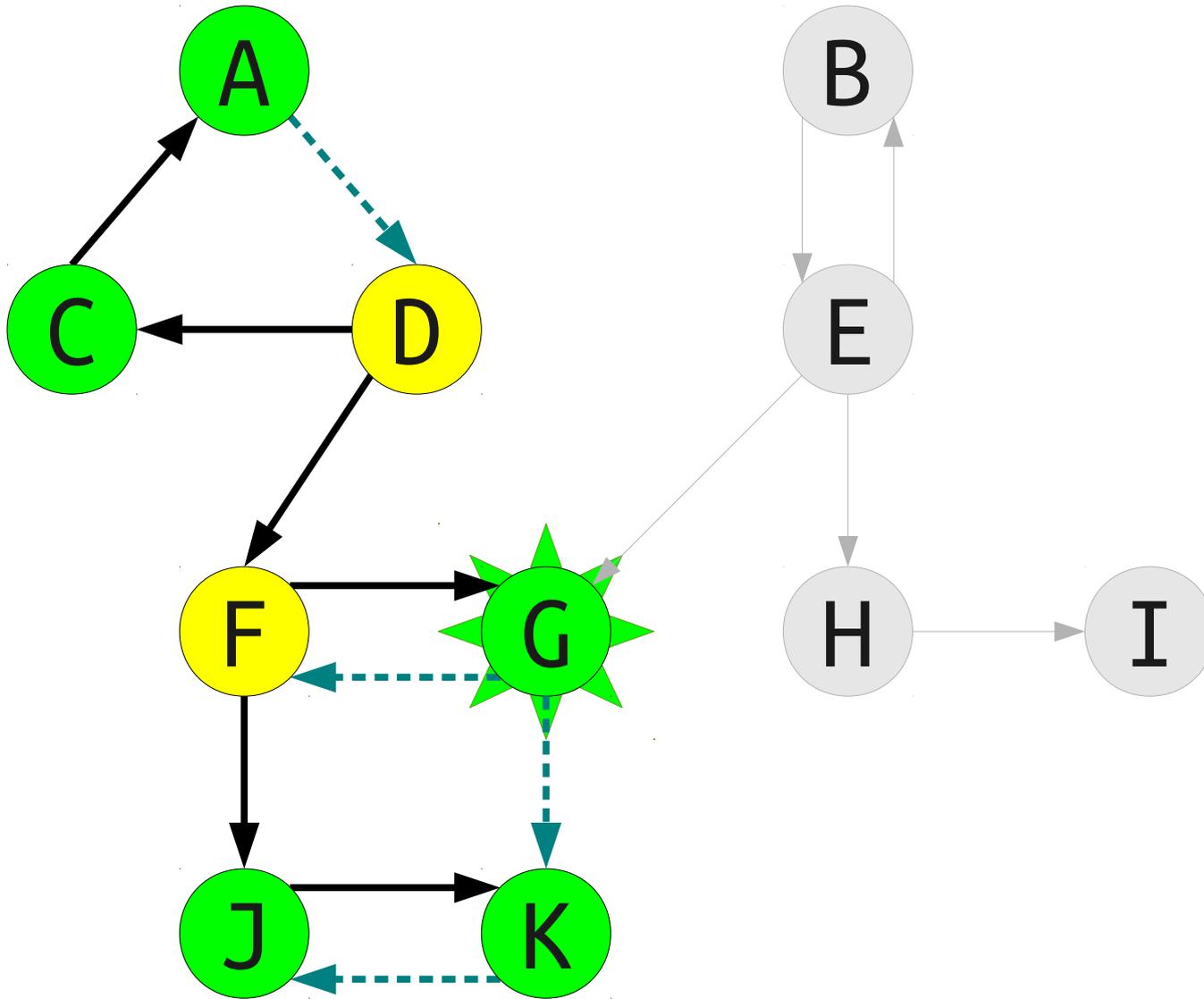


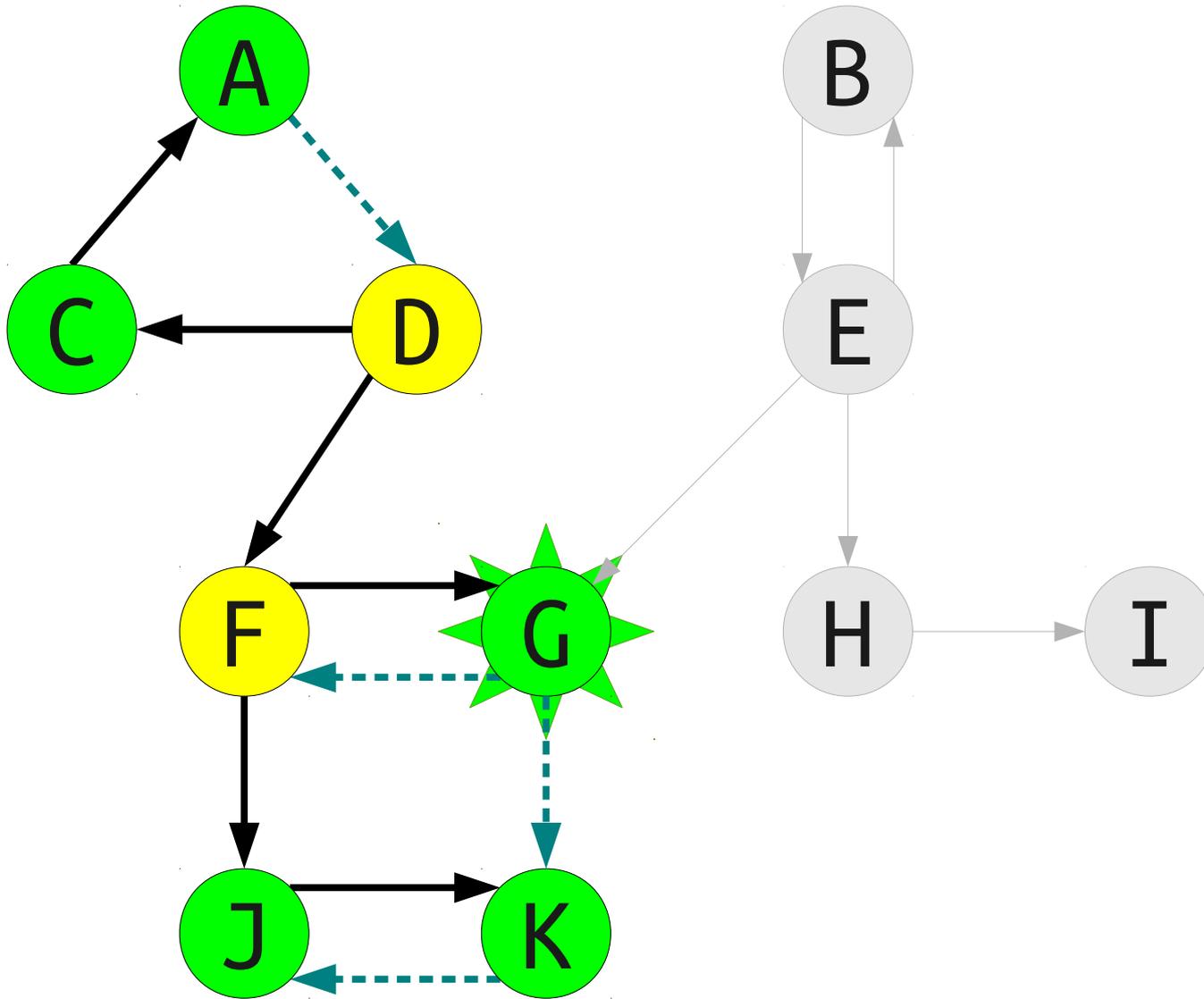
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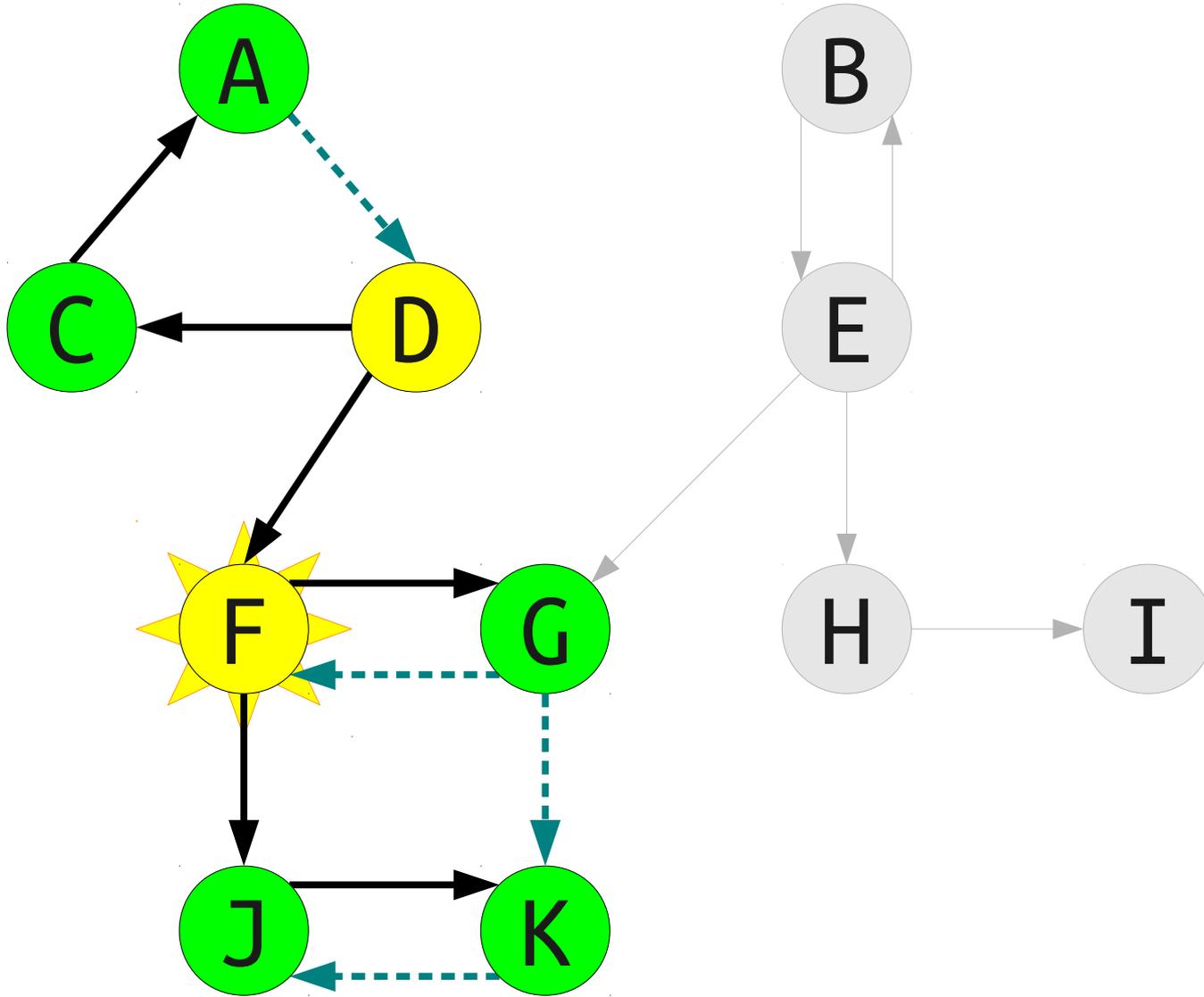
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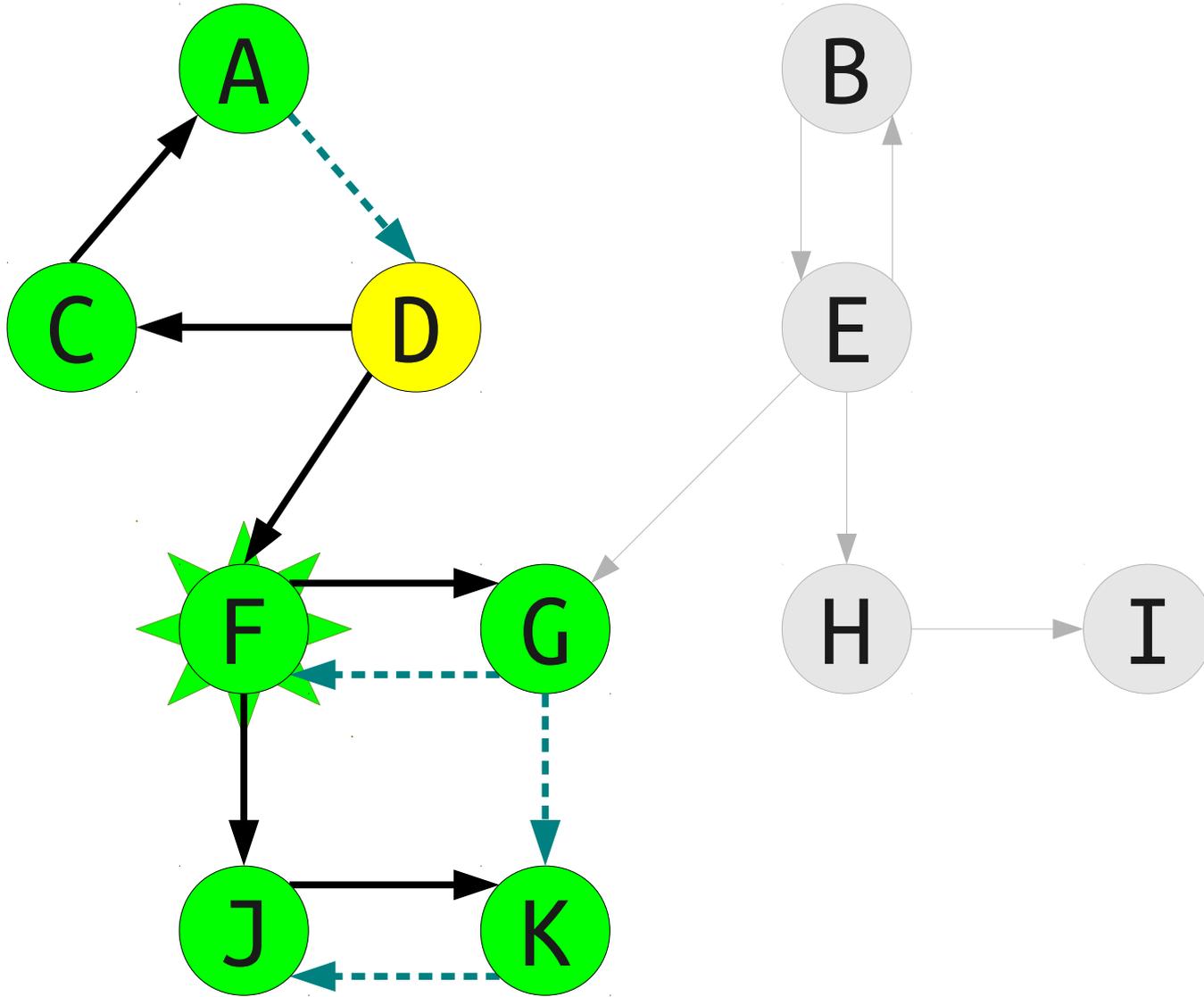




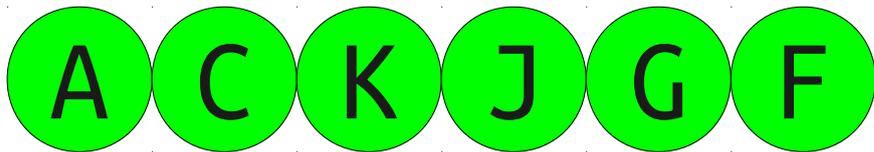
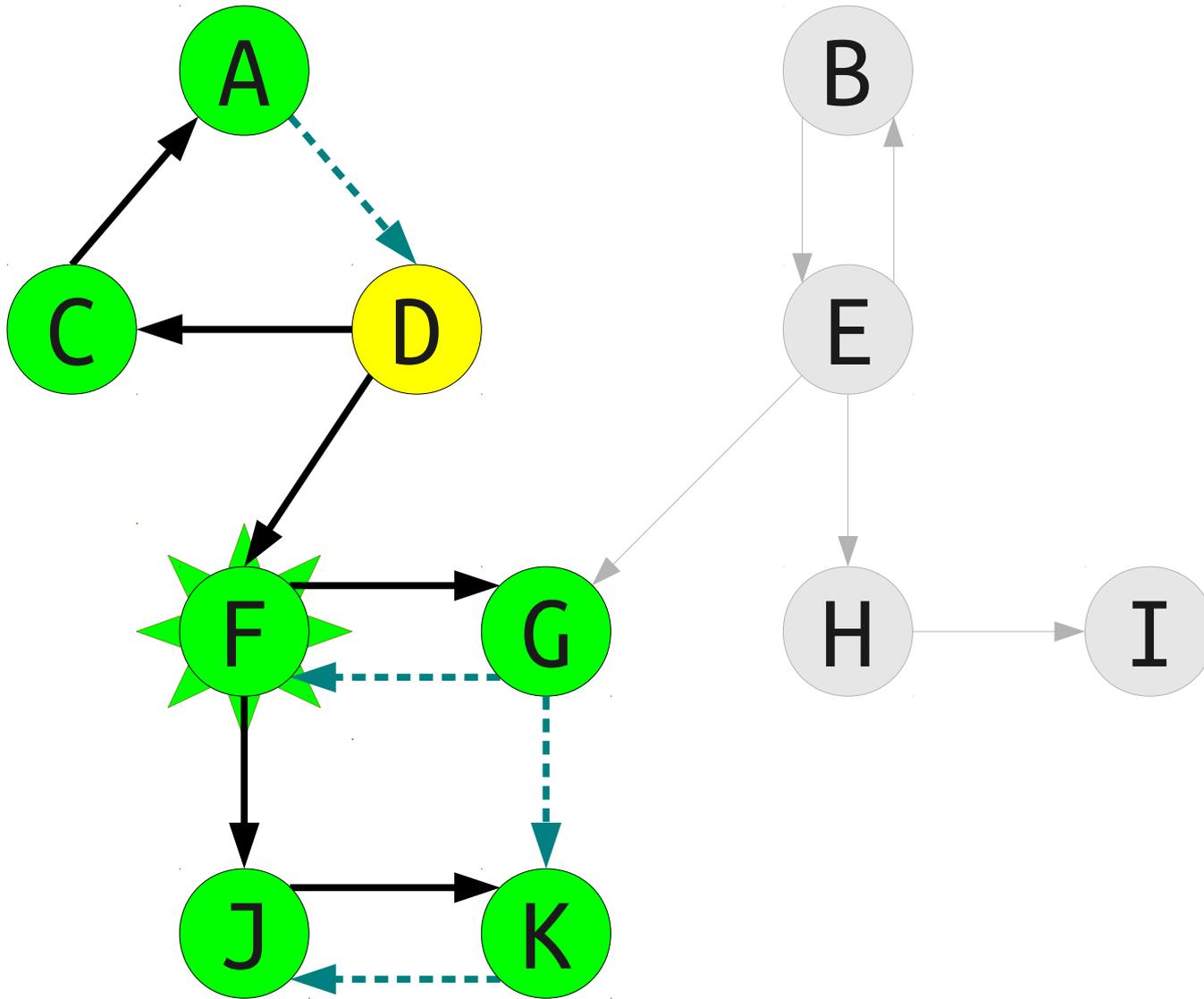
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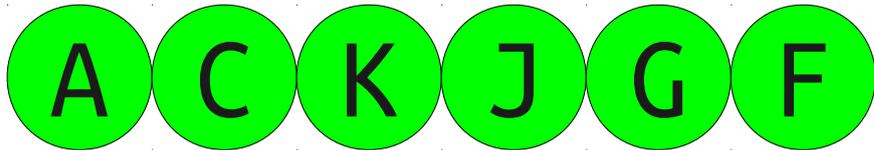
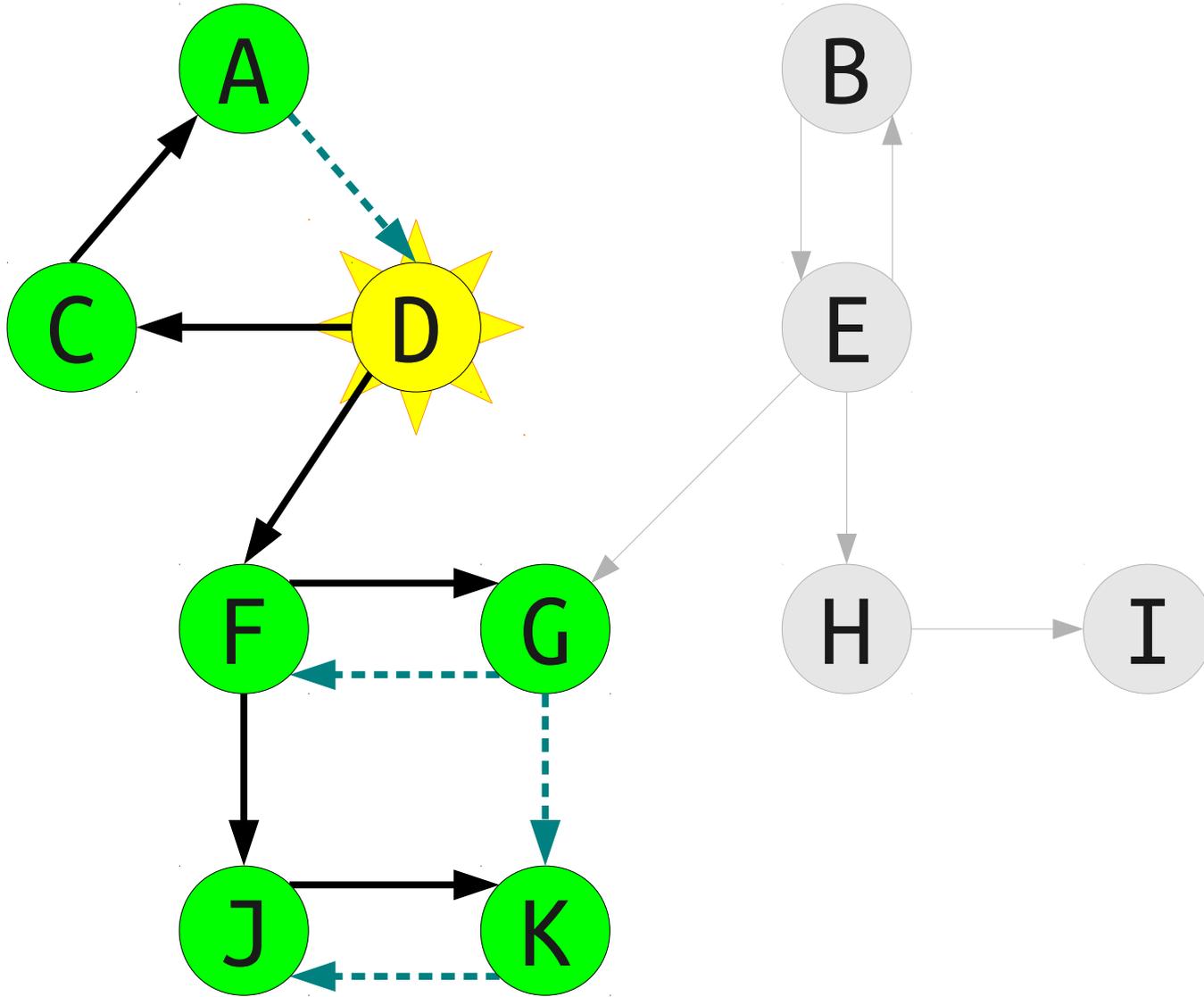


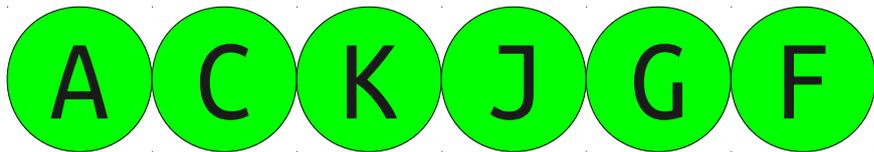
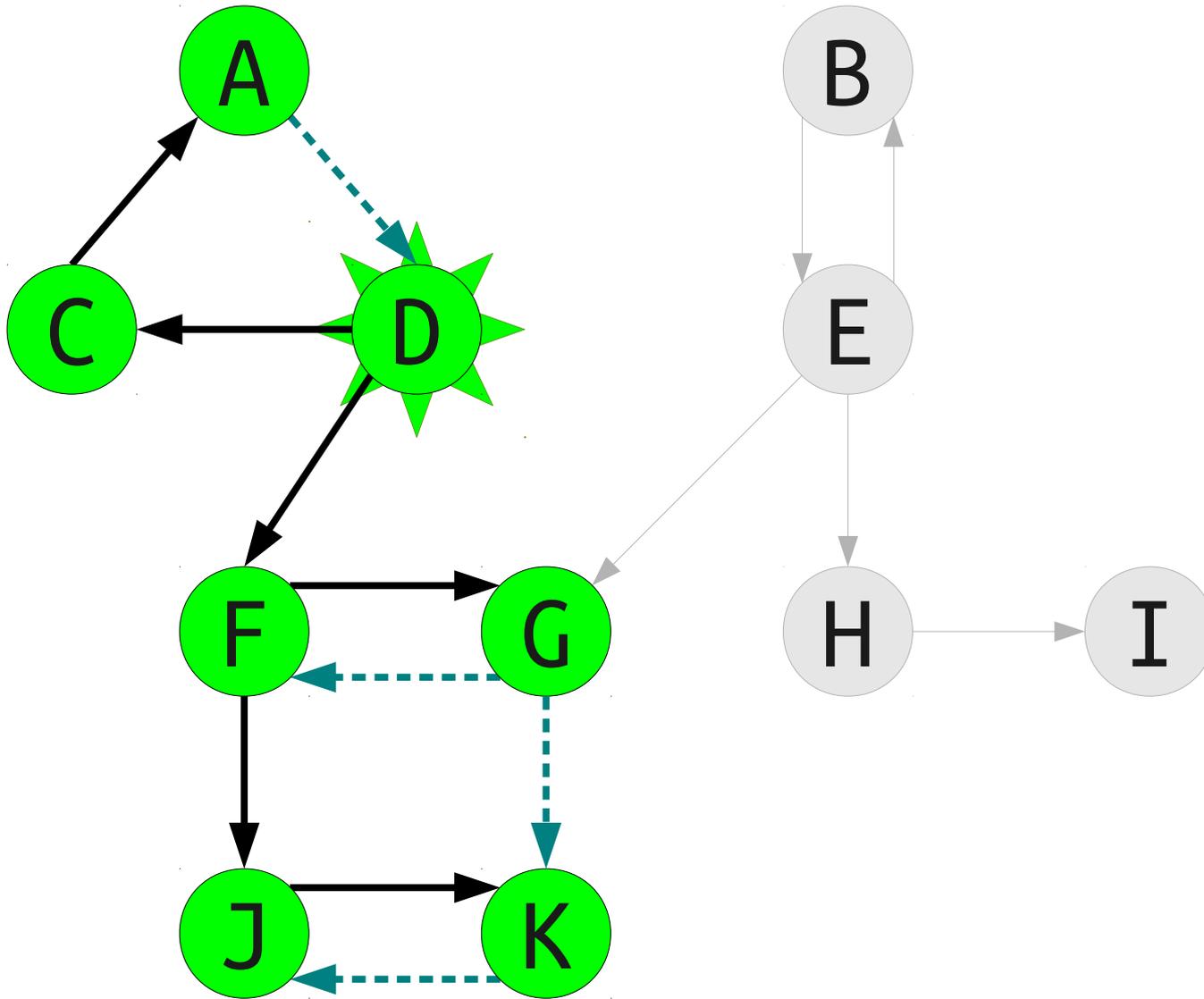
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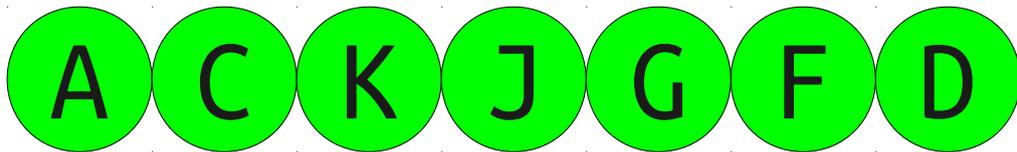
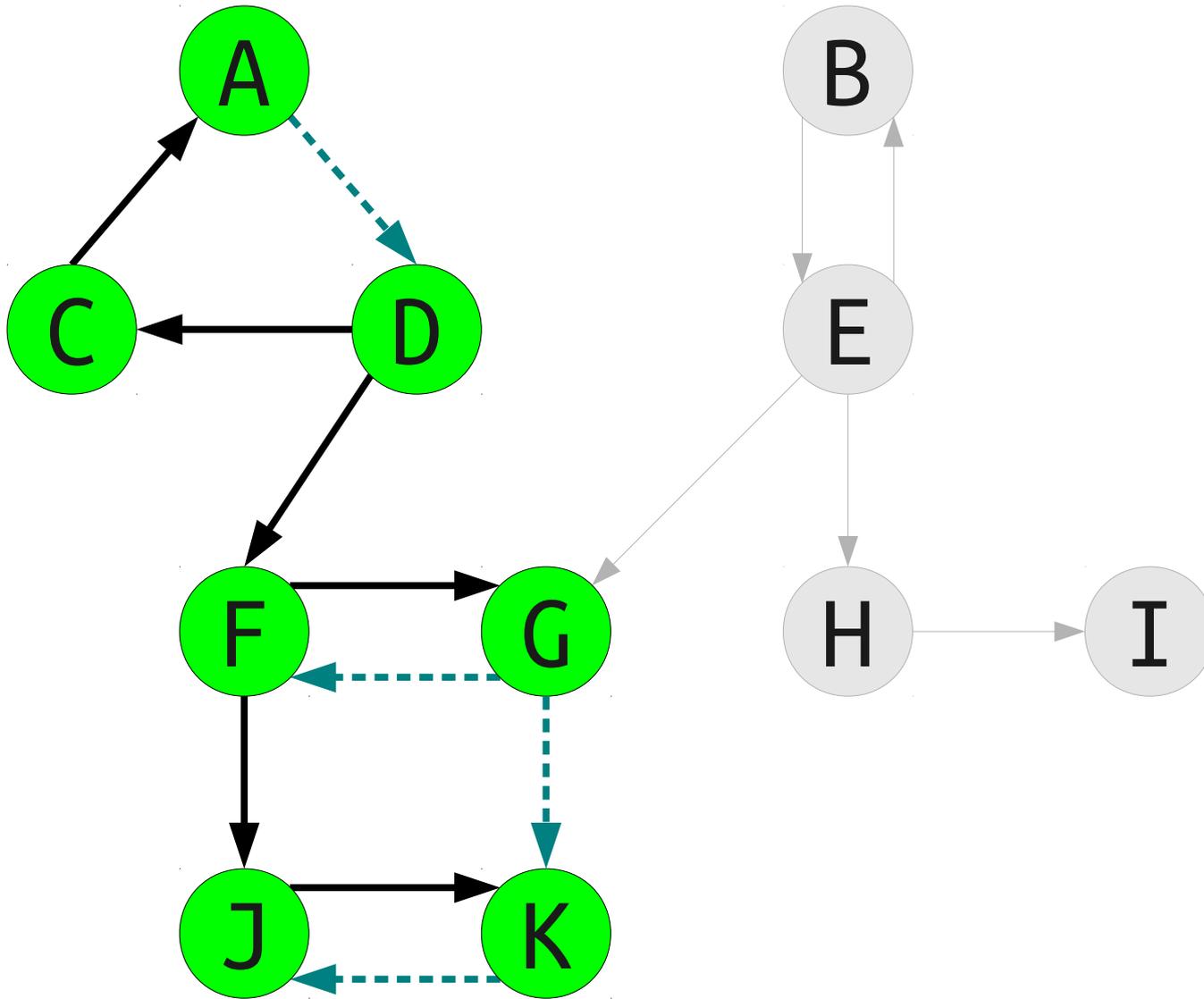


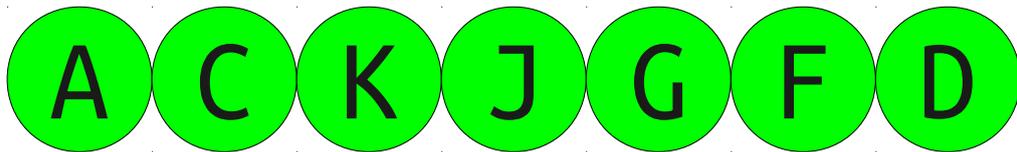
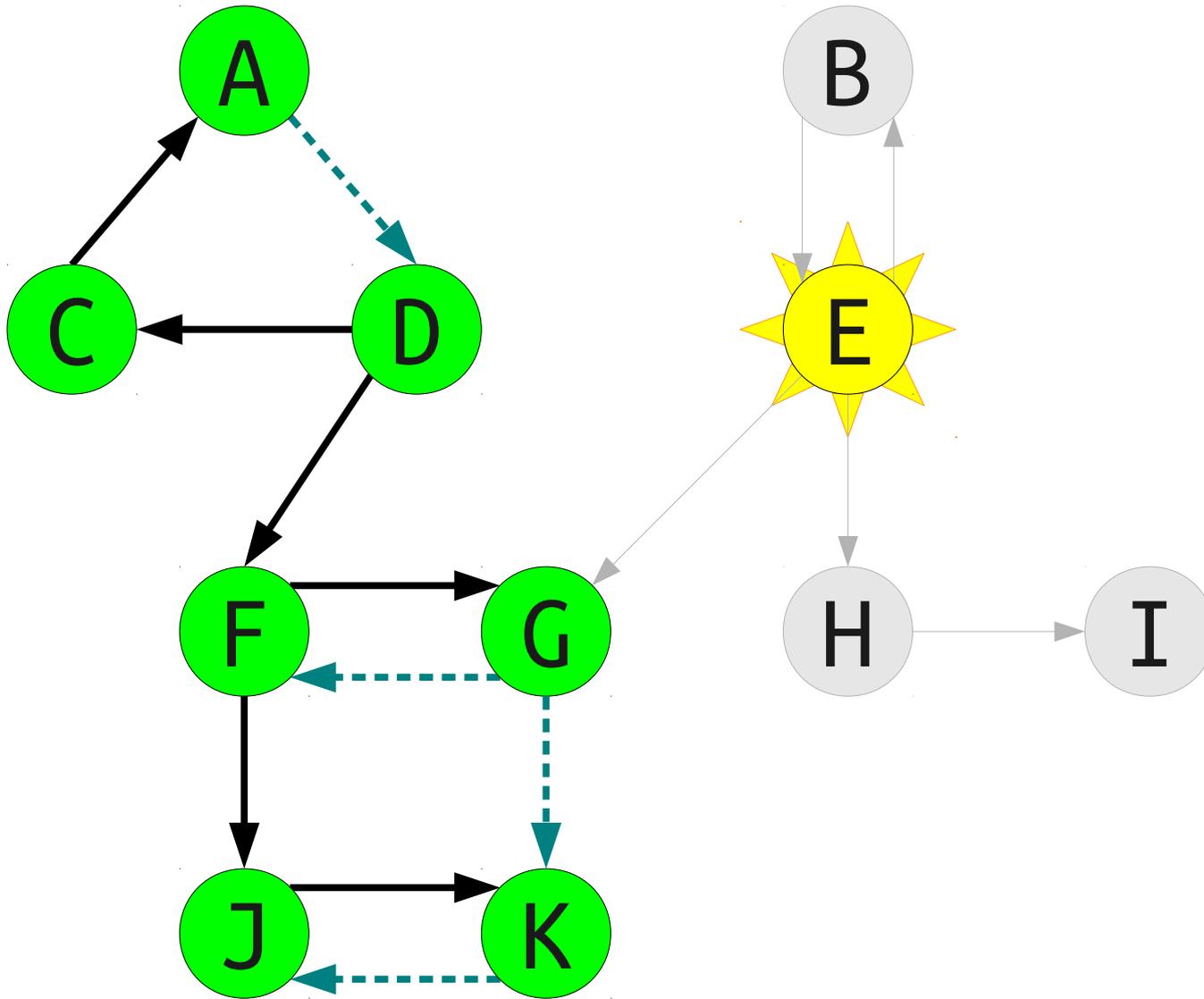
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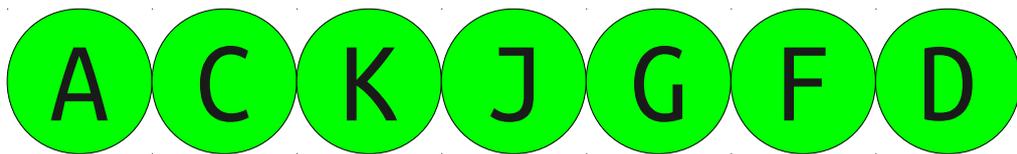
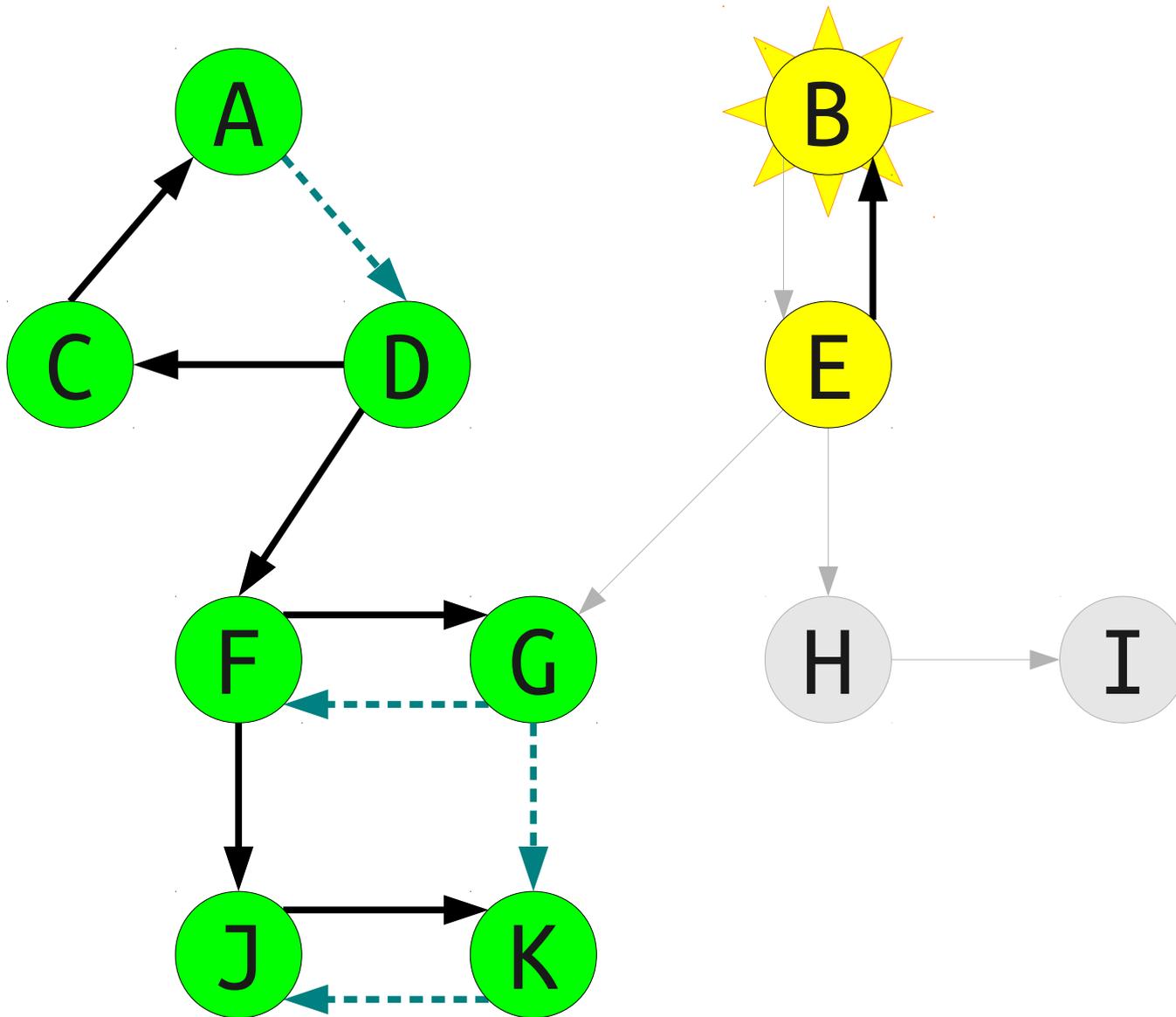


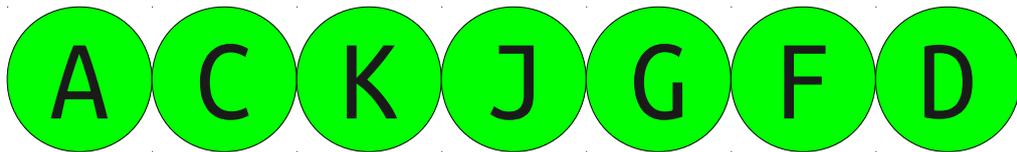
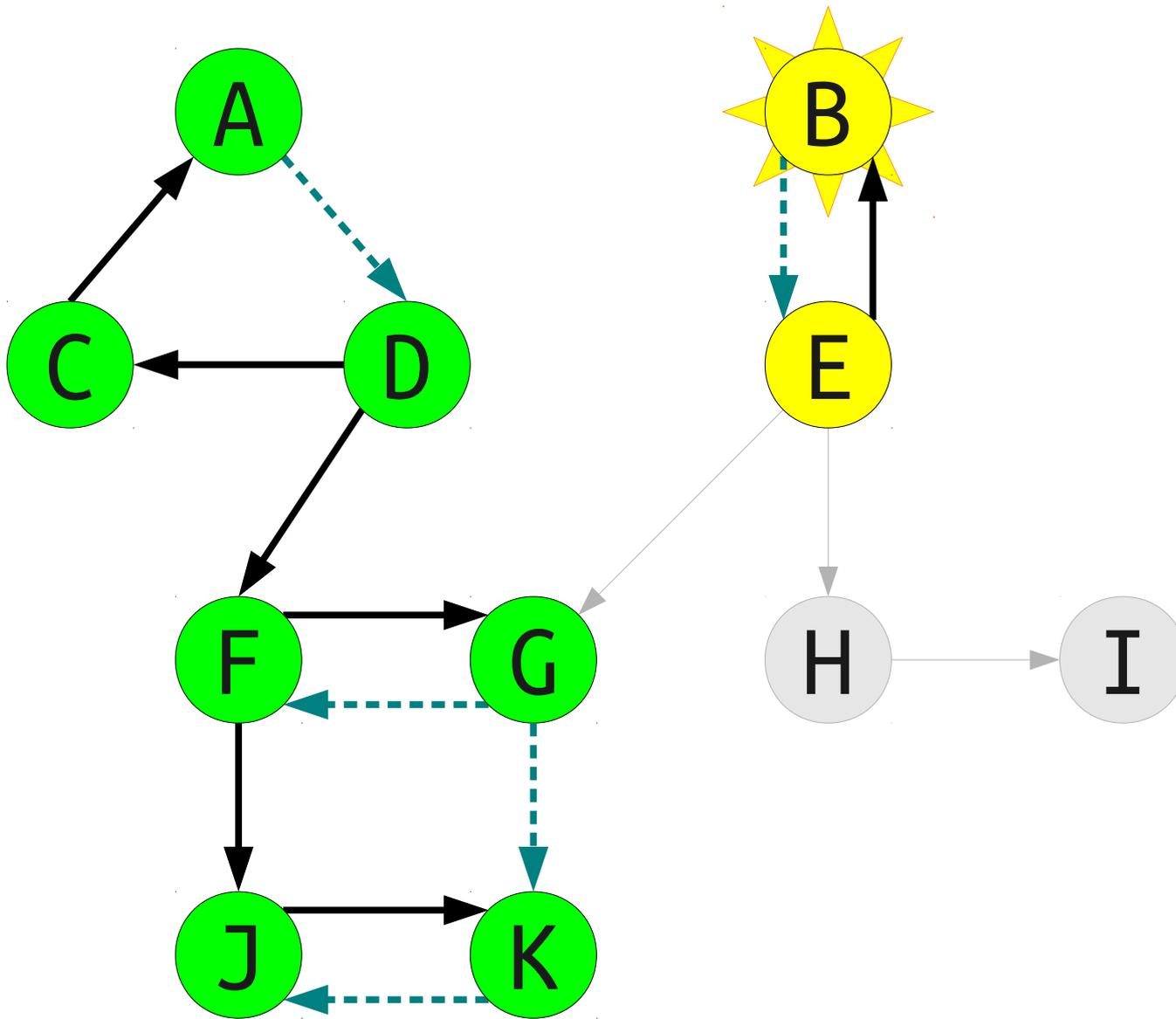


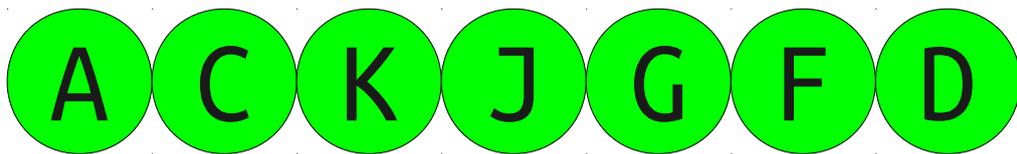
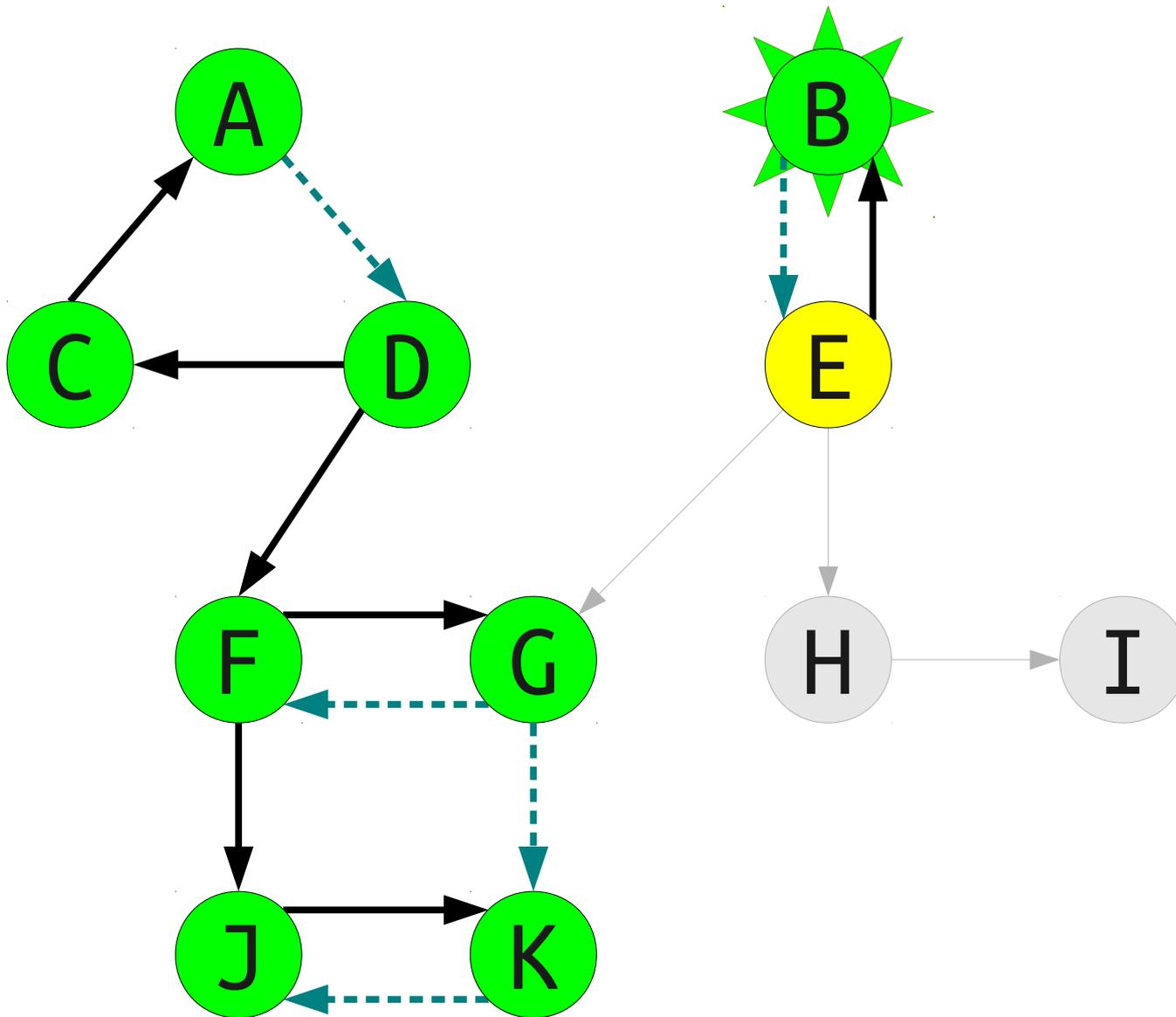


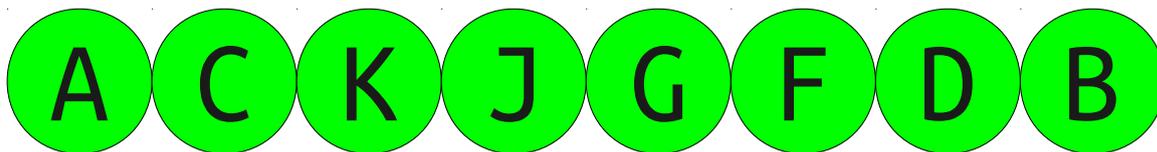
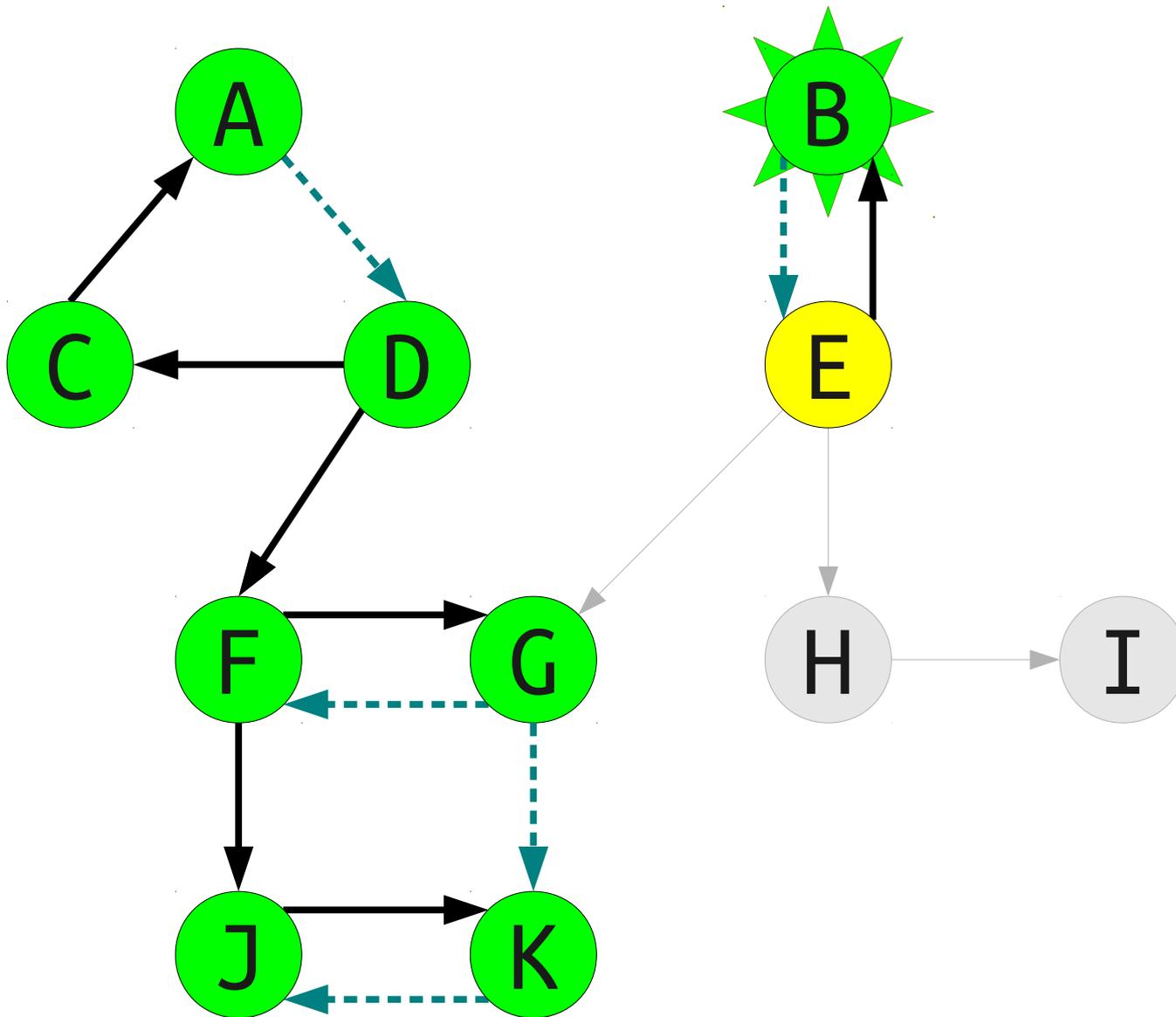


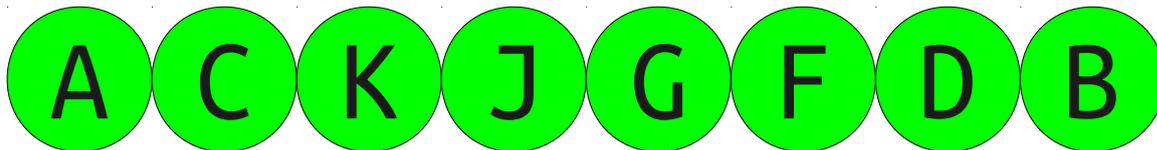
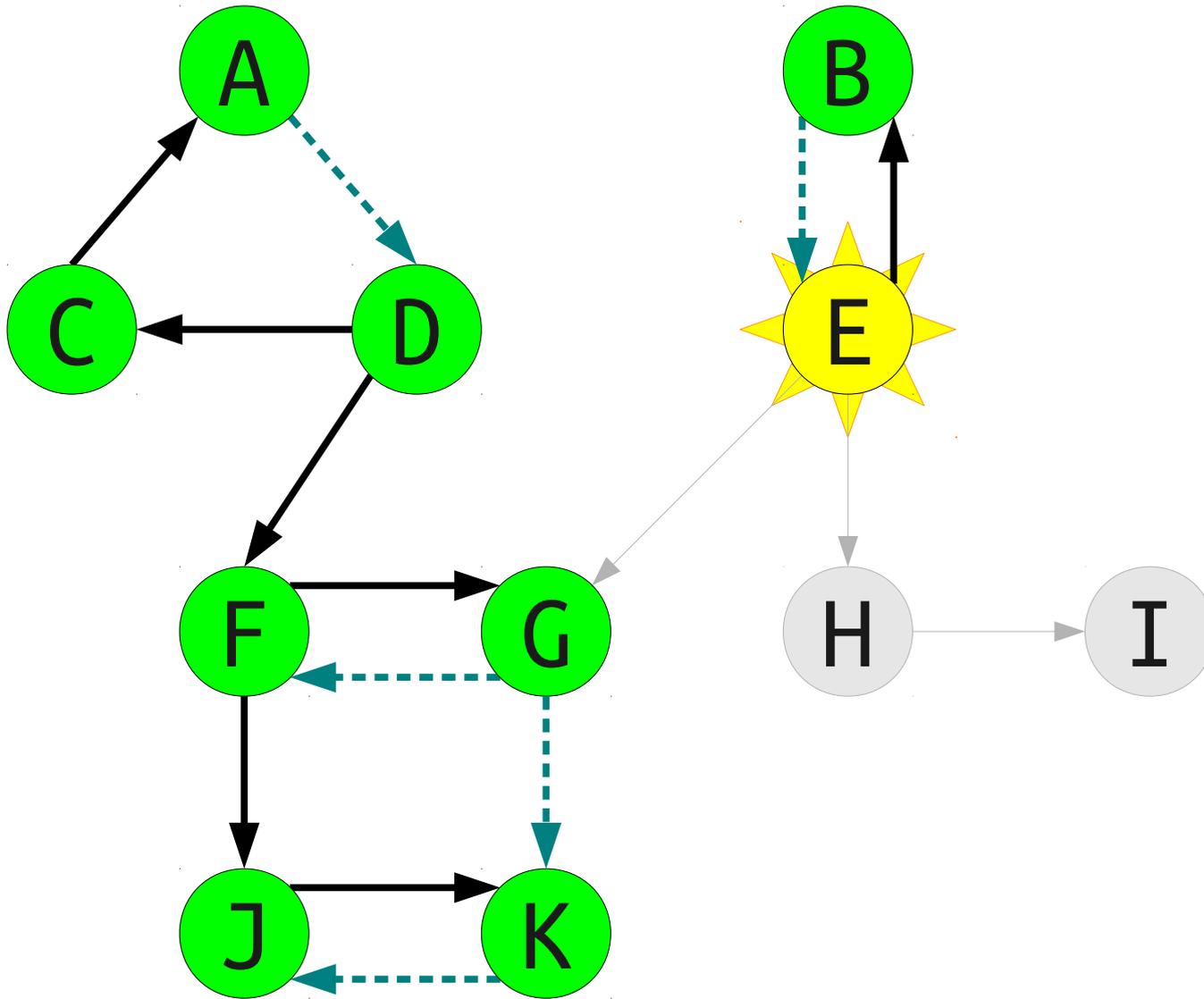


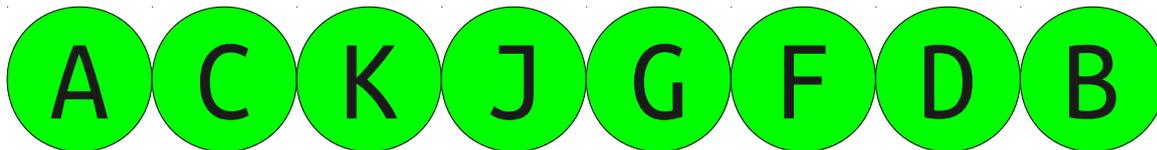
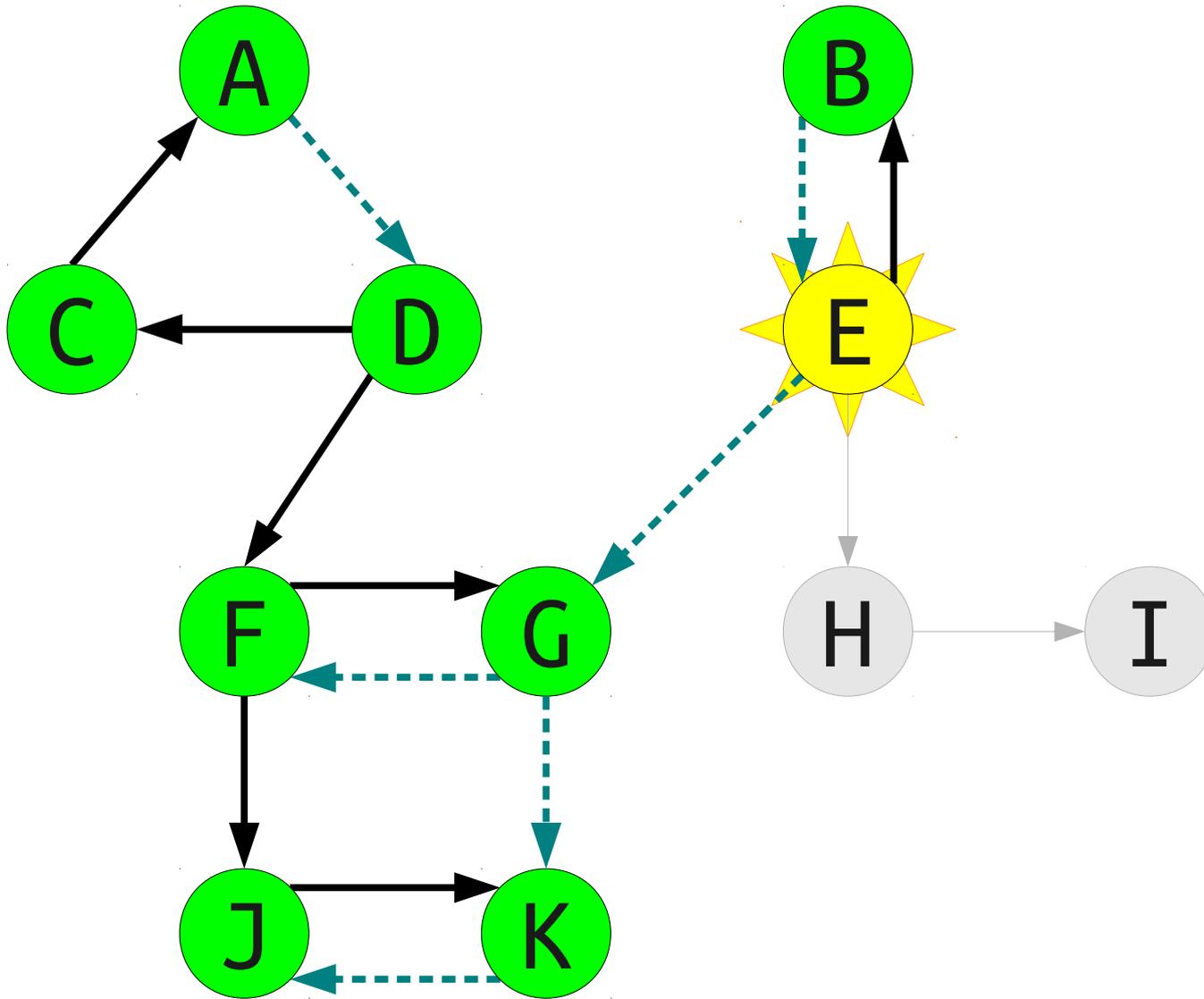


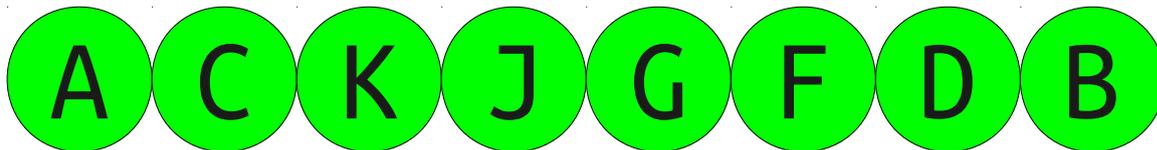
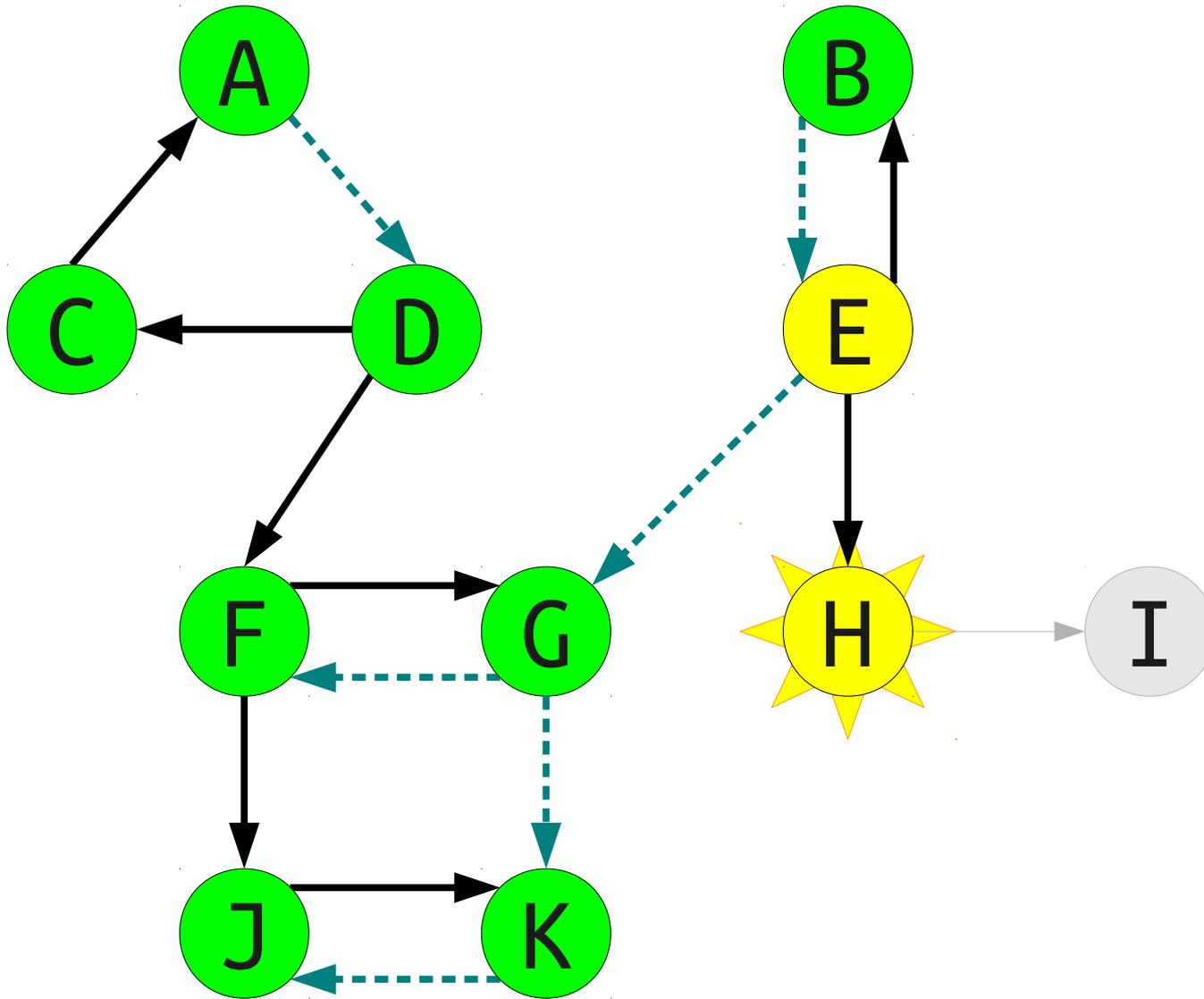


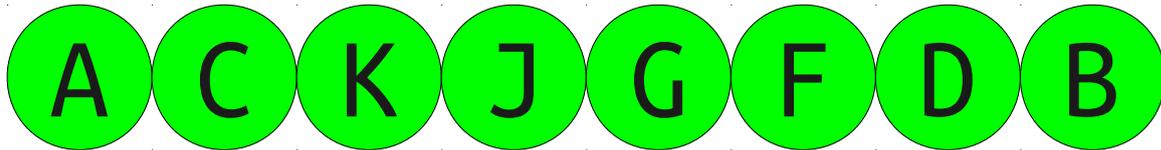
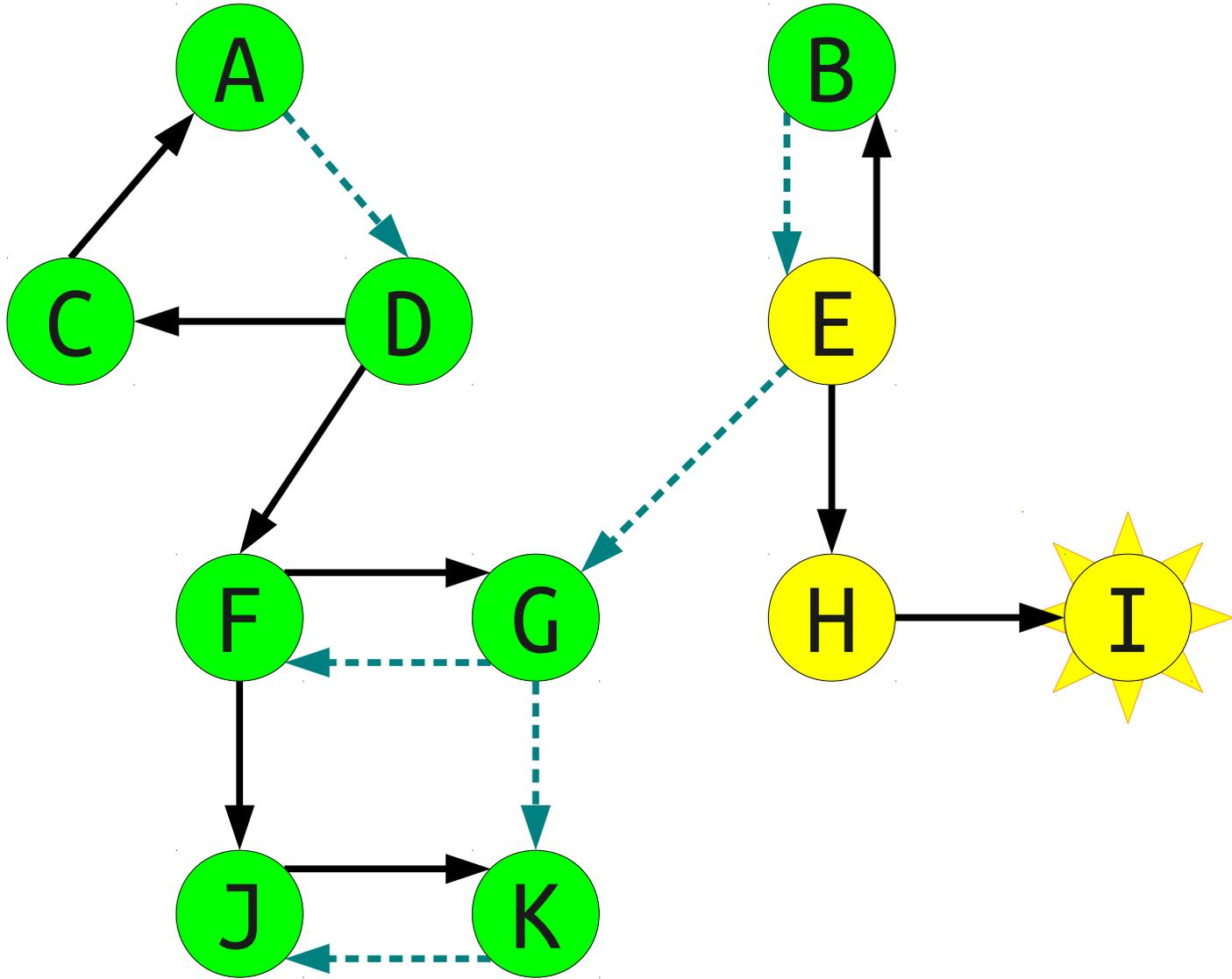


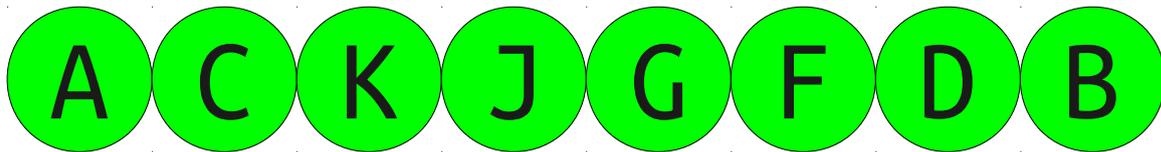
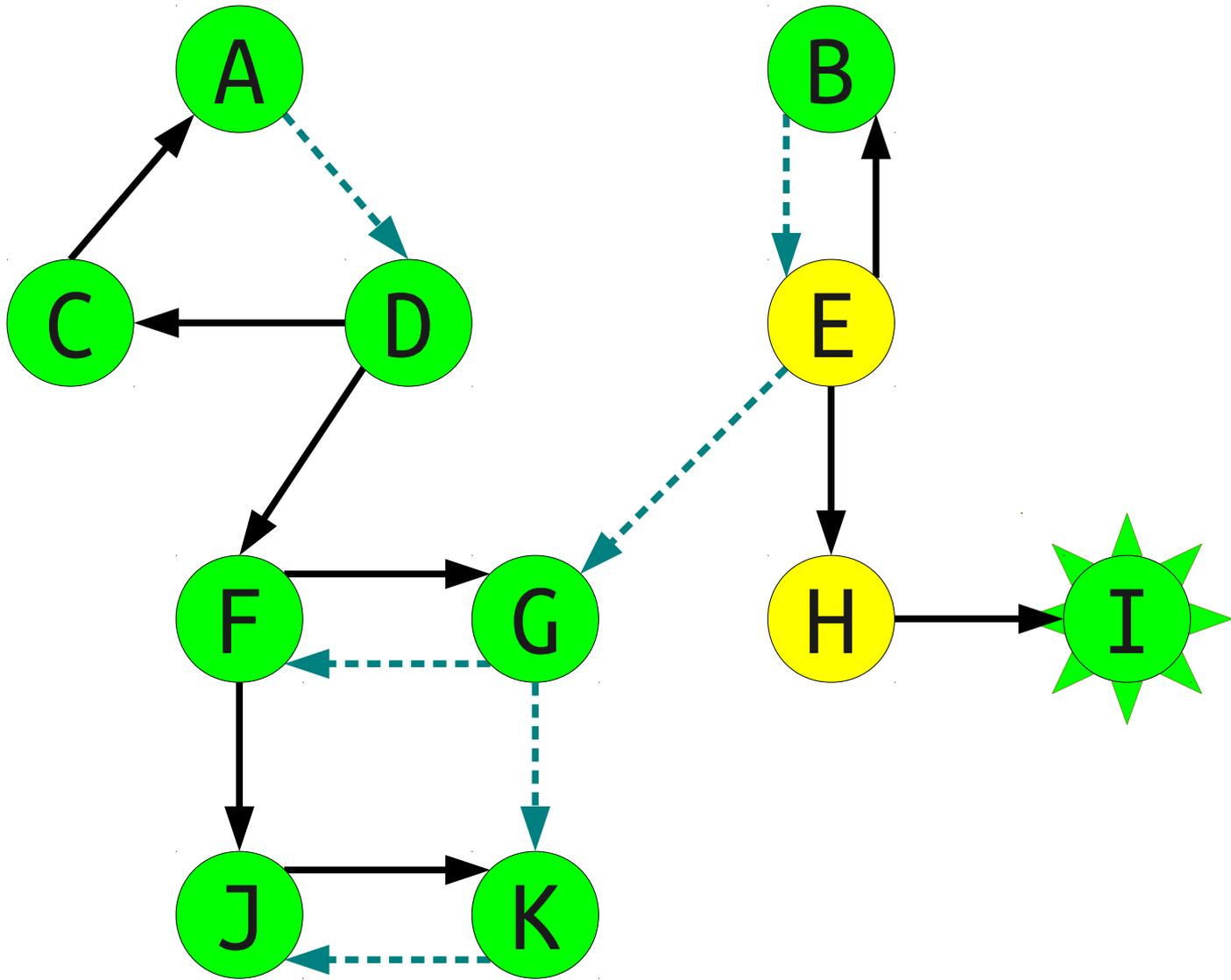


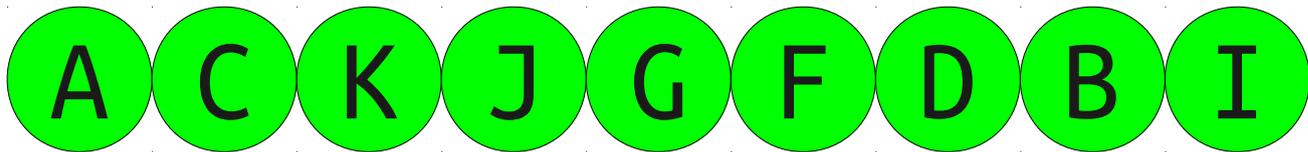
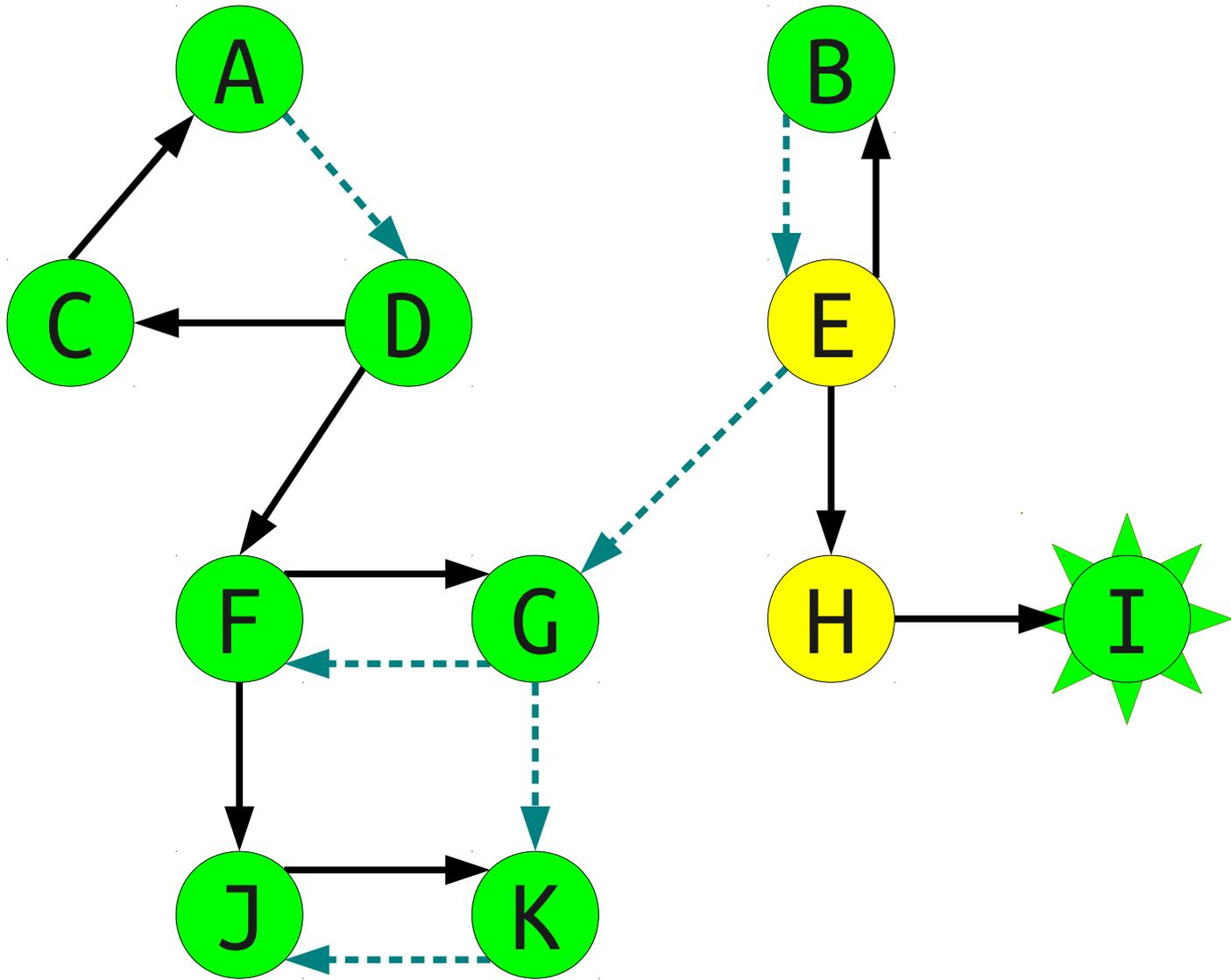


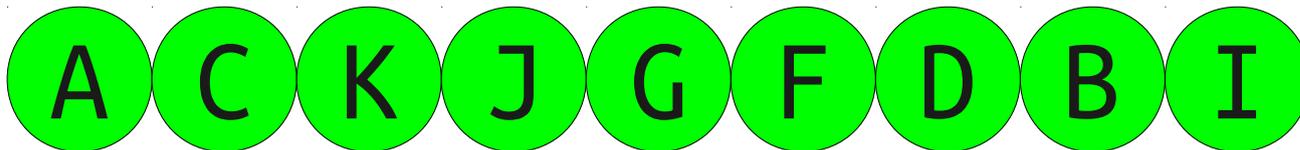
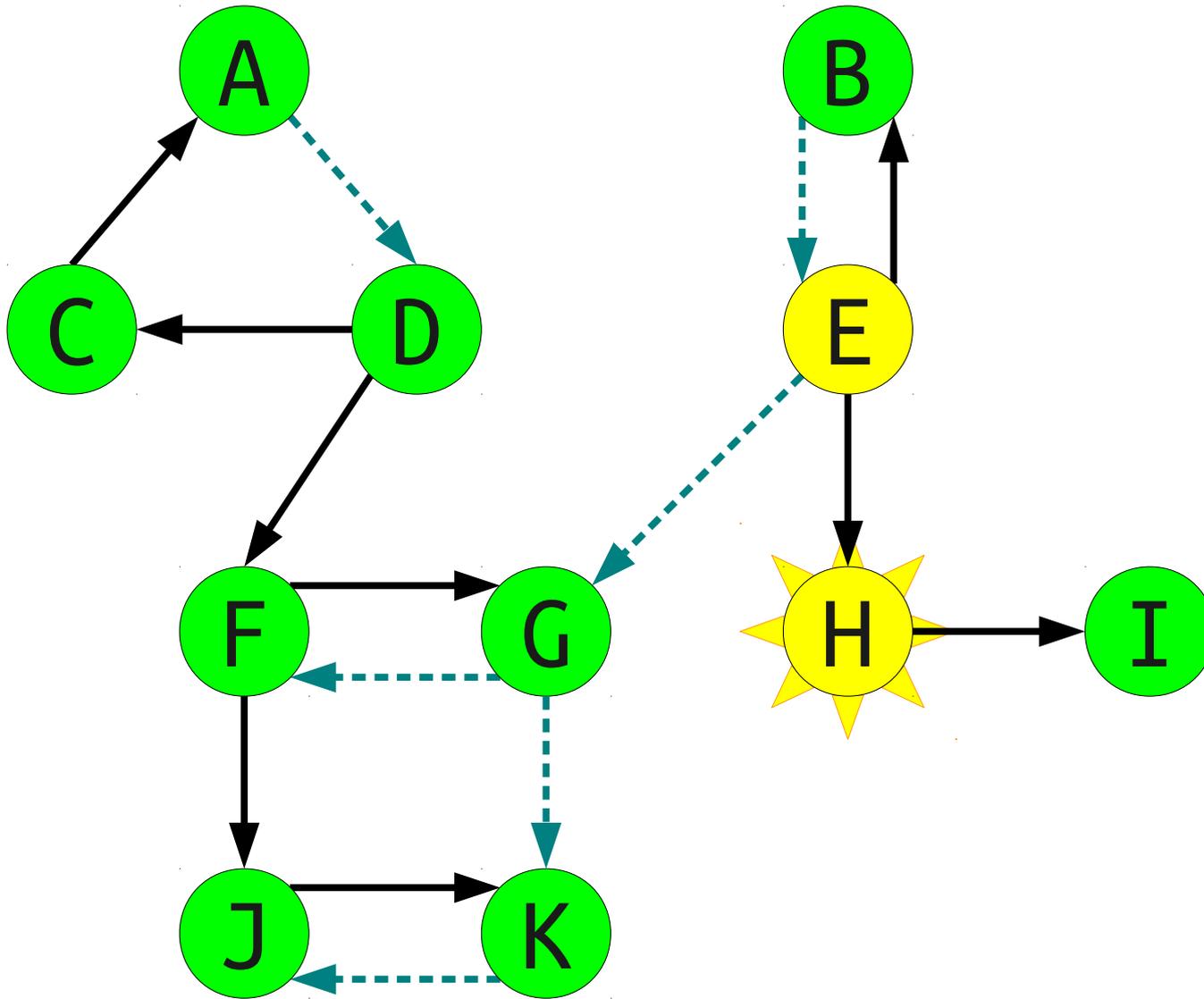


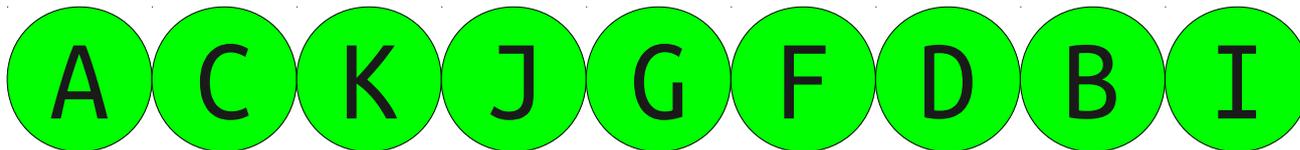
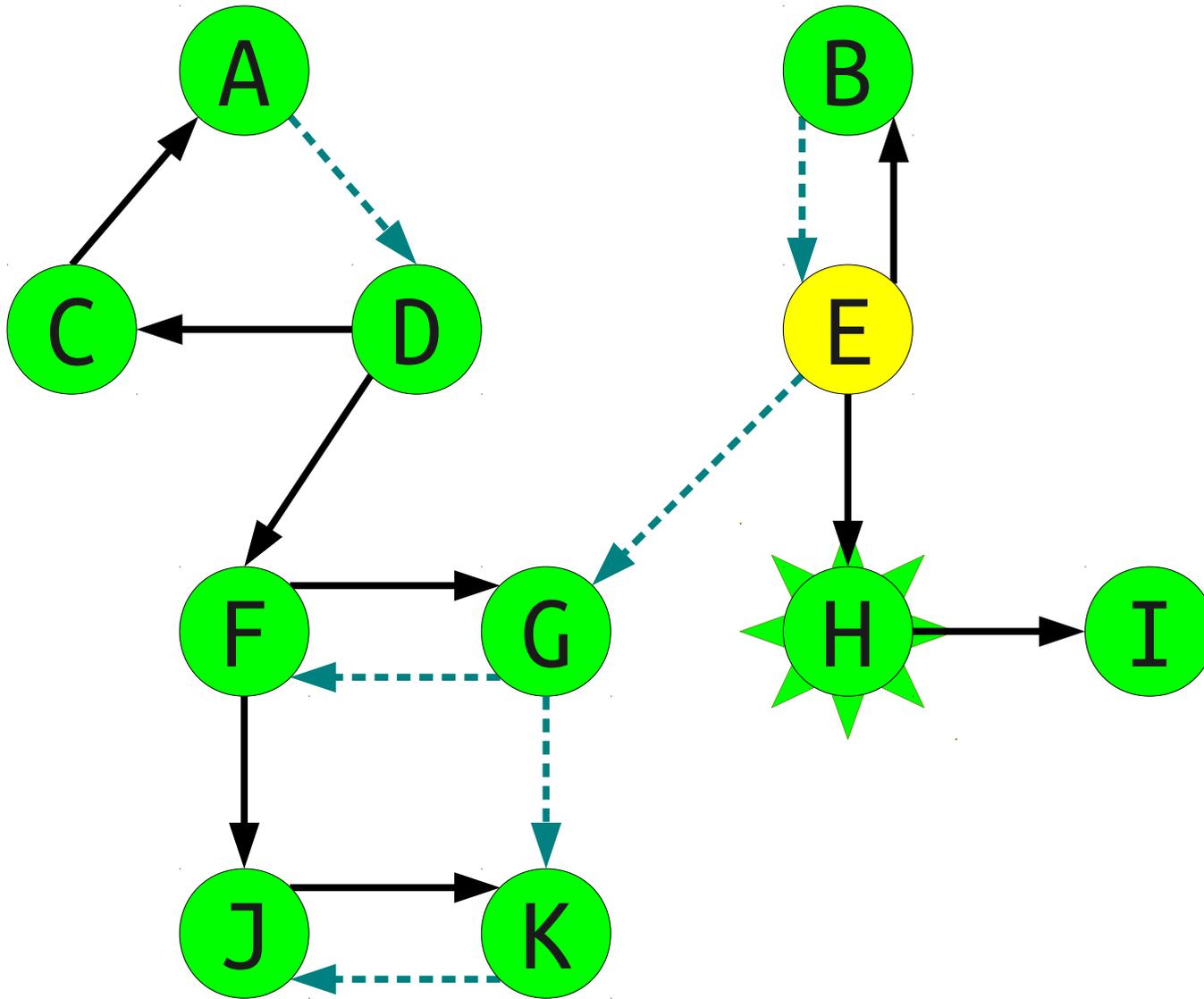


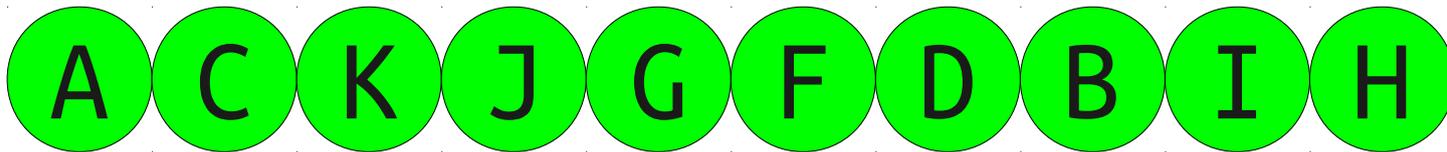
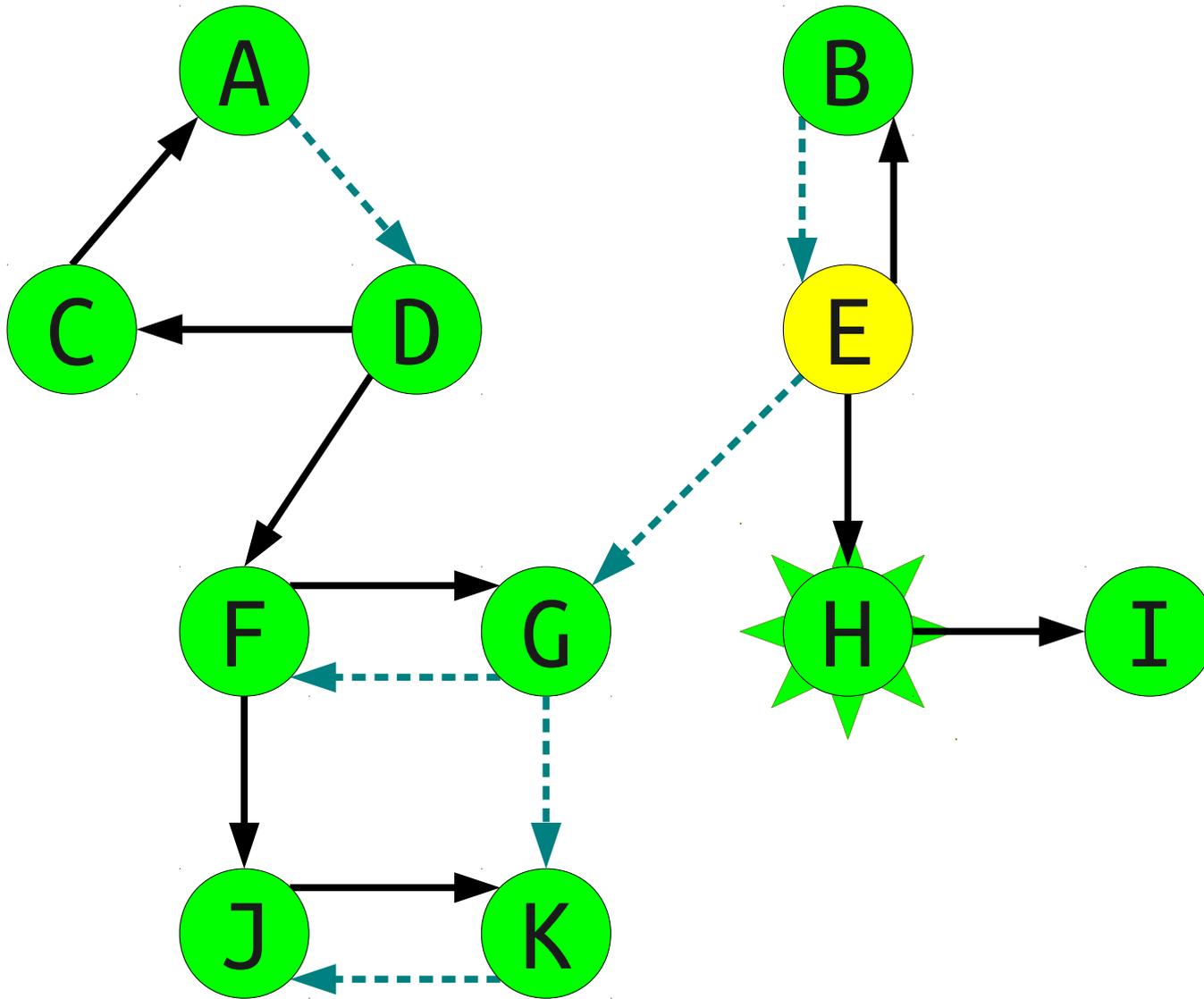


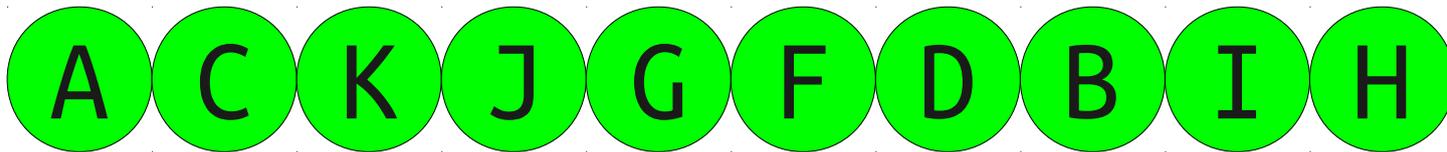
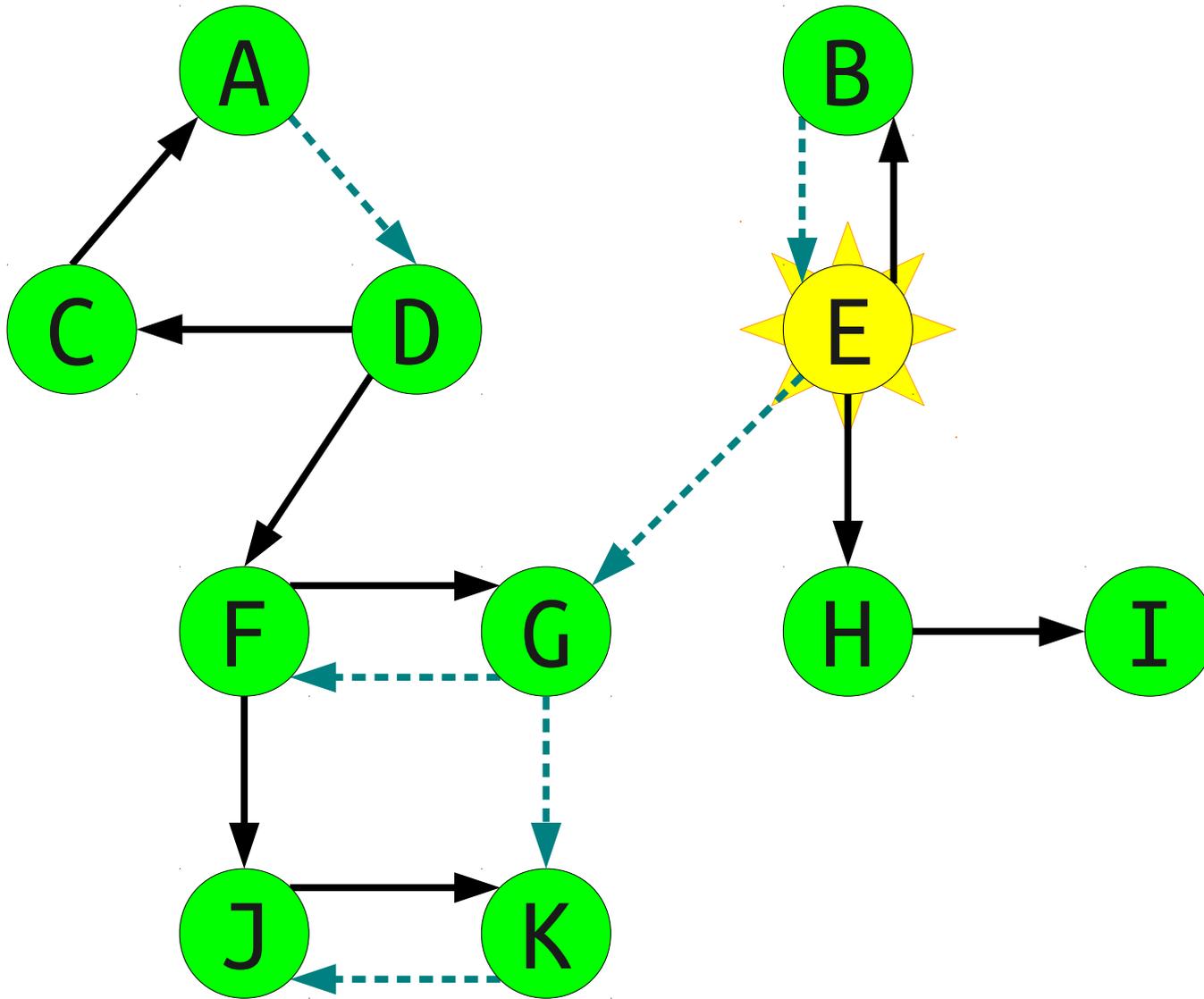


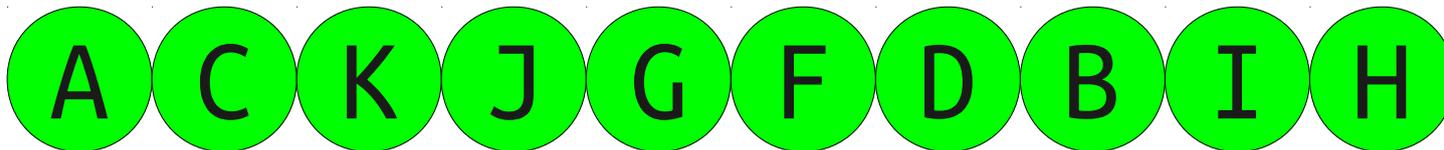
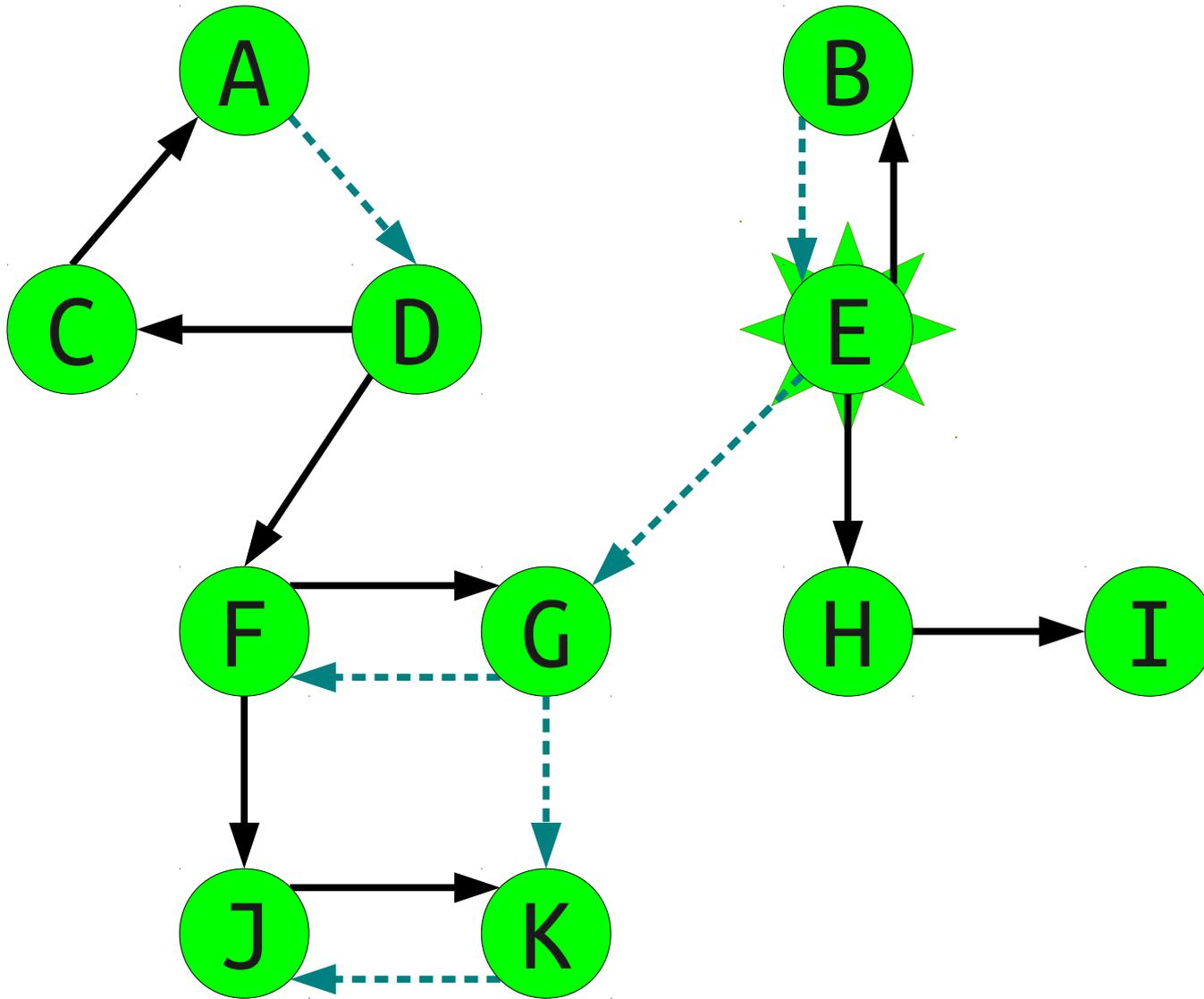


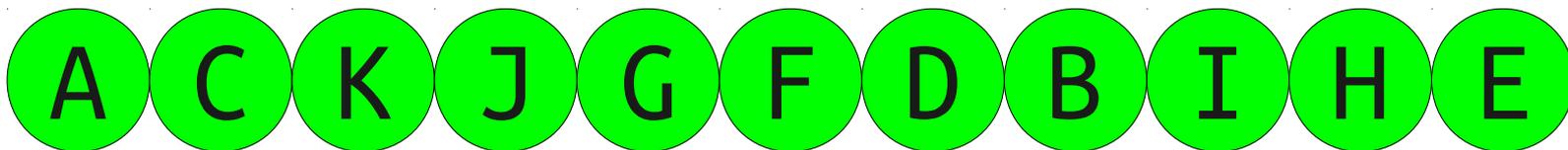
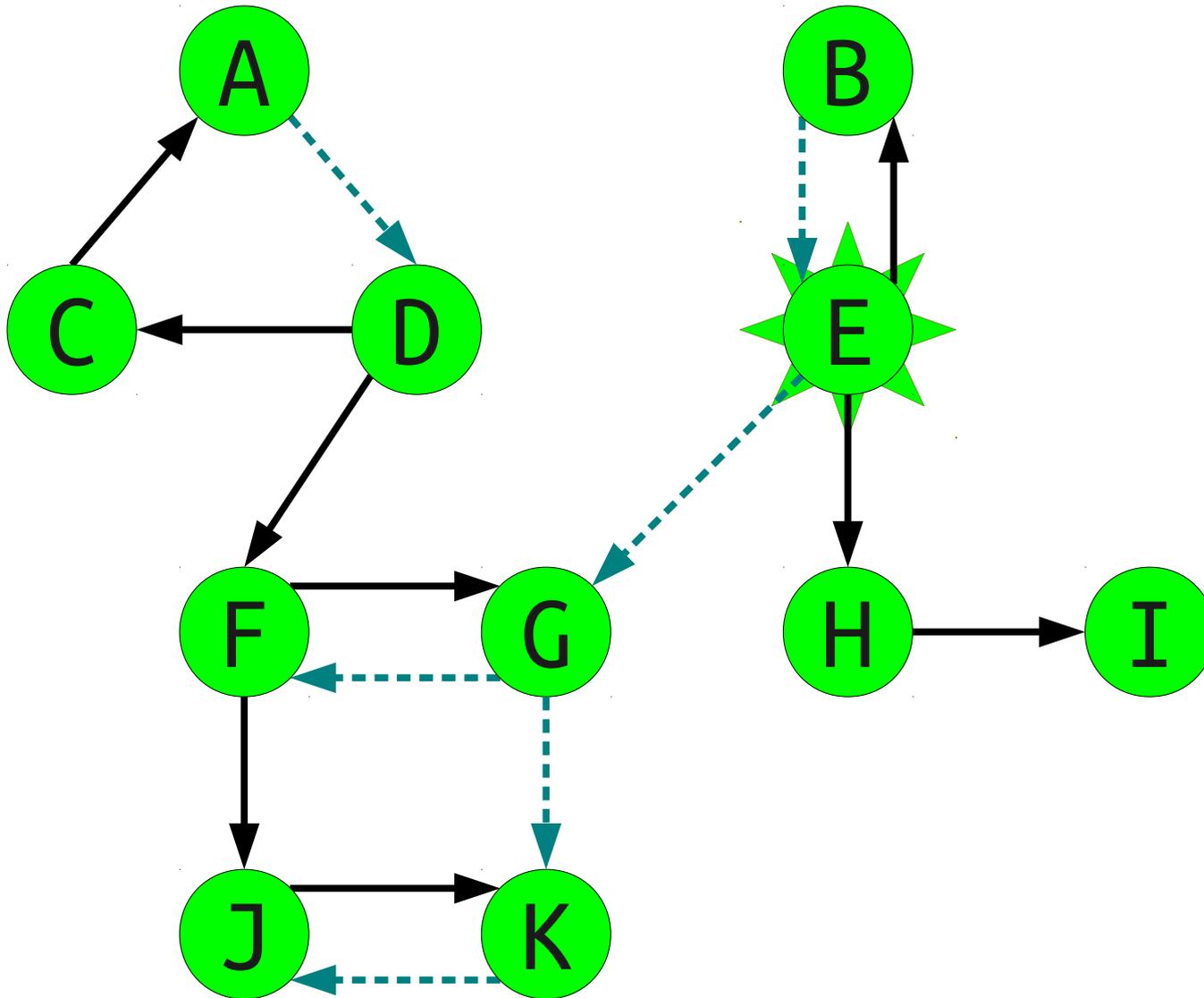


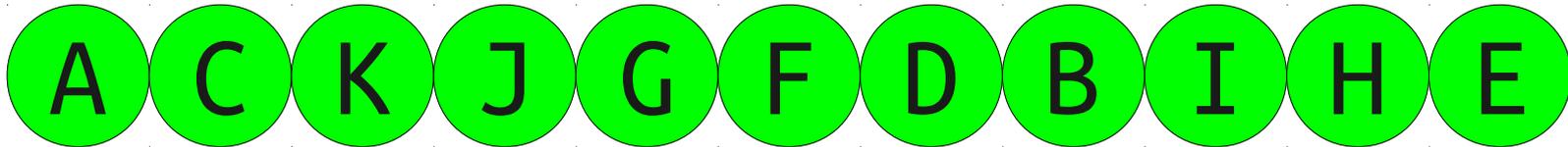
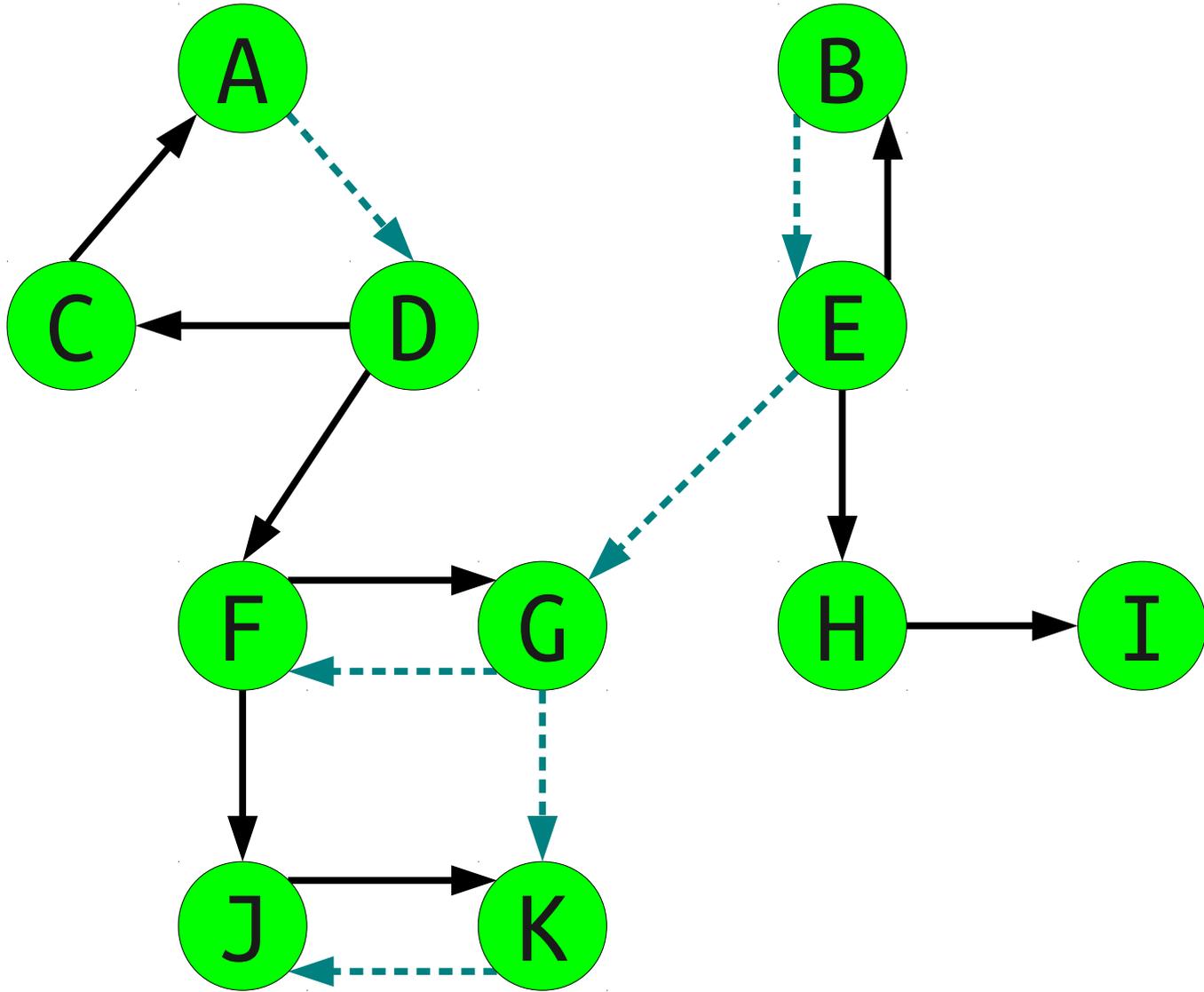


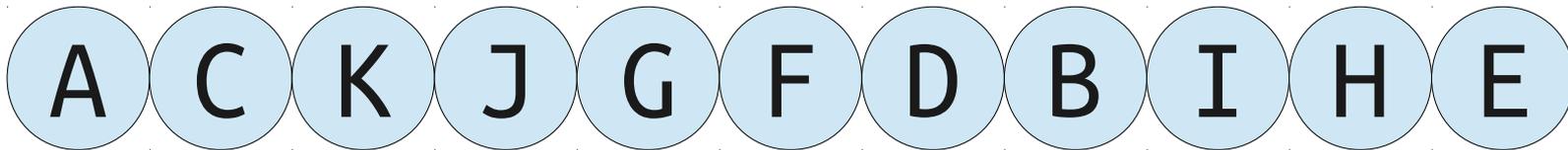
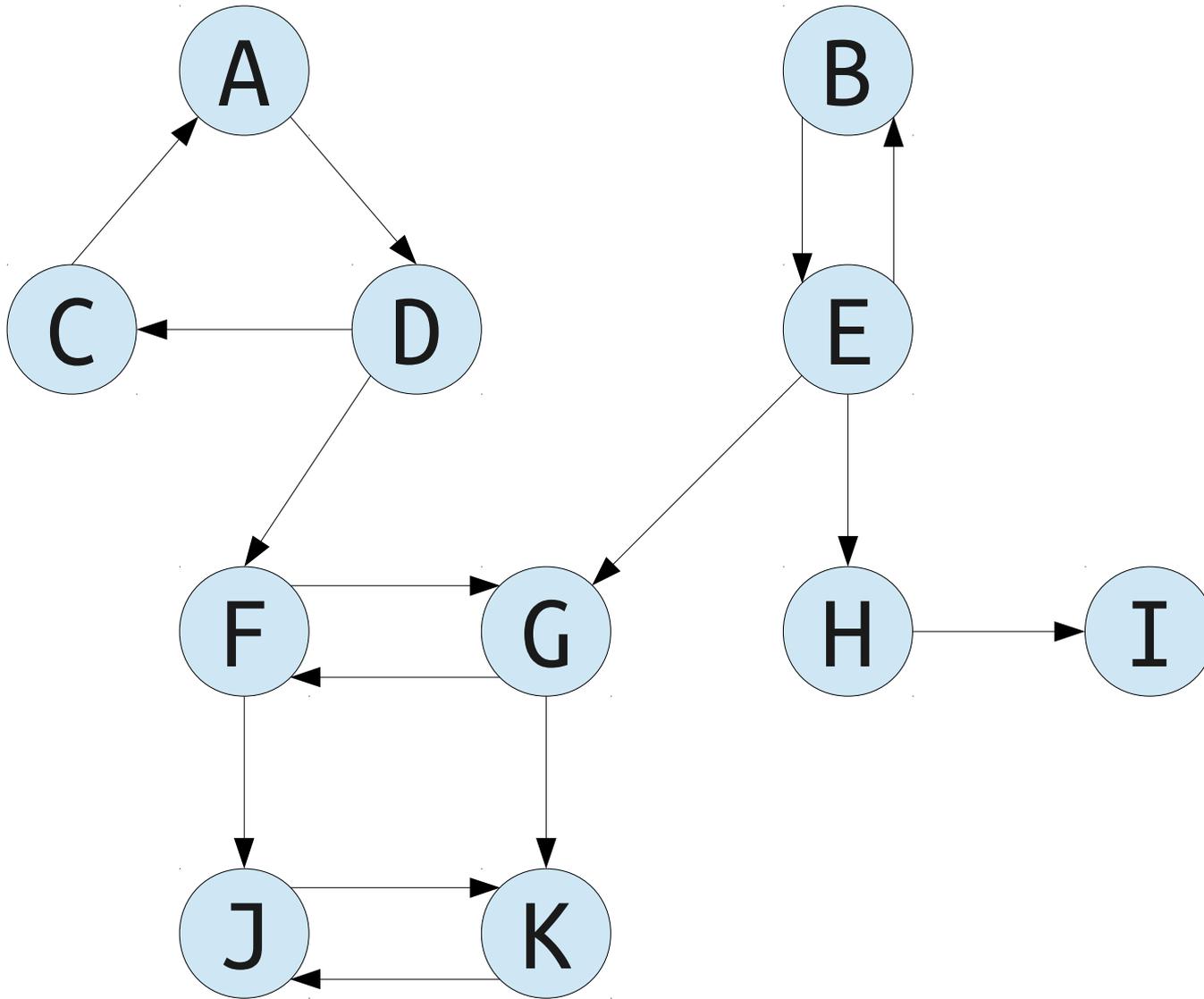


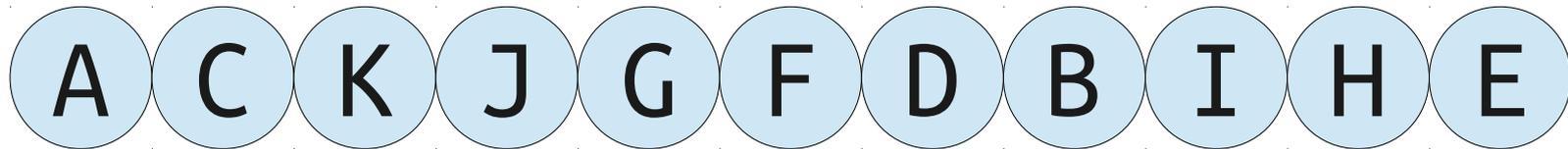
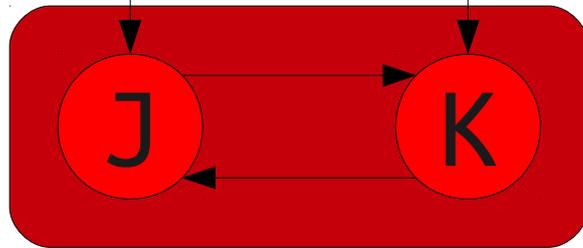
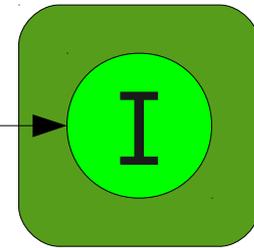
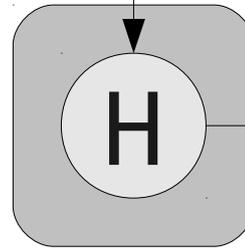
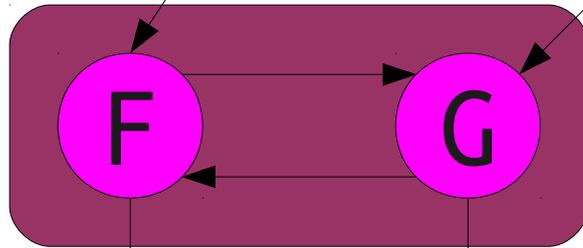
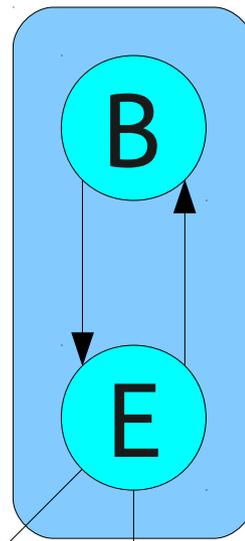
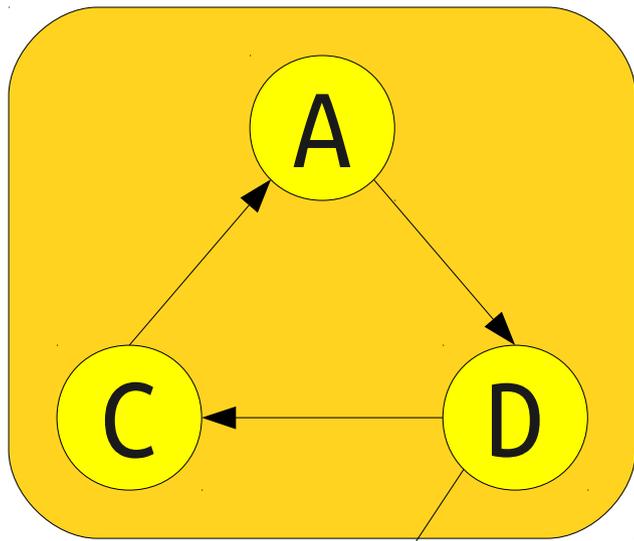


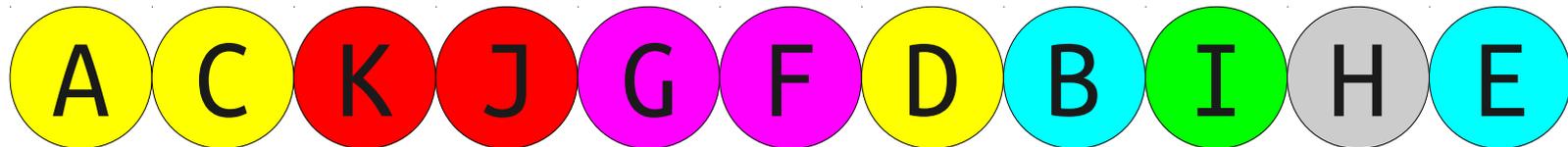
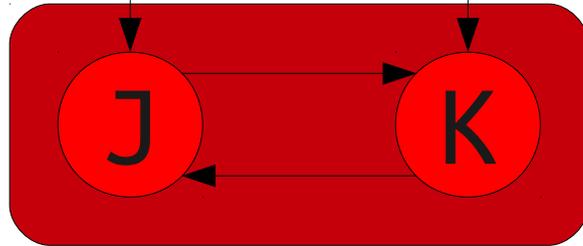
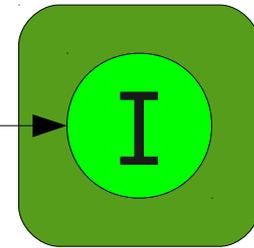
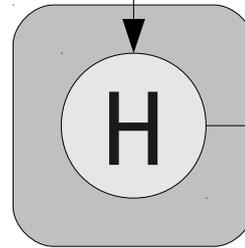
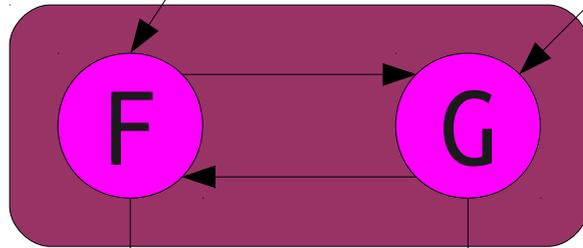
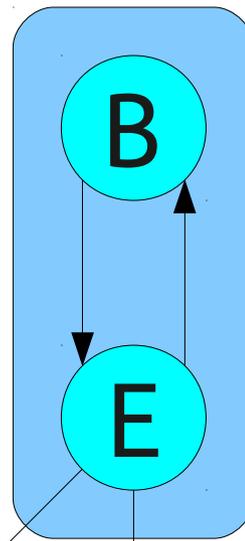
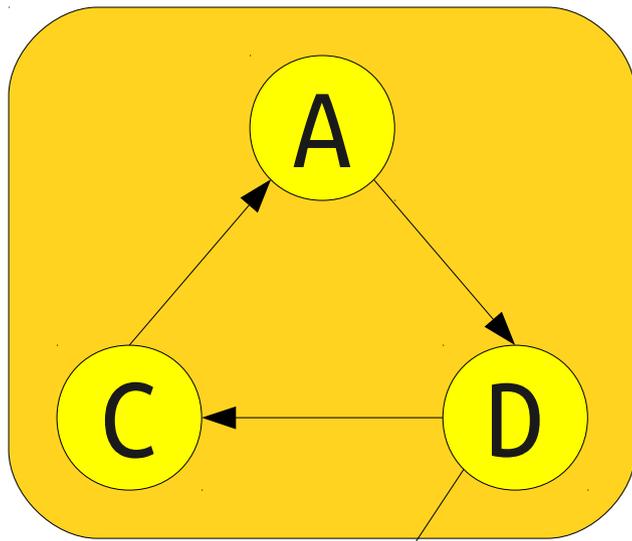


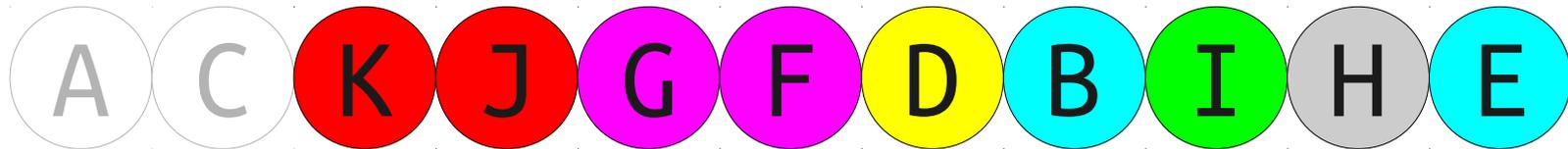
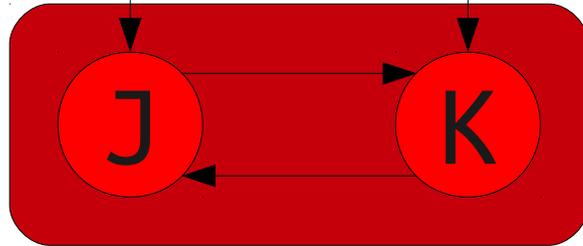
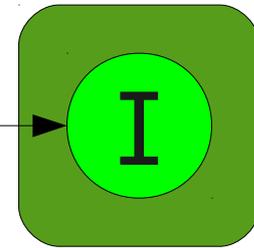
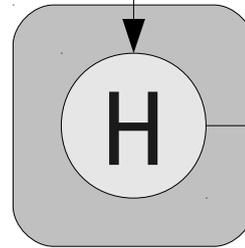
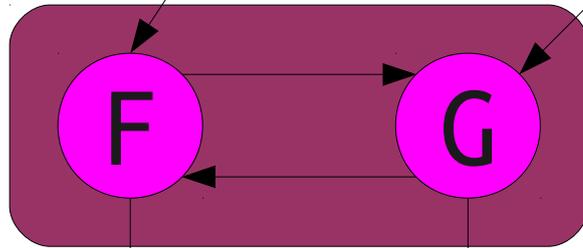
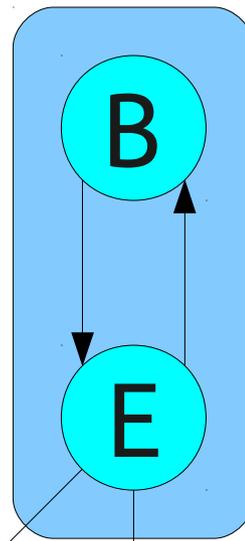
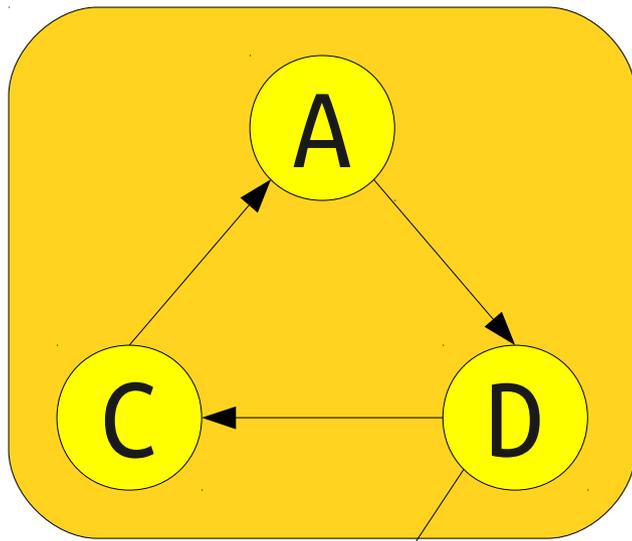


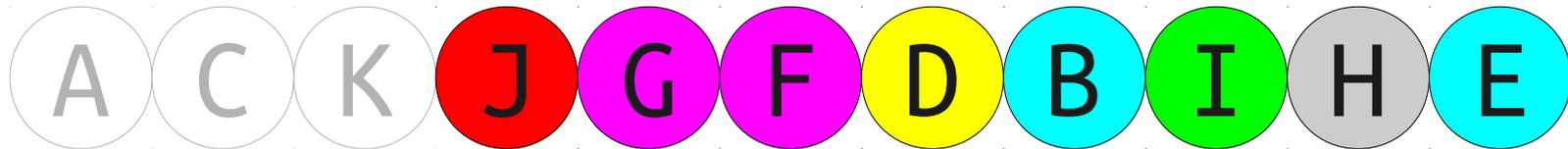
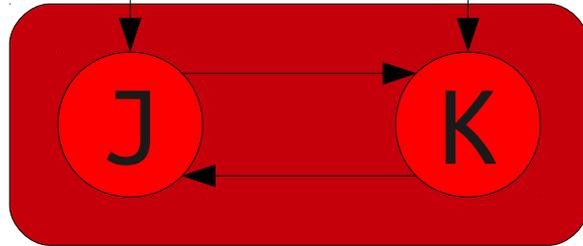
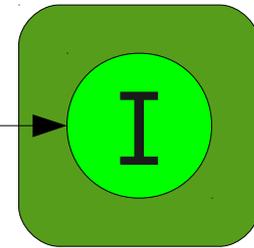
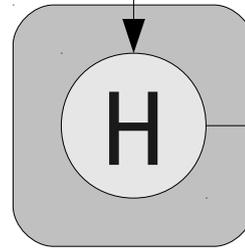
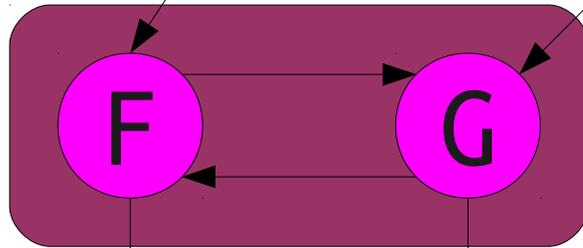
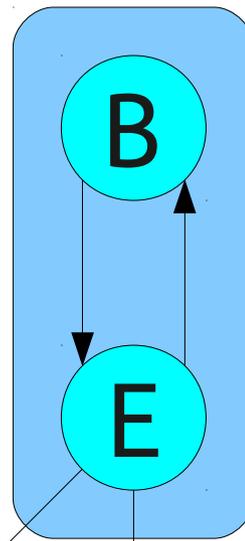
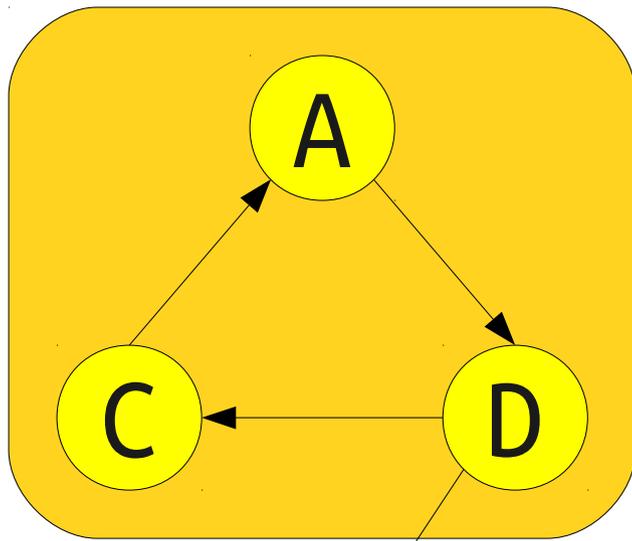


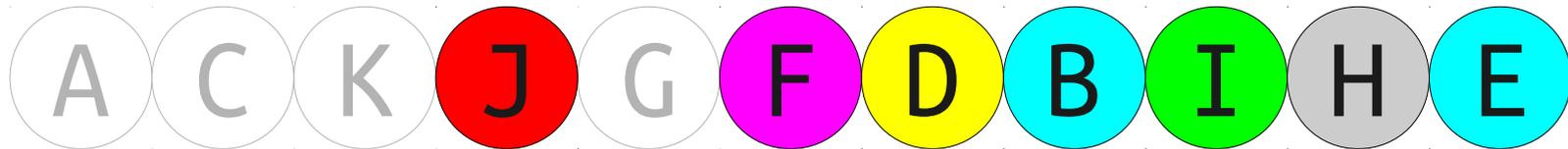
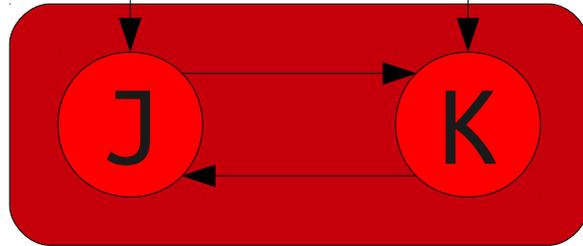
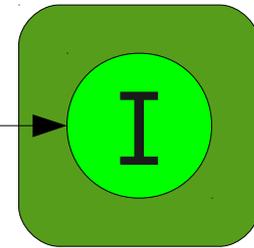
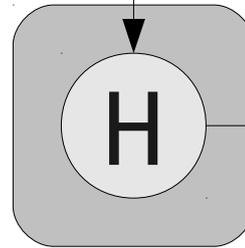
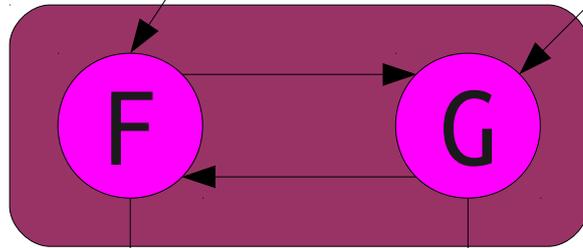
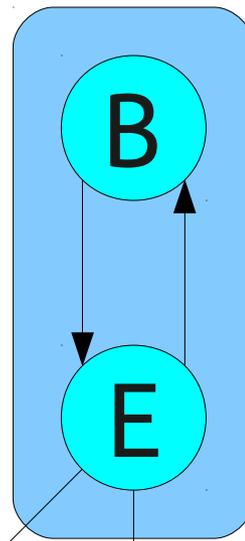
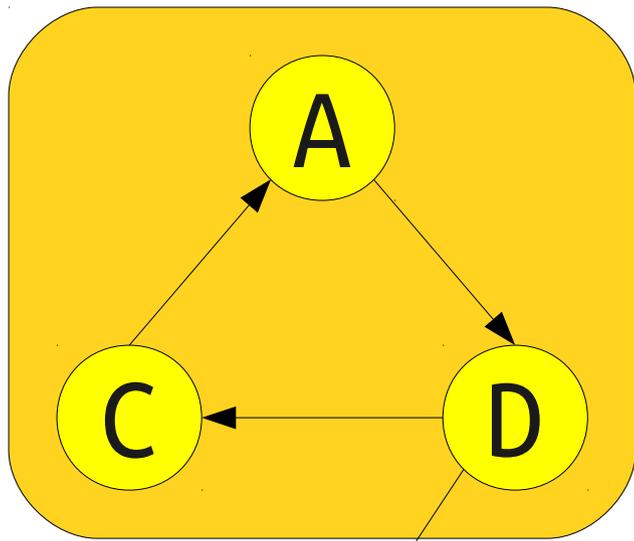


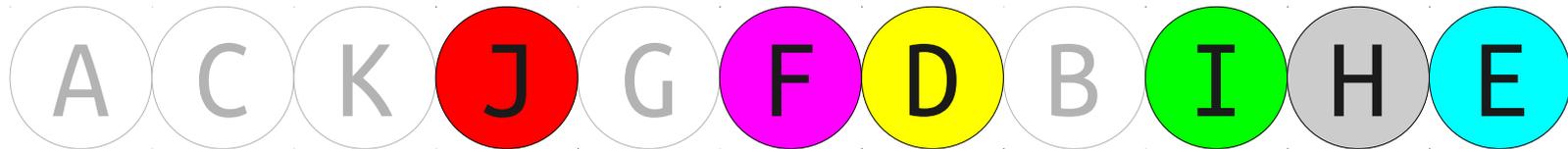
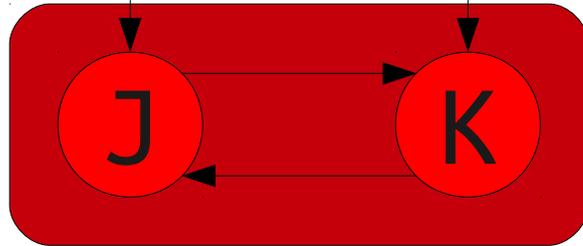
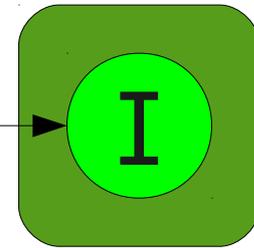
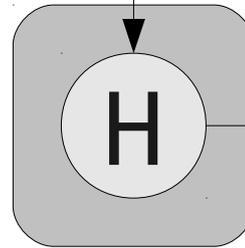
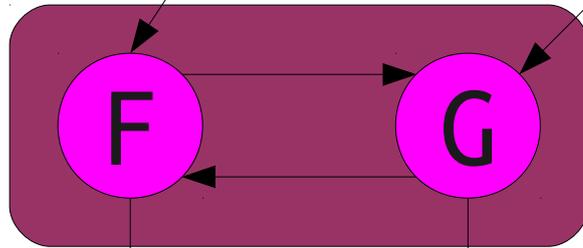
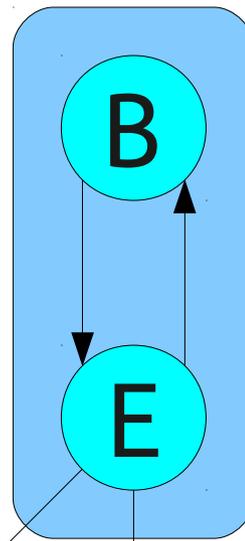
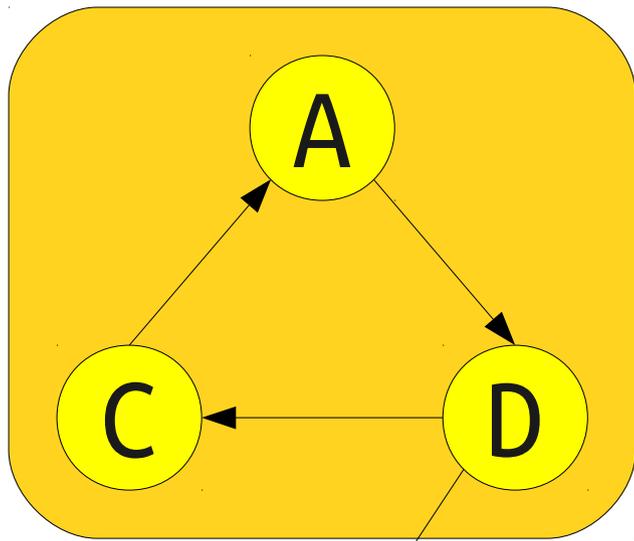


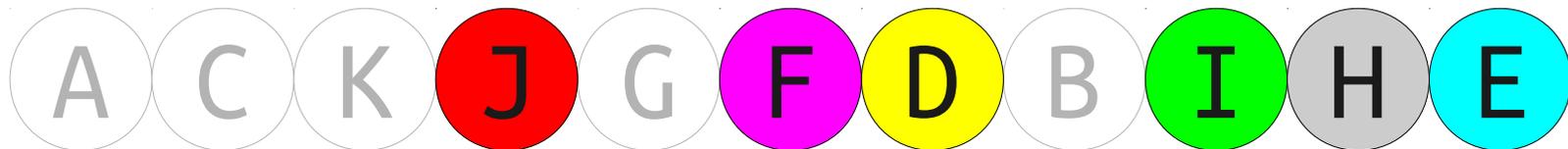
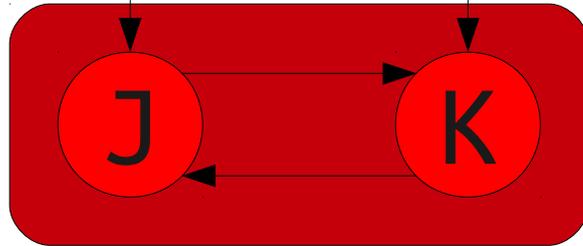
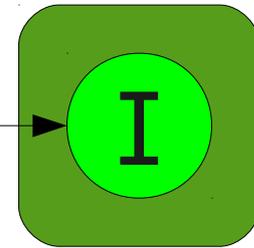
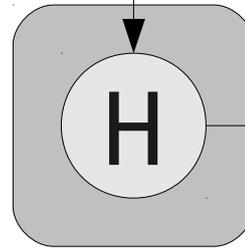
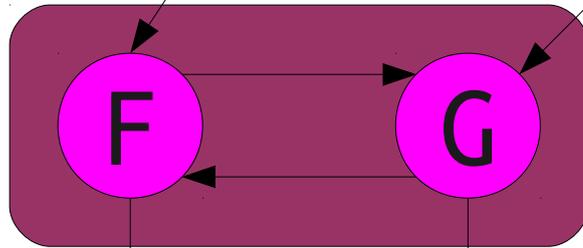
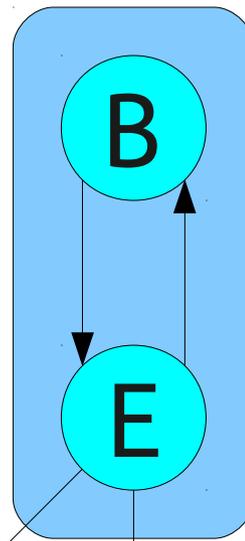
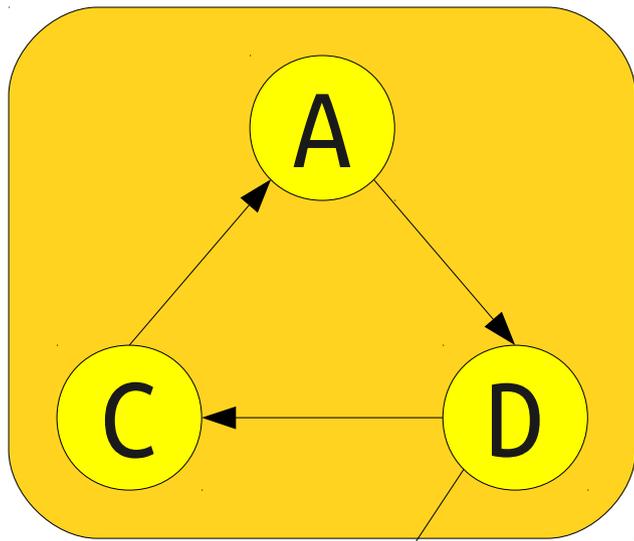


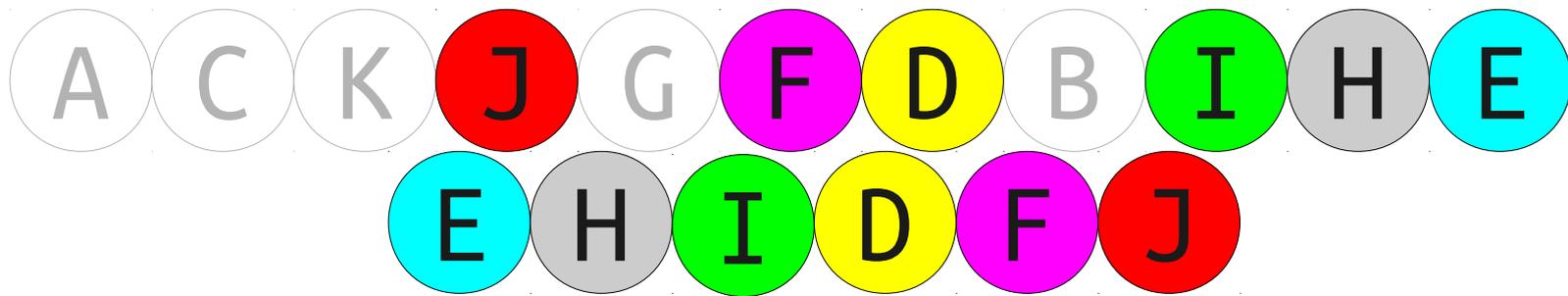
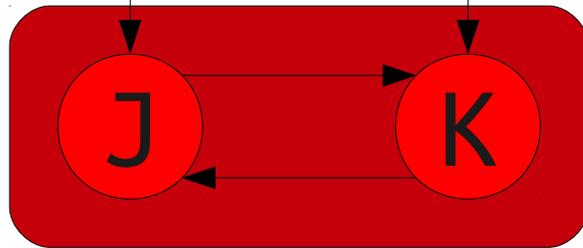
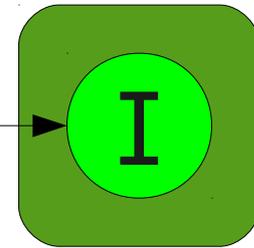
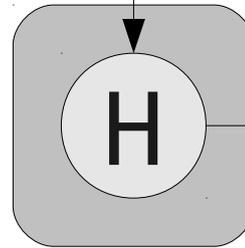
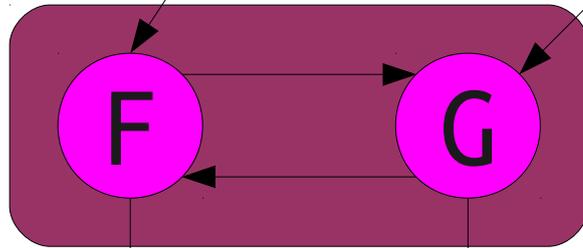
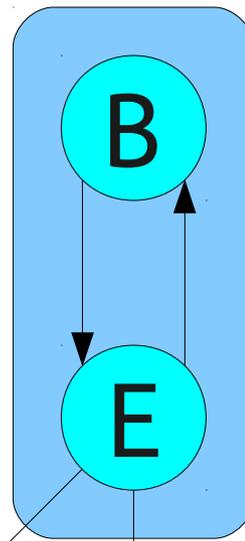
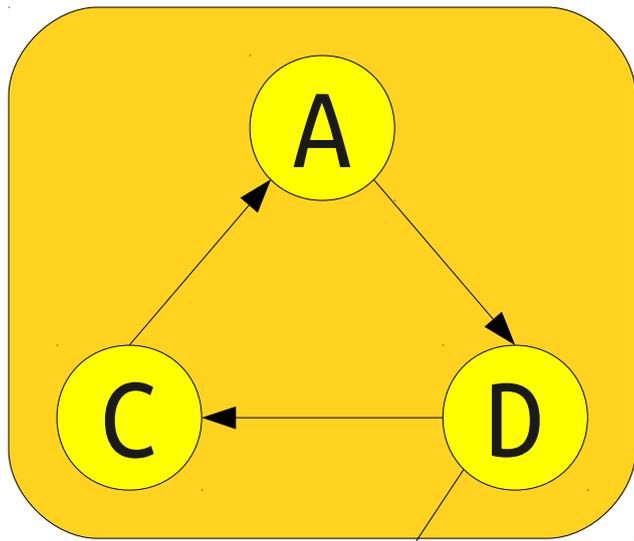












What's Going On?

- It looks like if we look purely at the *last* node from each SCC to turn green, we get a topological sort of G^{SCC} in reverse.
 - Here, each SCC is represented by a single node.
- This helps us get a better sense for how the SCCs are interlinked!
- However, we still don't have a reliable way to determine which node is the last node in each SCC to turn green...
- For starters, let's convince ourselves that this isn't a coincidence.

Some Notation

- We'll denote by $f(v)$ the time at which node v is colored green by the algorithm.
 - $f(u) < f(v)$ means “node u was colored green before node v was colored green.”
- Note that every node is eventually colored green, so this notation is well-defined.
- Let C be an SCC. Define

$$f(C) = \max_{v \in C} f(v)$$

- In other words, $f(C)$ is the time at which the last node in C was colored green.

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In other words:
"The last node in an SCC to turn green is the first node DFS visits in that SCC."

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Proof: At the time DFS(s) is called, since s is the first node in C visited by DFS, all nodes in C are gray.

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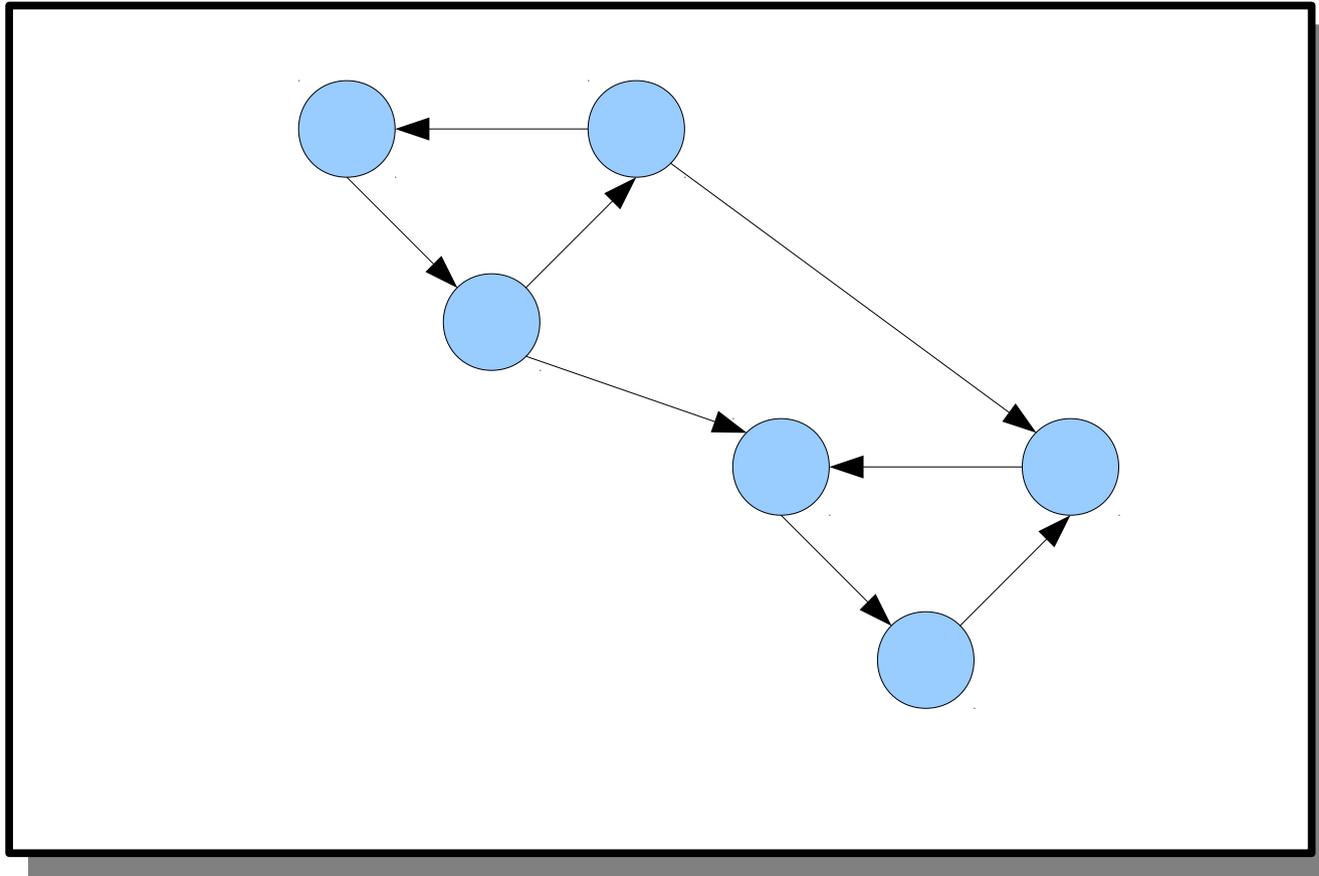
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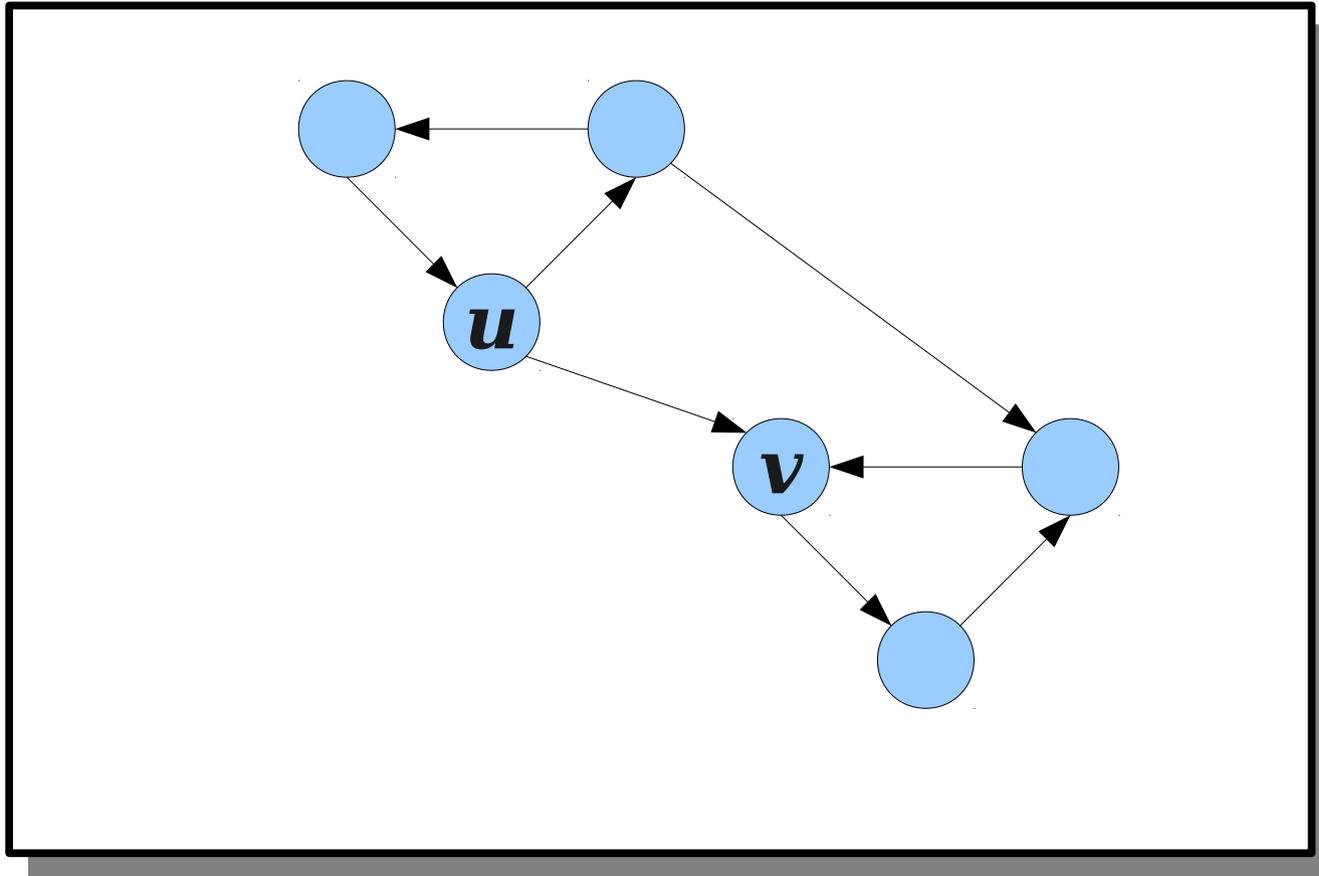
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Theorem: Suppose we run DFS starting at each node in G . Let C_1 and C_2 be SCCs in G . If (u, v) is an edge in G where $u \in C_1$ and $v \in C_2$, then $f(C_2) < f(C_1)$.

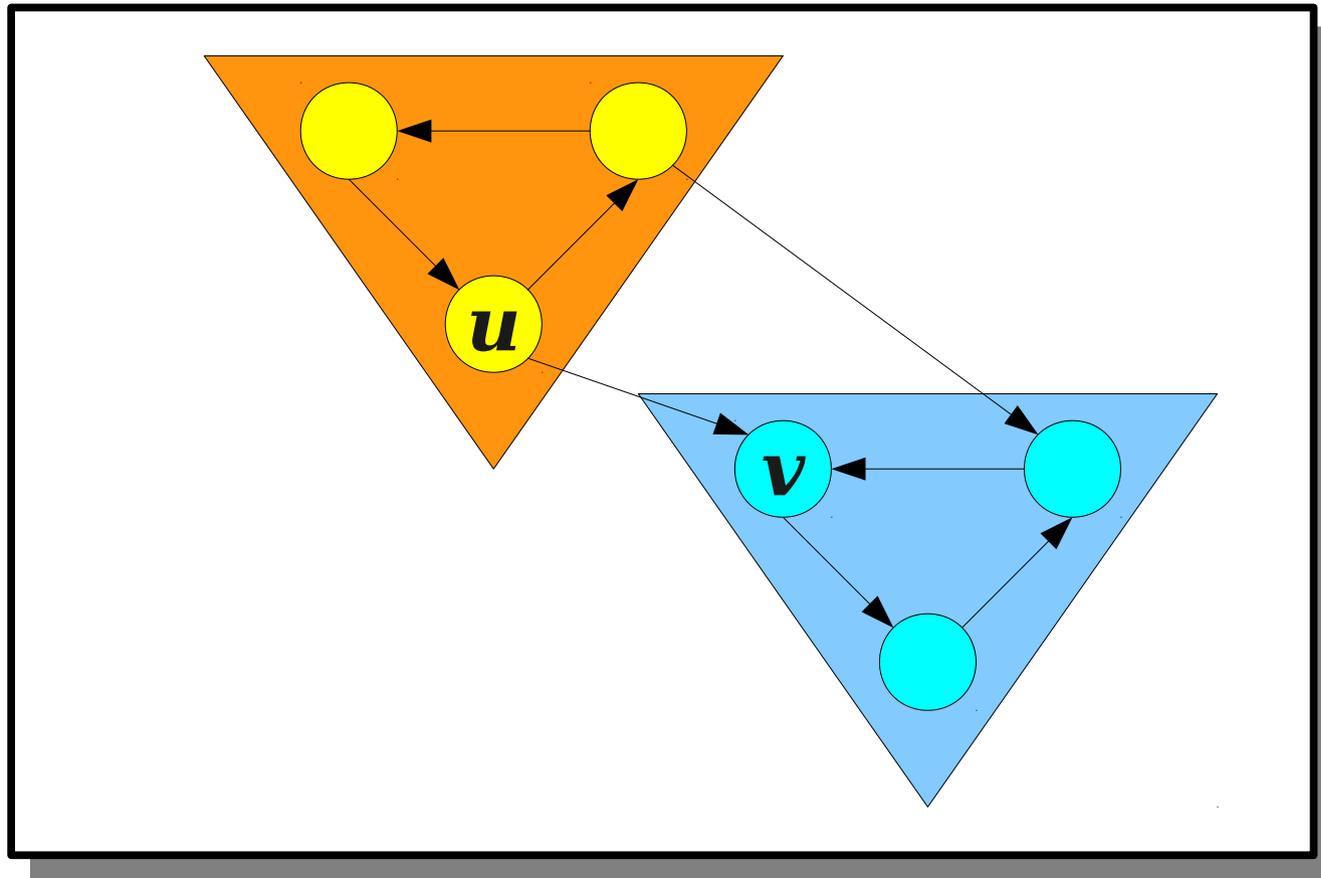
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Note x_2 is reachable from x_1 , since we can go from x_1 to u , across (u, v) , and from v to x_2 .

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