Lévy Processes in Finance: Theory, Numerics, and Empirical Facts

Dissertation zur Erlangung des Doktorgrades
der Mathematischen Fakultät
der Albert-Ludwigs-Universität Freiburg i. Br.

vorgelegt von

Sebastian Raible

Januar 2000

Dekan: Prof. Dr. Wolfgang Soergel

Referenten: Prof. Dr. Ernst Eberlein
Prof. Tomas Björk, Stockholm School of Economics

Datum der Promotion: 1. April 2000

Institut für Mathematische Stochastik
Albert-Ludwigs-Universität Freiburg
Eckerstraße 1
D–79104 Freiburg im Breisgau

Preface

Lévy processes are an excellent tool for modelling price processes in mathematical finance. On the one hand, they are very flexible, since for any time increment Δt any infinitely divisible distribution can be chosen as the increment distribution over periods of time Δt. On the other hand, they have a simple structure in comparison with general semimartingales. Thus stochastic models based on Lévy processes often allow for analytically or numerically tractable formulas. This is a key factor for practical applications.

This thesis is divided into two parts. The first, consisting of Chapters 1, 2, and 3, is devoted to the study of stock price models involving exponential Lévy processes. In the second part, we study term structure models driven by Lévy processes. This part is a continuation of the research that started with the author's diploma thesis Raible (1996) and the article Eberlein and Raible (1999).

The content of the chapters is as follows. In Chapter 1, we study a general stock price model where the price of a single stock follows an exponential Lévy process. Chapter 2 is devoted to the study of the Lévy measure of infinitely divisible distributions, in particular of generalized hyperbolic distributions. This yields information about what changes in the distribution of a generalized hyperbolic Lévy motion can be achieved by a locally equivalent change of the underlying probability measure. Implications for option pricing are discussed. Chapter 3 examines the numerical calculation of option prices. Based on the observation that the pricing formulas for European options can be represented as convolutions, we derive a method to calculate option prices by fast Fourier transforms, making use of bilateral Laplace transformations. Chapter 4 examines the Lévy term structure model introduced in Eberlein and Raible (1999). Several new results related to the Markov property of the short-term interest rate are presented. Chapter 5 presents empirical results on the non-normality of the log returns distribution for zero bonds. In Chapter 6, we show that in the Lévy term structure model the martingale measure is unique. This is important for option pricing. Chapter 7 presents an extension of the Lévy term structure model to multivariate driving Lévy processes and stochastic volatility structures. In theory, this allows for a more realistic modelling of the term structure by addressing three key features: Non-normality of the returns, term structure movements that can only be explained by multiple stochastic factors, and stochastic volatility.

I want to thank my advisor Professor Dr. Eberlein for his confidence, encouragement, and support. I am also grateful to Jan Kallsen, with whom I had many extremely fruitful discussions ever since my time as an undergraduate student. Furthermore, I want to thank Roland Averkamp and Martin Beibel for their advice, and Jan Kallsen, Karsten Prause and Heike Raible for helpful comments on my manuscript. I very much enjoyed my time at the Institut für Mathematische Stochastik.

I gratefully acknowledge financial support by Deutsche Forschungsgemeinschaft (DFG), Graduiertenkolleg “Nichtlineare Differentialgleichungen: Modellierung, Theorie, Numerik, Visualisierung.”

Contents

Chapter 1

Exponential Lévy Processes in Stock Price Modeling

1.1 Introduction

Lévy processes have long been used in mathematical finance. In fact, the best known of all Lévy processes—Brownian motion—was originally introduced as a stock price model (see Bachelier (1900).) Osborne (1959) refined Bachelier's model by proposing the exponential[1] $\exp(B_t)$ of Brownian motion as a stock price model. He justified this approach by a psychological argument based on the Weber-Fechner law, which states that humans perceive the intensity of stimuli on a log scale rather than a linear scale. In a more systematic manner, the same process $\exp(B_t)$, which is called exponential—or geometric—Brownian motion, was introduced as a stock price model by Samuelson (1965).

One of the first to propose an exponential non-normal Lévy process was Mandelbrot (1963). He observed that the logarithm of relative price changes on financial and commodities markets exhibit a long-tailed distribution. His conclusion was that Brownian motion in $\exp(B_t)$ should be replaced by symmetric *α-stable Lévy motion* with index $\alpha < 2$. This yields a pure-jump stock-price process. Roughly speaking, one may envisage this process as changing its values only by jumps. Normal distributions are α-stable distributions with $\alpha = 2$, so Mandelbrot's model may be seen as a complement of the Osborne (1959) or Samuelson (1965) model. A few years later, an exponential Lévy process model with a non-stable distribution was proposed by Press (1967). His log price process is a superposition of a Brownian motion and an independent compound Poisson process with normally distributed jumps. Again the motivation was to find a model that better fits the empirically observed distribution of the changes in the logarithm of stock prices.

More recently, Madan and Seneta (1987) have proposed a Lévy process with *variance gamma* distributed increments as a model for log prices. This choice was justified by a statistical study of Australian stock market data. Like α-stable Lévy motions, variance gamma Lévy processes are pure jump processes. However, they possess a moment generating function, which is convenient for modeling purposes. In particular, with a suitable choice of parameters the expectation of stock prices exists in the Madan and

[1] One should be careful not to confuse this with the *stochastic*—or *Doléans-Dade*—exponential. For Brownian motion, the exponential and the stochastic exponential differ only by a deterministic factor; for Lévy processes with jumps, the difference is more fundamental.

Seneta (1987) model. Variance Gamma distributions are limiting cases of the family of *generalized hyperbolic distributions*. The latter were originally introduced by Barndorff-Nielsen (1977) as a model for the grain-size distribution of wind-blown sand. We give a brief summary of its basic properties in Appendix A.

Two subclasses of the generalized hyperbolic distributions have proved to provide an excellent fit to empirically observed log return distributions: Eberlein and Keller (1995) introduced exponential *hyperbolic* Lévy motion as a stock price model, and Barndorff-Nielsen (1995) proposed an exponential *normal inverse Gaussian* Lévy process. Eberlein and Prause (1998) and Prause (1999) finally study the whole family of generalized hyperbolic Lévy processes.

In this chapter, we will be concerned with a general exponential Lévy process model for stock prices, where the stock price process $(S_t)_{t\in\mathbb{R}_+}$ is assumed to have the form

$$S_t = S_0 \exp(rt) \exp(L_t), \tag{1.1}$$

with a Lévy process L that satisfies some integrability condition. This class comprises all models mentioned above, except for the Mandelbrot (1963) model, which suffers from a lack of integrability.

The chapter is organized as follows. In Section 1.2, we formulate the general framework for our study of exponential Lévy stock price models. The remaining sections are devoted to the study of Esscher transforms for exponential Lévy processes and to option pricing. The class of Esscher transforms is an important tool for option pricing. Section 1.3 introduces the concept of an Esscher transform and examines the conditions under which an Esscher transform that turns the discounted stock price process into a martingale exists. Section 1.4 examines option pricing by Esscher transforms. We show that the option price calculated by using the Esscher transformed probability measure can be interpreted as the expected payoff of a modified option under the original probability measure. In Section 1.5, we derive an integro-differential equation satisfied by the option pricing function. In Section 1.6, we characterize the Esscher transformed measure as the only equivalent martingale measure whose density process with respect to the original measure has a special simple form.

1.2 Exponential Lévy Processes as Stock Price Models

The following basic assumption is made throughout the thesis.

Assumption 1.1. *Let $(\Omega, \mathfrak{A}, (\mathfrak{A}_t)_{t\in\mathbb{R}_+}, P)$ be a filtered probability space satisfying the* usual conditions, *that is, $(\Omega, \mathfrak{A}, P)$ is complete, all the null sets of $\mathfrak{A}$ are contained in $\mathfrak{A}_0$, and $(\mathfrak{A}_t)_{t\in\mathbb{R}_+}$ is a* right-continuous filtration:

$$\mathfrak{A}_s \subset \mathfrak{A}_t \subset \mathfrak{A} \text{ are } \sigma\text{-algebras for } s,t \in \mathbb{R}_+,\ s \le t, \quad \text{and} \quad \mathfrak{A}_s = \bigcap_{t>s} \mathfrak{A}_t \quad \text{for all } s \in \mathbb{R}_+.$$

Furthermore, we assume that

$$\mathfrak{A} = \sigma\big(\cup_{t\in\mathbb{R}_+} \mathfrak{A}_t\big).$$

This allows us to specify a change of the underlying probability measure P to a measure Q by giving a density process $(Z_t)_{t\in\mathbb{R}_+}$. *That is, we specify the measure Q by giving, for each $t \in \mathbb{R}_+$, the density $Z_t = dQ_t/dP_t$. Here Q_t and P_t denote the restrictions of Q and P, respectively, to the σ-algebra $\mathfrak{A}_t$. If $Z_t > 0$ for all $t \in \mathbb{R}_+$, the measures Q and P are then called* locally equivalent, $Q \overset{\text{loc}}{\sim} P$.

We cite the following definition from Protter (1992), Chap. I, Sec. 4.

Definition 1.2. *An adapted process* $X = (X_t)_{0 \le t < \infty}$ *with* $X_0 = 0$ *a.s. is a* Lévy process *if*

- *(i)* X *has* increments independent of the past: *that is,* $X_t - X_s$ *is independent of* $\mathcal{F}_s, 0 \le s < t < \infty$*;*
- *(ii)* X *has* stationary increments: *that is,* $X_t - X_s$ *has the same distribution as* $X_{t-s}, 0 \le s < t < \infty$*;*
- *(iii)* X_t *is* continuous in probability: *that is,* $\lim_{t \to s} X_t = X_s$*, where the limit is taken in probability.*

Keller (1997) notes on page 21 that condition (iii) follows from (i) and (ii), and so may be omitted here. Processes satisfying (i) and (ii) are called *processes with stationary independent increments (PIIS)*. (See Jacod and Shiryaev (1987), Definition II.4.1.)

The distribution of a Lévy processes is uniquely determined by any of its one-dimensional marginal distributions P^{L_t}, say, by P^{L_1}. From the property of independent and stationary increments of L, it is clear that P^{L_1} is infinitely divisible. Hence its characteristic function has the special structure given by the *Lévy-Khintchine formula.*

$$E\left[\exp(iuL_1)\right] = \exp\left(iub - \frac{c}{2}u^2 + \int \left(e^{iux} - 1 - iux\right) F(dx)\right).$$

Definition 1.3. *The* Lévy-Khintchine triplet (b, c, F) *of an infinitely divisible distribution consists of the constants* $b \in \mathbb{R}$ *and* $c \ge 0$ *and the measure* $F(dx)$*, which appear in the Lévy-Khintchine representation of the characteristic function*

We consider stock price models of the form

$$S_t = S_0 \exp(rt) \exp(L_t), \tag{1.2}$$

with a constant deterministic interest rate r and a Lévy process L.

Remark 1: In the stock price model (1.2), we could as well omit the interest rate r, since the process $\widetilde{L}$ with $\widetilde{L}_t := rt + L_t$ is again a Lévy process. This would lead to a simpler form $S_t = S_0 \exp(\widetilde{L}_t)$ of the stock price process. However, in the following we often work with *discounted* stock prices, that is, stock prices divided by the factor $\exp(rt)$. These have a simpler form with representation (1.2).

Remark 2: Stochastic processes in mathematical finance are often defined by stochastic differential equations (SDE). For example, the equation corresponding to the classical Samuelson (1965) model has the form

$$dS_t = S_t(\mu dt + \sigma dW_t), \tag{1.3}$$

with constant coefficients $\mu \in \mathbb{R}$ and $\sigma > 0$. W is a standard Brownian motion. ("Standard" means $E[W_1] = 0$ and $E[W_1^2] = 1$ here.) The solution of (1.3) is

$$S_t = S_0 \exp\left(\mu t - \frac{\sigma^2}{2} t + \sigma W_t\right).$$

Comparing this formula with (1.2), we see that the Samuelson model is a special case of (1.2). The Lévy process L in this case is given by $L_t = (\mu - \sigma^2/2 - r)t + \sigma W_t$. Apart from the constant factor σ, this differs from the driving process W of the stochastic differential equation (1.3) only by a deterministic drift term.

One may ask whether the process defined in equation (1.2) could equivalently be introduced by a stochastic differential equation (SDE) analogous to (1.3). This is indeed the case. However, unlike the situation in (1.3), the driving Lévy process of the stochastic differential equation differs considerably from the process L in (1.2). More precisely, for each Lévy process L the ordinary exponential $S_t = S_0 \exp(L_t)$ satisfies a stochastic differential equation of the form

$$dS_t = S_{t-} d\widetilde{L}_t, \tag{1.4}$$

where $\widetilde{L}$ is a Lévy process whose jumps are strictly larger than -1. On the other hand, if $\widetilde{L}$ is a Lévy process with jumps strictly larger than -1, then the solution S of (1.4), i. e. the stochastic exponential of the process $\widetilde{L}$, is indeed of the form

$$S_t = S_0 \exp(L_t)$$

with a Lévy process L. This connection is shown in Goll and Kallsen (2000), Lemma 5.8.

This relation between the ordinary exponential and the stochastic exponential of Lévy processes does not seem to be aware to some authors. For example, in a recent publication Chan (1999) compares the direct approach via (1.1) and his own approach via an SDE (1.4) as if they were completely different.

Note that, in particular, the restriction that the jumps of $\widetilde{L}$ are bounded below does not mean that the jumps of L are bounded. For technical reasons, we impose the following conditions.

Assumption 1.4. *The random variable L_1 is non-degenerate and possesses a moment generating function $\mathrm{mgf} : u \mapsto E[\exp(uL_1)]$ on some open interval (a, b) with $b - a > 1$.*

Assumption 1.5. *There exists a real number $\theta \in (a, b - 1)$ such that*

$$\mathrm{mgf}(\theta) = \mathrm{mgf}(1 + \theta). \tag{1.5}$$

Assumption 1.5 will be used below to prove the existence of a suitable Esscher transform.

Remark: One may wonder if 1.5 follows from 1.4 if the interval (a, b) is the maximal open interval on which the moment generating function exists. In fact, this is true if the moment generating function tends to infinity as $u \downarrow a$ and as $u \uparrow b$. However, in general assumption 1.4 is not sufficient for assumption 1.5. This can be seen from the following example.

Example 1.6. *Consider normal inverse Gaussian (NIG) distributions.*[2] *The moment generating function of NIG is given by*

$$\mathrm{mgf}(u) = \exp(\mu u) \frac{\exp(\delta\sqrt{\alpha^2 - \beta^2})}{\exp(\delta\sqrt{\alpha^2 - (\beta + u)^2})}.$$

For the parameters, choose the values $\alpha = 1$, $\beta = -0.1$, $\mu = 0.006$, and $\delta = 0.005$. Figure 1.1 shows the corresponding moment generating function. Its range of definition is $[-\alpha - \beta, \alpha - \beta] = [-0.9, 1.1]$, so the maximal open interval on which the moment generating function exists is $(-0.9, 1.1)$. Hence assumption 1.4 is satisfied, but assumption 1.5 is not. For clarity, figure 1.2 shows the same moment generating function on the range $(-0.9, 0)$. There are no two points θ, $\theta + 1$ in the range of definition such that the values of the moment generating function at these values are the same.

[2] See Section A.2.2. NIG distributions belong to the family of generalized hyperbolic distributions. They are infinitely divisible and thus can appear as the distributions of L_1 where L is a Lévy process.

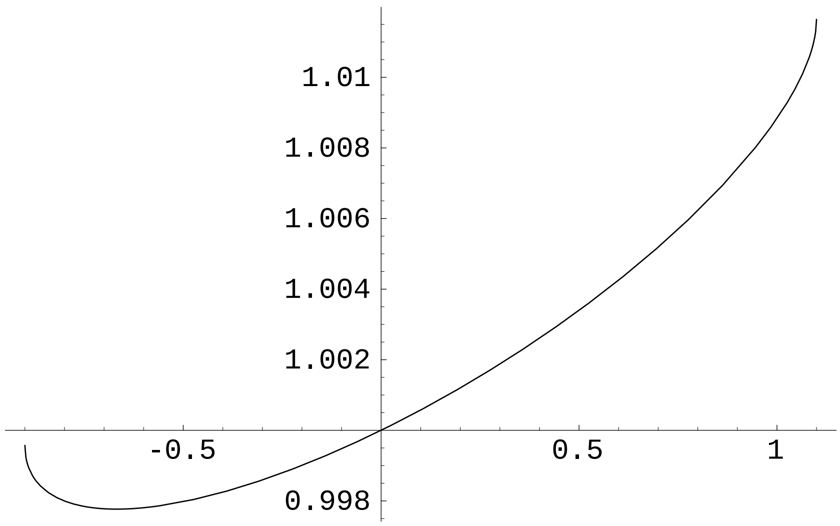

Figure 1.1: Moment generating function of a NIG distribution with parameters $\alpha = 1$, $\beta = -0.1$, $\mu = 0.006$, and $\delta = 0.005$.

Remark: Note that in the example $\mathrm{mgf}(u)$ stays bounded as u approaches the boundaries of the range of existence of the moment generating function. This is no contradiction to the fact that the boundary points are singular points of the analytic characteristic function (cf. Lukacs (1970), Theorem 7.1.1), since "singular point" is not the same as "pole".

1.3 Esscher Transforms

Esscher transforms have long been used in the actuarial sciences, where one-dimensional distributions P are modified by a density of the form

$$z(x) = \frac{e^{\theta x}}{\int e^{\theta x} P(dx)},$$

with some suitable constant θ.

In contrast to the one-dimensional distributions in classical actuarial sciences, in mathematical finance one encounters stochastic processes, which in general are infinite-dimensional objects. Here it is tempting to describe a transformation of the underlying probability measure by the transformation of the one-dimensional marginal distributions of the process. This naive approach can be found in Gerber and Shiu (1994). Of course, in general the transformation of the one-dimensional marginal distributions does not *uniquely* determine a transformation of the distribution of the process itself. But what is worse, in general there is no locally absolutely continuous change of measure at all that corresponds to a given set of absolutely continuous changes of the marginals. We give a simple example: Consider a normally distributed random variable N_1 and define a stochastic process N as follows.

$$N_t(\omega) := tN_1(\omega) \quad (t \in \mathbb{R}_+).$$

All paths of N are linear functions, and for each $t \in \mathbb{R}_+$, N_t is distributed according to $N(0, t^2)$. Now

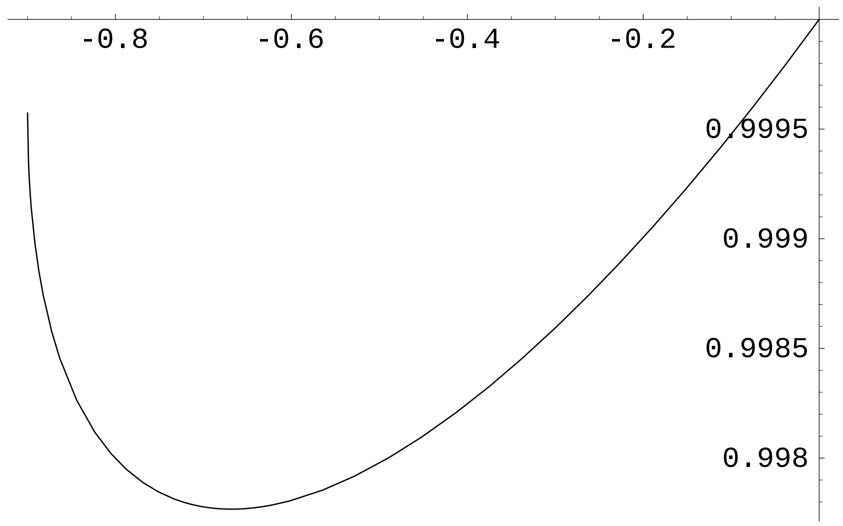

Figure 1.2: The moment generating function from figure 1.1, drawn on the interval $(-0.9, 0)$.

we ask whether there is a measure Q locally equivalent to P such that the one-dimensional marginal distributions transform as follows.

1. for $0 \leq t \leq 1$, N_t has the same distribution under Q as under P.
2. for $1 < t < \infty$, $Q^{N_t} = P^{2N_t}$, that is, $Q^{N_t} = N(0, 4t^2)$.

Obviously, these transformations of the one-dimensional marginal distributions are absolutely continuous. But a measure Q, locally equivalent to P, with the desired properties cannot exist, since the relation $N_t(\omega) = tN_1(\omega)$ holds irrespectively of the underlying probability measure: It reflects a path property of all paths of N. This property cannot be changed by changing the probability measure, that is, the probabilities of the paths. Hence for all $t \in \mathbb{R}_+$—and hence, in particular, for $1 < t < \infty$—we have $Q^{N_t} = Q^{tN_1}$, which we have assumed to be $N(0, t^2)$ by condition 1 above. This contradicts condition 2.[3]

Gerber and Shiu (1994) were lucky in considering Esscher transforms, because for Lévy processes there is indeed a (locally) equivalent transformation of the basic probability measure that leads to Esscher transforms of the one-dimensional marginal distributions.[4] The concept—but not the name—of Esscher transforms for Lévy processes had been introduced to finance before (see e. g. Madan and Milne (1991)), on a mathematically profound basis.

Definition 1.7. *Let L be a Lévy process on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, P)$. We call* Esscher transform *any change of P to a locally equivalent measure Q with a density process $Z_t = \frac{dQ}{dP}\big|_{\mathcal{F}_t}$ of the form*

$$Z_t = \frac{\exp(\theta L_t)}{\mathrm{mgf}(\theta)^t}, \tag{1.6}$$

[3]In Chapter 2, will encounter more elaborate examples of the importance of path properties. There again we will discuss the question whether the distributions of two stochastic processes can be locally equivalent.

[4]However, it is not clear whether this transformation is uniquely determined by giving the transformations of the one-dimensional marginal distributions alone.

where $\theta \in \mathbb{R}$, and where $\mathrm{mgf}(u)$ denotes the moment generating function of L_1.

Remark 1: Observe that we interpret the Esscher transform as a transformation of the underlying probability measure rather than as a transformation of the (distribution of) the process L. Thus we do not have to assume that the filtration is the canonical filtration of the process L, which would be necessary if we wanted to construct the measure transformation $P \to Q$ from a transformation of the distribution of L.

Remark 2: The Esscher density process, which formally looks like the density of a one-dimensional Esscher transform, indeed leads to one-dimensional Esscher transformations of the marginal distributions, with the same parameter θ: Denoting the Esscher transformed probability measure by P^θ, we have

$$\begin{aligned} P^\theta[L_t \in B] &= \int 1\!\!1_B(L_t) \frac{e^{\theta L_t}}{\mathrm{mgf}(\theta)^t} dP \\ &= \int 1\!\!1_B(x) \frac{e^{\theta x}}{\mathrm{mgf}(\theta)^t} P^{L_t}(dx) \end{aligned}$$

for any set $B \in \mathcal{B}^1$.

The following proposition is a version of Keller (1997), Proposition 20. We relax the conditions imposed there on the range of admissible parameters θ, in the way that we do not require that $-\theta$ also lies in the domain of existence of the moment generating function. Furthermore, our elementary proof does not require that the underlying filtration is the canonical filtration generated by the Lévy process.

Proposition 1.8. *Equation (1.6) defines a density process for all $\theta \in \mathbb{R}$ such that $E[\exp(\theta L_1)] < \infty$. L is again a Lévy process under the new measure Q.*

Proof. Obviously Z_t is integrable for all t. We have, for $s < t$,

$$\begin{aligned} E[Z_t|\mathcal{F}_s] &= E[\exp(\theta L_t)\mathrm{mgf}(\theta)^{-t}|\mathcal{F}_s] \\ &= \exp(\theta L_s)\mathrm{mgf}(\theta)^{-s} E[\exp(\theta(L_t - L_s))\mathrm{mgf}(\theta)^{-(t-s)}|\mathcal{F}_s] \\ &= \exp(\theta L_s)\mathrm{mgf}(\theta)^{-s} E[\exp(\theta L_{t-s})]\mathrm{mgf}(\theta)^{-(t-s)} \\ &= \exp(\theta L_s)\mathrm{mgf}(\theta)^{-s}. \\ &= Z_s \end{aligned}$$

Here we made use of the stationarity and independence of the increments of L, as well as of the definition of the moment generating function $\mathrm{mgf}(u)$. We go on to prove the second assertion of the Proposition. For any Borel set B, any pair $s < t$ and any $F_s \in \mathcal{F}_s$, we have the following

1. $L_t - L_s$ is independent of the σ-field $\mathcal{F}_s$, so $1\!\!1_{\{L_t - L_s \in B\}} \frac{Z_t}{Z_s}$ is independent of $1\!\!1_{F_s} Z_s$.

2. $E[Z_s] = 1$.

3. Again because of the independence of $L_t - L_s$ and $\mathcal{F}_s$, we have independence of $1\!\!1_{\{L_t - L_s \in B\}} \frac{Z_t}{Z_s}$ and Z_s.

Consequently, the following chain of equalities holds.

$$
\begin{aligned}
Q(\{L_t - L_s \in B\} \cap F_s) &= E\left[\mathbb{1}_{\{L_t - L_s \in B\}} \mathbb{1}_{F_s} Z_t\right] \\
&= E\left[\mathbb{1}_{\{L_t - L_s \in B\}} \frac{Z_t}{Z_s} \mathbb{1}_{F_s} Z_s\right] \\
&\overset{1.}{=} E\left[\mathbb{1}_{\{L_t - L_s \in B\}} \frac{Z_t}{Z_s}\right] E\left[\mathbb{1}_{F_s} Z_s\right] \\
&\overset{2.}{=} E\left[\mathbb{1}_{\{L_t - L_s \in B\}} \frac{Z_t}{Z_s}\right] E\left[Z_s\right] E\left[\mathbb{1}_{F_s} Z_s\right]. \\
&\overset{3.}{=} E\left[\mathbb{1}_{\{L_t - L_s \in B\}} \frac{Z_t}{Z_s} Z_s\right] E\left[\mathbb{1}_{F_s} Z_s\right]. \\
&= Q(\{L_t - L_s \in B\}) Q(F_s).
\end{aligned}
$$

For the stationarity of the increments of L under Q, we show

$$
\begin{aligned}
Q(\{L_t - L_s \in B\}) &= E[\mathbb{1}_{\{L_t - L_s \in B\}} Z_t] \\
&= E[\mathbb{1}_{\{L_t - L_s \in B\}} \frac{Z_t}{Z_s} Z_s] \\
&= E[\mathbb{1}_{\{L_t - L_s \in B\}} \exp(\theta(L_t - L_s)) \mathrm{mgf}(\theta)^{s-t}] E[Z_s] \\
&= E[\mathbb{1}_{\{L_{t-s} \in B\}} \exp(\theta(L_{t-s})) \mathrm{mgf}(\theta)^{s-t}] \\
&= E[\mathbb{1}_{\{L_{t-s} \in B\}} Z_{t-s}] \\
&= Q(\{L_{t-s} \in B\}),
\end{aligned}
$$

by similar arguments as in the proof of independence. □

In stock price modeling, the Esscher transform is a useful tool for finding an equivalent probability measure under which discounted stock prices are martingales. We will use this so-called *martingale measure* below when we price European options on the stock.

Lemma 1.9. *Let the stock price process be given by (1.2), and let Assumptions 1.4 and 1.5 be satisfied. Then the basic probability measure P is locally equivalent to a measure Q such that the discounted stock price $\exp(-rt)S_t = S_0 \exp(L_t)$ is a Q-martingale. A density process leading to such a martingale measure Q is given by the Esscher transform density*

$$
Z_t^{(\theta)} = \frac{\exp(\theta L_t)}{\mathrm{mgf}(\theta)^t}, \tag{1.7}
$$

with a suitable real constant θ. The value θ is uniquely determined as the solution of

$$
\mathrm{mgf}(\theta) = \mathrm{mgf}(\theta + 1), \quad \theta \in (a, b).
$$

Proof. We show that a suitable parameter θ exists and is unique. $\exp(L_t)$ is a Q-martingale iff $\exp(L_t)Z_t$ is a P-martingale. (This can be shown using Lemma 1.10 below.) Proposition 1.8 guarantees that L is a Lévy process under any measure $P^{(\theta)}$ defined by

$$
\left.\frac{dP^{(\theta)}}{dP}\right|_{\mathcal{F}_t} = Z_t^{(\theta)}, \tag{1.8}
$$

as long as $\theta \in (a, b)$. Choose a solution θ of the equation $\mathrm{mgf}(\theta) = \mathrm{mgf}(\theta + 1)$, which exists by Assumption 1.5.

Because of the independence and stationarity of the increments of L, in order to prove the martingale property of e^L under Q we only have to show that $E_Q\left[e^{L_1}\right] = 1$. We have

$$\begin{aligned} E_Q\left[e^{L_1}\right] &= E\left[e^{L_1} e^{\theta L_1} \mathrm{mgf}(\theta)^{-1}\right] \\ &= E\left[e^{(\theta+1)L_1}\right] \mathrm{mgf}(\theta)^{-1} \\ &= \frac{\mathrm{mgf}(\theta+1)}{\mathrm{mgf}(\theta)}. \end{aligned}$$

Thus $E_Q\left[e^{L_1}\right] = 1$ iff

$$\mathrm{mgf}(\theta + 1) = \mathrm{mgf}(\theta). \tag{1.9}$$

But the last equation is satisfied by our our choice of θ. On the other hand, there can be no other solution θ to this equation, since the logarithm $\ln[\mathrm{mgf}(u)]$ of the moment generating function is strictly convex for a non-degenerate distribution. This can be proved by a refinement of the argument in Billingsley (1979), Sec. 9, p. 121, where only convexity is proved. See Lemma 2.9. □

1.4 Option Pricing by Esscher Transforms

The locally absolutely continuous measure transformations appearing in mathematical finance usually serve the purpose to change the underlying probability measure P—the *objective probability measure*—to a so-called *risk-neutral* measure $Q \overset{\mathrm{loc}}{\sim} P$.[5] Under the measure Q, all discounted[6] price processes such that the prices are Q-integrable are assumed to be martingales. Therefore such a measure is also called *martingale measure*. By virtue of this assumption, prices of certain securities (called *derivatives*) whose prices at some future date T are known functions of other securities (called *underlyings*) can be calculated for all dates $t < T$ just by taking conditional expectations. For example, a so-called *European call option* with a *strike price* K is a derivative security that has a value of $(S_T - K)^+$ at some fixed future date T, where $S = (S_t)_{t \in \mathbb{R}}$ is the price process of another security (which consequently is the underlying in this case.) Assuming that the savings account process is given by $B_t = e^{rt}$, the process $e^{-rt} S_t$ is a martingale under Q, since Q was assumed to be a risk-neutral measure. The same holds true for the value process V of the option, for which we only specified the final value $V(T)$. $(e^{-rt} V(t))_{t \geq 0}$ is a Q-martingale, so

$$e^{-rt} V(t) = E_Q\left[e^{-rT} V(T) \middle| \mathcal{F}_t\right] = E_Q\left[e^{-rT} (S_T - K)^+ \middle| \mathcal{F}_t\right],$$

and hence

$$V(t) = e^{rt} E_Q\left[e^{-rT} (S_T - K)^+ \middle| \mathcal{F}_t\right] = E_Q\left[e^{-r(T-t)} (S_T - K)^+ \middle| \mathcal{F}_t\right]. \tag{1.10}$$

[5] Local equivalence of two probability measures Q and P on a filtered probability space means that for each t the restrictions $Q_t := Q|_{\mathcal{F}_t}$ and $P_t := P|_{\mathcal{F}_t}$ are equivalent measures.

[6] *Discounted* here means that prices are not measured in terms of currency units, but rather in terms of units of a security called the *savings account*. The latter is the current value of a savings account on which one currency unit was deposed at time 0 and that earns continuously interest with the short-term interest rate $r(t)$. For example, if $r(t) \equiv r$ is constant as in our case, the value of the savings account at time t is e^{rt}.

In this way, specification of the final value of a derivative security uniquely determines its price process up to the final date if one knows the risk-neutral measure Q.

We start with an auxiliary result.

Lemma 1.10. *Let Z be a density process, i.e. a non-negative P-martingale with $E[Z_t] = 1$ for all t. Let Q be the measure defined by $dQ/dP\big|_{\mathcal{F}_t} = Z_t$, $t \geq 0$. Then an adapted process $(X_t)_{t\geq 0}$ is a Q-martingale iff $(X_t Z_t)_{t\geq 0}$ is a P-martingale.*

If we further assume that $Z_t > 0$ for all $t \geq 0$, we have the following. For any pair $t < T$ and any Q-integrable $\mathcal{F}_T$-measurable random variable X,

$$E_Q\left[X \middle| \mathcal{F}_t\right] = E_P\left[X \frac{Z_T}{Z_t} \middle| \mathcal{F}_t\right].$$

Proof. The first part is a rephrasing of Jacod and Shiryaev (1987), Proposition III.3.8a, without the condition that X has càdlàg paths. (This condition is necessary in Jacod and Shiryaev (1987) because there a martingale is required to possess càdlàg paths.) We reproduce the proof of Jacod and Shiryaev (1987): For every $A \in \mathcal{F}_t$ (with $t < T$), we have $E_Q\left[1\!\!1_A X_T\right] = E_P\left[Z_T 1\!\!1_A X_T\right]$ and $E_Q\left[1\!\!1_A X_t\right] = E_P\left[Z_t 1\!\!1_A X_t\right]$. Therefore $E_Q\left[X_T - X_t \middle| \mathcal{F}_t\right] = 0$ iff $E_Q\left[Z_T X_T - Z_t X_t \middle| \mathcal{F}_t\right] = 0$, and the equivalence follows.

The second part follows by considering the Q-martingale $(X_t)_{0\leq t\leq T}$ generated by X:

$$X_t := E_Q\left[X|\mathcal{F}_t\right] \quad (0 \leq t \leq T).$$

By what we have shown above, $(Z_t X_t)_{0\leq t\leq T}$ is a P-martingale, so

$$Z_t X_t = E_P\left[Z_T X_T \middle| \mathcal{F}_t\right].$$

Division by Z_t yields the desired result. $\square$

Consider a stock price model of the form (1.2), that is, $S_t = S_0 \exp(rt)\exp(L_t)$ for a Lévy process L. We assume that there is a risk-neutral measure Q that is an Esscher transform of the objective measure P: For a suitable value $\theta \in \mathbb{R}$,

$$Q = P^{(\theta)},$$

with $P^{(\theta)}$ as defined in (1.8). All suitable discounted price processes are assumed to Q-martingales. In particular, Q is then a martingale measure for the options market consisting of Q-integrable European options on S. These are modeled as derivative securities paying an amount of $w(S_T)$, depending only on the stock price S_T, at a fixed time $T > 0$. We call $w(x)$ the *payoff function* of the option.[7] Assume that $w(x)$ is measurable and that $w(S_T)$ is Q-integrable. By (1.10), the option price at any time $t \in [0, T]$ is given by

$$\begin{aligned}
V(t) &= E_Q\left[e^{-r(T-t)} w(S_T) \middle| \mathcal{F}_t\right] \\
&= e^{-r(T-t)} E\left[w(S_T)\frac{Z_T}{Z_t} \middle| \mathcal{F}_t\right] \\
&= e^{-r(T-t)} E\left[w\left(S_t \frac{S_T}{S_t}\right)\frac{Z_T}{Z_t} \middle| \mathcal{F}_t\right] \\
&= e^{-r(T-t)} E\left[w\left(S_t e^{r(T-t)} \exp\left(L_T - L_t\right)\right)\frac{\exp(\theta(L_T - L_t))}{\mathrm{mgf}(\theta)^{T-t}} \middle| \mathcal{F}_t\right].
\end{aligned}$$

[7] This function is also called *contract function* in the literature.

By stationarity and independence of the increments of L we thus have

$$V(t) = e^{-r(T-t)} E\left[w\big(y e^{r(T-t)+L_{T-t}}\big) \frac{\exp(\theta(L_{T-t}))}{\mathrm{mgf}(\theta)^{T-t}} \right]\Big|_{y=S_t}. \tag{1.11}$$

Remark: The payoff $w(S_T)$ has to be Q-integrable for this to hold. This is true if $w(x)$ is bounded by some affine function $x \mapsto a + bx$, since by assumption, S is a Q-martingale and hence integrable. However, one would have to impose additional conditions to price *power options*, for which the payoff function w is of the form $w(x) = ((x-K)^+)^2$. (See Section 3.4 for more information on power options and other exotic options.)

The following proposition shows that we can interpret the Esscher transform price of the contingent claim in terms of a transform of payoff function w and interest rate r.[8]

Proposition 1.11. *Let the parameter θ of the Esscher transform be chosen such that the discounted stock price is a martingale under $Q = P^{(\theta)}$. Assume, as before, that Q is a martingale measure for the option market as well. Fix $t \in [0,T]$. Then the price of a European option with payoff function $w(x)$ (that is, with the value $w(S_T)$ at expiration) is the expected discounted value of another option under the objective measure P. This option has a payoff function*

$$\widetilde{w}_{S_t}(x) := w(x)\Big(\frac{x}{S_t}\Big)^{\theta},$$

which depends on the current stock price S_t. Also, discounting takes place under a different interest rate $\widetilde{r}$.

$$\widetilde{r} := r(\theta+1) + \ln \mathrm{mgf}(\theta).$$

Proof. In the proof of formula (1.11) for the price of a European option, we only used the fact that the density process Z is a P-martingale. Setting $\theta = 0$ in this formula we obtain the expected discounted value of a European option *under the measure* P. Calling the payoff function $\overline{w}$ and the interest rate $\overline{r}$, we get

$$\begin{aligned} E(t;\overline{r},\overline{w}) &\equiv E\Big[\exp(-\overline{r}(T-t))\overline{w}(S_T)\Big|\mathcal{F}_t\Big] \\ &= \exp(-\overline{r}(T-t))E\Big[\overline{w}\big(y\exp(\overline{r}(T-t)+L_{T-t})\big)\Big]\Big|_{y=S_t} \\ &= \exp(-\overline{r}(T-t))E\Big[\overline{w}\big(y\frac{S_T}{S_t}\big)\Big]\Big|_{y=S_t}. \end{aligned} \tag{1.12}$$

On the other hand, by (1.11) the price of the option considered before is

$$V(t) = \exp(-r(T-t))E\Big[w\big(y\frac{S_T}{S_t}\big)\frac{Z_T}{Z_t}\Big]\Big|_{y=S_t}. \tag{1.13}$$

Because of the special form of the Esscher density process Z we can express $\frac{Z_T}{Z_t}$ in terms of the stock

[8] This result is has an aesthetic value rather than being useful in practical applications: If we actually want to calculate option prices, we can always get the density of the Esscher transformed distribution by multiplying the original density by the function $\exp(\theta x - \kappa(\theta)t)$.

price:

$$\begin{aligned}
\frac{Z_T}{Z_t} &= \frac{\exp(\theta L_T - \ln \mathrm{mgf}(\theta)^T)}{\exp(\theta L_t - \ln \mathrm{mgf}(\theta)^t)} \\
&= \frac{\exp(\theta L_T + \theta r T)}{\exp(\theta L_t + \theta r t)} \exp(-(T-t)\theta r) \exp(-(T-t)\ln \mathrm{mgf}(\theta)) \\
&= \Big(\frac{S_T}{S_t}\Big)^\theta \exp(-(T-t)(\theta r + \ln \mathrm{mgf}(\theta))) \\
&= \frac{1}{y^\theta}\Big(y\frac{S_T}{S_t}\Big)^\theta \exp\big(-(T-t)(\theta r + \ln \mathrm{mgf}(\theta))\big),
\end{aligned}$$

for any real number $y > 0$. Inserting this expression for the density ratio into (1.13), we get

$$V(t) = \exp\big(-(r + \theta r + \ln \mathrm{mgf}(\theta))(T-t)\big)\frac{1}{y^\theta} E\Big[w\Big(y\frac{S_T}{S_t}\Big)\Big(y\frac{S_T}{S_t}\Big)^\theta\Big]\Big|_{y=S_t}. \tag{1.14}$$

Comparing this with the expected value (1.12) of an option characterized by the payoff function $\overline{w}$, as given in the statement of the proposition, we see that (1.14) is the P-expectation of the discounted price of an option with payoff function $\widetilde{w}$, discounted by the interest rate $\widetilde{r}$. □

1.5 A Differential Equation for the Option Pricing Function

Equation (1.11) shows that for a simple European option, one can express the option price at time t as a function of the time t and the stock price S_t. $V(t) = g(S_t, t)$, with the function

$$g(y,t) := e^{-r(T-t)} E_Q\Big[w\big(ye^{r(T-t)+L_{T-t}}\big)\Big].$$

Note that unlike (1.11), the expectation here is taken under the martingale measure Q. This formula is valid not only for option pricing by Esscher transforms, but moreover for all option pricing methods for which the log price process is a Lévy process under the martingale measure used for pricing. In what follows, it will turn out to be convenient to express the option price as a function of the log forward price. $X_t := \ln\big(e^{r(T-t)}S_t\big)$. This yields $V(t) = f\big(X_t, t\big)$, with

$$f(x,t) := e^{-r(T-t)} E_Q\big[w\big(e^{x+L_{T-t}}\big)\big] \tag{1.15}$$

In the following, denote by $\partial_i f$ the derivative of the function f with respect to its i-th argument. Likewise, $\partial_{ii} f$ shall denote the second derivative.

Proposition 1.12. *Assume that the function $f(x,t)$ defined in (1.15) is of class $C^{(2,1)}(\mathbb{R} \times \mathbb{R}_+)$, that is, it is twice continuously differentiable in the variable x and once continuously differentiable in the variable t. Assume further that the law of L_t has support $\mathbb{R}$. Then $f(x,t)$ satisfies the following integro-differential equation.*

$$\begin{aligned}
0 = &-rf\big(x,t\big) + (\partial_2 f)\big(x,t\big) + (\partial_1 f)\big(x,t\big)b + \frac{1}{2}(\partial_{11} f)\big(x,t\big)c \\
&+ \int \Big(f(x+y,t) - f(x,t) - (\partial_1 f)\big(x,t\big)y\Big)F(dy), \\
w\big(e^x\big) = &\; f(x,T) \qquad\qquad (x \in \mathbb{R},\ t \in (0,T)).
\end{aligned}$$

The only parameters entering here are the short-term interest rate r and the Lévy-Khintchine triplet (b,c,F) of the Lévy process L under the pricing measure Q.

Proof. The log forward price process $(X_t)_{0\leq t\leq T}$ introduced above satisfies the following relation.

$$X_t = \ln\left(e^{r(T-t)}S_t\right) = \ln S_t + r(T-t) = \ln S_0 + rT + L_t. \tag{1.16}$$

By assumption, the discounted option price process $e^{-rt}V(t) = e^{-rt}f(X_t,t)$ is a Q-martingale. Hence it is a special semimartingale, and any decomposition $e^{-rt}V(t) = V(0)+M_t+A_t$, with a local martingale M and a predictable process A with paths of bounded variation, has to satisfy $A_t \equiv 0$. In the following, we derive such a representation. The condition that A vanishes will then yield the desired integro-differential equation.

By Ito's formula, we have

$$\begin{aligned}
d(e^{-rt}V(t)) &= -re^{-rt}V(t)dt + e^{-rt}dV(t)\\
&= -re^{-rt}V(t)dt\\
&\quad + e^{-rt}\Big\{(\partial_2 f)\big(X_{t-},t\big)dt + (\partial_1 f)\big(X_{t-},t\big)dX_t + \frac{1}{2}(\partial_{11}f)\big(X_{t-},t\big)d\langle X^c,X^c\rangle_t\\
&\qquad + \int_{\mathbb{R}}\Big(f(X_{t-}+y,t) - f(X_{t-},t) - (\partial_1 f)\big(X_{t-},t\big)y\Big)\mu^{(X)}(dy,dt)\Big\},
\end{aligned}$$

where $\mu^{(X)}$ is the random measure associated with the jumps of X_t. By equation (1.16), the jumps of X and those of the Lévy process L coincide, and so do the jump measures. Furthermore, the stochastic differentials of X and $\langle X^c, X^c\rangle$ coincide with the corresponding differentials for the Lévy process L. Hence we get

$$\begin{aligned}
d(e^{-rt}V(t)) = &- re^{-rt}V(t)dt\\
&+ e^{-rt}\Big\{(\partial_2 f)\big(X_{t-},t\big)dt + (\partial_1 f)\big(X_{t-},t\big)dL_t + \frac{1}{2}(\partial_{11}f)\big(X_{t-},t\big)c\,dt\\
&\qquad + \int_{\mathbb{R}}\Big(f(X_{t-}+y,t) - f(X_{t-},t) - (\partial_1 f)\big(X_{t-},t\big)y\Big)\mu^{(L)}(dy,dt)\Big\}
\end{aligned}$$

The right-hand side can be written as the sum of a local martingale and a predictable process of bounded variation, whose differential is given by

$$\begin{aligned}
-re^{-rt}f\big(X_{t-},t\big)\,dt + e^{-rt}\Big\{&(\partial_2 f)\big(X_{t-},t\big)dt + (\partial_1 f)\big(X_{t-},t\big)b\,dt + \frac{1}{2}(\partial_{11}f)\big(X_{t-},t\big)c\,dt\\
&+ \int_{\mathbb{R}}\Big(f(X_{t-}+y,t) - f(X_{t-},t) - (\partial_1 f)\big(X_{t-},t\big)y\Big)\nu^{(L)}(dy,dt)\Big\},
\end{aligned}$$

where $\nu^{(L)}(dy,dt) = F(dy)dt$ is the compensator of the jump measure $\mu^{(L)}$. By the argument above, this process vanishes identically. By continuity, this means that for all values x from the support of $Q^{X_{t-}}$ (that is, by assumption, for all $x \in \mathbb{R}$) we have

$$\begin{aligned}
0 = -rf\big(x,t\big) &+ (\partial_2 f)\big(x,t\big) + (\partial_1 f)\big(x,t\big)b + \frac{1}{2}(\partial_{11}f)\big(x,t\big)c\\
&+ \int\Big(f(x+y,t) - f(x,t) - (\partial_1 f)\big(x,t\big)y\Big)F(dy).
\end{aligned}$$

□

Relation to the Feynman–Kac Formula

Equation (1.15) is the analogue of the Feynman–Kac formula. (See e. g. Karatzas and Shreve (1988), Chap. 4, Thm. 4.2.) The difference is that Brownian motion is replaced by a general Lévy process, L_t. The direction taken in the Feynman-Kac approach is the opposite of the one taken in Proposition 1.12: Feynman–Kac starts with the solution of some parabolic partial differential equation. If the solution satisfies some regularity condition, it can be represented as a conditional expectation.

A generalization of the Feynman-Kac formula to the case of general Lévy processes was formulated in Chan (1999), Theorem 4.1. The author states the this formula can be proven exactly in the same way as the classical Feynman-Kac formula. We have some doubts whether this is indeed the case. For example, an important step in the proof given in Karatzas and Shreve (1988) is to introduce a stopping time S_n that stops if the Brownian motion leaves some interval $[-n, n]$. Then on the stochastic interval $[0, S]$ the Brownian motion is bounded by n. But this argument cannot be easily transferred to a Lévy process having unbounded jumps: At the time S, the value of the Lévy process usually lies outside the interval $[-n, n]$. Furthermore, it is not clear which regularity condition has to be imposed on the solution of the integro-differential equation.

1.6 A Characterization of the Esscher Transform

In the preceding section, we have seen the importance of martingale measures for option pricing. In some situations, there is no doubt what the risk-neutral measure is: It is already determined by the condition that the discounted price processes of a set of basic securities are martingales. A famous example is the Samuelson (1965) model. Here the price of a single stock is described by a geometric Brownian motion.

$$S_t = S_0 \exp\big((\mu - \sigma^2/2)t + \sigma W_t\big) \quad (t \geq 0), \tag{1.17}$$

where S_0 is the stock price at time $t = 0$, $\mu \in \mathbb{R}$ and $\sigma \geq 0$ are constants, and where W is a standard Brownian motion. This model is of the form (1.2), with a Lévy process $L_t = (\mu - r - \sigma^2/2)t + \sigma W_t$. If the filtration in the Samuelson (1965) model is assumed to be the one generated by the stock price process,[9] there is only one locally equivalent measure under which $e^{-rt}S_t$ is a martingale. (See e. g. Harrison and Pliska (1981).) This measure is given by the following density process Z with respect to P.

$$Z_t = \frac{e^{\big((r-\mu)/\sigma\big)W_t}}{E\left[e^{\big((r-\mu)/\sigma\big)W_t}\right]}. \tag{1.18}$$

The fact that there is only one such measure implies that derivative prices are uniquely determined in this model: If the condition that $e^{-rt}S_t$ is a martingale is already sufficient to determine the measure Q, then necessarily Q must be the risk-neutral measure for any market that consists of the stock S and derivative securities that depend only on the underlying S. The prices of these derivative securities are then determined by equations of the form (1.10). For European call options, these expressions can be evaluated analytically, which leads to the famous Black and Scholes (1973) formula.

[9]This additional assumption is necessary since obviously the condition that $\exp(L_t)$ be a martingale can only determine the measure of sets in the filtration generated by L.

Note that the uniquely determined density process (1.18) is of the form (1.6) with $\theta = (r - \mu)/\sigma^2$. That is, it is an Esscher transform.

If one introduces jumps into the stock price model (e. g. by choosing a more general Lévy process instead of the Brownian motion W in (1.17)), this in general results in losing the property that the risk-neutral measure is uniquely determined by just one martingale condition. Then the determination of prices for derivative securities becomes difficult, because one has to figure out how the market chooses prices. Usually, one assumes that the measure Q—that is, prices of derivative securities—is chosen in a way that is optimal with respect to some criterion. For example, this may involve minimizing the L^2-norm of the density as in Schweizer (1996). (Unfortunately, in general the optimal density dQ/dP may become negative; so, strictly speaking, this is no solution to the problem of choosing a martingale measure.) Other approaches start with the problem of optimal hedging, where the aim is maximizing the expectation of a utility function. This often leads to derivative prices that can be interpreted as expectations under some martingale measure. In this sense, the martingale measure can also be chosen by maximizing expected utility. (Cf. Kallsen (1998), Goll and Kallsen (2000), Kallsen (2000), and references therein.)

For a large class of exponential Lévy processes other than geometric Brownian motion, Esscher transforms are still a way to satisfy the martingale condition for the stock price, albeit not the only one. They turn out to be optimal with respect to some utility maximization criterion. (See Keller (1997), Section 1.4.3, and the reference Naik and Lee (1990) cited therein.) However, the utility function used there depends on the Esscher parameter. Chan (1999) proves that the Esscher transform minimizes the relative entropy of the measures P and Q under all equivalent martingale transformations. But since his stock price is the stochastic exponential (rather than the ordinary exponential) of the Lévy process employed in the Esscher transform, this result does not hold in the context treated here.

Below we present another justification of the Esscher transform: If Conjecture 1.16 holds, then the Esscher transform is the only transformation for which the density process does only depend on the current stock price (as opposed to the entire stock price history.)

Martingale Conditions

The following proposition shows that the parameter θ of the Esscher transform leading to an equivalent martingale measure satisfies a certain integral equation. Later we show that an integro-differential equation of similar form holds for any function $f(x,t)$ for which $f(L_t, t)$ is another density process leading to an equivalent martingale measure. Comparison of the two equations will then lead to the characterization result for Esscher transforms.

Proposition 1.13. *Let L be a Lévy process for which L_1 possesses a finite moment generating function on some interval (a,b) containing 0. Denote by $\kappa(v)$ the cumulant generating function, i. e. the logarithm of the moment generating function. Then there is at most one $\theta \in \mathbb{R}$ such that e^{L_t} is a martingale under the measure $dP^\theta = (e^{\theta L_t}/E[e^{\theta L_1}]^t)dP$. This value θ satisfies the following equation.*

$$b + \theta c + \frac{c}{2} + \int \Big(e^{\theta x}(e^x - 1) - x \Big) F(dx) = 0, \tag{1.19}$$

where (b, c, F) is the Lévy-Khintchine triplet of the infinitely divisible distribution P^{L_1}.

Remark: Here we do not need to introduce a truncation function $h(x)$, since the existence of the moment generating function implies that $\int_{\{|x|>1\}} |x| F(dx) < \infty$, and hence L is a special semimartingale according to Jacod and Shiryaev (1987), Proposition II.2.29 a.

Proof of the proposition. It is well known that the moment generating function is of the Lévy-Khintchine form (1.2), with u replaced by $-iv$. (See Lukacs (1970), Theorem 8.4.2.) Since L has stationary and independent increments, the condition that e^{L_t} be a martingale under P^θ is equivalent to the following.

$$E\Big[e^{L_1}\frac{e^{\theta L_1}}{E[e^{\theta L_1}]}\Big] = 1.$$

In terms of the cumulant generating function $\kappa(v) = \ln E\left[\exp(vL_1)\right]$, this condition may be stated as follows.

$$\kappa(\theta+1) - \kappa(\theta) = 0.$$

Equation (1.19) follows by inserting the Lévy-Khintchine representation of the cumulant generating function, that is,

$$\kappa(u) = ub + \frac{c}{2}u^2 + \int \Big(e^{ux} - 1 - ux\Big)F(dx). \tag{1.20}$$

□

The density process of the Esscher transform is given by Z with

$$Z_t = \frac{\exp(\theta L_t)}{\exp(\kappa(\theta)t)}.$$

Hence it has a special structure: It depends on ω only via the current value $L_t(\omega)$ of the Lévy process itself. By contrast, even on a filtration generated by L, the value of a general density process at time t may depend on the whole history of the process, that is, on the path $L_s(\omega)$, $s \in [0,t]$.

Definition 1.14. *Let $\tau(dx)$ be a measure on $(\mathbb{R}, \mathcal{B}^1)$ with $\tau(\mathbb{R}) \in (0,\infty)$. Then $\mathcal{G}_\tau$ denotes the class of continuously differentiable functions $g : \mathbb{R} \to \mathbb{R}$ that have the following two properties*

$$\text{For all } x \in \mathbb{R}, \int_{\{|h|>1\}} \frac{g(x+h)}{|h|}\tau(dh) < \infty. \tag{1.21}$$

$$\text{If } \int \frac{g(x+h)-g(x)}{h}\tau(dh) = 0 \text{ for all } x \in \mathbb{R}, \text{ then } g \text{ is constant.} \tag{1.22}$$

In (1.22), we define the quotient $\frac{g(x+h)-g(x)}{h}$ to be $g'(x)$ for $h = 0$.

Lemma 1.15. *a) Assume that the measure $\tau(dx)$ has a support with closure $\mathbb{R}$. For monotone continuously differentiable functions $g(x)$, property (1.21) implies property (1.22).*

b) If the measure $\tau(dx)$ is a multiple of the Dirac measure δ_0, then $\mathcal{G}_\tau$ contains all continuously differentiable functions.

Proof. a) Without loss of generality, we can assume that g is monotonically increasing. Then $\frac{g(x+h)-g(x)}{h} \geq 0$ for all $x, h \in \mathbb{R}$. (Keep in mind that we have set $\frac{g(x+0)-g(x)}{0} = g'(x)$.) Hence $\int \frac{g(x+h)-g(x)}{h}\tau(dh) = 0$ implies that $g(x+h) = g(x)$ for $\tau(dh)$-almost every h. Since the closure of the support of $\tau(dx)$ was assumed to be $\mathbb{R}$, continuity of $g(x)$ yields the desired result.

b) Now assume that $\tau(dx) = \alpha\delta_0$ for some $\alpha > 0$. Condition (1.21) is trivially satisfied. For the proof of condition (1.22), we note that $\int \frac{g(x+h)-g(x)}{h}\alpha\delta_0(dh) = \alpha g'(x)$. But if the derivative of the continuously differentiable function $g(x)$ vanishes for almost all $x \in \mathbb{R}$, then obviously this function is constant. □

Conjecture 1.16. *In the definition above, if the support of τ has closure $\mathbb{R}$, then property (1.21) implies property (1.22).*

Remark: Unfortunately, we are not able to prove this conjecture. It bears some resemblance with the *integrated Cauchy functional equation* (see the monograph by Rao and Shanbhag (1994).) This is the integral equation

$$H(x) = \int_{\mathbb{R}} H(x+y)\tau(dy) \qquad \text{(almost all } x \in \mathbb{R}\text{)}, \tag{1.23}$$

where τ is a σ-finite, positive measure on $(\mathbb{R}, \mathcal{B}^1)$. If τ is a probability measure, then the function H satisfies (1.23) iff

$$\int_{\mathbb{R}} \big(H(x+y) - H(x)\big)\tau(dy) = 0 \qquad \text{(almost all } x \in \mathbb{R}\text{)}.$$

According to Ramachandran and Lau (1991), Corollary 8.1.8, this implies that H has every element of the support of τ as a period. In particular, the cited Corollary concludes that H is constant if the support of τ contains two incommensurable numbers. This is of course the case if the support of τ has closure $\mathbb{R}$, which we have assumed above. However, we cannot apply this theorem since we have the additional factor $1/h$ here.

As above, denote by $\partial_i f$ the derivative of the function f with respect to its i-th argument. Furthermore, the notation $X \cdot Y$ means the stochastic integral of X with respect to Y. Note that Y can also be the deterministic process t. Hence $X \cdot t$ denotes the Lebesgue integral of X, seen as a function of the upper boundary of the interval of integration.

We are now ready to show the following theorem, which yields the announced uniqueness result for Esscher transforms.

Theorem 1.17. *Let L be a Lévy process with a Lévy-Khintchine triplet $(b, c, F(dx))$ satisfying one of the following conditions*

1. *$F(dx)$ vanishes and $c > 0$.*
2. *The closure of the support of $F(dx)$ is $\mathbb{R}$, and $\int_{\{|x|\geq 1\}} e^{ux} F(dx) < \infty$ for $u \in (a, b)$, where $a < 0 < b$.*

Assume that $\theta \in (a, b-1)$ is a solution of $\kappa(\theta+1) = \kappa(\theta)$, where κ is the cumulant generating function of the distribution of L_1. Set $G(dx) := c\delta_0(dx) + x(e^x - 1)e^{\theta x} F(dx)$. Define a density process Z by

$$Z_t := \frac{\exp(\theta L_t)}{\exp(t\kappa(\theta))} \quad (t \in \mathbb{R}_+).$$

Then under the measure $Q \overset{\text{loc}}{\sim} P$ defined by the density process Z with respect to P, $\exp(L_t)$ is a martingale.[10] *Z is the only density process with this property that has the form $Z_t = f(L_t, t)$ with a function $f \in C^{(2,1)}(\mathbb{R} \times \mathbb{R}_+)$ satisfying the following: For every $t > 0$, $g(x,t) := f(x,t)e^{-\theta x}$ defines a function $g(\cdot, t) \in \mathcal{G}_G$.*

[10] See Assumption 1.1 for a remark why a change of measure can be specified by a density process.

Proof. First, note that the condition on $F(dx)$ implies that the distribution of every L_t has support $\mathbb{R}$ and possesses a moment generating function on (a, b). (The latter is a consequence of Wolfe (1971), Theorem 2.)

We have already shown that e^{L_t} indeed is a Q-martingale.

Let $f \in C^{(2,1)}(\mathbb{R} \times \mathbb{R}_+)$ be such that $f(L_t, t)$ is a density process. Assume that under the transformed measure, e^{L_t} is a martingale. Then $f(L_t, t)$ as well as $f(L_t, t)e^{L_t}$ are strictly positive martingales under P. By Ito's formula for semimartingales, we have

$$\begin{aligned} f(L_t, t) = f(L_0, 0) &+ \partial_2 f(L_{t-}, t) \cdot t \\ &+ \partial_1 f(L_{t-}, t) \cdot L_t \\ &+ (1/2)\partial_{11} f(L_{t-}, t) \cdot \langle L^c, L^c \rangle_t \\ &+ \Big(f(L_{t-} + x, t) - f(L_{t-}, t) - \partial_1 f(L_{t-}, t)x\Big) * \mu_t^L \end{aligned}$$

and

$$\begin{aligned} f(L_t, t)e^{L_t} =& f(L_0, 0)e^{L_0} + \partial_2 f(L_{t-}, t) \exp(L_{t-}) \cdot t \\ &+ (\partial_1 f(L_{t-}, t) + f(L_{t-}, t)) \exp(L_{t-}) \cdot L_t \\ &+ (1/2)\big(\partial_{11} f(L_{t-}, t) + 2\partial_1 f(L_{t-}, t) + f(L_{t-}, t)\big) \cdot \langle L^c, L^c \rangle_t \\ &+ \Big(\Big(f(L_{t-} + x, t)e^x - f(L_{t-}, t) - (\partial_1 f(L_{t-}, t) + f(L_{t-}, t))x\Big)e^{L_{t-}}\Big) * \mu_t^L. \end{aligned}$$

Since both processes are martingales, the sum of the predictable components of finite variation has to be zero for each process. So we have

$$\begin{aligned} 0 =& \partial_2 f(L_{t-}, t) \cdot t + \partial_1 f(L_{t-}, t)b \cdot t + (1/2)\partial_{11} f(L_{t-}, t)c \cdot t \\ &+ \iint \Big(f(L_{t-} + x, t) - f(L_{t-}, t) - \partial_1 f(L_{t-}, t)x\Big)F(dx)dt \end{aligned}$$

and

$$\begin{aligned} 0 =& \partial_2 f(L_{t-}, t) \exp(L_{t-}) \cdot t + \big(\partial_1 f(L_{t-}, t) + f(L_{t-}, t)\big)b \exp(L_{t-}) \cdot t \\ &+ (1/2)\big(\partial_{11} f(L_{t-}, t) + 2\partial_1 f(L_{t-}, t) + f(L_{t-}, t)\big)c \cdot t \\ &+ \iint \Big(\Big(f(L_{t-} + x, t)e^x - f(L_{t-}, t) - (\partial_1 f(L_{t-}, t) + f(L_{t-}, t))x\Big)e^{L_{t-}}\Big)F(dx)dt. \end{aligned}$$

By continuity, we have for any $t > 0$ and y in the support of $\mathcal{L}(L_{t-})$ (which is equal to the support of $\mathcal{L}(L_t)$, which in turn is equal to $\mathbb{R}$)

$$0 = \partial_2 f(y, t) + \partial_1 f(y, t)b + (1/2)\partial_{11} f(y, t)c + \int \Big(f(y + x, t) - f(y, t) - \partial_1 f(y, t)x\Big)F(dx)$$

and

$$\begin{aligned} 0 = \partial_2 f(y, t) + (\partial_1 f(y, t) + f(y, t))b + (1/2)(\partial_{11} f(y, t) + 2f_1(y, t) + f(y, t))c& \\ + \int \Big(f(y + x, t)e^x - f(y, t) - (\partial_1 f(y, t) + f(y, t))x\Big)F(dx).& \end{aligned}$$

Subtraction of these integro-differential equations yields

$$0 = f(y,t)b + f_1(y,t)c + f(y,t)c/2 + \int \Big(f(y+x,t)(e^x - 1) - f(y,t)x\Big) F(dx) \quad (y \in \mathbb{R}).$$

Division by $f(y,t)$ results in the equation

$$0 = b + \frac{f_1(y,t)}{f(y,t)} c + \frac{c}{2} + \int \Big(\frac{f(y+x,t)}{f(y,t)}(e^x - 1) - x\Big) F(dx). \quad (y \in \mathbb{R}) \tag{1.24}$$

By Proposition 1.13, the Esscher parameter θ satisfies a similar equation, namely

$$0 = b + \theta c + \frac{c}{2} + \int \Big(e^{\theta x}(e^x - 1) - x\Big) F(dx). \tag{1.25}$$

Subtracting (1.25) from (1.24) yields

$$0 = \Big(\frac{f_1(y,t)}{f(y,t)} - \theta\Big)c + \int \Big(\frac{f(y+x,t)}{f(y,t)} - e^{\theta x}\Big)(e^x - 1) F(dx) \quad (y \in \mathbb{R}).$$

For the ratio $g(y,t) := f(y,t)/e^{\theta y}$, this implies[11]

$$0 = \frac{g_1(y,t)}{g(y,t)} c + \int \Big(\frac{g(y+x,t)}{g(y,t)} - 1\Big)(e^x - 1) e^{\theta x} F(dx) \quad (y \in \mathbb{R}).$$

Multiplication by $g(y,t)$ finally yields

$$\begin{aligned} 0 &= g_1(y,t)c + \int \big(g(y+x,t) - g(y,t)\big)(e^x - 1)e^{\theta x} F(dx) \\ &= g_1(y,t)c + \int \frac{g(y+x,t) - g(y,t)}{x} x(e^x - 1) e^{\theta x} F(dx) \\ &= \int \frac{g(y+x,t) - g(y,t)}{x} G(dx) \quad (y \in \mathbb{R}), \end{aligned} \tag{1.26}$$

where we set again $\frac{g(y+0,t)-g(y,t)}{0} := g'(y)$. The measure $G(dx) = c\delta_0 + x(e^x - 1)e^{\theta x} F(dx)$ is finite on every finite neighborhood of $x = 0$. Furthermore, $G(dx)$ is non-negative and has support $\mathbb{R}\backslash\{0\}$. Since we have assumed that θ lies in the interval $(a, b-1)$ (where (a,b) is an interval on which the moment generating function of L_1 is finite), we can find $\epsilon > 0$ such that the interval $(\theta - \epsilon, \theta + 1 + \epsilon)$ is a subset of (a,b). Using the estimation $|x| \le (e^{-\epsilon x} + e^{\epsilon x})/\epsilon$, which is valid for all $x \in \mathbb{R}$, it is easy to see that one can form a linear combination of the functions $e^{(\theta-\epsilon)x}$, $e^{(\theta+\epsilon)x}$, $e^{(\theta+1-\epsilon)x}$, and $e^{(\theta+1+\epsilon)x}$ that is an upper bound for the function $x(e^x - 1)e^{\theta x}$. Choosing $\epsilon > 0$ small enough, all the coefficients in the exponents lie in (a,b). Therefore the measure $G(dx)$ is finite. Since by assumption $g \in \mathcal{G}_G$, equation (1.26) implies that $g(\cdot, t)$ is a constant for every fixed t, say $g(x,t) = c(t)$ for all $x \in \mathbb{R}, t > 0$. By definition of g, this implies $f(x,t) = c(t)e^{\theta x}$ for all $x \in \mathbb{R}, t > 0$. It follows from the relation $E[f(L_t, t)] = 1$ that $c(t) = 1/E\left[e^{\theta L_t}\right] = \exp(-t\kappa(\theta))$. □

[11] Note that

$$\frac{g_1(y,t)}{g(y,t)} = \frac{(\partial_1 f(y,t) - \theta f(y,t))e^{-\theta y}}{g(y,t)} = \frac{\partial_1 f(y,t)}{f(y,t)} - \theta$$

and

$$\frac{g(y+x,t)}{g(y,t)} \exp(\theta x) = \frac{f(y+x,t)}{f(y,t)}.$$

Chapter 2

On the Lévy Measure of Generalized Hyperbolic Distributions

2.1 Introduction

The Lévy measure determines the jump behavior of discontinuous Lévy processes. This measure is interesting from a practical as well as from a theoretical point of view. First, one can simulate a purely discontinuous Lévy process by approximating it by a compound Poisson process. The jump distribution of the approximating process is a normalized version of the Lévy measure truncated in a neighborhood of zero. This approach was taken e. g. in Rydberg (1997) for the simulation of normal inverse Gaussian (NIG) Lévy motions and in Wiesendorfer Zahn (1999) for the simulation of hyperbolic and NIG Lévy motions.[1] Of course, simulating the Lévy process in this way requires the numerical calculation of the Lévy density. We present a generally applicable method to get numerical values for the Lévy density based on Fourier inversion of a function derived form the characteristic function. We refine the method for the special case of generalized hyperbolic Lévy motions.[2] This class of Lévy processes matches the empirically observed log return behavior of financial assets very accurately. (See e. g. Eberlein and Prause (1998) for the general case, and Eberlein and Keller (1995), Barndorff-Nielsen (1997) for a study of some special cases of this class.)

The second important area where knowledge of the Lévy measure is essential is the study of singularity and absolute continuity of the distribution of Lévy processes. Here the density ratio of the Lévy measures under different probability measures is a key element. For the case of generalized hyperbolic Lévy processes, we study the local behavior of the Lévy measure near $x = 0$. This is the region that is most interesting for the study of singularity and absolute continuity. We apply this knowledge to a problem in option pricing: Eberlein and Jacod (1997b) have shown that with stock price models driven by pure-jump Lévy processes with paths of infinite variation, the option price is completely undetermined. Their proof relied on showing that the class of equivalent probability transformations that transform the driving Lévy process into another Lévy process is sufficiently large to generate almost arbitrary option prices consistent with no-arbitrage. For the class of generalized hyperbolic Lévy processes, we are able to

[1] These processes are Lévy processes L such that the unit increment L_1 has a hyperbolic respectively NIG distribution. See Appendix A.

[2] For a brief account of generalized hyperbolic distributions and the class of Lévy processes generated by them, see Appendix A.

specialize this result: It is indeed sufficient to consider only those measure transformations that transform the driving generalized hyperbolic Lévy process into a generalized hyperbolic Lévy process.

The chapter is structured as follows. In Section 2.2 we show how one can calculate the Fourier transform of the Lévy measure once the characteristic function of the corresponding distribution is known. The only requirement is that the distribution possesses a second moment. Section 2.3 considers the class of Lévy processes possessing a moment generating function. Here one can apply Esscher transforms to the basic probability measure. We study the effect of Esscher transforms on the Lévy measure and show how the Lévy measure is connected with the derivative of the characteristic function by means of Fourier transforms. Section 2.4 considers the class of generalized hyperbolic distributions. For a suitable modification of the Lévy measure, we calculate an explicit expression of its Fourier transform. It is shown how the Fourier inversion of this function, which yields the density of the Lévy measure, can be performed very efficiently by adding terms that make the Fourier transform decay more rapidly for $|u| \to \infty$.[3] In Section 2.5 we examine the question whether there are changes of probability that turn one generalized hyperbolic Lévy process into another. Proposition 2.20 identifies those changes of the parameters of the generalized hyperbolic distribution that can be achieved by changing the probability measure. The key to this is the behavior of the Lévy measures near $x = 0$. With the same methodology, we show in Proposition 2.21 that for the CGMY distributions a similar result holds. Due to the simpler structure of the Lévy density, the proof is much easier than for the generalized hyperbolic case. In Section 2.6, we demonstrate that the parameters δ and μ are indeed path properties of generalized hyperbolic paths, just as the volatility is a path property of the path of a Brownian motion. Section 2.7 studies implications for the problem of option pricing in models where the stock price is an exponential generalized hyperbolic Lévy motion.

2.2 Calculating the Lévy Measure

Let $\chi(u)$ denote the characteristic function of an infinitely divisible distribution. Then $\chi(u)$ possesses a Lévy-Khintchine representation.

$$\chi(u) = \exp\Big(iub - \frac{u^2}{2}c + \int_{\mathbb{R}\backslash\{0\}} \big(e^{iux} - 1 - iuh(x)\big) K(dx)\Big). \tag{2.1}$$

(See also Chapter 1.) Here $b \in \mathbb{R}$ and $c \geq 0$ are constants, and $K(dx)$ is the *Lévy measure*. This is a σ-finite measure on $\mathbb{R}\backslash\{0\}$ that satisfies

$$\int_{\mathbb{R}\backslash\{0\}} (x^2 \wedge 1)\; K(dx) < \infty. \tag{2.2}$$

It is convenient to extend $K(dx)$ to a measure on $\mathbb{R}$ by setting $K(\{0\}) = 0$. Unless stated otherwise, by $K(dx)$ we mean this extension. The function $h(x)$ is a *truncation function*, that is, a measurable bounded function with bounded support that that satisfies $h(x) = x$ in a neighborhood of $x = 0$. (See Jacod and Shiryaev (1987), Definition II.2.3.) We will usually use the truncation function

$$h(x) = x 1\!\!1_{\{|x|\leq 1\}}.$$

[3] The author has developed an S-Plus program based on this method. This was used by Wiesendorfer Zahn (1999) for the simulation of hyperbolic Lévy motions.

The proofs can be repeated with any other truncation function, but they are simpler with this particular choice of $h(x)$.

In general, the Lévy measure may have infinite mass. In this case the mass is concentrated around $x = 0$. However, condition (2.2) imposes restrictions on the growth of the Lévy measure around $x = 0$.

Definition 2.1. *Let $K(dx)$ be the Lévy measure of an infinitely divisible distribution. Then we call* modified Lévy measure *the measure $\widetilde{K}$ on $(\mathbb{R}, \mathcal{B})$ defined by $\widetilde{K}(dx) := x^2 K(dx)$.*

Lemma 2.2. *Let $\widetilde{K}$ be the modified Lévy measure corresponding to the Lévy measure $K(dx)$ of an infinitely divisible distribution that possesses a second moment. Then $\widetilde{K}$ is a finite measure.*

Proof. Since $x \mapsto x^2 \wedge 1$ is $K(dx)$ integrable, it is clear that $\widetilde{K}$ puts finite mass on every bounded interval. Moreover, by Wolfe (1971), Theorem 2, if the corresponding infinitely divisible distribution has a finite second moment, x^2 is integrable over any whose closure does not contain $x = 0$. So $\widetilde{K}$ assigns finite mass to any such set. Hence

$$\widetilde{K}(\mathbb{R}) = \widetilde{K}\big([-1,1]\big) + \widetilde{K}\big((-\infty,-1) \cup (1,\infty)\big) < \infty.$$

□

The following theorem shows how the Fourier transform of the modified Lévy measure $x^2 K(dx)$ is connected with the characteristic function of the corresponding distribution. This theorem is related to Bar-Lev, Bshouty, and Letac (1992), Theorem 2.2a, where the corresponding statement for the bilateral Laplace transform is given.[4]

Theorem 2.3. *Let $\chi(u)$ denote the characteristic function of an infinitely divisible distribution on $\mathbb{R}$ possessing a second moment. Then the Fourier transform of the modified Lévy measure $x^2 K(dx)$ is given by*

$$\int_{\mathbb{R}} e^{iux} x^2 K(dx) = -c - \frac{d}{du}\Big(\frac{\chi'(u)}{\chi(u)}\Big). \tag{2.3}$$

Proof. Using the Lévy-Khintchine representation, we have

$$\frac{d}{du}\chi(u) = \chi(u) \cdot \Big(ib - uc + \frac{d}{du}\int_{\mathbb{R}} \big(e^{iux} - 1 - iuh(x)\big) K(dx)\Big).$$

The integrand $e^{iux} - 1 - iuh(x)$ is differentiable with respect to u. Its derivative is

$$\partial_u\big(e^{iux} - 1 - iuh(x)\big) = ix\, e^{iux} - ih(x).$$

This is bounded by a $K(dx)$-integrable function as we will presently see. First, for $|x| \le 1$ we have

$$\begin{aligned} |ix\, e^{iux} - ih(x)| &= |x| \cdot |e^{iux} - 1| \\ &\le |x| \cdot (|\cos(ux) - 1| + |\sin(ux)|) \\ &\le |x| \cdot 2|ux| = |u| \cdot |x|^2. \end{aligned}$$

[4]However, Bar-Lev, Bshouty, and Letac (1992) do not give a proof. They say “The following result does not appear clearly in the literature and seems rather to belong to folklore.”

For u from some bounded interval, this is uniformly bounded by some multiple of $|x|^2$. For $|x| > 1$,

$$|ix\, e^{iux} - ih(x)| = |ix\, e^{iux}| = |x|.$$

From Wolfe (1971), Theorem 2, it follows that $\int_{\{|x|>1\}} |x| K(dx) < \infty$ iff the distribution possesses a finite first moment. Hence for each $u \in \mathbb{R}$ we can find some neighborhood U such that $\sup_{u \in U} |ix\, e^{iux} - ih(x)|$ is integrable, Therefore the integral is a differentiable function of u, and we can differentiate under the integral sign. (This follows from the differentiation lemma; see e. g. Bauer (1992), Lemma 16.2.) Consequently, we have

$$\frac{\chi'(u)}{\chi(u)} = ib - uc + \int_{\mathbb{R}} \big(ix\, e^{iux} - ih(x)\big) K(dx).$$

Again by the differentiation lemma, differentiating a second time is possible if the integrand $ix\, e^{iux} - ih(x)$ has a derivative with respect to u that is bounded by some $K(dx)$-integrable function $f(x)$, uniformly in a neighborhood of any $u \in \mathbb{R}$. Here this is satisfied with $f(x) = x^2$, since

$$\Big|\frac{\partial}{\partial u}\big(ix\, e^{iux} - ih(x)\big)\Big| = |-x^2\, e^{iux}| = x^2 \quad \text{for all } u \in \mathbb{R}.$$

Again by Wolfe (1971), Theorem 2, this is integrable with respect to $K(dx)$ because by assumption the second moment of the distribution exists. Hence we can again differentiate under the integral sign, getting

$$\frac{d}{du}\Big(\frac{\chi'(u)}{\chi(u)}\Big) = -c + \int_{\mathbb{R}} e^{iux} \cdot x^2\, K(dx).$$

This completes the proof. $\square$

Corollary 2.4. *Let $\chi(u)$ be the characteristic function of an infinitely divisible distribution on $(\mathbb{R}, \mathcal{B})$ that integrates x^2. Assume that there is a constant $\widetilde{c} \in \mathbb{R}$ such that the function*

$$\widehat{\rho}(u) := -\widetilde{c} - \frac{d}{du}\Big(\frac{\chi'(u)}{\chi(u)}\Big) \tag{2.4}$$

is integrable with respect to Lebesgue measure. Then $\widetilde{c}$ is equal to the Gaussian coefficient c in the Lévy-Khintchine representation, and $\widehat{\rho}(u)$ is the Fourier transform of the modified Lévy measure $x^2 K(dx)$. This measure has a continuous Lebesgue density on $\mathbb{R}$ that can be recovered from the function $\widehat{\rho}(u)$ by Fourier inversion.

$$x^2 \frac{dK}{d\lambda}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-iux} \widehat{\rho}(u) du.$$

Consequently, the measure $K(dx)$ has a continuous Lebesgue density on $\mathbb{R} \backslash \{0\}$:

$$\frac{dK}{d\lambda}(x) = \frac{1}{2\pi x^2} \int_{\mathbb{R}} e^{-iux} \widehat{\rho}(u) du.$$

For the proof, we need the following lemma.

Lemma 2.5. *Let $G(dx)$ be a finite Borel measure on $\mathbb{R}$. Assume that the characteristic function $\widehat{G}(u)$ of G tends to a constant c as $|u| \to \infty$. Then $G(\{0\}) = c$.*

Proof of Lemma 2.5. For any Lebesgue integrable function $g(u)$ with Fourier transform $\widehat{g}(x) = \int e^{iux} g(u)\, du$, we have by Fubini's theorem that

$$
\begin{aligned}
\int \widehat{g}(x)\, G(dx) &= \int\!\!\int e^{iux} g(u)\, du\, G(dx) \\
&= \int\!\!\int e^{iux}\, G(dx)\, g(u)\, du = \int \widehat{G}(u) g(u)\, du.
\end{aligned} \tag{2.5}
$$

Setting $\varphi(u) := (2\pi)^{-1/2} e^{-u^2/2}$, we get the Fourier transform $\widehat{\varphi}(x) = e^{-x^2/2}$. Now we consider the sequence of functions $g_n(u) := \varphi(u/n)/n, \quad n \geq 1$. We have $\widehat{g_n}(x) = \widehat{\varphi}(nx) \to 1\!\!1_{\{0\}}(x)$ as $n \to \infty$, for any $x \in \mathbb{R}$. By dominated convergence, this implies

$$
\int \widehat{g_n}(x)\, G(dx) \to \int 1\!\!1_{\{0\}}(x)\, G(dx) = G(\{0\}) \qquad (n \to \infty). \tag{2.6}
$$

On the other hand, setting $\widehat{G}_n(u) := \widehat{G}(nu), \quad n \geq 1$, we have $\widehat{G}_n(u) \to 1\!\!1_{\{x=0\}} + c 1\!\!1_{\{x \neq 0\}}$ pointwise for $u \in \mathbb{R}$. Hence, again by dominated convergence,

$$
\begin{aligned}
\int \widehat{G}(u) g_n(u)\, du &= \int \widehat{G}(u) \varphi(u/n)/n\, du \\
&= \int \widehat{G}(nu) \varphi(u)\, du \to \int (1\!\!1_{\{x=0\}} + c 1\!\!1_{\{x \neq 0\}}) \varphi(u)\, du = c.
\end{aligned} \tag{2.7}
$$

Since we have $\int \widehat{g_n}(x)\ G(dx) = \int \widehat{G}(u) g_n(u)\ du$ by (2.5), now relations (2.6) and (2.7) yield the desired result:

$$
\begin{array}{ccc}
\int \widehat{g_n}(x)\, G(dx) & \overset{(2.6)}{\longrightarrow} & G\big(\{0\}\big) \\
|| & & \\
\int \widehat{G}(u) g_n(u)\, du & \overset{(2.7)}{\longrightarrow} & c.
\end{array}
$$

□

Proof of Corollary 2.4. By Lemma 2.2, $x^2 K(dx)$ is a finite measure under the hypotheses of Corollary 2.4. By Theorem 2.3, its Fourier transform is given by

$$
\int_{\mathbb{R}} e^{iux} x^2 K(dx) = -c - \frac{d}{du}\Big(\frac{\chi'(u)}{\chi(u)}\Big).
$$

On the other hand, the assumed integrability of $\widehat{\rho}(u)$ implies that $\widehat{\rho}(u) \to 0$ as $|u| \to \infty$. So $\widehat{\rho}(u) + \widetilde{c} - c$, which is just the Fourier transform of $x^2 K(dx)$, converges to the value $\widetilde{c} - c$ as $|u| \to \infty$. Lemma 2.5 now yields that the limit $\widetilde{c} - c$ is just the modified Lévy measure of the set $\{0\}$. But this is zero, so indeed $\widetilde{c} = c$.

The remaining statements follow immediately from Theorem 2.3 and the fact that integrability of the Fourier transform implies continuity of the original function. (See e. g. Chandrasekharan (1989), I.(1.6).)

□

2.3 Esscher Transforms and the Lévy Measure

Lemma 2.6. *Let $G(dx)$ be a distribution on $\mathbb{R}$ with a finite moment generating function on some interval $(-a, b)$ with $-a < 0 < b$. Denote by $\chi(z)$ the analytic characteristic function of G. Then the Esscher transform of $G(dx)$ with parameter $\theta \in (-b, a)$, i. e. the probability measure*

$$G_\theta(dx) := \frac{1}{\chi(-i\theta)} e^{\theta x} G(dx),$$

has the analytic characteristic function

$$\chi_\theta(z) = \frac{\chi(z - i\theta)}{\chi(-i\theta)}.$$

Proof. This is immediately clear from the definition of the characteristic function. □

The Esscher transform of an infinitely divisible distribution is again infinitely divisible. The parameter θ changes the drift coefficient b, the coefficient c, and the Lévy measure $K(dx)$. This is shown in the following proposition.

Proposition 2.7. *Let $G(dx)$ be an infinitely divisible distribution on $\mathbb{R}$ with a finite moment generating function on some interval $(-a, b)$ with $-a < 0 < b$. Let the Lévy-Khintchine representation of the corresponding characteristic function $\chi(u)$ be given by*

$$\chi(u) = \exp\Big(iub - \frac{u^2}{2}c + \int_{\mathbb{R}\setminus\{0\}} \big(e^{iux} - 1 - iuh(x)\big) K(dx) \Big).$$

Then for any parameter $\theta \in (-b, a)$ the Esscher transform $G_\theta(dx)$ is infinitely divisible, with the parameters given by

$$\begin{aligned} b_\theta &= b + \theta c + \int_{\mathbb{R}\setminus\{0\}} h(x)(e^{\theta x} - 1) K(dx), \\ c_\theta &= c, \\ \textit{and} \qquad K_\theta(dx) &= e^{\theta x} K(dx). \end{aligned}$$

Proof. By Lukacs (1970), Theorem 8.4.2, we know that the Lévy-Khintchine representation of the characteristic function still holds if we replace u by a complex value z with $\mathrm{Im} z \in (-b, a)$. Hence by

Lemma 2.6 the characteristic function of the Esscher-transformed distribution is given by $\chi_\theta(u)$, with

$$\begin{aligned}
\chi_\theta(u) &= \frac{\exp\Big(i(u-i\theta)b - \frac{(u-i\theta)^2}{2}c + \int_{\mathrm{I\!R}\setminus\{0\}}\big(e^{i(u-i\theta)x} - 1 - i(u-i\theta)h(x)\big)K(dx)\Big)}{\chi(-i\theta)} \\
&= \exp\Big(iub + \theta b - \frac{u^2}{2}c + iu\theta c + \frac{\theta^2}{2}c \\
&\qquad + \int_{\mathrm{I\!R}\setminus\{0\}}\Big(e^{iux}e^{\theta x} - 1 - iuh(x) - \theta h(x)\Big)K(dx) \\
&\qquad - \theta b - \frac{\theta^2}{2}c - \int_{\mathrm{I\!R}\setminus\{0\}}\big(e^{\theta x} - 1 - \theta h(x)\big)K(dx)\Big) \\
&= \exp\Big(iu(b+\theta c) - \frac{u^2}{2}c + \int_{\mathrm{I\!R}\setminus\{0\}}\big(e^{iux} - 1 - iuh(x)\big)e^{\theta x}K(dx) \\
&\qquad - iu\int_{\mathrm{I\!R}\setminus\{0\}} h(x)(1-e^{\theta x})K(dx)\Big).
\end{aligned}$$

But this is again a Lévy-Khintchine representation of a characteristic function, with parameters as given in the proposition. □

In Chapter 1, we saw that in mathematical finance, Esscher transforms are used as a means of finding an equivalent martingale measure. The following proposition examines the question of existence and uniqueness of a suitable Esscher transform. It generalizes Lemma 1.9.

Proposition 2.8. *Consider a probability measure G on $(\mathrm{I\!R}, \mathcal{B})$ for which the moment generating function mgf(u) exists on some interval $(-a, b)$ with $a, b \in (0, \infty]$. Then we have the following.*

(a) If $G(dx)$ is non-degenerate, then for each $c > 0$ there is at most one value $\theta \in \mathrm{I\!R}$ such that

$$\frac{\textit{mgf}(\theta+1)}{\textit{mgf}(\theta)} = c. \tag{2.8}$$

(b) If mgf$(u) \to \infty$ as $u \to -a$ and as $u \to b$, and if $b + a > 1$, then equation (2.8) has exactly one solution $\theta \in (-a, b-1)$ for each $c > 0$.

For the proof of the proposition, we will need the following lemma. The statement is well known, but for the sake of completeness we give an elementary proof here.

Lemma 2.9. *Let $\mu(dx)$ be a non-degenerate probability measure on $(\mathrm{I\!R}, \mathcal{B}^1)$ which possesses a moment-generating function. Then the logarithm of this moment generating function possesses a strictly positive second derivative on the interior of its range of existence. In particular, it is strictly convex on its range of existence.*

Proof. Consider an arbitrary value u from the interior of the range of existence of the moment generating function. The second derivative of the log of the moment generating function, taken at u, is

$$\begin{aligned}
\Big(\ln \mathrm{mgf}\Big)''(u) &= \frac{\mathrm{mgf}''(u)\mathrm{mgf}(u) - \mathrm{mgf}'(u)^2}{\mathrm{mgf}(u)^2} \\
&= \frac{\int x^2 e^{ux}\mu(dx)}{\int e^{ux}\mu(dx)} - \left(\frac{\int x e^{ux}\mu(dx)}{\int e^{ux}\mu(dx)}\right)^2 > 0,
\end{aligned}$$

since obviously the last expression is the variance of the non-degenerate distribution $\frac{e^{ux}}{\int e^{ux}\mu(dx)}\mu(dx)$. $\square$

Proof of Proposition 2.8. Part (a). Equation (2.8) is equivalent to $\ln \mathrm{mgf}(\theta+1) - \ln \mathrm{mgf}(\theta) = \ln c$. The left-hand side is a strictly increasing function of θ, since by the mean value theorem there is a value $\xi \in (\theta, \theta+1)$ such that

$$\frac{d}{d\theta}\Big(\ln \mathrm{mgf}(\theta+1) - \ln \mathrm{mgf}(\theta)\Big) = \big(\ln \mathrm{mgf}\big)''(\xi).$$

The last term is strictly positive by Lemma 2.9. But a strictly increasing function possesses an inverse.

Part (b). By assumption, the function $\theta \mapsto \ln \mathrm{mgf}(\theta+1) - \ln \mathrm{mgf}(\theta)$ is well-defined for θ from the non-empty interval $(-a, b-1)$. It tends to $+\infty$ as θ reaches the boundaries of this interval. By continuity, there is a value θ where this function takes the value $\ln c$. This solves equation (2.8). Obviously, a distribution which satisfies the assumptions cannot be degenerate, so uniqueness follows by part (a). $\square$

Corollary 2.10. *Let G be a non-degenerate distribution on $(\mathbb{R}, \mathcal{B}^1)$ for which the moment generating function $\mathrm{mgf}(u)$ exists on some interval $(-a, b)$ with $a, b \in (0, \infty]$.*

(a) For each $c > 0$, there is at most one value θ such that the Esscher-transformed distribution $G_\theta(dx) := \frac{e^{\theta x}G(dx)}{\int e^{\theta x}G(dx)}$ satisfies $\mathrm{mgf}_{G_\theta}(1) = c$.

(b) If, in addition, we have $a + b > 1$, and if the moment generating function tends to infinity as one approaches the boundaries of the interval $(-a, b)$, then for each $c > 0$ there is exactly one $\theta \in \mathbb{R}$ such that the Esscher transformed distribution $G_\theta(dx)$ has a moment generating function equal to c at $u = 1$.

Proof. This follows at once from Proposition 2.8 and Lemma 2.6. $\square$

2.4 Fourier Transform of the Modified Lévy Measure

The characteristic function of a generalized hyperbolic distribution with parameters $\lambda \in \mathbb{R}$, $\alpha > 0$, $-\alpha < \beta < \alpha$, $\delta > 0$, and $\mu \in \mathbb{R}$ is given by

$$\chi(u) = e^{i\mu u}\frac{(\delta\sqrt{\alpha^2-\beta^2})^\lambda}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})} \cdot \frac{K_\lambda\big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\big)}{\big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\big)^\lambda}. \tag{2.9}$$

(See Appendix A.) This distribution possesses a moment generating function on the interval $(-\alpha-\beta, \alpha-\beta)$.[5] Therefore there is an analytic extension of the characteristic function to the strip $S := \mathbb{R} - i(-\alpha-\beta, \alpha-\beta) \subset \mathbb{C}$. Since the expression given above for the characteristic function can be shown to yield an analytic function if one replaces the real variable u by a complex variable $z \in S$, by standard arguments it coincides with the analytic characteristic function.

[5]For parameter values $\lambda < 0$, the moment generating function exists on the endpoints of this interval as well. However, most results concerning the analytic extension of the characteristic function only deal with the interior of the interval of existence. This is because by Lukacs (1970), Theorem 7.1.1, the endpoints are singular points of the moment generating function, which means that the usual arguments involving analyticity are not valid there.

For further reference, we formulate the following statements as propositions. Proposition 2.11 generalizes Keller (1997), Lemma 52, which considers hyperbolic distributions. Moreover, using characteristic functions we are able to give a more elegant proof.

Proposition 2.11. *An Esscher transform of a generalized hyperbolic distribution $GH(\lambda, \alpha, \beta, \delta, \mu)$ (with Esscher parameter $\theta \in (-\alpha - \beta, \alpha - \beta)$) corresponds to a change of parameter $\beta \rightsquigarrow \beta + \theta$ and vice versa. The same holds for any member of a generalized hyperbolic convolution semigroup.*

Proof. By Lemma 2.6, the following connection exists between the characteristic function χ_θ of an Esscher transformed distribution and the (analytic) characteristic function of the original distribution.

$$\chi_\theta(z) = \frac{\chi(z - i\theta)}{\chi(-i\theta)}. \tag{2.10}$$

By the argument above, the analytic characteristic function of a generalized hyperbolic distribution has the form

$$\chi(z) = \frac{\exp\big(i\mu(z - i\beta)\big)}{\exp\big(i\mu(-i\beta)\big)} \frac{(\delta\sqrt{\alpha^2 + (i\beta)^2})^\lambda}{K_\lambda(\delta\sqrt{\alpha^2 + (i\beta)^2})} \cdot \frac{K_\lambda\big(\delta\sqrt{\alpha^2 + (z - i\beta)^2}\big)}{\big(\delta\sqrt{\alpha^2 + (z - i\beta)^2}\big)^\lambda}. \tag{2.11}$$

Clearly the parameter β acts exactly as the Esscher parameter θ in (2.10). Taking the t-th power yields the statement for arbitrary $t > 0$. $\square$

Proposition 2.12. *If $\lambda \geq 0$, then the moment generating function of a generalized hyperbolic distribution tends to $+\infty$ as $u \to \alpha - \beta$ or $u \to -\alpha - \beta$. On the other hand, if $\lambda < 0$, then the moment generating function converges to a finite limit as u approaches $\alpha - \beta$ or $-\alpha - \beta$.*

Proof. If u tends to one of the values $\alpha - \beta$ or $-\alpha - \beta$, the expression $\delta\sqrt{\alpha^2 - (\beta - u)^2}$ converges to zero. Consequently, in this case the behavior of the moment generating function coincides with the behavior of the expression $\frac{K_\lambda(x)}{x^\lambda}$ for $x \downarrow 0$. From Abramowitz and Stegun (1968), equations 9.6.6, 9.6.8, and 9.6.9, we know that, as $z \to 0$,

$$K_\lambda(z) \sim \begin{cases} \frac{\Gamma(\lambda)2^{\lambda-1}}{z^\lambda} & \text{for } \lambda > 0, \\ -\ln z, & \text{for } \lambda = 0, \\ \frac{\Gamma(-\lambda)}{2^{\lambda+1}} z^\lambda & \text{for } \lambda < 0. \end{cases}$$

Hence if $x \downarrow 0$, $\frac{K_\lambda(x)}{x^\lambda}$ converges to a finite limit iff $\lambda < 0$. Otherwise it tends to $+\infty$. This completes the proof. $\square$

Corollary 2.13. *Let $\lambda \geq 0$ and $\alpha > 1/2$, and choose arbitrary values $\delta > 0$, $\mu \in \mathbb{R}$, and $c > 0$. Then there is a unique value β such that the moment generating function $\mathrm{mgf}_{GH(\lambda,\alpha,\beta,\delta,\mu)}(1) = c$.*

Proof. The distribution $GH(\lambda, \alpha, 0, \delta, \mu)$ possesses a moment generating function on the set $(-\alpha, \alpha)$. Since $\alpha > 1/2$, the conditions of Corollary 2.10 (b) are satisfied, and so there is a unique Esscher parameter θ such that the moment generating function of the Esscher transformed distribution has the value c at $u = 1$. By Proposition 2.11, the Esscher parameter is exactly the increment of the parameter β due to an Esscher transformation. $\square$

2.4.1 The Lévy Measure of a Generalized Hyperbolic Distribution

The Lévy measure of a generalized hyperbolic distribution depends on the parameters λ, α, β, and δ. It is continuous with respect to Lebesgue measure. The following formulas for the densities are proven in Prause (1999), Theorem 1.43 (for the case $\lambda \geq 0$), and in Wiesendorfer Zahn (1999), Anhang C (for the case of general λ).

$$\frac{dK(x)}{dx} = \begin{cases} \dfrac{e^{\beta x}}{|x|}\left(\displaystyle\int_0^\infty \frac{\exp\big(-\sqrt{2y+\alpha^2}|x|\big)}{\pi^2 y \big(J_\lambda^2(\delta\sqrt{2y}) + Y_\lambda^2(\delta\sqrt{2y})\big)} dy + \lambda e^{-\alpha|x|}\right), & \lambda \geq 0 \\ \dfrac{e^{\beta x}}{|x|} \displaystyle\int_0^\infty \frac{\exp\big(-\sqrt{2y+\alpha^2}|x|\big)}{\pi^2 y \big(J_{-\lambda}^2(\delta\sqrt{2y}) + Y_{-\lambda}^2(\delta\sqrt{2y})\big)} dy, & \lambda < 0. \end{cases} \tag{2.12}$$

However, numerical evaluation of these expressions is cumbersome, especially for small values of $|x|$, where the decay of the exponential term in the numerator becomes very slow. Because of the two Bessel functions J_λ and Y_λ appearing in the denominator, it is also difficult to examine the density analytically. Therefore we do not use representation (2.12) for our studies of the Lévy measure. Instead, we represent the Lévy density as a Fourier back-transform as in Corollary 2.4.

The following considerations show that we can limit our examinations to the case $\beta = 0$. From (2.11) it is clear that changing the value of the parameter β from β_1 to β_2 amounts to shifting the argument u of the characteristic function by $-i(\beta_2 - \beta_1)$ while multiplying by a normalizing factor. By Lemma 2.6, this corresponds to an Esscher transform with parameter $\beta_2 - \beta_1$. By Proposition 2.7, this only introduces a factor $\exp((\beta_2 - \beta_1)x)$ into the Lévy measure. Hence we only have to derive a formula for the Lévy measure for the case $\beta = 0$: If we know the Lévy density $K_{\beta=0}(dx)/dx$ for vanishing β, then the Lévy measure for $\beta \neq 0$, with the other parameters unchanged, is given by the relation

$$K_\beta(dx) = \exp(\beta x) K_{\beta=0}(dx).$$

Corollary 2.4 provides a way to get a formula for the density of the Lévy measure that is analytically and computationally more tractable than (2.12). We first derive an expression for the Fourier transform of the modified Lévy measure $x^2 K(dx)$ as given in (2.3) and then apply numerical methods to invert the Fourier transform. In the following proposition, we examine the Lévy measure $K(dx)$ and the Gaussian coefficient c of generalized hyperbolic distributions.

Proposition 2.14. *a) Let $K(dx)$ be the Lévy measure of a generalized hyperbolic distribution with parameters α, δ, and λ, and with skewness parameter $\beta = 0$. Then the Fourier transform of the modified measure $x^2 K(dx)$ is given by*

$$\int e^{iux} x^2 \, K(dx) = \left[\Big(-1 - 2\frac{\lambda + 1/2}{v} k + k^2 \Big)\Big(\delta^2 - \frac{\alpha^2\delta^4}{v^2}\Big) + k \cdot \frac{\alpha^2 \delta^4}{v^3} \right] \Bigg|_{\substack{k = \frac{K_{\lambda+1}(v)}{K_\lambda(v)} \\ v = \delta\sqrt{\alpha^2+u^2}}}, \tag{2.13}$$

where the subscripts "$k = \cdots$" and "$v = \cdots$" mean that the given expressions are to be substituted in the term in square brackets. The variable k has to be substituted first.

b) The Gaussian coefficient c in the Lévy-Khintchine representation of any generalized hyperbolic distribution vanishes.

Proof. By Theorem 2.3, we have to calculate the derivative of $-\chi'(u)/\chi(u)$. For a generalized hyperbolic distribution, the derivative of the characteristic function (2.9) is

$$\chi'(u) = i\mu\chi(u) + e^{i\mu u}\frac{(\delta\sqrt{\alpha^2-\beta^2})^\lambda}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})} \cdot \left[\frac{d}{dv}\frac{K_\lambda(v)}{v^\lambda}\right]\Bigg|_{v=\delta\sqrt{\alpha^2+u^2}} \cdot \frac{\partial}{\partial u}\big(\delta\sqrt{\alpha^2+u^2}\big).$$

Therefore

$$\frac{\chi'(u)}{\chi(u)} = i\mu + \left[\frac{\frac{d}{dv}\frac{K_\lambda(v)}{v^\lambda}}{\frac{K_\lambda(v)}{v^\lambda}}\right]\Bigg|_{v=\delta\sqrt{\alpha^2+u^2}} \cdot \frac{\partial}{\partial u}\big(\delta\sqrt{\alpha^2+u^2}\big).$$

We have

$$\frac{d}{dv}\frac{K_\lambda(v)}{v^\lambda} = \frac{K_\lambda{}'(v)\cdot v^\lambda - K_\lambda(v)\cdot\lambda\, v^{\lambda-1}}{v^{2\lambda}} = \frac{K_\lambda{}'(v)}{v^\lambda} - \frac{\lambda}{v}\cdot\frac{K_\lambda(v)}{v^\lambda}.$$

Hence

$$\frac{\frac{d}{dv}\frac{K_\lambda(v)}{v^\lambda}}{\frac{K_\lambda(v)}{v^\lambda}} = \frac{K_\lambda{}'(v)}{K_\lambda(v)} - \frac{\lambda}{v} = -\frac{K_{\lambda+1}(v)}{K_\lambda(v)} + \frac{\lambda}{v} - \frac{\lambda}{v} = -\frac{K_{\lambda+1}(v)}{K_\lambda(v)},$$

where for the second equality we have used the following relation for modified Bessel functions (see Abramowitz and Stegun (1968), equation 9.6.26):

$$K_\lambda{}'(v) = -K_{\lambda+1}(v) + \frac{\lambda}{v}K_\lambda(v).$$

Consequently, we have

$$\frac{\chi'(u)}{\chi(u)} = i\mu - \left[\frac{K_{\lambda+1}(v)}{K_\lambda(v)}\right]\Bigg|_{v=\delta\sqrt{\alpha^2+u^2}} \frac{\partial}{\partial u}\big(\delta\sqrt{\alpha^2+u^2}\big). \tag{2.14}$$

Taking the negative derivative of this expression yields

(2.15)

$$-\frac{d}{du}\Big(\frac{\chi'(u)}{\chi(u)}\Big) = \left[\frac{d}{dv}\frac{K_{\lambda+1}(v)}{K_\lambda(v)}\cdot\left(\frac{\partial}{\partial u}\big(\delta\sqrt{\alpha^2+u^2}\big)\right)^2 + \frac{K_{\lambda+1}(v)}{K_\lambda(v)}\cdot\frac{\partial^2}{\partial u^2}\big(\delta\sqrt{\alpha^2+u^2}\big)\right]\Bigg|_{v=\delta\sqrt{\alpha^2+u^2}}.$$

We go on to calculate the derivatives appearing in (2.15). First,

$$\begin{aligned}\frac{d}{dv}\frac{K_{\lambda+1}(v)}{K_\lambda(v)} &= \frac{K'_{\lambda+1}(v)K_\lambda(v) - K_{\lambda+1}(v)K_\lambda{}'(v)}{K_\lambda(v)^2}\\ &= \frac{K'_{\lambda+1}(v)}{K_\lambda(v)} - K_{\lambda+1}(v)\frac{K_\lambda{}'(v)}{K_\lambda(v)^2}.\end{aligned} \tag{2.16}$$

Again we can use Abramowitz and Stegun (1968), equation 9.6.26, to eliminate the derivatives of the Bessel functions: By this relation, we have

$$K'_{\lambda+1}(v) = -K_\lambda(v) - \frac{\lambda+1}{v}K_{\lambda+1}(v)$$

$$\text{and}\qquad K_\lambda{}'(v) = -K_{\lambda+1}(v) + \frac{\lambda}{v}K_\lambda(v).$$

Substituting these expressions for the derivatives of the Bessel functions in (2.16) yields

$$
\begin{aligned}
\frac{d}{dv}\frac{K_{\lambda+1}(v)}{K_\lambda(v)} &= \frac{-K_\lambda(v)-\frac{\lambda+1}{v}K_{\lambda+1}(v)}{K_\lambda(v)} - K_{\lambda+1}(v)\frac{-K_{\lambda+1}(v)+\frac{\lambda}{v}K_\lambda(v)}{K_\lambda(v)^2} \\
&= -1-\frac{\lambda+1}{v}\cdot\frac{K_{\lambda+1}(v)}{K_\lambda(v)}+\frac{K_{\lambda+1}(v)^2}{K_\lambda(v)^2}-\frac{\lambda}{v}\cdot\frac{K_{\lambda+1}(v)}{K_\lambda(v)} \qquad (2.17) \\
&= \Big[-1-2\frac{\lambda+1/2}{v}k+k^2\Big]\Big|_{k=\frac{K_{\lambda+1}(v)}{K_\lambda(v)}}.
\end{aligned}
$$

Next, we calculate the two derivatives of $\delta\sqrt{\alpha^2+u^2}$ appearing in (2.15).

$$
\frac{\partial}{\partial u}\big(\delta\sqrt{\alpha^2+u^2}\big) = \frac{\delta u}{\sqrt{\alpha^2+u^2}}, \qquad (2.18)
$$

$$
\begin{aligned}
\text{and}\quad \frac{\partial^2}{\partial u^2}\Big(\delta\sqrt{\alpha^2+u^2}\Big) &= \frac{\partial}{\partial u}\Big(\frac{\delta u}{\sqrt{\alpha^2+u^2}}\Big) \\
&= \frac{\delta}{\sqrt{\alpha^2+u^2}}-\frac{\delta u^2}{\sqrt{\alpha^2+u^2}^3}. \qquad (2.19)
\end{aligned}
$$

It will turn out to be convenient to write the square of the first derivative (2.18), as well as the second derivative (2.19), as a function of $v=\delta\sqrt{\alpha^2+u^2}$. This gives

$$
\begin{aligned}
\Big(\frac{\partial}{\partial u}\big(\delta\sqrt{\alpha^2+u^2}\big)\Big)^2 &= \frac{\delta^2u^2}{\sqrt{\alpha^2+u^2}^2} \\
&= \frac{\big(\delta\sqrt{\alpha^2+u^2}\big)^2-\alpha^2\delta^2}{\big(\delta\sqrt{\alpha^2+u^2}\big)^2/\delta^2} = \Big[\delta^2-\frac{\alpha^2\delta^4}{v^2}\Big]\Big|_{v=\delta\sqrt{\alpha^2+u^2}} \qquad (2.20)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial u^2}\big(\delta\sqrt{\alpha^2+u^2}\big) &= \frac{\delta}{\sqrt{\alpha^2+u^2}}-\frac{\delta^2u^2}{\delta\sqrt{\alpha^2+u^2}^3} \\
&= \Big[\frac{\delta^2}{v}-\frac{v^2-\alpha^2\delta^2}{v^3/\delta^2}\Big]\Big|_{v=\delta\sqrt{\alpha^2+u^2}} \\
&= \Big[\frac{\delta^2}{v}-\frac{\delta^2}{v}+\frac{\alpha^2\delta^4}{v^3}\Big]\Big|_{v=\delta\sqrt{\alpha^2+u^2}} = \Big[\frac{\alpha^2\delta^4}{v^3}\Big]\Big|_{v=\delta\sqrt{\alpha^2+u^2}}. \qquad (2.21)
\end{aligned}
$$

Substituting relations (2.17), (2.20), and (2.21) into (2.15) we get

$$
-\frac{d}{du}\Big(\frac{\chi'(u)}{\chi(u)}\Big) = \Big[\Big(-1-2\frac{\lambda+1/2}{v}k+k^2\Big)\Big(\delta^2-\frac{\alpha^2\delta^4}{v^2}\Big)+k\cdot\frac{\alpha^2\delta^4}{v^3}\Big]\Big|_{\substack{k=\frac{K_{\lambda+1}(v)}{K_\lambda(v)}\\ v=\delta\sqrt{\alpha^2+u^2}}}. \qquad (2.22)
$$

Denoting by $\widehat{\rho}(u)$ the function on the right-hand side, we have that $\widehat{\rho}(u)$ is continuous. Moreover, in Corollary 2.16 below we shall see that the modulus $|\widehat{\rho}(u)|$ decays at least as fast as $|u|^{-2}$ for $|u|\to\infty$. Therefore $\widehat{\rho}(u)$ is absolutely integrable, and the conditions of Corollary 2.4 are satisfied with $\widetilde{c}=0$. This yields that Gaussian coefficient in the Lévy-Khintchine representation of a generalized hyperbolic distribution vanishes, and that $\widehat{\rho}(u)$ is the Fourier transform of the modified Lévy measure. This proves part a) of the Proposition.

As a consequence of Propositions 2.7 and 2.11, the Gaussian coefficient c in the case $\beta\neq 0$ is the same as in the case $\beta=0$, with the other parameters unchanged. Above we have shown that $c=0$ for $\beta=0$. Therefore c vanishes for all generalized hyperbolic distributions, which proves part b). □

2.4.2 Asymptotic Expansion

For subsequent numerical calculations as well as for our theoretical considerations, it will turn out to be useful to know the first few terms of the asymptotic expansion of the Fourier transform of $\int e^{iux}x^2K(dx)$ for $|u| \to \infty$. By Proposition 2.14,

$$\int_{\mathbb{R}} e^{iux}x^2K(dx) = -\frac{d}{du}\Big(\frac{\chi'(u)}{\chi(u)}\Big) = f\big(\delta\sqrt{\alpha^2+u^2}\big),$$

with the function $f(v)$ defined for $v > 0$ by

$$f(v) := \bigg[\Big(-1-2\frac{\lambda+1/2}{v}k+k^2\Big)\Big(\delta^2-\frac{\alpha^2\delta^4}{v^2}\Big)+k\cdot\frac{\alpha^2\delta^4}{v^3}\bigg]\bigg|_{k=\frac{K_{\lambda+1}(v)}{K_\lambda(v)}}. \tag{2.23}$$

In order to find an asymptotic expansion of the Fourier transform $\int e^{iux}x^2K(dx)$ for $|u| \to \infty$, we expand the function $f(v)$ in powers of $1/v$.

Proposition 2.15. *The function $f(v)$ defined in (2.23) has the asymptotic expansion*

$$f(v) = \sum_{n=2}^{N} c_n\Big(\frac{\delta}{v}\Big)^n + O\Big(\frac{1}{v^{N+1}}\Big) \qquad \textit{as } v \to \infty.$$

for $N = 2, \dots, 6$, where

$$\begin{aligned}
c_2 &:= -(\lambda+1/2),\\
c_3 &:= \frac{\alpha^2\delta^2-(\lambda+1/2)(\lambda-1/2)}{\delta},\\
c_4 &:= \frac{(\lambda+1/2)\big((3/2)(\lambda-1/2)+2\alpha^2\delta^2\big)}{\delta^2},\\
c_5 &:= \frac{(\lambda+1/2)(\lambda-1/2)\big((\lambda+5/2)(\lambda-5/2)+3\alpha^2\delta^2\big)}{2\delta^3},\\
\textit{and} \qquad c_6 &:= -\frac{(\lambda+1/2)(\lambda-1/2)\big(5(\lambda+5/2)(\lambda-5/2)+4\alpha^2\delta^2+15\big)}{2\delta^4}.
\end{aligned}$$

Proof. For the modified Bessel functions of the third kind we have by Abramowitz and Stegun (1968), eq. 9.7.2:

(2.24)
$$K_\nu(z)\sqrt{\frac{2z}{\pi}}e^z = 1 + \sum_{n=1}^{N}\frac{(4\nu^2-1^2)\cdots(4\nu^2-(2n-1)^2)}{n!(8z)^n} + O\Big(\frac{1}{z^{N+1}}\Big) \quad \text{as } |z| \to \infty, |\arg z| < \frac{3\pi}{2}.$$

for arbitrary $N \in \mathbb{N}$. From this, one can derive the asymptotic expansion for the quotient of Bessel functions $K_\nu(z)$ and $K_{\nu'}(z)$ with different indices. For further reference, we note that in particular

$$\frac{K_{\lambda+1}(z)}{K_\lambda(z)} = 1 + O\Big(\frac{1}{z}\Big), \qquad \frac{K_{\lambda+2}(z)}{K_\lambda(z)} = 1 + O\Big(\frac{1}{z}\Big) \qquad \text{as } |z| \to \infty. \tag{2.25}$$

Using (2.24) to expand $f(v)$ in powers of $1/v^n$, we get, after a long and tedious calculation,

$$f(v) = \sum_{n=2}^{N} c_n\Big(\frac{\delta}{v}\Big)^n + O\Big(\frac{1}{v^{N+1}}\Big) \qquad \text{as } v \to \infty. \tag{2.26}$$

with c_2 through c_6 as given in the statement of the proposition. □

Corollary 2.16. *Let $K(dx)$ be the Lévy measure of a symmetric generalized hyperbolic distribution with parameters α, δ, and λ. Then for $N = 2, \ldots, 6$ the Fourier transform of the modified measure $x^2 K(dx)$ has the following expansion.*

$$\int e^{iux} x^2 \; K(dx) = \sum_{n=2}^{N} c_n \Big(\frac{1}{\sqrt{\alpha^2 + u^2}} \Big)^n + O\Big(\frac{1}{|u|^{N+1}} \Big) \qquad (|u| \to \infty), \tag{2.27}$$

with c_2 through c_6 as in Proposition 2.15.

Proof. This follows from Proposition 2.15, since $\sqrt{\alpha^2 + u^2} \sim |u|$ as $|u| \to \infty$. □

Remark: One may ask why we chose to expand up to order $1/|u|^6$, that is, why the maximal allowed value of N is 6. The answer is that further expansion of the function $f(v)$ is not advisable since we have only an asymptotic expansion here. This seems to diverge for small v if we expand further and further, while of course the decay for $v \to \infty$ becomes faster and faster. Choosing the order of expansion is therefore a tradeoff between good behavior for small v and good behavior as $v \to \infty$. An order of $N = 4$ seemed to be a good compromise for the parameters we used in our numerical studies, so we presume that providing formulas for orders up to 6 should be enough for other applications.

2.4.3 Calculating the Fourier Inverse

By Corollary 2.16, the Fourier transform of the modified Lévy density is of the order $O(1/u^2)$ as $|u| \to \infty$. This implies that it is absolutely integrable. Hence the Fourier inversion integral converges. Since the Fourier transform (2.13) is real and symmetric,[6] this integral reduces to the following cosine transform integral.

$$\rho(x) = \frac{1}{\pi} \int_0^\infty \cos(ux) \Big[\sum_{n=2}^{N} c_n \Big(\frac{\delta}{v} \Big)^n + R_{N+1}(v) \Big] \Big|_{v=\delta\sqrt{\alpha^2+u^2}} du, \tag{2.28}$$

with the remainder term

$$R_{N+1}(v) := f(v) - \sum_{n=2}^{N} c_n \Big(\frac{\delta}{v} \Big)^n = O\Big(\frac{1}{v^{N+1}} \Big). \tag{2.29}$$

The integral in (2.28) converges slowly, since in general the modulus of the integrand is only of the order $O(1/|u|^2)$. Therefore this formula does not directly allow for a fast numerical calculation of the Lévy density. But below we will explicitly calculate the Fourier inverse of the summands in the integrand for $n = 2, \ldots, 6$. Then Corollary 2.16 yields that for $N = 2, \ldots, 6$ the integrand $\cos(ux) \cdot R_{N+1}(\delta\sqrt{\alpha^2 + u^2})$ of the remaining Fourier inversion integral tends to zero like $|u|^{-N+1}$ as $|u| \to \infty$. Apart from speeding up the convergence of the integral, this will allow us to gain insights into the behavior of the Lévy density near $x = 0$.

[6] Remember that we have chosen $\beta = 0$.

We calculate explicitly the Fourier inversion integrals of the first five summands in (2.13). For $x = 0$, $n = 2, \dots, 6$, we have

$$\begin{aligned}
\frac{1}{\pi}\int_0^\infty \frac{c_n}{(\alpha^2+u^2)^{n/2}}du &= \frac{1}{\pi}\cdot\frac{c_n}{\alpha^n}\cdot\int_0^\infty \frac{1}{(1+(u/\alpha)^2)^{n/2}}du \\
&= \frac{1}{\pi}\cdot\frac{c_n}{\alpha^{n-1}}\cdot\int_0^\infty \frac{1}{(1+w^2)^{n/2}}dw \\
&= \frac{1}{\pi}\cdot\frac{c_n}{\alpha^{n-1}}\cdot\frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma(n/2-1/2)}{\Gamma(n/2)} = \frac{c_n}{2\sqrt{\pi}\alpha^{n-1}}\cdot\frac{\Gamma((n-1)/2)}{\Gamma(n/2)},
\end{aligned}$$

where the third equality follows from Abramowitz and Stegun (1968), eqs. 6.2.1 and 6.2.2. For the case $x \neq 0$, we observe that the integrand is a symmetric function of x. Therefore we can limit our examinations to the case $x > 0$. There we have

$$\frac{1}{\pi}\int_0^\infty \cos(ux)\frac{c_n}{(\alpha^2+u^2)^{n/2}}\,du = \frac{c_n}{\sqrt{\pi}}\cdot\frac{x^{(n-1)/2}}{\Gamma(n/2)(2\alpha)^{(n-1)/2}}K_{(n-1)/2}(\alpha x), \tag{2.30}$$

where we have used an integral representation of the modified Bessel functions $K_\nu(z)$ that can be found e. g. in Abramowitz and Stegun (1968), equation 9.6.25. Summing up the results of this subsection, we get the following

Proposition 2.17. *Let N be from the range $\{2, \dots, 6\}$. Then for a generalized hyperbolic distribution with $\beta = 0$ the modified Lévy measure $x^2K(dx)$ has the density*

(2.31)

$$\rho(x) = \begin{cases} \displaystyle\sum_{n=2}^{N}\frac{c_n}{2\sqrt{\pi}\alpha^{n-1}}\cdot\frac{\Gamma((n-1)/2)}{\Gamma(n/2)} + \int R_{N+1}\big(\delta\sqrt{\alpha^2+u^2}\big)\,du & (x=0) \\ \displaystyle\sum_{n=2}^{N}\frac{c_n}{\sqrt{\pi}}\cdot\frac{|x|^{(n-1)/2}}{\Gamma(n/2)(2\alpha)^{(n-1)/2}}K_{(n-1)/2}(\alpha|x|) + \int \cos(ux)R_{N+1}\big(\delta\sqrt{\alpha^2+u^2}\big)\,du & (x\neq 0) \end{cases}$$

with $c_2, \dots, c_6$ as defined in Proposition 2.15 and $R_{N+1}(x)$ as defined in (2.29).

Here is an example of the numerical calculation of the Lévy density with the approach presented above. We calculate the modified Lévy density of a hyperbolic distribution. In order to have realistic parameters, we take a log return dataset and fit a hyperbolic distribution by the method of maximum likelihood. The dataset consists of log returns on the German stock index DAX from September 28, 1959 to April 16, 1999. The parameters are estimated by the C-program `hyp` of Blæsild and Sørensen (1992).

α	β	δ	μ
157	-1.82	0.00610	0.000458

Figure 2.1 shows the Fourier transform of the modified Lévy density and the functions R_3 to R_6 defined in (2.29). Note that due to the log-log scale, one can easily see the speed of the decay as $u \to \infty$. The slopes of the functions R_3 to R_6 at the right end of the plot are approximately $3, 4, 5$, and 6, respectively. This reflects the increase in the speed of decay that is achieved by subtracting the terms in the asymptotic expansion. Note that at the left end of the plot, that is, for small values of u, the term R_6 is larger than the term R_5. This indicates that one cannot expect the remainder function $R_{N+1}(u)$ to become uniformly small as $N \to \infty$.

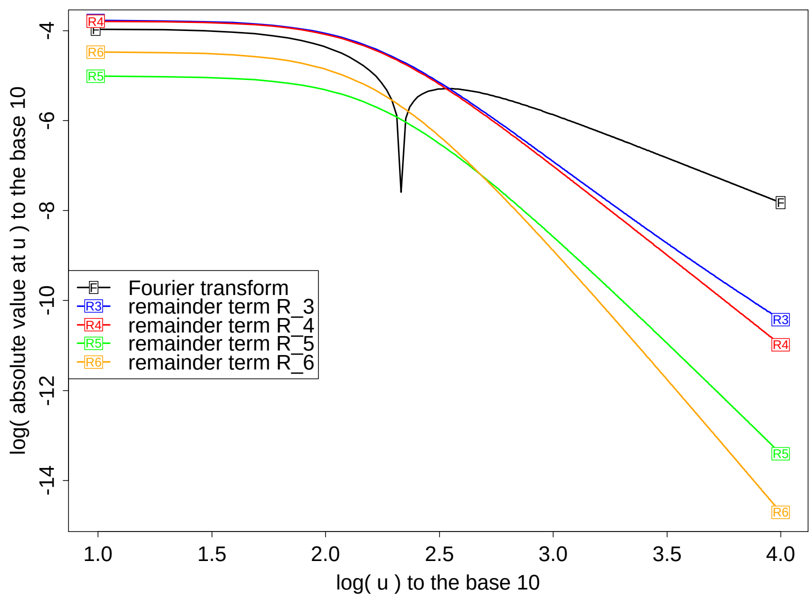

Figure 2.1: Fourier transform of modified Lévy density for hyperbolic distribution and corresponding terms R_3 to R_7 according to equation (2.29).

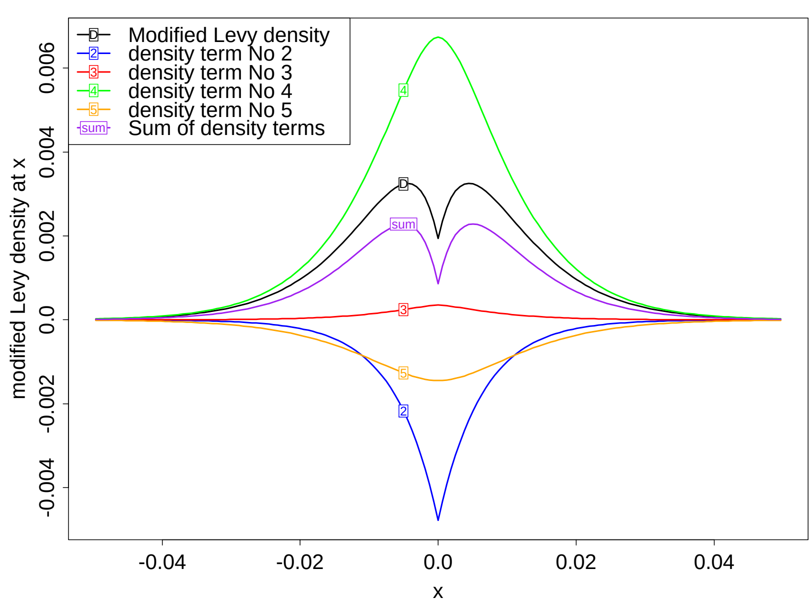

Figure 2.2: Modified Lévy density for hyperbolic distribution and first four summands of representation (2.31).

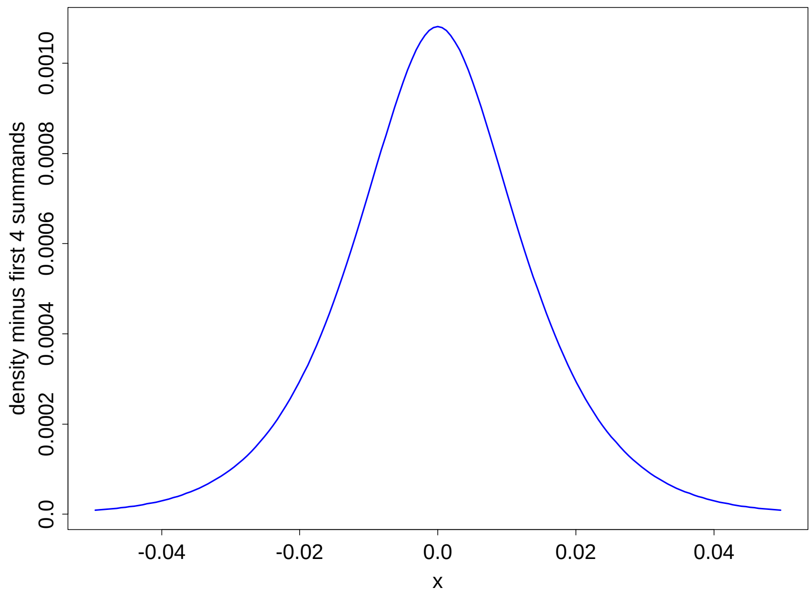

Figure 2.3: Difference between modified Lévy density and first four summands of representation (2.31).

Figure 2.2 shows the modified Lévy density, together with the first four summands from representation (2.31), here called "density term No 2" to "density term No 5". The sum of these density terms differs from the correct density by the four times continuously differentiable function shown in figure 2.3. Note that the discontinuity of the modified density at $x = 0$ is captured entirely by the density term No. 2, that is, in the $n = 2$ term of representation (2.31).

2.4.4 Sum Representations for Some Bessel Functions

By Watson (1944), Section 3.71, p. 80, eq. (12), we know that the modified Bessel functions $K_\nu(z)$ can be represented as finite sums of elementary functions if $\nu = m + 1/2$ with $m = 0, 1, 2, \ldots$:

$$K_{m+1/2}(z) = \sqrt{\pi/(2z)}e^{-z}\sum_{r=0}^{m}\frac{(m+r)!}{r!(m-r)!(2z)^r}. \tag{2.32}$$

Hence we can speed up the calculation of the Bessel functions $K_{1/2}(z)$, $K_{3/2}(z)$, and $K_{5/2}(z)$ in (2.31) by using the relations

$$K_{1/2}(z) = \sqrt{\pi/(2z)}e^{-z}, \tag{2.33}$$

$$K_{3/2}(z) = \sqrt{\pi/(2z)}e^{-z}\left(1 + \frac{1}{z}\right),$$

and $$K_{5/2}(z) = \sqrt{\pi/(2z)}e^{-z}\left(1 + \frac{3}{z} + \frac{3}{z^2}\right).$$

2.4.5 Explicit Expressions for the Fourier Backtransform

For further reference, we write down the Fourier backtransform given in (2.30) explicitly for $n = 2$ and $n = 3$. For $n = 2$, we can use $\Gamma(1) = 1$ and (2.33) to get

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty \cos(ux)\frac{c_2}{(\alpha^2+u^2)^{2/2}}\,du &= \frac{c_2}{\sqrt{\pi}}\cdot\frac{|x|^{1/2}}{\Gamma(1)(2\alpha)^{1/2}}K_{1/2}(\alpha|x|)\\ &= \frac{c_2}{\sqrt{\pi}}\cdot\frac{|x|^{1/2}}{(2\alpha)^{1/2}}\cdot\sqrt{\frac{\pi}{2\alpha|x|}}e^{-\alpha|x|}\\ &= \frac{c_2}{2\alpha}e^{-\alpha|x|} = -\frac{\lambda+1/2}{2\alpha}e^{-\alpha|x|}.\end{aligned} \tag{2.34}$$

For $n = 3$, we use $\Gamma(3/2) = \sqrt{\pi}/2$.

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty \cos(ux)\frac{c_3}{(\alpha^2+u^2)^{3/2}}\,du &= \frac{c_3}{\sqrt{\pi}}\cdot\frac{|x|^1}{(\sqrt{\pi}/2)(2\alpha)^1}K_1(\alpha|x|)\\ &= \frac{\alpha^2\delta^2-(\lambda+1/2)(\lambda-1/2)}{\pi\alpha\delta}\cdot|x|\cdot K_1(\alpha|x|).\end{aligned} \tag{2.35}$$

2.4.6 Behavior of the Density around the Origin

For the study of change-of-measure problems (see Section 2.5 below), it is important to know how the Lévy measure $K(dx)$ behaves around $x = 0$. Up to now, we know that for any generalized hyperbolic distribution, the modified Lévy measure $x^2K(dx)$ has a continuous Lebesgue density $\rho(x)$ since its Fourier transform is absolutely integrable. Studying the behavior of $\rho(x)$ of for $x \to 0$ yields information about the behavior of the Lévy measure in a neighborhood of $x = 0$. The following proposition gives the first-order behavior of $\rho(x)$ around $x = 0$.

Proposition 2.18. *Let $K(dx)$ be the Lévy measure of a generalized hyperbolic distribution with parameters $(\lambda,\alpha,\beta,\delta,\mu)$. Then $K(dx) = \rho(x)/x^2\,dx$, with*

$$\rho(x) = \frac{\delta}{\pi} + \frac{\lambda+1/2}{2}|x| + \frac{\delta\beta}{\pi}x + o(|x|) \qquad (x \to 0).$$

Proof. First consider the case $\beta = 0$. By equation (2.28), the value of the density at $x = 0$ is given by

$$\begin{aligned}\rho(0) &= \frac{1}{\pi}\int_0^\infty\Big(-\frac{d}{du}\Big(\frac{\chi'(u)}{\chi(u)}\Big)\Big)\,du\\ &= \frac{1}{\pi}\lim_{U\to\infty}\int_0^U\Big(-\frac{d}{du}\Big(\frac{\chi'(u)}{\chi(u)}\Big)\Big)\,du = \frac{1}{\pi}\lim_{U\to\infty}\Big(\frac{\chi'(0)}{\chi(0)}-\frac{\chi'(U)}{\chi(U)}\Big)\\ &\overset{(2.14),(2.18)}{=} \frac{1}{\pi}\lim_{U\to\infty}\left(\frac{K_{\lambda+1}\big(\delta\sqrt{\alpha^2+U^2}\big)}{K_\lambda\big(\delta\sqrt{\alpha^2+U^2}\big)}\cdot\frac{\delta U}{\sqrt{\alpha^2+U^2}}\right)\\ &= \frac{1}{\pi}\lim_{V\to\infty}\left(\frac{K_{\lambda+1}(V)}{K_\lambda(V)}\cdot\frac{\delta^2\sqrt{V^2/\delta^2-\alpha^2}}{V}\right)\\ &= \frac{\delta}{\pi}\lim_{V\to\infty}\left(\frac{K_{\lambda+1}(V)}{K_\lambda(V)}\cdot\sqrt{1-\frac{\alpha^2\delta^2}{V^2}}\right) = \frac{\delta}{\pi},\end{aligned}$$

since $K_{\lambda+1}(V)/K_\lambda(V) \to 1 \quad (V \to \infty)$ by the asymptotic relation (2.25).

Now that we have established the value of $\rho(x)$ at $x = 0$, we go on to study the behavior in the neighborhood of $x = 0$. The central idea here is to use the fact that $\rho(x)$ is an even function for $\beta = 0$. It will turn out that for $\lambda \neq -1/2$, $\rho(x)$ is not differentiable at $x = 0$. But below we will give an even function $g(x)$ such that the sum $\rho(x) + g(x)$ is differentiable. Then we know that the derivative of the sum has to vanish at $x = 0$, because the sum is an even function as well. Consequently,

$$\rho(x) + g(x) = \rho(0) + g(0) + o(|x|) \qquad (x \to 0),$$

by the definition of differentiability. This will allow us to prove the statement of the proposition, since it means that around $x = 0$ the behavior of the functions $\rho(x) - \rho(0)$ and $g(x) - g(0)$ is the same up to terms of order $o(|x|)$.

From Fourier analysis, we know that differentiability of an original function $f(x)$ is closely linked with the integrability of its Fourier transform $\widehat{f}(u)$. More precisely, Chandrasekharan (1989), p. 18, Theorem 3.A, together with the remark following it on p. 20, yields that if $u^n \widehat{f}(u)$ is absolutely integrable for some $n \in \mathbb{N}$, then $f(x)$ is continuously differentiable n times. The Fourier transform (2.13) of the density $\rho(x)$ does not satisfy an integrability condition of this kind if $\lambda \neq -1/2$. But as outlined above, we can find another function $g(x)$ so that the sum of the Fourier transforms satisfies the integrability condition. Then we can use the linearity of the Fourier transformation to gain information on the behavior of $\rho(x)$.

First, assume $\beta = 0$. We choose

$$g(x) := -\frac{\lambda + 1/2}{2\alpha} e^{-\alpha|x|},$$

which by (2.34) is just the Fourier transform of

$$u \mapsto \frac{c_2}{(\alpha^2 + u^2)}.$$

This is the first term in the asymptotic expansion (2.27) for the Fourier transform $\int e^{iux} x^2\, G(dx)$. By this expansion, we have

$$\int e^{iux} x^2\, G(dx) - \frac{c_2}{(\alpha^2 + u^2)} = O(1/|u|^3) \qquad \text{as } |u| \to \infty.$$

By the relation between integrability of the Fourier transform and differentiability of the original function mentioned above, this means that the inverse Fourier transform of the l. h. s. is continuously differentiable. But this inverse Fourier transform is given by

$$\rho(x) + g(x) = \rho(x) - \frac{\lambda + 1/2}{2\alpha} e^{-\alpha|x|}.$$

Consequently

$$\begin{aligned} \rho(x) &= \rho(0) + g(0) - g(x) + o(|x|) \\ &= \frac{\delta}{\pi} - \frac{\lambda + 1/2}{2\alpha} + \frac{\lambda + 1/2}{2\alpha} e^{-\alpha|x|} + o(|x|) \\ &= \frac{\delta}{\pi} + \frac{\lambda + 1/2}{2}|x| + o(|x|) \qquad (|x| \to 0). \end{aligned} \tag{2.36}$$

For $\beta \neq 0$, Proposition 2.7 yields that the density of the Lévy measure is $e^{\beta x}$ times the density for $\beta = 0$. Combining expansion (2.36), which is was derived for the case $\beta = 0$, and the expansion $e^{\beta x} = 1 + \beta x + o(|x|)$, we get

$$\begin{aligned}\rho(x) &= \big(1 + \beta x + o(|x|)\big)\Big(\frac{\delta}{\pi} + \frac{\lambda + 1/2}{2}|x| + o(|x|)\Big)\\ &= \frac{\delta}{\pi} + \frac{\lambda + 1/2}{2}|x| + \frac{\delta}{\pi}\beta x + o(|x|) \qquad (|x| \to 0).\end{aligned}$$

$\square$

Remark 1: The proof of the preceding proposition depends crucially on the fact that an even, differentiable function has derivative 0 at $x = 0$. One cannot use the same method to gain knowledge about the higher-order behavior of the density $\rho(x)$ around $x = 0$. Fortunately, knowing the first-order behavior is sufficient to tackle the problem of absolute continuity and singularity of the distributions of generalized hyperbolic Lévy processes.

Remark 2: One must be aware that the series expansion for the modified Bessel functions $K_\nu(z)$ is only an asymptotic series. Therefore, driving the expansion (2.26) further and further and calculating explicitly more and more terms of the cosine transformation (2.28) will in general not yield a convergent series.

2.4.7 NIG Distributions as a Special Case

By Abramowitz and Stegun (1968), Eq. 9.6.6, we have

$$K_{-\nu}(z) = K_\nu(z).$$

Consequently $K_{\lambda+1}(z)/K_\lambda(z) = K_{1/2}(z)/K_{-1/2}(z) = 1$ if $\lambda = -1/2$. The generalized hyperbolic distributions with $\lambda = -1/2$ are called *normal inverse Gaussian* (NIG) distributions.[7] Using $k = 1$ and $\lambda = -1/2$ in (2.14), we get

$$\begin{aligned}\int_{\mathbb{R}} e^{iux} x^2 K_{\mathrm{NIG}}(dx) &= \left[\delta^2(k^2 - 1) - \frac{\delta^2 k(2\lambda + 1)}{v} - \frac{\alpha^2\delta^4(k^2 - 1)}{v^2} + \frac{\alpha^2\delta^4 k(2\lambda + 2)}{v^3}\right]\Bigg|_{\substack{k=1,\ \lambda=-1/2\\ v=\delta\sqrt{\alpha^2+u^2}}}\\ &= \left[\frac{\alpha^2\delta^4}{v^3}\right]\Bigg|_{v=\delta\sqrt{\alpha^2+u^2}}.\end{aligned}$$

This is just the $n = 3$ term in the expansion (2.27). Hence the Lévy measure can be calculated with the help of equation (2.35).

$$K_{NIG}(dx) = e^{\beta x}\frac{\delta\alpha}{\pi|x|}K_1(\alpha|x|)\,dx. \tag{2.37}$$

This coincides with the formula derived in Barndorff-Nielsen (1997), eq. (3.15).

[7] See Barndorff-Nielsen (1997) and references therein for a thorough account of the class of NIG distributions.

2.5 Absolute Continuity and Singularity for Generalized Hyperbolic Lévy Processes

2.5.1 Changing Measures by Changing Triplets

In this subsection, we shall be concerned with the question which changes of the distribution of a Lévy process can be achieved by an equivalent change of the underlying probability measure. The probability shall be changed in such a way that the property of stationary and independent increments is preserved. In other words, the Lévy process stays a Lévy process under the change of probability.

The following proposition answers this question in terms of the Lévy-Khintchine triplet $\big(b, c, K(dx)\big)$ of the process under the two measures. An important point that—in my opinion—is missing in a similar statement to be found in Keller (1997), Proposition 15, is the following: On a general stochastic basis, do the conditions in part (b) indeed imply the existence of a suitable density process? The question is *yes*, but—as may be seen in the proof—proving this requires some deep results from stochastic analysis.

Proposition 2.19. *Let X be a Lévy process with Lévy-Khintchine triplet (b, c, K) under some probability measure P. Then the following two conditions are equivalent.*

(a) There is a probability measure $Q \overset{\text{loc}}{\sim} P$ such that X is a Q-Lévy process with triplet (b', c', K').

(b) All of the following conditions hold.

(i) $K'(dx) = k(x)K(dx)$ *for some Borel function* $k : \mathbb{R} \to (0, \infty)$.

(ii) $b' = b + \int h(x)(k(x) - 1)K(dx) + \sqrt{c}\beta \qquad$ *for some* $\beta \in \mathbb{R}$.

(iii) $c' = c$.

(iv) $\int (1 - \sqrt{k(x)})^2 K(dx) < \infty$.

Proof. *(a) $\Longrightarrow$ (b).* This is Jacod and Shiryaev (1987), Theorem IV.4.39 c.

(b) $\Longrightarrow$ (a). We define a local martingale N according to Jacod and Shiryaev (1987), III.5.11.[8]

$$N := \beta \cdot X^c + (k(x) - 1) * (\mu^X - \nu). \tag{2.38}$$

Now choose some deterministic time T and consider the stopped process N^T. With this local martingale N^T, the process $B(3, N^T)^p$ from Jacod (1979), (5.15), is

$$\begin{aligned} B(3, N^T)^p_t &= \beta^2 < X^c, X^c >_{(t\wedge T)} + (1 - \sqrt{k(x)})^2 * \nu_{(t\wedge T)} \\ &= \beta^2 c \cdot (t \wedge T) + (t \wedge T) \int (1 - \sqrt{k(x)})^2 K(dx). \end{aligned}$$

By condition (iv) this process is bounded. Hence Jacod (1979), Corollaire (8.30), yields that the stochastic exponential $\mathcal{E}(N^T) = \mathcal{E}(N)^T$ is a uniformly integrable martingale. Jacod and Shiryaev (1987), Lemma III.5.27, now yields that if we define $Z_t := \mathcal{E}(N)_t$, then under the measure Q defined by $Q_T = Z_T P_t$ the process X is a semimartingale with characteristics $(b't, c't, K'(dx)dt)$. Hence it is a Lévy process with Lévy-Khintchine triplet $(b', c', K'(dx))$. □

[8] The formula given in Jacod and Shiryaev (1987), III.5.11 simplifies considerably because of the property of stationary and independent increments.

2.5.2 Allowed and Disallowed Changes of Parameters

Proposition (2.19) connects the change of the Lévy measure with the change of the underlying probability measure. In particular, for generalized hyperbolic Lévy processes it allows us to analyze whether a certain change of parameters can be achieved by a change of the probability measure.

Proposition 2.20. *Let X be a generalized hyperbolic Lévy process under the measure P. Let $(\lambda, \alpha, \beta, \delta, \mu)$ denote the parameters of the generalized hyperbolic distribution of X_1. Then there is another measure $P' \overset{\text{loc}}{\sim} P$ under which X is again a generalized hyperbolic Lévy process, with parameters $(\lambda', \alpha', \beta', \delta', \mu')$, if and only if $\delta' = \delta$ and $\mu' = \mu$.*

Proof. Since generalized hyperbolic Lévy processes are purely discontinuous, the change of measure is determined by the density $k(x) := dK'/dK$, where $K'(dx)$ and $K(dx)$ are the Lévy measures of the generalized hyperbolic distributions under P' and P, respectively. By Proposition (2.19), there exists a measure P' under which X is a generalized hyperbolic Lévy process with parameters $(\lambda', \alpha', \beta', \delta', \mu')$ if and only if this density satisfies the conditions

$$\int \left(1 - \sqrt{k(x)}\right)^2 K(dx) < \infty \tag{2.39}$$

$$\text{and} \qquad b + \int h(x)(k(x) - 1)K(dx) = b'. \tag{2.40}$$

Since $K(dx)$ and $K'(dx)$ are both continuous with respect to the Lebesgue measure on $\mathbb{R}\backslash\{0\}$, the density $k(x)$ is equal to the quotient of the respective (modified) Lévy densities.

$$k(x) = \frac{\frac{1}{x^2}\rho_{(\lambda',\alpha',\beta',\delta',\mu')}(x)}{\frac{1}{x^2}\rho_{(\lambda,\alpha,\beta,\delta,\mu)}(x)} = \frac{\rho_{(\lambda',\alpha',\beta',\delta',\mu')}(x)}{\rho_{(\lambda,\alpha,\beta,\delta,\mu)}(x)}.$$

For the case of generalized hyperbolic Lévy processes, the integrability condition (2.39) is therefore

$$\begin{aligned}\int \left(1 - \sqrt{k(x)}\right)^2 K(dx) &= \int \left(1 - \frac{\sqrt{\rho_{(\lambda',\alpha',\beta',\delta',\mu')}(x)}}{\sqrt{\rho_{(\lambda,\alpha,\beta,\delta,\mu)}(x)}}\right)^2 \frac{1}{x^2}\rho_{(\lambda,\alpha,\beta,\delta,\mu)}(x)\; dx \\ &= \int \left(\sqrt{\rho_{(\lambda',\alpha',\beta',\delta',\mu')}(x)} - \sqrt{\rho_{(\lambda,\alpha,\beta,\delta,\mu)}(x)}\right)^2 \frac{dx}{x^2} < \infty.\end{aligned}$$

Since the Lévy densities—modified or not—of generalized hyperbolic distributions satisfying $|\beta| < \alpha$ always decay exponentially as $|x| \to \infty$, the only critical region for the boundedness of the integral is the neighborhood of $x = 0$. Here we have, by Proposition 2.18,

$$\rho(x) = \frac{\delta}{\pi} + \frac{\lambda + 1/2}{2}|x| + \frac{\delta\beta}{\pi}x + o(|x|) = \frac{\delta}{\pi} + O(|x|) \qquad (|x| \to 0).$$

Therefore

$$\sqrt{\rho_{(\lambda,\alpha,\beta,\delta,\mu)}(x)} = \frac{\delta}{\pi}\sqrt{1 + O(|x|)} = \frac{\delta}{\pi}\big(1 + O(|x|)\big) \qquad (|x| \to 0).$$

since $\sqrt{1+y} = 1 + y/2 + o(|y|)$ for $|y| \to 0$. Using this, the integrability condition (2.39) becomes

$$\begin{aligned}\int \left(1-\sqrt{k(x)}\right)^2 K(dx) &= \int \left(\frac{\delta'}{\pi}\big(1+O(|x|)\big) - \frac{\delta}{\pi}\big(1+O(|x|)\big)\right)^2 \frac{dx}{x^2} \\ &= \int \left(\frac{\delta'-\delta}{\pi} + O(|x|)\right)^2 \frac{dx}{x^2} \\ &= \int \left(\frac{(\delta'-\delta)^2}{\pi^2 x^2} + \frac{2(\delta'-\delta)}{\pi} O(1/|x|) + O(1)\right) dx < \infty.\end{aligned}$$

Clearly, this is satisfied if and only if $\delta' = \delta$.

Now we consider the second condition (2.40). Under the assumption $\delta' = \delta$, this will turn out to be equivalent to $\mu' = \mu$, thus providing the second constraint stated above.

By Jacod and Shiryaev (1987), Theorem II.4.15, the constant b from the Lévy-Khintchine representation is indeed the drift coefficient from the characteristic triplet. We have

$$b = E[L_1] + \int (h(x) - x) K(dx) \tag{2.41}$$

$$\text{and} \qquad b' = E'[L_1] + \int (h(x) - x) K'(dx), \tag{2.42}$$

where $E'[\cdot]$ denotes the expectation under the measure P'. On the other hand, condition (2.40) requires

$$b' = b + \int h(x)(k(x) - 1) K(dx),$$

which, in view of (2.41), is equivalent to

$$\begin{aligned}b' &= E[L_1] + \int (h(x) - x) K(dx) + \int h(x)(k(x)-1) K(dx) \\ &= E[L_1] + \int (h(x)k(x) - x) K(dx).\end{aligned}$$

Comparison with equation (2.42) yields that condition (2.40) is satisfied if and only if

$$E'[L_1] - E[L_1] = \int x(1 - k(x)) K(dx). \tag{2.43}$$

For the generalized hyperbolic distribution, the expectation is known explicitly. So

$$E[L_1] = \mu + \frac{\beta\delta \cdot K_{\lambda+1}(\delta\sqrt{\alpha^2-\beta^2})}{\sqrt{\alpha^2-\beta^2} K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}.$$

Hence if L_1 has a generalized hyperbolic distribution with equal parameter δ under both the measure P and P', we have

$$E'[L_1] - E[L_1] = \mu' - \mu + \frac{\beta'\delta \cdot K_{\lambda'+1}(\delta\sqrt{\alpha'^2-\beta'^2})}{\sqrt{\alpha'^2-\beta'^2} K_{\lambda'}(\delta\sqrt{\alpha'^2-\beta'^2})} - \frac{\beta\delta \cdot K_{\lambda+1}(\delta\sqrt{\alpha^2-\beta^2})}{\sqrt{\alpha^2-\beta^2} K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}.$$

Hence we can rewrite condition (2.43) (and hence condition (2.40)) as a condition on the change of the parameter μ:

(2.44)
$$\mu' - \mu = \int x(1-k(x))K(dx) + \frac{\beta\delta \cdot K_{\lambda+1}(\delta\sqrt{\alpha^2-\beta^2})}{\sqrt{\alpha^2-\beta^2}K_\lambda(\delta\sqrt{\alpha^2-\beta^2})} - \frac{\beta'\delta \cdot K_{\lambda'+1}(\delta\sqrt{\alpha'^2-\beta'^2})}{\sqrt{\alpha'^2-\beta'^2}K_{\lambda'}(\delta\sqrt{\alpha'^2-\beta'^2})}.$$

It will turn out that the right-hand side vanishes, so that, under the assumption $\delta' = \delta$, condition (2.40) is equivalent to $\mu' = \mu$.

We consider two cases: **Case 1:** $\beta' = \beta = 0$. Then $(1-k(x))K(dx)$ is symmetric with respect to $x = 0$, and the integral in (2.44) vanishes. On the other hand, the remaining expressions on the right-hand side of (2.44) also vanish. Hence (2.44) is equivalent to $\mu' - \mu = 0$.

Case 2: β arbitrary and $\beta' = 0$, with the remaining parameters $\lambda' = \lambda$ and $\alpha' = \alpha$. Then also $\mu' = \mu$. Before we prove this, we will make clear why these two cases indeed suffice to complete the proof. We can decompose a change of parameters $(\lambda, \alpha, \beta, \delta, \mu) \rightsquigarrow (\lambda', \alpha', \beta', \delta, \mu')$ into three steps:

$$\begin{aligned} &(\lambda, \alpha, \beta, \delta, \mu) \rightsquigarrow (\lambda, \alpha, 0, \delta, \mu_2), \\ &(\lambda, \alpha, 0, \delta, \mu_2) \rightsquigarrow (\lambda', \alpha', 0, \delta, \mu_3), \\ \text{and} \qquad &(\lambda', \alpha', 0, \delta, \mu_3) \rightsquigarrow (\lambda', \alpha', \beta', \delta, \mu'). \end{aligned}$$

In the second step the parameter μ does not change by what we have proved for case 1 above, so $\mu_2 = \mu_3$. If we can, in addition, show that setting the parameter β to zero does not change μ—this is the statement in case 2—, then we also have $\mu = \mu_2$ and, by symmetry, $\mu_3 = \mu'$. So indeed the situation considered in case 2 above is sufficiently general.

Now we prove the statement for case 2. We have

$$\begin{aligned} \int x(1-k(x))K(dx) &= \int x(1-k(x))K(dx) \\ &= \int x\Big(\frac{1}{x^2}\rho_\beta(x) - \frac{1}{x^2}\rho_{\beta=0}(x)\Big)dx \\ &= \int \frac{1}{x}\Big(\rho_\beta(x) - \rho_{\beta=0}(x)\Big)dx \\ &= -\int \frac{1}{2\pi}\int \frac{1}{i}\cdot\frac{i}{x}\cdot e^{\frac{x}{i}u}\cdot\frac{d}{du}\Big(\frac{{\chi_\beta}'(u)}{\chi_\beta(u)} - \frac{{\chi_{\beta=0}}'(u)}{\chi_{\beta=0}(u)}\Big)du\ dx \\ &= \frac{i}{2\pi}\int\int \frac{i}{x}\cdot e^{\frac{x}{i}u}\cdot\frac{d}{du}\Big(\frac{{\chi_\beta}'(u)}{\chi_\beta(u)} - i\mu - \frac{{\chi_{\beta=0}}'(u)}{\chi_{\beta=0}(u)} + i\mu'\Big)du\ dx \\ &= \frac{i}{2\pi}\int\Big(\Big[\frac{i}{x}\cdot e^{\frac{x}{i}u}G(u)\Big]_{u=-\infty}^{u=\infty} - \int e^{\frac{x}{i}u}G(u)du\Big)dx, \end{aligned} \tag{2.45}$$

with

$$G(u) := \frac{{\chi_\beta}'(u)}{\chi_\beta(u)} - i\mu - \frac{{\chi_{\beta=0}}'(u)}{\chi_{\beta=0}(u)} + i\mu' \tag{2.46}$$

By (2.14) and (2.18),

$$\begin{aligned}\frac{\chi_{\beta=0}'(u)}{\chi_{\beta=0}(u)} &= i\mu - \frac{\partial}{\partial u}\big(\delta\sqrt{\alpha^2+u^2}\big)\cdot\frac{K_{\lambda+1}(\delta\sqrt{\alpha^2+u^2})}{K_\lambda(\delta\sqrt{\alpha^2+u^2})}\\ &= i\mu - \frac{\delta u}{\sqrt{\alpha^2+u^2}}\cdot\frac{K_{\lambda+1}(\delta\sqrt{\alpha^2+u^2})}{K_\lambda(\delta\sqrt{\alpha^2+u^2})}\end{aligned} \tag{2.47}$$

Analogously one can derive the relation

$$\frac{\chi_\beta'(u)}{\chi_\beta(u)} = i\mu' + \frac{i\delta(\beta+iu)}{\sqrt{\alpha^2-(\beta+iu)^2}}\frac{K_{\lambda+1}(\delta\sqrt{\alpha^2-(\beta+iu)^2})}{K_\lambda(\delta\sqrt{\alpha^2-(\beta+iu)^2})}. \tag{2.48}$$

Using relations (2.47) and (2.48), we can rewrite the expression for $G(u)$ given in (2.46).

$$\begin{aligned}G(u) &= \Big(\frac{\chi_\beta'(u)}{\chi_\beta(u)} - i\mu\Big) - \Big(\frac{\chi_{\beta=0}'(u)}{\chi_{\beta=0}(u)} - i\mu'\Big)\\ &= \frac{\delta i(\beta+iu)}{\sqrt{\alpha^2-(\beta+iu)^2}}\cdot\frac{K_{\lambda+1}(\delta\sqrt{\alpha^2-(\beta+iu)^2})}{K_\lambda(\delta\sqrt{\alpha^2-(\beta+iu)^2})} - \frac{\delta u}{\sqrt{\alpha^2+u^2}}\cdot\frac{K_{\lambda+1}(\delta\sqrt{\alpha^2+u^2})}{K_\lambda(\delta\sqrt{\alpha^2+u^2})}\\ &= \frac{i\beta}{2}\cdot(\lambda+1/2)(\lambda+3/2)\cdot\frac{1}{u^2} + O\Big(\frac{1}{|u|^3}\Big) \qquad (|u|\to\infty).\end{aligned}$$

The last equality follows from the asymptotic expansion of the Bessel functions $K_{\lambda+1}$ and K_λ. Around $u=0$, $G(u)$ is bounded because it is a differentiable (and hence continuous) function. This, together with the limit behavior as $|u|\to\infty$, shows that in (2.45) the term in square brackets vanishes and the integral converges absolutely. Consequently we can continue the chain of equalities in (2.45).

$$\begin{aligned}\int x(1-k(x))K(dx) &= -\frac{i}{2\pi}\int\int e^{\frac{x}{i}u}G(u)du\,dx\\ &= -\frac{i}{2\pi}\int\int e^{\frac{x}{i}u}G(u)du\,e^{i\cdot 0\cdot x}dx\\ &= -\Big(\frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}\cdot\frac{K_{\lambda+1}(\delta\sqrt{\alpha^2+\beta^2})}{K_\lambda(\delta\sqrt{\alpha^2+\beta^2})} - \frac{\delta}{\alpha}\cdot\frac{K_{\lambda+1}(\delta\alpha)}{K_\lambda(\delta\alpha)}\Big)\end{aligned}$$

Here we have used the fact that the integration with respect to x can be interpreted as a Fourier integral taken at the point $u=0$. So since the function $G(u)$ is continuous, the Fourier transform of the inverse Fourier transform of $G(u)$ coincides with $G(u)$. Substituting this into (2.44) completes the proof. □

The main difficulty in the proof of Proposition 2.20 was to derive the local behavior of the generalized hyperbolic Lévy measure near $x=0$. For a distribution with a Lévy measure of a simpler structure, it is much easier to derive an analogous result. As an example, consider the class of CGMY Lévy processes. (See Section A.3.) For this class the Lévy measure is known explicitly and has a simple form.[9]

Proposition 2.21. *Let L be a CGMY(C,G,M,Y) Lévy process under the measure P. Then the following statements are equivalent.*

(i) There is a measure $Q \overset{\text{loc}}{\sim} P$ under which L is a CGMY(C',G',M',Y') Lévy process.

[9] In fact, the CGMY distributions are *defined* by giving their Lévy measure.

(ii) Either the CGMYparameters satisfy the relations $C' = C$ and $Y' = Y$, or $Y, Y' < 0$.

Proof. The *CGMY*(C, G, M, Y) Lévy measure has Lebesgue density

$$k_{CGMY}(x) = \frac{C}{|x|^{1+Y}} \exp\Big(\frac{G-M}{2}x - \frac{G+M}{2}|x|\Big). \tag{2.49}$$

(Cf. eq. (A.9).) Conditions *(i)*, *(ii)*, and *(iii)* of Proposition 2.19 are obviously satisfied if we set

$$k(x) := \frac{k_{CGMY(C',G',M',Y')}(x)}{k_{CGMY(C,G,M,Y)}(x)}.$$

Hence a measure $Q \overset{\text{loc}}{\sim} P$ with the desired property exists if and only if condition *(iv)* of Proposition 2.19 holds, that is, iff

$$\int \big(1 - \sqrt{k(x)}\big)^2 K(dx) < \infty.$$

Using the explicit form (2.49) of the Lévy density, this condition reads

$$\int \Big(\sqrt{\frac{C'}{|x|^{1+Y'}} \exp\Big(\frac{G'-M'}{2}x - \frac{G'+M'}{2}|x|\Big)} - \sqrt{\frac{C}{|x|^{1+Y}} \exp\Big(\frac{G-M}{2}x - \frac{G+M}{2}|x|\Big)}\Big)^2 dx < \infty.$$

It is easy to see that this condition holds if $C = C'$ and $Y = Y'$. On the other hand, it does not hold if any of these equalities is violated: First, it is clear that the finiteness of the integral depends only on the local behavior of the integrand around $x = 0$. Since the exponential factor tends to 1 for $x \to 0$, we can skip it for this discussion. So all amounts to the question whether

$$\int_{-1}^{1} \Big(\frac{\sqrt{C'}}{|x|^{(1+Y')/2}} - \frac{\sqrt{C}}{|x|^{(1+Y)/2}}\Big)^2 dx < \infty. \tag{2.50}$$

For $Y, Y' < 0$, this integral is obviously finite. If any of the relations $Y \geq 0$, $Y' \geq 0$ holds, then the integral can only be finite if $C = C'$ and $Y = Y'$, as was stated above:

- If $0 \leq Y'$, $Y < Y'$, then for $|x| \to 0$ the first summand in (2.50) grows faster than the second. Therefore we have

  $$\frac{1}{2} \cdot \frac{\sqrt{C'}}{|x|^{(1+Y')/2}} \geq \frac{\sqrt{C}}{|x|^{(1+Y)/2}}$$

  for $|x| \leq \epsilon$, $\epsilon \in (0, 1)$ small enough. Hence we can estimate

  $$\int_{-1}^{1} \Big(\frac{\sqrt{C'}}{|x|^{(1+Y')/2}} - \frac{\sqrt{C}}{|x|^{(1+Y)/2}}\Big)^2 dx \geq \int_{-\epsilon}^{\epsilon} \Big(\frac{1}{2} \cdot \frac{\sqrt{C'}}{|x|^{(1+Y')/2}}\Big)^2 dx = \frac{C'}{4} \int_{-\epsilon}^{\epsilon} \frac{1}{|x|^{1+Y'}} dx = \infty.$$

  The case $0 \leq Y$, $Y' < Y$ may be treated analogously.

- If $0 \leq Y' = Y$, but $C' \neq C$, then

  $$\int_{-1}^{1} \Big(\frac{\sqrt{C'}}{|x|^{(1+Y')/2}} - \frac{\sqrt{C}}{|x|^{(1+Y)/2}}\Big)^2 dx = \big(\sqrt{C'} - \sqrt{C}\big)^2 \int_{-1}^{1} \frac{1}{|x|^{1+Y}} dx = \infty.$$

$\square$

2.6 The GH Parameters δ and μ as Path Properties

In the preceding section, we have seen that the distributions of two generalized hyperbolic Lévy processes are (locally) equivalent if and only if the parameters satisfy $\delta = \delta'$ and $\mu = \mu'$. This suggests that the parameters δ and μ should be determinable from properties of a typical path of a generalized hyperbolic Lévy process. Indeed, this is the case here, as we will show below. Moreover, we present methods with which one can—at least in principle—determine the parameters δ and μ by inspecting a typical path of the Lévy process. This yields a converse statement to the property of absolute continuity mentioned above: Since we can determine δ and μ from the restriction of every path to a finite interval, these parameters cannot change during a change of measure. For the distributions (on the Skorokhod space $\mathbb{D}$ of càdlàg functions) of two generalized hyperbolic Lévy processes with different parameters δ or μ this implies the following. The restrictions of these distributions to every finite time interval are singular.[10]

2.6.1 Determination of δ

For a Lévy process with very many small jumps whose Lévy measure behaves like a/x^2 around the origin, the constant factor a can be determined by counting the small jumps of an arbitrary path. More precisely, we have the following proposition.

Proposition 2.22. *Let X be a Lévy process with a finite second moment such that the Lévy measure has a density $\rho(x)$ with the asymptotic behavior $\rho(x) = a/x^2 + o(1/x^2)$ as $x \downarrow 0$. Fix an arbitrary time $t > 0$ and consider the sequence of random variables*

$$Y_n := \frac{1}{t} \cdot \#\Big\{ s \le t : \Delta X_s \in \Big[\frac{1}{n+1}, \frac{1}{n}\Big)\Big\}.$$

Then with probability one the sequence $(S_k)_{k\ge 1}$ with

$$S_k := \frac{1}{k}\sum_{n=1}^{k} Y_n = \frac{1}{k}\#\Big\{ s \le t : \Delta X_s \in \Big[\frac{1}{k+1}, 1\Big)\Big\}, \quad k \ge 1, \tag{2.51}$$

converges to the value a.

Proof. The random measure of jumps μ^X of the process X is defined by

$$\mu^X(\omega, A) := \#\Big\{ s \le t : (s, \Delta X_s) \in A\Big\}$$

for any measurable set $A \subset \mathbb{R}_+ \times \mathbb{R}$. We have

$$Y_n = \frac{1}{t}\mu^X\left(\omega; [0,t] \times \Big[\frac{1}{n+1}, \frac{1}{n}\Big)\right) = \frac{1}{t}\mathbb{1}_{[0,t]\times[1/(n+1),1/n)} * \mu^X,$$

where the star denotes the integral with respect to a random measure. (See Jacod and Shiryaev (1987), II.1.5, for a definition.)

By Jacod and Shiryaev (1987), Corollary II.4.19 and Theorem II.4.8, the fact that X is a process with independent and stationary increments implies that μ is a Poisson random measure. Jacod and Shiryaev

[10] Of course, the distributions itself are also singular, but this is a trivial consequence of Jacod and Shiryaev (1987), Theorem IV.4.39a.

(1987), II.4.10 yields that any for any finite family $(A_i)_{1\le i\le d}$ of pairwise disjoint, measurable sets $A_i \subset \mathbb{R}_+ \times \mathbb{R}$ the random variables $\mu^X(\omega, A_i), 1 \le i \le d$, are independent. In particular, the random variables

$$Y_n = \frac{1}{t} 1\!\!1_{[0,t]\times[1/(n+1),1/n)} * \mu^X, \qquad n \ge 1$$

form an independent family. By the definition of the compensator ν of the random measure μ^X (cf. Jacod and Shiryaev (1987), Theorem II.1.8), we have

$$E[Y_n] = \frac{1}{t} 1\!\!1_{[0,t]\times[1/(n+1),1/n)} * \nu, \qquad n \ge 1.$$

Jacod and Shiryaev (1987), Corollary II.4.19, yields that ν can be chosen deterministic, with $\nu(dt, dx) = dt\ K(dx)$, where $K(dx)$ is the Lévy measure of the process X. Hence

$$\begin{aligned} E\left[Y_n\right] &= \int_{[1/(n+1),1/n)} 1\ K(dx) \\ &= \int_{1/(n+1)}^{1/n} \left(\frac{a}{x^2} + o\Big(\frac{1}{x^2}\Big)\right) dx = a + o(1) \qquad \text{as } n \to \infty. \end{aligned}$$

Furthermore, we have

$$\begin{aligned} \mathrm{Var}(Y_n) &= E\left[(Y_n - E[Y_n])^2\right] \\ &= \frac{1}{t^2} E\left[1\!\!1_{[0,t]\times[1/(n+1),1/n)} * (\mu^X - \nu)_t\right] \\ &= \frac{1}{t}\int_{1/(n+1)}^{1/n} 1\ K(dx) = \frac{1}{t}\big(a + o(1)\big). \end{aligned}$$

Therefore the sequence $(Y_n - E[Y_n])_{n\ge1}$ satisfies Kolmogorov's criterion for the strong law of large numbers (cf. Billingsley (1979), Theorem 22.4.) Hence we conclude that with probability 1 we have $S_k \to a$. $\square$

Remark: Obviously, an analogous result holds if one considers the behavior of the density $K(dx)/dx$ as $x \uparrow 0$ instead of $x \downarrow 0$.

Corollary 2.23. *Consider a generalized hyperbolic Lévy process X with parameters $(\lambda, \alpha, \beta, \delta, \mu)$. Then with probability 1 the re-normed number of jumps*

$$N_n = \frac{1}{nt} \# \Big\{ s \le t : \Delta X_s \ge 1/n \Big\}$$

converges to the value δ/π.

Proof. Since we always assume that a Lévy process has càdlàg paths, the number of jumps larger than 1 is finite for each path. Hence the sequence N_n and the sequence S_n from Proposition 2.23 converge to the same limit, viz the coefficient a of $1/x^2$ in the Lévy density. By Proposition 2.18, we have $a = \delta/\pi$. $\square$

2.6.2 Determination of μ

In the preceding subsection we have seen how the parameter δ can be derived from almost every path of a generalized hyperbolic Lévy motion. The key idea was to count the small jumps of the path. In the current subsection, we will show how the drift parameter μ can be derived from an arbitrarily short section of almost every path. Note that this is completely different from the case of a Brownian motion, where the drift coefficient can only be "seen" by observing the whole path.

Proposition 2.24. *Let X be a generalized hyperbolic Lévy process with parameters $(\lambda, \alpha, \beta, \delta, \mu)$. Fix an arbitrary time $t > 0$. Then with probability 1 the random variables*

$$Y_n := X_t - \sum_{0 \le s \le t} \Delta X_s 1\!\!1_{|\Delta X_s| \ge 1/n}$$

converge to the limit $\mu \cdot t$ as $n \to \infty$.

Remark: Note that in spite of the similar notation, the GH parameter μ and the random measure μ^X are completely different concepts. But since both notations are standard, we do not consider it useful to change any of them.

Proof. First we note that it suffices to consider the case $\beta = 0$: Assume that the statement is proved for this special case. Then consider a general parameter vector $(\lambda, \alpha, \beta, \delta, \mu)$. By Proposition 2.20, we can change the underlying probability measure P to an equivalent probability measure P' such that only the parameter β changes, with the new parameter $\beta' = 0$. Since we have assumed that the statement is proven for the case $\beta = 0$, we then have $Y_n \to \mu \cdot t$ P'-a.s.. Obviously this implies $Y_n \to \mu \cdot t$ P-a.s., so it is indeed sufficient to consider the symmetric case.

Since X_1 possesses a finite first moment, by Wolfe (1971) we have $(x^2 \wedge |x|) * \nu_t < \infty$. So Jacod and Shiryaev (1987), Proposition II.2.29 a yields that X is a special semimartingale. Therefore we can decompose the generalized hyperbolic Lévy process according to Jacod and Shiryaev (1987), Corollary II.2.38:

$$X_t = X_0 + X_t^c + x * (\mu^X - \nu)_t + A_t = x * (\mu^X - \nu)_t + \mu \cdot t.$$

So

$$\begin{aligned} Y_n &= X_t - (x 1\!\!1_{|x| \ge 1/n}) * \mu_t^X \\ &= (x 1\!\!1_{|x| < 1/n}) * (\mu^X - \nu)_t + (x 1\!\!1_{|x| \ge 1/n}) * (\mu^X - \nu)_t + \mu \cdot t - (x 1\!\!1_{|x| \ge 1/n}) * \mu_t^X \\ &= (x 1\!\!1_{|x| < 1/n}) * (\mu^X - \nu)_t - (x 1\!\!1_{|x| \ge 1/n}) * \nu + \mu \cdot t \\ &= (x 1\!\!1_{|x| < 1/n}) * (\mu^X - \nu)_t + \mu \cdot t, \end{aligned}$$

where the last equality holds because $(x 1\!\!1_{|x| \ge 1/n}) * \nu = 0$ by symmetry of the Lévy measure. The process $(x 1\!\!1_{|x| < 1/n}) * (\mu^X - \nu)_t,\ t \in \mathbb{R}_+$, is a martingale by Jacod and Shiryaev (1987), Theorem II.1.33. Since this martingale starts in 0 at $t = 0$, we have $E[Y_n] = \mu \cdot t$. Furthermore, still by Theorem II.1.33 we know that

$$\mathrm{Var}(Y_n) = E\Big[\Big((x 1\!\!1_{|x| < 1/n}) * (\mu^X - \nu)_t\Big)^2\Big] = (x^2 1\!\!1_{|x| < 1/n}) * \nu_t.$$

Since $\nu(dt, dx) = dt \times K(dx)$ with $\int_{[-1,1]} x^2 K(dx) < \infty$, the last term above tends to zero as $n \to \infty$. Hence the sequence $(Y_n)_{n \ge 1}$ converges to $\mu \cdot t$ in L^2 and a fortiori in probability. It remains to show that convergence indeed takes place with probability one. To this end, observe that the sequence $(Y_{-n})_{n\in-\mathbb{N}}$ is a martingale: Because μ^X is a Poisson random measure, we have that $Y_n - Y_{n+1} = (x1\!\!1_{\{1/(n+1)\le|x|<1/n\}}) * (\mu^X - \nu)_t$ is independent of Y_{n+1}. Furthermore, $E[Y_n - Y_{n+1}] = 0$.

Doob's second convergence theorem (see Bauer (1991), Corollary 19.10), yields that the martingale $(Y_{-n})_{n\in-\mathbb{N}}$ (and hence the sequence $(Y_n)_{n\in\mathbb{N}}$) converges with probability one. □

2.6.3 Implications and Visualization

Corollary 2.23 and Proposition 2.24 give rise to two simple methods for determining the generalized hyperbolic parameters δ and μ, respectively, from inspection of a typical path over a time interval of finite length. For clarity, we assume that the interval has unit length and starts at time 0.

- δ is the limit of the number of jumps larger than $1/n$, multiplied by π/n.
- μ is the limit of the increment $X_{t+1} - X_t$ minus all jumps of magnitude larger than $1/n$, as one lets $n \to \infty$.

So by counting jumps and summing up the jump heights one can determine the two parameters δ and μ. One could say that the parameters δ and μ are imprinted on almost every path of the generalized hyperbolic Lévy process, in the same way as the volatility is imprinted on almost any path of a Brownian motion. Remarkably, the drift parameter μ is a path property for the GH Lévy motion, but not for the Brownian motion.

In what follows, we give an example to illustrate the methods for the determination of δ and μ from a path of a generalized hyperbolic Lévy motion. In order to consider realistic parameters, we estimate the parameters of a NIG distribution from observed log returns on the German stock index DAX. This yields the following parameters.

α	β	δ	μ
99.4	-1.79	0.0110	0.000459

We generate approximations to the sample paths of a NIG Lévy motion by the compound-Poisson approach. Let $K(dx)$ denote the Lévy measure of the NIG distribution of L_1. Given a boundary $\epsilon > 0$, we simulate a compound Poisson process that has only jumps with jump heights $\ge \epsilon$. The jump intensity $I^{(\epsilon)}$ of this process is determined by the measure of the set $\{|x| \ge \epsilon\}$:

$$I^{(\epsilon)} := K\big((-\infty, -\epsilon] \cup [\epsilon, \infty)\big).$$

Given that a path $L_\cdot(\omega)$ of L jumps at time t, the jump height has the distribution

$$K^{(\epsilon)}(dx) := \frac{1}{I^{(\epsilon)}} K\big(dx \cap \big((-\infty, -\epsilon] \cup [\epsilon, \infty)\big)\big).$$

Denoting this compound Poisson process by $N^{(\epsilon)}$, the NIG Lévy process is approximated by

$$L_t^{(\epsilon)} := \mu t + N_t^{(\epsilon)}.$$

The NIG parameter μ enters only by the drift term μt, and the three parameters α, β, and δ enter by the compound Poisson process $N^{(\epsilon)}$.

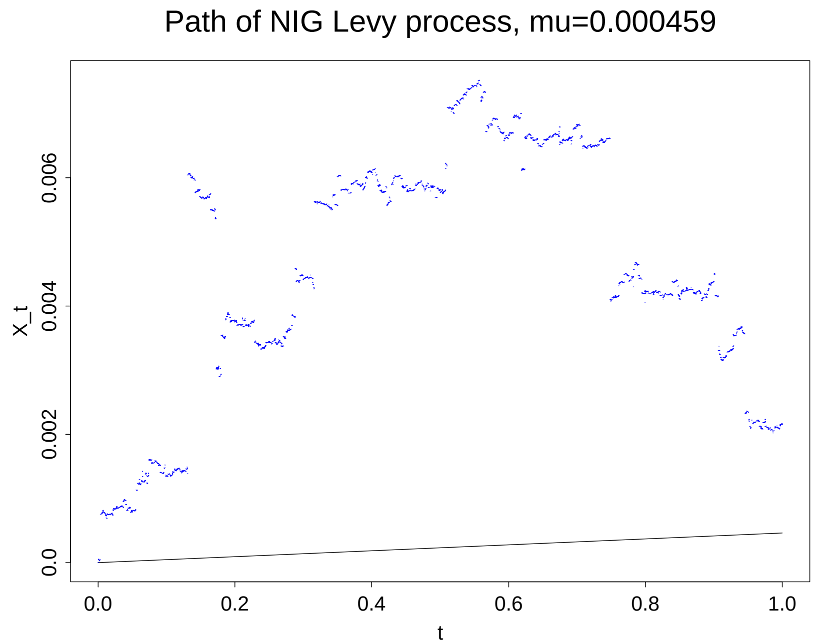

Figure 2.4: Sample path of NIG Lévy motion, determined by compound Poisson approximation $L^{(\epsilon)}$ with $\epsilon = 10^{-8}$. The line gives the drift component μt.

Since the density of the NIG Lévy measure is known in closed form (see equation (2.37),) simulation of the approximating Lévy process $L^{(\epsilon)}$ is straightforward: Given a time horizon T, independent exponentially distributed random variates τ_i $(i = 1, \dots, N)$ with parameter $I^{(\epsilon)}$ are generated. The number N of variates is determined by the condition that $\sum_{i=1}^{N-1} \tau_i < T \leq \sum_{i=1}^{N} \tau_i$. For $i = 1, \dots, N-1$, the value τ_i is the waiting time between the $(i-1)$-th and the i-th jump. Then the $N-1$ jump heights are generated by inserting iid $U(0,1)$-distributed (pseudo-)random variables into the inverse of the cumulative distribution function of $K^{(\epsilon)}$. This inverse has to be determined numerically from the density of $K^{(\epsilon)}(dx)$.

Figure 2.4 shows a sample path of the process $L^{(\epsilon)}$, which we take to be a sample path of the NIG Lévy motion L itself. At $\epsilon = 10^{-8}$, such a path has around 700,000 jumps on the interval $[0,1]$.

Figure 2.5 shows how the normed jump count S_k defined in subsection 2.6.1, equation (2.51) converges against the value δ/π. For the x-axis, we have chosen a log scale (with basis 10). x denotes the lower boundary $1/(k+1)$. That is, at $x = 10^{-5}$ we give the value S_{10^5}.

To illustrate the determination of μ by the method described above, we plot again the path given in Figure 2.4. But this time, we subtract the process of jumps with magnitude greater than 10^{-5} (Fig. 2.6) respectively greater than 10^{-7} (Fig. 2.7). The convergence of the path against a straight line with slope μ is obvious.

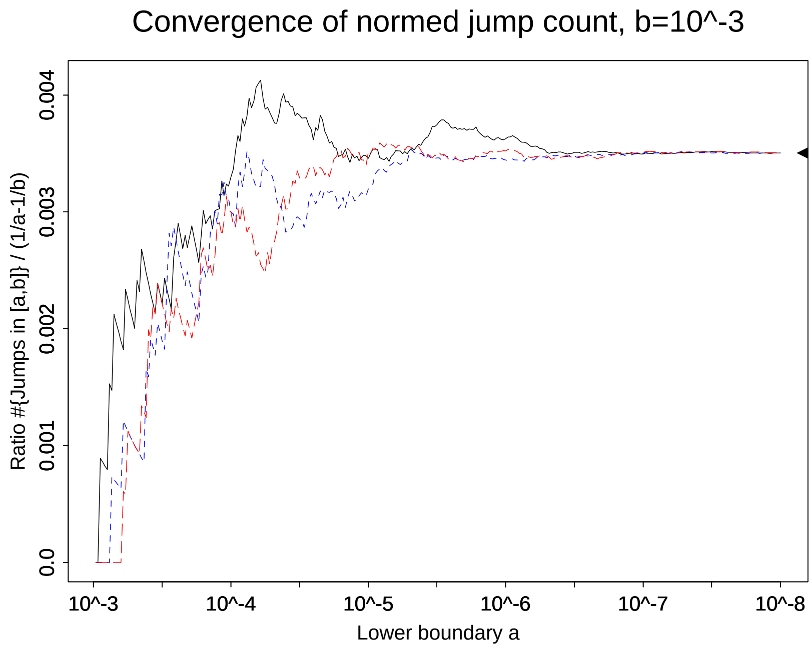

Figure 2.5: Convergence of normed jump count against the true value $\delta/\pi = 0.00350$ (marked on the right side of the plot). The three curves represent three different paths.

2.7 Implications for Option Pricing

In stock price models driven by generalized hyperbolic Lévy processes, one models the stock price as

$$S_t = S_0 e^{rt+L_t}, \tag{2.52}$$

where L is a generalized hyperbolic Lévy process. (See Chapter 1.)

For option pricing, one changes the probability measure such that the discounted stock price process $S_t^* := e^{-rt}S_t$ becomes a martingale. Then one prices European options by calculating the conditional expectations of discounted payoffs. In incomplete models, there are usually many different equivalent probability measures turning the discounted stock price process into a martingale. In general this leads to many different possible prices for European options. By Eberlein and Jacod (1997b), for infinite-variation stock price models of the form (2.52), the range of call option prices that can be calculated this way is the whole no-arbitrage interval.[11]

If one applies the pricing method described above to the generalized hyperbolic stock price model, the distribution of the generalized hyperbolic Lévy process changes with the change of probability. In particular, with the changes of probability used in Eberlein and Jacod (1997b), the process does not stay a generalized hyperbolic Lévy process. One might ask if one can narrow the range of option prices by imposing the additional condition that the process L is again a generalized hyperbolic Lévy process under the transformed measure. But in Proposition 2.28 we will show that even with this severe restriction, the

[11]The boundaries of this interval are given by the following condition. If the price lies beyond either of these boundaries, there is a simple buy/sell-and-hold strategy that allows a riskless arbitrage.

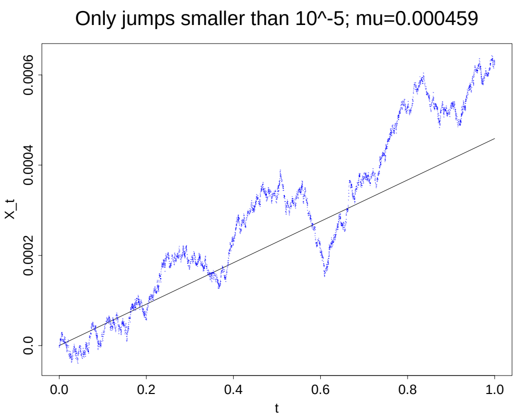

Figure 2.6: Path of NIG Lévy motion minus jumps of magnitude $> 10^{-5}$.

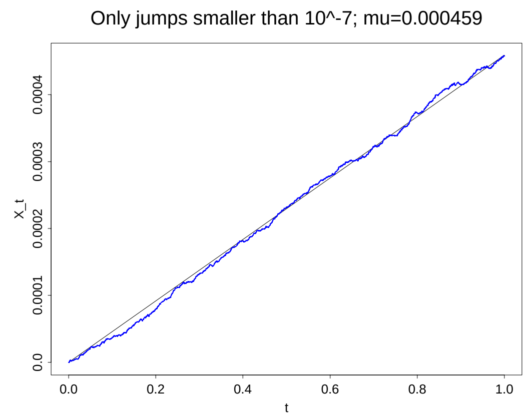

Figure 2.7: Path of NIG Lévy motion minus jumps of magnitude $> 10^{-7}$.

range of possible option prices does not shrink. Before we can prove this, we have to prove the following proposition.[12] We consider functions $g(x)$ that satisfy the following conditions.[13]

$$0 \le g(x) < x \text{ for } x > 0, \qquad \frac{g(x)}{x} \to 1 \text{ as } x \to \infty, \qquad g \text{ convex.} \tag{2.53}$$

Proposition 2.25. *Let (π_n) be a sequence of probability measures on $(\mathbb{R}, \mathcal{B}^1)$ with $\pi_n((-\infty, 0]) = 0$ for all n. Assume that there is a constant $c < \infty$ such that*

$$\int x \, \pi_n(dx) = c \qquad \text{for all } n \ge 1.$$

(a) If $\pi_n \to \pi$ weakly for some probability measure π on $(\mathbb{R}, \mathcal{B}^1)$ satisfying $\int x \, \pi(dx) = c$, then

$$\int g(x) \, \pi_n(dx) \to \int g(x) \, \pi(dx)$$

for any function $g(x)$ satisfying (2.53).

(b) The following conditions are equivalent.

(i) $\int x^d \, \pi_n(dx) \to 0$ for all $d \in (0, 1)$.
(ii) $\int x^d \, \pi_n(dx) \to 0$ for some $d \in (0, 1)$.
(iii) (π_n) converges weakly to the Dirac measure δ_0.
(iv) $\int g(x) \, \pi_n(dx) \to c$ for all g with (2.53).
(v) $\int g(x) \, \pi_n(dx) \to c$ for some g with (2.53).

(c) Furthermore, there is equivalence between the following conditions.

[12] Points (b) [(iii) and (iv)] and part (c) [(i) and (ii)] are essentially contained in Frey and Sin (1999), Proposition 2.3 or Eberlein and Jacod (1997a), Theorem 1-1. The former source considers only call payoff functions.

[13] Up to a constant factor, $g(S_T)$ will be the payoff of the option at expiration. The class of payoff functions covered here is the same that was used in Eberlein and Jacod (1997b).

(i) (π_n) *converges weakly to the Dirac measure* δ_c.

(ii) $\int g(x)\ \pi_n(dx) \to g(c)$ *for all functions* $g(x)$ *with (2.53).*

(iii) $\int g(x)\ \pi_n(dx) \to g(c)$ *for some function* $g(x)$ *with (2.53) that satisfies* $g(c) = \alpha c + \beta$ *and* $g(x) > \alpha x + \beta$ *for* $x \neq c$*, where* α *and* β *are real constants.*

(d) If $\int \ln x\ \pi_n(dx) \to \ln c$*, then* (π_n) *converges weakly to the Dirac measure* δ_c.

Remark: When applying this proposition to mathematical finance, the measures π_n, the constant c, and the function $g(x)$ will have the following significance:

π_n	is the distribution of the stock price at expiration, S_T, under some martingale measure.
c	is the expectation of S_T under any of those martingale measures. That is, $c = e^{rT} S_0$.
$g(x)$	is the *payoff function* of the option. That is, the option pays an amount of $g(S_T)$ at time T. For a European call option, this means $g(x) = (x - K)^+$, where $K > 0$ is the strike price. Obviously, this payoff function satisfies condition (2.53).
$\int g(x)\ \pi_n(dx)$	is the expected payoff of the option, calculated under some martingale measure. The option price would be the *discounted* expected payoff, that is, $e^{-rT} \int g(x)\ \pi_n(dx)$.

Proof of Proposition 2.25. We recall the following definition from Chow and Teicher (1978), Sec. 8.1, p. 253.

Definition 2.26. *If* $(G_n)_{n \geq 1}$ *is a sequence of distribution functions on* $\mathbb{R}$*, and* g *is a real, continuous function on* $(-\infty, \infty)$*, the* g *is called* uniformly integrable relative to (G_n) *if*

$$\sup_{n \geq 1} \int_{\{|y| \geq a\}} \big|g(y)\big|\ dG_n(y) = o(1) \quad \textit{as } a \to \infty.$$

Note that—unlike the usual notion of uniform integrability of a sequence of functions with respect to a fixed probability measure—here the function is fixed and a sequence of probability measures is considered.

Obviously, uniform integrability of a function g implies uniform integrability of all real, continuous functions f with $|f| \leq |g|$.

Uniform integrability is tightly linked with the convergence of integrals under weak convergence of the integrating measures. This is shown in the following theorem, which we cite from Chow and Teicher (1978), Sec. 8.1, Theorem 2.

Theorem 2.27. *If* $(G_n)_{n \geq 1}$ *is a sequence of distribution functions on* $\mathbb{R}$ *with* $G_n \to G$ *weakly, and* g *is a nonnegative, continuous function on* $(-\infty, \infty)$ *for which* $\int_{-\infty}^{\infty} g\ dG_n < \infty$, $n \geq 1$*, then*

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} g\ dG_n = \int_{-\infty}^{\infty} g\ dG < \infty$$

if and only if g *is uniformly integrable with respect to* (G_n).

Remark: We will be concerned with functions defined on the nonnegative real axis $\mathbb{R}_+$. Obviously, Theorem 2.27 holds in this context as well if the function g satisfies the additional condition $g(0) = 0$. Then one can extend it continuously to the negative real axis by setting $g(x) := 0, \quad x < 0$.

Now we can start to prove Proposition 2.25.

Part (a). Since $\int x \ \pi_n(dx) = c$ for all n and $\int x \ \pi(dx) = c$, we trivially have the convergence $\int x \ \pi_n(dx) \to \int x \ \pi(dx)$. Together with the weak convergence $\pi_n \to \pi$, this implies the uniform integrability of x with respect to the sequence (π_n). (See Theorem 2.27.) The boundedness condition $0 \leq g(x) < x$ thus implies the uniform integrability of the function $g(x)$ with respect to the same sequence. Another application of Theorem 2.27 yields the desired convergence.

Part (b). $(i) \Rightarrow (ii)$ is trivial.

$(ii) \Rightarrow (iii)$: We have

$$\pi_n(\{|x-c| > \epsilon\}) = \int \mathbb{1}_{\{x>\epsilon\}} \pi_n(dx) \leq \frac{1}{\epsilon^d} \int x^d \pi_n(dx) \to 0,$$

which implies weak convergence.

$(iii) \Rightarrow (i)$: For any fixed $d \in (0,1)$ and $x_0 > 0$ we have

$$\int x^d \mathbb{1}_{\{x>x_0\}} \pi_n(dx) \leq \int \frac{x}{x_0^{1-d}} \mathbb{1}_{\{x>x_0\}} \pi_n(dx) \leq \frac{c}{x_0^{1-d}}.$$

The last expression tends to 0 as $x_0 \to \infty$, and so the function $x \mapsto x^d$ is uniformly integrable with respect to the sequence (π_n). By Theorem 2.27, weak convergence $\pi_n \to \delta_0$ then implies $\int x^d \pi_n(dx) \to 0$.

$(iii) \Rightarrow (iv)$: Fix an arbitrary function $g(x)$ satisfying (2.53). We have

$$c - \int g(x) \ \pi_n(dx) = \int (x - g(x)) \ \pi_n(dx).$$

The function $x \mapsto x - g(x)$ is nonnegative and continuous.[14] In addition, it is uniformly integrable with respect to the sequence (π_n): Because $g(x)/x \to 1$ as $x \to \infty$, for any $\epsilon > 0$ there exists a value $x_\epsilon < \infty$ such that $x - g(x) \leq \epsilon x$ as $x \geq x_\epsilon$. So

$$\int (x - g(x)) \mathbb{1}_{\{x>x_\epsilon\}} \ \pi_n(dx) \leq \int \epsilon x \ \pi_n(dx) \leq \epsilon \int x \ \pi_n(dx) = \epsilon c,$$

which implies uniform integrability. Hence, again by Theorem 2.27, weak convergence $\pi_n \to \delta_0$ implies convergence of the expectations. Thus

$$c - \int g(x) \ \pi_n(dx) = \int (x - g(x)) \ \pi_n(dx) \to \int (x - g(x)) \ \delta_0(dx) = 0.$$

$(iv) \Rightarrow (v)$: Trivial.

$(v) \Rightarrow (iii)$: Convexity of $g(x)$ together with $0 \leq g(x) < x$ implies that $x \mapsto x - g(x)$ is non-decreasing and strictly positive. Hence for any $\epsilon > 0$ we have

$$\pi_n(\{x \geq \epsilon\}) = \int \mathbb{1}_{\{x\geq\epsilon\}} \ \pi_n(dx) \leq \int \frac{x - g(x)}{\epsilon - g(\epsilon)} \ \pi_n(dx) = \frac{1}{\epsilon - g(\epsilon)} \int (x - g(x)) \ \pi_n(dx)$$

[14] Remember that we extend all functions by setting them equal to zero on the negative real axis.

By assumption, for fixed $\epsilon > 0$ the last expression tends to 0 as $n \to \infty$. Weak convergence $\pi_n \to \delta_0$ follows.

Proof of part (c). $(i) \Rightarrow (ii)$: Since $\int x \; \delta_c(dx) = c$, we can apply part (a). This yields

$$\int g(x) \; \pi_n(dx) \to \int g(x) \; \delta_c(dx) = g(c).$$

$(ii) \Rightarrow (iii)$: Trivial.

$(iii) \Rightarrow (i)$: By assumption, we have

$$\int (\alpha x + \beta) \; \pi_n(dx) = \alpha \int x \; \pi_n(dx) + \beta = \alpha c + \beta = g(c), \quad \text{for all } n \geq 1.$$

Hence $\int g(x) \; \pi_n(dx) \to g(c)$ implies

$$\int \big(g(x) - (\alpha x + \beta)\big)\pi_n(dx) \to 0 \qquad \text{as } n \to \infty.$$

Because $g(x) - (\alpha x + \beta) > 0$ for $x \neq c$, and because this function is convex, for each $\epsilon \in (0, 1)$ there is a constant C_ϵ such that

$$1\!\!1_{(0,c(1-\epsilon)] \cup [c(1+\epsilon),\infty)}(x) \leq C_\epsilon \cdot \big(g(x) - (\alpha x + \beta)\big) \qquad \text{for } x \in (0, \infty).$$

This implies that for every fixed $\epsilon > 0$, $\pi_n((0, c(1-\epsilon)] \cup [c(1+\epsilon, \infty)) \to 0$ as $n \to \infty$, and hence the weak convergence $\pi_n \to \delta_c$.

Proof of part (d). By our standing assumption (2.4) concerning the sequence (π_n), we have $\int \big(1 - x/c\big) \; \pi_n(dx) = 0$. The function $x \mapsto \ln(x/c) + 1 - x/c$ is strictly convex and takes on its minimum value 0 at $x = c$. Hence for any $\epsilon > 0$ there is a constant $C_\epsilon < \infty$ such that $1\!\!1_{\{|x-c| \geq \epsilon c\}} \leq C_\epsilon \cdot (\ln(x/c) + 1 - x/c)$ for $x > 0$. Consequently

$$\begin{aligned}
\pi_n\big(\{|x - c| \geq \epsilon c\}\big) &= \int 1\!\!1_{\{|x-c| \geq \epsilon c\}} \pi_n(dx) \\
&\leq C_\epsilon \cdot \int \big(\ln(x/c) + 1 - x/c\big)\pi_n(dx) \\
&= C_\epsilon \cdot \int \big(\ln(x/c)\big) \; \pi_n(dx) + C_\epsilon \cdot \int \big(1 - x/c\big) \; \pi_n(dx) \\
&= C_\epsilon \cdot \Big(\int \ln x \; \pi_n(dx) - \ln c\Big),
\end{aligned}$$

which by assumption tends to 0 as $n \to \infty$ for fixed ϵ. $\square$

In what follows, we show that for generalized hyperbolic distributions the constraint that the generalized hyperbolic Lévy process should stay a generalized hyperbolic Lévy process under the martingale measure is not sufficient to make the interval of possible option prices smaller. The key to the proof will be to notice that by changing the basic probability measure we can transform a given generalized hyperbolic Lévy process into a generalized hyperbolic Lévy process with arbitrary values of α and β. If $\lambda > 0$, we can, for any given α, satisfy the martingale condition by choosing β in a suitable way. We will show that changing α continuously will result in a weakly continuous change of the distribution of L_t for arbitrary fixed t.

Proposition 2.28. *Fix arbitrary numbers $\lambda > 0$, $\delta > 0$, and $\mu \in \mathbb{R}$. Consider a convex function $g(x)$ on $(0, \infty)$ satisfying (2.53). Fix arbitrary constants $c, t \in (0, \infty)$. Then for each $p \in (g(c), c)$ we can find $\alpha > 0$ and β with $|\beta| < \alpha$ such that for the time-t member H^{*t} of the generalized hyperbolic convolution semigroup with parameters $(\lambda, \alpha, \beta, \delta, \mu)$, the following two conditions are satisfied.*

1. $\int e^x \; H^{*t}(dx) = c$ *and*
2. $\int g(x) \; H^{*t}(dx) = p$.

Before we prove the theorem, we want to highlight its significance for option pricing.

Corollary 2.29. *Let $S_t = S_0 \exp(rt + L_t)$ describe the evolution of a stock price, where L is a generalized hyperbolic Lévy process. Then we have the following.*

1. *If $\lambda > 0$, the range of possible call option prices is the whole no-arbitrage interval $((S_0 - e^{-rt}K)^+, S_0)$ even if one restricts the set of possible equivalent pricing measures to those measures that make $L_t := \ln(S_t) - rt$ again a generalized hyperbolic Lévy process with the same parameters λ, δ, and μ.*

2. *If $\lambda \leq 0$, then one has an analogous result, but one is only allowed to keep the parameters δ and μ fixed.*

Proof of Corollary 2.29. Part 1 follows at once from Propositions 2.28 and 2.20, since $E[e^{L_1}] = 1$ iff $S_0 e^{L_t}$ is a martingale. Part 2 is reduced to part 1 by first changing the parameter λ to a positive value, say $\lambda = 1$, using Proposition 2.20. □

Proof of Proposition 2.28. Since $\lambda > 0$, we can always satisfy condition 1 by varying the parameter β alone. This follows at once from Corollary 2.10 and Proposition 2.12.

Given α, the corresponding value $\beta = \beta(\alpha)$ is determined as the unique zero of the strictly monotonic function $\beta \mapsto e^{-r}\mathrm{mgf}_{(\alpha,\beta=0)}(\beta + 1) - \mathrm{mgf}_{(\alpha,\beta=0)}(\beta)$. We will now show that the mapping $\alpha \mapsto (\alpha, \beta(\alpha)) \mapsto \int (e^x - K)^+ \; GH_{(\lambda,\alpha,\beta(\alpha),\delta,\mu)}(dx)$ is continuous: Since $\mathrm{mgf}_{(\alpha,\beta=0)}(\beta)$ depends continuously on α, so does the solution $\beta(\alpha)$. By inspection of the characteristic function (2.9), one sees that for any sequence (α_n) with $\alpha_n > 1/2$ and $\alpha_n \to \alpha \in (1/2, \infty)$, the following convergence holds for the characteristic functions.

$$\chi_{(\lambda,\alpha_n,\beta(\alpha_n),\delta,\mu)}(u)^t \to \chi_{(\lambda,\alpha,\beta(\alpha),\delta,\mu)}(u)^t \qquad \text{for all } u \in \mathbb{R}.$$

(The exponent t of the characteristic function denotes that we consider the t-fold convolution.) By the Lévy continuity theorem, this implies weak convergence of the distributions. An application of Proposition 2.25 (a) yields

$$\int (e^x - K)^+ \; GH^{*t}_{(\lambda,\alpha_n,\beta(\alpha_n),\delta,\mu)}(dx) \to \int (e^x - K)^+ \; GH^{*t}_{(\lambda,\alpha,\beta(\alpha),\delta,\mu)}(dx).$$

By standard properties of continuous functions, the function

$$\alpha \mapsto \int (e^x - K)^+ \; GH^{*t}_{(\lambda,\alpha,\beta(\alpha),\delta,\mu)}(dx)$$

maps the interval $(1/2, \infty)$ onto an interval. The statement of the proposition follows if we can show that the image of $(1/2, \infty)$ contains values arbitrarily close to the boundaries of the interval $(g(c), c)$. More precisely, we will show the following: If one lets $\alpha \downarrow 1/2$, then the expectation tends to the upper boundary c. On the other hand, if $\alpha \uparrow \infty$ then the expectation tends to the lower boundary, $g(c)$.

The case $\alpha \downarrow 1/2$.
Since $\beta(\alpha) \in (-\alpha, \alpha)$ and $[0,1] \subset (-\alpha - \beta, \alpha - \beta)$, along with $\alpha \downarrow 1/2$ the corresponding values of $\beta(\alpha)$ have to tend to $-1/2$. By equation (2.9) and the remark following this equation, we have

$$\text{(2.54)} \qquad \operatorname{mgf}(d) = \chi(-id) = e^{\mu d t} \left(\frac{(\delta\sqrt{\alpha^2 - \beta(\alpha)^2})^\lambda}{K_\lambda(\delta\sqrt{\alpha^2 - \beta(\alpha)^2})} \cdot \frac{K_\lambda\big(\delta\sqrt{\alpha^2 - (\beta(\alpha)+d)^2}\big)}{\big(\delta\sqrt{\alpha^2 - (\beta(\alpha)+d)^2}\big)^\lambda} \right)^t.$$

We show that the moment generating function, taken at $d = 1/2$, tends to zero as $\alpha \downarrow 1/2$ and, consequently, $\beta(\alpha) \to -1/2$. By Abramowitz and Stegun (1968), equation 9.6.9, we have the following asymptotic relation for the Bessel function $K_\lambda(z)$.

$$K_\lambda(z) \sim \frac{\Gamma(\lambda)}{2}\Big(\frac{z}{2}\Big)^{-\lambda} \text{ and hence } \frac{z^\lambda}{K_\lambda(z)} \sim \frac{z^{2\lambda}}{2^{\lambda-1}\Gamma(\lambda)} \qquad (\lambda > 0 \text{ fixed, } z \to 0.)$$

Hence the first fraction in (2.54) tends to zero as $\alpha \downarrow 1/2, \quad \beta(\alpha) \to -1/2$.

$$\text{(2.55)} \qquad \frac{(\delta\sqrt{\alpha^2 - \beta(\alpha)^2})^\lambda}{K_\lambda(\delta\sqrt{\alpha^2 - \beta(\alpha)^2})} \sim \frac{(\delta\sqrt{\alpha^2 - \beta(\alpha)^2})^{2\lambda}}{2^{\lambda-1}\Gamma(\lambda)} \to 0 \qquad \text{as } \alpha \downarrow 1/2,$$

because $\delta\sqrt{\alpha^2 - \beta(\alpha)^2} \to 0$ $(\alpha \downarrow 0)$. For the second fraction in (2.54), we note that

$$\delta\sqrt{\alpha^2 - (\beta(\alpha)+d)^2} \to \delta\sqrt{d - d^2} \qquad \text{as } \alpha \downarrow 1/2.$$

Hence for $d = 1/2$ the second fraction in (2.54) tends to a finite constant.

$$\text{(2.56)} \qquad \frac{K_\lambda\big(\delta\sqrt{\alpha^2 - (\beta(\alpha)+1/2)^2}\big)}{\big(\delta\sqrt{\alpha^2 - (\beta(\alpha)+1/2)^2}\big)^\lambda} \to \frac{K_\lambda(\delta/4)}{(\delta/4)^\lambda} < \infty.$$

Taking together (2.55) and (2.56), we see that indeed the moment generating function (2.54), taken at $d = 1/2$, tends to 0 as $\alpha \downarrow 1/2$. By Proposition 2.25 (b), this is equivalent to saying that the expectation $\int g(e^x)\, GH^{*t}_{(\lambda,\alpha,\beta(\alpha),\delta,\mu)}(dx)$ tends to the upper boundary, c, of the interval given in the proposition.

The case $\alpha \uparrow \infty$.
First, we show that as $\alpha \uparrow \infty$, $\beta(\alpha) \to \infty$ in such a way that $\beta(\alpha)/\alpha$ tends to a value $\gamma^* \in (-1, 1)$.

The martingale condition 1 is equivalent to

$$\chi_{\beta(\alpha)}(-i) = c^{1/t},$$

where we have indicated the parameter $\beta(\alpha)$ as a subscript. Since changing $\beta(\alpha)$ corresponds to an Esscher transform, Lemma 2.6 yields the following equivalence.

$$\chi_{\beta(\alpha)}(-i) = c^{1/t} \iff e^{-(\ln c)/t}\chi_{\beta=0}(-i \cdot (\beta(\alpha)+1)) = \chi_{\beta=0}(-i \cdot \beta(\alpha)),$$

where $\chi_{\beta=0}$ denotes the characteristic function of the generalized hyperbolic distribution with parameter $\beta = 0$ and the other parameters unchanged. If we further change the parameter μ to the value $\widetilde{\mu} := \mu - (\ln c)/t$, then the condition above takes the form

$$\chi_{\beta=0,\mu=\widetilde{\mu}}(-i \cdot (\beta(\alpha)+1)) = \chi_{\beta=0,\mu=\widetilde{\mu}}(-i \cdot \beta(\alpha)). \tag{2.57}$$

Because of the complicated structure of the characteristic function $\chi_{\beta=0,\mu=\widetilde{\mu}}(u)$, it is difficult to analyze the properties of the function $\beta(\alpha)$ directly using (2.57). Therefore we consider a modification of condition (2.57). Since the moment generating function $u \mapsto \chi_{\beta=0,\mu=\widetilde{\mu}}(-iu)$ is strictly convex, relation (2.57) implies that the unique minimum of $\chi_{\beta=0,\mu=\widetilde{\mu}}(-iu)$ is attained for some $u^* \in (\beta(\alpha), \beta(\alpha)+1)$. As we will see, the quotient u^*/α converges to a limit in $(-1,1)$ as $\alpha \uparrow \infty$. (This implies the convergence of $\beta(\alpha)/\alpha$ to the same limit.)

We have

$$\frac{\frac{d}{du}\chi_{\beta=0,\mu=\widetilde{\mu}}(-iu)}{\chi_{\beta=0,\mu=\widetilde{\mu}}(-iu)} = \widetilde{\mu} + \frac{\delta u}{\sqrt{\alpha^2-u^2}} \cdot \frac{K_{\lambda+1}(\delta\sqrt{\alpha^2-u^2})}{K_\lambda(\delta\sqrt{\alpha^2-u^2})}. \tag{2.58}$$

Because of the strict convexity of the moment generating function, this function has only one zero, which is located at the minimum of the moment generating function. Denoting the position of the minimum by u again, we have

$$-\frac{\widetilde{\mu}}{\delta} = \frac{u}{\sqrt{\alpha^2-u^2}} \cdot \frac{K_{\lambda+1}(\delta\sqrt{\alpha^2-u^2})}{K_\lambda(\delta\sqrt{\alpha^2-u^2})}. \tag{2.59}$$

Obviously for $\widetilde{\mu} = 0$ the unique solution of this equation is $u = 0$, so we only have to study the cases $\widetilde{\mu} > 0$ and $\widetilde{\mu} < 0$. These can be treated analogously, and so we only consider the case $\widetilde{\mu} < 0$. It is clear that in this case the solution satisfies $u > 0$. Letting $\gamma = \frac{u}{\alpha} > 0$, condition (2.59) is equivalent to

$$-\frac{\widetilde{\mu}}{\delta} = \frac{1}{\sqrt{\frac{1}{\gamma^2}-1}} \cdot \frac{K_{\lambda+1}(\delta\alpha\sqrt{1-\gamma^2})}{K_\lambda(\delta\alpha\sqrt{1-\gamma^2})}. \tag{2.60}$$

The Bessel function quotient in this condition tends to 1, uniformly for γ from any fixed compact interval $I \subset (-1,1)$. This is clear from the asymptotic relation (2.25). Therefore it is easy to see that the solution γ of (2.60) tends to the solution γ^* of the equation

$$-\frac{\widetilde{\mu}}{\delta} = \frac{1}{\sqrt{\frac{1}{(\gamma^*)^2}-1}}, \text{ which is given by } \gamma^* = \frac{1}{\sqrt{\frac{\delta^2}{\widetilde{\mu}^2}+1}}. \tag{2.61}$$

So $\gamma^* \in (0,1)$, which had to be proved. (The analogous proof for the case $\widetilde{\mu} > 0$ would yield $\gamma^* \in (-1,0)$.)

Using these results we examine the behavior of the mean of the convoluted generalized hyperbolic distribution $\mathrm{GH}^{*t}_{(\lambda,\alpha,\beta(\alpha),\delta,\mu)}$ as $\alpha \uparrow \infty$. We show that the expectation tends to zero in this case.

By insertion of $u = \beta(\alpha)$ into the right-hand side of equation (2.58) we get an expression for the mean value of a generalized hyperbolic distribution with parameters α, $\beta(\alpha)$, and $\widetilde{\mu}$. As the quotient $\beta(\alpha)/\alpha$ tends to the limit γ^* solving (2.61), locally uniform convergence of the right-hand side of equation

(2.60) yields that the mean value of the distribution tends to zero. Consequently the mean value of the distribution with μ instead of $\widetilde{\mu}$ tends to $\ln c$. Now let π_n denote the law of e^x under the measure $\mathrm{GH}^{*t}_{(\lambda,\alpha=n,\beta(n),\delta,\mu)}(dx)$. Then $\int x\ \pi_n(dx) = c$. Proposition 2.25 (d) and (c) yields that the convergence $\int \ln x\ \pi_n(dx) \to \ln c$ implies convergence of $\int g(e^x)\ \mathrm{GH}^{*t}_{(\lambda,\alpha,\beta(\alpha),\delta,\mu)}(dx)$ to $g(c)$. $\square$

Chapter 3

Computation of European Option Prices Using Fast Fourier Transforms

3.1 Introduction

Suppose a stock price is modeled by a stochastic process S_t with discrete or continuous time parameter t. The evolution of the stock price is governed by the underlying probability measure P. As mentioned in Chapter 1, when pricing European options on the stock one usually looks for a so-called *equivalent martingale measure*, that is a probability measure $Q \stackrel{\text{loc}}{\sim} P$ such that $\exp\big(-\int_0^t r(s)ds\big)S_t$ is a Q-martingale. Then the option price is calculated as the Q-conditional expectation of the discounted payoff. Q is also called *risk-neutral measure*.

In general, there is more than one equivalent martingale measure. As mentioned in Chapter 1, this induces the problem of choosing one of these measures. Here we do not want to dive into the theory of choosing the martingale measure. Instead, we assume that the choice has happened, and that we are given the measure Q.

The remaining task for option pricing is the calculation of the expected final value of the option. This requires knowledge of the distribution of the stock price under the martingale measure Q. What is known in many cases is the characteristic function of the stock return to expiration. Exponential Lévy processes are a typical example. Here the logarithm of the stock price is assumed to follow a process with stationary independent increments. Consequently, the characteristic function of the n-period return is just the n-th power of the characteristic function of the one-period return, which is known in most of the models.

In the current chapter, we present an elegant way to calculate the prices of European options once the pricing measure Q is chosen. It uses the characteristic function of the log return on the stock from now till the expiration date of the option. There is no need to know the density of the stock price distribution explicitly. This is particularly favorable in the Lévy stock price models mentioned above: Generally, here the characteristic functions are easier to compute than the corresponding probability densities. In some cases, closed-form expressions for the densities do not exist at all.

Our approach applies to a large class of European options that depend only on the price of the underlying at expiration. It is based on the observation that one can represent the pricing formula for European options as a convolution. This allows us to profit from the fact that the bilateral (or, two-sided) Laplace transform of a convolution is the product of the bilateral Laplace transforms of the factors. Usually the latter transformations are known explicitly.

Inversion of the bilateral Laplace transformation yields the option prices as a function of the current price of the underlying. Approximate values for the inverse Laplace transform can be calculated by discretizing the inversion integral and applying the Fast Fourier Transform (FFT) algorithm. This has the advantage that one can simultaneously calculate the option prices for a range of current underlying prices.

The approach presented here turns out to be a generalization of the approach of Carr and Madan (1999) to the pricing of European call options. In this article, the pricing of call options was reduced to the calculation of a discrete Fourier transform by introducing the *dampened call value*. Applying our approach to the pricing of European call options reproduces their formula. In fact, the Fourier transform of the dampened call value is exactly the bilateral Laplace transform of the call price function itself.

The rest of the chapter is organized as follows. In Section 3.2, we give the basic assumptions about the price model for the underlying and about the option to be priced. Section 3.3 shows that the general formula for the price of a European option may be interpreted as a convolution of the payoff function and the distribution of the log of the price/forward-price ratio. This yields a Laplace back-transformation representation of the option pricing formula. In section 3.4 we calculate the Laplace transforms of power call and put payoff functions. For these options a simple transformation yields a formula for the option price as a function of the strike price rather than the current underlying price. Section 3.5 presents a way to discretize the inversion integral. This allows us to apply the FFT algorithm to the calculation of the option price. Section 3.6 outlines an algorithm that implements the option price calculation by FFT.

3.2 Definitions and Basic Assumptions

In the following, we assume that the probability measure Q is a martingale measure (or *risk neutral* measure) for the stock and options market. This means that discounted stock and option prices are martingales under this measure.

We place ourselves at time 0 and want to price European options with some fixed expiration date T. These options are characterized by their payoff at time T. We assume that the payoff is given by some measurable function $w(S_T)$ of the stock price at time T. As in Chapter 1, we call $w(x)$ the *payoff function*. S_T is assumed to be given by the random variable

$$S_T = S_0 \exp(rT) \exp(X_T), \tag{3.1}$$

where S_0 is the stock price at time 0. We assume that the interest rate r is deterministic. Then equation (3.1) represents the time-T stock price as the product of the *forward price* $S_0 \exp(rT)$ and a random factor $\exp(X_T)$. The random variable

$$X_T = \ln\left(S_T/(e^{rT}S_0)\right)$$

is the log of the ratio between the actually realized price at time T and the forward price for time T, contracted at time 0. Thus X_T is the log return on a *forward contract* to buy the stock at time T. Some readers might prefer a model of the form

$$S_T = S_0 \exp(Y_T) \tag{3.2}$$

instead. Appendix B.2 shows how one can easily gain the characteristic function of X_T from the characteristic function of Y_T. Since the characteristic function is the only information we need about the stock price model, it effectively makes no difference whether we start with a stock price model of type (3.1) or with a model of type (3.2).

Since by assumption Q is a martingale measure, S_T must be integrable with respect to Q. This implies the existence of the first exponential moment of X_T:

$$E_Q\left[e^{X_T}\right] < \infty.$$

Consequently, the moment generating function $\mathrm{mgf}(u) = E_Q\left[e^{uX_T}\right]$ exists at least on the interval $[0,1]$. The following lemma shows how risk neutrality of Q corresponds to a property of the moment generating function.

Lemma 3.1. *Assume that $E^Q[e^{X_T}] < \infty$. If Q is a martingale measure for the stock market, then the moment generating function m of Q^{X_T} satisfies $\mathrm{mgf}(1) = 1$.*

Proof.

$$\mathrm{mgf}(1) \equiv E^Q[e^{X_T}] = E^Q[e^{-rT}S_T]/S_0 = 1.$$

□

We sum up the required properties of the distribution of X_T.

Assumption [Distribution]: The distribution of the log return $X_T = \ln\{e^{-rT}S_T/S_0\}$ satisfies the following.

1. It is continuous with respect to Lebesgue measure, with density function $\rho(x)$.

2. Its extended characteristic function $\chi(z) = E\left[\exp(izX_T)\right]$ is defined for all $z \in \mathbb{R} - i[0,1] \subset \mathbb{C}$, with $\chi(-i) = 1$.

Concerning the option to be priced, we make the following

Assumption [Option]: We are given a European option which pays its holder the amount $w(S_T)$ at time T. The payoff $w(S_T)$ is integrable with respect to Q. In accordance with martingale pricing, the option price at time 0 is given by the discounted expectation $e^{-rT}E^Q[w(S_T)]$.

3.3 Convolution Representation for Option Pricing Formulas

Since we have assumed that Q is martingale measure for the options market, we can calculate the price of a European option with Q-integrable payoff $w(S_T)$ at time T by taking the expectation. The following theorem shows how this expectation may be calculated by a Laplace transformation.

Theorem 3.2. *Consider a European option with payoff $w(S_T)$ at time T. Let $v(x) := w(e^{-x})$ denote the* modified payoff function. *Assume that $x \mapsto e^{-Rx}|v(x)|$ is bounded and integrable for some $R \in \mathbb{R}$ such that the moment generating function $\mathrm{mgf}(u)$ of X_T satisfies $\mathrm{mgf}(-R) < \infty$.*

Letting $V(\zeta)$ denote the time-0 price of this option, taken as a function of the negative log forward price $\zeta := -\ln\{e^{rT}S_0\}$, we have

$$V(\zeta) = \frac{e^{\zeta R - rT}}{2\pi} \int_{-\infty}^{\infty} e^{iu\zeta} \cdot L[v](R+iu) \cdot \chi(iR-u)\, du, \tag{3.3}$$

whenever the integral on the r. h. s. exists at least as the limit $\lim_{M\to\infty}\int_{-M}^{M}\cdots du$. Here $L[v](z)$ is the bilateral *(or, two-sided) Laplace transform*[1] *of v for $z \in \mathbb{C}, \mathrm{Re}\; z = R$:*

$$L[v](z) := \int_{-\infty}^{\infty} e^{-zx} v(x)\, dx.$$

Proof. The current forward price is given by $e^{-\zeta}$. By definition of X_T the stock price at expiration is $S_T = e^{-\zeta}e^{X_T}$.

For the option price, this implies

$$V(\zeta) = e^{-rT} E_Q\left[w(e^{-\zeta + X_T})\right] = e^{-rT} E_Q\left[v(\zeta - X_T)\right] = e^{-rT}\int_{\mathbb{R}} v(\zeta - x)\rho(x)\, dx.$$

Apart from the discounting factor e^{-rT}, the last expression is the convolution of the functions $v(x)$ and $\rho(x)$, taken at the point ζ. By Theorem B.2, the bilateral Laplace transform of a convolution equals the product of the bilateral Laplace transforms of the factors. Denoting the bilateral Laplace transforms of V, v, and ρ by $L[V]$, $L[v]$, and $L[\rho]$, respectively, we thus have

$$L[V](R+iu) = e^{-rT} L[v](R+iu) \cdot L[\rho](R+iu) \qquad \text{for all } u \in \mathbb{R}. \tag{3.4}$$

We can apply Theorem B.2 to the functions $F_1(x) := v(x)$ and $F_2(x) := \rho(x)$.[2] This shows that the bilateral Laplace integral defining $L[V](z)$ converges absolutely and that $\zeta \mapsto V(\zeta)$ is a continuous function. Hence we can invert the bilateral Laplace transformation by Theorem B.3:

$$\begin{aligned} V(\zeta) &= \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} e^{\zeta z} L[V](z) dz \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\zeta(R+iu)} L[V](R+iu) du \\ &= \frac{e^{\zeta R}}{2\pi}\lim_{M\to\infty}\int_{-M}^{M} e^{iu\zeta} L[V](R+iu) du, \end{aligned} \tag{3.5}$$

where we have made explicit the Cauchy principal value. The bilateral Laplace transform $L[\rho]$ of the density ρ is given by $L[\rho](z) = \int_{\mathbb{R}} e^{-zx}\rho(x)dx$. Obviously, we have the identity $L[\rho](R+iu) = \chi(iR-u)$. Hence substituting (3.4) into (3.5) completes the proof. $\square$

Remark: The integral in (3.3) is a Fourier transformation. Hence we will be able to apply FFT methods to its calculation.

[1] For a thorough account of the bilateral Laplace transform, see Doetsch (1950).

[2] The conditions imposed by Theorem B.2 on the function F_1 are clearly satisfied. Moreover, as required by Theorem B.2, the function $x \mapsto e^{-Rx}|F_2(x)|$ is integrable because of the assumption $\mathrm{mgf}(-R) < \infty$.

3.4 Standard and Exotic Options

3.4.1 Power Call Options

Consider the problem of calculating the value of a *European call option* on the underlying. At the expiration time T, the holder of this option has the right to buy one share of the underlying for the price $K(> 0)$. K is called the *strike price* or simply *strike* of the option. Assuming frictionless markets without transaction costs, this right to buy is worth $(S_T - K)^+$ at the expiration date T. Therefore we can identify the option with a contract that pays its holder the amount $(S_T - K)^+$ at time T. We consider at once the more general case of *power call options*. These are characterized by the payoff

$$w(S_T) = \left((S_T - K)^+\right)^d$$

at maturity.[3] The exponent d is positive, with $d = 1$ corresponding to the standard European call described above. The case $d = 2$ can be visualized as follows: It corresponds to a contract that pays at time T the quantity $(S_T - K)^+$ of standard European call options with the same strike price.

The general Laplace approach described above is applicable here. It yields the option price as a function of the negative log forward price $\zeta = -\ln(e^{rT} S_0)$. The FFT method described below calculates approximate values of this function for argument values $\zeta_k = k\ \Delta\zeta$, where the integer k ranges from $-N/2$ to $N/2$. This corresponds to forward prices around 1. But in general the interesting values for the forward price are not the ones close to 1, but rather values around the strike price K. The remedy for this issue is a simple transformation of the call price function.

Lemma 3.3. *Let $C^d(\zeta; K)$ denote the price of a power call, expressed as a function of the negative log forward price $\zeta := -\ln e^{rT} S_0$ and the strike price $K > 0$. Then we have the following relation between prices for strike K and prices for strike 1.*

$$C^d(\zeta; K) = K^d\, C^d(\zeta + \ln K; 1), \tag{3.6}$$

where the second argument of the function C^d denotes the strike price.

Proof.

$$\begin{aligned} C^d(\zeta; K) &\equiv e^{-rT} E_Q\left[\left((S_T - K)^+\right)^d\right] \\ &= e^{-rT} E_Q\left[\left((e^{-\zeta} e^{X_T} - K)^+\right)^d\right] \\ &= K^d\, e^{-rT}\, E_Q\left[\left((e^{-\zeta - \ln K} e^{X_T} - 1)^+\right)^d\right] \\ &= K^d\, C^d(\zeta + \ln K; 1). \end{aligned}$$

□

Hence we only need to calculate approximate values for the function $C(\cdot; 1)$ in order to be able to price call options for all strike prices. The argument $\zeta + \ln K$ in (3.6) is exactly the log forward-price ratio, so its interesting values lie around 0.

[3] Cf. Eller and Deutsch (1998), p. 167.

Another advantage is that we can gain insight into the behavior of power call prices as a function of the strike rather than the forward price. This is achieved by fixing ζ and varying K in (3.6), using the FFT approximation of the function $C(\cdot;1)$.

In order to apply the Laplace inversion formula deduced in Theorem 3.2, we have to know the bilateral Laplace transform of the modified payoff function

$$c^d(x) := \left((e^{-x}-1)^+\right)^d \quad (d>0, x\in\mathbb{R}).$$

The bilateral Laplace transform exists for $z\in\mathbb{C}$ with Re $z<-d$. For these values of z and d we have

$$\begin{aligned}\int_{\mathbb{R}} e^{-zx}v^d(x)dx &= \int_{-\infty}^{0} e^{-zx}(e^{-x}-1)^d dx\\ &= \int_0^1 t^{-z}(1/t-1)^d\frac{dt}{t}\\ &= \int_0^1 t^{-z-d-1}(1-t)^d dt\\ &= B(-z-d,d+1)\\ &= \frac{\Gamma(-z-d)\Gamma(d+1)}{\Gamma(-z+1)}.\end{aligned}$$

Here $B(\cdot,\cdot)$ and $\Gamma(\cdot)$ denote the Euler Beta and Gamma functions respectively.[4] We give some brief comments on the chain of equalities above: In the second line, we have substituted $t=e^x$. The fourth equality follows directly from the definition of the Beta function (cf. Abramowitz and Stegun (1968), Formula 6.2.1):

$$B(z,w) := \int_0^1 t^{z-1}(1-t)^{w-1}dt \qquad (\text{Re } z>0,\ \text{Re}w>0)$$

The last line is a consequence of the relation between the Beta and Gamma function (cf. Abramowitz and Stegun (1968), Formula 6.2.2):

$$B(z,w)=\frac{\Gamma(z)\Gamma(w)}{\Gamma(z+w)}.$$

For practical purposes, writing the Beta function as a quotient of Gamma functions may be necessary if the Beta function is not implemented in the programming environment you use.[5]

The practically relevant cases of power calls have exponents $d=1$ or $d=2$. Here we can simplify the expression given above, using the relation $\Gamma(z+n+1)/\Gamma(z)=(z+n)(z+n-1)\cdots z$ for $n=0,1,2,\dots$ (cf. Abramowitz and Stegun (1968), Formula 6.1.16). For $d=1$, that is the standard European call, we have

$$L[c^1](z)=\frac{\Gamma(2)}{(-z)(-z-1)}=\frac{1}{z(z+1)}.$$

For $d=2$,

$$L[c^2](z)=\frac{\Gamma(3)}{(-z)(-z-1)(-z-2)}=\frac{-2}{z(z+1)(z+2)}.$$

[4] Properties of these functions may be found e. g. in Abramowitz and Stegun (1968), Chapter 6.

[5] This is the case for S-Plus 3.4: Here, the Gamma function is available, but the Beta function is not.

3.4.2 Power Put Options

Now consider the case of the *power put* with payoff

$$w(S_T) := \big((K - S_T)^+\big)^d$$

for some constant $d > 0$. The choice $d = 1$ corresponds to the standard European put. By a completely analogous proof as in Lemma 3.3, one shows the following relation for the put price function $P^d(\zeta; K)$.

Lemma 3.4. *Letting $P^d(\zeta; K)$ denote the price of a power put, expressed as a function of the negative log forward price $\zeta = -\ln\{e^{rT} S_0\}$, we have*

$$P^d(\zeta; K) = K^d \cdot P^d(\zeta + \ln K; 1).$$

The modified payoff function for $K = 1$ is

$$p^d(x) := \big((1 - e^{-x})^+\big)^d \quad (d > 0, x \in \mathbb{R}).$$

Its Laplace transform exists for Re $z > 0$:

$$\begin{aligned}\int_{\mathbb{R}} e^{-zx} v^d(x) dx &= \int_0^\infty e^{-zx} (1 - e^{-x})^d dx \\ &= \int_0^1 t^{z-1} (1-t)^d dt \\ &= B(z, d+1) \\ &= \frac{\Gamma(z)\Gamma(d+1)}{\Gamma(z+d+1)}.\end{aligned}$$

Again, $B(\cdot, \cdot)$ and $\Gamma(\cdot)$ denote the Euler Beta and Gamma functions respectively.[6]

The practically relevant cases are again $d = 1$—the standard European put—and $d = 2$. The bilateral Laplace transform for $d = 1$ is

$$L[p^1](z) = \frac{1}{z(z+1)} \qquad (\text{Re } z > 0),$$

and for the case $d = 2$ we have

$$L[p^2](z) = \frac{2}{z(z+1)(z+2)} \qquad (\text{Re } z > 0).$$

Remark: There is no put-call parity for power calls with $d \neq 1$, so explicit calculation of both put and call values is required here.

3.4.3 Asymptotic Behavior of the Bilateral Laplace Transforms

Below we will encounter integrals in which the integrands contain a factor $L[v](R + iu)$. In order to determine absolute integrability, we will need information about the asymptotic behavior of this term for

[6]See e. g. Abramowitz and Stegun (1968), Chapter 6.

large $|u|$. This amounts to studying the asymptotic behavior of the Beta function, which in turn can be deduced from the behavior of the Gamma function. For $|z| \to \infty$, the Gamma function behaves in the following way (cf. Abramowitz and Stegun (1968), Formula 6.1.39):

$$\Gamma(az+b) \sim \sqrt{2\pi} e^{-az} (az)^{az+b-1/2} \qquad (|\arg z| < \pi, a > 0).$$

From this relation we can derive the asymptotic behavior if the Beta function $B(z,w)$ for fixed w.

$$\begin{aligned} B(z,w) &\sim \Gamma(w) \frac{\sqrt{2\pi} e^{-z} z^{z-1/2}}{\sqrt{2\pi} e^{-(z+w)} (z+w)^{z+w-1/2}} \\ &= \Gamma(w) e^{w} \Big(\frac{z}{z+w} \Big)^{z+w-1/2} z^{-w} \\ &\sim \Gamma(w) z^{-w} \qquad (|\arg z| < \pi). \end{aligned}$$

Hence we get the following Lemma.

Lemma 3.5. *For fixed w, the asymptotic behavior of the Beta function $B(z,w)$ appearing in the bilateral Laplace transforms for power calls and puts is as follows.*

$$B(z,w) \sim \frac{\Gamma(w)}{z^w} \qquad (|\arg z| < \pi).$$

In particular,

$$\begin{aligned} |B(R+iu, d+1)| &= O\Big(\frac{1}{|u|^{d+1}} \Big) \qquad \textit{and} \\ |B(-(R+iu)-d, d+1)| &= O\Big(\frac{1}{|u|^{d+1}} \Big) \qquad (|u| \to \infty), \end{aligned}$$

if R lies in the respective allowed range.

3.4.4 Self-Quanto Calls and Puts

A *self-quanto call* has the payoff function

$$w_K(S_T) = (S_T - K)^+ S_T.$$

This cash flow may be visualized as follows: At exercise, the buyer of the call receives the quantity $(S_T - K)^+$ of *shares of the underlying*. This contrasts with the usual European call where $(S_T - K)^+$ is the amount of *currency units* received.

Writing the value of a self-quanto call as a function $C^S(\zeta; K)$ of $\zeta = -\ln\{e^{rT} S_T\}$, we have

$$\begin{aligned} C^S(\zeta; K) &= E_Q \Big[(e^{-\zeta + X_T} - K)^+ e^{-\zeta + X_T} \Big] \\ &= K^2 E_Q \Big[(e^{-\zeta - \ln(K) + X_T} - 1)^+ e^{-\zeta - \ln(K) + X_T} \Big] \\ &= K^2 C^S(\zeta + \ln K; 1). \end{aligned}$$

Hence we can again limit our studies to the case $K = 1$. The modified payoff function for a self-quanto call with $K = 1$ is

$$v(x) = e^{-x} (e^{-x} - 1)^+.$$

Its bilateral Laplace transform exists for Re $z < -2$ and is given by

$$\begin{aligned}\int_{\mathbb{R}} e^{-zx} e^{-x} (e^{-x} - 1)^+ dx &= \int_{\mathbb{R}} e^{-(z+1)x} (e^{-x} - 1)^+ dx \\ &= \frac{1}{(z+2)(z+1)}.\end{aligned}$$

Here the last equality is based on the following observation: The expression in the second line is nothing else than the bilateral Laplace transform of the modified payoff function of a standard call, taken at $z+1$.

Analogous relations hold for a *self-quanto put* with payoff function

$$w(S_T) = (K - S_T)^+ S_T.$$

The bilateral Laplace transform in this case exists for Re $z > -1$ and is equal to the bilateral Laplace transform of the standard European put, taken at the point $z + 1$.

$$\int_{\mathbb{R}} e^{-zx} (1 - e^{-x})^+ e^{-x} dx = \frac{1}{(z+1)(z+2)} \qquad (\text{Re } z > -1).$$

Remark: Of the call and put options considered here, this is the only one where the bilateral Laplace transform exists for Re $z = 0$.

3.4.5 Summary

Table 3.1 summarizes the results for standard and exotic call and put options. The second column shows the payoff at expiration as a function $w(S_T; K)$ of stock price S_T and strike K. The third column gives the bilateral Laplace transform of the modified payoff function $v(x) := w(e^{-x}; 1)$. The fourth column gives the range of existence of the Laplace transform. The fifth column gives the option price for arbitrary strike price $K > 0$, expressed by the option price function for strike $K = 1$.

3.5 Approximation of the Fourier Integrals by Sums

In order to compute the value of the Fourier integral in (3.3), we approximate the integral by a sum.

Consider the problem of calculating

$$\int_{-\infty}^{\infty} e^{iux} g(u) du$$

for some continuous complex-valued function g with the property $g(-u) = \overline{g(u)}$.[7] The integral encountered in (3.3) is of this form, with $g(u) = L[v](R + iu)\chi(iR - u)$. In particular, the symmetry condition on g is satisfied: $L[v](z)$ is the bilateral Laplace transform of a real-valued function (namely, the modified payoff function v), and so $L[v](\overline{z}) = \overline{L[v](z)}$. An analogous relation holds for the function $z \mapsto \chi(iz)$.

[7] $\overline{z}$ denotes the complex conjugate of a complex number z.

Option	Payout $w(S_T;K)$	Laplace transform of modif. payoff $(K=1)$	Region of existence	Option price for arbitrary strike $K>0$
classical call	$(S_T-K)^+$	$\frac{1}{z(z+1)}$	Re $z<-1$	$K\ C(\zeta+\ln K;1)$
power call	$[(S_T-K)^+]^d$	$\frac{\Gamma(-z-d)\Gamma(d+1)}{\Gamma(-z+1)}$	Re $z<-d$	$K^d\ C^d(\zeta+\ln K;1)$
power call $(d=2)$	$[(S_T-K)^+]^2$	$\frac{-2}{z(z+1)(z+2)}$	Re $z<-2$	$K^2\ C^2(\zeta+\ln K;1)$
self-quanto call	$(S_T-K)^+S_T$	$\frac{1}{(z+1)(z+2)}$	Re $z<-2$	$K^2\ C^S(\zeta+\ln K;1)$
classical put	$(K-S_T)^+$	$\frac{1}{z(z+1)}$	Re $z>0$	$K\ P(\zeta+\ln K;1)$
power put	$[(K-S_T)^+]^d$	$\frac{\Gamma(z)\Gamma(d+1)}{\Gamma(z+d+1)}$	Re $z>0$	$K^d\ P^d(\zeta+\ln K;1)$
power put $(d=2)$	$[(K-S_T)^+]^2$	$\frac{2}{z(z+1)(z+2)}$	Re $z>0$	$K^2\ P^2(\zeta+\ln K;1)$
self-quanto put	$(K-S_T)^+S_T$	$\frac{1}{(z+1)(z+2)}$	Re $z>-1$	$K^2\ P^S(\zeta+\ln K;1)$

Table 3.1: Standard and exotic calls and puts.

We choose a number $2N-1$ of intervals and a step size Δu. We truncate the integral at the points $-(N-1/2)\cdot\Delta u$ and $(N-1/2)\cdot\Delta u$. Subsequent application of the midpoint rule[8] yields

$$\begin{aligned}\int_{-\infty}^{\infty} e^{iux} g(u)\ du &\approx \int_{-(N-1/2)\cdot\Delta u}^{(N-1/2)\cdot\Delta u} e^{iux} g(u)\ du \\ &\approx \Delta u \cdot \sum_{n=-(N-1)}^{n=N-1} e^{i\cdot n\cdot\Delta u\cdot x} g\big(n\cdot\Delta u\big).\end{aligned}$$

By making use of the property $g(-u)=\overline{g(u)}$ for the integrand, we can reduce the number of summands to N. In fact, we have

$$e^{i\cdot n\cdot\Delta u\cdot x} g\big(n\ \Delta u\big) = \overline{e^{i\cdot(-n)\cdot\Delta u\cdot x} g\big((-n)\cdot\Delta u\big)},$$

and hence adding the terms for $-n$ and n ($n=1,\dots,N-1$) in the sum eliminates the imaginary parts.

$$e^{i\ (-n)\ \Delta u\ x} g\big((-n)\ \Delta u\big) + e^{i\ n\ \Delta u\ x} g\big(n\ \Delta u\big) = 2\ \mathrm{Re}\left(e^{i\ n\ \Delta u\ x} g\big(n\ \Delta u\big)\right).$$

This leads to the following formula for the sum approximation:

$$\int_{-\infty}^{\infty} e^{iux} g(u)\ du \approx \Delta u \cdot \mathrm{Re}\left(\sum_{n=0}^{N-1} e^{i\cdot n\cdot\Delta u\cdot x} g_n\right), \tag{3.7}$$

with $g_0 := g(0)/2$ and $g_n := g(n\cdot\Delta u)$ $\quad (n=1,\dots,N-1)$. The sum appearing here is called *discrete Fourier transform* of the complex numbers $(g_n)_{n=0,\dots,N-1}$.

[8] One might wonder if some more elaborate scheme for numerical integration might improve the approximation. In the opinion of the author, this is not the case if one restricts oneself to equidistant node points, which is essential for the application of FFT techniques. Any improvements gained by employing methods like the Simpson method have to do with the better behavior at the boundaries. But if the truncation is done correctly, boundary effects can be neglected here.

3.5.1 Fast Fourier Transform

Fast Fourier Transform (FFT)[9] is an algorithm for the calculation of the discrete Fourier transform for a range of parameter values x simultaneously. It calculates the sum appearing in (3.7) for parameter values $x = x_k = k\ \Delta x$ $(k = 0, .., N-1)$ with $\Delta x = \frac{2\pi}{N\Delta u}$. For the FFT to be most efficient, N has to be an integer power of 2.

Given a sequence $(g_n)_{n=0,..,N-1}$ of complex numbers, the FFT computes

$$\sum_{n=0}^{N-1} e^{in\Delta u \cdot x_k} g_n = \sum_{n=0}^{N-1} e^{2\pi i \frac{nk}{N}} g_n =: G_k \qquad (k = -N/2, \dots, N/2).$$

Because the values for $k = -N/2$ and $k = N/2$ are equal, it suffices that the algorithm computes one of them. Hence the FFT takes N complex numbers as input and gives back N complex numbers as output.

The approximation of the Fourier integral is best when k/N is close to 0. On the other hand, when $k = -N/2$ or $k = +N/2$, there are only two sampling points for each period of $u \mapsto e^{-2\pi iux}$. Therefore the approximation is not reliable there. Increasing N by a factor 2 while keeping $N\Delta u$ constant leads to a doubling of the sampling frequency, thereby improving the quality of the approximation. Keeping $N\Delta u$ constant implies that the discretization step Δx stays the same.

3.6 Outline of the Algorithm

We summarize the steps we have to take in order to calculate the option price by the method described above.

Let χ be the (extended) characteristic function of the log return $X_T = \ln(e^{-rT} S_T/S_0)$. Consider a European option that pays its holder the amount $w(S_T)$ at time T. Let $L[v](z)$ be the bilateral Laplace transform of the modified payoff function $v(x) := w(e^{-x})$.

- Choose $R \in \mathbb{R}$ with $\chi(iR) < \infty$ and $L[v](R) < \infty$. Choose the discretization step width Δu and the number of intervals N. N has to be a power of 2.

- Calculate the sequence $(g_n)_{n=0,..,N-1}$ defined by

$$g_n := \begin{cases} \frac{1}{2}\chi(iR)\ L[v](R) & (n = 0) \\ \chi(iR - n\ \Delta u)\ L[v](R + in\ \Delta u) & (n = 1, \dots, N-1). \end{cases}$$

- Apply the FFT to $(g_n)_{n=0,\dots,N-1}$ to get the transformed sequence $(G_k)_{k=-N/2,..,N/2}$.

- Set $\Delta\zeta := 2\pi/(N\ \Delta u)$. Then the approximate option price for a negative log forward price of $\zeta_k = k\ \Delta\zeta$ $(k = -N/2, .., N/2)$ — that means, a forward price of $e^{rT} S_0 = e^{-k\ \Delta\zeta}$ — is given by

$$C_k := \Delta u \frac{e^{\zeta_k R - rT}}{\pi} \mathrm{Re}(G_k). \tag{3.8}$$

[9] See e. g. Brigham (1974).

- For standard and exotic calls and puts, you may use the Laplace transform $L[v](z)$ of the payoff function for $K = 1$ in the steps above. This yields approximate values C_k of the function $\zeta \mapsto C^d(\zeta; 1)$ at the points $\zeta = k\ \Delta\zeta$ ($k = -N/2, \ldots, N/2$). Since the price for the option with arbitrary K is given by $K^d\ C^d(\zeta + \ln K; 1)$, $K^d\ C_k$ approximates the price of this option for a negative log forward price of $\zeta = k\ \Delta\zeta - \ln K$.

3.7 Applicability to Different Stock Price Models

In order to calculate option prices, one has to specify two things, namely the option to be priced and the stock price model under which one wants to price it.

The pricing method presented above applies to European options that do only depend on the spot price of the stock at expiration. Hence they are completely specified by giving the expiration date T and the payoff function $w(S_T)$. For the new pricing method to be applicable, the bilateral Laplace transform of the payoff function has to exist on a non-degenerate interval. In Section 3.4, we have presented a number of standard and exotic options satisfying these conditions. The corresponding bilateral Laplace transforms can be found in Table 3.1.

The second step is the specification of the stock price model. This should be done by giving the (extended) characteristic function $\chi(z) := E_Q\left[\exp(izX_T)\right]$ of the random variable $X_T = \ln(S_T/(e^{rT}S_0))$ which we have identified as the log return on the forward contract to buy the stock at time T. Below, we give examples of stock price models, together with the corresponding extended characteristic function $\chi(z)$ and the strip of regularity of this characteristic function.

The algorithm developed above is applicable to any valid combination of option and stock price models. Here "valid" means that there exists a constant $R \in \mathbb{R}$ as in Theorem 3.2, such that R lies in the strip of regularity of the bilateral Laplace transform $L[v](z)$ and that iR lies in the strip of regularity of the extended characteristic function $\chi(z)$.

The method was successfully tested with the stock price models given in Table 3.2. All of these models are of the exponential Lévy type, that is, they assume that the log return process $(X_t)_{t\in\mathbb{R}^+}$ is a process with stationary and independent increments. Hence the distribution of the process is uniquely characterized by each of its one-dimensional marginal distributions. We choose $t = 1$, that is, we characterize the Lévy process by the distribution of X_1. The first column in Table 3.2 gives the type of this distribution, and the second column gives its characteristic function.

- A normally distributed log return X_1 corresponds to the famous geometric Brownian motion model introduced by Samuelson (1965).

- Generalized hyperbolic (GH) distributions were introduced by Barndorff-Nielsen (1978). Eberlein and Prause (1998) used these class of distributions to model log returns on stocks. This generalized earlier work by Eberlein and Keller (1995), where hyperbolically distributed log returns were considered.

- The class of normal inverse Gaussian (NIG) distributions was proposed in Barndorff-Nielsen (1995) and Barndorff-Nielsen (1998) as a model for log returns. NIG distributions constitute a subclass of the class of GH distributions. See also Barndorff-Nielsen (1997).

Type	extended characteristic function $\chi(z)$	$\chi(iR) < \infty$ if...
normal	$\exp\left(i\mu z t - \frac{\sigma^2}{2} z^2 t\right)$	$-\infty < R < \infty$
GH	$e^{i\mu z t} \frac{(\delta\sqrt{\alpha^2-\beta^2})^{\lambda t}}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})^t} \cdot \frac{K_\lambda\left(\delta\sqrt{\alpha^2-(\beta+iz)^2}\right)^t}{\left(\delta\sqrt{\alpha^2-(\beta+iz)^2}\right)^{\lambda t}}$	$\beta - \alpha < R < \beta + \alpha$
NIG	$\frac{\exp(izt\mu + t\delta\sqrt{\alpha^2-\beta^2})}{\exp(t\delta\sqrt{\alpha^2-(\beta+iz)^2})}$	$\beta - \alpha < R < \beta + \alpha$
VG	$\frac{\exp(izt\mu)}{(1 - i\theta\nu z + (\sigma^2\nu/2)z^2)^{t/\nu}}$	$R > \frac{\theta}{\sigma^2}\left(1 - \sqrt{1 + \frac{2\sigma^2}{\nu\theta^2}}\right)$ $R < \frac{\theta}{\sigma^2}\left(1 + \sqrt{1 + \frac{2\sigma^2}{\nu\theta^2}}\right)$

Table 3.2: Different models for the stock price: Characteristic functions and admissible values for R.

- Variance gamma (VG) distributions were first proposed by Madan and Seneta (1987) for the modeling of log returns on stocks. Madan, Carr, and Chang (1998) generalize this approach to non-symmetric VG distributions.

All of the non-normal models cited above have been shown to capture the distribution of observed market price movements significantly better than the classical geometric Brownian motion model. Moreover, Madan, Carr, and Chang (1998) and Eberlein and Prause (1998) observed a substantial reduction of the smile effect in call option pricing with the non-symmetrical VG and the GH model, respectively.

As a benchmark example, we have used the method described above to calculate the prices of European call options with one day to expiration. The log return distributions employed were those displayed in Table 3.2. The parameters shown in Table 3.3 were generated as follows. First, the parameters of the respective distribution were estimated by maximum likelihood from a dataset of log returns on the German stock index DAX, from June 1, 1997 to June 1, 1999. Then, an Esscher transform was performed on each of these distributions so as to make e^{L_t} a martingale. For simplicity, we have always assumed that the interest rate vanishes, $r = 0$. Hence the results should be not viewed as reasonable option prices, but rather as an illustration of the algorithm. Figure 3.1 shows the prices of a European call option, displayed as a function of the strike price at a fixed stock price of $S_0 = 1$. The option prices were calculated by means of the algorithm described above. Note that this algorithm yields the option price only at a discrete set of values of K. But we have chosen this set so dense that—for the limited resolution of the plot—the discrete point set looks like a solid line.

Because of the efficient calculation of option prices by the FFT method, it becomes easier to study the behavior of the option pricing function. Figure 3.2 shows the difference of call option prices from the standard Black-Scholes model. Here the W-shape that usually appears for these differences is distorted.

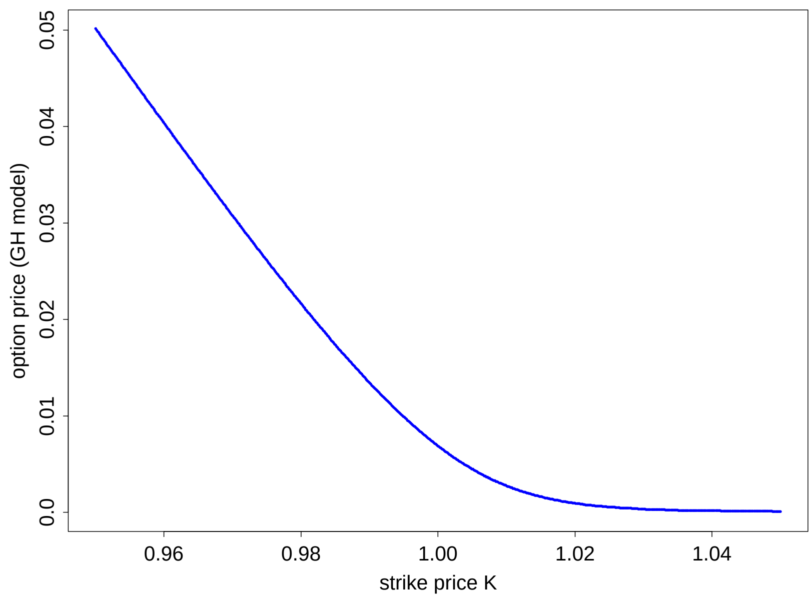

Figure 3.1: Call option prices for the generalized hyperbolic stock price model. Parameters are given in row “GH” of Table 3.3. Fixed current stock price $S_0 = 1$. One day to expiration.

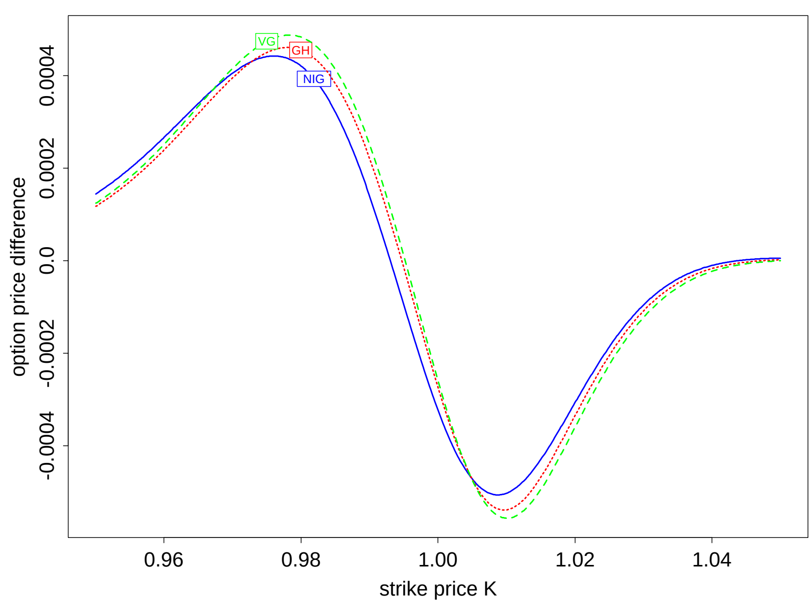

Figure 3.2: Difference of call option prices between alternative stock price models and Black Scholes prices. The alternative models are GH (exponential generalized hyperbolic Lévy motion), NIG (normal inverse Gaussian), and VG (variance gamma) with parameters as given in Table 3.3. Fixed current stock price $S_0 = 1$. One day to expiration.

Type	Parameters				
normal	$\sigma = 0.01773,$	$\mu = -1.572 \cdot 10^{-4}$			
GH	$\alpha = 127.827,$	$\beta = -31.689,$	$\delta = 7.07 \cdot 10^{-31},$	$\mu = 0.0089,$	$\lambda = 2.191$
NIG	$\alpha = 85.312,$	$\beta = -27.566,$	$\delta = 0.0234,$	$\mu = 0.00784$	
VG	$\sigma = 0.0168,$	$\nu = 0.4597,$	$\theta = -0.00962,$	$\mu = 0.009461$	

Table 3.3: Estimated Parameters for stock price models in Table 3.2

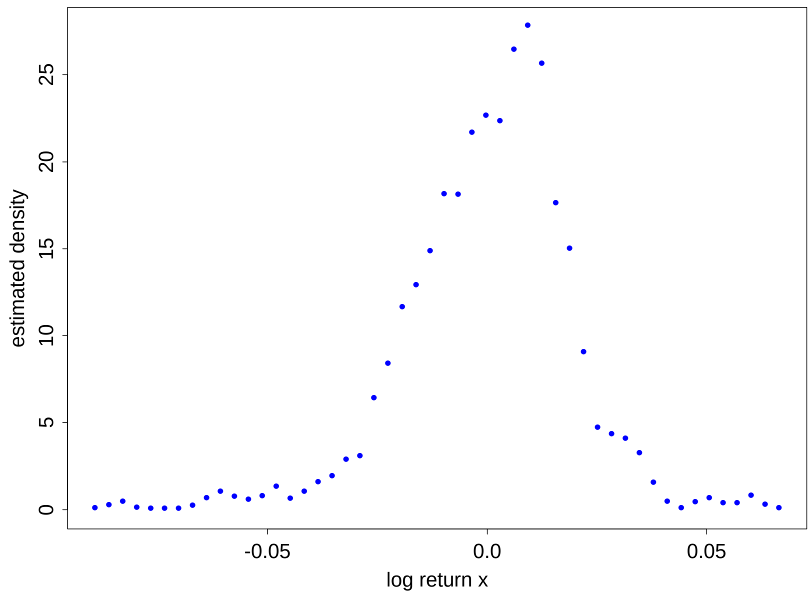

Figure 3.3: Empirical density of daily log returns on the German stock index DAX, 1 June 1997 to 1 June 1999. Gauss kernel estimate by S-Plus function `density`.

This distortion is a result of the fact that the empirically observed return distribution for the DAX in the interval 1 June 1997 till 1 June 1999 was relatively asymmetric. See Figure 3.3.

Figure 3.4 shows the difference of NIG and Black-Scholes option prices, seen as a function of the strike price K and the time to expiration. Note how the difference grows as time to maturity increases. Also note that the shape of the difference curve changes as time to expiration increases. For larger τ, it becomes increasingly bell-shaped. These effects are due to the different standard deviations of the NIG and the normal distribution under the martingale measure: The standard deviation of the NIG distribution is 0.0180, while that of the normal distribution is 0.0177 (see Table 3.3.)

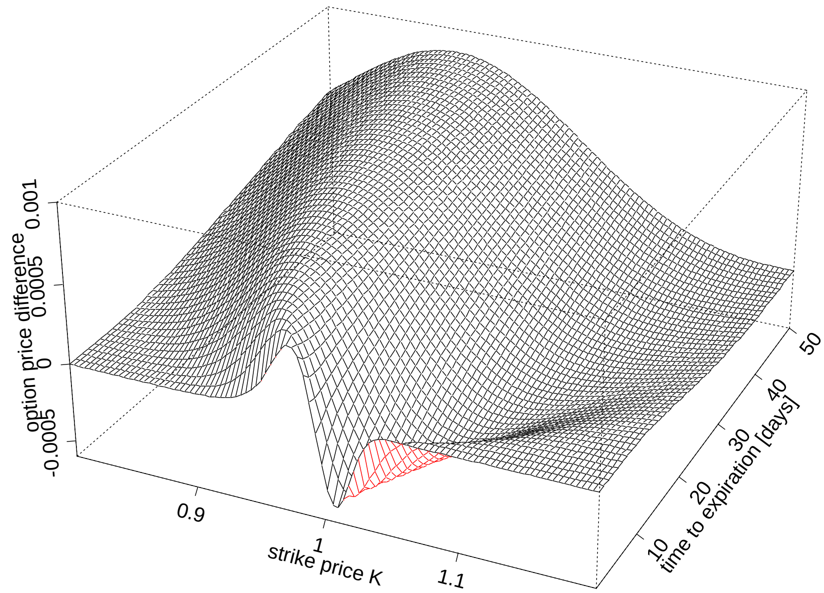

Figure 3.4: Difference between call option prices in the NIG model minus call option prices in the Black-Scholes model. Fixed initial stock price $S_0 = 1$; time to maturities between 1 and 50 (trading) days. Strike prices between 0.8 and 1.2.

On the Choice of the Rate of Decay, R

It turns out that the precision of the calculation crucially depends on the choice of the rate of decay, R. We have found that for usual parameter values $R = -25$ works best for call options. This coincides with the findings of Carr and Madan (1999).

3.8 Conclusion

We have developed and presented an algorithm that prices general non-path-dependent European options in a wide variety of stock price models. This method is based on first calculating the bilateral Laplace transform of the option price and then inverting this transform. Calculating the bilateral Laplace transform is easy in all models where one knows explicitly the (extended) characteristic function of the log return to expiration. We have given some examples where this holds. The inversion of the bilateral Laplace transform can be done with the aid of the Fast Fourier transform (FFT) algorithm. We have given a detailed description of the calculation of option prices by the new method.

Chapter 4

The Lévy Term Structure Model

This chapter is devoted to the further study of the term structure model proposed in Eberlein and Raible (1999). This model can be driven by a wide range of Lévy processes. Hence we call it the *Lévy term structure model*.

The chapter is organized as follows. Section 4.1 introduces some basic concepts of term structure modeling. In Section 4.2, we give a short overview of the Lévy term structure model. Section 4.3 studies the decay behavior of the generalized hyperbolic characteristic function. This has implications for the Markov property of the short-term interest rate in a Lévy term structure model driven by a generalized hyperbolic Lévy motion. Section 4.4 shows that the Markov property of the short rate implies the existence of a so-called *affine term structure* for the Lévy term structure model. Finally, in Section 4.5 we derive an integro-differential equation for the price of simple interest-rate contingent claims.

4.1 Introduction

In this section, we give a very brief introduction to the terminology and methodology of term structure modeling. A more detailed account of interest rate theory can be found in Björk (1998), Chapters. 15–19, or Musiela and Rutkowski (1997), Part II.

When money is lent or borrowed, the interest rate received on this contract usually depends on the length of the period for which money is transferred. The *term structure of interest rates* captures this dependence. It gives the interest rate as a function of the length of the borrowing/lending contract.

Zero coupon bonds are the simplest contracts of this type. The issuer of a zero coupon bond will pay the owner of the bond a fixed amount of money, the *face value*, at a pre-specified date, T. We call this date *maturity date* of the bond. There are no intermediate payments here. This is in contrast to so-called *coupon bonds*, whose issuer pays the holder a certain percentage, p, of the face value, V, each year up to the final repayment of the face value itself. A coupon bond can be thought of as a bundle of zero coupon bonds: n bonds with face value pV paying back $1, \dots, n$ years from now, and one bond paying back V n years from now.[1] This decomposition shows that it is sufficient to build a model for zero coupon bonds, since the price of a coupon bond is just the sum of the prices of its constituents.

[1] In fact, this is not merely a theoretical construction: On many bond markets there is a procedure called *stripping*, where a coupon bond is decomposed into these zero coupon bonds. Each of these bonds can then be traded separately.

For many applications, the focus is more on interest rates than on bond prices. Given a zero coupon bond maturing after a time τ, with a current price of $P(t, t+\tau)$. one introduces its *yield* as the interest rate at which an initial investment of $P(t, t+\tau)$ would grow to a capital of V, including continuously compounded interest, in a time period of length τ.

$$Y(t, t+\tau) := \frac{1}{\tau} \ln \frac{V}{P(t, t+\tau)}.$$

The derivative of $\ln\big(V/P(t, t+\tau)\big)$ with respect to τ is called the *forward rate*.

$$f(t, t+\tau) : S = -\frac{\partial}{\partial \tau} \ln P(t, t+\tau).$$

This is the interest rate one gets on a contract (contracted now, at time t) to lend money at time $t+\tau$ for an infinitesimal period of length $d\tau$.

In particular, if $\tau = 0$, then lending takes place immediately at the time of the contract. The corresponding interest rate is called the *short rate*, $r(t)$.

$$r(t) := f(t, t).$$

For each fixed time horizon $T^* \in (t, \infty]$, bond prices $P(t, T)$ $(t \le T \le T^*)$ and forward rates $f(t, T)$ $(t \le T \le T^*)$ carry the same information. Hence it does not matter whether one models forward rates or bond prices in the first place: Every forward rate model implies a bond price model and vice versa.

In order to simplify notation, we will always assume that zero bonds have a face value of 1, $V = 1$.

Historically, the first term structure models were based on models of the short rate $r(t)$. In these models, the bond prices are calculated as conditional expectations under some martingale measure Q.

$$P(t, T) = E_Q\left[\exp\Big(-\int_t^T r(s)\,ds\Big)\,\Big|\,\mathcal{F}_t\right].$$

Note that if one starts with a short rate model alone, then *every* measure Q satisfying

$$E_Q\left[\exp\Big(-\int_t^T r(s)\,ds\Big)\right] < \infty \quad \text{for all } t < T$$

is a martingale measure for the bond market. There is no martingale condition here simply because one starts with a market without assets other than the numéraire.

One of the simplest short rate models is the one introduced by Ho and Lee (1986), which was formulated in a discrete-time context. According to Hull and White (1990), its continuous-time analogue is due to Dybvig (1988)[2] and Jamshidian (1988):

$$dr(t) = \theta(t)\,dt + \widehat{\sigma}\,dW_t,$$

with a deterministic function $\theta(t)$ and a constant $\widehat{\sigma}$. Another simple short rate model was introduced by Vasicek (1977). He assumed that the short rate follows an Ornstein–Uhlenbeck process. This was later generalized by Hull and White (1990), whose model allows for a time-dependence of the level of mean reversion, ρ.[3]

$$dr(t) = \big(\rho(t) - r(t)\big)a\,dt - \widehat{\sigma}\,dW_t, \tag{4.1}$$

[2]Meanwhile, this working paper has been published in Dempster and Pliska (1997).

[3]Actually, Hull and White (1990) allow all three parameters ρ, a, and $\widehat{\sigma}$ to be time-dependent.

with a deterministic function $\rho(t)$ and constants a and $\widehat{\sigma}$. This is called the *extended Vasicek model*. There are other proposals for the dynamics of $r(t)$, which introduce diffusion coefficients depending on $r(t)$. The most important are the Cox, Ingersoll, and Ross (1985) model (with a diffusion term $\sigma\sqrt{r(t)}dW_t$) and the Dothan (1978) model (with a diffusion term $\sigma r(t)\ dW_t$.) For a deeper discussion of these and other short rate models, we refer the reader to Musiela and Rutkowski (1997), Sec. 12.3.

A more recent approach to term structure modeling is based on modeling forward rates. Heath, Jarrow, and Morton (1992) present a very general model of this kind. Their forward rates are diffusion processes driven by a vector Brownian motion $\big(W^{(1)},\dots,W^{(n)}\big)$.

$$f(t,T) = f(0,T) + \int_0^t \alpha(v,T,\omega)\ dv + \sum_{i=1}^n \int_0^t \widetilde{\sigma}^{(i)}(v,T,\omega)\ dW_v^{(i)}. \tag{4.2}$$

(Cf. Heath, Jarrow, and Morton (1992), eq. (4).) If the bond prices corresponding to these forward rate processes are free of arbitrage, then the drift coefficient $\alpha(v,T,\omega)$ and the diffusion coefficients $\widetilde{\sigma}^{(i)}(v,T,\omega),\ i=1,\dots,n$ are connected by the so-called *Heath-Jarrow-Morton (HJM) drift condition*. See Björk (1998), Theorem 18.1. In particular, if (4.2) describes the dynamics under a martingale measure, then $\alpha(v,T,\omega)$ are related by

$$\int_t^T \alpha(t,s)\ ds = \frac{1}{2}\left\|\int_t^T \widetilde{\sigma}(t,s)\ ds\right\|^2.$$

The Gaussian HJM Model

The special case where the coefficients $\alpha(v,T,\omega)$ and $\widetilde{\sigma}(v,T,\omega)$ in (4.2) are deterministic yields the so-called Gaussian Heath-Jarrow-Morton (HJM) model. In the risk-neutral setting, zero-bond prices are then given by

$$P(t,T) = P(0,T)\exp\Big(\int_0^t r(s)ds\Big)\exp\Big(\sum_{i=1}^n\int_0^t \sigma^{(i)}(v,T)\ dW_v^{(i)} - \frac{1}{2}\sum_{i=1}^n\int_0^t \sigma^{(i)}(v,T)\ dW_v^{(i)}\Big), \tag{4.3}$$

where the bond price volatility structure $\sigma^{(i)}(t,T)$ is defined by

$$\sigma^{(i)}(t,T) := -\int_t^T \widetilde{\sigma}^{(i)}(t,s)ds.$$

(Cf. Heath, Jarrow, and Morton (1992), eq. 8.) Note that the bond price volatility $\sigma^{(i)}$ and the forward rate volatility $\widetilde{\sigma}^{(i)}$ have opposite signs. This reflects the fact that bond prices fall as interest rates rise and vice versa.

4.2 Overview of the Lévy Term Structure Model

Eberlein and Raible (1999) generalize the Gaussian Heath-Jarrow-Morton term structure model by replacing the Brownian motion W by a general Lévy process L in a suitable way. In this section, we describe their approach and main results.

Fix a finite time horizon T^*. Denote by $L = (L_s)_{s \geq 0}$ a Lévy process. The law of L_1, $\mathcal{L}(L_1)$, is infinitely divisible and hence by the Lévy-Khintchine formula is characterized by the Lévy-Khintchine triplet $\big(b, c, F(dx)\big)$. its Lévy measure F. In order to guarantee the existence of the expectations that appear in the term structure model, Eberlein and Raible (1999) impose the following integrability assumption on the Lévy measure F:
There are constants $M, \epsilon > 0$ such that

$$\int_{\{|x|>1\}} \exp(vx)F(dx) < \infty \qquad \forall\, |v| \leq (1+\epsilon)M. \tag{4.4}$$

As the underlying filtration, the completed canonical filtration of the Lévy process L is chosen.

A zero bond maturing at time $T \in [0, T^\star]$ is assumed to have the following price process.

$$P(t,T) = P(0,T) \cdot \beta(t) \cdot \frac{\exp\left(\int_0^t \sigma(s,T)dL_s\right)}{\mathrm{E}\left[\exp\left(\int_0^t \sigma(s,T)dL_s\right)\right]}, \tag{4.5}$$

where $\beta(t)$ denotes a process—called the *numéraire*— that is later determined by the boundary condition $P(t,t) = 1$, $t \in [0, T^*]$.

Besides (4.4), the following standard assumptions are made throughout the chapter. Recall that $T^\star$ is the fixed time horizon.

Assumption 4.1. *The initial bond prices are given by a deterministic, positive, and twice continuously differentiable function $T \mapsto P(0,T)$ on the interval $[0, T^\star]$.*

Assumption 4.2. $P(T,T) = 1$ *for all* $T \in [0, T^\star]$.

From this boundary condition, the explicit form of the process $\beta(t)$ is derived.

Assumption 4.3. *$\sigma(s,T)$ is defined on the triangle $\Delta := \{(s,T) : 0 \leq s \leq T \leq T^\star\}$. This function is twice continuously differentiable in both variables, and $\sigma(s,T) \leq M$ for all $(s,T) \in \Delta$, where M is the constant from (4.4). Furthermore, $\sigma(s,T) > 0$ for all $(s,T) \in \Delta$, $s \neq T$, and $\sigma(T,T) = 0$ for all $T \in [0, T^\star]$.*

For fixed $t \in [0, T^\star]$, introduce the *forward rate* with maturity T, contracted at time t, $f(t,T)$, and the *short rate* $r(t)$ as $\mathcal{F}_t$-measurable random variables

$$f(t,T) := -\frac{\partial}{\partial T} \ln P(t,T), \quad \text{and} \quad r(t) := f(t,t). \tag{4.6}$$

It is then shown that the short rate process satisfies

$$r(t) = f(0,t) + \int_0^t \frac{\partial}{\partial t}\kappa(\sigma(s,t))ds - \int_0^t \partial_2\sigma(s,t)dL_s, \tag{4.7}$$

where $\kappa(u) := \ln E[\exp(uL_1)]$ denotes the cumulant generating function of the distribution of L_1. The basic model (4.5) is proved to be equivalent to the following.

$$P(t,T) = P(0,T) \cdot \exp\left(\int_0^t \Big(r(s) - \kappa(\sigma(s,T))\Big)ds + \int_0^t \sigma(s,T)dL_s\right), \tag{4.8}$$

which generalizes the Gaussian HJM model (4.3). Section 4 of Eberlein and Raible (1999) examines the special case where the short-rate process r—which is defined by (4.6)—is a Markov process. It turns out that the subclass of the family of Lévy term structure models that has a Markovian short-rate process is characterized by a special form of the volatility structure: The partial derivative $\partial_2\sigma(t,T)$ can be factorized as a product of two functions depending only on t respectively T. For the case of stationary volatility structures—that is, functions $\sigma(t,T)$ that depend only on the time to maturity, $(T-t)$—the Markov property of the short rate implies that σ can only have a very special structure, namely,

$$\sigma(t,T) = \frac{\widehat{\sigma}}{a}\cdot\Big(1-e^{-a\cdot(T-t)}\Big) \qquad \text{(Vasicek volatility structure)}$$

$$\text{or}\quad \sigma(t,T) = \widehat{\sigma}\cdot(T-t) \qquad \text{(Ho-Lee volatility structure)},$$

with real constants $\widehat{\sigma} > 0$ and $a \neq 0$. (See Eberlein and Raible (1999), Theorem 4.4.) The proof of theses results concerning the Markov property requires an additional assumption on the rate of decay of the characteristic function for $|u| \to \infty$. (See Section 4.4 below.)

Section 5 of Eberlein and Raible (1999) examines the special case of the Lévy term structure model where the driving Lévy process L is a hyperbolic Lévy motion. For this case, the above-mentioned decay condition concerning the characteristic function is verified. Option prices are calculated numerically by first calculating a joint density of two stochastic integrals via a two-dimensional fast Fourier transform.

4.3 The Markov Property of the Short Rate: Generalized Hyperbolic Driving Lévy Processes

In this section, we show that the characteristic function of a generalized hyperbolic distribution *GH*$(\lambda,\alpha,\beta,\mu,\delta)$ is dominated by an exponential expression of some special form. This implies that the results of Eberlein and Raible (1999), Section 4, not only hold for a model driven by a hyperbolic Lévy motion, but as well for a model driven by a *generalized* hyperbolic Lévy motion. Thus the short-term interest rate r is a Markov process in this case iff the partial derivative $\partial_2\sigma(t,T)$ of the volatility structure can be factorized into functions depending only on t respectively T.

Proposition 4.4. *Let $\chi(u)$ denote the characteristic function of a generalized hyperbolic distribution GH$(\lambda,\alpha,\beta,\mu,\delta)$. Then there exist constants $C,\gamma,\eta > 0$ such that*

$$|\chi(u)| \le C\exp\big(-\gamma|u|^{\eta}\big) \quad \forall u \in \mathbb{R}.$$

More precisely, we can choose $\gamma := \delta/2$ and $\eta := 1$.

Application of Eberlein and Raible (1999), Theorem 4.3, then directly yields the following.

Corollary 4.5. *Let L be a generalized hyperbolic Lévy process with $\mathcal{L}(L_1) = GH(\lambda,\alpha,\beta,\mu,\delta)$. Assume that bond prices evolve according to the Lévy term structure model (4.8), where we additionally assume $\partial_2\sigma(t,T) > 0$.*

Then the short rate process derived from the Bond price dynamics via relation (4.6) has the Markov property iff the partial derivative $\partial_2\sigma(t,T)$ has a representation

$$\partial_2\sigma(t,T) = \tau(t)\cdot\zeta(T) \qquad \forall\ (t,T) \in \Delta,$$

where $\tau:\ [0,T^\star] \to \mathbb{R}$ and $\zeta:\ [0,T^\star] \to (0,\infty)$ are continuously differentiable functions. □

Proof of Proposition 4.4. We cite the following definition from Abramowitz and Stegun (1968), 3.6.15:

Definition 4.6. *A series* $\sum_{k=0}^{\infty} a_k x^{-k}$ *is said to be an* asymptotic expansion *of a function* $f(x)$ *if*

$$f(x) - \sum_{k=0}^{n-1} a_k x^{-k} = O(x^{-n}) \quad \textit{as } x \to \infty$$

for every $n = 1, 2, \ldots$. *We write*

$$f(x) \sim \sum_{k=0}^{\infty} a_k x^{-k}.$$

The series itself may be either convergent or divergent.

The characteristic function of a generalized hyperbolic distribution is given by

$$\chi_{GH(\lambda,\alpha,\beta,\delta,\mu)}(u) = e^{i\mu u} \frac{(\delta\sqrt{\alpha^2-\beta^2})^\lambda}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})} \cdot \frac{K_\lambda\big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\big)}{\big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\big)^\lambda}.$$

(See equation (A.3).) The modulus of this function is given by

$$\big|\chi_{GH(\lambda,\alpha,\beta,\delta,\mu)}(u)\big| = C_1 \cdot \frac{\big|K_\lambda\big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\big)\big|}{\big|\delta\sqrt{\alpha^2-(\beta+iu)^2}\big|^\lambda},$$

with a constant $C_1 > 0$ that depends only on the parameters λ, α, β, and δ (and not on the variable u.) Abramowitz and Stegun (1968), 9.7.2, provides the following asymptotic expansion for the modified Bessel function of the third kind:

$$\begin{aligned} K_\nu(z) &\sim \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{n=0}^{\infty} \left(\prod_{k=1}^{n} \Big(\frac{4\nu^2-(2k-1)^2}{8k} \Big) \right) \frac{1}{z^n} \\ &= \sqrt{\frac{\pi}{2z}} e^{-z} \Big\{ 1 + \frac{4\nu^2-1}{8z} + \frac{(4\nu^2-1)(4\nu^2-9)}{2!(8z)^2} + \cdots \Big\} \quad \Big(|\arg z| < \frac{3}{2}\pi\Big). \end{aligned} \tag{4.9}$$

We deduce that

$$\frac{\big|K_\lambda(z)\big|}{|z|^\lambda} \sim \frac{\big|\sqrt{\pi/(2z)}e^{-z}\big|}{|z|^\lambda} = \sqrt{\frac{\pi}{2}} \frac{e^{-\mathrm{Re} z}}{|z|^{\lambda+1/2}} \quad \Big(|\arg z| < \frac{3}{2}\pi\Big). \tag{4.10}$$

For a generalized hyperbolic distribution, the argument z has the value $z = \delta\sqrt{\alpha^2-(\beta+iu)^2}$. For this function, we have the following estimates.

Lemma 4.7. *a) For* $x \in \mathbb{R}$, *we have*

$$\mathrm{Re}(\sqrt{1+ix}) \geq 1,$$

where the square root of a complex number $z \notin (-\infty, 0]$ *denotes the complex number* z' with $\mathrm{Re}(z') > 0$ *such that* ${z'}^2 = z$.

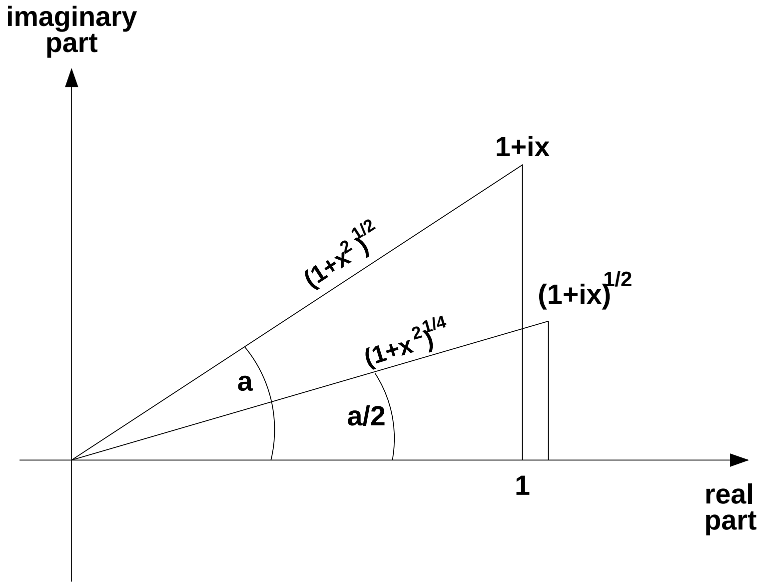

Figure 4.1: Illustration for Lemma 4.7a.

b) Let $\delta > 0$, $\alpha > 0$, and $\beta \in (-\alpha, \alpha)$. Then

$$\operatorname{Re}\Big(\delta\sqrt{\alpha^2 - (\beta + iu)^2}\Big) \geq \delta|u|, \tag{4.11}$$

and

$$\Big|\delta\sqrt{\alpha^2 - (\beta + iu)^2}\Big| \geq \delta|u|. \tag{4.12}$$

Furthermore, there is a constant C_2 (depending on the parameters δ, α, and β) such that

$$\Big|\delta\sqrt{\alpha^2 - (\beta + iu)^2}\Big| \leq C_2\sqrt{\alpha^2 - \beta^2} + C_2|u|. \quad (u \in \mathbb{R}).$$

Proof. Part a. Elementary trigonometry shows that the real part of $\sqrt{1 + ix}$ is given by $(1 + x^2)^{1/4}\cos(a/2)$, where a is the angle spanned by 1 and $1 + ix$ in the complex plane, that is, $a = \arctan(x)$.

From the angle addition formula

$$\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y),$$

we deduce that

$$\cos(2y) \leq \cos(y)^2 \qquad (y \in \mathbb{R}).$$

Since $\cos(2y) > 0$ for $|y| < \pi/4$, we thus have

$$\begin{aligned} 1 &\leq \frac{\cos(y)^4}{\cos(2y)^2} \\ &= \cos(y)^4\frac{\sin(2y)^2 + \cos(2y)^2}{\cos(2y)^2} \\ &= \cos(y)^4(\tan(2y)^2 + 1) \quad (|y| < \pi/4). \end{aligned}$$

Inserting $y = \arctan(x)/2$ yields

$$1 \leq \cos(\arctan(x)/2)^4(x^2+1)$$

and hence

$$1 \leq \cos(\arctan(x)/2)(x^2+1)^{1/4}.$$

But as was shown above, the right-hand side is nothing else than the real part of $\sqrt{1+ix}$. For part b), we note that

$$\begin{aligned}
\mathrm{Re}\Big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\Big) &= \delta\mathrm{Re}\sqrt{\alpha^2-\beta^2-2\beta iu+u^2} \\
&= \delta\sqrt{\alpha^2-\beta^2+u^2}\,\mathrm{Re}\sqrt{1-\frac{2\beta u}{\alpha^2-\beta^2+u^2}i} \\
&\geq \delta\sqrt{\alpha^2-\beta^2+u^2} \\
&\geq \delta|u|,
\end{aligned}$$

where we have used the result of part a). Estimation (4.12) follows trivially because $|z| \geq |\mathrm{Re} z|$ for $z \in \mathbb{C}$. Furthermore, we have

$$\begin{aligned}
\Big|\delta\sqrt{\alpha^2-(\beta+iu)^2}\Big| &= \delta\sqrt{\alpha^2-\beta^2+u^2}\left|\sqrt{1-\frac{2\beta u}{\alpha^2-\beta^2+u^2}i}\right| \\
&\leq \delta\sqrt{\alpha^2-\beta^2+u^2}C_2 \\
&\leq C_2\sqrt{\alpha^2-\beta^2}+C_2|u|
\end{aligned}$$

for some suitable constant $C_2 > 0$, because $u \mapsto \frac{2\beta u}{\alpha^2-\beta^2+u^2}$ is bounded. □

With Lemma 4.7 at hand, we can proceed with the proof of Proposition 4.4. As was shown above, the key to the estimation of the characteristic function is the estimation of the expression on the right-hand side of (4.10), viz

$$\sqrt{\frac{\pi}{2}}\frac{e^{-\mathrm{Re} z}}{|z|^{\lambda+1/2}}.$$

With $\delta\sqrt{\alpha^2-(\beta+iu)^2}$ substituted for z, this expression is asymptotically equivalent to (a multiple of) the modulus of the characteristic function.

$$\begin{aligned}
\left|\chi_{GH(\lambda,\alpha,\beta,\delta,\mu)}(u)\right| &\sim C_1\frac{e^{-\mathrm{Re}(\delta\sqrt{\alpha^2-(\beta+iu)^2})}}{|\delta\sqrt{\alpha^2-(\beta+iu)^2}|^{\lambda+1/2}} \quad (|u|\to\infty) \\
&\leq C_1\frac{e^{-\delta|u|}}{|\delta\sqrt{\alpha^2-(\beta+iu)^2}|^{\lambda+1/2}} \\
&= C_1\frac{e^{-\frac{\delta}{2}|u|}}{|\delta\sqrt{\alpha^2-(\beta+iu)^2}|^{\lambda+1/2}}e^{-\frac{\delta}{2}|u|}.
\end{aligned}$$

By Lemma 4.7b,

$$\delta|u| \leq \left|\delta\sqrt{\alpha^2-(\beta+iu)^2}\right| \leq C_2\sqrt{\alpha^2-\beta^2}+C_2|u|$$

for some constant C_2. Hence for $|u| \to \infty$,

$$\frac{e^{-\frac{\delta}{2}|u|}}{|\delta\sqrt{\alpha^2 - (\beta + iu)^2}|^{\lambda+1/2}} \to 0,$$

no matter what the value of λ is. The statement of the proposition follows. □

4.4 Affine Term Structures in the Lévy Term Structure Model

In some term structure models, the prices of zero coupon bonds can be written as exponential-affine functions of the current level of the short rate.

$$P(t, T) = \exp\big(A(t, T) - B(t, T)r(t)\big),$$

with the deterministic functions $A(t, T)$ and $B(t, T)$. If this is the case, one says that the model possesses an *affine term structure*. (See e. g. Björk (1998), Definition 17.1.) Existence of an affine term structure implies that for fixed times $t < T$ the yields on zero-coupon bonds at time t are affine functions of the current short rate.

$$Y(t, T) = -\frac{\ln P(t, T)}{T - t} = -\frac{A(t, T)}{T - t} + \frac{B(t, T)}{T - t}\, r(t).$$

The same holds for the forward rates.

$$f(t, T) = -\partial_2 \ln P(t, T) = -\partial_2 A(t, T) + \partial_2 B(t, T)\, r(t).$$

For Lévy term structure models satisfying a certain integrability condition, Eberlein and Raible (1999) prove a connection between the Markov property of the short rate and a factorization property of the volatility structure. (See Section 4.3.) The following theorem shows that the same factorization property is also sufficient for the existence of an affine term structure.

Theorem 4.8. *Assume that in a Lévy term-structure model there is a factorization*

$$\partial_2 \sigma(t, T) = \tau(t)\zeta(T)$$

of the derivative of the volatility structure, with a strictly positive function ζ. Then for any dates $t < T$ the bond price is

$$P(t, T) = \exp(A(t, T) - B(t, T)r(t)),$$

with the deterministic functions $A(t, T)$ and $B(t, T)$ given by

$$\begin{aligned}
A(t, T) &= \ln\Big(\frac{P(0, T)}{P(0, t)}\Big) + B(t, T)\Big(f(0, t) + \int_0^t \frac{\partial}{\partial t}\kappa(\sigma(s, t))ds\Big) \\
&\quad - \int_0^t \Big[\kappa\big(\sigma(s, T)\big) - \kappa\big(\sigma(s, t)\big)\Big]ds \\
B(t, T) &= \frac{1}{\zeta(t)} \int_t^T \zeta(u)du
\end{aligned}$$

Proof. We already know the following representation for the short rate (see (4.7)):

$$r(t) = f(0,t) + \int_0^t \frac{\partial}{\partial t}\kappa\big(\sigma(s,t)\big)ds - \int_0^t \partial_2\sigma(s,t)dL_s. \tag{4.13}$$

On the other hand, by (4.8),

$$P(t,T) = P(0,T)\exp\left(\int_0^t \Big(r(s) - \kappa\big(\sigma(s,T)\big)\Big)ds + \int_0^t \sigma(s,T)dL_s\right),$$

where we can substitute for $\exp\big(\int_0^t r(s)ds\big)$ by making use of the equality $P(t,t) = 1$:

$$\exp\Big(\int_0^t r(s)ds\Big) = \frac{1}{P(0,t)}\exp\left(-\int_0^t \sigma(s,t)dL_s + \int_0^t \kappa(\sigma(s,t))ds\right).$$

Hence we have

$$P(t,T) = \frac{P(0,T)}{P(0,t)}\cdot\frac{\exp\Big(\int_0^t \big[\sigma(s,T)-\sigma(s,t)\big]dL_s\Big)}{\exp\Big(\int_0^t \big[\kappa\big(\sigma(s,T)\big)-\kappa\big(\sigma(s,t)\big)\big]ds\Big)} \tag{4.14}$$

Observe that

$$\begin{aligned}\sigma(s,T)-\sigma(s,t) &= \int_t^T \partial_2\sigma(s,u)du\\ &= \partial_2\sigma(s,t)\cdot\frac{1}{\zeta(t)}\int_t^T \zeta(u)du.\end{aligned} \tag{4.15}$$

Set

$$B(t,T) := \frac{1}{\zeta(t)}\int_t^T \zeta(u)du.$$

Now consider the product $-B(t,T)\cdot r(t)$. We have

$$\begin{aligned}-B(t,T)\cdot r(t) &\overset{(4.13)}{=} -B(t,T)\left(f(0,t)+\int_0^t \frac{\partial}{\partial t}\kappa\big(\sigma(s,t)\big)ds\right) + B(t,T)\int_0^t \partial_2\sigma(s,t)dL_s\\ &\overset{(4.15)}{=} -B(t,T)\left(f(0,t)+\int_0^t \frac{\partial}{\partial t}\kappa\big(\sigma(s,t)\big)ds\right) + \int_0^t \big[\sigma(s,T)-\sigma(s,t)\big]\,dL_s.\end{aligned}$$

The only stochastic term appearing on the right-hand side is the integral $\int_0^t \big[\sigma(s,T)-\sigma(s,t)\big]\,dL_s$. But this is also the only stochastic term in (4.14). Therefore the bond price can be written as a deterministic function, applied to $-B(t,T)\cdot r(t)$ (and hence as a deterministic function, applied to $r(t)$.)

$$\begin{aligned}P(t,T) = \frac{P(0,T)}{P(0,t)}\exp\Big[&-B(t,T)r(t) + B(t,T)\Big(f(0,t)+\int_0^t \frac{\partial}{\partial t}\kappa(\sigma(s,t))ds\Big)\\ &-\int_0^t \big(\kappa(\sigma(s,T))-\kappa(\sigma(s,t))\big)ds\Big].\end{aligned}$$

Defining

$$\begin{aligned}A(t,T) := \ln\Big(\frac{P(0,T)}{P(0,t)}\Big) &+ B(t,T)\Big(f(0,t)+\int_0^t \frac{\partial}{\partial t}\kappa(\sigma(s,t))ds\Big)\\ &-\int_0^t \big(\kappa(\sigma(s,T))-\kappa(\sigma(s,t))\big)ds\end{aligned}$$

yields the desired result. $\square$

Remark: Observe that the functions A and B defined above satisfy $A(T,T)=0, B(T,T)=0$.

Implications for Term Structure Movements

Above we have noted that in a term structure model with an affine term structure, the forward rates are given by

$$f(t,T) = -\partial_2 A(t,T) + \partial_2 B(t,T)\, r(t).$$

This means that for fixed t every possible forward rate curve is a member of the family

$$\big\{T \mapsto -\partial_2 A(t,T) + \partial_2 B(t,T) r \big| r \in R\big\}.$$

The possible range R of the parameter r depends on the kind of interest rate model considered. In a Gaussian term structure model or the general Lévy model, we have $R = \mathbb{R}$, while in the Cox, Ingersoll, and Ross (1985) model, it would be $R = \mathbb{R}_+$.

4.5 Differential Equations for the Option Price

In what follows, we study one-factor interest rate models where the short-term interest rate r constitutes the only stochastic factor determining bond prices. We will see that the question of option pricing leads to an integro-differential equation. The reasoning here is analogous to the case of stock price models treated in Chapter 1.

Let r be given as the solution of a stochastic differential equation

$$r(t) = r(0) + \sum_{\alpha=1}^{d} \int f_\alpha(t-, r(t-))\, dL_t^\alpha, \tag{4.16}$$

driven by a vector $(L^1, \dots, L^d)$ of independent Lévy processes. Assume that each f_α is a *Lipschitz function* in the sense of the definition given in Protter (1992), Chapter V.3, p. 194:

Definition 4.9. *A function* $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$ *is* Lipschitz *if there exists a (finite) constant* k *such that*

(i) $|f(t,x) - f(t,y)| \le k|x-y|$*, each* $t \in \mathbb{R}_+$*;*

(ii) $t \mapsto f(t,x)$ *is right-continuous with left limits, each* $x \in \mathbb{R}^n$*.*

f *is* autonomous *if* $f(t,x) = f(x)$*, all* $t \in \mathbb{R}_+$*.*

As a consequence of Protter (1992), Theorem V.7, equation (4.16) has a unique solution.[4] Protter (1992), Theorem V.32, shows that r is a Markov process. For this, the independence of the increments of L^α is essential.

Remark: The above considerations are still valid for stochastic differential equations with an additional drift term "$\dots dt$". This is because one can take account of the drift term by considering the deterministic Lévy process $L_t^{d+1} = t$. This yields an equation of the form (4.16).

[4] The cited theorem is valid for so-called functional Lipschitz coefficient functions. According to a remark in Protter (1992), p. 195, the functional Lipschitz functions f_α induce functional Lipschitz coefficients.

Assumption 4.10. *The short rate follows the diffusion (with jumps) given by*

$$dr_t = k(r_{t-}, t)dt + f(r_{t-}, t)dL_t, \tag{4.17}$$

with Lipschitz coefficient functions $k(r,t)$ *and* $f(r,t)$.

For example, the Lévy version of the extended Vasicek model (4.1) is of this form. In this case, the functions k and f are given by

$$\begin{cases} k(r,t) = (\rho(t) - r)a \\ f(r,t) = -\widehat{\sigma}, \end{cases} \tag{4.18}$$

with a deterministic function $\rho(t)$ and positive constants $\widehat{\sigma}$, a.

Proposition 4.11. *Assume that we are given a European option with a price process* $V(t)$ *satisfying the following conditions.*

(i) The payoff at the maturity date T *is* $V(T) = v\big(r(T)\big)$*, with a deterministic function* $v(x)$.

(ii) $\exp\big(-\int_0^t r(s)\ ds\big)V(t)$ *is a* Q*-martingale. That is,* Q *is a martingale measure for this option.*

(iii) $\exp\big(-\int_t^T r(s)\ ds\big)V(T)$ *is* Q*-integrable for all* $t \in [0,T]$.

Then there is a function $g(x,t)$ *such that the option price at time* $t \in [0,T]$ *is given by* $V(t) = g\big(r(t),t\big)$.

Remark. If the term structure model possesses an affine term structure, condition *(i)* is satisfied for all simple European options on a zero bond.[5] This is because the zero bond price $P(T,S)$ itself can be written as a deterministic (exponential-affine) function of the short rate $r(T)$.

Proof of Proposition 4.11. By assumptions *(i)* and *(ii)*, the option price at any time $t \in [0,T]$ can be obtained by taking conditional expectations.

$$\begin{aligned} \exp\Big(-\int_0^t r(s)ds\Big)V(t) &= E_Q\left[\exp\Big(-\int_0^T r(s)ds\Big)v\big(r(T)\big)\middle|\,\mathcal{F}_t\right], \\ \text{hence}\quad V(t) &= \frac{1}{\exp\big(-\int_0^t r(s)ds\big)}E_Q\left[\exp\Big(-\int_0^T r(s)ds\Big)v\big(r(T)\big)\middle|\,\mathcal{F}_t\right] \\ &= E_Q\left[\exp\Big(-\int_t^T r(s)ds\Big)v\big(r(T)\big)\middle|\,\mathcal{F}_t\right]. \end{aligned}$$

For the last equality, we have used condition *(iii)* from above. The last conditional expectation only depends on the conditional distribution of $(r(s))_{t\le s\le T}$ given $\mathcal{F}_t$. But because of the Markov property of r, this is equal to the conditional distribution given $r(t)$, and hence

$$V(t) = E_Q\left[\exp\Big(-\int_t^T r(s)ds\Big)v\big(r(T)\big)\middle|\,r(t)\right].$$

For each $t \in [0,T]$, this conditional expectation can be factorized. $V(t) = g\big(r(t),t\big)$. □

If we impose additional differentiability assumptions on the function $g(x,t)$, we can deduce that this function satisfies a linear integro-differential equation.

[5] *Simple* here means that the value of the option at its expiration date T can be written as a deterministic function, of the bond price.

Proposition 4.12. *We make the following assumptions.*

(i) The value process $V(t)$ of a security can be represented in the form $V(t) = g\big(r(t), t\big)$ with a deterministic function $g(x, t)$.

(ii) The function $g(x, t)$ from (i) is of class $C^{2,1}(\mathbb{R}\times\mathbb{R}_+)$, that is, it is twice continuously differentiable in the first variable and once continuously differentiable in the second variable.

(iii) The short-rate process r satisfies Assumption 4.10. For each $t > 0$, the distribution of r_{t-} has support I, where I is a bounded or unbounded interval.

Then the function $g(r, t)$ satisfies the integro-differential equation

$$
\begin{aligned}
(4.19)\qquad & 0 = -g(r,t)r + \partial_2 g(r,t) + (\partial_1 g)(r,t)\big(f(r,t)b + k(r,t)\big) + \frac{1}{2}(\partial_{11}g)(r,t)\cdot c\cdot f(r,t)^2 \\
&\qquad + \int \Big(g\big(r + f(r,t)x, t\big) - g\big(r,t\big) - (\partial_1 g)\big(r,t\big)f(r,t)x\Big)F(dx) \quad \big(r \in I, t \in (0,T)\big) \\
& v(r) = g(r,T),
\end{aligned}
$$

where $(b, c, F(dx))$ is the Lévy-Khintchine triplet of the Lévy process driving the stochastic differential equation (4.10).

Remark. We are not aware of suitable results about continuity and differentiability of the function $g(x, t)$ from Proposition 4.11 in the case where r follows a jump-diffusion. Hence we introduce assumption *(ii)* in the proposition in order to guarantee that the differential equation (4.19) makes sense.

Proof of Proposition 4.12. Consider the discounted option price process

$$\widetilde{V}(t) := \exp\Big(-\int_0^t r(s)ds\Big)V(t) = \exp\Big(-\int_0^t r(s)ds\Big)g\big(r(t), t\big).$$

The discount factor process $\gamma(t) := \exp(-\int_0^t r(s)ds)$ is continuous and of finite variation, and so the quadratic co-variation $[\gamma, V]$ vanishes. (See e. g. Jacod and Shiryaev (1987), Proposition I.4.49 d.) Hence

$$(4.20)\qquad d(\gamma\, V)_t = \gamma(t-)dV(t) + V(t-)d\gamma(t).$$

Ito's formula provides the stochastic differential of V:

$$
\begin{aligned}
dV(t) = g_t(r_{t-}, t)dt + g_r(r_{t-}, t)dr_t + \frac{1}{2}g_{rr}(r_{t-}, t)d\langle r^c, r^c\rangle_t& \\
+ \big(g(r_{t-} + \Delta r_t, t) - g(r_{t-}, t) - g_r(r_{t-}, t)\Delta r_t\big)&.
\end{aligned}
$$

The predictable quadratic variation of r^c (which appears as an integrator here) is given by $d\langle r^c, r^c\rangle_t = f(r_{t-}, t)^2 c\, dt$. The process γ has differentiable paths. Therefore it is of bounded variation.

$$d\gamma(t) = -r_t\gamma(t)dt.$$

Hence V is the sum of a local martingale and the following predictable process of finite variation.

$$
\begin{aligned}
(4.21)\qquad & g_t(r_{t-}, t)dt + g_r(r_{t-}, t)\big(f(r_{t-}, t)b + k(r_{t-}, t)\big)dt + \frac{1}{2}g_{rr}(r_{t-}, t)\cdot c\cdot f(r_{t-}, t)^2 dt \\
&\qquad + \Big[\int \Big(g(r_{t-} + f(r_{t-}, t)x, t) - g(r_{t-}, t) - g_r(r_{t-}, t)f(r_{t-}, t)x\Big)F(dx)\Big]\, dt.
\end{aligned}
$$

By (4.20), this means that the process γV is the sum of a local martingale and the following predictable process of finite variation starting in 0.

$$
\begin{aligned}
&\gamma(t)\Big\{ -g(r_{t-},t)r(t) + g_t(r_{t-},t) + g_r(r_{t-},t)\big(f(r_{t-},t)b + k(r_{t-},t)\big) \\
&\qquad\qquad + \frac{1}{2} g_{rr}(r_{t-},t)\cdot c \cdot f(r_{t-},t)^2 \\
&\qquad\qquad + \Big[\int \big(g(r_{t-} + f(r_{t-},t)x,t) - g(r_{t-},t) - g_r(r_{t-},t)f(r_{t-},t)x\big) F(dx)\Big]\Big\} dt.
\end{aligned}
\tag{4.22}
$$

This decomposition is the special semimartingale decomposition of γV in the sense of Jacod and Shiryaev (1987), Definition I.4.22. Since the decomposition of a special semimartingale into a local martingale and a predictable process of finite variation is unique, we conclude that the process (4.22) vanishes identically: Otherwise there would be two special semimartingale decompositions, because of course $\gamma V = \gamma V + 0$, where γV is a (local) martingale by assumption. Hence we have

$$
\begin{aligned}
0 = &-g(r_{t-},t)r(t) + g_t(r_{t-},t) + g_r(r_{t-},t)\big(f(r_{t-},t)b + k(r_{t-},t)\big) + \frac{1}{2} g_{rr}(r_{t-},t)\cdot c \cdot f(r_{t-},t)^2 \\
&+ \int \big(g(r_{t-} + f(r_{t-},t)x,t) - g(r_{t-},t) - g_r(r_{t-},t)f(r_{t-},t)x\big) F(dx).
\end{aligned}
$$

Since the distribution of r_{t-} has support I, we conclude that for every $r \in I$ the following equation holds.

$$
\begin{aligned}
0 = &-g(r,t)r + g_t(r,t) + g_r(r,t)\big(f(r,t)b + k(r,t)\big) + \frac{1}{2} g_{rr}(r,t)\cdot c \cdot f(r,t)^2 \\
&+ \int \Big(g\big(r + f(r,t)x,t\big) - g\big(r,t\big) - g_r\big(r,t\big)f(r,t)x\Big) F(dx).
\end{aligned}
$$

This is the desired integro-differential equation for the option pricing function $g(r,t)$. $\square$

The Fourier Transform of the Option Pricing Equation

We briefly sketch a possible method for solving the integro-differential equation for $g(x,t)$. Of course, further studies in this direction are necessary in order to come up with numerically tractable methods. However, this would go far beyond the scope of this thesis. We assume that $g(x,t)$ is sufficiently regular for the Fourier inversion to make sense. Assume that we have coefficient functions (4.18).

We have the following identities for sufficiently regular functions f:

$$
\begin{aligned}
\int f(x) x \exp(iux) dx &= \frac{1}{i} \int f(x) \frac{\partial}{\partial u} \exp(iux) dx \\
&= -i \frac{\partial}{\partial u} \int f(x) \exp(iux) dx,
\end{aligned}
$$

and

$$
\begin{aligned}
\int f'(x) \exp(iux) dx &= - \int f(x) \frac{\partial}{\partial x} \exp(iux) dx \\
&= -iu \int f(x) \exp(iux) dx,
\end{aligned}
$$

The transform of $f'(x)x$ is therefore

$$\begin{aligned} \widehat{f'(x)x}(u) &= -i\widehat{f'(x)}_u(u) \\ &= -(u\widehat{f}(u))_u = -\widehat{f}(u) - u\widehat{f}_u(u). \end{aligned}$$

Thus by Fourier transforming (4.19) with respect to the variable x, we get the following equation for the Fourier transform $\widehat{g}(u,t) = \int \exp(iur)g(r,t)\ dr$.

$$\begin{aligned} 0 = i\partial_1\widehat{g} + \partial_2\widehat{g} - iu(\widehat{\sigma}b + a\rho(t))\widehat{g} + a\widehat{g} + au\partial_1\widehat{g} - \frac{1}{2}c\widehat{\sigma}^2u^2\widehat{g} \\ + \widehat{g}\int \left(e^{-iu\widehat{\sigma}x} - 1 + iu\widehat{\sigma}x\right)F(dx). \end{aligned}$$

The sum of the third and the last two terms on the right-hand side is readily identified as $\widehat{g}\ln\phi(-\widehat{\sigma}u)$, where $\phi(u)$ denotes the exponent in the Lévy-Khintchine representation of the characteristic function of L_1. Hence

$$\widehat{g}_t = -(au+i)\widehat{g}_u + (iua\rho(t) - a)\widehat{g} - \widehat{g}\cdot\ln\phi(-\widehat{\sigma}u) \tag{4.23}$$

This is a partial differential equation involving only the first derivatives of the Fourier transform $\widehat{g}$. Furthermore, in contrast to the original integro-differential equation, integration with respect to the Lévy measure $F(dx)$ is not required here. This could be an advantage, since the calculation of the density of the Lévy measure often is computationally demanding.

Chapter 5

Bond Price Models: Empirical Facts

5.1 Introduction

Since the introduction of exponential Brownian motion as a stock price model by Samuelson (1965) and its application in the famous work by Black and Scholes (1973) leading to the Black-Scholes formula, Brownian motion has become the basic building block of mathematical finance. In spite of this overwhelming success, it has long been known that normal distributions provide only a poor fit to empirically observed distributions of stock returns. We show that this is also true for zero-coupon bonds. As in the case of stocks, generalized hyperbolic distributions provide a much better fit for the empirical distributions of log returns on zero-coupon bonds.

5.2 Log Returns in the Gaussian HJM Model

Heath, Jarrow, and Morton (1992) propose a general model for the term structure, which is driven by a Brownian motion. In their model, volatility is allowed to vary stochastically. The special case of deterministic volatility, which was considered e. g. in El Karoui, Lepage, Myneni, Roseau, and Viswanathan (1991) and Carverhill (1994), leads to a Gaussian process for the logarithm of zero bond prices. Therefore we call it the *Gaussian HJM term structure model*. Under very general assumptions, the martingale—or, risk-neutral—measure is uniquely determined in the HJM model (see Heath, Jarrow, and Morton (1992), Proposition 2; for the case of a Gaussian HJM model with a one-dimensional driving Brownian motion, this also follows from Theorem 6.9.) Under the martingale measure, the bond price dynamics in the Gaussian HJM model have the form

$$P(t,T) = P(0,T)\exp\Big(\int_0^t r(s)ds\Big)\exp\Big(\int_0^t \sigma(s,T)dB_s - \int_0^t \frac{\sigma(s,T)^2}{2}ds\Big). \tag{5.1}$$

(See El Karoui, Lepage, Myneni, Roseau, and Viswanathan (1991) or Eberlein and Raible (1999); the latter gives a theorem for more general driving Lévy processes.) The volatility structure $\sigma(t,T)$, which gives the volatility at time t of a zero bond maturing at time $T \geq t$, is deterministic. By equation (5.1), the log return (between times t and $t+\Delta t$) on a zero-coupon bond maturing at time T is given by

$$\ln P(t+\Delta t,T) - \ln P(t,T) = \int_t^{t+\Delta t} r(s)\ ds + \int_t^{t+\Delta t} \sigma(s,T)\ dB_s - \int_t^{t+\Delta t} \frac{\sigma(s,T)^2}{2} ds.$$

For $\Delta t \to 0$, the integrals may be replaced by the product of the value of the integrand at the left endpoint times the increment of the integrator. Hence if we assume that Δt is small enough, we have

$$\ln P(t+\Delta t, T) - \ln P(t,T) \approx r(t)\Delta t + \sigma(t,T)(B_{t+\Delta t} - B_t) - \frac{\sigma(t,T)^2}{2}\Delta t.$$

Assume that the volatility is stationary, that is, $\sigma(s,T)$ depends only on the time to maturity given by the difference $T-s$. For brevity, we will use the same symbol σ for the function depending only on the difference: $\sigma(s,T) = \sigma(T-s)$. Then the *discounted* log-return is approximately normally distributed under the martingale measure.

$$\mathcal{L}^{\text{martingale measure}}\Big(\ln \frac{P(t+\Delta t, T)}{P(t,T)} - r(t)\Delta t\Big) \approx N\Big(-\frac{\sigma(T-t)^2}{2}\Delta t, \sigma(T-t)^2\Delta t\Big)$$

Since we have assumed that the risk-neutral dynamics of bond prices are described by (5.1), there has to be an equivalent probability measure that describes the statistically observable dynamics on the market.[1] As in Chapter 1, we call this measure the *objective probability measure*. By Girsanov's theorem we know that under any equivalent probability measure the Brownian motion B becomes a Brownian motion with a (generally random) drift $\mu(t)$ (see e. g. Jacod and Shiryaev (1987), Theorem III.3.24.)

$$dB_t = dW_t + \mu(t)dt,$$

with some optional stochastic process μ and a process W that is a Brownian motion under the equivalent measure.

We assume that μ is a deterministic constant. Then under the objective measure, again discounted log returns are (approximately) normally distributed.

$$\mathcal{L}^{\text{objective measure}}\Big(\ln \frac{P(t+\Delta t, T)}{P(t,T)} - r(t)\Delta t\Big) \approx N\Big(\Big[\mu - \frac{\sigma(T-t)^2}{2}\Big]\Delta t, \sigma(T-t)^2\Delta t\Big).$$

5.3 The Dataset and its Preparation

We work with a dataset containing daily data on zero-coupon bonds on the German market. For each of the 2343 trading day between 31-Dec-1985 and 10-May-1995, it lists the yields of zero-coupon bonds with maturities ranging from one to ten years, in steps of one year. In the covered period, government zero-coupon bonds did not exist on the German market.[2] Therefore, the quoted yields have been calculated from the known prices of government coupon bonds. Although we do not have information on the actual algorithm used for this calculation, we will assume that the given yields are those seen by market participants.

As usual, we adopt a trading day time scale here. That is, we assume that the time variable t counts trading days rather than calendar days. All interest rates are calculated relative to this time scale. For example, at an interest rate of $r = 0.0002$, an initial capital of 1000 will earn an interest of $10 \cdot 1000 \cdot 0.0002 = 2$ during a period of 10 trading days (which is corresponds to two weeks in calendar time.)

[1] The usual statement is that there exists an equivalent martingale measure for the market; here we go the other way around and start with the martingale measure, assuming that it is in fact the uniquely determined martingale measure belonging to a certain objective probability measure.

[2] Only recently, so-called *stripped* government bonds have been introduced for trade in Germany. These are government bonds whose coupons are traded separately. This means that the annual payments are traded as separate securities, while the stripped bond is effectively a zero bond. That is, it pays its holder the face value at the maturity date.

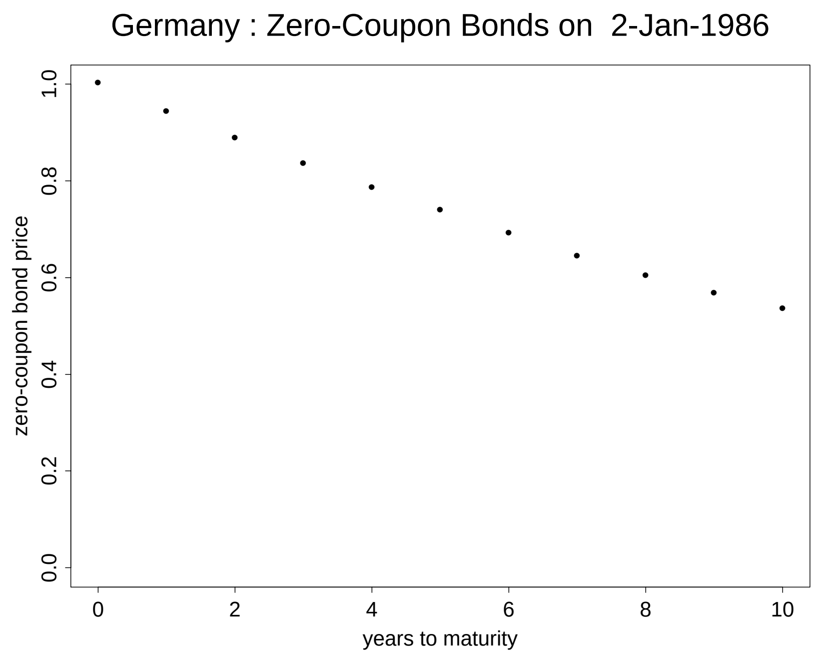

Figure 5.1: Implied prices for zero-coupon bonds on the German market on 2-Jan-1986. (Source: Zero coupon yields dataset)

5.3.1 Calculating Zero Coupon Bond Prices and Log Returns From the Yields Data

The zero-coupon yield for a maturity of n years is the number y_n for which the expression $\big(1+y_n\big)^{-n}$ equals the price $P(t, t+(n \text{ years}))$ of a zero-coupon bond with n years to maturity. Thus for each day, we can recover zero-coupon prices from zero yields:

$$P(t, t+(n \text{ years})) = \big(1+y_n\big)^{-n}. \tag{5.2}$$

For $t =$2-Jan-1986, these bond prices are given in Figure 5.1. Bond prices for a time to maturity of up to 10 years can be interpolated from these prices. In order to be able to interpolate bond prices with a maturity of less than one year, we use the fact that a zero-coupon bond has a value of 1 at its maturity date. Hence we can extend the bond price "curve" in Figure 5.1 by setting the bond price to 1 for a time-to-maturity of 0 years. Even with a constant interest rate, bond prices fall exponentially as a function of the time to maturity. Therefore the curvature of the bond price curve is usually positive, which leads to errors when interpolating linearly to find bond prices for intermediate times to maturity. Therefore we transform zero bond prices by taking the logarithm of the inverse bond price, or, equivalently, taking the negative of the logarithm of the bond price. In the case of a constant interest rate r, this transformation yields a linear function, namely $\tau \mapsto r\tau$. So linear interpolation is likely to introduce smaller errors when working with the transformed bond price curve. We approximate the negative of the logarithm of the zero-bond price by fitting a cubic smoothing spline, generated with the S-Plus command

```
s <- smooth.spline(0:10,-log(zeroBond.price[1:11,i]),df=10).
```

Figure 5.2 shows the negative log bond prices as well as the fitted smoothing spline.

Figure 5.2: Negative of the log bond prices for the German market, 2-Jan-1986.

As usual, the forward rate is the derivative of the negative logarithm of the bond price, with respect to the maturity date T.

$$f(t,T) = \frac{\partial}{\partial T}\Big(- \ln P(t,T)\Big).$$

They can be estimated from our discretely sampled bond price curve by taking the derivative of the smoothing spline. We will only need the derivative at $t = T$, which yields the short-term interest rate $r(t)$.

$$r(t) = f(t,t) = -\frac{\partial \ln P(t,T)}{\partial T}\Big|_{T=t}. \tag{5.3}$$

We estimate this by the S-Plus command

```
predict.smooth.spline( s, 0, 1 ),
```

which gives the first derivative at $n = 0$ of the smoothing spline fitted above.

One may wonder why we do not determine the short-term interest rate directly by taking a time series of short-term money market rates such as three-month LIBOR rates. The reason for this is the following. The money market is a market for short term lending and borrowing only. As such, it is intrinsically different from the bond market, which is a market for long-term investment and financing. In particular, the usances on the two market are different. We prefer to stay within one market, namely the bond market. Therefore it makes sense for us to use the short-term interest rate (5.3), which is the one implied by the bond market itself.

At time t, the price of a bond with an integer number n of years to maturity can be calculated directly from the data, using equation (5.2). But one trading day later, at time $t + \Delta t$, the same bond will have

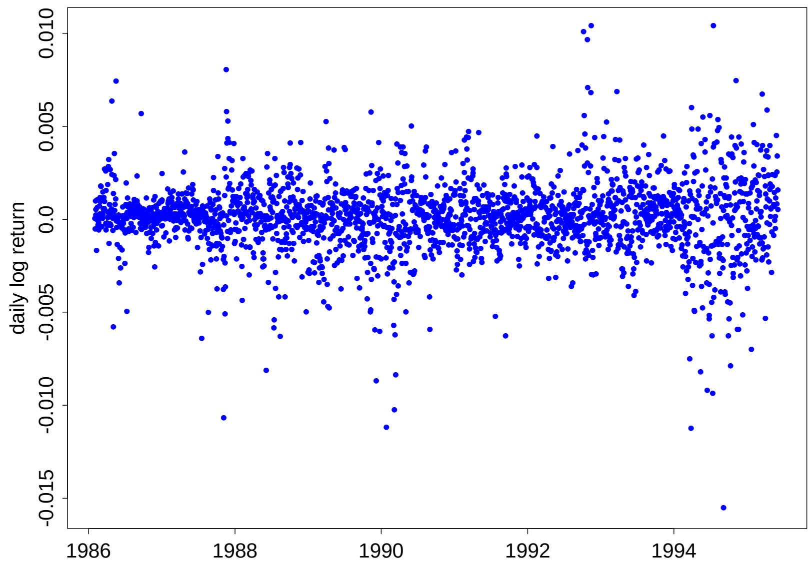

Figure 5.3: Estimated daily discounted log-returns on zero bonds with five years to maturity on the German market, January 1986 to May 1995.

a time to maturity of n *years minus* Δt, which cannot be calculated directly from the dataset; hence we have to use the interpolation method described above to determine its market price. With the prices of the same bond today and tomorrow, we calculate the log return on a bond with a time to maturity of n years. This log return is then discounted with the short term interest rate estimated above. Hence we approximate the discounted log return over a the period $[t, t + \Delta t]$ by the expression

$$\ln P(t, t + (n \text{ years})) - \ln P(t + \Delta t, t + (n \text{ years}))_{\text{interpolated}} - \Delta t \cdot r(t)_{\text{derived from bond prices}}.$$

In this way, we build up a dataset of daily log returns for zero-coupon bonds with times to maturity of $n = 1, \ldots, 10$ years. For each value of n, we have a time series of 2342 log returns.

5.3.2 A First Analysis

Figure 5.3 shows the log returns on zero bonds with five years to maturity, from January 1986 to May 1995. Figure 5.4 shows the log returns on the German stock index DAX for comparison. Both time series do not seem to be perfectly stationary over a range of ten years. For the practical use of models with a stationary volatility structure (in the bond price case) or a constant volatility parameter (in the stock price case), this means that one should better concentrate on shorter time horizons. The log returns on bonds are markedly smaller than those on the stock index. Table 5.1 shows this quantitatively. Note how volatility (that is, the standard deviation of the discounted log return) increases with time to maturity. Hence bonds that are closer to maturity are less risky. The dependence of the volatility on the time to maturity is displayed graphically in Figure 5.5.

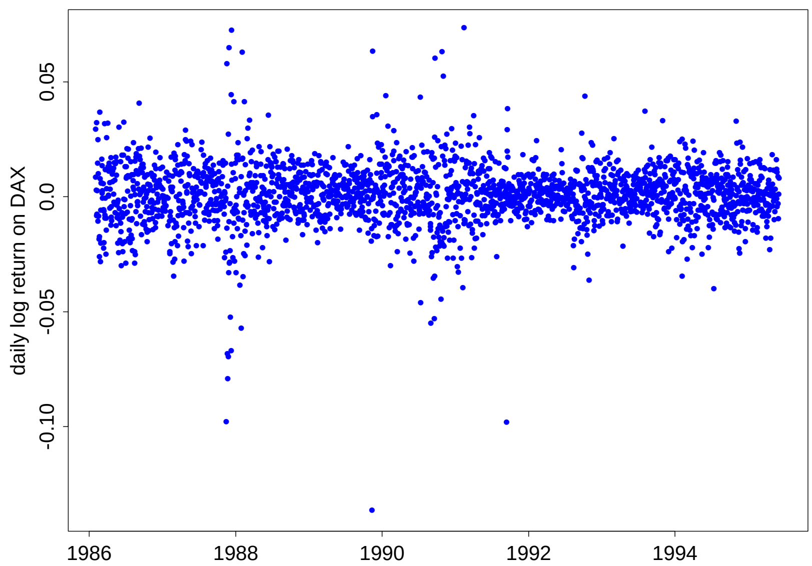

Figure 5.4: Daily log-returns on the German stock index DAX, January 1986 to May 1995.

	DAX	Zero coupon bonds with time to maturity [in years]						
		1	2	3	4	5	7	10
mean abs.	0.91%	0.041%	0.068%	0.089%	0.11%	0.13%	0.21%	0.36%
std. dev.	1.3%	0.062%	0.1%	0.13%	0.17%	0.19%	0.31%	0.54%

Table 5.1: Mean absolute log returns and standard deviation of log returns on the German stock index DAX and on German zero-coupon bonds with maturities of up to ten years. January 1986 to May 1995.

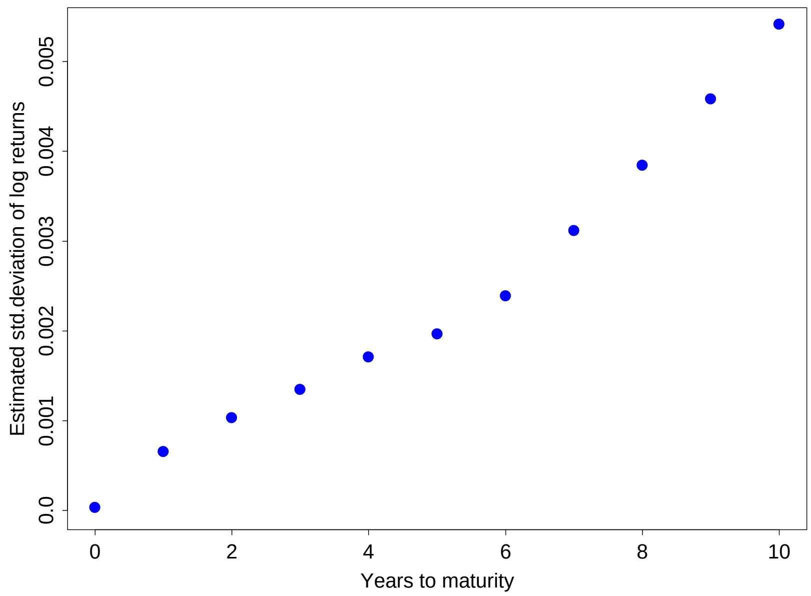

Figure 5.5: Standard deviation of zero-bond log returns: Dependence on time to maturity.

5.4 Assessing the Goodness of Fit of the Gaussian HJM Model

5.4.1 Visual Assessment

Quantile-quantile plots and density plots are powerful tools for the visual assessment of the goodness of fit for an empirical distribution. A *quantile-quantile plot* of an ordered sample $y = (y_1 \leq \ldots \leq y_n)$ plots y_j (that is, the empirical $(j - 1/2)/n$-quantile of the data) against the $(j - 1/2)/n$-quantile of the fitted distribution, which we assume to have a continuous distribution function. If the fit is good, then the points (x_j, y_j), $j = 1, \ldots, n$, should lie close to the line $y = x$. Figure 5.6 shows the strong deviation from normality of the log return distribution for 5-year bonds. The fact that the points lie far below the line $x = y$ for small quantiles and far above this line for large quantiles shows that the empirical distribution has fatter tails than the fitted normal distribution.[3]

In *density plots*, the empirical density of the sample, that is, a Gaussian kernel estimation of the density from the sample, is compared with the density of the fitted distribution. Figure 5.7 shows the two densities. It can be clearly seen that the empirical distribution is leptokurtic, that is, it puts more mass around the origin and in the tails than a normal distribution with the same mean and standard deviation. In terms of bond prices, this means that relatively small daily price changes and relatively large daily price changes take place more frequently than the Gaussian HJM model predicts. On the other hand, price changes of medium size are observed less frequently than in the Gaussian model. Choosing the log scale for the y-axis allows us to study the tail behavior of the distributions. Figure 5.8 compares the log densities of the empirical distribution and the log density of the fitted normal distribution. The log density of the normal distribution is a parabola, while the log of the empirical density resembles a hyperbola,

[3]The normal distribution was fitted by choosing the sample mean and the sample standard deviation.

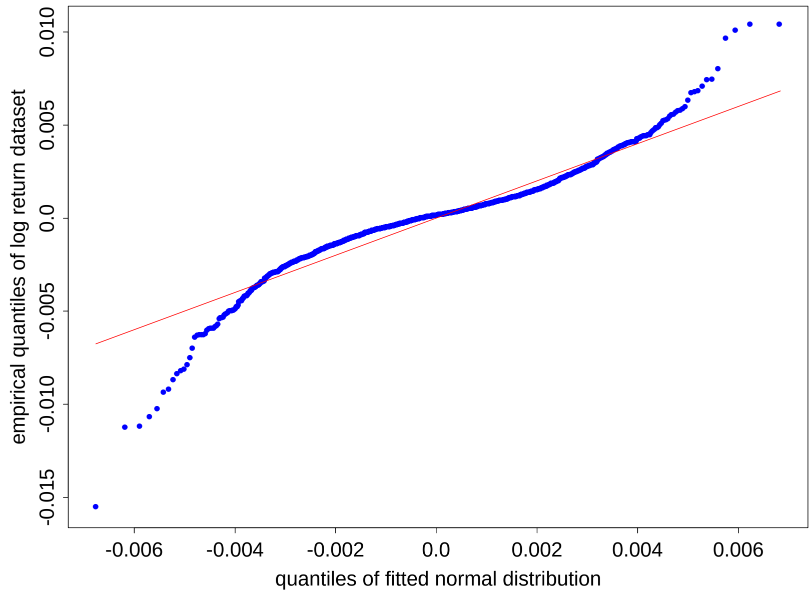

Figure 5.6: Quantile-quantile plot: Empirical quantiles of log returns on 5-year bonds against quantiles of fitted normal distribution. January 1986 to May 1995.

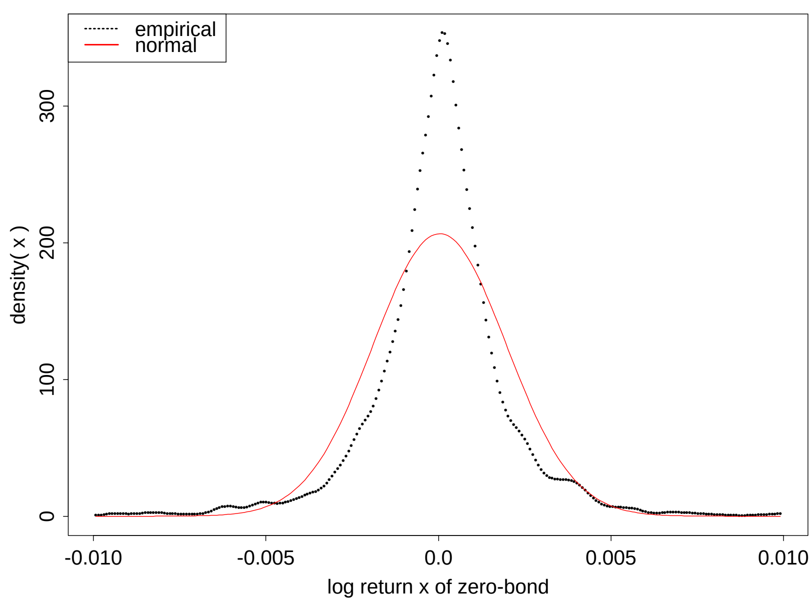

Figure 5.7: Density plot: Empirical density of log returns on 5-year bonds and density of fitted normal distribution. January 1986 to May 1995.

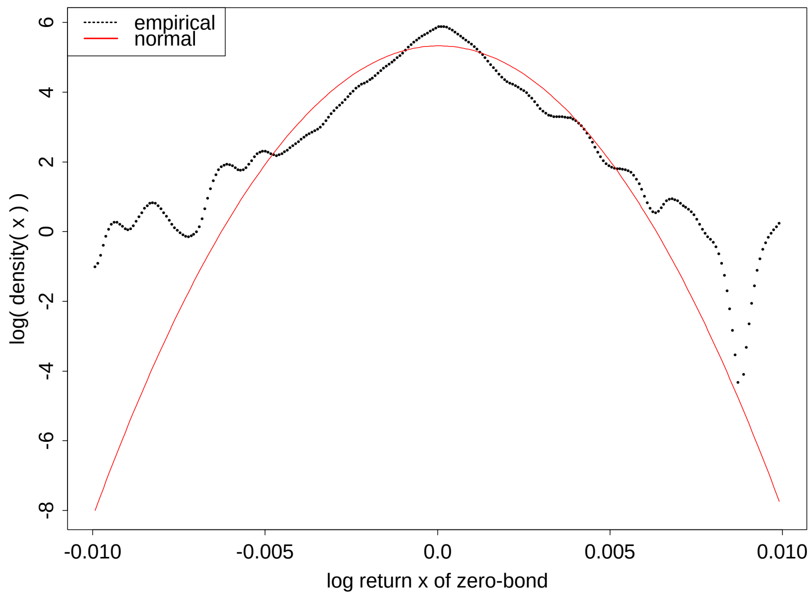

Figure 5.8: Log-density plot: Logarithm of empirical density of log returns on 5-year bonds against log density of fitted normal distribution. German bond market, January 1986 to May 1995.

at least in the central region. We see that the empirical distribution puts considerably more mass into the tails than one would expect if log returns were normally distributed. As a conclusion, we can say that, judged by the visual measures of quantile-quantile plot and density plot, the Gaussian model performs poorly as a description of the empirically observed daily movements of bond prices.

5.4.2 Quantitative Assessment

In this subsection, we apply two common goodness-of-fit tests to test the null hypothesis that the discounted log returns on zero-coupon bonds are normally distributed. The *Kolmogorov distance* of two probability distributions on $(\mathbb{R}, \mathcal{B})$ (given by their distribution functions F and G) is defined by

$$d_K(F,G) := \sup_{x\in\mathbb{R}} |F(x) - G(x)|.$$

If G is the empirical distribution $G_{\overline{x}}$ of a sample $\overline{x} = (x_1, \dots, x_n)$ of size n, this distance can be written as follows.

$$d_K(F, G_{\overline{x}}) = \max_{1\leq k\leq n} \Big\{ F(x_k-) - \frac{k-1}{n}, \frac{k}{n} - F(x_k) \Big\},$$

where $F(x-)$ denotes the left limit of the distribution function F at the point x, i. e. $F(x-)$ is the measure assigned to the open interval $(-\infty, x)$. For a distribution F without point masses we have $F(x-) = F(x)$, and consequently

$$d_K(F, G_{\overline{x}}) = \max_{1\leq k\leq n} \Big\{ F(x_k) - \frac{k-1}{n}, \frac{k}{n} - F(x_k) \Big\}.$$

α	20%	10%	5%	2%	1%
$\lambda_{1-\alpha}$	1.08	1.23	1.36	1.52	1.63

Table 5.2: Critical values for the Kolmogorov-Smirnov test if $n > 40$. (From: Hartung (1986))

$1-\alpha$	85%	90%	95%	97.5%	99%
$Q_{1-\alpha}$	0.775	0.819	0.895	0.995	1.035

Table 5.3: Approximate quantiles of the distribution of the modified Kolmogorov-Smirnov statistic D^{mod} for the test against a normal distribution whose parameters are estimated from the sample via the standard estimators. (From: Stephens (1986))

The *Kolmogorov-Smirnov test* uses the Kolmogorov distance of the empirical distribution function $G_{\overline{x}}$ and a given continuous distribution function F to test whether $\overline{x}$ was sampled from the distribution F. It rejects this hypothesis if the Kolmogorov distance is too large, that is, if

$$D_n := \sqrt{n} d_K(F, G_{\overline{x}}) \geq \lambda_{1-\alpha},$$

with a value $\lambda_{1-\alpha}$ that depends on the significance level α.

The situation is somewhat different if one wants to test whether the sample $\overline{x}$ was drawn from a distribution from a parameterized class $\mathcal{F} = \{F^\theta : \theta \in \Theta\}$, where $\Theta \subset \mathrm{I\!R}^d$ for some dimension d. Then usually one first estimates the unknown parameter θ from the sample $\overline{x}$, say, by maximum likelihood. Then one calculates the Kolmogorov distance between the empirical distribution $G_{\overline{x}}$ and the estimated distribution F^θ. However, since one has used the sample $\overline{x}$ to determine the distribution F^θ, the distribution of the Kolmogorov distance is not known in general. For the Kolmogorov-Smirnov test on normal distribution, a formula for the tail probability was derived in Tyurin (1985). Another approach by Stephens (1974) (see Stephens (1986)) uses the fact that the modified Kolmogorov-Smirnov statistic

$$D_n^{\text{mod}} := D_n \cdot (\sqrt{n} - 0.01 + 0.85/\sqrt{n}) \tag{5.4}$$

has a distribution that exhibits a very weak dependence on n. Approximate quantiles of this distribution are given in Stephens (1986), Table 4.7. We reproduce them in Table 5.3.

We analyze the log return data for different maturities. The values of D_n that we get if F is a normal distribution fitted to the respective sample $\overline{x}$ are displayed in Table 5.4. (There is virtually no difference between the values of D and D^{mod} here because the additional term $-0.01 + 0.85/\sqrt{n}$ is close to zero for $n = 2342$.) Comparison with the critical values given in Table 5.3 yields that the Kolmogorov-Smirnov

	Zero coupon bonds with time to maturity [in years]									
	1	2	3	4	5	6	7	8	9	10
D	4.37	3.86	4.2	4.82	4.56	4.08	4.3	4.15	4.01	3.87
D^{mod}	4.37	3.86	4.2	4.82	4.56	4.08	4.3	4.15	4.01	3.87

Table 5.4: Values of Kolmogorov-Smirnov test statistic: Test of the normal fit of the log-return distribution for zero bonds with maturities of up to ten years. January 1986 to May 1995.

	Zero coupon bonds with time to maturity [in years]									
	1	2	3	4	5	6	7	8	9	10
normal fit	391	316	366	518	465	351	386	375	348	331

Table 5.5: Values of χ^2 test statistic: Test of the normal fit of the log-return distribution for zero bonds with maturities of up to ten years. The number of classes was 45; the 90%-, 95%-, 98%, and 99%-quantiles of the $\chi^2(44)$-distribution are 56.4, 60.5, 65.3, 68.7, respectively.

test clearly rejects the assumption of normality.

The χ^2 test for goodness of fit counts the number of sample points falling into certain intervals and compares these counts with the expected number in these intervals under the null hypothesis. Following the recommendation in Moore (1986), Section 3.2.4, we choose a number $M := \lceil 2n^{2/5} \rceil$ of equiprobable classes.[4]

The log-return datasets under consideration have length $n = 2342$, so $M = 45$. We choose the j/M-quantiles ($j = 1, \ldots , M - 1$) of the fitted distribution as the boundary points of the classes.

Table 5.5 shows the values of the χ^2-test statistic for the null hypothesis of normality. As the number of degrees of freedom, we have chosen 44, which is the number of classes minus one. The exact distribution of the test statistic under the null hypothesis is now known. However, the correct quantiles lie between those of $\chi^2(44)$ and $\chi^2(42)$ if the distribution has two unknown parameters that are estimated via the maximum likelihood. (See Moore (1986), Section 3.2.2.) Choosing the quantiles of $\chi^2(44)$ thus yields a test that is too conservative. But even this conservative test rejects the assumption of normality.

5.5 Normal Inverse Gaussian as Alternative Log Return Distribution

In the preceding section, we have seen that normal distributions do not provide a good description of the log return distributions for zero bonds. In the current section, we study the class of normal inverse Gaussian (NIG) distributions as an alternative model. For a brief overview of NIG distributions, see Section A.2.2.

For bonds with different times to maturity, we estimate the parameters of the NIG distribution by the method of maximum likelihood. Table 5.6 shows the estimated values.

5.5.1 Visual Assessment of Fit

The quantiles of the NIG distributions can be calculated by numerical integration of the density, which is explicitly known for NIG distributions. This is done by the function `qgh` provided by Karsten Prause (see Eberlein, Ehret, Lübke, Özkan, Prause, Raible, Wirth, and Wiesendorfer Zahn (1999)). Figure 5.9 shows the quantile-quantile plot for a maturity of 5 years. The fit is excellent; in particular, comparison with Figure 5.6 shows the vast improvement that can be achieved by replacing the class of normal distributions with the class of NIG distributions.

[4]This is also the default value used by the S-Plus built-in function `chisq.gof`, which performs χ^2-goodness-of-fit tests for various distributions.

time to maturity [years]	α	β	δ	μ
1	1030	-7.53	0.000384	$0.17 \cdot 10^{-4}$
2	735	-26.4	0.000712	$0.53 \cdot 10^{-4}$
3	492	-21.8	0.000851	$0.71 \cdot 10^{-4}$
4	327	-18.2	0.000937	$0.85 \cdot 10^{-4}$
5	309	-25.4	0.00116	0.000132
6	291	-25.1	0.00159	0.000188
7	220	-18	0.00203	0.000214
8	184	-12.1	0.00258	0.000199
9	159	-6.55	0.00319	0.000143
10	133	-2.15	0.00378	$0.56 \cdot 10^{-4}$

Table 5.6: Estimated parameters of NIG distribution: Maximum likelihood estimates produced by the S-Plus function `ghe` by Karsten Prause (see Eberlein et al (1999)).

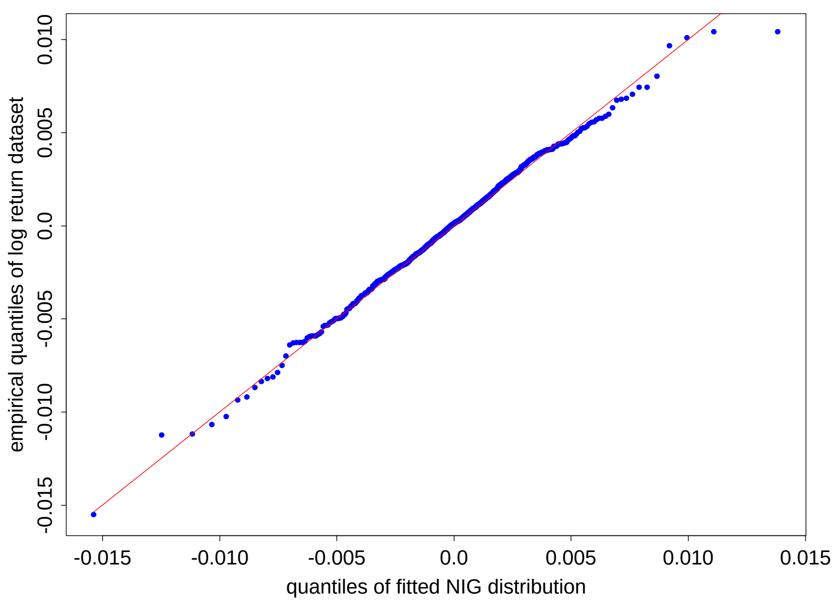

Figure 5.9: Quantile-quantile plot: Empirical quantiles of log returns on 5-year zero bonds against quantiles of fitted NIG distribution. January 1986 to May 1995.

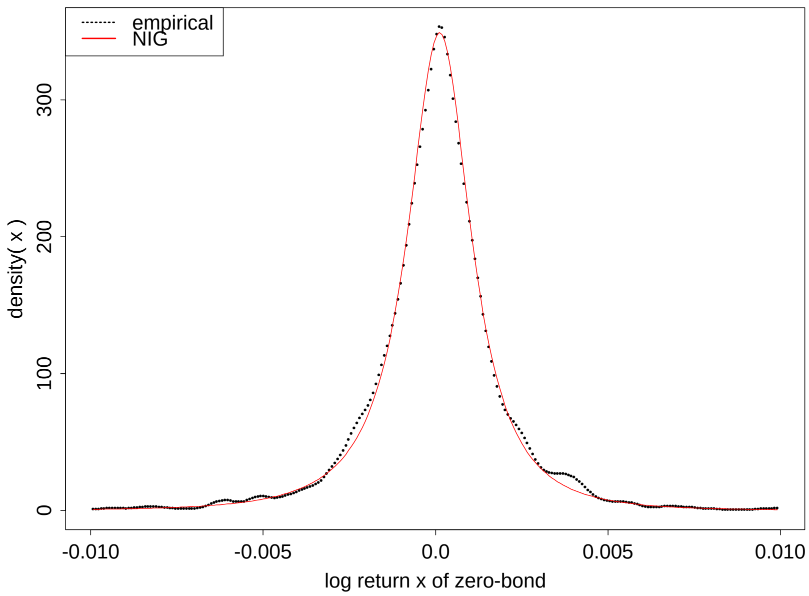

Figure 5.10: Density plot: Empirical density of log returns on 5-year bonds and density of fitted NIG distribution. January 1986 to May 1995.

Figures 5.10 (resp., 5.11) show the empirical density (resp., log density) compared with the density (resp., log density) of the fitted NIG distribution for log-returns on zero bonds with 5 years to maturity. Obviously the NIG distribution provides a very good fit to the empirical distribution in the center as well as in the tails.

5.5.2 Quantitative Assessment of Fit

As in the case of the normal fit, we apply the Kolmogorov-Smirnov test and the χ^2-test. The values of the Kolmogorov-Smirnov test statistic are shown in Table 5.7. Comparison with the values for the fitted normal distribution shows an enormous improvement in the quality of the fit. Unfortunately, the distribution of the test statistic of the Kolmogorov-Smirnov test is unknown for the case of NIG distributions with unknown parameters.[5] Therefore we cannot apply a Kolmogorov-Smirnov test here. Furthermore, when comparing the values of the Kolmogorov-Smirnov statistic in Table 5.7, one must take into account that the class of NIG distributions has four parameters, that is, two more than the class of normal distributions. Nevertheless, the values indicate the superiority of the NIG distributions.

Table 5.8 shows the values of the χ^2 test statistic. We see that at the 2% level, we cannot reject the hypothesis that the log returns are NIG distributed, except for the maturities of 4 and 5 years.

[5]The problem is that the parameters of the NIG distribution have been estimated from the sample. Therefore the standard Kolmogorov-Smirnov test is not appropriate here. See Kulinskaya (1995) for a deeper discussion of this issue. Of course, one could try to estimate the quantiles of this statistic by Monte Carlo methods. However, this would involve an enormous effort since the maximum likelihood estimation of the four NIG parameters is computationally very demanding.

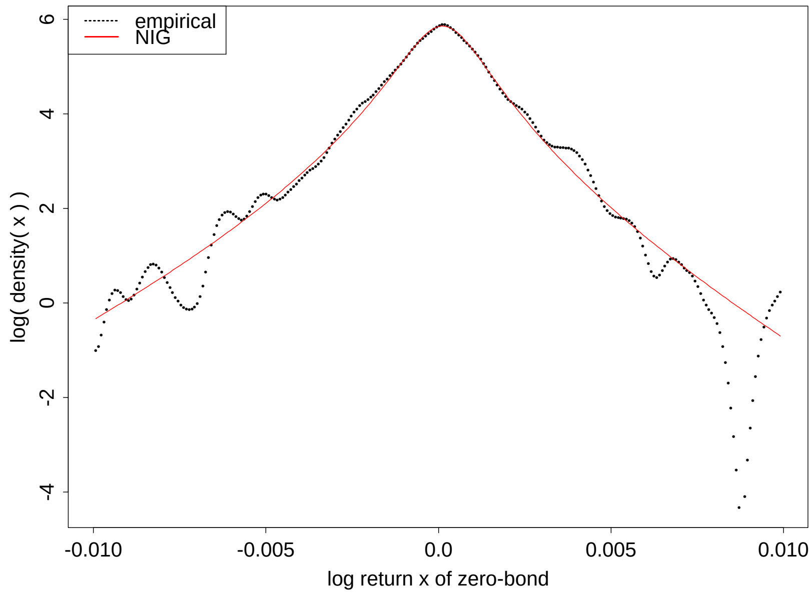

Figure 5.11: Log density plot: Logarithm of empirical density of log returns on 5-year bonds and log density of fitted NIG distribution. January 1986 to May 1995.

	Zero coupon bonds with time to maturity [in years]									
	1	2	3	4	5	6	7	8	9	10
normal fit	4.37	3.86	4.2	4.82	4.56	4.08	4.3	4.15	4.01	3.87
NIG fit	0.46	0.56	0.42	0.6	0.84	0.5	0.56	0.73	0.65	0.61

Table 5.7: Values of Kolmogorov-Smirnov statistic: Normal fit vs. NIG fit of the log-return distribution for zero bonds with maturities of up to ten years. January 1986 to May 1995.

	Zero coupon bonds with time to maturity [in years]									
	1	2	3	4	5	6	7	8	9	10
NIG fit	28.4	46	39.3	64.6	75	55.9	49.7	52.7	55.7	42.2

Table 5.8: Values of χ^2 test statistic: Test of the NIG fit of the log-return distribution for zero bonds with maturities of up to ten years. The number of classes was 45; the 90%-, 95%-, 98%, and 99%-quantiles of the $\chi^2(40)$-distribution are 51.8, 55.8, 60.4, 63.7, respectively.

5.6 Conclusion

We have shown that the Gaussian HJM model under some simple assumptions predicts approximately normally distributed daily log returns for discounted zero-coupon bonds. However, empirically observed log returns on bonds turn out to have a leptokurtic distribution. Visually, the inadequacy of the normal distribution assumption was shown by quantile-quantile plots and by comparing the empirical density with the density of the fitted normal distribution. Kolmogorov-Smirnov tests and χ^2-tests for goodness of fit clearly reject the hypothesis of normality for the log return distribution. In contrast to this, normal inverse Gaussian (NIG) distributions provide a much better fit for the distribution of log returns. This was shown visually by quantile-quantile plots and density plots. The Kolmogorov-Smirnov test statistic was calculated and turned out to be much smaller than for the normal distribution fit. However, since the limiting distribution of this statistic is not known for samples from a NIG distribution with unknown parameters, we were not able to determine the critical region for this test. At significance level 2%, the χ^2 test does not reject the assumption of a NIG distribution for eight of the ten times to maturity considered. From the point of view of data analysis, we therefore recommend replacing the driving Brownian motion in term structure models by more general Lévy processes such as the NIG Lévy process.

Chapter 6

Lévy Term Structure Models: Uniqueness of the Martingale Measure

6.1 Introduction

In Chapter 1, we studied a generalization of the Samuelson (1965) stock-price model. Brownian motion was replaced by a more general exponential Lévy motion. Every non-Brownian Lévy process has jumps, and if the jump-height distribution allows for infinitely many jump heights while there is only a finite number of underlying securities, the model is incomplete. Incompleteness is reflected by non-uniqueness of the martingale measure. In other words, the condition that discounted security prices be (local) martingales is not sufficient to fix a unique equivalent probability measure Q. Since derivative prices are formed as expectations under an equivalent martingale measure, non-uniqueness of the martingale measure implies non-uniqueness of derivative prices. This means that the model for the underlying alone does not suffice to uniquely determine derivative prices. An extreme example was given by Eberlein and Jacod (1997b). They examined stock price models driven by general pure-jump exponential Lévy processes with infinite variation. In this context, the price of European call options is completely undetermined even though the price dynamics of the underlying are known: The range of prices one may get by choosing an equivalent martingale measure is identical with the trivial no-arbitrage interval.

The underlying securities in term structure modeling are zero coupon bonds. These are characterized by a real number T which marks the time when they are paid back. At this time, the holder of the zero coupon bond receives an amount of one currency unit. Since generally the set of maturity dates T is an interval, we have an infinitude of securities in the market. Consequently, we may expect that even when the driving Lévy process has infinitely many possible jump heights, the martingale measure is unique. We will show below that in this is indeed the case for the Lévy term structure model introduced in Chapter 4. This is the statement of the main Theorem 6.9 proved below. Thus once the model is fixed, derivative prices are fixed as well. This parallels the situation in the Samuelson (1965) stock price model that led to the famous option pricing formula of Black and Scholes (1973).

The chapter is structured as follows. Section 6.2 reviews a general term structure model presented in Björk, Di Masi, Kabanov, and Runggaldier (1997). In Section 6.3, we show that the term structure model presented by Eberlein and Raible (1999) is a special case of this general term structure model. Section 6.4 presents some results from the general theory of stochastic processes as displayed in Jacod and Shiryaev (1987). Using the generalization of the Heath, Jarrow, and Morton (1992) drift condition by Björk et al. (1997), we prove the central result of this chapter, namely Theorem 6.9. This is done in Section 6.5. Section 6.6 concludes with a short remark on the relationship between completeness and uniqueness of the martingale measure.

6.2 The Björk/Di Masi/Kabanov/Runggaldier Framework

In Section 5 of Björk, Di Masi, Kabanov, and Runggaldier (1997), the following forward rate dynamics are considered.

$$d_t f(t,T) = \alpha(t,T)dt + \sigma(t,T)dW_t + \int_X \delta(t,x,T)(\mu(dt,dx) - \nu(dt,dx)). \tag{6.1}$$

Here, W is a standard Brownian motion in $\mathbb{R}^n$, $(X,\mathcal{X})$ is a Lusin space, and μ is a $\mathcal{P}\otimes\mathcal{X}$-$\sigma$-finite optional random measure such that its compensator[1] is continuous. It is assumed that the coefficient functions $\alpha(\omega,t,T)$, $\sigma(\omega,t,T)$, and $\delta(\omega,t,x,T)$ are continuous functions of T. Furthermore, $\alpha(\omega,t,T)$ and $\sigma(\omega,t,T)$ have to be $\mathcal{P}\otimes\mathcal{B}_+$-measurable, and δ is assumed to be $\mathcal{P}\otimes\mathcal{X}\otimes\mathcal{B}_+$-measurable.

In addition, the following integrability conditions have to be satisfied: For all finite t and $T\geq t$,

$$\int_0^T\int_t^T |\alpha(u,s)|\; ds\; du < \infty, \quad \int_0^T\int_t^T |\sigma(u,s)|^2\; ds\; du < \infty,$$

and

$$\int_0^T\int_X\int_t^T |\delta(u,x,s)|^2 ds\; \nu(du,dx) < \infty.$$

By Björk et al. (1997), Proposition 5.2, the bond price processes in this model are given by

$$\begin{aligned}P(t,T) = P(0,T)\exp\left(\int_0^t r(s)ds\right)\exp\bigg(\int_0^t A_s(T)ds + \int_0^t S_s(T)dW_s \\ + \int_0^t\int_X D(s,x,T)(\mu-\nu)(ds,dx)\bigg),\end{aligned}$$

with the notation

$$A_t(T) := -\int_t^T \alpha(t,s)ds, \tag{6.2}$$

$$D(t,x,T) := -\int_t^T \delta(t,x,s)ds, \tag{6.3}$$

$$S_t(T) := -\int_t^T \sigma(t,s)ds. \tag{6.4}$$

[1] For a definition of the compensator of an optional $\mathcal{P}\otimes\mathcal{X}$-$\sigma$-finite random measure, see Jacod and Shiryaev (1987), Theorem II.1.8, where a proof of existence is given.

6.3 The Lévy Term Structure Model as a Special Case

6.3.1 General Assumptions

Let L be a Lévy process, that is, an adapted stochastic process with stationary independent increments which is continuous in probability. According to Protter (1992), Theorem I.30, for every Lévy process there is a unique càdlàg version. We will always work with this version. This enables us to use the general theory of stochastic processes as presented in Jacod and Shiryaev (1987), where semimartingales and their components, i. e. local martingales and processes of finite variation, are always assumed to have càdlàg paths.

Assume that L_1 possesses a moment generating function on some open interval $(-a, b)$ containing $[-M, M]$, with $M > 0$. By Lukacs (1970), Section 7.2, this implies that L_1 has an *analytic characteristic function*. Thus, the characteristic function $\chi(u)$ of L_1 can be extended to an analytic function on the vertical strip $\mathbb{R} - i(-a, b) \subset \mathbb{C}$. We will denote this extended characteristic function by the same symbol χ. The *cumulant generating function* $\kappa(u) := \ln E[\exp(uL_1)]$ is a (real) analytic function, because it is the restriction of the analytic function $z \mapsto \ln \chi(-iz)$ to the real line.[2]

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and let $\mathcal{N}$ be the σ-field generated by the null sets. Assume that the filtration $(\mathcal{F}_t)_{0 \leq t < \infty}$ is generated by L and $\mathcal{N}$ in the sense of Jacod and Shiryaev (1987), III.2.12.

6.3.2 Classification in the Björk/Di Masi/Kabanov/Runggaldier Framework

In Chapter 4, we have discussed some properties of the Lévy term structure model of Eberlein and Raible (1999). In this model, the price of a zero coupon bond maturing at time T with $T \in (0, T^\star]$ is given by the stochastic process

$$P(t,T) = P(0,T) \exp\left[\int_0^t \big(r(s) - \kappa(\Sigma(s,T))\big) ds + \int_0^t \Sigma(s,T) dL_s\right], \tag{6.5}$$

where L is a Lévy process with finite moment generating function on some interval $(-a, b)$ with $a, b > 0$. The volatility structure $\Sigma(s, T)$ is assumed to be deterministic and continuously differentiable, with $\Sigma(t, t) = 0\ \forall t$ and $\Sigma(t, s) > 0 \quad (t < s)$. It is required to be bounded by the constant $M < \min(a, b)$.

Remark: We use a slightly different notation here than in Chapter 4: The volatility structure of zero bonds is denoted by $\Sigma(t, T)$ instead of $\sigma(t, T)$. This is because we want to refer to Björk, Di Masi, Kabanov, and Runggaldier (1997), where $\sigma(t, T)$ is the symbol for the forward rate volatility.

The bond price model (6.5) can be equivalently stated in terms of forward rates. For $T \in (0, T^\star)$ the forward rate process $f(\cdot, T)$ has to satisfy the stochastic differential equation

$$d_t f(t,T) = \kappa'\big(\Sigma(t,T)\big)\partial_2\Sigma(t,T)dt - \partial_2\Sigma(t,T)dL_t, \tag{6.6}$$

where $\partial_2\Sigma$ denotes the derivative of the function $\Sigma(t, T)$ with respect to the second argument. This forward rate formulation allows us to prove the next lemma.

Lemma 6.1. *The class of term structure models (6.5) is a subclass of the class considered in Björk et al. (1997).*

[2] By $\ln \chi(z)$ we understand that determination of $\ln \chi(z)$ which is continuous and vanishes at $z = 0$.

Proof. Write the integrable Lévy process L in its *canonical decomposition*, that is,[3]

$$L_t = bt + \sqrt{c}W_t + x * (\mu^L - \nu^L)_t. \tag{6.7}$$

We get the forward rate version (6.6) of the Lévy term structure model of Eberlein and Raible (1999) by the following specification of the model (6.1):

$$\begin{aligned}
\alpha(t,T) &:= \partial_2\Sigma(t,T)\big[\kappa'\big(\Sigma(t,T)\big) - b\big],\\
\sigma(t,T) &:= -\partial_2\Sigma(t,T)\sqrt{c},\\
\delta(t,x,T) &:= -\partial_2\Sigma(t,T)x,\\
(X,\mathcal{X}) &= (\mathbb{R},\mathcal{B}^1),\\
\mu &= \mu^L, \quad \text{and}\\
\nu &= \nu^L.
\end{aligned}$$

By definition, the compensator ν^L of the random measure of jumps, μ^L, is the third characteristic of the Lévy process L. It is given by $\nu^L(dt,dx) = dtF(dx)$, where F is the Lévy measure of the infinitely divisible distribution of L_1.

Using the above specifications for $\alpha, \sigma, \delta, \mu$, and ν and the condition $\Sigma(t,t) = 0\ \forall t$, the functions A, D, and S defined in (6.2)-(6.4) above become

$$\begin{aligned}
A_t(T) &\equiv -\int_t^T \alpha(t,s)ds = \Sigma(t,T)b - \kappa\big(\Sigma(t,T)\big),\\
D(t,x,T) &\equiv -\int_t^T \delta(t,x,s)ds = \Sigma(t,T)x,\\
S_t(T) &\equiv -\int_t^T \sigma(t,s)ds = \Sigma(t,T)\sqrt{c}.
\end{aligned}$$

□

6.4 Some Facts from Stochastic Analysis

Let $X = (X^i)_{i\le d}$ be a d-dimensional semimartingale with characteristics (B, C, ν) relative to a given truncation function h, with continuous martingale part X^c relative to P, and with $C^{ij} = c^{ij} \cdot A$, A increasing and predictable. Then we have the following Girsanov theorem for semimartingales (Jacod and Shiryaev (1987), III.3.24):

Theorem 6.2. *Assume that $Q \overset{\text{loc}}{\ll} P$, and let X be as above. There exist a $\mathcal{P}\otimes\mathcal{B}^1_+$-measurable nonnegative function Y and a predictable process $\beta = (\beta^i)_{i\le d}$ satisfying*

$$|h(x)(Y-1)| * \nu_t < \infty \ \textit{Q-a.s. for } t \in \mathbb{R}_+$$

[3] The canonical decomposition of a special semimartingale X is the unique decomposition $X = X_0 + M + A$, with the initial value X_0, a local martingale M and a predictable process of bounded variation A, where both M and A start in 0, that is, $M_0 = 0$ and $A_0 = 0$. By Jacod and Shiryaev (1987), Corollary II.2.38, we have $M = X^c + x * (\mu^X - \nu^X)$, where X^c is the continuous local martingale part of X, μ^X is the random measure of jumps of X and ν^X is the compensator of μ^X. The stochastic integral with respect to the compensated random measure exists because X is a special semimartingale and so by Jacod and Shiryaev (1987), Proposition 2.29a, $(|x|^2 \wedge |x|) * \nu^X$ has locally integrable variation.

$$\Big|\sum_{j\le d} c^{ij}\beta^j\Big| \cdot A_t < \infty \text{ and } \Big(\sum_{j,k\le d}\beta^j c^{jk}\beta^k\Big)\cdot A_t < \infty\ Q\text{-a.s. for } t \in \mathbb{R}_+,$$

and such that a version of the characteristics of X relative to Q are

$$\begin{cases} B'^i = B^i + \Big(\sum_{j\le d} c^{ij}\beta^j\Big)\cdot A + h^i(x)(Y-1) * \nu \\ C' = C \\ \nu' = Y \cdot \nu \end{cases}$$

By Theorem 6.2, Y and β tell us how the characteristics of the semimartingale transform under the change of measure $P \rightsquigarrow Q$. Since we assumed that the filtration is generated by the Lévy process L, Y and β tell us even more—they completely determine the change of the underlying probability measure. The reason for this is displayed below. By Jacod and Shiryaev (1987), Theorem III.4.34, the independence of increments of L implies that *every local martingale M has the representation property relative to L,* in the sense of Jacod and Shiryaev (1987), Definition III.4.22. This means that every local martingale M can be represented in the form

(6.8) $$M = M_0 + H \cdot L^c + W * (\mu^L - \nu^L)$$

with $H \in L^2_{\text{loc}}(L^c)$ and $W \in G_{\text{loc}}(\mu^L)$.[4]

If $Q \overset{\text{loc}}{\ll} P$, there is a unique (càdlàg) P-martingale Z such that $Z_t = dQ_t/dP_t$ for all t (cf. Jacod and Shiryaev (1987), Theorem III.3.4.) It is called the *density process of Q, relative to P*. Since Z is a martingale, it can be represented in the form (6.8). Jacod and Shiryaev (1987), Theorem III.5.19, provide explicit expressions for the integrands H and W in this case.

(6.9) $$Z = Z_0 + (Z_-\beta)\cdot L^c + Z_-\Big(Y - 1 + \frac{\widehat{Y} - a}{1-a}1\!\!1_{\{a<1\}}\Big) * (\mu^L - \nu^L).$$

Here, β (resp., Y) are the predictable process (resp., function) whose existence follows by the Girsanov theorem for semimartingales. The predictable processes a and $\widehat{Y}$ are defined by

(6.10) $$a_t(\omega) := \nu^L(\omega; \{t\} \times \mathbb{R}^d),$$

(6.11) $$\widehat{Y}_t(\omega) := \begin{cases} \int_{\mathbb{R}} Y(\omega,t,x)\nu^L(\omega;\{t\}\times dx) & \text{(if the integral is well-defined)} \\ +\infty & \text{otherwise.}\end{cases}$$

In our case, L is a Lévy process. This implies a considerable simplification of equation (6.9).

Lemma 6.3. *Let L be a Lévy process under the measure P. Consider another measure $Q \overset{\text{loc}}{\ll} P$. Then the density process $Z_t = dQ/dP$ has the following representation as a stochastic integral.*

(6.12) $$Z = Z_0 + (Z_-\beta)\cdot L^c + \big(Z_-(Y-1)\big) * (\mu^L - \nu^L).$$

[4] These two set of integrands are defined in Jacod and Shiryaev (1987), III.4.3 resp. Definition II.1.27 a.

Proof. L has stationary and independent increments, so

$$\nu(\{t\} \times \mathbb{R}) = 0 \quad \text{for all } t \in \mathbb{R}_+. \tag{6.13}$$

This follows from Jacod and Shiryaev (1987), II.4.3, and the fact that for processes with independent increments, quasi-left-continuity is equivalent to condition (6.13). Thus $a \equiv 0$ and $\widehat{Y} \equiv 0$, and equation (6.9) simplifies to (6.12). □

If $\beta \equiv 0$ and $Y \equiv 1$, then obviously the density process (6.12) vanishes up to indistinguishability. The conditions on β and Y can be relaxed somewhat, admitting them to be different from zero on some null sets. We will make this precise in Lemma 6.7 below. In order to prove this, we need the following two propositions. [$\mathcal{H}^2_{\text{loc}}$ denotes the class of locally square integrable local martingales; see Jacod and Shiryaev (1987), Definitions I.1.33 and I.1.41.]

Proposition 6.4. *Consider a local martingale $X \in \mathcal{H}^2_{loc}$. Let H be a predictable process such that $H^2 \cdot \langle X, X \rangle = 0$ up to indistinguishability. Then the stochastic integral $H \cdot X$ exists and*

$$H \cdot X = 0$$

up to indistinguishability.

Proof. According to Jacod and Shiryaev (1987), Theorem 4.40 d,

$$\langle H \cdot X, H \cdot X \rangle = H^2 \cdot \langle X, X \rangle,$$

and this process vanishes up to indistinguishability. This means that the local martingale $M := H \cdot X$ is orthogonal to itself, and we have $M = M_0 = 0$ up to indistinguishability. See Jacod and Shiryaev (1987), Lemma 4.13 a. □

Remark: Since continuous local martingales are locally bounded, the class of continuous local martingales is contained in $\mathcal{H}^2_{\text{loc}}$. Thus Proposition 6.4 applies to all continuous local martingales.

Proposition 6.5. *If $V \in G_{loc}(\mu)$ with $\nu(\{V \neq 0\}) = 0$ P-a.s., then for the stochastic integrals with respect to the compensated random measure $\mu - \nu$ we have*

$$V * (\mu - \nu) = 0$$

up to indistinguishability.

Proof. According to Jacod and Shiryaev (1987), Theorem II.1.33 a, we have $V * (\mu - \nu) \in \mathcal{H}^2_{\text{loc}}$, with

$$\langle V * (\mu - \nu), V * (\mu - \nu) \rangle = (V - \widehat{V})^2 * \nu_t + \sum_{s \leq t} (1 - a_s)(\widehat{V}_s)^2, \tag{6.14}$$

where a and $\widehat{V}$ are defined analogously to (6.10) and (6.11). For P-almost all ω, $a_\cdot(\omega) \equiv 0$ and $\widehat{V}(\omega, \cdot, \cdot) \equiv 0$. This implies that the predictable quadratic variation (6.14) of $M := V * (\mu - \nu)$ is equal to 0, that is, the local martingale M is orthogonal to itself. By Jacod and Shiryaev (1987), Lemma I.4.13 a, $M = M_0 = 0$ up to an evanescent set. □

Corollary 6.6. *If $V, W \in G_{loc}(\mu)$ with $\nu(\{V \neq W\}) = 0$ P-a.s., then*

$$V * (\mu - \nu) = W * (\mu - \nu)$$

up to indistinguishability.

Proof. $G_{\text{loc}}(\mu)$ is a linear space and the mapping $V \mapsto V * (\mu - \nu)$ is linear on $G_{\text{loc}}(\mu)$ up to indistinguishability (cf. Jacod and Shiryaev (1987), remark below II.1.27.) Hence $V - W \in G_{\text{loc}}(\mu)$, and

$$V * (\mu - \nu) - W * (\mu - \nu) = (V - W) * (\mu - \nu).$$

But by Proposition 6.5, this vanishes up to indistinguishability. □

The following lemma uses Propositions 6.4 and 6.5 to examine a change-of-measure problem.

Lemma 6.7. *Let L be a Lévy process with respect to the probability measure P. Assume that the stochastic basis is generated by L and the null sets. Let Q be a probability measure which is locally absolutely continuous w.r.t. P, and let β and Y be the predictable process (resp. function) associated according to Theorem 6.2 with the change of probability $P \rightsquigarrow Q$. If $\int_0^\infty \beta(\omega, s) d\langle L^c, L^c\rangle = 0$ P-almost-surely and $\nu(\omega; \{Y(\omega, \cdot, \cdot) \neq 1\}) = 0$ P-almost-surely, then $Q = P$.*

Proof. According to equation (6.12), the conditions imply that $Z \equiv Z_0$ up to indistinguishability. But $Z_0 = 1$ P-a.s., since under the assumptions above, $\mathcal{F}_0$ consists only of null-sets and their complements. Thus $Z \equiv Z_0 = 1$ P-a.s. □

In addition to this result from the theory of stochastic processes, we need the following lemma. It states that a measure on $\mathbb{R}$ is uniquely characterized by the values of its bilateral Laplace transform on an interval of the real line.

Lemma 6.8. *Let $G(dx)$ and $H(dx)$ be measures on $(\mathbb{R}, \mathcal{B}^1)$. If*

$$\int_{\mathbb{R}} \exp(ux) G(dx) = \int_{\mathbb{R}} \exp(ux) H(dx) < \infty$$

for all u from a non-empty finite interval $(a, b) \subset \mathbb{R}$, then $H = G$.

Proof. Set $c := (a + b)/2$, $d := (b - a)/2$, and define measures G' and H' by

$$G'(dx) := \exp(cx) G(dx), \qquad H'(dx) := \exp(cx) H(dx).$$

Then

$$\int_{\mathbb{R}} \exp(vx) G'(dx) = \int_{\mathbb{R}} \exp(vx) H'(dx) < \infty \tag{6.15}$$

for all $v \in (-d, d)$. In particular, taking $v = 0$ shows that G' and H' are finite (positive) measures with equal mass, and without loss of generality we can assume that this mass is 1. Thus we can apply the theory of probability distributions. Equation (6.15) says that the moment generating functions of G' and H' coincide on the interval $(-d, d)$. By well-known results about moment generating functions (cf. Billingsley (1979), p. 345), this implies that $G' = H'$ and hence $G = H$. □

6.5 Uniqueness of the Martingale Measure

Theorem 6.9. *In the Lévy term structure model (6.5), there is no measure $Q \neq P$ with $Q \overset{\text{loc}}{\sim} P$ such that all bond prices, when expressed in units of the money account, are local martingales. Thus—under the paradigm of martingale pricing—prices of integrable contingent claims are uniquely determined by this model and can be calculated by taking conditional expectations under P.*

Proof. We proceed as follows: We assume $Q \overset{\text{loc}}{\sim} P$ is a (local) martingale measure for the bond market. Using Proposition 5.6 of Björk et al. (1997), we show that the characteristic triplet of the process L, which is a Lévy process under the measure P, is the same under P and under Q. Then Lemma 6.7 shows that the two measures are equal.

Björk et al. (1997), Proposition 5.6, states the following: If a measure Q is a (local) martingale measure which is locally equivalent to P, then there exist a predictable process φ with values in $\mathbb{R}^n$ (that is, the state space of the Brownian motion W) and a $\mathcal{P} \otimes \mathcal{X}$-measurable function $Y = Y(\omega, t, x) > 0$ satisfying the following integrability conditions (6.16) and (6.17) as well as conditions 1 through 4 below. (The compensator of the jump measure is assumed to be continuous with respect to Lebesgue measure, $\nu(ds, dx) = \lambda_s(dx)ds$.) The integrability conditions on φ and Y are

$$\int_0^t |\varphi_s|^2 ds < \infty \quad \text{for } t < \infty \quad \text{P-a.s.,} \tag{6.16}$$

$$\text{and} \quad \int_0^t \int_X \left(\sqrt{Y(s,x)} - 1\right)^2 \lambda_s(dx) ds < \infty \quad \text{for } t < \infty \quad \text{P-a.s.} \tag{6.17}$$

The martingale conditions are as follows.

1. The process
$$W'_s := W_s - \int_0^t \varphi_s ds$$
is a standard Brownian motion with respect to Q.

2. The random measure $\nu' := Y\ \nu$ is the Q-compensator of μ.

3. The following integrability condition is satisfied for all finite t and T:
$$\int_0^t \int_X \left(\exp(D(s,x,T)) - 1\right) 1\!\!1_{\{D(s,x,T) > \ln 2\}} Y(s,x) \lambda_s(dx) ds < \infty \quad P\text{-a.s.} \tag{6.18}$$

4. For any T it holds that $dPdt$-almost everywhere
$$a_t(T) + S_t(T)\varphi_t + \int_X \left[(\exp(D(s,x,T)) - 1)Y(t,x) - D(s,x,T)\right] \lambda_t(dx) = 0$$
Here $a_t(T)$ is defined by $a_t(T) = A_t(T) + \frac{1}{2}\left|S_t(T)\right|^2$. This is a generalization of the Heath, Jarrow, and Morton (1992) drift condition to the case of jump-diffusion models.

The functions φ and Y come from the Girsanov theorems for semimartingales and random measures (see Theorems 6.2 and Jacod and Shiryaev (1987), Theorem III.3.17, respectively.) Björk et al. (1997)

consider very general random measures μ and ν, so they need the more general Girsanov theorem. In the case of the Lévy model presented in Eberlein and Raible (1999), μ^L is the jump-measure of a semimartingale. Consequently we can rely entirely on the Girsanov theorem for semimartingales here. The characteristic C can be represented as $C = \tilde{c} \cdot A$, with $\tilde{c} \geq 0$ predictable and A increasing. Since L is a Lévy process, we can choose $A_t = t$ and $\tilde{c} = c$, with the constant c from (6.7).

The proof of Björk et al. (1997), Proposition 5.6, examines the process $Z(\vartheta)_t$ that describes the discounted price process of the ϑ-bond. By assumption, this process is a local martingale under the measure Q, and so is the stochastic integral $M := Z(\vartheta)_- \cdot Z(\vartheta)$. Trivially, M is a special semimartingale, and the predictable finite-variation process in the canonical decomposition is zero. Björk et al. (1997) give an explicit formula for this process:

$$\int_0^t [a_s(\vartheta) + S_s(\vartheta)\varphi_s]ds + \int_0^t \int_X [(e^{D(s,x,\vartheta)} - 1)Y(s,x) - D(s,x,\vartheta)]\nu(ds,dx) = 0. \tag{6.19}$$

As usual in the theory of stochastic processes, this equality is to be understood up to indistinguishability: For each ϑ, there is a set $N(\vartheta) \subset \Omega$ of P-measure zero such that the paths of the process are zero for all $\omega \in \Omega \backslash N(\vartheta)$. Below it will turn out to be convenient to differentiate this equation with respect to the variable ϑ while keeping ω fixed. But for this to be possible, relation (6.19) must hold for any ϑ, which is not necessarily true because the exception set $N(\vartheta)$ may depend on ϑ. It is tempting to work around this problem by removing the "null set" $\cup N(\vartheta)$ from Ω. But since the set of repayment dates ϑ is uncountable, uniting all $N(\vartheta)$ might result in a set of strictly positive measure, or in a set that is not measurable at all. Therefore we will have to use a different approach below, using the continuity of Σ.

Given any $\vartheta \in [0, T^\star]$, equation (6.19) is valid for $t \in [0, \vartheta]$ and $\omega \in \Omega \backslash N(\vartheta)$. In order to avoid the problems mentioned above, we choose a countable dense subset Θ of $[0, T^\star]$ and define $N := \cup_{\vartheta \in \Theta} N(\vartheta)$. *For the arguments to come, we fix an arbitrary* $\omega \in \Omega \backslash N$.

With the coefficients of the Lévy term structure model inserted into equation (6.19), and with ω fixed as described, we get

$$\int_0^t \Big[\Sigma(s,\vartheta)b - \kappa\big(\Sigma(s,\vartheta)\big) + \frac{1}{2}\Sigma(s,\vartheta)^2 c + \Sigma(s,\vartheta)\sqrt{c}\varphi_s(\omega) \\ + \int_{\mathbb{R}} [(e^{\Sigma(s,\vartheta)x} - 1)Y(s,x,\omega) - \Sigma(s,\vartheta)x]F(dx)\Big]ds = 0$$

for all $\vartheta \in \Theta$ and all $t \in [0, \vartheta]$. Since the integral with respect to s is zero for all t, the integrand must be zero for λ^1-almost every s. We can choose a Lebesgue null set $N' \subset [0, T^\star]$, not depending on ϑ, such that

$$\Sigma(s,\vartheta)b - \kappa\big(\Sigma(s,\vartheta)\big) + \frac{1}{2}\Sigma(s,\vartheta)^2 c + \Sigma(s,\vartheta)\sqrt{c}\varphi_s(\omega) \\ + \int_{\mathbb{R}} [(e^{\Sigma(s,\vartheta)x} - 1)Y(s,x,\omega) - \Sigma(s,\vartheta)x]F(dx) = 0 \tag{6.20}$$

for all $\vartheta \in \Theta$ and all $s \in [0, \vartheta] \backslash N'$. Note that N' depends on the value ω we have fixed above.

Equation (6.20) may be written in the form

$$f\big(\Sigma(s,\vartheta), s, \omega\big) + \int_{\mathbb{R}} g\big(\Sigma(s,\vartheta), x, Y(s,x,\omega)\big)F(dx) = 0 \tag{6.21}$$

for all $\vartheta \in \Theta$ and all $s \in [0, \vartheta] \backslash N'$, with the functions f and g defined by

$$f(\sigma,s,\omega) := \sigma b - \kappa\big(\sigma\big) + \frac{1}{2}\sigma^2 c + \sigma\sqrt{c}\varphi_s(\omega),$$
$$g(\sigma,x,y) := (e^{\sigma x}-1)y - \sigma x.$$

Since $T \mapsto \Sigma(s,T)$ is continuous for fixed s, it maps the set Θ onto a dense subset $D_s \subset \Sigma(s,[s,T^\star])$. By assumption, $\Sigma(s,s)=0$ and $\Sigma(s,t)>0$ $(s<t\le T^\star)$, so that $\operatorname{int}(\Sigma(s,[s,T^\star])) = (0,a)$ for some $a=a_s>0$. For any fixed $s\in[0,T^\star)\backslash N'$ we have, by equation (6.21),

$$f\big(d,s,\omega\big) + \int_{\mathbb{R}} g\big(d,x,Y(s,x,\omega)\big)F(dx) = 0 \qquad \forall\,\vartheta\in[s,T^\star]\cap\Theta, \quad d\in D_s.$$

The next proposition shows that the function

$$\sigma\mapsto f\big(\sigma,s,\omega\big) + \int_{\mathbb{R}} g\big(\sigma,x,Y(s,x,\omega)\big)F(dx)$$

is twice continuously differentiable and that we can interchange differentiation and integration. As we will see below, this implies the equality of the bilateral Laplace transforms of the measures $x^2F(dx)$ and $Y(s,x,\omega)x^2F(dx)$.

Proposition 6.10. *Let $F(dx)$ be the Lévy measure of an infinitely divisible distribution. Let $y(x)$ be a strictly positive function on $\mathbb{R}$ such that there exist an open interval $(0,a)$ and a dense subset $D\subset(0,a)$ with*

$$\int_{\mathbb{R}} \big(\sqrt{y(x)}-1\big)^2 F(dx) < \infty, \tag{6.22}$$

$$\int_{\mathbb{R}} (|x|\wedge|x|^2)F(dx) < \infty, \tag{6.23}$$

$$\textit{and}\ \int_{\mathbb{R}} (e^{ux}-1)\mathbb{1}_{\{ux>\ln 2\}}y(x)F(dx) < \infty \quad \forall u\in D. \tag{6.24}$$

Then for any constants b, c, and φ, the function

$$u\mapsto ub-\kappa(u)+\frac{1}{2}u^2c+u\varphi+\int_{\mathbb{R}}[(e^{ux}-1)y(x)-ux]F(dx) \tag{6.25}$$

is twice continuously differentiable on $(0,a)$. Its first and second derivative is given by

$$u\mapsto b-\kappa'(u)+uc+\varphi+\int_{\mathbb{R}}[e^{ux}\cdot x\cdot y(x)-x]F(dx) \tag{6.26}$$

$$\textit{and}\quad u\mapsto -\kappa''(u)+c+\int_{\mathbb{R}} x^2e^{ux}y(x)\,F(dx), \quad \textit{respectively.} \tag{6.27}$$

Proof. Obviously, all the terms in (6.25) but the integral are twice continuously differentiable. Below we will prove that the integral term is twice continuously differentiable, and that differentiation and integration can be interchanged there. This proves that the first two derivatives are given by (6.26) and (6.27), respectively.

We have to show that $u\mapsto\int_{\mathbb{R}}[(e^{ux}-1)y(x)-ux]F(dx)$ is twice continuously differentiable on $(0,a)$. Since differentiability is a local property, it suffices to prove differentiability on any subset $(\underline{u},\overline{u})\subset(0,a)$ with $0<\underline{u}<\overline{u}<a$. For the rest of the proof, we fix such an interval.

The first derivative. It is well known that an integral whose integrand depends on a parameter is differentiable with respect to this parameter, with the derivative being the integral of the derivative of the integrand, if the following three conditions are satisfied (cf. Bauer (1992), Lemma 16.2.)

(D-1) For all parameter values, the integrand has to be integrable.

(D-2) The integrand has to be differentiable in the parameter.

(D-3) There has to exist an integrable bound for the derivative of the integrand that does not depend on the parameter.

The first two points are clearly satisfied here. We denote by $g(u,x)$ the integrand in (6.25). The first derivative of $g(u,t)$ with respect to u is given by

$$\partial_1 g(u,x) = x \cdot (\exp(ux)y(x) - 1).$$

In order to verify (D-3), we have to find an $F(dx)$-integrable function $G(x)$ that satisfies

$$\sup_{u\in(\underline{u},\overline{u})} |\partial_1 g(u,x)| \leq G(x).$$

Below, in equation (6.29), we define a function $H(x,y)$ such that $H(x,y(x))$ is $F(dx)$-integrable and

$$\sup_{u\in(\underline{u},\overline{u})} \left|x\cdot(e^{ux}y-1)\right| \leq H(x,y) \qquad \text{for } x\in\mathbb{R},\ y>0.$$

Then $G(x) := H(x,y(x))$ is the desired bound for $\partial_1 g(u,x)$. The following lemma will be used to prove that the function $H(x,y)$ defined in (6.29) indeed satisfies the condition that $H(x,y(x))$ be $F(dx)$-integrable.

Lemma 6.11. *Let $F(dx)$ and $y(x)$ be as in Proposition 6.10. Then the following functions are $F(dx)$-integrable over any bounded interval:*

$$x\mapsto x^2, \qquad x\mapsto y(x)x^2.$$

On the other hand, the following functions are $F(dx)$-integrable over any interval that is bounded away from zero.

$$x\mapsto |x|, \qquad x\mapsto y(x).$$

Finally, for every u from the dense set D the function

$$x\mapsto \exp(ux)y(x)$$

is $F(dx)$-integrable over any interval of the form (ξ,∞) with $\xi>0$.

Proof. The integrability of x^2 and $|x|$ over the respective intervals is trivially implied by condition (6.23).

For arbitrary numbers $v\in\mathbb{R}, w>0$, the following estimation holds.

$$\begin{aligned}|(w-1)v| &= |(\sqrt{w}-1)^2 + 2\sqrt{w}-2||v| \\ &\leq (\sqrt{w}-1)^2|v| + 2|\sqrt{w}-1||v| \\ &\leq (\sqrt{w}-1)^2|v| + (\sqrt{w}-1)^2 + v^2 \\ &\leq (\sqrt{w}-1)^2(|v|+1) + v^2. \end{aligned} \tag{6.28}$$

Hence we have

$$y(x)x^2 = (y(x)-1)x^2 + x^2 \leq \big(\sqrt{y(x)}-1\big)^2(x^2+1) + x^4 + x^2.$$

Since the functions $x \mapsto (x^2+1)$, $x \mapsto x^4$, and $x \mapsto x^2$ are bounded on any bounded interval, the integrability of $x \mapsto y(x)x^2$ follows by condition (6.22).

For the function $y(x)$, we have

$$y(x) \leq 4\mathbb{1}_{\{y(x)<4\}} + (\sqrt{y(x)}-1)^2\mathbb{1}_{\{y(x)\geq 4\}},$$

which by (6.22) and (6.23) is $F(dx)$-integrable over any set bounded away from zero.

Finally,

$$e^{ux}y(x) = (e^{ux}-1)\mathbb{1}_{\{ux>\ln 2\}}y(x) + (e^{ux}-1)\mathbb{1}_{\{ux\leq \ln 2\}}y(x) + y(x),$$

with $0 < (e^{ux}-1)\cdot\mathbb{1}_{\{ux\leq\ln 2\}} \leq 1$ for $x > 0$. Hence

$$\big|e^{ux}y(x)\big| = (e^{ux}-1)\mathbb{1}_{\{ux>\ln 2\}}y(x) + 2y(x),$$

which is integrable over (ξ,∞) because of condition (6.24) and because $y(x)$ was already shown to be integrable over any interval bounded away from zero. □

Now we proceed with the proof of Proposition 6.10. We define a function $H(x,y)$ that is a bound for the function $(x,y,u) \mapsto x\cdot(e^{ux}y-1)$, uniformly for $u \in (\underline{u},\overline{u})$. For this, we choose some $\delta > 0$ such that $\overline{u}+\delta \in D$. The function $H(x,y)$ will be defined piecewise for $x \in (-\infty,-1/\underline{u})$, $x \in [-1/\underline{u}, 1/\overline{u}]$, and $x \in (1/\overline{u},\infty)$, respectively. We use the following estimations.

1. For $x \in (-\infty,-1/\underline{u})$,

$$\begin{aligned}|x(\exp(ux)y-1)| &\leq |x|\exp(ux)y + |x|\\ &\leq |x|\exp(-\underline{u}|x|)y + |x|\\ &\leq C_1 y + |x|,\end{aligned}$$

   since $\underline{u} > 0$ implies that $|x|\exp(-\underline{u}|x|)$ is bounded by some constant C_1.

2. For $x \in [-1/\underline{u}, 1/\overline{u}]$, we have $|x| \leq 1/\underline{u}$ and $ux \leq u/\overline{u} < 1$. Hence with the aid of relation (6.28), we get

$$\begin{aligned}|x(\exp(ux)y-1)| &= |x||\exp(ux)-1+(y-1)\exp(ux)|\\ &\leq |x||\exp(ux)-1| + |y-1||x|\exp(ux)\\ &\leq |x||e\cdot ux| + (\sqrt{y}-1)^2\big(\frac{1}{\underline{u}}+1\big) + |x|^2\\ &\leq (\overline{u}e+1)|x|^2 + (\sqrt{y}-1)^2\big(\frac{1}{\underline{u}}+1\big).\end{aligned}$$

3. For $x \in (1/\overline{u},\infty)$,

$$\begin{aligned}|x(\exp(ux)y-1)| &\leq |x|\exp(ux)y + |x| \leq |x|\exp(\overline{u}x)y + |x|\\ &\leq |x|\exp(-\delta x)\exp((\overline{u}+\delta)x)y + |x|\\ &\leq C_2\exp((\overline{u}+\delta)x)y + |x|,\end{aligned}$$

   where $C_2 > 0$ is a bound for $x \mapsto x\exp(-\delta x)$ on $x > 0$.

Now we define $H(x,y)$ by

$$H(x,y) := \begin{cases} C_1 y + |x| & (x \in (-\infty, -1/\underline{u})), \\ (\overline{u}e+1)|x|^2 + (\sqrt{y}-1)^2\big(\frac{1}{\underline{u}}+1\big) & (x \in [-1/\underline{u}, 1/\overline{u}]), \\ C_2 \exp((\overline{u}+\delta)x)y + |x| & (x \in (1/\overline{u}, \infty)). \end{cases} \tag{6.29}$$

Lemma 6.11 yields that $x \mapsto H(x, y(x))$ is $F(dx)$-integrable. Hence we have proved that the integral in (6.25) is continuously differentiable, and that we can interchange differentiation and integration.

The second derivative. The proof here is completely analogous to the proof for the first derivative. Again we use the fact that an integral is differentiable with respect to a parameter of the integrand if the three conditions (D-1), (D-2), and (D-3) hold. The first two conditions are obviously satisfied. For the proof of (D-3), we only have to find some uniform bound on the second derivative. In order to do this, we show that $g_{11}(u,x) = x^2\ e^{ux} y(x)$ is bounded, uniformly in u for $u \in (\underline{u}, \overline{u})$, by a function $H(x,y)$ that turns into an $F(dx)$-integrable function when we substitute $y(x)$ for y..

Again fix a value $\delta > 0$ with $\overline{u} + \delta \in D$. We define $H(x,y)$ piecewise in x, using the following estimations.

1. For $x \in (-\infty, -1/\underline{u})$,

$$|x^2 \exp(ux) y| \le |x|^2 e^{-\underline{u}|x|} y \le C_3 y,$$

   where $x \mapsto |x|^2 e^{-\underline{u}|x|}$ is bounded by some constant $C_3 > 0$ because $\underline{u} > 0$.

2. For $x \in [-1/\underline{u}, 1/\overline{u}]$, we have

$$-1 < -\frac{u}{\underline{u}} \le ux \le \frac{u}{\overline{u}} < 1,$$

   because $u \in (\underline{u}, \overline{u})$ by assumption. Hence $e^{ux} < e$, and thus

$$x^2 \exp(ux) y \le x^2 \cdot e \cdot y.$$

3. For $x \in (1/\overline{u}, \infty)$,

$$\begin{aligned} x^2 \exp(ux) y &\le x^2 \exp(\overline{u}x) \cdot y \\ &\le x^2 \exp(-\delta x) \exp((\overline{u}+\delta)x) \cdot y \\ &\le C_4 \cdot \exp\big((\overline{u}+\delta)x\big) \cdot y, \end{aligned}$$

   where $C_4 > 0$ is a bound for $x \mapsto x^2 e^{-\delta x}$ on $\{x > 0\}$.

The function $H(x,y)$ is defined as follows.

$$H(x,y) := \begin{cases} C_3 y & (x \in (-\infty, -1/\underline{u})), \\ x^2 e y & (x \in [-1/\underline{u}, 1/\overline{u}]), \\ C_4 \cdot \exp\big((\overline{u}+\delta)x\big) \cdot y & (x \in (1/\overline{u}, \infty)). \end{cases}$$

Again Lemma 6.11 yields that $x \mapsto H(x, y(x))$ is $F(dx)$-integrable. Hence the integral in (6.25) is twice continuously differentiable, and we can interchange differentiation and integration. This completes the proof. $\square$

We now apply Proposition 6.10 to our change-of-measure problem $P \rightsquigarrow Q$. Condition (6.16) implies that for P-a. e. ω, $\varphi_s(\omega) < \infty$ for $\lambda(ds)$-a. e. $s \in \mathbb{R}$. Condition (6.17) implies that for P-a. e. ω

$$\int_{\mathbb{R}} \big(\sqrt{Y(s,x,\omega)} - 1\big)^2 F(dx) < \infty \qquad \text{for } \lambda(ds)\text{-a. e. } s \in \mathbb{R}.$$

Equation (6.18) implies that for $\vartheta \in \Theta$ and P-a. e. ω,

$$\int_{\mathbb{R}} \big(\exp(\Sigma(s,\vartheta)x) - 1\big) \mathbb{1}_{\{\Sigma(s,\vartheta)x > \ln 2\}} Y(s,x,\omega) F(dx) < \infty \quad \text{for } \lambda(ds)\text{-a. e. } s.$$

Fixing $\omega \in \Omega \backslash N$ outside the three null sets corresponding to the three conditions above, and fixing s outside the corresponding Lebesgue-null sets, we can apply Proposition 6.10 with the function $y(x)$ defined by

$$y(x) := Y(s,x,\omega).$$

This yields that the function

$$\sigma \mapsto \sigma b - \kappa(\sigma) + \frac{1}{2}|\sigma|^2 c + \sigma\sqrt{c}\varphi_s(\omega) + \int_{\mathbb{R}} \Big[(e^{\sigma x} - 1)Y(s,x,\omega) - \sigma x\Big] F(dx) \tag{6.30}$$

is twice continuously differentiable. By equation (6.20) the function vanishes for $\sigma \in D_s$. Since D_s is dense in some interval $(0,a)$, the function (6.30) has to vanish on the whole interval $(0,a)$. Hence the first and second derivative of this function are zero on this interval:

$$-\kappa'(\sigma) + \sigma c + \varphi_s(\omega) + \int_{\mathbb{R}} (e^{\sigma x} Y(s,x,\omega) - 1) x F(dx) = 0 \tag{6.31}$$

$$\text{and } -\kappa''(\sigma) + c + \int_{\mathbb{R}} e^{\sigma x} x^2 Y(s,x,\omega) F(dx) = 0 \tag{6.32}$$

for all $\sigma \in (0,a)$, respectively.

By assumption, the measure P itself is a martingale measure. Consequently, the choices $Y(s,x,\omega) \equiv 1$ and $\varphi_s(\omega) = 0$, corresponding to the trivial change of measure $P \rightsquigarrow P$, satisfy equations (6.31) and (6.32). Equation (6.32) yields

$$\int_{\mathbb{R}} e^{\sigma x} x^2 F(dx) = \kappa''(\sigma) - c,$$

and hence

$$\int_{\mathbb{R}} e^{\sigma x} x^2 Y(s,x,\omega) F(dx) = \int_{\mathbb{R}} e^{\sigma x} x^2 F(dx) \quad \text{for } \sigma \in (0,a).$$

Because the measure $x^2 F(dx)$ is uniquely characterized by the values of its bilateral Laplace transform on any non-degenerate interval (see Lemma 6.8), we have $Y(s,x,\omega) = 1$ for $F(dx)$-almost all $x \in \mathbb{R}$ and for all $s \in [0,T^\star] \backslash N'$. Equation (6.31) then yields $\varphi_s(\omega) = 0$ for $s \in [0,T^\star] \backslash N'$.

With these conditions satisfied, we get $Q = P$ by Lemma 6.7. Hence there is no measure $Q \neq P$ with $Q \overset{\text{loc}}{\sim} P$ such that all bond prices are local martingales when expressed in units of the money account. $\square$

6.6 Conclusion

Using a drift condition of the Heath, Jarrow, and Morton (1992) type, we have shown that the martingale measure is unique in the Lévy term structure model of Eberlein and Raible (1999). However, as was noted by Björk, Di Masi, Kabanov, and Runggaldier (1997), in models with a continuum of securities there is a marked difference between completeness and uniqueness of the martingale measure, the latter being a strictly weaker condition in general: Even with measure-valued strategies as introduced in Björk et al. (1997), uniqueness of the martingale measure does not imply that one can hedge every contingent claim. Instead, the right concept to replace completeness in this context seems to be *approximate completeness* as defined in Björk et al. (1997), section 6.3. Approximate completeness means that every (bounded) contingent claim can be approximated in L^2 by final values of admissible strategies. Under certain conditions on the compensator of the driving random measure, uniqueness of the martingale measure is equivalent to approximate completeness (see Björk et al. (1997), Theorem 6.11.) In this sense, the Lévy term structure model of Eberlein and Raible (1999) is approximately complete.

Chapter 7

Lévy Term-Structure Models: Generalization to Multivariate Driving Lévy Processes and Stochastic Volatility Structures

7.1 Introduction

The Lévy term structure model of Eberlein and Raible (1999), which was presented in Chapter 4, is driven by a one-dimensional general Lévy process. This allows for a more realistic modeling of the return distribution of zero coupon bonds. The volatility structure employed in this model is deterministic.

Empirical evidence suggests that term structure movements can be captured even better by a multi-factor model. Furthermore volatility seems to fluctuate randomly in financial markets, so there is need for term structure models featuring a multivariate driving process and stochastic volatility. In what follows, we present such a model as a generalization of the Lévy term structure model described in Chapter 4. As in this chapter, the model is based on a driving Lévy process, which preserves analytical tractability because of the simple structure of Lévy processes.

The chapter is structured as follows. In section 7.2, we show in general how one can construct martingales of an exponential form. In particular, this is applied to martingales driven by Lévy processes. As a corollary, we verify the martingale condition for a multivariate stochastic-volatility term structure model. Section 7.3 shows how one can construct the model in such a way that forward rates are well defined. An explicit expression for the forward rate processes is derived. Section 7.4 concludes.

7.2 Constructing Martingales of Exponential Form

Definition 7.1. *Let X be an adapted stochastic process.* X belongs to class D *if the family of random variables $(X_T)_{T \text{ stopping time}}$ is uniformly integrable.* X belongs to class LD *if for each $t \in \mathbb{R}_+$, $(X_T)_{T \leq t \text{ stopping time}}$ is uniformly integrable.*

For the convenience of the reader, we cite the following lemma, which does not seem to be standard textbook material.

Lemma 7.2 (Loève (1963), Sec. 25.1.2). *Let $X_n \to X$ in L^p for $p \geq 1$. Then for any σ-algebra $\mathfrak{F} \subset \mathfrak{A}$ we have $E\,[X_n |\, \mathfrak{F}] \to E\,[X |\, \mathfrak{F}]$ in L^p.*

Proof. Jensen's inequality for conditional expectations yields

$$\begin{aligned} E\,[\,|E\,[X_n|\,\mathfrak{F}] - E\,[X|\,\mathfrak{F}]\,|^p\,] &= E\,[\,|E\,[X_n - X|\,\mathfrak{F}]\,|^p\,] \\ &\leq E\,[\,E\,[\,|X_n - X|^p \mid \mathfrak{F}\,]\,] = E\,[\,|X_n - X|^p\,]\,, \end{aligned}$$

where the right hand side tends to zero by assumption. □

The following proposition was formulated in Elworthy, Li, and Yor (1999), Proposition 2.2. We feel that the proof given there is incomplete because it uses a formula (namely, formula (2) in their article) that was only proven for *continuous* local martingales. We give an alternative proof here.

Proposition 7.3. *A local martingale (M_t) such that $E\,[|M_0|] < \infty$ and that its negative part M^- belongs to class LD is a supermartingale. It is a martingale if and only if $E\,[M_t] = E\,[M_0]$ for all $t > 0$.*

Proof. Let $(T_n)_{n\in\mathbb{N}}$ be a localizing sequence of stopping times, that is, $T_n \uparrow \infty$ almost surely, with $M_{t\wedge T_n}$ being a uniformly integrable martingale for all $n \in \mathbb{N}$. Fix an arbitrary pair $s, t \in \mathbb{R}_+$ with $s \leq t$. Then obviously the following sequences converge almost surely as $n \to \infty$.

$$M^+_{t\wedge T_n} \to M^+_t, \quad M^-_{t\wedge T_n} \to M^-_t, \quad \text{and} \quad M_{s\wedge T_n} \to M_s. \tag{7.1}$$

For all $n \in \mathbb{N}$, we have by assumption

$$M_{s\wedge T_n} = E\,[M_{t\wedge T_n}|\,\mathcal{F}_s] = E\left[M^+_{t\wedge T_n}\middle|\,\mathcal{F}_s\right] - E\left[M^-_{t\wedge T_n}\middle|\,\mathcal{F}_s\right]. \tag{7.2}$$

The stopping times $(t \wedge T_n)_{n\in\mathbb{N}}$ are bounded by $t < \infty$. Since M^- belongs to LD, this implies that $(M^-_{t\wedge T_n})_{n\in\mathbb{N}}$ is a uniformly integrable sequence. Hence almost sure convergence entails convergence in L^1, which by Lemma 7.2 implies L^1-convergence of the conditional expectations.

$$E\left[M^-_{t\wedge T_n}\middle|\,\mathcal{F}_s\right] \to E\left[M^-_t\middle|\,\mathcal{F}_s\right] \quad \text{in } L^1 \text{ and hence in probability.} \tag{7.3}$$

Without loss of generality, we can assume that we have almost sure convergence here. (Otherwise we would repeat the proof with a suitable subsequence $(T_{n_k})_{k\in\mathbb{N}}$ of stopping times.) On the other hand, Fatou's Lemma for conditional expectations (see e. g. Chow and Teicher (1997), Section 7.1, Theorem 2(ii)) yields

$$\liminf_{n\to\infty} E\left[M^+_{t\wedge T_n}\middle|\,\mathcal{F}_s\right] \geq E\left[\,\liminf_{n\to\infty} M^+_{t\wedge T_n} \mid \mathcal{F}_s\,\right] \overset{(7.1)}{=} E\left[M^+_t\middle|\,\mathcal{F}_s\right]. \tag{7.4}$$

Combining (7.1), (7.2), (7.3), and (7.4) yields the almost sure relations

$$\begin{aligned} M_s \overset{(7.1)}{=} \lim_{n\to\infty} M_{s\wedge T_n} &\overset{(7.2)}{=} \liminf_{n\to\infty} E\left[M^+_{t\wedge T_n}\middle|\,\mathcal{F}_s\right] - \lim_{n\to\infty} E\left[M^-_{t\wedge T_n}\middle|\,\mathcal{F}_s\right] \\ &\overset{(7.3),\,(7.4)}{\geq} E\left[M^+_t\middle|\,\mathcal{F}_s\right] - E\left[M^-_t\middle|\,\mathcal{F}_s\right] = E\,[M_t|\,\mathcal{F}_s]\,. \end{aligned}$$

The second part of the proposition is a well-known result. The "only if" part is trivially true. On the other hand, if M is a supermartingale that is not a martingale, then there is at least one pair $s, t \in \mathbb{R}_+$, $s < t$, and a set $A_s \in \mathfrak{A}_s$ such that

$$\int_{A_s} M_s dP > \int_{A_s} M_t dP.$$

But since the complement A_s^c is also contained in $\mathfrak{A}_s$, the supermartingale property of M implies

$$\int_{A_s^c} M_s dP \geq \int_{A_s^c} M_t dP.$$

Adding these inequalities yields

$$E\left[M_s\right] = \int_{A_s} M_s dP + \int_{A_s^c} M_s dP > \int_{A_s} M_t dP + \int_{A_s^c} M_t dP = E\left[M_t\right].$$

□

Corollary 7.4. *a) Any local martingale belonging to LD is a martingale.*
b) Let M be a local martingale. If for any $t \geq 0$ there is an integrable random variable $B^{(t)}(\omega)$ such that $|M_s(\omega)| \leq B^{(t)}(\omega)$ for all $\omega \in \Omega$ and for all $s \leq t$, then M is a martingale.

Proof. a) Obviously, M^- and $(-M)^- = M^+$ belong to LD if M does. Application of Proposition (7.3) yields that M as well as $-M$ are supermartingales. Hence M must be a martingale.
b) If T is a stopping time that is bounded by some constant $t < \infty$, then we have

$$|M_T(\omega)| = |M_{T(\omega)}(\omega)| \leq B^{(t)}(\omega) \qquad \text{for all } \omega \in \Omega.$$

Consequently, the family $(M_T)_{T \text{ stopping time with } T \leq t}$ is bounded by the integrable random variable $B^{(t)}$ and hence is uniformly integrable. This implies that the local martingale M belongs to LD and thus is a martingale. (See part a.) □

The following proposition shows how one can use arguments from complex analysis to prove the martingale property for a larger class of stochastic processes when it is known for a subclass.

Proposition 7.5. *Let $M(s, \omega; z)$ be a family of adapted stochastic processes parameterized by a complex variable z. Assume that the mapping $z \mapsto M(s, \omega; z)$ is analytic for $z \in S$, with a horizontal strip $S := \mathbb{R} + i(a, b) \subset \mathbb{C}$ where $a < 0 < b$, $a, b \in \mathbb{R}$. Assume further that the partial derivatives $\partial_z M(s, \cdot; z)$ are bounded by integrable functions, locally in $z \in S$; that is, assume that for each $s \in \mathbb{R}_+, z_0 \in S$ there is an open neighborhood $N(z_0)$ and an integrable random variable $B^{(s,z_0)}(\omega)$ such that*

$$|\partial_z M(s, \omega; z)| \leq B^{(s,z_0)}(\omega) \qquad \textit{for all } z \in N(z_0), \omega \in \Omega.$$

Under these conditions, if $M(\cdot, \cdot; u)$ is a martingale for each $u \in \mathbb{R}$, then all $M(\cdot, \cdot; z)$, $z \in S$, are martingales as well.

Proof. First, we have to show integrability of $\omega \mapsto M(s,\omega;z)$ for arbitrary fixed $s \in \mathbb{R}_+$, $z \in S$. To this end, we note that the compact set $\mathrm{Re}(z) + i[0, \mathrm{Im}(z)] \subset \mathbb{C}$ is covered by a finite number of neighborhoods, say $N(z_j), j = 1, \dots, k$. Hence

$$|M(s,\omega;z)| \leq |M(s,\omega;\mathrm{Re}(z))| + |\mathrm{Im}(z)| \sum_{j=1}^{k} B^{(s,z_j)}(\omega),$$

where the right-hand side is integrable by assumption.

Next, consider an arbitrary set $A_s \in \mathfrak{A}_s$. Then for any t with $t \geq s$ we have that

$$z \mapsto \int \mathbb{1}_{A_s} M(t,\omega;z)\ P(d\omega) \tag{7.5}$$

is a differentiable function on $S \subset \mathbb{C}$, with

$$\partial_z \int_{A_s} M(t,\omega;z)\ P(d\omega) = \int_{A_s} \partial_z M(t,\omega;z)\ P(d\omega).$$

This is true because the integrand on the right-hand side is by assumption bounded locally around each $z \in S$ by an integrable function $B^{(t,z_0)}(\omega)$. Obviously, the Cauchy-Riemann differential equations are satisfied by the function $z \mapsto \mathbb{1}_{A_s} M(t,\omega;z)$ in the integrand, and interchanging differentiation and integration yields the validity of these equations for the integral. Hence the function (7.5) is analytic for each fixed $t \geq s$. In particular, it is analytic for $t = s$. Taking an arbitrary pair $t \geq s$, we have

$$\int_{A_s} M(s,\omega;z)\ P(d\omega) = \int_{A_s} M(t,\omega;z)\ P(d\omega) \quad \text{for all } z \in \mathbb{R},\ A_s \in \mathfrak{A}_s, \tag{7.6}$$

because $M(s;z)$ was assumed to be a martingale for real values of z. Since both sides in (7.6) depend on z in an analytic way, this equality carries over to all $z \in S$, in virtue of the identity theorem for analytic functions. Hence indeed $M(s;z)$ is a martingale for each $z \in S$. □

Proposition 7.6. *Let X be a d-dimensional locally bounded predictable process and let Y be a d-dimensional special semimartingale that has characteristics B^Y, C^Y, and ν^Y with respect to a truncation function h. Then the stochastic integral process*

$$X \cdot Y := \int X\ dY := \sum_{i=1}^{d} \int X^i\ dY^i$$

is a special semimartingale as well and has the following characteristics with respect to the truncation function h.

$$B^{X\cdot Y} = X \cdot B^Y + (h(Xx) - Xh(x)) * \nu^Y,$$

$$C^{X\cdot Y} = \sum_{i,j=1}^{d} \int (X^i X^j)_s d((C^Y)^{ij})_s,$$

and $\nu^{X\cdot Y}$ *with* $W(\omega,t,x) * \nu^{X\cdot Y} = W(\omega,t,X_t(\omega)x) * \nu^Y$
for all non-negative predictable functions $W(\omega,t,x)$.

Proof. Consider the canonical representation of the d-dimensional special semimartingale Y (see Jacod and Shiryaev (1987), Corollary II.2.38.)

$$Y = Y_0 + Y^c + x * (\mu^Y - \nu^Y) + A^Y, \tag{7.7}$$

where Y^c is the continuous local martingale part of Y and $A^Y = B^Y + (x - h(x)) * \nu^Y$, according to Jacod and Shiryaev (1987), Proposition II.2.29 a. [In order to stay within the framework set by Jacod and Shiryaev (1987), we have to use a truncation function $h(x)$ here even though this is not necessary for *special* semimartingales.] From (7.7), it is obvious that

$$\int X\, dY = \int X\, dY^c + \int X\, d\Big(x * (\mu^Y - \nu^Y)\Big) + \int X\, dA^Y. \tag{7.8}$$

Hence $\int X dY^c = (\int X dY)^c$ is the continuous local martingale part of $\int X dY$, and $\int X\ d(x * (\mu^Y - \nu^Y))$ is the purely discontinuous local martingale part. Since X is locally bounded and predictable, $\int X\ dA^Y$ is locally integrable and predictable. Therefore $X \cdot Y$ is indeed a special semimartingale. By Jacod and Shiryaev (1987), Corollary II.2.38, x^i belongs to $G_{\text{loc}}(\mu^Y)$. Consequently we can use Jacod and Shiryaev (1987), Proposition II.1.30 b, to get

$$\int X\ d(x * (\mu^Y - \nu^Y)) = (X_t x) * (\mu^Y - \nu^Y).$$

Since the jump process of $X \cdot Y$ is $X \Delta Y$, the jump measure of the process $X \cdot Y$ satisfies

$$W(\omega, t, x) * \mu^{X \cdot Y} = W(\omega, t, X_t(\omega)x) * \mu^Y. \tag{7.9}$$

for all predictable (i. e., $\widetilde{\mathcal{P}} \otimes \mathcal{B}^1$-measurable), non-negative functions W. The characteristic $\nu^{X \cdot Y}$ is defined to be the compensator of the random measure associated with the jumps of $X \cdot Y$. In general, the *compensator* of an optional, $\widetilde{\mathcal{P}} \otimes \mathcal{B}^1$-$\sigma$-finite random measure μ is defined to be the unique predictable random measure ν satisfying

$$E\left[W(\omega, t, x) * \mu\right] = E\left[W(\omega, t, x) * \nu\right]$$

for all predictable, non-negative functions W. (See Jacod and Shiryaev (1987), Theorem II.1.8.) From this definition, we can directly derive the form of the compensator $\nu^{X \cdot Y}$: For all predictable, non-negative functions $W(\omega, t, x)$, $W(\omega, t, X_t(\omega)x)$ is again predictable and non-negative. Since ν^Y is the compensator of μ^Y, we have

$$\begin{aligned} E\left[W(\omega, t, x) * \mu^{X \cdot Y}\right] &\overset{(7.9)}{=} E\left[W(\omega, t, X_t(\omega)x) * \mu^Y\right] \\ &= E\left[W(\omega, t, X_t(\omega)x) * \nu^Y\right]. \end{aligned}$$

Hence the natural candidate for the compensator $\nu^{X \cdot Y}$ is the optional random measure defined by

$$V(\omega, t, x) * \nu^{X \cdot Y} := V(\omega, t, X_t(\omega)x) * \nu^Y \tag{7.10}$$

for all optional functions V. This measure is indeed predictable: By definition (see Jacod and Shiryaev (1987), Definition 1.6 a), a random measure μ is called predictable iff for every predictable function W the integral process $W * \mu$ is predictable. But the definition (7.10) shows that for predictable V, the integral process $V * \nu^{X \cdot Y}$ is equal to an integral of a predictable function (namely, $V(\omega, t, X_t(\omega)x)$)

with respect to the compensator ν^Y. Since ν^Y is predictable by definition, this integral is a predictable process. Hence indeed $\nu^{X\cdot Y}$ is a predictable random measure.

The quadratic characteristic C of a d-dimensional semimartingale Z is defined component-wise:

$$C^{ij} := \langle (Z^i)^c, (Z^j)^c \rangle,$$

where $(Z^i)^c$ is the continuous local martingale part of the i-th component of Z $(i = 1, \dots, d)$. The two semimartingales $X \cdot Y$ and Y that we consider here have continuous local martingale parts $X \cdot Y^c = \sum_{i=1}^d X^i \cdot (Y^i)^c$ and Y^c, respectively. Hence we can use the relation

$$\langle X^i \cdot (Y^i)^c, X^j \cdot (Y^j)^c \rangle = (X^i X^j) \cdot \langle (Y^i)^c, (Y^j)^c \rangle,$$

which is valid by Jacod and Shiryaev (1987), Theorem I.4.40 d, to get

$$C^{X\cdot Y} = \sum_{i,j=1}^d \langle X^i \cdot (Y^i)^c, X^j \cdot (Y^j)^c \rangle = \sum_{i,j=1}^d (X^i X^j) \cdot \langle (Y^i)^c, (Y^j)^c \rangle = \sum_{i,j=1}^d (X^i X^j) \cdot C^{ij},$$

as was stated above. Finally, the drift component can be derived from the locally integrable, predictable summand $A^{X\cdot Y}$ in the canonical decomposition (7.8) of the special semimartingale $X \cdot Y$:

$$\begin{aligned} B^{X\cdot Y} &= A^{X\cdot Y} - (x - h(x)) * \nu^{X\cdot Y} \\ &= X \cdot B^Y + X \cdot ((x - h(x)) * \nu^Y) - (Xx - h(Xx)) * \nu^Y \\ &= X \cdot B^Y + (h(Xx) - Xh(x)) * \nu^Y. \end{aligned}$$

□

Corollary 7.7. *Let L be a $\mathbb{R}^d$-valued Lévy process. Then for any $\mathbb{R}^d$-valued bounded predictable process X the stochastic integral $X \cdot L$ has the characteristic triplet*

$$\begin{aligned} B_t^{X\cdot L} &= \int_0^t \Big(bX_s + \int (h(X_s x) - X_s h(x)) F(dx) \Big) ds, \\ C_t^{X\cdot L} &= \int X_s^T c X_s \, ds, \\ \nu^{X\cdot L}(\omega, ds, dx) &= ds \; F^{x \mapsto X_s(\omega)x}(dx), \end{aligned} \tag{7.11}$$

where $F^{x\mapsto X_s(\omega)x}(dx)$ denotes the image of $F(dx)$ under the mapping $x \mapsto X_s(\omega)x$, that is, $F^{x\mapsto X_s(\omega)x}(A) = \int \mathbb{1}_A(X_s(\omega)x) F(dx)$ for $A \in \mathcal{B}^1$, $s \in \mathbb{R}_+$, $\omega \in \Omega$.

Proof. By Jacod and Shiryaev (1987), Corollary II.4.19, the characteristic triplet of L can be chosen deterministic: It is given by

$$B_t^L(\omega) := bt, \quad C_t^L(\omega) := ct, \quad \nu^L(\omega; dt, dx) := dt \; F(dx),$$

where the constant $b \in \mathbb{R}^d$, the constant non-negative definite matrix c, and the σ-finite measure $F(dx)$ on $(\mathbb{R}^d, \mathcal{B}^d)$ with $\int (|x|^2 \wedge 1) F(dx) < \infty$ and $F(\{x \in \mathbb{R}^d : x^j = 0 \text{ for at least one } j \in \{1, \dots, d\}\}) = 0$ appear in the Lévy-Khintchine representation of the characteristic function of L_1 (see Jacod and Shiryaev (1987), II.4.21):

$$E\left[e^{iu\cdot L_1}\right] = \exp\Big(iu \cdot b - \frac{1}{2} u^T c u + \int (e^{iu\cdot x} - 1 - iu \cdot h(x)) F(dx) \Big). \tag{7.12}$$

Proposition 7.6, then yields the stated expressions for the characteristic triplet of the process $X \cdot L$. □

Proposition 7.8. *Let X be a d-dimensional predictable bounded process and let L be a d-dimensional Lévy process. For each $u \in \mathbb{R}$, define a process $A(u)_t$ as in Jacod and Shiryaev (1987), Eq. II.2.40:*

$$A(u)_t := iuB_t^{X\cdot L} - \frac{u^2}{2}C_t^{X\cdot L} + \int (e^{iux} - 1 - iuh(x))\nu^{X\cdot L}([0,t]\times dx). \tag{7.13}$$

Then

$$A(u)_t = \int_0^t \psi(u\cdot X_s)\,ds \equiv \int_0^t \psi\Big(\sum_{j=1}^d u^j X_s^j\Big)\,ds,$$

where $\psi(u)$ is the exponent of the Lévy-Khintchine representation (7.12) of the characteristic function of L_1, i. e. the uniquely determined continuous function with $\psi(0) = 0$ that satisfies $E[\exp(iuL_1)] = \exp(\psi(u)), u \in \mathbb{R}$. Furthermore, for each $u \in \mathbb{R}$ the process $(M(t;u))_{t\in\mathbb{R}_+}$ defined by

$$M(t;u) := \frac{\exp(iuX\cdot L_t)}{\exp\Big(\int_0^t \psi(u\cdot X_s)\,ds\Big)} \qquad (t\in\mathbb{R}_+) \tag{7.14}$$

is a martingale.

Proof. In virtue of (7.11), we have

$$\begin{aligned}
A(u)_t &= iub\int_0^t X_s ds - \frac{u^2}{2}\int_0^t X_s^T c X_s\,ds + \int_0^t\int_{\mathbb{R}^d}(e^{iuX_s\cdot x} - 1 - iuh(X_s x))\,F(dx)\,ds\\
&= \int_0^t \Big(iubX_s ds - \frac{u^2}{2}X_s^T cX_s + \int_{\mathbb{R}^d}(e^{iuX_s\cdot x} - 1 - iuh(X_s x))\,F(dx)\Big)ds\\
&= \int_0^t \psi(uX_s)\,ds,
\end{aligned}$$

Obviously $A(u)$ has continuous paths. In particular, we always have $\Delta A(u)_t(\omega) \neq -1$, and hence the conditions of Jacod and Shiryaev (1987), Corollary II.2.48, are satisfied. This means that

$$\frac{\exp(iuX\cdot L)}{\mathcal{E}(A(u))} \tag{7.15}$$

is a local martingale for all $u \in \mathbb{R}$, where $\mathcal{E}(A(u))$ denotes the Doléans-Dade exponential of the process $A(u)$. A formula for the exponential of a general real-valued semimartingale Z is given in Jacod and Shiryaev (1987), I.4.64:

$$\mathcal{E}(Z)_t = e^{Z_t - Z_0 - 1/2\langle Z^c, Z^c\rangle_t}\prod_{s\le t}(1+\Delta Z_s)e^{-\Delta Z_s}.$$

In Jacod and Shiryaev (1987), below Eq. II.2.40, it is noted that $A(u)$ is of finite variation. In addition, in our case it is continuous. Hence the continuous local martingale part $A(u)^c$ as well as the jump process $\Delta A(u)$ vanish identically, and the stochastic exponential turns out to be the ordinary exponential,

$$\mathcal{E}(A(u))_t = e^{A(u)_t - A(u)_0} = e^{A(u)_t},$$

since obviously $A(u)_0 = 0$. In order to show that (7.15) is actually a martingale (and not only a local martingale), we show that it is uniformly bounded on each interval $[0,t]$, $t\in\mathbb{R}_+$. Then Corollary 7.4 b yields the desired result. The numerator $\exp(iuX \cdot L)$ in (7.15) satisfies $|\exp(iuX \cdot L)| \equiv 1$. The modulus of the denominator is

$$\Big|\exp\Big(\int \psi(uX_s)ds\Big)\Big| = \exp\Big(\int \text{Re}\ \psi(uX_s)\ ds\Big).$$

But $\psi(x)$ (and hence Re $\psi(x)$) is continuous by assumption. In particular, it is bounded over bounded subsets of $\mathbb{R}^d$. So we can find a constant $C > -\infty$ such that

$$\text{Re}\ \psi(ux) \geq C \qquad \text{for all } x \text{ in the range of } X_{\cdot\wedge t}.$$

Therefore

$$\int_0^s \text{Re}\ \psi(uX_s)\ ds \geq \int_0^s C\ ds \geq t(C \wedge 0) > 0,$$

for any $s \leq t < \infty$, and so

$$\left|\frac{\exp(iuX \cdot L_s)}{\exp(A(u)_s)}\right| \leq \frac{1}{\exp(t(C \wedge 0))} < \infty.$$

□

We are now ready to prove the main result of the chapter.

Theorem 7.9. *Let X be an adapted $\mathbb{R}^d$-valued process with left-continuous paths. Assume that X takes only values in a d-dimensional rectangle $[a,b] := [a^1,b^1] \times \cdots \times [a^d,b^d] \subset \mathbb{R}^d$ with $a^i < 0 < b^i, i = 1,\dots,n$. Let L be a Lévy process. Assume that L_1 possesses a finite moment generating function on an open neighborhood U of $[a,b]$. Then the process N with*

$$N_t := \frac{X \cdot L_t}{\exp(\int_0^t \kappa(X_s)ds)} = \frac{\exp(\sum_{j=1}^d \int_0^t X_s^j\ dL_s^j)}{\exp(\int_0^t \kappa(X_s)ds)} \qquad (t \in \mathbb{R}_+)$$

is a martingale, where

$$\begin{aligned}\kappa(u) &= \ln E\,[\,\exp(uL_1)\,]\\ &= b \cdot u + \frac{1}{2}u^T cu + \int \big(e^{u\cdot x} - 1 - u \cdot x\big)F(dx) \qquad (u \in U)\end{aligned}$$

is the cumulant generating function of the infinitely divisible distribution of L_1.

Proof. First we note that X is predictable and bounded, and so the stochastic integral $X \cdot L$ is well defined. It was shown in Proposition 7.8 that for each $u \in \mathbb{R}$ the process

$$\frac{\exp(iuX \cdot L)}{\exp(\int \psi(uX_s)ds)},$$

is a martingale, where $\psi : \mathbb{R}^d \to \mathbb{R}$ is the exponent of the Lévy-Khintchine representation of the characteristic function of L_1.

Since the moment generating function of L_1 exists on an open neighborhood of the compact set $[a,b]$, there is a d-dimensional rectangle $(a_*,b_*) = (a_*^1,b_*^1) \times \cdots \times (a_*^d,b_*^d) \subset \mathbb{R}^d$ with $[a,b] \subset (a_*,b_*) \subset U$. The function $\psi(u)$ can be extended to an analytic function on the complex d-dimensional rectangle

$$\mathbb{R}^d - i(a_*,b_*) := \{(x^1 - ir^1,\dots,x^d - ir^d) \in \mathbb{C}^d : x^i \in \mathbb{R}, a_*^j < r^j < b_*^j\} \subset \mathbb{C}^d$$

We denote this extension by the symbol ψ again. The functions κ and ψ are then connected by the relation

$$\kappa(u) = \psi(-iu) \qquad \text{for } u \in (a_*, b_*).$$

Define the set $Z \subset \mathbb{C}$ by

$$Z := \left\{ z \in \mathbb{C} : \max\left\{ \frac{a_*^j}{b^j}, \frac{b_*^j}{a^j} \right\} < -\mathrm{Im}(z) < \min\left\{ \frac{a_*^j}{a^j}, \frac{b_*^j}{b^j} \right\} \text{ for } j = 1, \dots, d \right\}$$

For $z \in Z$, we have $zX_s(\omega) \in \mathbb{R}^d - i(a_*, b_*)$ for all $s \in \mathbb{R}_+$, $\omega \in \Omega$. Hence the process $A(z)$ with $A(z)_t(\omega) := \int \psi(zX_s(\omega))ds$ is well defined. For fixed ω and t, the function $z \mapsto A(z)_t(\omega), z \in Z$, is an analytic extension of the function $u \mapsto A(u)_t(\omega)$ defined in (7.13). (Analyticity follows because the partial derivative $\partial_z \kappa(zX_s(\omega)) = X_s(\omega)\kappa'(zX_s(\omega))$ is bounded, locally in $z \in Z$, if X is bounded. Therefore we can interchange integration with respect to ds and differentiation with respect to z.)

Furthermore, the function

$$z \mapsto \exp(izX \cdot L) \qquad (z \in Z)$$

is an analytic extension of $u \mapsto \exp(iuX \cdot L)$. Define

$$M(t, \omega; z) := \frac{\exp(izX \cdot L)}{\exp\Big(\int \psi(zX_s(\omega))ds \Big)}.$$

Then it follows from what was said above that

$$z \mapsto M(t, \omega; z) \qquad (z \in Z)$$

is an analytic extension of $u \mapsto M(t, \omega; u)$, with $M(t, \omega; u)$ as defined for $u \in \mathbb{R}$ in (7.14). The derivative of this analytic function is given by

$$\partial_z M(t, \omega; z) = \Big(iX \cdot L_t - \int_0^t X_s \psi'(zX_s)\, ds \Big) M(t, \omega; z).$$

We want to show that this is bounded, locally uniformly in z, by an integrable function of ω. To this end, we estimate

$$|\partial_z M(t, \omega; z)| \leq \Big(|X \cdot L_t| + \Big| \int_0^t X_s \psi'(zX_s)\, ds \Big| \Big) \frac{\exp(-\mathrm{Im}(z) X \cdot L)}{\exp\Big(\int \mathrm{Re}\psi(zX_s(\omega))ds \Big)}.$$

- For any $\varepsilon > 0$, we have

$$|X \cdot L_t| \leq \frac{\exp(-\varepsilon X \cdot L_t)}{\varepsilon} + \frac{\exp(\varepsilon X \cdot L_t)}{\varepsilon},$$

  since the relation $|x| \leq \frac{\exp(-\varepsilon x)}{\varepsilon} + \frac{\exp(\varepsilon x)}{\varepsilon}$ holds for all $x \in \mathbb{R}$.

- For z from any bounded set whose closure is contained in Z, we have that $|\int_0^t X_s \psi'(zX_s)\, ds| \leq \int_0^t |X_s| \cdot |\psi'(zX_s)|\, ds$ is bounded by an expression of the form $t \cdot \mathrm{const}$ because X is bounded and $\psi'(w)$ is analytic and hence bounded over compact subsets of its domain of regularity.

- For z from any bounded set W whose closure is contained in Z, we have with $\alpha := \inf_{z\in W}(-\mathrm{Im}(z))$ and $\beta := \sup_{z\in W}(-\mathrm{Im}(z))$ that

  $$\exp(-\mathrm{Im}(z)X\cdot L)\leq \exp(\alpha X\cdot L)+\exp(\beta X\cdot L)$$

  Clearly, $-i\alpha\in Z$ and $-i\beta\in Z$, and so the right hand side is an integrable function.

- For z from any bounded set W whose closure is contained in Z, we have that zX takes only values in a compact subset of $\mathbb{R}^d - i(a_*, b_*)$, and hence

  $$\exp\Big(\int_0^t \mathrm{Re}\ \psi(zX_s(\omega))ds\Big)\geq \exp(t\cdot C)>0\quad (t\in\mathbb{R}_+)$$

  for some finite constant C.

Taking these points together we see that the conditions of Proposition 7.5 are satisfied. Hence $(M(t;z))_{t\in\mathbb{R}_+}$ is a martingale for each $z\in Z$. Setting $z=-i$, which is indeed an element of Z in virtue of the relations $a_*^j < a^j$ and $b^j < b_*^j$ $(j=1,\dots,d)$, yields the statement that was to be shown. □

Corollary 7.10. *Let $\sigma:\Omega\times\Delta\to\mathbb{R}^d$ with $n\in\mathbb{N}$ be a d-dimensional stochastic volatility structure such that for each fixed ω, T the function $s\mapsto\sigma(\omega,s,T)$ is continuous to the left and that σ^i is globally bounded by constants $a^i<0<b^i$, $i=1,\dots,d$. Assume that a d-dimensional Lévy process L is given that possesses a moment generating function on some open neighborhood of $[a,b] := [a^1,b^1]\times\cdots\times[a^d,b^d]$. Let the price of a bond that matures at time T be given by the stochastic process*

$$P(t,T)=P(0,T)\exp\Big(\int_0^t r(s)ds\Big)\frac{\exp\Big(\int_0^t\sigma(\omega,s,T)dL_s\Big)}{\exp\Big(\int_0^t\kappa(\sigma(\omega,s,T))ds\Big)}.$$

Then for each T the discounted bond price process

$$\exp\Big(-\int_0^t r(s)ds\Big)P(t,T)$$

is a martingale.

Proof. For fixed T, the discounted bond price process is

$$\exp\Big(-\int_0^t r(s)ds\Big)P(t,T)=P(0,T)\frac{\exp\Big(\int_0^t\sigma(\omega,s,T)dL_s\Big)}{\exp\Big(\int_0^t\kappa(\sigma(\omega,s,T))ds\Big)},$$

which is—up to a constant—a process of the form treated in Theorem 7.9. Since the conditions of this theorem are satisfied here, the statement follows. □

7.3 Forward Rates

Proposition 7.11. *Assume that the conditions for the multivariate stochastic-volatility term structure model as described in Corollary 7.10 are given. Additionally, assume that $T \mapsto P(0,T) \in (0,\infty)$ is a continuously differentiable initial bond price structure. Further, assume that $\sigma(t,T)$ be twice differentiable in the second variable, with a bounded derivative $\partial_{22}\sigma(t,T;\omega) := (\partial_T)^2\sigma(t,T;\omega)$ that is a continuous function of t. Then the stochastic integrals $\int \sigma(s,T)dL_s$ appearing in the definition of the bond price processes can be chosen such that the (instantaneous) forward rates*

$$f(t,T) = -\frac{\partial}{\partial T}\ln P(t,T) \qquad (t \leq T).$$

are well defined and are given by

$$f(t,T) = f(0,T) - \int_0^t \partial_2\sigma(s,T)dL_s + \int_0^t \partial_2\sigma(s,T)\kappa'(\sigma(s,T))ds. \tag{7.16}$$

Remark: The instantaneous forward rate $f(t,T)$, contracted at time t for time $T \geq t$, is the interest rate one will get on an investment over the infinitesimal period $[T, T+dT]$ if this investment is contractually fixed at time t. The term "instantaneous" reflects the fact that money is lent only over an infinitesimal period.

Proof of the proposition. The first and the third summand in (7.16) are the negative logarithmic derivatives of $T \mapsto P(0,T)$ and $T \mapsto \exp(-\int_0^t \kappa(\sigma(s,T))ds$, respectively. (For the latter term, this follows by standard arguments used before, since κ is an analytic function and σ, $\partial_2\sigma$ are bounded.)

So we only have to prove that the second term in (7.16), namely $\int_0^t \partial_2\sigma(s,T)dL_s$, is the logarithmic derivative of $T \mapsto \int_0^t \sigma(s,T)dL_s$. Since we want to differentiate a family of stochastic integrals with respect to a continuous parameter (namely, T), it is essential to choose the stochastic integrals in the right way in order to avoid trouble with null sets. (Remember that each stochastic integral is defined up to a null set only, which leads to problems when considering an uncountable family of stochastic processes.)

Consider the function $\partial_{22}\sigma(\omega;t,T)$. Define $\partial_{22}\sigma(\omega;t,T) := 0$ for $t > T$. The function $\partial_{22}\sigma(\omega;t,T)$ is $\mathcal{P}\otimes\mathcal{B}^1$-measurable and bounded by assumption. Hence by Protter (1992), Theorem IV.44, there is a $\mathfrak{A}\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}^1$-measurable function $Z(\omega;t,T)$ such that for each $T \in \mathbb{R}_+$, $Z(\omega;t,T)$ is a càdlàg, adapted version of the stochastic integral $\int_0^t \partial_{22}\sigma(\omega;s,T)dL_s$.

Now define for each $S \in \mathbb{R}_+$ the finite measure $\kappa^{(S)} := 1\!\!1_{[0,S]}\lambda^1(dT)$ on $\mathcal{B}^1$. By Fubini's Theorem for stochastic integrals (see Protter (1992), Theorem IV.45), we have that for each $S \in \mathbb{R}_+$, the process $Y^{(S)}$ defined by

$$Y_t^{(S)}(\omega) := \int_{\mathbb{R}} Z(\omega;t,T)\,\kappa^S(dT) = \int_0^S Z(\omega;t,T)\,dT$$

is a càdlàg version of

$$\int \left(\int_{\mathbb{R}} \partial_2\sigma(\omega;s,T)\,\kappa^S(dT)\right)dL_s = \int \left(\int_s^S \partial_2\sigma(\omega;s,T)\,dT\right)dL_s = \int \sigma(s,S)\,dL_s.$$

The function $Y(\omega,t;S) := Y_t^{(S)}(\omega)$ is $\mathfrak{A}\otimes\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}^1$-measurable, and for each $S \in \mathbb{R}_+$ it is a càdlàg adapted version of the stochastic integral $\int \partial_2\sigma(s,S)\,dL_s$, where the integrand $\partial_2\sigma$ is $\mathcal{P}\otimes\mathcal{B}^1$-measurable and bounded. Hence we can apply Protter (1992), Theorem IV.44, a second time and get that

$$\int Y(\omega, t; S)\ \kappa^{(T)}(dS) = \int_0^T Y(\omega, t; S)\ dS$$

is a càdlàg version of

$$\int \int \partial_2 \sigma(s, S)\ \kappa^{(T)}(dS)\ dL_s = \int \int_0^T \partial_2 \sigma(s, S)\ dS\ dL_s = \int \sigma(s, T)\ dL_s.$$

Hence we have shown that one can choose the stochastic integrals $\int \sigma(s, S)\ dL_s$, $S \in \mathbb{R}_+$, by

$$\int \sigma(s, T)\ dL_s := \int_0^T \int \partial_2 \sigma(s, S)\ dL_s\ dS, \qquad (\omega \in \Omega), \tag{7.17}$$

with $\int \partial_2 \sigma(s, S)\ dL_s$ defined by

$$\int \partial_2 \sigma(s, S)\ dL_s := \int_0^S Z(\omega, s; U)\ dU \quad (\omega \in \Omega).$$

Since the latter is obviously continuous (in fact, absolutely continuous) as a function of S, we have that the right-hand side (and hence the left-hand side) of (7.17) is continuously differentiable with respect to T, with the derivative given by $\int \partial_2 \sigma(s, T)\ dL_s$. □

7.4 Conclusion

We have shown how one can construct a Lévy-driven multi-factor term structure model with a stochastic volatility structure. In this way, it is possible to capture three key features of the empirical behavior of the term structure: Non-normal return behavior, multi-factor movement, and stochastic volatility.

As in most cases, more precise modeling requires a more complex model here, so in practice one will have to trade off precision and numerical tractability. In particular, it would be illustrative to check which of the three features named above yields the biggest improvements for the modeling of real term structure movements.

Appendix A

Generalized Hyperbolic and CGMY Distributions and Lévy Processes

In the following, we give a brief description of generalized hyperbolic distributions, CGMY distributions, and the Lévy processes generated by this class of distributions. For a more detailed account of generalized hyperbolic distributions see Eberlein (1999) or Prause (1999). More information on CGMY distributions can be found in Carr, Geman, Madan, and Yor (1999).

A.1 Generalized Hyperbolic Distributions

Generalized hyperbolic distributions constitute a five-parameter class of Lebesgue-continuous, infinitely divisible distributions $GH(\lambda, \alpha, \beta, \delta, \mu)$. The Lebesgue density is given by $\rho_{GH(\lambda,\alpha,\beta,\delta,\mu)}$, where[1]

(A.1)

$$\rho_{GH(\lambda,\alpha,\beta,\delta,\mu)}(x+\mu) = (2\pi)^{-1/2}\delta^{-1/2}\alpha^{-\lambda+1/2}(\alpha^2-\beta^2)^{\lambda/2}K_\lambda\big(\delta\sqrt{\alpha^2-\beta^2}\big)^{-1} \cdot \sqrt{1+\frac{x^2}{\delta^2}}^{\lambda-1/2} K_{\lambda-1/2}\left(\delta\alpha\sqrt{1+\frac{x^2}{\delta^2}}\right)\exp(\beta x)$$

$$= \frac{e^{\beta x}}{\sqrt{2\pi}\alpha^{2\lambda-1}\delta^{2\lambda}} \cdot \frac{\big(\delta\sqrt{\alpha^2-\beta^2}\big)^\lambda}{K_\lambda\big(\delta\sqrt{\alpha^2-\beta^2}\big)} \cdot \big(\alpha\sqrt{\delta^2+x^2}\big)^{\lambda-1/2}K_{\lambda-1/2}\big(\alpha\sqrt{\delta^2+x^2}\big). \tag{A.2}$$

The domain of variation of the parameters is as follows. $\lambda \in \mathbb{R}$, $\alpha > 0$, $\beta \in (-\alpha, \alpha)$, $\delta > 0$, and $\mu \in \mathbb{R}$.[2] The functions K_λ and $K_{\lambda-1/2}$ are the modified Bessel functions of the third kind with orders λ and $\lambda - 1/2$, respectively.[3]

[1]Representation (A.1) can be found in e. g. in Barndorff-Nielsen (1997), eq. (4.3).

[2]Some authors include the limits into the parameter ranges (e.g. $\delta = 0$, leading to the variance gamma distributions; see Section A.3.) Hence they inlcude certain limiting distributions into the definition. However, often the behavior of the limiting distributions differs considerably from the behavior of the generalized hyperbolic distributions as defined here. Therefore we do not include limiting distributions.

[3]See Abramowitz and Stegun (1968), Section 9.6.

The characteristic function of $GH(\lambda,\alpha,\beta,\delta,\mu)$ was calculated in Prause (1999), Lemma 1.18:[4]

$$\chi_{(\lambda,\alpha,\beta,\delta,\mu)}(u) = e^{i\mu u}\frac{(\delta\sqrt{\alpha^2-\beta^2})^\lambda}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}\cdot\frac{K_\lambda\big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\big)}{\big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\big)^\lambda}. \tag{A.3}$$

This is a real-analytic function, which can be extended to a holomorphic function on the strip

$$S := \big\{z : -\alpha < \beta - \mathrm{Im}(z) < \alpha\big\}.$$

The form of the expression for the characteristic function is preserved under the extension, because all functions appearing here are restrictions of analytic functions. This means that one can calculate the extended characteristic function at some point $z \in S$ by just plugging in z instead of u into expression (A.3). In particular, we get the moment-generating function

$$u \mapsto \chi_{(\lambda,\alpha,\beta,\delta,\mu)}(-iu).$$

(See Prause (1999), Lemma 1.13.) Taking derivatives at $u = 0$ then yields the algebraic moments of a random variable $X \sim GH(\lambda,\alpha,\beta,\delta,\mu)$ (Prause (1999), Lemma 1.14.)

$$E[X] = \mu + \delta\cdot\frac{\rho}{\sqrt{1-\rho^2}}\cdot\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)}, \tag{A.4}$$

$$Var[X] = \delta^2\left(\frac{K_{\lambda+1}(\zeta)}{\zeta K_\lambda(\zeta)} + \frac{\rho^2}{1-\rho^2}\left(\frac{K_{\lambda+2}(\zeta)}{K_\lambda(\zeta)} - \left[\frac{K_{\lambda+1}(\zeta)}{K_\lambda(\zeta)}\right]^2\right)\right). \tag{A.5}$$

Here we have used the new parameters $\zeta := \delta\sqrt{\alpha^2-\beta^2}$ and $\rho := \beta/\alpha$.

The characteristic function of the time-t element of the convolution semigroup generated by $GH(\lambda,\alpha,\beta,\delta,\mu)$ is

$$\chi_{(\lambda,\alpha,\beta,\delta,\mu;t)}(u) = e^{i\mu t u}\frac{(\delta\sqrt{\alpha^2-\beta^2})^{\lambda t}}{K_\lambda(\delta\sqrt{\alpha^2-\beta^2})^t}\cdot\frac{K_\lambda\big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\big)^t}{\big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\big)^{\lambda t}}.$$

Note that we have be careful when taking the "t-th power" of the characteristic function; The main branch of the t-th power function, applied to the complex number $\chi_{(\lambda,\alpha,\beta,\delta,\mu)}(u)$, in general does *not* yield the desired characteristic function. [Cf. Bauer (1991), p. 250, Bemerkung 1.]

A.2 Important Subclasses of GH

A.2.1 Hyperbolic Distributions

Setting the first parameter $\lambda = 1$ yields the four-parameter class of *hyperbolic distributions*. For this subclass, the density takes the form

$$\begin{aligned}\rho_{\mathrm{HYP}(\alpha,\beta,\delta,\mu)}(x+\mu) &= \frac{e^{\beta x}}{\sqrt{2\pi}\alpha\delta^2}\cdot\frac{\delta\sqrt{\alpha^2-\beta^2}}{K_1\big(\delta\sqrt{\alpha^2-\beta^2}\big)}\cdot\big(\alpha\sqrt{\delta^2+x^2}\big)^{1/2}K_{1/2}\big(\alpha\sqrt{\delta^2+x^2}\big)\\ &= \frac{e^{\beta x}}{2\alpha\delta^2}\cdot\frac{\delta\sqrt{\alpha^2-\beta^2}}{K_1\big(\delta\sqrt{\alpha^2-\beta^2}\big)}\cdot\exp\big(-\alpha\sqrt{\delta^2+x^2}\big),\end{aligned}$$

[4]The square root sign in (A.3) is taken to mean the main branch of the square root function. The same holds for the λ-th power function $z \mapsto z^\lambda$ etc.

where we have used the relation

$$K_{1/2}(z) = \sqrt{\pi/(2z)}e^{-z} \tag{A.6}$$

from Watson (1944), Section 3.71, p. 80, eq. (12). Note that—in contrast to the general form (A.2)—the Bessel function appears only in the norming factor. Hence the hyperbolic density is numerically more tractable than the generalized hyperbolic density: When calculating the density at a number n of different points, e. g. for numerical integration, the Bessel function has to be evaluated only once. This considerably reduces the computational effort compared with the generalized hyperbolic case, where $2n$ evaluations of Bessel functions have to be calculated.

The log density of a hyperbolic distribution is a hyperbola, which is the origin of the name *hyperbolic distribution*.

A.2.2 Normal Inverse Gaussian (NIG) Distributions

Setting $\lambda := -1/2$ leads to the class of Normal Inverse Gaussian (NIG) distributions. The name of this class stems from the fact that a NIG distribution can be represented as a variance-mean mixture of normal distributions, where the mixing distribution is inverse Gaussian. (See e. g. Barndorff-Nielsen (1998), Section 2.)

In contrast to the case of hyperbolic distributions, the characteristic function of NIG is expressible by elementary functions: The Bessel function $K_{-1/2}(z)$ is equal to $K_{1/2}(z)$ by Abramowitz and Stegun (1968), 9.6.6, which in turn can be reduced to elementary functions via relation (A.6).

$$\begin{aligned}
\chi_{\mathrm{NIG}(\alpha,\beta,\delta,\mu;t)}(u) &= e^{i\mu t u}\frac{(\delta\sqrt{\alpha^2-\beta^2})^{-(1/2)t}}{K_{-1/2}(\delta\sqrt{\alpha^2-\beta^2})^t}\cdot\frac{K_{-1/2}\big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\big)^t}{\big(\delta\sqrt{\alpha^2-(\beta+iu)^2}\big)^{-(1/2)t}} \\
&= e^{i\mu t u}\exp\Big(t\delta\sqrt{\alpha^2-\beta^2}\Big)\cdot\exp\Big(-t\delta\sqrt{\alpha^2-(\beta+iu)^2}\Big).
\end{aligned}$$

Obviously,

$$\chi_{\mathrm{NIG}(\alpha,\beta,\delta,\mu;t)}(u) = \chi_{\mathrm{NIG}(\alpha,\beta,t\delta,t\mu;1)}(u).$$

This yields another favorable property of NIG distributions: The convolution semigroup generated by a NIG distribution only contains NIG distributions. Hence the Lévy process generated by a NIG distribution in the sense of Section A.6 possesses only NIG distributed increments. Thus the density of the increment distribution is known, which constitutes an important advantage over other generalized hyperbolic distributions.

A.3 The Carr-Geman-Madan-Yor (CGMY) Class of Distributions

In Carr, Geman, Madan, and Yor (1999), a new class of infinitely divisible probability distributions—called CGMY—is introduced as a model for log returns on financial markets. This class is an extension of the class of variance gamma distributions, which date back to Madan and Seneta (1987).

A.3.1 Variance Gamma Distributions

The class of *variance gamma* distributions was introduced in Madan and Seneta (1987) as a mode for stock returns. There, as in the succeeding publications (Madan and Seneta 1990) and (Madan and Milne 1991), the symmetric case (i. e. $\theta = 0$ in the parameterization given below) was considered. In (Madan, Carr, and Chang 1998), the general case with skewness is treated. Variance gamma distributions are limiting cases of generalized hyperbolic distributions as $\delta \to 0$ in the parameterization by $(\lambda, \alpha, \beta, \delta, \mu)$, as we will see below.

In the literature, variance gamma distributions always appear as the one-dimensional marginal distributions of *variance gamma Lévy processes*. These are time-changed Brownian motions with drift. The time change process is itself a Lévy process, namely a gamma process, characterized by having a gamma distribution as its increment distribution. More precisely, the increment of the gamma process over a time interval of length t has a distribution given by the probability density

$$\rho_{(\widetilde{\mu},\widetilde{\nu})}(\tau) = \left(\frac{\mu}{\nu}\right)^{\frac{\widetilde{\mu}^2}{\widetilde{\nu}}} \frac{\tau^{\frac{\widetilde{\mu}^2}{\widetilde{\nu}}-1} \exp\left(-\frac{\widetilde{\mu}}{\widetilde{\nu}}\tau\right)}{\Gamma\left(\frac{\widetilde{\mu}^2}{\widetilde{\nu}}\right)} \qquad (\tau > 0)$$

Here $\widetilde{\mu} = t\mu$ and $\widetilde{\nu} = t\nu$, where $\mu > 0$ and $\nu > 0$ are the parameters of the distribution for $t = 1$.

The characteristic function of the Gamma(μ, ν) distribution is

$$\chi_{\text{Gamma}(\mu,\,\nu)}(u) = \left(\frac{1}{1 - iu\frac{\nu}{\mu}}\right)^{\frac{\mu^2}{\nu}},$$

where the exponential is well-defined because $-\pi < \arg(\frac{1}{1-iu\frac{\nu}{\mu}}) < \pi$. Consequently, the characteristic function of the time-t element of the gamma convolution semigroup is

$$\chi_{\text{Gamma}(\mu,\,\nu)}(u)^t = \left(\frac{1}{1 - iu\frac{\nu}{\mu}}\right)^{\frac{t\mu^2}{\nu}},$$

which is again the characteristic function of a gamma distribution, with parameters $\widetilde{\mu}$ and $\widetilde{\nu}$ as defined above. (Of course, this was already clear from the behavior of the densities.)

The variance gamma Lévy process $X^{(\sigma,\theta,\nu)}$ is defined as a time-changed Brownian motion with drift:

$$X_t^{(\sigma,\theta,\nu)} = \theta\gamma^{(1,\nu)}(t) + \sigma W_\gamma(t),$$

where W is a standard Brownian motion and $\gamma^{(1,\nu)}$ is a gamma process with $\mu = 0$, independent of W. In contrast to the exposition in (Madan, Carr, and Chang 1998), we would like to modify the definition of the process by adding a linear drift μt.[5] Hence our variance gamma Lévy process is

$$X_t^{(\sigma,\theta,\nu,\mu)} = \mu t + \theta\gamma^{(1,\nu)}(t) + \sigma W_\gamma(t),$$

Consequently, the distribution of $X_t^{(\sigma,\theta,\nu)}$ is a variance-mean mixture of normals, with a gamma distribution as mixing distribution: It is the marginal distribution of x in a pair (x, z) where z is distributed as $\gamma(t)$ and, conditionally on z, x is distributed as $N(\mu + \theta z, \sigma^2 z)$.

[5]Note that the parameter μ is *not* the parameter of the Gamma distribution, which will not be used in the following.

The characteristic function of the distribution $VG(\sigma, \theta, \nu, \mu)$ is given by

$$\chi_{VG(\sigma,\theta,\nu,\mu)}(u) = \exp(i\mu u) \left(\frac{1}{1 - i\theta\nu u + (\sigma^2\nu/2)u^2} \right)^{1/\nu}.$$

(See (Madan, Carr, and Chang 1998), eq. 7.)

Consequently, the time-t element of the $VG(\sigma, \theta, \nu, \mu)$ convolution semigroup has the characteristic function

$$\chi_{VG(\sigma,\theta,\nu,\mu)}(u)^t = \exp(it\mu u) \left(\frac{1}{1 - i\theta\nu u + (\sigma^2\nu/2)u^2} \right)^{t/\nu},$$

which is just the characteristic function of $VG(\sqrt{t}\sigma, t\theta, \nu/t, t\mu)$:

$$\begin{aligned}
\chi_{VG(\sqrt{t}\sigma,t\theta,\nu/t,t\mu)}(u) &= \exp(it\mu u) \left(\frac{1}{1 - it\theta\nu/tu + (t\sigma^2\nu/2t)u^2} \right)^{1/(\nu/t)} \\
&= \exp(it\mu u) \left(\frac{1}{1 - i\theta\nu u + (\sigma^2\nu/2)u^2} \right)^{t/\nu}.
\end{aligned}$$

Therefore the convolution semigroup of a particular variance gamma distribution is nested in the set of all variance gamma distributions. This is the same situation as in the case of the NIG distributions. Therefore, these two classes of distributions are analytically more tractable than a generalized hyperbolic distribution with $\lambda \neq -1/2$, as for example the hyperbolic distributions studied in (Eberlein and Keller 1995), (Keller 1997) and (Eberlein, Keller, and Prause 1998).

The density of $VG(\sigma, \theta, \nu, \mu)$ can be calculated by making use of the mixture representation.[6] It is given by $\rho_{(\sigma,\theta,\nu,\mu)}(x)$, with

$$\text{(A.7)} \qquad \rho_{(\sigma,\theta,\nu,\mu)}(x+\mu) = \frac{2\exp(\theta x/\sigma^2)}{\nu^{1/\nu}\sqrt{2\pi\sigma^2}\,\Gamma(\frac{1}{\nu})} \left(\frac{x^2}{2\sigma^2/\nu + \theta^2} \right)^{\frac{1}{2\nu} - \frac{1}{4}} K_{\frac{1}{\nu} - \frac{1}{2}} \left(\frac{\sqrt{x^2(2\sigma^2/\nu + \theta^2)}}{\sigma^2} \right).$$

As a consequence of our considerations above, the density of the time-t element of the convolution semigroup is of the same form, with the parameters $(\sigma, \theta, \nu, \mu)$ replaced by $(\sqrt{t}\sigma, t\theta, \nu/t, t\mu)$.[7]

The Lévy measure of the variance gamma distribution $VG(\sigma, \theta, \nu, \mu)$ is

$$\text{(A.8)} \qquad K_{VG(\sigma,\theta,\nu,\mu)}(dx) = \frac{\exp\left(\theta x/\sigma^2\right)}{\nu|x|} \exp\left(-\frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}}{\sigma}|x| \right) dx.$$

(See Madan, Carr, and Chang (1998), eq. 14.) This measure has infinite mass, and hence a variance gamma Lévy process has infinitely many jumps in any finite time interval. Since the function $x \mapsto |x|$ is integrable with respect to $K_{VG(\sigma,\theta,\nu,\mu)}(dx)$, a variance gamma Lévy process has paths of finite variation.

A.3.2 CGMY Distributions

The class of CGMY distributions is a class of infinitely divisible distributions that contains the variance gamma distributions as a subclass. It is defined in Carr, Geman, Madan, and Yor (1999) by giving its

[6] See (Madan, Carr, and Chang 1998), pp. 87 and 98.

[7] Actually, it was this density which was calculated in (Madan, Carr, and Chang 1998).

Lévy-Khintchine triplet $(b, c, K(dx))$ with respect to a truncation function $h(x)$.

$$b = \int \Big(h(x) k_{CGMY}(x) - x \mathbb{1}_{\{|x| \le 1\}} \frac{C}{|x|^{1+Y}} e^{-|x|} \Big) dx, \quad c = 0, \quad K(dx) = k_{CGMY}(x)\, dx,$$

with the four-parameter Lévy density

$$k_{CGMY}(x) := \begin{cases} C \frac{\exp(-G|x|)}{|x|^{1+Y}} & \text{for } x < 0 \\ C \frac{\exp(-M|x|)}{|x|^{1+Y}} & \text{for } x > 0 \end{cases} = \frac{C}{|x|^{1+Y}} \exp\Big(\frac{G-M}{2} x - \frac{G+M}{2} |x| \Big). \tag{A.9}$$

The range of parameters is not made explicit in Carr, Geman, Madan, and Yor (1999), but natural choices would be $C, G, M > 0$ and $Y \in (-\infty, 2)$. Choosing $Y \ge 2$ does not yield a valid Lévy measure, since it violates the condition that any Lévy measure must integrate the function $x \mapsto 1 \wedge |x|^2$.[8] For $Y < 1$, the Lévy measure integrates the function $x \mapsto |x|$; hence we could choose the "truncation function" $h(x) \equiv 0$. This would let the first component of the Lévy-Khintchine triplet vanish: $b = 0$. But in order to preserve generality, we always use a truncation function here.

Like for the variance gamma distribution, one could introduce an additional location parameter $\mu \in \mathbb{R}$ here.

For $Y < 0$, the characteristic function of *CGMY* is given by

$$\chi_{CGMY}(u) = \exp\Big\{ C\Gamma(-Y) \big[(M - iu)^Y - M^Y + (G + iu)^Y - G^Y \big] \Big\}.$$

This formula was derived in Carr, Geman, Madan, and Yor (1999), Theorem 1.[9]

The CGMY Lévy Process

As described below in Section A.6, every infinitely divisible distribution generates a Lévy process. The CGMY Lévy process is pure-jump, that is, it contains no Brownian part. As shown in Carr, Geman, Madan, and Yor (1999), Theorem 2, the path behavior of this process is determined by the parameter Y: The paths have infinitely many jumps in any finite time interval iff $Y \in [0, 2)$, and they have infinite variation iff $Y \in [1, 2)$.

Variance Gamma as a Subclass of CGMY

Variance gamma distributions constitute the subclass of the class of CGMY distributions where $Y = 0$. (See Carr, Geman, Madan, and Yor (1999), Sec. 2.2.) One can easily see this by comparing formula (A.9) with the variance gamma Lévy density (A.8). The parameters are related as follows.

$$C = \frac{1}{\nu}, \quad \frac{G-M}{2} = \frac{\theta}{\sigma}, \quad \text{and} \quad \frac{G+M}{2} = \frac{\sqrt{\frac{2}{\nu} + \frac{\theta^2}{\sigma^2}}}{\sigma}.$$

[8] However, Carr, Geman, Madan, and Yor (1999) also consider the case $Y > 2$.

[9] This theorem does not mention a restriction on the range of Y. However, examination of the proof reveals that it can only work for $Y < 0$. Otherwise at least one of the integrals appearing there does not converge.

A.3.3 Reparameterization of the Variance Gamma Distribution

We think that it is useful to change the parameterization of the variance gamma distribution in order to compare it to the generalized hyperbolic distribution: Let

$$\begin{aligned}
\lambda &:= \frac{1}{\nu}, \\
\alpha &:= \frac{\sqrt{2\sigma^2/\nu + \theta^2}}{\sigma^2} = \sqrt{\frac{2}{\nu\sigma^2} + \left(\frac{\theta}{\sigma^2}\right)^2}, \\
\beta &:= \frac{\theta}{\sigma^2}.
\end{aligned}$$

Then we have $0 \leq |\beta| < \alpha$ and $\lambda > 0$, and

$$\begin{aligned}
\sigma^2 &= \frac{2\lambda}{\alpha^2 - \beta^2}, \\
\theta &= \beta\sigma^2 = \frac{2\beta\lambda}{\alpha^2 - \beta^2}, \\
\nu &= \frac{1}{\lambda}.
\end{aligned}$$

The parameter transformation[10] $(\sigma, \theta, \nu, \mu) \to (\sqrt{t}\sigma, t\theta, \nu/t, t\mu)$ has the following effect on the parameters $(\lambda, \alpha, \beta, \mu)$:

$$\begin{aligned}
\lambda &\to t \cdot \lambda, \\
\alpha &\to \alpha, \\
\beta &\to \beta, \\
\mu &\to t \cdot \mu.
\end{aligned}$$

Therefore this parameterization seems to be useful for the study of the convolution semigroup.

In the new parameterization, the characteristic function of the variance gamma distribution takes the form $\chi_{VG(\lambda,\alpha,\beta,\mu)}(u)$, with

$$\begin{aligned}
\chi_{VG(\lambda,\alpha,\beta,\mu)}(u) &= \exp(i\mu u) \left(\frac{1}{1 - i\theta\nu u + (\sigma^2\nu/2)u^2}\right)^{1/\nu} \\
&= e^{i\mu u} \left(\frac{1}{1 - i\frac{2\beta\lambda}{\alpha^2 - \beta^2} \cdot \frac{1}{\lambda}u + \frac{2\lambda}{\alpha^2 - \beta^2} \cdot \frac{1}{2\lambda}u^2}\right)^{1/\nu} \\
&= e^{i\mu u} \left(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + iu)^2}\right)^{\lambda}.
\end{aligned}$$

Note how the structure of the characteristic function becomes clearer in this parameterization.

[10]Note that this is the transformation that we need in order to get the variance gamma parameters of the time-t element of the convolution semigroup.

The variance gamma density takes the form $\rho_{(\lambda,\alpha,\beta,\mu)}(x)$, with

$$\begin{aligned}
\rho_{(\lambda,\alpha,\beta,\mu)}(x+\mu) &= \frac{2\exp(\theta x/\sigma^2)}{\nu^{1/\nu}\sqrt{2\pi\sigma^2}\Gamma(\frac{1}{\nu})}\left(\frac{\sigma^2|x|}{\sigma^2\sqrt{2\sigma^2/\nu+\theta^2}}\right)^{\frac{1}{\nu}-\frac{1}{2}} K_{\frac{1}{\nu}-\frac{1}{2}}\left(\frac{\sqrt{x^2(2\sigma^2/\nu+\theta^2)}}{\sigma^2}\right)\\
&= \frac{2\lambda^\lambda\exp(\beta x)}{\sqrt{2\pi}\Gamma(\lambda)}\left(\frac{\alpha^2-\beta^2}{2\lambda}\right)^\lambda\left(\frac{|x|}{\alpha}\right)^{\lambda-1/2} K_{\lambda-1/2}\big(\alpha|x|\big)\\
&= \sqrt{\frac{2}{\pi}}\frac{\exp(\beta x)}{2^\lambda\Gamma(\lambda)}\left(\alpha^2-\beta^2\right)^\lambda\left(\frac{|x|}{\alpha}\right)^{\lambda-1/2} K_{\lambda-1/2}\big(\alpha|x|\big).
\end{aligned}$$

This is the pointwise limit of the generalized hyperbolic density as $\delta\to 0$:

$$\begin{aligned}
\rho_{GH(\lambda,\alpha,\beta,\delta,\mu)}(x+\mu) &= (2\pi)^{-1/2}\delta^{-1/2}\alpha^{-\lambda+1/2}(\alpha^2-\beta^2)^{\lambda/2}K_\lambda\big(\delta\sqrt{\alpha^2-\beta^2}\big)^{-1}\cdot\\
&\qquad \sqrt{1+\frac{x^2}{\delta^2}}^{\lambda-1/2} K_{\lambda-1/2}\left(\delta\alpha\sqrt{1+\frac{x^2}{\delta^2}}\right)\exp(\beta x)\\
&= \frac{\exp(\beta x)(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\delta^{1/2}\alpha^{\lambda-1/2}K_\lambda\big(\delta\sqrt{\alpha^2-\beta^2}\big)}\frac{1}{\delta^{\lambda-1/2}}\sqrt{\delta^2+x^2}^{\lambda-1/2}K_{\lambda-1/2}\big(\alpha\sqrt{\delta^2+x^2}\big)\\
&= \frac{\exp(\beta x)(\alpha^2-\beta^2)^{\lambda/2}}{\sqrt{2\pi}\alpha^{\lambda-1/2}}\frac{1}{\delta^\lambda K_\lambda\big(\delta\sqrt{\alpha^2-\beta^2}\big)}\sqrt{\delta^2+x^2}^{\lambda-1/2}K_{\lambda-1/2}\big(\alpha\sqrt{\delta^2+x^2}\big)
\end{aligned}$$

But from (Abramowitz and Stegun 1968), formula 9.6.9, we know that for fixed $\lambda>0$,

$$K_\lambda(z)\sim\frac{1}{2}\Gamma(\lambda)\left(\frac{z}{2}\right)^{-\lambda}\qquad (z\to 0).$$

Inserting $z=\delta\sqrt{\alpha^2-\beta^2}$, we conclude

$$\frac{1}{\delta^\lambda K_\lambda(\delta\sqrt{\alpha^2-\beta^2})}\to\frac{2^{1-\lambda/2}(\alpha^2-\beta^2)^{\lambda/2}}{\Gamma(\lambda)}\qquad (\delta\to 0),$$

and hence for fixed $x\neq 0$

$$\begin{aligned}
&\frac{1}{\delta^\lambda K_\lambda\big(\delta\sqrt{\alpha^2-\beta^2}\big)}\sqrt{\delta^2+x^2}^{\lambda-1/2}K_{\lambda-1/2}\big(\alpha\sqrt{\delta^2+x^2}\big)\\
&\qquad\to\frac{2^{1-\lambda/2}(\alpha^2-\beta^2)^{\lambda/2}}{\Gamma(\lambda)}\sqrt{x^2}^{\lambda-1/2}K_{\lambda-1/2}\big(\alpha\sqrt{x^2}\big).
\end{aligned}$$

For $\lambda>1/2$, convergence holds also for $x=0$.

In the new parameterization, the Lévy measure of the variance gamma distribution *VG*$(\lambda,\alpha,\beta,\mu)$ has the form

$$
\begin{aligned}
K_{VG(\lambda,\alpha,\beta,\mu)}(dx) &= \frac{\exp\left(\theta x/\sigma^2\right)}{\nu|x|}\exp\left(-\sqrt{\frac{2}{\nu\sigma^2}+\frac{\theta^2}{\sigma^4}}|x|\right)dx \\
&= \frac{\exp\left(\beta\sigma^2 x/\sigma^2\right)}{(1/\lambda)|x|}\exp\left(-\sqrt{\frac{2}{(1/\lambda)\frac{2\lambda}{\alpha^2-\beta^2}}+\frac{(\beta\sigma^2)^2}{\sigma^4}}|x|\right)dx \\
&= \frac{\lambda\exp\left(\beta x\right)}{|x|}\exp\left(-\sqrt{\alpha^2-\beta^2+\beta^2}|x|\right)dx \\
&= \frac{\lambda}{|x|}\exp\left(\beta x-\alpha|x|\right)dx.
\end{aligned}
$$

A.4 Generation of (Pseudo-)Random Variables

For simulation purposes, it is essential to know how to generate (pseudo-)random variables with a given distribution. The standard random-number generators provide pseudo-random numbers with a uniform distribution on $(0,1)$. The aim is now to find a transformation that turns a sequence of independent, identically distributed (iid) random variables with a uniform distribution on $(0,1)$ into an iid sequence of random variables with the given distribution. One approach which always works is to invert the cumulative distribution function (cdf) $F(x)$ of the distribution in question. This yields a function $F^{-1}(x)$, and if U is uniformly distributed on $(0,1)$, then the random variable $F^{-1}(U)$ has a distribution with cdf $F(x)$. In practice, this is often cumbersome since inversion of $F(x)$ is often only possible numerically. However, for some special distributions $F(x)$ there are other possibilities. For normal distributions, Box and Muller (1958) showed how one can generate two iid $N(0,1$-distributed random variables by applying a simple transformation to two iid $U(0,1)$-distributed random variables: If U_1 and U_2 are iid with distribution $U(0,1)$, then the random variables N_1 and N_2 defined by

(A.10) $$N_1 := \sqrt{-2\ln U_1}\cos(2\pi U_2)$$

(A.11) $$\text{and}\quad N_2 := \sqrt{-2\ln U_1}\sin(2\pi U_2)$$

are iid with distribution $N(0,1)$.

With the same approach, one can generate chi-square distributed random numbers: For an even number $2n$ of degrees of freedom,

(A.12) $$X_{(n)} = \sum_{k=1}^{n}(-2\ln U_k)$$

is $\chi^2_{(n)}$-distributed, while for an odd number $2n+1$, one simply has to add $X_{(n)}$ and the square of an independent normally distributed random variable.

Inverse Gaussian (IG) distributions can be generated by a method introduced in Michael, Schucany, and Haas (1976) which we describe below. Since NIG distributions are variance-mean mixtures of normals with an IG mixing distribution, once we have a IG-distributed random variable, we can take an independent $N(0,1)$ random variate and generate a NIG random variate. Below we sketch the resulting algorithm for the generation of NIG random variables.

First, take two independent random variables U_1, U_2 that are uniformly distributed on $(0,1)$. Then N_1 and N_2 defined by (A.10) and (A.11) are independent normally distributed variables. Hence $V := N_1^2$ is $\chi^2_{(1)}$-distributed. That is, it has the same distribution as $(\gamma Z - \delta)^2/Z$ with an arbitrary *IG*(γ, δ)-distributed random variable Z, and we can hope to construct such a random variable Z from V.

Because we want to get a *NIG*$(\alpha, \beta, \mu, \delta)$-distributed random variable in the end, we have to choose $\gamma = \sqrt{\alpha^2 - \beta^2}$, as we will see below.

If $\gamma = 0$, then $Z = \delta^2/V$ is already *IG*$(0, \delta)$ distributed.

For $\gamma > 0$, the value of Z is not uniquely determined by $\widetilde{V} := (\gamma Z - \delta)^2/Z$. Instead, given $\widetilde{V} = v$ with $v > 0$ the variable Z has a binomial distribution with values z_1 and z_2, where

$$\begin{aligned} z_1 &= \delta/\gamma + (v/\gamma^2) - \sqrt{2v\delta/\gamma^3 + (v/\gamma^2)^2}, \\ z_2 &= (\delta/\gamma)^2/z_1 \end{aligned}$$

are the roots of the equation $(\gamma z - \delta)^2/z = v$. The probability of $Z = z_1(v)$ given $\widetilde{V} = v$ is

$$p_{z_1}(v) = \frac{\delta}{\delta + \gamma z_1},$$

according to Michael, Schucany, and Haas (1976).[11]

Let U_3 be a third uniform random variable, independent of U_1 and U_2, and define Z by

$$\widetilde{Z} := \begin{cases} z_1(V) & \text{on } \{U_3 < p_{z_1}(V)\}, \\ z_2(V) & \equiv (\delta/\gamma)^2/z_1(V). \end{cases} \tag{A.13}$$

Then the distribution of $\widetilde{Z}$ is *IG*(δ, γ).

Once we have an *IG*(δ, γ)-distributed random variable, we can use the mixture representation of *NIG*$(\alpha, \beta, \delta, \mu)$: With the standard normal random variable N_2, which is independent of N_1 and U_3, we define

$$X = \mu + \widetilde{Z}\beta + \sqrt{\widetilde{Z}}N_2.$$

Conditionally on $\widetilde{Z} = z$, X has distribution $N(\mu + \beta z, z)$. Because Z has distribution *IG*$(\delta, \gamma = \sqrt{\alpha^2 - \beta^2})$, X is distributed according to *NIG*$(\alpha, \beta, \mu, \delta)$. (See e. g. Barndorff-Nielsen (1997), p. 2.)

Remark 1
Using three independent $U(0,1)$-distributed random variables, we have got a *NIG*-distributed random variable. Consequently, plugging into this algorithm three "independent" pseudo-random numbers from a $U(0,1)$ population produces a pseudo-random number from a *NIG* population.

Remark 2
Normal inverse Gaussian distributed variates can be very efficiently generated by the approach described above. This algorithm is perfect for list-oriented statistics packages such as S-Plus, since it can be given in terms of listable functions. In other words, there appear only simple functions such as addition, multiplication, square root and logarithm. In S-Plus, these functions can be applied to a whole vector of arguments at once, generating a vector of results. Other approaches to random number generation generally use loops. For example, the acceptance-rejection method is an iterative procedure. However, it is well known that loops considerably slow down S-Plus programs.

[11] To be strict, one would have to show that *IG*(γ, δ) has a representation as a mixture of the binomial distributions described above, with a $\chi^2_{(1)}$ as the mixing distribution.

A.5 Comparison of NIG and Hyperbolic Distributions

Analytically, NIG is easier to handle for us than the hyperbolic distribution, because we are working primarily with the log moment-generating function. In the NIG case, this function is very simple. For the hyperbolic distribution, it contains the modified Bessel function K_1, which makes numerical evaluation difficult.

It is interesting to observe the different behavior of the moment generating function when u tends towards the boundary of the interval $[-\alpha - \beta, \alpha - \beta]$: For the hyperbolic distribution, the moment generating function

$$u \mapsto e^{u\mu}\Big(\frac{\alpha^2 - \beta^2}{\alpha^2 - (\beta + u)^2}\Big)^{1/2} \frac{K_1(\delta\sqrt{\alpha^2 - (\beta + u)^2})}{K_1(\delta\sqrt{\alpha^2 - \beta^2})}$$

diverges because $K_1(z) \sim \frac{1}{z}$ for small z, according to Abramowitz and Stegun (1968), 9.6.9.

In contrast, the moment generating function of a *NIG*$(\alpha, \beta, \mu, \delta)$ distribution,

$$u \mapsto \exp\big(\delta(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2}) + \mu u\big),$$

stays finite when $|\beta + u| \uparrow \alpha$, while its derivative becomes infinite.

Another striking difference becomes apparent if we examine, for the classes of hyperbolic and normal inverse Gaussian distributions, the subclasses of symmetric distributions with variance 1. In both cases, symmetry of the distribution is equivalent to $\beta = 0$. In the hyperbolic case, the condition of unit variance means

$$1 = \frac{\delta\alpha}{\alpha^2} \frac{K_2(\delta\alpha)}{K_1(\delta\alpha)},$$

so given the value $\zeta = \delta\alpha$ we have to choose

$$\alpha = \sqrt{\frac{\zeta K_2(\zeta)}{K_1(\zeta)}}.$$

Since the parameter restrictions for hyperbolic distributions allow $\zeta = \alpha\delta$ to vary in the interval $(0, \infty)$, α can take on only values in the interval $(\sqrt{2}, \infty)$, where $\sqrt{2} = \lim_{\zeta \downarrow 0} \frac{\zeta K_2(\zeta)}{K_1(\zeta)}$ because of the limiting form of the modified Bessel functions K_ν for $z \to 0$ with fixed Re$(\nu) > 0$:

$$K_\nu(z) \sim \frac{2^{\nu-1}\Gamma(\nu)}{z^\nu}.$$

(Remember $\Gamma(1) = \Gamma(2) = 1$.) So there is a positive lower bound for α, which means that the exponential decay of the tails of the distribution takes place with a positive minimum rate.

On the other hand, from the expression for the variance of a normal inverse Gaussian distribution,

$$\frac{\delta}{\alpha \cdot \big(1 - \big(\frac{\beta}{\alpha}\big)^2\big)^{3/2}},$$

we see that $\alpha = \delta$ is the choice which leads to unit variance. Thus the admissible range of α is the whole interval $(0, \infty)$, and the exponential decay of the tails can take place with arbitrarily low rates.

This different behavior of NIG and hyperbolic distributions is illustrated by Figures A.1 and A.2. Both show the log densities of three symmetric and centered distributions with variance 1. There remains one

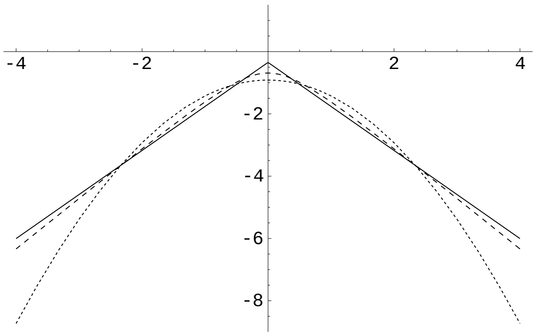

Figure A.1: Log densities of normalized hyperbolic distributions for parameters $\zeta = 100$ (dotted line), $\zeta = 1$ (dashed line), $\zeta = 0.01$ (solid line).

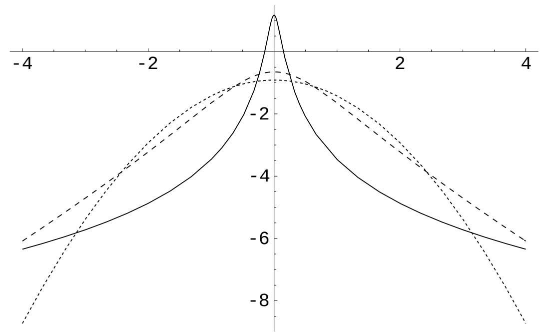

Figure A.2: Log densities of normalized normal inverse Gaussian distributions for parameters $\zeta = 100$ (dotted line), $\zeta = 1$ (dashed line), $\zeta = 0.01$ (solid line).

free parameter, ζ, in both classes. When changing ζ from large values to values near 0, we observe the following. For large values of ζ, both log densities look very much like a parabola near $x = 0$. For $\zeta \downarrow 0$, the log density of the hyperbolic distribution converges (pointwise) to the function $x \mapsto const. - \sqrt{2}|x|$. By contrast, for the NIG distribution there is no finite limit function. Instead, the log density becomes increasingly pointed around $x = 0$ as $\zeta \downarrow 0$.

There is some connection between this difference of hyperbolic and normal inverse Gaussian distributions and another point: Hyperbolic log-densities, being hyperbolas, are strictly concave everywhere Therefore they cannot form any sharp tips near $x = 0$ without losing too much mass in the tails to have variance 1. In contrast, normal inverse Gaussian log-densities are concave only in an interval around $x = 0$, and convex in the tails. Therefore they can form very sharp tips in the center and yet have variance 1.

A.5.1 Implications for Maximum Likelihood Estimation

The program “hyp”, which does maximum likelihood estimation for hyperbolic distributions (see Blæsild and Sørensen (1992)), often fails to find an estimate or runs towards $\delta = 0$. This behavior is probably due to the above-mentioned property of the family of hyperbolic distributions. Obviously it is not favorable for maximum likelihood estimation to have convergence of the densities when parameters tend towards the boundary of the allowed domain. “hyp” does not seem to tackle these boundary problems.

A.6 Generalized Hyperbolic Lévy Motion

Every infinitely divisible distribution has a characteristic function of the form

$$\chi(u) = \exp(\phi(u))$$

with some continuous function $\phi(u)$ satisfying $\phi(0) = 0$. (See Chow and Teicher (1997), Section 12.1, Proposition 2 and Lemma 1. The Lévy-Khintchine formula gives the explicit form of the function $\phi(u)$, but this is not needed here.) For every $t \geq 0$, one can form the exponential $\chi(u)^t := \exp(t\phi(u))$. $\chi(u)^t$

is again a characteristic function. The corresponding probability measures $P^{(t)}$ form a convolution semi-group, for which can construct a *canonical process* with stationary, independent increments according to Bauer (1991), §§35, 36. The increment of this process over a period of length Δt has the distribution $P^{(\Delta t)}$.

In this sense, every infinitely divisible distribution D on $(\mathbb{R}, \mathcal{B})$ *generates* a Lévy process L with $L_1 \sim D$. As in Eberlein (1999), Section 4, we denote by *generalized hyperbolic Lévy motion* the Lévy processes corresponding to generalized hyperbolic distributions. Analogously, we define the terms *hyperbolic Lévy motion* and *NIG Lévy motion*.

Appendix B

Complements to Chapter 3

B.1 Convolutions and Laplace transforms

For the convenience of the reader, we present here some easy consequences of theory of the Laplace transformation as displayed e. g. in Doetsch (1950).

Theorem B.1. *Let F_1 and F_2 be measurable complex-valued functions on the real line. If $|F_1(x)|$ is bounded and if $F_2(x)$ is absolutely integrable, then the convolution $F_1 * F_2$, defined by*

$$F_1 * F_2(x) := \int_{\mathbb{R}} F_1(x-y)F_2(y)dy,$$

is a well-defined function on $\mathbb{R}$. $F_1 * F_2$ is bounded and uniformly continuous.

Proof. Existence and boundedness follow from Doetsch (1950), p. 108, Satz 1. Uniform continuity follows by Doetsch (1950), p. 111, Satz 3. □

Theorem B.2. *Let F_1 and F_2 be measurable complex-valued functions on the real line. Let $z \in \mathbb{C}$ and $R :=$ Re z. If*

$$\int_{\mathbb{R}} e^{-Rx}|F_1(x)|dx < \infty \qquad \textit{and} \qquad \int_{\mathbb{R}} e^{-Rx}|F_2(x)|dx < \infty,$$

and if $x \mapsto e^{-Rx}|F_1(x)|$ is bounded, then the convolution $F(x) := F_1 * F_2(x)$ exists and is continuous for all $x \in \mathbb{R}$, and we have

$$\int_{\mathbb{R}} e^{-Rx}|F(x)|dx < \infty \qquad \textit{and} \qquad \int_{\mathbb{R}} e^{-zx}F(x)dx = \int_{\mathbb{R}} e^{-zx}F_1(x)dx \cdot \int_{\mathbb{R}} e^{-zx}F_2(x)dx.$$

Proof. Except for the statement of continuity, this is a part of the statements proven in Doetsch (1950), p. 123, Satz 3. For the continuity, note that

$$\widetilde{F}_1(x) := e^{-Rx}F_1(x) \qquad \text{and} \qquad \widetilde{F}_2(x) := e^{-Rx}F_2(x)$$

satisfy the conditions of Theorem B.1. Thus their convolution

$$\begin{aligned}\widetilde{F}(x) :&= \int_{\mathbb{R}} \widetilde{F}_1(x-y)\widetilde{F}_2(y)dy \\ &= \int_{\mathbb{R}} e^{-R(x-y)}F_1(x-y)e^{-Ry}F_2(y)dy.\end{aligned}$$

is uniformly continuous. But we have

$$
\begin{aligned}
F(x) &\equiv \int_{\mathbb{R}} F_1(x-y)F_2(y)dy \\
&= e^{Rx}\int_{\mathbb{R}} e^{-R(x-y)}F_1(x-y)e^{-Ry}F_2(y)dy \\
&= e^{Rx}\widetilde{F}(x),
\end{aligned}
$$

which proves the continuity of F. □

Remark: Theorem B.2 shows that the Laplace transform of a convolution is the product of the Laplace transforms of the factors. This is a generalization of the well-known analogous result for Fourier transforms.

The next theorem shows how one can invert the Laplace transformation. Together with Theorem B.2, this enables us to calculate the convolution if we know the Laplace transforms of the factors.

Theorem B.3. *Let F be a measurable complex-valued function on the real line. Let $R \in \mathbb{R}$ such that*

$$
f(z) = \int_{\mathbb{R}} e^{-zx}F(x)dx \qquad (z \in \mathbb{C}, \mathrm{Re}\ z = R),
$$

with the integral converging absolutely for $z = R$.[1] *Let $x \in \mathbb{R}$ such that the integral*

$$
\int_{R-i\infty}^{R+i\infty} e^{zx}f(z)dz
$$

exists as a Cauchy principal value. Assume that F is continuous at the point x. Then

$$
F(x) = \frac{1}{2\pi i}\int_{R-i\infty}^{R+i\infty} e^{zx}f(z)dz,
$$

where the integral is to be understood as the Cauchy principal value if the integrand is not absolutely integrable.

Proof. Cf. Doetsch (1950), p. 216, Satz 5. □

B.2 Modeling the Log Return on a Spot Contract Instead of a Forward Contract

In the text, we assume that χ is the characteristic function of the distribution of $X_T := \ln(e^{-rT}S_T/S_0)$. This corresponds to a stock price model of the form

$$
S_T = S_0 e^{rT+X_T},
$$

where X_T is the log return on a forward contract to buy the stock at the forward date T. In some contexts, models of the form

$$
S_T = S_0 e^{Y_T}
$$

[1] Obviously, then the integral converges absolutely for all $z \in \mathbb{C}$ with Re $z = R$.

are used instead. Here Y_T is the log return on a spot contract in which one buys the stock today and sells it at date T. Equating the stock prices leads to the relation

$$rT + X_T = Y_T.$$

Consequently, if we are given the characteristic function $\psi(u)$ of Y_T, we can calculate the characteristic function $\chi(u)$ of X_T as

$$\chi(u) = E[e^{iuX_T}] = e^{-iurT}E[e^{iuY_T}] = e^{-iurT}\psi(u).$$

Therefore if we know the characteristic function ψ, we at once have an expression for the characteristic function $\chi(u)$. This can then be used to price the options as described in the text.

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