Ackermann's function

Definition. Ackermann's function is recursively defined as follows:

$$\begin{array}{rcll} \alpha(m,0) & = & m+1 & \text{(i)} \\ \alpha(0,n+1) & = & \alpha(1,n) & \text{(ii)} \\ \alpha(m+1,n+1) & = & \alpha(\alpha(m,n+1),n) & \text{(iii)} \end{array}$$

The Ackermann function is well defined, i.e. we can prove the following lemma:

Lemma 0. For all $y, x \in N$, there exists a $z \in N$ such that $\alpha(x,y) = z$.

Proof. By a main induction on y and a secondary induction on x:

- Base Case [Main Induction]:

$$\alpha(x,0) = x + 1 \qquad \text{Def. of } \alpha$$

- Inductive Step [Main Induction]:

$$\forall x \in N$$
$$\alpha(x,k) = z_{x,k} \rightarrow \exists z_{x,k+1}\colon \alpha(x,k+1) = z_{x,k+1}$$

  - Base Case [2nd Induction]:

$$\alpha(0,k+1) = \alpha(1,k) = \qquad \text{Def. of } \alpha$$
$$z_{1,k} \qquad \text{Main Ind. Hypo.}$$

  - Inductive Step [2nd Induction]:

$$\alpha(t,k+1) = z_{t,k+1} \rightarrow \exists z_{t+1,k+1}\colon \alpha(t+1,k+1) = z_{t+1,k+1}$$

$$\alpha(t+1,k+1) = \alpha(\alpha(t,k+1),k) = \qquad \text{Def. of } \alpha$$
$$\alpha\left(z_{t,k+1},k\right) = \qquad \text{2nd Ind. Hypo.}$$
$$z_{z_{t,k+1},k} \qquad \text{Main Ind. Hypo.}$$

Lemma 1. For all $m, n \in N$, $\alpha(m, n) > m$.

Proof. By a main induction on n and a secondary induction on m:

- Base Case [Main Induction]:

$$\alpha(m, 0) = m + 1 > m \qquad \text{Def. of } \alpha$$

- Inductive Step [Main Induction]:

$$\forall m \in N$$
$$\alpha(m, k) > m \to \alpha(m, k + 1) > m$$

  - Base Case [2nd Induction]:

$$\begin{aligned} &\alpha(0, k + 1) = \alpha(1, k) && \text{Def. of } \alpha \\ &> 1 && \text{Main Ind. Hyp.} \\ &> 0 && \end{aligned}$$

  - Inductive Step [2nd Induction]:

$$\alpha(t, k + 1) > t \to \alpha(t + 1, k + 1) > t + 1$$

$$\begin{aligned} &\alpha(t + 1, k + 1) = \alpha(\alpha(t, k + 1), k), && \text{Def. of } \alpha \\ &p = \alpha(t, k + 1) > t && \text{2nd Ind. Hyp.} \\ &\Rightarrow \alpha(p, k) > p, \quad p \geq t + 1 && \text{Main Ind. Hyp.} \\ &\Rightarrow \alpha(p, k) > t + 1 && \end{aligned}$$

Lemma 2.1. For all $y, z, x \in N$, if $x < z$ then $\alpha(x, y) < \alpha(z, y)$.

Proof. By a main induction on y and a secondary induction on z:

- Base Case [Main Induction]:

$$\alpha(x, 0) = x + 1, \qquad \text{Def. of } \alpha$$
$$\alpha(z, 0) = z + 1 \qquad \text{Def. of } \alpha$$
$$x < z \Rightarrow \alpha(x, 0) < \alpha(z, 0)$$

- Inductive Step [Main Induction]:

$$\forall x, z \in N: x < z$$
$$\alpha(x, k) < \alpha(z, k) \rightarrow \alpha(x, k + 1) < \alpha(z, k + 1)$$

  - Base Case [2nd Induction]:

$$\alpha(0, k + 1) = \alpha(1, k), \qquad \text{Def. of } \alpha$$
$$\alpha(1, k + 1) = \alpha(\alpha(0, k + 1), k) \qquad \text{Def. of } \alpha$$
$$= \alpha(\alpha(1, k), k) \qquad \text{Def. of } \alpha$$
$$> \alpha(1, k) \qquad \text{Lemma 1}$$
$$\Rightarrow \alpha(0, k + 1) < \alpha(1, k + 1)$$

  - Inductive Step [2nd Induction]:

$$\forall x < t,$$
$$\alpha(x, k + 1) < \alpha(t, k + 1) \rightarrow \alpha(x, k + 1) < \alpha(t + 1, k + 1)$$

$$\alpha(t + 1, k + 1) = \alpha(\alpha(t, k + 1), k), \qquad \text{Def. of } \alpha$$
$$> \alpha(t, k + 1) \qquad \text{Lemma 1}$$
$$> \alpha(x, k + 1) \qquad \text{2nd Ind. Hyp.}$$
$$\Rightarrow \alpha(x, k + 1) < \alpha(t + 1, k + 1)$$

Lemma 2.2. For all $y, z, x \in N$, if $y < z$ then $\alpha(x, y) < \alpha(x, z)$.

Proof. By a main induction on z and secondary induction on x:

- Base Case [Main Induction]:

$$\forall x \in N \;\; \alpha(x, 0) < \alpha(x, 1)$$

  - Base Case [2nd Induction]:

$$\begin{aligned}
&\alpha(0,0) = 1, && \text{Def. of } \alpha \\
&\alpha(0,1) = \alpha(1,0) = 2 && \text{Def. of } \alpha \\
&\Rightarrow \alpha(0,0) < \alpha(0,1) &&
\end{aligned}$$

  - Inductive Step [2nd Induction]:

$$\alpha(k, 0) < \alpha(k, 1) \rightarrow \alpha(k + 1,0) < \alpha(k + 1,1)$$

$$\begin{aligned}
&\alpha(k + 1,0) = k + 2, && \text{Def. of } \alpha \\
&\alpha(k + 1,1) = \alpha(\alpha(k, 1), 0) = \alpha(k, 1) + 1, && \text{Def. of } \alpha \\
&k + 1 < \alpha(k, 1) && \text{2nd Ind. Hyp.} \\
&\Rightarrow \alpha(k, 1) + 1 > k + 2 = \alpha(k + 1,0) &&
\end{aligned}$$

- Inductive Step [Main Induction]:

$$\forall x, y \in N\colon y < k$$
$$\alpha(x, y) < \alpha(x, k) \rightarrow \alpha(x, y) < \alpha(x, k + 1)$$

  - Base Case [2nd Induction]:

$$\begin{aligned}
&\alpha(0, k + 1) = \alpha(1, k) && \text{Def. of } \alpha \\
&> \alpha(0, k) && \text{Lemma 2.1} \\
&> \alpha(0, y) && \text{Main Ind. Hyp.}
\end{aligned}$$

  - Inductive Step [2nd Induction]:

$$\alpha(t, y) < \alpha(t, k + 1) \rightarrow \alpha(t + 1, y) < \alpha(t + 1, k + 1)$$

$$\begin{aligned}
&\alpha(t + 1, k + 1) = \alpha(\alpha(t, k + 1), k) && \text{Def. of } \alpha \\
&> \alpha(\alpha(t, y), k) && \text{Lemma 1, 2nd Ind. Hyp.} \\
&> \alpha(\alpha(t, y), y - 1) && \text{Main Ind. Hyp.} \\
&= \alpha(t + 1, y) && \text{Def. of } \alpha
\end{aligned}$$

Lemma 3. For all $m, n \in N$, $\alpha(m, n+1) \geq \alpha(m+1, n)$.

Proof. By a man induction on n and a secondary induction on m:

- Base Case [Main Induction]:

$$\forall m \in N \;\; \alpha(m, 1) \geq \alpha(m+1, 0)$$

  - Base Case [2nd Induction]:

$$\alpha(0,1) = \alpha(1,0) \qquad \text{Def. of } \alpha$$

  - Inductive Step [2nd Induction]:

$$\alpha(m, 1) \geq \alpha(m+1, 0) \rightarrow \alpha(m+1, 1) \geq \alpha(m+2, 0)$$

$$\begin{aligned} &\alpha(m+1,1) = \alpha(\alpha(m,1), 0) = \alpha(m,1) + 1 && \text{Def. of } \alpha \\ &\geq \alpha(m+1,0) + 1 && \text{2nd Ind. Hyp.} \\ &= m + 3 = \alpha(m+2,0) && \text{Def. of } \alpha \end{aligned}$$

- Inductive Step [Main Induction]:

$$\forall m \in N$$
$$\alpha(m, k+1) \geq \alpha(m+1, k) \rightarrow \alpha(m, k+2) \geq \alpha(m+1, k+1)$$

  - Base Case [2nd Induction]:

$$\alpha(0, k+2) = \alpha(1, k+1) \qquad \text{Def. of } \alpha$$

  - Inductive Step [2nd Induction]:

$$\alpha(t, k+2) \geq \alpha(t+1, k+1) \rightarrow \alpha(t+1, k+2) \geq \alpha(t+2, k+1)$$

$$\begin{aligned} &\alpha(t+1, k+2) = \alpha(\alpha(t, k+2), k+1) && \text{Def. of } \alpha \\ &\geq \alpha(\alpha(t+1, k+1), k+1), && \text{Lemma 2.1, 2nd Ind. Hyp.} \\ &\alpha(t+1, k+1) > t+1 && \text{Lemma 1} \\ &\Rightarrow \alpha(t+1, k+2) \geq \alpha(t+2, k+1) \end{aligned}$$

Majorization Lemma. For every primitive recursive function $f(x_1, \dots, x_k)$ there exists an $c \in N$ such that

$$f(x_1, \dots, x_k) < \alpha(\max(x_1, \dots, x_k), c)$$

for all values of $x_1, \dots, x_k$.

Proof. By induction on definition of primitive recursive functions. There are five cases:

1. Null Function

$$n(x) = 0 < x + 1 = \alpha(x, 0) \Rightarrow c = 0$$

2. Successor Function

$$s(x) = x + 1 < x + 2 = \alpha(x + 1, 0) \le \alpha(x, 1) \Rightarrow c = 1$$

3. Projection Function

$$u_i^k(x_1, \dots, x_k) = x_i < x_i + 1 \le \alpha(\max(x_1, \dots, x_k), 0) \Rightarrow c = 0$$

4. Composition

Let $g(y_1, \dots, y_k)$ and $f_i(x_1, \dots, x_m)$ for $i = 1, \dots, k$ be primitive recursive functions and let $h(x_1, \dots, x_m) = g(f_1(x_1, \dots, x_m), \dots, f_k(x_1, \dots, x_m))$. Assume that there are d and c_i such that

$$g(y_1, \dots, y_k) < \alpha(\max(y_1, \dots, y_k), d),$$

$$f_i(x_1, \dots, x_m) < \alpha(\max(x_1, \dots, x_m), c_i) \qquad i = 1, \dots, k$$

$$g(f_1, \dots, f_k) < \alpha(\max(f_1, \dots, f_k), d) \qquad \text{Hyp.}$$

Assume that $\max(f_1, \dots, f_k) = f_j$

$$\begin{aligned} g(f_1, \dots, f_k) &< \alpha(f_j, d) \\ &< \alpha(\alpha(\max(x_1, \dots, x_m), c_j), d) && \text{Lemma 2.1} \end{aligned}$$

Put $c_{max} = \max(c_1, \dots, c_k, d)$

$$\begin{aligned} &\alpha(\alpha(\max(x_1, \dots, x_m), c_j), d) \\ &\le \alpha(\alpha(\max(x_1, \dots, x_m), c_j), c_{max}) && \text{Lemma 2.2} \\ &< \alpha(\alpha(\max(x_1, \dots, x_m), c_{max} + 1), c_{max}) && \text{Lemma 2.1} \\ &= \alpha(\max(x_1, \dots, x_m) + 1, c_{max} + 1) && \text{Def. of } \alpha \\ &\le \alpha(\max(x_1, \dots, x_m), c_{max} + 2) && \text{Lemma 3} \\ &\Rightarrow h(x_1, \dots, x_k) < \alpha(\max(x_1, \dots, x_k), c_{max} + 2) \\ &\Rightarrow c = c_{max} + 2 \end{aligned}$$

5. Primitive Recursion

Let f be a primitive recursive function and h is defined by recursion:

$$h(0) = 0, \qquad h(t+1) = f(t, h(t))$$

Suppose there exists a c_f such that $f(a, b) < \alpha(\max(a, b), c_f)$ for all a, b. Let $c_h = c_f + 1$. We prove by induction that $h(x) < \alpha(x, c_h)$ for all x.

- Base Case

$$h(0) = 0 < 1 = \alpha(0,0) < \alpha(0, c_h) \qquad \text{Def. of } \alpha\text{, Lemma 2.2}$$

- Inductive Step

$$h(t) < \alpha(t, c_h) \Rightarrow h(t+1) < \alpha(t+1, c_h)$$

$$h(t+1) = g\big(t, h(t)\big) < \alpha\big(\max\big(t, h(t)\big), c_g\big)$$

If $\max\big(t, h(t)\big) = t$

$$\alpha\big(\max\big(t, h(t)\big), c_g\big) = \alpha\big(t, c_g\big) < \alpha\big(t+1, c_g\big) < \alpha(t+1, c_h) \qquad \text{Lemma 2.1, 2.2}$$

If $\max\big(t, h(t)\big) = h(t)$

$$\alpha\big(\max\big(t, h(t)\big), c_g\big) = \alpha\big(h(t), c_g\big) < \alpha\big(\alpha(t, c_h), c_g\big) = \alpha(t+1, c_h) \qquad \text{Lemma 2.1, Hyp. Def. of } \alpha$$

Theorem. The Ackerman function is not primitive recursive.

Proof. Define $\beta(x) = \alpha(x, x) + 1$ and suppose α (Ackerman function) is primitive recursive, then also β is primitive recursive. So, by the lemma 3, there exists a k such that $\beta(x) < \alpha(x, k)$ for all x.

$$\beta(k) = \alpha(k, k) + 1 < \alpha(k, k)$$

Is a contradiction. Therefore α is not a primitive recursive function.