Solution Manual
for
Modern Electrodynamics

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A Note from the Author

This manual provides solutions to the end-of-chapter problems for the author's *Modern Electrodynamics*. The chance that all these solutions are correct is zero. Therefore, I will be pleased to hear from readers who discover errors. I will also be pleased to hear from readers who can provide a better solution to this or that problem than I was able to construct. I urge readers to suggest that this or that problem *should not* appear in a future edition of the book and (equally) to propose problems (and solutions) they believe *should* appear in a future edition.

At a fairly advanced stage in the writing of this book, I decided that a source should be cited for every end-of-chapter problem in the book. Unfortunately, I had by that time spent a decade accumulating problems from various places without always carefully noting the source. For that reason, I encourage readers to contact me if they recognize a problem of their own invention or if they can identify the (original) source of any particular problem in the manual. An interesting issue arises with problems I found on instructor or course websites which were taken down after the course they serviced had concluded. My solution has been to cite the source of these problems as a "public communication" between myself and the course instructor. This contrasts with problems cited as a true "private communication" between myself and an individual.

Chapter 1: Mathematical Preliminaries

1.1 Levi-Cività Practice I

(a) $\epsilon_{123} = \hat{\mathbf{e}}_1 \cdot (\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_3) = \hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 = 1$. The cyclic property of the triple scalar product guarantees that $\epsilon_{231} = \epsilon_{312} = 1$ also. Similarly, $\epsilon_{132} = \hat{\mathbf{e}}_1 \cdot (\hat{\mathbf{e}}_3 \times \hat{\mathbf{e}}_2) = -\hat{\mathbf{e}}_1 \cdot \hat{\mathbf{e}}_1 = -1$ with $\epsilon_{321} = \epsilon_{213} = -1$ also. Finally, $\epsilon_{122} = \hat{\mathbf{e}}_1 \cdot (\hat{\mathbf{e}}_2 \times \hat{\mathbf{e}}_2) = 0$ and similarly whenever two indices are equal.

(b) Expand the determinant by minors to get

$$\mathbf{a} \times \mathbf{b} = \hat{\mathbf{e}}_1(a_2 b_3 - a_3 b_2) - \hat{\mathbf{e}}_2(a_1 b_3 - a_3 b_1) + \hat{\mathbf{e}}_3(a_1 b_2 - a_2 b_1).$$

Using the Levi-Cività symbol to supply the signs, this is the same as the suggested identity because

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \epsilon_{123}\hat{\mathbf{e}}_1 a_2 b_3 + \epsilon_{132}\hat{\mathbf{e}}_1 a_3 b_2 \\ &\quad + \epsilon_{213}\hat{\mathbf{e}}_2 a_1 b_3 + \epsilon_{231}\hat{\mathbf{e}}_2 a_3 b_1 \\ &\quad + \epsilon_{312}\hat{\mathbf{e}}_3 a_1 b_2 + \epsilon_{321}\hat{\mathbf{e}}_3 a_2 b_1. \end{aligned}$$

(c) To get a non-zero contribution to the sum, the index i must be different from the unequal indices j and k, and also different from the unequal indices s and t. Therefore, the pair (i, j) and the pair (s, t) are the same pair of different indices. There are only two ways to do this. If $i = s$ and $j = t$, the ϵ terms are identical and their square is 1. This is the first term in the proposed identity. The other possibility introduces a transposition of two indices in one of the epsilon factors compared to the previous case. This generates an overall minus sign and thus the second term in the identity.

(d) The scalar of interest is $S = \hat{L}_m a_m \hat{L}_p b_p - \hat{L}_q b_q \hat{L}_s a_s$. Using the given commutation relation,

$$\begin{aligned} S &= a_m b_p \hat{L}_m \hat{L}_p - a_p b_m \hat{L}_m \hat{L}_p \\ &= a_m b_p \hat{L}_m \hat{L}_p - a_m b_p \hat{L}_p \hat{L}_m \\ &= a_m b_p [\hat{L}_m, \hat{L}_p] \\ &= i\hbar \epsilon_{mpi} \hat{L}_i a_m b_p \\ &= i\hbar \hat{L}_i \epsilon_{imp} a_m b_p \\ &= i\hbar \hat{\mathbf{L}} \cdot (\mathbf{a} \times \mathbf{b}). \end{aligned}$$

1.2 Levi-Civitá Practice II

(a) $\delta_{ii} = 1 + 1 + 1 = 3$

(b) $\delta_{ij}\epsilon_{ijk} = \epsilon_{iik} = 0$

(c) $\epsilon_{ijk}\epsilon_{\ell jk} = \epsilon_{jki}\epsilon_{jk\ell} = \delta_{kk}\delta_{i\ell} - \delta_{k\ell}\delta_{ik} = 3\delta_{i\ell} - \delta_{i\ell} = 2\delta_{i\ell}$

(d) $\epsilon_{ijk}\epsilon_{ijk} = \delta_{jj}\delta_{kk} - \delta_{jk}\delta_{kj} = 9 - \delta_{kk} = 6$

1.3 Vector Identities

(a)
$$\begin{aligned}(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) &= \epsilon_{ijk}A_jB_k\epsilon_{imp}C_mD_p = \epsilon_{ijk}\epsilon_{imp}A_jB_kC_mD_p \\ &= (\delta_{jm}\delta_{kp} - \delta_{jp}\delta_{km})A_jB_kC_mD_p \\ &= A_mC_mB_kD_k - A_jD_jB_kC_k = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})\end{aligned}$$

(b)
$$\begin{aligned}\nabla\cdot(f\times g) &= \partial_i\epsilon_{ijk}f_jg_k = \epsilon_{ijk}f_j\partial_ig_k + \epsilon_{ijk}g_k\partial_if_j = f_j\epsilon_{jki}\partial_ig_k + g_k\epsilon_{kij}\partial_if_j \\ &= g_k\epsilon_{kij}\partial_if_j - f_j\epsilon_{jik}\partial_ig_k = g\cdot(\nabla\times f) - f\cdot(\nabla\times g)\end{aligned}$$

(c)
$$\begin{aligned}[(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D})]_i &= \epsilon_{ijk}\left\{\mathbf{A}\times\mathbf{B}\right\}_j\left\{\mathbf{C}\times\mathbf{D}\right\}_k = \epsilon_{ijk}\epsilon_{jmp}\epsilon_{kst}A_mB_pC_sD_t \\ &= \epsilon_{jki}\epsilon_{jmp}\epsilon_{kst}A_mB_pC_sD_t = (\delta_{km}\delta_{ip} - \delta_{kp}\delta_{im})\epsilon_{kst}A_mB_pC_sD_t \\ &= \epsilon_{kst}A_kB_iC_sD_t - \epsilon_{kst}A_iB_kC_sD_t = A_k\epsilon_{kst}C_sD_tB_i - B_k\epsilon_{kst}C_sD_tA_i \\ &= \mathbf{A}\cdot(\mathbf{C}\times\mathbf{D})B_i - \mathbf{B}\cdot(\mathbf{C}\times\mathbf{D})A_i\end{aligned}$$

(d) $(\boldsymbol{\sigma}\cdot\mathbf{a})(\boldsymbol{\sigma}\cdot\mathbf{b}) = \sigma_ia_i\sigma_jb_j = \sigma_i\sigma_ja_ib_j = (\delta_{ij}+i\epsilon_{ijk}\sigma_k)a_ib_j = a_ib_i+i\epsilon_{kij}\sigma_ka_ib_j = \mathbf{a}\cdot\mathbf{b}+i\boldsymbol{\sigma}\cdot(\mathbf{a}\times\mathbf{b})$

1.4 Vector Derivative Identities

(a) $\nabla\cdot(f\mathbf{g}) = \partial_i(fg_i) = f\partial_ig_i + g_i\partial_if = f\nabla\cdot\mathbf{g} + (\mathbf{g}\cdot\nabla)f$

(b) $\left\{\nabla\times(f\mathbf{g})\right\}_i = \epsilon_{ijk}\partial_j(fg_k) = f\epsilon_{ijk}\partial_jg_k + \epsilon_{ijk}(\partial_jf)g_k = f\left[\nabla\times\mathbf{g}\right]_i + \left[\nabla f\times\mathbf{g}\right]_i$

(c)

$$\begin{aligned}
[\nabla \times (\mathbf{g} \times \mathbf{r})]_i &= \epsilon_{ijk}\partial_j \epsilon_{k\ell m} g_\ell r_m \\
&= \epsilon_{kij}\epsilon_{k\ell m}\partial_j (g_\ell r_m) \\
&= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})\partial_j (g_\ell r_m) \\
&= \partial_j (g_i r_j) - \partial_j (g_j r_i) \\
&= r_j g_j g_i + g_i(\nabla \cdot \mathbf{r}) - r_i(\nabla \cdot \mathbf{g}) - \delta_{ij} g_j \\
&= (\mathbf{r} \cdot \nabla) g_i + 3g_i - r_i(\nabla \cdot \mathbf{g}) - g_i
\end{aligned}$$

Therefore,

$$\nabla \times (\mathbf{g} \times \mathbf{r}) = 2\mathbf{g} + r\frac{\partial \mathbf{g}}{\partial r} - \mathbf{r}(\nabla \cdot \mathbf{g}).$$

1.5 Delta Function Identities

(a) Let $f(x)$ be an arbitrary function. Then, if $a > 0$, a change of variable to $y = ax$ gives

$$\int_{-\infty}^{\infty} dx f(x)\delta(ax) = \frac{1}{a}\int_{-\infty}^{\infty} dy f(y/a)\delta(y) = \frac{1}{a}f(0).$$

However, if $a < 0$,

$$\int_{-\infty}^{\infty} dx f(x)\delta(ax) = \frac{1}{a}\int_{\infty}^{-\infty} dy f(y/a)\delta(y) = -\frac{1}{a}\int_{-\infty}^{\infty} dy f(y/a)\delta(y) = -\frac{1}{a}f(0).$$

These two results are summarized by $\delta(ax) = \dfrac{1}{|a|}\delta(x)$.

(b) If $g(x_0) = 0$, $\delta[g(x)]$ is singular at $x = x_0$. Very near this point, $g(x) \approx (x - x_0)g'(x_0)$. Therefore, using the identity in part (a),

$$\int_{-\infty}^{\infty} dx f(x)\delta[g(x)] \approx \int_{-\infty}^{\infty} dx f(x) g[(x - x_0)g'(x_0)] = \frac{1}{|g'(x_0)|}\delta(x - x_0).$$

A similar contribution comes from each distinct zero x_m. Adding these together gives the advertised result.

(c) We use the result of part (b). The zeroes of $\cos x$ occur at $x = (2n+1)\pi/2$. At these points, $|-\sin x| = 1$; therefore,

$$I = \sum_{n=0}^{\infty} \exp([-(2n+1)\pi/2] = \exp(-\pi/2)\sum_{n=0}^{\infty} \exp(-n\pi) = \frac{\exp(-\pi/2)}{1-\exp(\pi)} = \frac{1}{2\sinh(\pi/2)}.$$

1.6 Radial Delta Functions

(a) We need to show that $\delta(r)/r$ and $-\delta'(r)$ have the same effect when multiplied by an arbitrary function and integrated over the radial part of a volume integral. If we call the arbitrary function $f(r)$, one of these integrals vanishes identically because $r\delta(r) = 0$:

$$\int_0^{\infty} dr\, r^2 f(r)\frac{\delta(r)}{r} = \int_0^{\infty} dr\, r\, f(r)\,\delta(r) = 0$$

This tells us we need to represent the arbitrary function in a smarter way. One possibility is $f(r)/r$. This gives

$$\int_0^{\infty} dr\, r^2 \frac{f(r)}{r}\frac{\delta(r)}{r} = \int_0^{\infty} dr\, f(r)\,\delta(r) = f(0).$$

An integration by parts shows that the proposed identity is correct:

$$\begin{aligned} -\int_0^{\infty} dr\, r^2 \frac{f(r)}{r}\delta'(r) &= -\int_0^{\infty} dr\, r\, f(r)\,\delta'(r) = \int_0^{\infty} dr\,\delta(r)\,[rf(r)]' \\ &= \int_0^{\infty} dr\,\delta(r)\,[f(r) + rf'(r)] = f(0). \end{aligned}$$

(b) By direct calculation,

$$\nabla\cdot[\delta(r-a)\hat{\mathbf{r}}] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\delta(r-a)\right] = \frac{2}{r}\delta(r-a) + \delta'(r-a). \tag{1}$$

Let us look at the effect of $\delta'(r-a)$ on an arbitrary test function:

$$\begin{aligned} \int_0^{\infty} dr r^2 f(\mathbf{r})\frac{d}{dr}\delta(r-a) &= \int_0^{\infty} dr\frac{d}{dr}\left[\delta(r-a)r^2 f\right] - \int_0^{\infty} \delta(r-a)\frac{d}{dr}\left[r^2 f\right] \\ &= -\int_0^{\infty} dr\delta(r-a)\left[2rf + r^2\frac{df}{dr}\right] = -2af(a,\theta,\phi) - a^2\left.\frac{df}{dr}\right|_{r=a}. \end{aligned}$$

This shows that

$$\delta'(r-a) = -\frac{2}{a}\delta(r-a) + \frac{a^2}{r^2}\delta'(r-a).$$

Combining this with (1) shows that

$$\nabla\cdot[\delta(r-a)\hat{\mathbf{r}}] = (a^2/r^2)\delta'(r-a).$$

Source: R. Donnelly, *Journal of the Optical Society of America A* **10**, 680 (1993).

1.7 A Representation of the Delta Function

The calculation involves a change of variable,

$$\begin{aligned}\int_{-\infty}^{\infty} dx f(x)D(x) &= \lim_{m\to\infty}\int_{-\infty}^{\infty} dx f(x)\frac{\sin mx}{\pi x} = \lim_{m\to\infty}\int_{-\infty}^{\infty}\frac{dy}{m}f(\frac{y}{m})\frac{\sin y}{\pi}\frac{m}{y} \\ &= f(0)\frac{1}{\pi}\int_{-\infty}^{\infty} dy\frac{\sin y}{y}.\end{aligned}$$

The assertion is proved if the integral on the far right side is equal to π. You can look up the integral or use this trick:

$$\begin{aligned}\int_{-\infty}^{\infty} dy\frac{\sin y}{y} &= 2\int_0^{\infty} dy\frac{\sin y}{y} = \int_0^{\infty} dy\sin y\int_0^{\infty} d\nu e^{-\nu y} = 2\int_0^{\infty} d\nu\int_0^{\infty} dy\, e^{-\nu y}\sin y \\ &= 2\int_0^{\infty}\frac{d\nu}{1+\nu^2} = \pi.\end{aligned}$$

1.8 An Application of Stokes' Theorem

(a) Let $\mathbf{p} = \nabla\times(\mathbf{c}\times\mathbf{F})$. Then, because $\mathbf{c}$ is a constant vector,

$$p_i = \epsilon_{ijk}\partial_j\epsilon_{kst}c_sF_t = \epsilon_{kij}\epsilon_{kst}c_s\partial_jF_t = (\delta_{is}\delta_{jt}-\delta_{it}\delta_{js})c_s\partial_jF_t = c_i\partial_jF_j - c_j\partial_jF_i.$$

This shows that $\nabla\times(\mathbf{c}\times\mathbf{F}) = \mathbf{c}(\nabla\cdot\mathbf{F}) - (\mathbf{c}\cdot\nabla)\mathbf{F}$. Inserting this into Stokes' Theorem as suggested gives

$$\int_S dS\,\hat{\mathbf{n}}\cdot\{\mathbf{c}(\nabla\cdot\mathbf{F})-(\mathbf{c}\cdot\nabla)\mathbf{F}\} = \oint_C d\mathbf{s}\cdot(\mathbf{c}\times\mathbf{F}) = \oint_C \mathbf{c}\cdot(\mathbf{F}\times d\mathbf{s})$$

or

$$\mathbf{c}\cdot\left[\int_S dS\,\{\hat{\mathbf{n}}(\nabla\cdot\mathbf{F}) - \hat{n}_i\nabla F_i\}\right] = \mathbf{c}\cdot\oint_C \mathbf{F}\times d\mathbf{s}.$$

This establishes the equality because $\mathbf{c}$ is arbitrary.

(b) Let $\mathbf{K} = \int_S dS(\hat{\mathbf{n}}\times\nabla)\times\mathbf{F}$. Then

$$\begin{aligned}K_i &= \int_S dS\,\epsilon_{ijk}(\hat{\mathbf{n}}\times\nabla)_jF_k = \int_S dS\,\epsilon_{ijk}\epsilon_{jst}\hat{n}_s\partial_tF_k \\ &= \int_S dS\,(\delta_{ks}\delta_{it}-\delta_{kt}\delta_{is})\hat{n}_s\partial_tF_k = \int_S dS\,(\hat{n}_k\partial_iF_k - \hat{n}_i\partial_kF_k).\end{aligned}$$

This proves that $\mathbf{K} = \int_S dS\,\{\hat{n}_i\nabla F_i - \hat{\mathbf{n}}(\nabla\cdot\mathbf{F})\}$, which was the second equality in question.

(c) This is a special case of the identity in part (a) with $\mathbf{F} = \mathbf{r}$. Therefore,

$$\oint_C \mathbf{r} \times d\mathbf{s} = -\int_S dS \left\{ \hat{n}_i \nabla r_i - \hat{\mathbf{n}}(\nabla \cdot \mathbf{r}) \right\}.$$

Now, $\nabla \cdot \mathbf{r} = 3$. Also, $\nabla r_i = \hat{e}_j \partial_j r_i = \hat{e}_j \delta_{ij} = \hat{e}_i$ so $\hat{n}_i \nabla r_i = \hat{n}_i \hat{e}_i = \hat{\mathbf{n}}$. Hence,

$$\oint_C \mathbf{r} \times d\mathbf{s} = -\int_S dS(\hat{\mathbf{n}} - 3\hat{\mathbf{n}}) = 2\int_S d\mathbf{S}.$$

1.9 Three Derivative Identities

(a) Consider the x-component of the gradient. We have

$$\frac{\partial}{\partial x} f(x - x', y - y', z - z') = -\frac{\partial}{\partial x'} f(x - x', y - y', z - z')$$

and similarly for the y and z components. This proves the assertion.

(b) Writing this out in detail,

$$\begin{aligned}
\nabla \cdot [\mathbf{A}(r) \times \mathbf{r}] &= \partial_i \epsilon_{ijk} A_j r_k = \epsilon_{ijk} [r_k A_j'(r) \partial_i r + A_j(r) \partial_i r_k] \\
&= \epsilon_{ijk} A_j' \frac{r_k r_i}{r} + \epsilon_{ijk} A_j \delta_{ik} \\
&= \hat{\mathbf{r}} \cdot \mathbf{A}'(r) \times \mathbf{r} + \epsilon_{iji} A_j(r) \\
&= \mathbf{A}'(r) \cdot \mathbf{r} \times \mathbf{r} + 0 = 0.
\end{aligned}$$

(c) By definition, $d\mathbf{A} = \dfrac{d\mathbf{A}}{dx} dx + \dfrac{d\mathbf{A}}{dy} dy + \dfrac{d\mathbf{A}}{dz}$. Therefore, since $d\mathbf{s} = \hat{\mathbf{x}} dx + \mathbf{y} dy + \mathbf{z} dz$,

$$d\mathbf{A} = dx \frac{d\mathbf{A}}{dx} + dy \frac{d\mathbf{A}}{dy} + dz \frac{d\mathbf{A}}{dz} = \left[x \frac{d}{dx} + dy \frac{d}{dy} + dz \frac{d}{dz} \right] \mathbf{A} = (d\mathbf{s} \cdot \nabla)\mathbf{A}.$$

1.10 Derivatives of $\exp(i\mathbf{k} \cdot \mathbf{r})$

As a preliminary, let $\psi(\mathbf{r}) = \exp(i\mathbf{k} \cdot \mathbf{r})$ and consider the derivative

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x} \left[e^{ikx} e^{iky} e^{iky} \right] = i k_x \psi.$$

The y and z derivatives are similar. We conclude from this that

$$\nabla\psi = i\mathbf{k}\psi.$$

Therefore, because $\mathbf{c}$ is a constant vector,

$$\nabla\cdot\mathbf{A} = \psi(\nabla\cdot\mathbf{c}) + \mathbf{c}\cdot\nabla\psi = i\mathbf{k}\cdot\mathbf{A}$$

$$\nabla\times\mathbf{A} = \psi(\nabla\times\mathbf{c}) - \mathbf{c}\times\nabla\psi = -i(\mathbf{c}\times\mathbf{k})\psi = i\mathbf{k}\times\mathbf{A}$$

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla\times(i\mathbf{k}\times\mathbf{A}) = -i(\mathbf{k}\cdot\nabla)\mathbf{A} + i\mathbf{k}(\nabla\cdot\mathbf{A}) = k^2\mathbf{A} - \mathbf{k}(\mathbf{k}\cdot\mathbf{A}) = -\mathbf{k}\times(\mathbf{k}\times\mathbf{A})$$

$$\nabla(\nabla\cdot\mathbf{A}) = i\nabla(\mathbf{k}\cdot\mathbf{A}) = i\left[\mathbf{k}\times(\nabla\times\mathbf{A}) + (\mathbf{k}\cdot\nabla)\mathbf{A}\right] = i\left[\mathbf{k}\times(i\mathbf{k}\times\mathbf{A}) + ik^2\mathbf{A}\right] = -\mathbf{k}(\mathbf{k}\cdot\mathbf{A})$$

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times(\nabla\times\mathbf{A}) = -\mathbf{k}(\mathbf{k}\cdot\mathbf{A}) + \mathbf{k}\times(\mathbf{k}\times\mathbf{A}) = -k^2\mathbf{A}.$$

1.11 Some Integral Identities

(a) By direction substitution,

$$\int d^3r\,\mathbf{F}\cdot\mathbf{G} = \int d^3r\,\nabla\varphi\cdot\mathbf{G} = \int d^3r\,\left[\nabla\cdot(\varphi\mathbf{G}) - \varphi\nabla\cdot\mathbf{G}\right] = \int d\mathbf{S}\cdot\mathbf{G}\varphi = 0.$$

The last integral is zero with the stated conditions at infinity.

(b) Following the example of part (a),

$$\int d^3r\,\mathbf{F}\times\mathbf{G} = \int d^3r\,\nabla\varphi\times\mathbf{G} = \int d^3r\,\left[\nabla\times(\mathbf{G}\varphi) - \varphi\nabla\times\mathbf{G}\right] = \int d\mathbf{S}\times\mathbf{G}\varphi = 0.$$

The last integral is zero with the stated conditions at infinity.

(c) The given vector is $\partial_j(P_j\mathbf{G}) = (\nabla\cdot\mathbf{P})\mathbf{G} + (\mathbf{P}\cdot\nabla)\mathbf{G}$. Integrate the given identity over a volume V to get

$$\int_V d^3r\,\partial_j(P_j\mathbf{G}) = \int_V d^3r\,(\nabla\cdot\mathbf{P})\mathbf{G} + \int_V d^3r\,(\mathbf{P}\cdot\nabla)\mathbf{G}.$$

Therefore,

$$\int_S dS(\hat{\mathbf{n}}\cdot\mathbf{P})\mathbf{G} = \int_V d^3r\,(\nabla\cdot\mathbf{P})\mathbf{G} + \int_V d^3r\,(\mathbf{P}\cdot\nabla)\mathbf{G}.$$

The choice $\mathbf{G} = \mathbf{r}$ produces the desired identity because $(\mathbf{P}\cdot\nabla)\mathbf{r} = \mathbf{P}$.

1.12 Unit Vector Practice

From Chapter 1,

$$\hat{\mathbf{r}} = \hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta \qquad \hat{\mathbf{x}} = \hat{\mathbf{r}}\sin\theta\cos\phi + \hat{\boldsymbol{\theta}}\cos\theta\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$$

$$\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta \qquad \hat{\mathbf{y}} = \hat{\mathbf{r}}\sin\theta\sin\phi + \hat{\boldsymbol{\theta}}\cos\theta\sin\phi + \hat{\boldsymbol{\phi}}\cos\phi$$

$$\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}\sin\phi + \hat{\mathbf{y}}\cos\phi \qquad \hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta.$$

By direct calculation,

$$\frac{\partial\hat{\mathbf{r}}}{\partial\theta} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta = \hat{\boldsymbol{\theta}}$$

$$\frac{\partial\hat{\mathbf{r}}}{\partial\phi} = -\hat{\mathbf{x}}\sin\theta\sin\phi + \hat{\mathbf{y}}\sin\theta\cos\phi = -\sin\theta\hat{\boldsymbol{\phi}}.$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} = -\hat{\mathbf{x}}\sin\theta\cos\phi - \hat{\mathbf{y}}\sin\theta\sin\phi - \hat{\mathbf{z}}\cos\theta = -\hat{\mathbf{r}}$$

$$\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\phi} = -\hat{\mathbf{x}}\cos\theta\sin\phi + \hat{\mathbf{y}}\cos\theta\cos\phi = \cos\theta\hat{\boldsymbol{\phi}}$$

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\theta} = 0$$

$$\frac{\partial\hat{\boldsymbol{\phi}}}{\partial\phi} = -\hat{\mathbf{x}}\cos\phi - \hat{\mathbf{y}}\sin\phi = -\sin\theta\hat{\mathbf{r}} - \cos\theta\hat{\boldsymbol{\theta}}.$$

1.13 Compute the Normal Vector

By definition,

$$\hat{\mathbf{n}} = \frac{\nabla\Phi}{|\nabla\Phi|} = \frac{(x/a^2)\hat{\mathbf{x}} + (y/b^2)\hat{\mathbf{y}} + (z/c^2)\hat{\mathbf{z}}}{\sqrt{(x^2/a^4) + (y^2/b^4) + (z^2/c^4)}}.$$

When $a = b = c$, the foregoing reduces to

$$\hat{\mathbf{n}} = \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{\sqrt{x^2 + y^2 + z^2}} = \frac{\mathbf{r}}{r} = \hat{\mathbf{r}}.$$

1.14 A Variant of the Helmholtz Theorem I

Following our proof of the Helmholtz theorem,

$$\varphi(\mathbf{r}) = \int_V d^3r'\,\varphi(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') = -\frac{1}{4\pi}\int_V d^3r'\,\varphi(\mathbf{r}')\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \nabla\cdot\frac{1}{4\pi}\int_V d^3r'\,\varphi(\mathbf{r}')\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

Using an elementary vector identity gives

$$\varphi(\mathbf{r}) = \nabla \cdot \frac{1}{4\pi} \int_V d^3r' \left\{ \nabla' \frac{\varphi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \frac{\nabla'\varphi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right\}.$$

On the other hand, for any scalar function ψ,

$$\int_V d^3r \, \nabla\psi = \int_S d\mathbf{S}\, \psi.$$

Using this to transform the first term above gives the desired result,

$$\varphi(\mathbf{r}) = -\nabla \cdot \frac{1}{4\pi} \int_V d^3r' \, \frac{\nabla'\varphi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \nabla \cdot \frac{1}{4\pi} \int_S d\mathbf{S}' \frac{\varphi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

Source: D.A. Woodside, *Journal of Mathematical Physics* **40**, 4911 (1999).

1.15 A Variant of the Helmholtz Theorem II

From the textbook discussion of the Helmholtz theorem,

$$\begin{aligned} \mathbf{Z}(\mathbf{r}) \quad = \quad & -\frac{1}{4\pi}\nabla \int_V d^3r' \, \frac{\nabla' \cdot \mathbf{Z}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi}\nabla \times \int_V d^3r' \, \frac{\nabla' \times \mathbf{Z}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\ & + \frac{1}{4\pi}\nabla \int_V d^3r' \, \nabla' \cdot \left[\frac{\mathbf{Z}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] - \frac{1}{4\pi}\nabla \times \int_V d^3r' \, \nabla' \times \left[\frac{\mathbf{Z}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right]. \end{aligned}$$

The first two integrals are zero because $\nabla \cdot \mathbf{Z} = 0$ and $\nabla \times \mathbf{Z} = 0$ in V. The divergence theorem transforms the third term into an integral over S. Chapter 1 of the text states a corollary of the divergence theorem that similarly transforms the fourth term into a surface integral. The final result is

$$\mathbf{Z}(\mathbf{r}) = \frac{1}{4\pi}\nabla \int_S dS' \, \frac{\hat{\mathbf{n}}(\mathbf{r}') \cdot \mathbf{Z}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \frac{1}{4\pi}\nabla \times \int_S dS' \, \frac{\hat{\mathbf{n}}(\mathbf{r}') \times \mathbf{Z}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

Knowledge of $\mathbf{Z}$ at every point of the surface S permits us to compute the required factors $\hat{\mathbf{n}}(\mathbf{r}') \cdot \mathbf{Z}(\mathbf{r}')$ and $\hat{\mathbf{n}}(\mathbf{r}') \times \mathbf{Z}(\mathbf{r}')$.

1.16 Densities of States

This problem exploits the delta function identity

$$\delta[g(x)] = \sum_n \frac{1}{|g'(x_n)|}\delta(x - x_n), \qquad \text{where} \qquad g(x_n) = 0, \quad g'(x_n) \neq 0.$$

(a) Here, $g(k_x) = E - k_x^2 = 0$ when $k_x = \pm\sqrt{E}$. Moreover, $g'(k_x) = -2k_x$. Therefore,

$$g_1(E) = \int_{-\infty}^{\infty} dk_x \, \frac{1}{2\sqrt{E}} \left[\delta(k_x - \sqrt{E}) + \delta(k_x + \sqrt{E})\right] = \frac{1}{\sqrt{E}}.$$

(b) It is simplest to switch to polar coordinates in two dimensions, so

$$g_2(E) = \int d^2k \, \delta(E - k^2) = \int_0^{2\pi} d\phi \int_0^{\infty} dk \, k\delta(E - k^2) = 2\pi \int_0^{\infty} dk \, \frac{k}{2\sqrt{E}} \, \delta(k - \sqrt{E}) = \pi.$$

(c) It is simplest to switch to spherical coordinates in three dimensions and write

$$g_3(E) = \int d^3k \, \delta(E - k^2) = 4\pi \int_0^{\infty} dk \, k^2 \delta(E - k^2) = \frac{4\pi}{2\sqrt{E}} \int_0^{\infty} dk \, k^2 \delta(k - \sqrt{E}) = 2\pi\sqrt{E}.$$

1.17 Dot and Cross Products

(a)

$$\begin{aligned} b_i &= b_j \hat{n}_j \hat{n}_i + \epsilon_{ijk} \hat{n}_j \epsilon_{k\ell m} b_\ell \hat{n}_m \\ &= b_j \hat{n}_j \hat{n}_i + \epsilon_{kij} \epsilon_{k\ell m} \hat{n}_j b_\ell \hat{n}_m \\ &= b_j \hat{n}_j \hat{n}_i + (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) \hat{n}_j b_\ell \hat{n}_m \\ &= b_j \hat{n}_j \hat{n}_i + \hat{n}_j b_i \hat{n}_j - \hat{n}_j b_j \hat{n}_i \\ &= b_i \hat{n}_j \hat{n}_j = b_i. \end{aligned}$$

(b) The given formula is $\mathbf{b} = \mathbf{b}_\parallel + \mathbf{b}_\perp$ where $\mathbf{b}_\parallel$ is a vector parallel to $\hat{\mathbf{n}}$ and $\mathbf{b}_\perp$ is a vector perpendicular to $\hat{\mathbf{n}}$.

(c)

$$\begin{aligned} (\mathbf{B} \times \mathbf{C})_i &= \epsilon_{ijk} B_j C_k / \omega^2 \\ &= \epsilon_{ijk} \epsilon_{j\ell m} \epsilon_{kst} c_\ell a_m a_s b_t / \omega^2 \\ &= (\delta_{kl}\delta_{im} - \delta_{km}\delta_{i\ell}) \epsilon_{kst} c_\ell a_m a_s b_t / \omega^2 \\ &= \epsilon_{kst} [c_k a_i - c_i a_k] a_s b_t / \omega^2. \end{aligned}$$

Therefore,

$$\omega^2 (\mathbf{B} \times \mathbf{C}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})\mathbf{a} - \mathbf{a} \cdot (\mathbf{a} \times \mathbf{c})\mathbf{c} = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})\mathbf{a}.$$

Hence,

$$\Omega = \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \frac{\mathbf{a} \cdot (\mathbf{b} \times \mathbf{b})\mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})}{\omega^3} = \frac{|\mathbf{a} \cdot (\mathbf{c} \times \mathbf{c})|^2}{\omega^3} = \frac{\omega^2}{\omega^3} = \frac{1}{\omega}.$$

1.18 S_{ij} and T_{ij}

(a)

$$\epsilon_{ijk}S_{ij} = \frac{1}{2}\epsilon_{ijk}S_{ij} + \frac{1}{2}\epsilon_{ijk}S_{ij}.$$

Relabel the dummy indices in the second term to get

$$\epsilon_{ijk}S_{ij} = \frac{1}{2}\epsilon_{ijk}S_{ij} + \frac{1}{2}\epsilon_{jik}S_{ji} = \frac{1}{2}\epsilon_{ijk}S_{ij} - \frac{1}{2}\epsilon_{ijk}S_{ji} = \frac{1}{2}\epsilon_{ijk}(S_{ij} - S_{ji}).$$

This will be zero if $S_{ij} = S_{ji}$.

(b) We have $y_i = b_k T_{ki} = \epsilon_{iks} b_k \omega_s$. Therefore, $T_{ki} = \epsilon_{iks}\omega_s$. Notice that this representation requires that $T_{ik} = -T_{ki}$. Now, multiply by ϵ_{ipq} and sum over i:

$$\begin{aligned} \epsilon_{ipq}T_{ki} &= \epsilon_{ipq}\epsilon_{iks}\omega_s \\ &= \omega_s(\delta_{pk}\delta_{qs} - \delta_{ps}\delta_{qk}) \\ &= \omega_q\delta_{pk} - \omega_p\delta_{qk}. \end{aligned}$$

This is true for all values of p, q, and k. Choose $p = k$ and sum over k:

$$\epsilon_{ikq}T_{ki} = \omega_q\delta_{kk} - \omega_p\delta_{qk} = 3\omega_q - \omega_q = 2\omega_q.$$

Therefore,

$$\omega_q = \frac{1}{2}\epsilon_{ikq}T_{ki}.$$

This is not an unreasonable result because $T_{ij} = -T_{ji}$ implies that $\mathbf{T}$ has only three independent components, just like $\boldsymbol{\omega}$:

$$\mathbf{T} = \begin{vmatrix} 0 & \omega_1 & \omega_2 \\ -\omega_1 & 0 & \omega_3 \\ -\omega_2 & -\omega_3 & 0 \end{vmatrix}.$$

1.19 Two Surface Integrals

(a) A corollary of the divergence theorem is $\int_S d\mathbf{S}\psi = \int_V d^3r\,\nabla\psi$. Put $\psi = const.$ to get the desired result.

(b) By the divergence theorem,

$$\int_S d\mathbf{S}\cdot\mathbf{r} = \int_V d^3r\,\nabla\cdot\mathbf{r} = 3\int_V d^3r = 3V.$$

1.20 Electrostatic Dot and Cross Products

Begin with

$$\varphi = \epsilon_{ijk}a_j r_k \epsilon_{ipt} b_p r_t = (\delta_{jp}\delta_{kt} - \delta_{jt}\delta_{pk})a_j r_k b_p r_t = (\mathbf{a}\cdot\mathbf{b}r^2 - (\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r}).$$

Therefore,

$$E_i = -\partial_i[r_s r_s(\mathbf{a}\cdot\mathbf{b}) - a_m b_p r_m r_p] = -2r_i(\mathbf{a}\cdot\mathbf{b}) + a_m b_p(\delta_{im}r_p + r_m\delta_{ip}),$$

or

$$\mathbf{E} = -2\mathbf{r}(\mathbf{a}\cdot\mathbf{b}) + \mathbf{a}(\mathbf{b}\cdot\mathbf{r}) + \mathbf{b}(\mathbf{a}\cdot\mathbf{r}).$$

Now, $\nabla\cdot\mathbf{r} = 3$ and

$$\nabla\cdot[\mathbf{a}(\mathbf{b}\cdot\mathbf{r})] = \partial_k([a_k b_i r_i]) = a_k b_i \delta_{ik} = \mathbf{a}\cdot\mathbf{b}.$$

Therefore,

$$\rho = \epsilon_0\nabla\cdot\mathbf{E} = \epsilon_0[-6(\mathbf{a}\cdot\mathbf{b}) + 2(\mathbf{a}\cdot\mathbf{b}) = -4\epsilon_0(\mathbf{a}\cdot\mathbf{b}).$$

1.21 A Decomposition Identity

$$\begin{aligned}
\frac{1}{2}\epsilon_{ijk}(\mathbf{A}\times\mathbf{B})_k &= \frac{1}{2}\epsilon_{ijk}\epsilon_{k\ell m}A_\ell B_m \\
&= \frac{1}{2}\epsilon_{kij}\epsilon_{k\ell m}A_\ell B_m \\
&= \frac{1}{2}(\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell})A_\ell B_m \\
&= \frac{1}{2}(A_i B_j - A_j B_i).
\end{aligned}$$

Therefore,

$$\frac{1}{2}\epsilon_{ijk}(\mathbf{A}\times\mathbf{B})_k + \frac{1}{2}(A_i B_j + A_j B_i) = A_i B_j.$$

Chapter 2: The Maxwell Equations

2.1 Measuring B

The Lorentz force on the particle moving with velocity $\boldsymbol{v}_1$ is

$$\mathbf{F}_1 = q\boldsymbol{v}_1 \times \mathbf{B}.$$

Taking the cross product with $\boldsymbol{v}_1$ gives

$$\boldsymbol{v}_1 \times \mathbf{F}_1 = q\boldsymbol{v}_1 \times (\boldsymbol{v}_1 \times \mathbf{B}) = q\left[\boldsymbol{v}_1(\boldsymbol{v}_1 \cdot \mathbf{B}) - \mathbf{B}v_1^2\right].$$

Therefore,

$$\mathbf{B} = -\frac{\boldsymbol{v}_1 \times \mathbf{F}_1}{qv_1^2} + \frac{(\boldsymbol{v}_1 \cdot \mathbf{B})\boldsymbol{v}_1}{v_1^2}. \tag{1}$$

Similarly,

$$\mathbf{B} = -\frac{\boldsymbol{v}_2 \times \mathbf{F}_2}{qv_2^2} + \frac{(\boldsymbol{v}_2 \cdot \mathbf{B})\boldsymbol{v}_2}{v_2^2}.$$

The dot product of $\boldsymbol{v}_1$ with the preceding equation is

$$\boldsymbol{v}_1 \cdot \mathbf{B} = -\frac{\boldsymbol{v}_1 \cdot (\boldsymbol{v}_2 \times \mathbf{F}_2)}{qv_2^2} + \frac{(\boldsymbol{v}_2 \cdot \mathbf{B})(\boldsymbol{v}_2 \cdot \boldsymbol{v}_1)}{v_2^2}.$$

The last term above vanishes if $\boldsymbol{v}_1 \perp \boldsymbol{v}_2$ and the result can be substituted into (1) to get an explicit formula for $\mathbf{B}$:

$$\mathbf{B} = -\frac{\boldsymbol{v}_1 \times \mathbf{F}_1}{qv_1^2} - \frac{\boldsymbol{v}_1 \cdot (\boldsymbol{v}_2 \times \mathbf{F}_2)}{qv_1^2 v_1^2}\boldsymbol{v}_1.$$

Source: J.R. Reitz and F.J. Milford, *Foundations of Electromagnetic Theory* (Addison-Wesley, Reading, MA, 1960).

2.2 The Coulomb and Biot-Savart Laws

(a) Use $\nabla(1/r) = -\mathbf{r}/r^3$ to write $\mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\int d^3r'\,\rho(\mathbf{r}')\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}$. Then,

$$\nabla \times \mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\int d^3r'\,\rho(\mathbf{r}')\nabla \times \nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 0.$$

Similarly, because $\nabla^2(1/r) = -4\pi\delta(\mathbf{r})$,

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\int d^3r'\,\rho(\mathbf{r}')\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{\epsilon_0}\int d^3r'\,\rho(\mathbf{r}'\delta(\mathbf{r}-\mathbf{r}') = \rho(\mathbf{r})/\epsilon_0.$$

(b) Here we write

$$\mathbf{B}(\mathbf{r}) = -\frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{j}(\mathbf{r}')\times\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0}{4\pi}\int d^3r'\,\nabla\times\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0}{4\pi}\nabla\times\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

This gives $\nabla\cdot\mathbf{B}(\mathbf{r}) = 0$ because $\nabla\cdot\nabla\times\mathbf{f} = 0$ for any $\mathbf{f}$. To compute the curl of $\mathbf{B}$, let

$$\mathbf{g}(\mathbf{r}-\mathbf{r}') = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{\mathbf{R}}{R^3},$$

so

$$\begin{aligned}\nabla\times\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int d^3r'\nabla\times[\mathbf{j}(\mathbf{r}')\times\mathbf{g}(\mathbf{r}-\mathbf{r}')] \\ &= -\frac{\mu_0}{4\pi}\int d^3r'\,[\mathbf{j}(\mathbf{r}')\cdot\nabla]\,\mathbf{g} + \frac{\mu_0}{4\pi}\int d^3r'\mathbf{j}(\mathbf{r}')\nabla\cdot\mathbf{g}. \qquad (1)\end{aligned}$$

Focus on the first integral. We know that $[\mathbf{j}(\mathbf{r}')\cdot\nabla]\,\mathbf{g}(\mathbf{r}-\mathbf{r}') = -\,[\mathbf{j}(\mathbf{r}')\cdot\nabla']\,\mathbf{g}(\mathbf{r}-\mathbf{r}')$. Therefore,

$$[\mathbf{j}(\mathbf{r}')\cdot\nabla']\,g_x(\mathbf{R}) = \nabla'\cdot[g_x(\mathbf{R})\mathbf{j}(\mathbf{r}')] - g_x(\mathbf{R})\nabla'\cdot\mathbf{j}(\mathbf{r}'). \qquad (2)$$

The charge and current density are time-independent so the continuity equation reads

$$\nabla\cdot\mathbf{j} = -\frac{\partial\rho}{\partial t} = 0.$$

Accordingly, the second term on the right-hand side of (2) vanishes. Therefore, using the divergence theorem, the x-component of the first integral in (1) is

$$-\frac{\mu_0}{4\pi}\int d^3r'\nabla'\cdot[g_x(\mathbf{R})\mathbf{j}(\mathbf{r}')] = -\frac{\mu_0}{4\pi}\int d\mathbf{S}'\cdot\mathbf{j}(\mathbf{r}')g_x(\mathbf{R}) = 0.$$

The integral is zero because $\mathbf{j}$ vanishes on the surface at infinity. The y- and z-components are zero similarly. Therefore, (1) becomes

$$\nabla\times\mathbf{B} = \frac{\mu_0}{4\pi}\int d^3r'\mathbf{j}(\mathbf{r}')\,[\nabla\cdot\mathbf{g}]\,.$$

But

$$\nabla\cdot\mathbf{g} = \nabla\cdot\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} = -\nabla\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} = -\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 4\pi\delta(\mathbf{r}-\mathbf{r}').$$

Therefore,

$$\nabla\times\mathbf{B}(\mathbf{r}) = \mu_0\int d^3r'\,\mathbf{j}(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}') = \mu_0\mathbf{j}(\mathbf{r}).$$

2.3 The Force between Current Loops

(a) $\oint_{C_1} d\mathbf{s}_1 \cdot \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} = -\oint_{C_2} d\mathbf{s}_1 \cdot \nabla_1 \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \oint_{C_1} ds_1 \frac{\partial}{\partial s_1} \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = \oint_{C_1} \partial \frac{1}{|\mathbf{r}_1 - \mathbf{r}_2|} = 0.$

(b) We use the identity $d\mathbf{s}_1 \times \left(d\mathbf{s}_2 \times \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \right) = d\mathbf{s}_2 \left(d\mathbf{s}_1 \cdot \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \right) - (d\mathbf{s}_1 \cdot d\mathbf{s}_2) \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}.$

Substituting this equation into the given expression for $\mathbf{F}_1$ generates two terms. One of them is zero by part (a). What remains is

$$\mathbf{F}_1 = \frac{\mu_0}{4\pi} \oint_{C_1} I_1 d\mathbf{s}_1 \times \oint_{C_2} I_2 d\mathbf{s}_2 \times \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}.$$

This is the desired formula because the magnetic field at point $\mathbf{r}_1$ produced by a current loop which carries a current I_2 is

$$\mathbf{B}_2(\mathbf{r}_1) = \frac{\mu_0}{4\pi} \oint_{C_2} I_2 d\mathbf{s}_2 \times \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3}.$$

2.4 Necessity of Displacement Current

The divergence of the suggested equation is

$$\nabla \cdot \nabla \times \mathbf{B} = \mu_0 \nabla \cdot \mathbf{j} + \nabla \cdot \mathbf{j}_{\rm D}.$$

The left side is identically zero so, using the continuity equation and $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$,

$$\nabla \cdot \mathbf{j}_{\rm D} = -\mu_0 \nabla \cdot \mathbf{j} = \mu_0 \frac{\partial \rho}{\partial t} = \mu_0 \epsilon_0 \frac{\partial}{\partial t} \nabla \cdot \mathbf{E} = \nabla \cdot \left[\mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right].$$

Since $\mu_0 \epsilon_0 = c^{-2}$, this equation is satisfied by the standard form of the displacement current,

$$\mathbf{j}_{\rm D} = \frac{1}{c^2} \frac{\partial \mathbf{E}}{\partial t}.$$

2.5 Prelude to Electromagnetic Angular Momentum

The time-changing magnetic field induces an electric field in accordance with the integral form of Faraday's law:

$$\oint_C d\mathbf{s} \cdot \mathbf{E} = -\frac{d}{dt} \int_S d\mathbf{S} \cdot \mathbf{B}.$$

By symmetry, the electric field is azimuthal. Specifically, if we choose C to be a circle of radius r coaxial with the z-axis,

$$\mathbf{E} = -\frac{1}{2\pi r}\frac{d\Phi}{dt}\hat{\boldsymbol{\phi}}.$$

The force $q\mathbf{E}$ on the particle produces a torque around the z-axis so the mechanical angular momentum of the particle is

$$\frac{d\mathbf{L}}{dt} = \mathbf{r} \times \mathbf{F} = -\frac{q}{2\pi}\frac{d\Phi}{dt}\hat{\mathbf{z}}.$$

Therefore, as suggested,

$$\frac{d}{dt}\left(\mathbf{L} + \frac{q\mathbf{\Phi}}{2\pi}\right) = 0.$$

2.6 Time-Dependent Charges at Rest

(a) The charge density is

$$\rho(\mathbf{r}, t) = \sum_k q_k(t)\delta(\mathbf{r} - \mathbf{r}_k).$$

We find the current density using the continuity equation, $\nabla \cdot \mathbf{j} + \dfrac{\partial \rho}{\partial t} = 0$. Specifically,

$$\frac{\partial \rho}{\partial t} = \sum_k \dot{q}_k(t)\delta(\mathbf{r} - \mathbf{r}_k) = -\nabla \cdot \mathbf{j}.$$

Since $\nabla \cdot \nabla|\mathbf{r} - \mathbf{r}'|^{-1} = -4\pi\delta(\mathbf{r} - \mathbf{r}')$, a current density which does the job is

$$\mathbf{j}(\mathbf{r}, t) = -\frac{1}{4\pi}\sum_k \dot{q}_k \frac{\mathbf{r} - \mathbf{r}_k}{|\mathbf{r} - \mathbf{r}_k|^3}.$$

(b) We begin with Gauss' law:

$$\nabla \cdot \mathbf{E} = \frac{1}{4\pi\epsilon_0}\sum_k q_k(t)\nabla \cdot \frac{\mathbf{r} - \mathbf{r}_k}{|\mathbf{r} - \mathbf{r}'|^3} = -\frac{1}{4\pi\epsilon_0}\sum_k q_k(t)\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\epsilon_0}\sum_k q_k(t)\delta(\mathbf{r} - \mathbf{r}_k) = \rho(\mathbf{r})/\epsilon_0.$$

The curl of the electric field is

$$\nabla \times \mathbf{E} = -\frac{1}{4\pi\epsilon_0}\sum_k q_k(t)\nabla \times \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} = 0.$$

Faraday's law is $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$. This will be satisfied if $\mathbf{B}(\mathbf{r}, t) = \mathbf{B}(\mathbf{r})$ is a time-independent vector field. Using this and $c^2 = 1/\mu_0\epsilon_0$, the Ampère-Maxwell law looks like

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = -\frac{\mu_0}{4\pi}\sum_k \dot{q}_k \frac{\mathbf{r} - \mathbf{r}_k}{|\mathbf{r} - \mathbf{r}_k|^3} + \frac{1}{4\pi\epsilon_0 c^2}\sum_k \dot{q}_k(t)\frac{\mathbf{r} - \mathbf{r}_k}{|\mathbf{r} - \mathbf{r}'|^3} = 0.$$

We satisfy the equation above and also $\nabla\cdot\mathbf{B}=0$ if $\mathbf{B}$ is a constant vector everywhere in space. Given the initial conditions, we conclude that

$$\mathbf{B}(\mathbf{r},t)\equiv 0.$$

(c) The current density $\mathbf{j}(\mathbf{r},t)$ in (b) shows that the changes in $q_k(t)$ at each point $\mathbf{r}_k$ occur because a radial stream of charge flows in and out of each point to and from infinity as needed.

2.7 Rotation of Free Fields in Vacuum

(a) By assumption,

$$\nabla\cdot\mathbf{E}=0 \qquad \nabla\cdot\mathbf{B}=0$$
$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t} \qquad \nabla\times\mathbf{B}=\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}.$$

Therefore,

$$\nabla\cdot\mathbf{E}'=(\nabla\cdot\mathbf{E})\cos\theta+c(\nabla\cdot\mathbf{B})\sin\theta=0$$
$$c\nabla\cdot\mathbf{B}'=-(\nabla\cdot\mathbf{E})\sin\theta+c(\nabla\cdot\mathbf{B})\cos\theta=0$$

$$\begin{aligned}\nabla\times\mathbf{E}' &= (\nabla\times\mathbf{E})\cos\theta+c(\nabla\times\mathbf{B})\sin\theta=-\frac{\partial\mathbf{B}}{\partial t}\cos\theta+c\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\sin\theta\\ &= -\frac{\partial}{\partial t}\left(\mathbf{B}\cos\theta-\frac{1}{c}\mathbf{E}\sin\theta\right)=-\frac{\partial\mathbf{B}'}{\partial t}\end{aligned}$$

$$\begin{aligned}c\nabla\times\mathbf{B}' &= -(\nabla\times\mathbf{E})\sin\theta+c(\nabla\times\mathbf{B})\cos\theta=\frac{\partial\mathbf{B}}{\partial t}\sin\theta+c\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\cos\theta\\ &= c\frac{1}{c^2}\frac{\partial}{\partial t}\left(c\mathbf{B}\sin\theta+\mathbf{E}\cos\theta\right)=c\frac{1}{c^2}\frac{\partial\mathbf{E}'}{\partial t}.\end{aligned}$$

(b) If $\mathbf{E}\perp\mathbf{B}$, the stated transformation simply rotates the two vectors by an angle θ while retaining their perpendicularity. Therefore, $\mathbf{E}'$ and $\mathbf{B}'$ also describe a plane wave in vacuum.

2.8 A Current Density Which Varies Linearly in Time

Let z be the symmetry axis of the solenoid. The simplest guess for the exterior magnetic field is that the magnetostatic field does not change; namely,

$$\mathbf{B}(\rho>b,t)=0.$$

Now consider the integral form of the Ampère-Maxwell law:

$$\int_C d\boldsymbol{\ell}\cdot\mathbf{B} = \mu_0 I_{\rm enc} + \frac{1}{c^2}\frac{d}{dt}\int_S d\mathbf{S}\cdot\mathbf{E}.$$

Symmetry suggests that $\mathbf{B}(\mathbf{r},t) = B(\rho,t)\hat{z}$ and $\mathbf{E}(\mathbf{r},t) = E(\rho,t)\hat{\boldsymbol{\phi}}$. Therefore, we choose a rectangular Ampèrian circuit C of the same kind used to solve the magnetostatic problem. One leg of length ℓ points along $-\hat{\mathbf{z}}$ and lies outside the solenoid. The other leg points along $+\hat{\mathbf{z}}$, has length ℓ, and lies at a radius $\rho < b$. The other two legs are aligned with $\hat{\boldsymbol{\rho}}$. In that case, the foregoing gives

$$B_z\ell = \mu_0 K_0(t/\tau)\ell + \frac{1}{c^2}\frac{d}{dt}\int_S d\mathbf{S}\cdot\mathbf{E}.$$

If we guess that the electric field does not depend on time, a solution of this equation is

$$\mathbf{B}(\rho < b, t) = \mu_0 K_0(t/\tau)\hat{\mathbf{z}}.$$

This magnetic field satisfies $\nabla\cdot\mathbf{B} = 0$ everywhere because the field lines end at infinity only. We turn next to Faraday's law and choose a circular loop C' lying in the x-y plane with radius ρ:

$$\int_{C'} d\boldsymbol{\ell}\cdot\mathbf{E} = -\frac{d}{dt}\int_{S'} d\mathbf{S}\cdot\mathbf{B}.$$

Evaluating this for $\rho < b$ and $\rho > b$ gives

$$\mathbf{E}(\rho,t) = \begin{cases} \dfrac{1}{2\tau}\mu_0 K_0 \rho\hat{\boldsymbol{\phi}} & \rho < b, \\[2ex] \dfrac{1}{2\tau}\mu_0 K_0 \dfrac{b^2}{\rho}\hat{\boldsymbol{\phi}} & \rho > b. \end{cases}$$

This electric field satisfies $\nabla\cdot\mathbf{E} = 0$ everywhere because the electric field lines close on themselves.

2.9 A Charge Density Which Varies Linearly in Time

There is no conflict because the origin of coordinates can be placed anywhere we please.

2.10 Coulomb Repulsion in One Dimension

Newton's equation of motion for the released particle is

$$m\ddot{x} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{x^2}.$$

If we let $A = q^2/4\pi\epsilon_0 m$, the equation of motion is

$$\ddot{x} = \frac{A}{x^2}.$$

To integrate this, we multiply by $\dot{x}$ to get

$$\dot{x}\ddot{x} = \frac{A}{x^2}\dot{x} \qquad \text{or} \qquad \frac{d}{dt}\left(\frac{1}{2}\dot{x}^2\right) = -\frac{d}{dt}\left(\frac{A}{x}\right).$$

Using the initial conditions, this integrates to

$$\frac{1}{2}\dot{x}^2 = \frac{A}{d} - \frac{A}{x}.$$

Therefore, as $x \to \infty$ the speed approaches $v = \sqrt{2A/d}$.

2.11 Ampère-Maxwell Matching Conditions

(a) The fields for this problem are

$$\mathbf{B}(\mathbf{r},t) = \Theta(z)\mathbf{B}_1(\mathbf{r},t) + \Theta(-z)\mathbf{B}_2(\mathbf{r},t)$$
$$\mathbf{E}(\mathbf{r},t) = \Theta(z)\mathbf{E}_1(\mathbf{r},t) + \Theta(-z)\mathbf{E}_2(\mathbf{r},t)$$
$$\mathbf{j}(\mathbf{r},t) = \Theta(z)\mathbf{j}_1(\mathbf{r},t) + \Theta(-z)\mathbf{j}_2(\mathbf{r},t) + \mathbf{K}(\mathbf{r}_S,t)\delta(z).$$

Then,

$$\nabla\times\mathbf{B} = \Theta(z)\nabla\times\mathbf{B}_1 - \mathbf{B}_1\times\nabla\Theta(z) + \Theta(-z)\nabla\times\mathbf{B}_2 - \mathbf{B}_2\times\nabla\Theta(-z)$$

and

$$\frac{\partial\mathbf{E}}{\partial t} = \Theta(z)\frac{\partial\mathbf{E}_1(\mathbf{r},t)}{\partial t} + \Theta(-z)\frac{\partial\mathbf{E}_2(\mathbf{r},t)}{\partial t}.$$

However, $\mathbf{B}_1$, $\mathbf{E}_1$, and $\mathbf{j}_1$ satisfy the Ampère-Maxwell law, as do $\mathbf{B}_2$, $\mathbf{E}_2$, and $\mathbf{j}_2$. Therefore, the time derivative disappears when we write out this law using the previous two equations to get

$$-\mathbf{B}_1\times\nabla\Theta(z) - \mathbf{B}_2\times\nabla\Theta(-z) = \mu_0\mathbf{K}\delta(z).$$

This simplifies because $\nabla\Theta(\pm z) = \pm\hat{\mathbf{z}}\delta(z)$. Using this information gives

$$[\mathbf{B}_2 - \mathbf{B}_1]\delta(z)\times\hat{\mathbf{z}} = \mu_0\mathbf{K}\delta(z).$$

If we use the square brackets to enforce the delta function evaluation of $\mathbf{B}_1$ and $\mathbf{B}_2$ infinitesimally near to (but on opposite sides of) $z = 0$, we get the matching condition,

$$\hat{\mathbf{z}}\times[\mathbf{B}_1 - \mathbf{B}_2] = \mu_0\mathbf{K}.$$

(b) We can apply this result to an arbitrary point $\mathbf{r}_S$ on a non-flat interface because the fields involved in the matching condition are evaluated infinitesimally close to $\mathbf{r}_S$. From that distance, the interface looks flat and the result proved in part (a) is applicable. Using our usual convention that $\hat{\mathbf{n}}_2$ is the outward normal from region 2, the matching condition of part (a) generalizes to

$$\hat{\mathbf{n}}_2\times[\mathbf{B}_1 - \mathbf{B}_2] = \mu_0\mathbf{K}(\mathbf{r}_S).$$

2.12 A Variation of Gauss' Law

(a) Because $\nabla \times \mathbf{E} = 0$, we can still define a scalar potential $\mathbf{E} = -\nabla\varphi$. Substituting this into the Podolsky-Gauss equation gives

$$\epsilon_0(1 - a^2\nabla^2)\nabla^2\varphi(\mathbf{r}) = -q\delta(\mathbf{r}).$$

Integrating this over a small spherical volume of radius R centered on the origin gives

$$-q = \epsilon_0 \int d^3r\, \nabla \cdot \{\nabla\varphi - a^2\nabla\nabla^2\varphi\} = \epsilon_0 \int d\mathbf{S} \cdot \{\nabla\varphi - a^2\nabla\nabla^2\varphi\}.$$

By symmetry, we may assume that $\varphi(\mathbf{r}) = \varphi(r)$. Therefore, writing out the gradient and Laplacian operators and doing the integral over $r = R$ gives

$$-q = 4\pi\epsilon_0 R^2 \left\{ \frac{d\varphi}{dR} - a^2 \frac{d}{dR}\left[\frac{1}{R^2}\frac{d}{dR}\left(R^2 \frac{d\varphi}{dR} \right)\right]\right\}$$

or

$$\frac{q}{4\pi\epsilon_0 R} = \varphi - a^2 \frac{1}{R^2}\frac{d}{dR}\left(R^2 \frac{d\varphi}{dR}\right).$$

Using the suggested ansatz, $\varphi(r) = qu(r)/4\pi\epsilon_0 R$, simplifies this equation to

$$a^2 \frac{d^2u}{dr^2} = u - 1.$$

Since $u(r)$ cannot diverge at infinity, this integrates immediately to $u(r) = 1 + B\exp(-r/a)$. Therefore,

$$\varphi(r) = \frac{q}{4\pi\epsilon_0 r}\{1 + B\exp(-ar)\}.$$

The entire point of this exercise was to eliminate the divergence of the field at the origin. The corresponding divergence of the potential disappears only if $B = -1$. Therefore, we conclude that

$$\varphi(r) = \frac{q}{4\pi\epsilon_0 r}\{1 - \exp(-r/a)\}$$

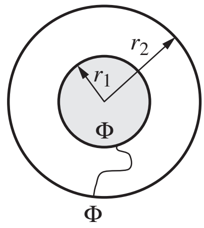
so

$$\mathbf{E}(r) = -\hat{\mathbf{r}}\frac{\partial\varphi}{\partial r} = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}\left[1 - (1 + r/a)\exp(-r/a)\right].$$

(b) The parameter a has dimensions of length so, by analogy with meson theory, we may regard it as the de Broglie wavelength of a particle which mediates the modified Coulomb interaction.

Source: B. Podolsky, *Physical Review* **62**, 68 (1942).

2.13 If the Photon Had Mass . . .



(a) The equation to be solved is $\nabla^2\varphi = \varphi/L^2$ because $\rho = 0$ between the shells. By spherical symmetry,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\varphi}{dr}\right) = \frac{\varphi}{L^2}.$$

The substitution $\varphi = u/r$ simplifies this equation to $d^2u/dr^2 = u/L^2$, which is solved by real exponentials. Therefore,

$$\varphi(r) = \frac{1}{r}\left[ae^{r/L} + be^{-r/L}\right].$$

This has the proposed form. The constants are determined by the boundary conditions $\varphi(r_1) = \varphi(2) = \Phi$. After a bit of algebra, we find

$$\varphi(r) = \Phi\left\{\frac{r_2}{r}\frac{\sinh[(r-r_1)/L]}{\sinh[(r_2-r_1)/L]} + \frac{r_1}{r}\frac{\sinh[(r_2-r)/L]}{\sinh[(r_2-r_1)/L]}\right\}.$$

If $\Delta = (r_2 - r_1)/L$, the associated electric field $\mathbf{E} = -\nabla\varphi$ is

$$\begin{aligned}\mathbf{E}(r) \quad &= \quad \frac{\Phi\,\hat{\mathbf{r}}}{\sinh\Delta}\left\{r_2\left[\frac{\sinh[(r-r_1)/L]}{r^2} - \frac{\cosh[(r-r_1)/L]}{rL}\right]\right. \\ &\quad\left. + \; r_1\left[\frac{\sinh[(r_2-r)/L]}{r^2} + \frac{\cosh[(r_2-r)/L]}{rL}\right]\right\}\end{aligned}$$

(b) The generalized Poisson equation can be written in the form

$$\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} - \frac{\varphi}{L^2}.$$

Integration over a volume V bounded by a surface S and using the divergence theorem gives the generalized Gauss' law:

$$\int_S d\mathbf{S}\cdot\mathbf{E} = \frac{Q_{\rm encl}}{\epsilon_0} - \frac{1}{L^2}\int_V d^3r\,\varphi(\mathbf{r}).$$

We will apply this to a Gaussian sphere of radius r_1. Using the field from part (a),

$$\mathbf{E}(r_1) = \Phi\left[\frac{1}{r_1} + \frac{\coth\Delta}{L} - \frac{r_2}{r_1 L}\text{csch}\Delta\right]\hat{\mathbf{r}}.$$

Moreover, the potential takes the constant value Φ in and on the inner sphere. Therefore,

$$Q = 4\pi\epsilon_0 r_1^2\Phi\left[\frac{1}{r_1} + \frac{\coth\Delta}{L} - \frac{r_2}{r_1 L}\text{csch}\Delta\right] + \frac{4\pi\epsilon_0}{3L^2}r_1^3\Phi.$$

(c) The limit $L \to \infty$ corresponds to $\Delta \to 0$ so we use

$$\text{csch}\Delta \approx \frac{1}{\Delta} - \frac{\Delta}{6} + \cdots$$

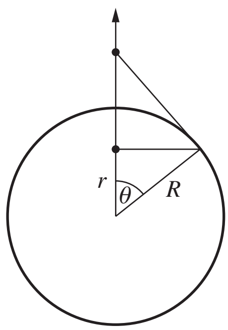
$$\coth\Delta \approx \frac{1}{\Delta} + \frac{\Delta}{3} + \cdots.$$

This gives

$$Q \approx 4\pi\epsilon_0 r_1^2\Phi\frac{(r_2 - r_1)(2r_1 + r_2)}{6L^2 r_1} + \frac{4\pi\epsilon_0}{3L^2}r_1^3\Phi = \frac{2\pi\epsilon_0}{3}\frac{r_1\Phi}{L^2}\left(\frac{r_2}{L}\right)^2\left(1 + \frac{r_1}{r_2}\right).$$

2.14 A Variation of Coulomb's Law

By symmetry, it is sufficient to find the potential at a point on the z-axis at a distance r from the center of the sphere.



The distance between the observation point and a typical point on the surface of the sphere is $\sqrt{r^2 + R^2 - 2rR\cos\theta}$. The charge contributed by a element of surface is $dQ = \sigma R^2 d\Omega$. Therefore, by superposition,

$$\varphi(r) = \frac{\sigma}{4\pi\epsilon_0}R^2\int_0^{2\pi} d\phi\int_{-1}^{1} d(\cos\theta)\frac{1}{(r^2 + R^2 - 2rR\cos\theta)^{(1+\eta)/2}}.$$

The integral we need to do is elementary:

$$\int_{-1}^{1}\frac{dx}{(a + bx)^{(1+\eta)/2}} = \frac{2}{1-\eta}\frac{1}{b}\left[\frac{1}{(a+b)^{(\eta-1)/2}} - \frac{1}{(a-b)^{(\eta-1)/2}}\right].$$

Therefore, we find immediately that

$$\varphi(r) = \frac{Q}{4\pi\epsilon_0}\frac{1}{1-\eta}\frac{1}{2Rr}\left\{|r+R|^{1-\eta} - |r-R|^{1-\eta}\right\}.$$

When $\eta \to 0$, this reduces to $Q/4\pi\epsilon_0 r$ when $r > R$ and $Q/4\pi\epsilon_0 R$ when $r < R$.

Chapter 3: Electrostatics

3.1 Charged Particle Refraction

(a) Let $\mathbf{p}_1$ and $\mathbf{p}_2$ be the linear momentum of the particle in the two regions. There is no impulsive force on the particle in the direction parallel to the interface. Therefore, the component of linear momentum along the interface is conserved:

$$p_1 \sin\theta_1 = p_2 \sin\theta_2. \tag{1}$$

Otherwise, we have conservation of energy in the form

$$E = \frac{p_1^2}{2m} + qV_1 = \frac{p_2^2}{2m} + qV_2.$$

Combining these two equations identifies (1) as "Snell's law" and the magnitude of the particle momentum as the "index of refraction" where

$$p_k = \sqrt{2m(E - qV_k)}, \qquad k = 1, 2.$$

(b) The electrostatic potential is continuous through an interface endowed with a simple charge distribution σ. A dipole layer oriented perpendicular to the interface is needed to produce a jump in potential like that envisioned for this problem.

3.2 Symmetric and Traceless

(a) The field in question must satisfy $\nabla \cdot \mathbf{E} = 0$ and $\nabla \times \mathbf{E} = 0$. The first condition gives

$$0 = \partial_k E_k = D_{jk}\partial_k r_j = D_{jk}\delta_{kj} = D_{kk} = 0.$$

The second condition gives

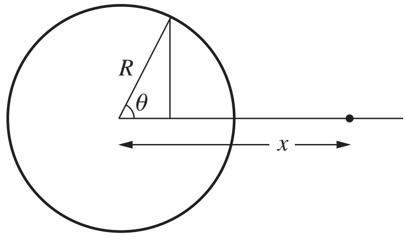
$$0 = \epsilon_{isk}\nabla_s E_k = \epsilon_{isk} D_{jk}\nabla_s r_j = \epsilon_{isk} D_{jk}\delta_{sj} = \epsilon_{ijk} D_{jk} = \epsilon_{ijk} D_{jk} + \epsilon_{kj} D_{kj} = \epsilon_{ijk}(D_{jk} - D_{kj}).$$

(b) We must have $E_k = -\partial_k \varphi$. Therefore,

$$\varphi = A - C_m r_m - \frac{1}{2} D_{sm} r_s r_m.$$

3.3 Practice Superposing Fields

(a) Let R be the radius of a shell centered at the origin with uniform charge/area $\sigma = Q/4\pi R^2$. Consider first $\mathbf{E}(x\hat{\mathbf{x}})$ when $x > R$ so every ring is a perpendicular distance $x - R\cos\theta > 0$ from the evaluation point and contributes a charge increment $dq = \sigma 2\pi R^2 \sin\theta d\theta$.



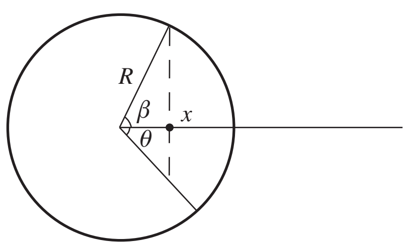
Then, by symmetry, $\mathbf{E}(x\hat{\mathbf{x}}) = E(x)\hat{\mathbf{x}}$ where

$$\begin{aligned}
E(x) &= \frac{\sigma 2\pi R^2}{4\pi\epsilon_0}\int_0^\pi d\theta \frac{(x - R\cos\theta)\sin\theta}{(R^2 + x^2 - 2Rx\cos\theta)^{3/2}} \\
&= -\sigma R^2\epsilon_0 \frac{d}{dx}\int_0^\pi d\theta \frac{\sin\theta}{\sqrt{R^2 + x^2 - 2Rx\cos\theta}} \\
&= -\sigma R^2\epsilon_0 \frac{d}{dx}\left[\frac{1}{Rx}\sqrt{R^2 + x^2 - 2Rx\cos\theta}\right]_0^\pi \\
&= -\sigma R^2\epsilon_0 \frac{d}{dx}\left[\frac{x+R}{Rx} - \frac{x-R}{Rx}\right] \\
&= \frac{Q}{4\pi\epsilon_0 x^2}.
\end{aligned}$$

The x-direction is not special. Therefore, by symmetry, we conclude that

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

Now consider points on the positive x-axis where $x < R$ and let $\cos\beta = x/R$ as indicated in the figure below.



The contribution to the field from rings that lie to the *left* of x ($\beta \leq \theta \leq \pi$) is the same as in the previous calculation. The contribution to the field from the rings that lie to the *right* of x ($0 \leq \theta \leq \beta$) point in the $-\hat{x}$ (rather than $+\hat{x}$) direction. But the distance from these rings to the evaluation point is $R\cos\theta - x$ (rather than $x - R\cos\theta$)

so the two minus signs cancel one another. The calculation is thus identical to the one above except that we now choose $R - x = \sqrt{(x-R)^2}$ to get

$$E(x) = -\frac{\sigma R^2}{2\epsilon_0}\frac{d}{dx}\left[\frac{x+R}{Rx} - \frac{R-x}{Rx}\right] = -\frac{\sigma R^2}{2\epsilon_0}\frac{d}{dx}\left[\frac{2}{R}\right] = 0.$$

By symmetry, The field is then zero everywhere within the shell.

(b) Let R be the radius of the spherical volume centered at the origin with uniform charge per unit volume $\rho = Q/(4\pi R^3/3)$. As above, the field is $E(x)\hat{\mathbf{x}}$ if we choose an evaluation point on the x-axis. If $x > R$, each disk with surface charge density $d\sigma = \rho R \sin\theta d\theta$ lies a distance $x - R\cos\theta > 0$ from the observation point. By superposition,

$$\begin{aligned}
E(x) &= \frac{\rho R}{2\epsilon_0}\int_0^\pi d\theta \sin\theta\left[1 - \frac{x - R\cos\theta}{\sqrt{R^2 + x^2 - 2xr\cos\theta}}\right] \\
&= \frac{\rho R}{2\epsilon_0}\left[2 - \frac{d}{dx}\int_0^\pi d\theta \sin\theta\sqrt{R^2 + x^2 - 2xr\cos\theta}\right] \\
&= \frac{\rho R}{2\epsilon_0}\left[2 - \frac{d}{dx}\left\{\frac{1}{3xR}(R^2 + x^2 - 2xr\cos\theta)^{3/2}\right\}_0^\pi\right] \\
&= \frac{\rho R}{2\epsilon_0}\left[2 - \frac{d}{dx}\left\{\frac{(x+R)^3 - (x-R)^3}{3xR}\right\}\right] \\
&= \frac{Q}{4\pi\epsilon_0 x^2}.
\end{aligned}$$

As before, symmetry guarantees that

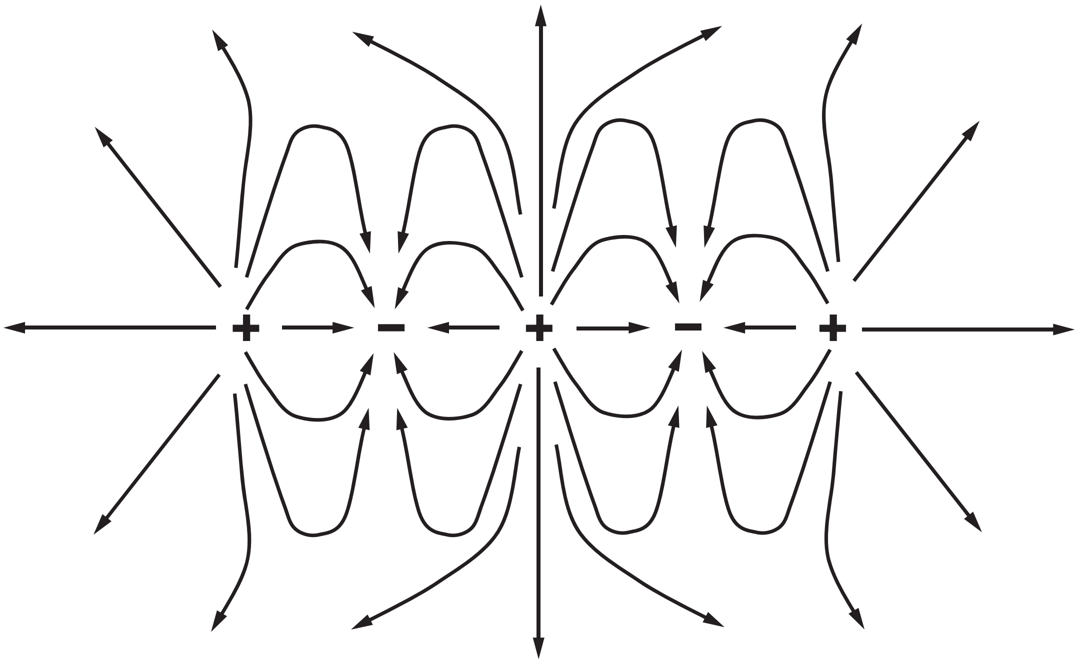
$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

When $x < R$, the contributions to the field from the disk to the left and right of the evaluation point change sign as in part (a). The distance factor $x - R\cos\theta$ in the first line of the calculation above changes sign too, so the *second* term in the integral above stays the same save for writing $(R-x)^3 = \left[(x-R)^2\right]^{3/2}$. However, the *first* integral above (from the "1" in the original square brackets) must be performed explicitly with a change of sign for contributions from angles less than or greater than $\beta = \cos^{-1}(x/R)$. Specifically,

$$\begin{aligned}
E(x) &= \frac{\rho R}{2\epsilon_0}\left[\int_\beta^\pi d\theta\sin\theta - \int_0^\beta d\theta \sin\theta - \frac{d}{dx}\left\{\frac{(x+R)^3 - (x-R)^3}{3xR}\right\}\right] \\
&= \frac{\rho R}{2\epsilon_0}\left[\frac{2x}{R} - \frac{4}{3}\frac{x}{R}\right] \\
&= \frac{\rho R}{3\epsilon_0}x.
\end{aligned}$$

By symmetry, we conclude that $\mathbf{E}(\mathbf{r}) = \dfrac{\rho R\mathbf{r}}{3\epsilon_0}$. This is the familiar Gauss' law result.

3.4 Five Charges in a Line



3.5 Gauss' Law Practice

We use Gauss' law in integral form.

(a) Write $\rho(x) = \rho_0 \exp(-\kappa|x|)$ in Cartesian coordinates. Like the charge density, the electric field must be translationally invariant along y and z. This implies that $\mathbf{E} = \hat{\mathbf{x}}E(x)$ where $E(-x) = -E(x)$. Then, for a rectangular Gaussian box which extends from $s = -x$ to $s = +x$ with an area A perpendicular to the x-axis:

$$\int_S d\mathbf{S}\cdot\mathbf{E} = 2E(x)A = \frac{2\rho_0}{\epsilon_0}A\int_0^x dse^{-\kappa s} = \frac{A\rho_0}{\kappa\epsilon_0}\left\{1 - e^{-\kappa x}\right\}. \qquad (x > 0)$$

Therefore,

$$\mathbf{E} = \hat{\mathbf{x}}\frac{\rho_0}{\kappa\epsilon_0}\mathrm{sgn}(x)\left[1 - e^{-\kappa|x|}\right].$$

(b) Write $\rho(s) = \rho_0 \exp(-\kappa s)$ in cylindrical coordinates (s, ϕ, z). By symmetry, $\mathbf{E} = \hat{\boldsymbol{\rho}}E(s)$, so we use a Gaussian cylinder of length L and radius ρ. This gives

$$\int_S d\mathbf{S}\cdot\mathbf{E} = 2\pi\rho LE = 2\pi\frac{\rho_0 L}{\epsilon_0}\int_0^\rho ds\, s\, e^{-\kappa s} = -2\pi\frac{\rho_0 L}{\epsilon_0}\frac{d}{d\kappa}\int_0^\rho ds\, e^{-\kappa s}.$$

Hence,

$$\mathbf{E} = \hat{\boldsymbol{\rho}}\frac{\rho_0}{\epsilon_0}\frac{1}{\rho}\frac{d}{d\kappa}\left[\frac{1}{\kappa}e^{-\kappa\rho} - \frac{1}{\kappa}\right] = \hat{\boldsymbol{\rho}}\frac{\rho_0}{\epsilon_0}\frac{1}{\kappa^2\rho}\left[1 - e^{-\kappa\rho} - \kappa\rho e^{-\kappa\rho}\right].$$

(c) Write $\rho(r) = \rho_0\exp(-\kappa r)$ in spherical coordinates (r,θ,ϕ). By symmetry, $\mathbf{E} = \hat{\mathbf{r}}E(r)$, so we use a Gaussian sphere of radius r. This gives

$$\int_S d\mathbf{S}\cdot\mathbf{E} = 4\pi r^2 E(r) = 4\pi\frac{\rho_0}{\epsilon_0}\int_0^r ds s^2\, e^{-\kappa s} = 4\pi\frac{\rho_0}{\epsilon_0}\frac{d^2}{d\kappa^2}\int_0^r ds\, e^{-\kappa s}.$$

Hence,

$$\mathbf{E}(r) = \hat{\mathbf{r}}\frac{\rho_0}{\epsilon_0}\frac{1}{r^2}\frac{d}{d\kappa}\left[\frac{1}{\kappa^2} - \frac{1}{\kappa^2}e^{-\kappa r} - \frac{r}{\kappa}e^{-\kappa r}\right] = \hat{\mathbf{r}}\frac{\rho_0}{\epsilon_0}\frac{2}{\kappa^3 r^2}\left\{1 - e^{-\kappa r}\left[1 + \kappa r + \tfrac{1}{2}\kappa^2 r^2\right]\right\}.$$

Source: P.C. Clemmow, *An Introduction to Electromagnetic Theory* (University Press, Cambridge, 1973).

3.6 General Electrostatic Torque

Let $\mathbf{E}'$ be the field produced by ρ'. Then, the torque on ρ is

$$\mathbf{N} = \int d^3r\, \mathbf{r}\times\rho(\mathbf{r})\mathbf{E}'(\mathbf{r}) = -\int d^3r\, \mathbf{r}\times\rho(\mathbf{r})\nabla\varphi'(\mathbf{r}).$$

Therefore,

$$N_k = -\epsilon_{k\ell m}\int d^3r\, r_\ell\, \rho\, \partial_m\varphi' = \epsilon_{k\ell m}\int d^3r\, \varphi'\partial_m(r_\ell\, \rho).$$

Since $\epsilon_{k\ell m}\delta_{\ell m} = 0$,

$$N_k = \epsilon_{k\ell m}\int d^3r\, \varphi' r_\ell\, \partial_m\rho = \epsilon_{k\ell m}\int d^3r\int d^3r'\, \frac{1}{4\pi\epsilon_0}\frac{\rho'(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}r_\ell\, \partial_m\rho(\mathbf{r}).$$

Integrating by parts gives

$$N_k = -\frac{1}{4\pi\epsilon_0}\epsilon_{k\ell m}\int d^3r\int d^3r'\, \rho\,\rho'\,\partial_m\left[\frac{r_\ell}{|\mathbf{r}-\mathbf{r}'|}\right],$$

so

$$N_k = -\frac{1}{4\pi\epsilon_0}\epsilon_{k\ell m}\int d^3r\int d^3r'\, \rho\rho'\left[\frac{\delta_{\ell m}}{|\mathbf{r}-\mathbf{r}'|} - \frac{r_\ell\,(r_m - r'_m)}{|\mathbf{r}-\mathbf{r}'|^3}\right].$$

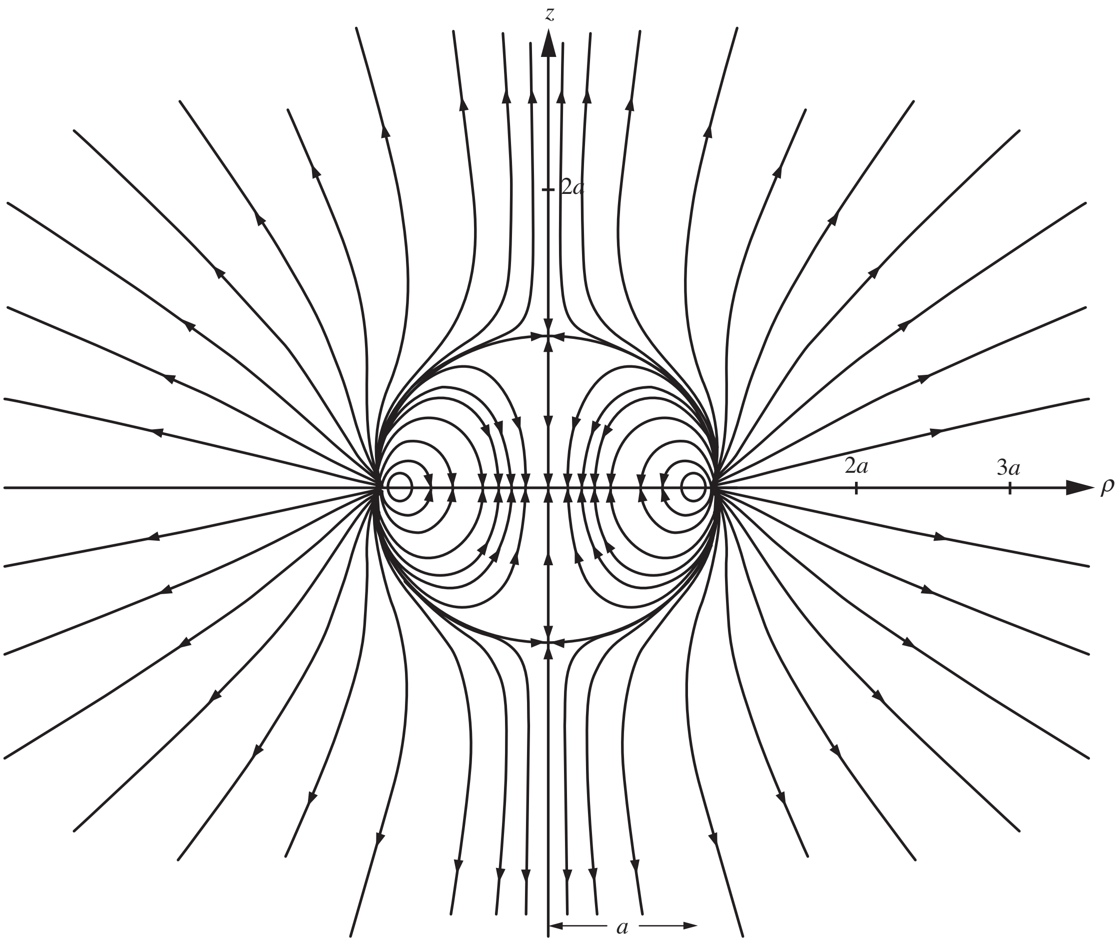
But $\mathbf{r}\times\mathbf{r} = 0$, so

$$N_k = -\frac{1}{4\pi\epsilon_0}\epsilon_{k\ell m}\int d^3r\int d^3r'\, \rho\rho'\left[\frac{r_\ell\, r'_m}{|\mathbf{r}-\mathbf{r}'|^3}\right].$$

This is the advertised formula. Notice that the torque on ρ' due to ρ is $\mathbf{N}' = -\mathbf{N}$, as it must be because their sum must be zero for an isolated system.

Source: P.C. Clemmow, *An Introduction to Electromagnetic Theory* (University Press, Cambridge, 1973).

3.7 Field Lines for a Non-Uniformly Charged Disk



The field lines above are drawn so that (1) the lines far from the disk resemble those of a positive point charge; (2) field lines very near the disk (but away from the rim) resemble the field lines near an infinite sheet of negative charge; and (3) the field lines very near the rim resemble the field lines near an infinitely long positive charge line. When these things are done, there are inevitably points in space where the field lines must cross. This is allowed if $\mathbf{E} = 0$ at those isolated points.

Source: C.L. Pekeris and K. Frankowski, *Physical Review A* **36**, 5118 (1987).

3.8 The Electric Field of a Charged Slab and a Charged Sheet

(a) By symmetry, the electric field is along $\hat{\mathbf{x}}$. For the sheet of charge at $x = 0$, we use a pillbox-shaped Gaussian surface centered at $x = 0$. This gives

$$\mathbf{E}_s(x) = \text{sgn}(x)\frac{\sigma_0}{2\epsilon_0}\hat{\mathbf{x}}.$$

For the slab of charge between $x = 0$ and $x = b$, we use a pillbox-shaped Gaussian surface centered at $x = 1/2$. This gives $\mathbf{E}_b = E_b\hat{\mathbf{x}}$ where

$$E_b(x) = \begin{cases} -\rho_0 b/2\epsilon_0 & x < 0, \\ \rho_0(x - b/2)/\epsilon_0 & 0 \le x \le b, \\ \rho_0 b/2\epsilon_0 & x \ge b. \end{cases}$$

Therefore, the total field is $\mathbf{E} = \mathbf{E}_s + \mathbf{E}_b = E\,\hat{\mathbf{x}}$ where

$$E(x) = \begin{cases} -(\sigma_0 + \rho_0 b)/2\epsilon_0 & x < 0, \\ (\sigma_0 + 2\rho_0 x - \rho_0 b)/2\epsilon_0 & 0 < x \le b, \\ (\sigma_0 + \rho_0 b)/2\epsilon_0 & x > b. \end{cases}$$

(b) If we write $\rho_s(x) = \sigma_0\delta(x)$ and $\rho_b(x) = \rho_0\theta(x) - \rho_0\theta(x - b)$, the force per unit area acting on $\rho(x)$ is directed in the $\hat{\mathbf{x}}$-direction with magnitude

$$f = \int_{-\infty}^{\infty} dx\rho_s(x)E_s(x) + \int_{-\infty}^{\infty} dx\rho_s(x)E_b(x) + \int_{-\infty}^{\infty} dx\rho_b(x)E_s(x) + \int_{-\infty}^{\infty} dx\rho_b E_b.$$

or

$$\begin{aligned} f &= \int_{-\infty}^{\infty} dx\sigma_0\delta(x)\mathrm{sgn}(x)\frac{\sigma_0}{2\epsilon} + \int_{-\infty}^{\infty} dx\sigma_0\delta(x)\left(-\frac{\rho_0 b}{2\epsilon_0}\right) \\ &\quad + \int_0^b dx\rho_0\mathrm{sgn}(x)\frac{\sigma_0}{2\epsilon_0} + \int_0^b dx\rho_0\rho_0(x - b/x)/\epsilon_0. \end{aligned}$$

The integrand of the first integral contains an electric field which is discontinuous at the surface of integration. Our prescription for this situation is to use the average value of the field, which is zero in this case. Therefore, the first integral is zero. The three terms which remain are

$$f = -\frac{\rho_0\sigma_0 b}{2\epsilon} + \frac{\rho_0\sigma_0 b}{2\epsilon_0} + \frac{\rho_0^2}{\epsilon_0}\left(\frac{b^2}{2} - \frac{b^2}{2}\right) = 0.$$

Source: E.M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1965).

3.9 The Electric Flux Through a Plane

By superposition, it is sufficient to consider a single point charge q in the $z < 0$ half-space. Pass a plane through q which is parallel to the plane $z = 0$. By the radial symmetry of $\mathbf{E}$ for a point charge, half the field lines pass through the plane $z = 0$. Therefore, half the total electric flux q/ϵ_0 passes through $z = 0$.

3.10 Two Electrostatic Theorems

(a) Our task is to produce $\varphi(0)$ from the volume integration part of Green's identity. This will happen if we choose $f = \varphi$ and $g = r^{-1}$ because $\nabla^2 f = \nabla^2\varphi = \rho/\epsilon_0 = 0$ throughout the volume and $\nabla^2(1/r) = -4\pi\delta(\mathbf{r})$. Specifically, one side of Green's identity becomes

$$\int_V d^3r\,(f\nabla^2 g - g\nabla^2 f) = -4\pi\int_V d^3r\,\varphi(\mathbf{r})\delta(\mathbf{r}) = -4\pi\varphi(0). \tag{1}$$

As for the other side of Green's identity, let the sphere have radius R. Then,

$$\int_S d\mathbf{S}\cdot(f\nabla g - g\nabla f) = \int d\mathbf{S}\cdot\left(\varphi\nabla\frac{1}{r} - \frac{1}{r}\nabla\varphi\right) = -\int d\mathbf{S}\cdot\varphi\frac{\hat{\mathbf{r}}}{r^2} + \frac{1}{R}\int d\mathbf{S}\cdot\mathbf{E}.$$

The last integral is zero by Gauss' law because there is no charge enclosed by S. Hence,

$$\int_S d\mathbf{S}\cdot(f\nabla g - g\nabla f) = -\frac{1}{R^2}\int dS\varphi. \tag{2}$$

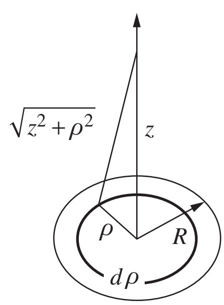
Setting (1) equal to (2) gives the desired result,

$$\varphi(0) = \frac{1}{4\pi R^2}\int_S dS\varphi = \langle\varphi\rangle_S.$$

(b) Shrink the sphere down to an arbitrarily small size. The average $\langle\varphi\rangle_S$ cannot be greater than the largest value of φ on the sphere, nor can it be less than the smallest value of φ on the sphere. Therefore, the largest and smallest values of φ lie on the surface of a sphere which encloses zero charge. This is Earnshaw's theorem.

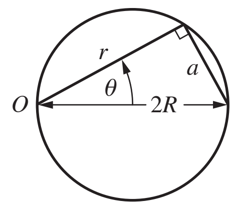
3.11 Potential, Field, and Energy of a Charged Disk

(a) Refer to the figure below. The contribution to the potential on the z-axis is the same for every bit of charge $dq = \sigma\rho d\rho d\phi$ on the annular ring at radius ρ. Therefore we need to integrate over ϕ and sum over rings to get the total potential. This gives



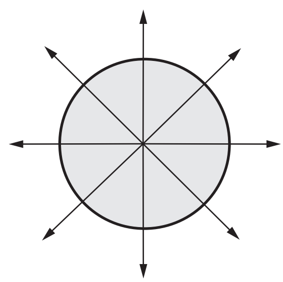
$$\varphi(z) = \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi} d\phi \int_0^R d\rho \frac{\rho}{\sqrt{\rho^2+z^2}} = \frac{\sigma}{2\epsilon_0}\left[\sqrt{R^2+z^2}-|z|\right].$$

(b) Refer to the figure below. We will find the potential at the point on the rim labeled O, which we choose as the origin of a polar coordinate system. From the geometry of a circle, the maximum value of r is $2R\cos\theta$. By symmetry, the upper half-disk and lower half-disk contribute equally. Therefore, at O,



$$\varphi(O) = \frac{2\sigma}{4\pi\epsilon_0}\int_0^{\pi/2} d\theta \int_0^{2R\cos\theta} \frac{rdr}{r} = \frac{\sigma}{2\pi\epsilon_0}\int_0^{\pi/2} d\theta 2R\cos\theta = \frac{\sigma R}{\pi\epsilon_0}.$$

(c) From part (a), the potential at the center of the disk ($z = 0$) is $\sigma R/2\epsilon_0$. This is greater than $\varphi(O)$ when $\sigma > 0$. Electric field lines point in the direction of decreasing potential everywhere. Therefore, by rotational symmetry, the field line pattern must be as shown below.



(d) U_E is the energy of assembly. At an intermediate stage, the disk has radius r. At the rim, where the potential is $\sigma r/\pi\epsilon_0$, we add an annulus of charge with radius r and thickness dr. This costs a potential energy $dq\varphi(r) = (\sigma 2\pi r dr)(\sigma r/\pi\epsilon_0) = 2\sigma^2 r^2 dr/\epsilon_0$. Therefore, we get U_E by integrating over r from zero to R. Hence,

$$U_E = \frac{2\sigma^2}{\epsilon_0}\int_0^R drr^2 = \frac{2\sigma^2 R^3}{3\epsilon_0}.$$

Source: O.D. Jefimenko, *Electricity and Magnetism* (Appleton-Century-Crofts, New York, 1966).

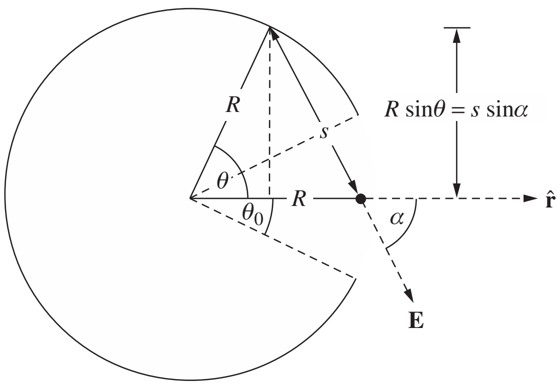
3.12 A Charged Spherical Shell with a Hole

(a) By symmetry, the field at P (black dot in diagram below) points in the $\hat{\mathbf{r}}$ direction indicated. From the geometry, $s = 2R\sin(\theta/2)$ and $\alpha = \pi/2 - \theta/2$. Therefore, the contribution to the radial field from the charge dq in an annular ring of radius $R\sin\theta$ whose points lie a distance s from P is

$$dE_r = \frac{1}{4\pi\epsilon_0}\frac{dq}{s^2}\cos\alpha = \frac{1}{4\pi\epsilon_0}\frac{dq}{4R^2\sin^2(\theta/2)}\cos(\pi/2-\theta/2) = \frac{1}{4\pi\epsilon_0}\frac{dq}{4R^2\sin(\theta/2)}.$$

But $dq = \sigma dA = \sigma(2\pi R\sin\theta)(Rd\theta)$. Therefore, if θ_0 is the angle where the hole ends, the magnitude of the radial electric field at P is

$$E(P) = \int dE_r = \frac{\sigma}{4\epsilon_0}\int_{\theta_0}^{\pi}\cos(\theta/2)d\theta = \frac{\sigma}{2\epsilon_0}[1-\sin(\theta_0/2)].$$



(b) The shell is essentially complete when $\theta_0 \ll 1$. In that case, Gauss' law gives the field inside the shell as zero and the field outside the shell as $\mathbf{E} = \hat{\mathbf{r}}Q/4\pi\epsilon_0 R^2 = \hat{\mathbf{r}}\sigma/\epsilon_0$. The field exactly on the shell is the average of these two.

Source: E.M. Purcell, *Electricity and Magnetism*, 2nd edition (McGraw-Hill, New York, 1985).

3.13 A Uniformly Charged Cube

Take one corner of the cube as the origin. The charge density ρ is a constant, so the potential at the corner farthest from the origin is

$$\varphi_1(s) = \int_0^s dx\int_0^s dy\int_0^s dz\frac{\rho}{4\pi\epsilon_0\sqrt{x^2+y^2+z^2}}.$$

Similarly, the potential at the corner of the cube with side length $s/2$ rather than s is

$$\varphi_1(s/2) = \int_0^{s/2} dx \int_0^{s/2} dy \int_0^{s/2} dz \frac{\rho}{4\pi\epsilon_0\sqrt{x^2+y^2+z^2}}.$$

However, changing integration variables to $x' = 2x$, $y' = 2y$, and $z' = 2z$ transforms the last integral to

$$\varphi_1(s/2) = \frac{1}{8}\int_0^{s} dx' \int_0^{s} dy' \int_0^{s} dz' \frac{\rho}{4\pi\epsilon_0\sqrt{\dfrac{x'^2+y'^2+z'^2}{4}}} = \frac{1}{4}\varphi_1(s).$$

Therefore, using superposition as suggested,

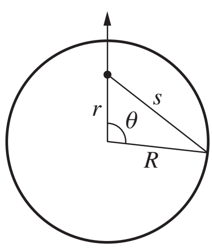
$$\varphi_0(s) = 8\varphi_1(s/2) = 2\varphi_1(s).$$

Source: E.M. Purcell, *Electricity and Magnetism*, 2nd edition (McGraw-Hill, New York, 1985).

3.14 A Variation on Coulomb's Law

(a) If the potential of a point charge at the origin is $\dfrac{1}{4\pi\epsilon_0}\dfrac{q}{r^{1+\epsilon}}$, superposition gives the potential at point r as

$$V(r) = \frac{\sigma R^2}{4\pi\epsilon_0}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \frac{1}{s^{1+\epsilon}}.$$



On the other hand, $s^2 = r^2 + R^2 - 2rR\cos\theta$, so $sds = rR\sin\theta d\theta$. Therefore,

$$V(r) = \frac{\sigma R^2}{4\pi\epsilon_0}\frac{2\pi}{rR}\int_{R-r}^{R=+r}\frac{sds}{s^{1+\epsilon}} = \frac{\sigma R}{2\epsilon_0 r}\frac{1}{1-\epsilon}\left[(R+r)^{1-\epsilon} - (R-r)^{1-\epsilon}\right].$$

Since $\sigma = Q/4\pi R^2$, the Coulomb limit of $\epsilon = 0$ gives the usual result that the potential is constant everywhere inside the sphere:

$$V(r) = \frac{\sigma R}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 R} = V(R).$$

(b) In the $\epsilon \ll 1$ limit, we use $s^{1-\epsilon} \approx s(1-\epsilon \ln s)$ to get

$$\frac{V(r)}{V(R)} \approx \frac{1}{2r}\left[\frac{2r+\epsilon(R-r)\ln(R-r)-\epsilon(R+r)\ln(R+r)}{1-\epsilon\ln 2R}\right].$$

Consistency to first order in ϵ requires that we use $(1-\epsilon\ln 2R)^{-1} \approx 1+\epsilon\ln 2R$. Hence, as the statement of the problem suggests,

$$\frac{V(r)}{V(R)} \approx 1+\frac{\epsilon}{2}\left[\frac{R}{r}\ln\frac{R-r}{R+r}+\ln\frac{4R^2}{R^2-r^2}\right].$$

3.15 Practice with Electrostatic Energy

(a) A straightforward application of Gauss' law for this situation gives $\mathbf{E} = E(r)\hat{\mathbf{r}}$ where

$$E(r) = \begin{cases} 0 & r \le a, \\ \dfrac{Q(r)}{4\pi\epsilon_0 r^2} & a \le r \le R, \\ \dfrac{Q}{4\pi\epsilon_0 r^2} & r > R, \end{cases}$$

and

$$Q(r) = \frac{r^3-a^3}{R^3-a^3}Q.$$

In terms of the variable $x = a/R$, the electrostatic total energy is

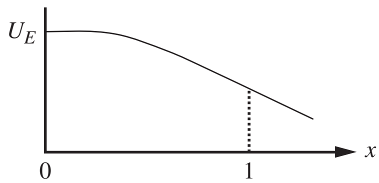
$$U_E = \frac{\epsilon_0}{2}\int d^3r\,|\mathbf{E}|^2 = \frac{Q^2}{4\pi\epsilon_0}\frac{1}{2}\left[\frac{1}{(R^3-a^3)^2}\int_a^R dr\frac{(r^3-a^3)^2}{r^2}+\int_R^\infty\frac{dr}{r^2}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\frac{Q^2}{2R}\left[1+\frac{1-5x^3+9x^5-5x^6}{5(1-x^3)^2}\right].$$

The $x=0$ limit gives $U_E = \dfrac{1}{4\pi\epsilon_0}\dfrac{3Q^2}{5R}$ which is the total energy of a uniform ball of charge.

The $x=1$ limit requires two applications of l'Hospital's rule. We find $\dfrac{1}{4\pi\epsilon_0}\dfrac{Q^2}{2R}$, which is the energy of a hollow shell.

(b) Straightforward differentiation shows that $dU_E/dx = 0$ at $x = 0$. However, we have just seen that $U_E(x = 0) > U_E(x < 1)$. Therefore, $x = 0$ must be the *maximum* of U_E. A quick sketch shows that the true minimum occurs outside the physical range of x. We conclude that the lowest energy results when all the charge resides on the sphere surface ($x = 1$). This is the situation for a perfect conductor.



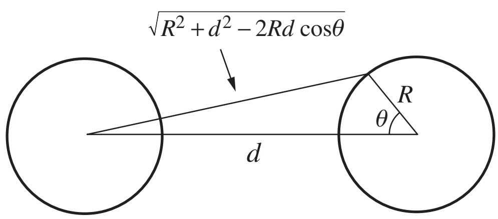
Source: L. Brito and M. Fiolhais, *European Journal of Physics* **23**, 427 (2002).

3.16 Interaction Energy of Spheres

(a) The total charge and potential have contributions from each sphere, so $\rho = \rho_1 + \rho_2$ and $\varphi = \varphi_1 + \varphi_2$. Substituting into the integral given in the problem statement, the interaction part of the energy is

$$V_E = \frac{1}{2}\int d^3r\, \rho_1(\mathbf{r})\varphi_2(\mathbf{r}) + \frac{1}{2}\int d^3r\, \rho_2(\mathbf{r})\varphi_1(\mathbf{r}).$$

By symmetry, the two contributions must be equal. Therefore, if S_2 is the surface of sphere 2,



$$V_E = \int_{S_2} dS \sigma_2(\mathbf{r})\varphi_1(\mathbf{r}) = \frac{Q}{4\pi R^2} R^2 \int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta) \frac{Q}{4\pi\epsilon_0} \frac{1}{\sqrt{R^2 + d^2 - 2R\cos\theta}} = \frac{Q^2}{4\pi\epsilon_0 d}.$$

The last equality is a consequence of

$$\int_{-1}^{1} \frac{dx}{\sqrt{1 - 2xy + y^2}} = \frac{2}{y}.$$

(c) By Gauss' law, a uniformly charged spherical shell of arbitrary radius R has the same potential (outside of itself) as a point charge at the center of the shell. Together with $V_E = \int d^3r\, \rho_1(\mathbf{r})\varphi_2(\mathbf{r})$, this implies that V_E between a uniformly charged spherical shell (1) of radius R_1 and another shell (2) of radius R_2 equals the interaction energy between the spherical shell (1) and a point charge equivalent to (2). This means that V_E is independent of R_2. By symmetry, V_E must be independent of R_1 also. Since $R_1 = R_2 = R$, we conclude that V_E is independent of R.

3.17 Electrostatic Interaction Energy

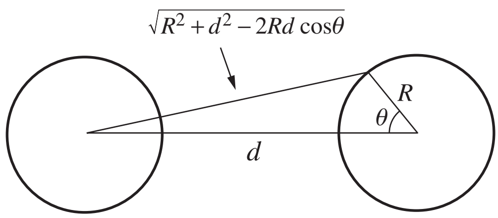
(a) The total charge and potential have contributions from each sphere, so $\rho = \rho_1 + \rho_2$ and $\varphi = \varphi_1 + \varphi_2$. Substituting into the given integral generates the interaction energy as

$$V_E = \frac{1}{2}\int d^3r\, \rho_1(\mathbf{r})\varphi_2(\mathbf{r}) + \frac{1}{2}\int d^3r\, \rho_2(\mathbf{r})\varphi_1(\mathbf{r}).$$

By symmetry, the two contributions must be equal. Therefore, if S_2 is the surface of sphere 2, we use the definite integral

$$\int_{-1}^{1} \frac{dx}{\sqrt{1-2xy+y^2}} = \frac{2}{y}$$

to get



$$V_E = \int_{S_2} dS\sigma_2(\mathbf{r})\varphi_1(\mathbf{r}) = \frac{Q}{4\pi R^2}R^2\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\frac{Q}{4\pi\epsilon_0}\frac{1}{\sqrt{R^2+d^2-2R\cos\theta}} = \frac{Q^2}{4\pi\epsilon_0 d}.$$

(b) By Gauss' law, the field of each sphere outside itself is the same as the field of a point charge. Therefore, we get the interaction energy between two point charges, independent of the sphere radius R.

3.18 Ionization Energy of a Model Hydrogen Atom

The total charge density of the atom is $\rho(\mathbf{r}) = \rho_+(\mathbf{r}) + \rho_-(\mathbf{r})$ where $\rho_+(\mathbf{r}) = |e|\delta(\mathbf{r})$. The ionization energy I is the *negative* of the total energy required to assemble the atom from its constituent parts. If we ignore the self-energy of the nucleus, this is the negative of the interaction energy of the nucleus with the electron and the self-energy of the electron:

$$I = -\frac{1}{4\pi\epsilon_0}\int d^3r \int d^3r' \frac{\rho_-(\mathbf{r})\rho_+(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} - \tfrac{1}{2}\int d^3r\, \varphi_-(\mathbf{r})\rho_-(\mathbf{r}).$$

In this expression, $\varphi_-(\mathbf{r})$ is the potential produced by $\rho_-(\mathbf{r})$. The first term is

$$
\begin{aligned}
I_1 &= -\frac{1}{4\pi\epsilon_0}\int d^3r\int d^3r'\,\frac{\rho_-(\mathbf{r})\rho_+(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\\
&= -|e|\frac{1}{4\pi\epsilon_0}\int d^3r\,\frac{\rho_-(\mathbf{r})}{|\mathbf{r}|} = \frac{e^2}{\pi\epsilon_0 a^2}\int_0^\infty dr\exp(-2r/a) = \frac{e^2}{2\pi\epsilon_0 a}.
\end{aligned}
$$

To find the second term, we begin with Gauss' law. This gives

$$
\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}} = \hat{\mathbf{r}}q(r)/4\pi\epsilon_0 r^2,
$$

where

$$
q(r) = 4\pi\int_0^r ds\,s^2\rho_-(s) = -\frac{4|e|}{a^2}\int_0^r ds\,s\exp(-2s/a) = |e|\exp(-2r/a)\,[1+2r/a] - |e|.
$$

Then,

$$
\begin{aligned}
\varphi_-(r) &= -\int_\infty^r ds\,E(s)\\
&= -\frac{1}{4\pi\epsilon_0}\int_r^\infty ds\,q(s)\frac{d}{ds}\frac{1}{s} = \frac{q(r)}{4\pi\epsilon_0 r} + \frac{1}{4\pi\epsilon_0}\int_r^\infty \frac{ds}{s}\frac{dq(s)}{ds} = \frac{q(r)}{4\pi\epsilon_0 r} + \frac{1}{\epsilon_0}\int_r^\infty ds\,s\rho_-(s)\\
&= \frac{q(r)}{4\pi\epsilon_0 r} - \frac{|e|}{2\pi\epsilon_0 a}\exp(-2r/a) = \frac{|e|}{4\pi\epsilon_0 r}\left\{\exp(-2r/a)-1\right\}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
I_2 &= -\tfrac{1}{2}\int d^3r\,\varphi_-(\mathbf{r})\rho_-(\mathbf{r})\\
&= \frac{e^2}{2\pi\epsilon_0 a^2}\left\{\int_0^\infty dr\exp(-4r/a) - \int_0^\infty dr\exp(-2r/a)\right\}\\
&= -\frac{e^2}{8\pi\epsilon_0 a}.
\end{aligned}
$$

The final result is

$$
I = I_1 + I_2 = \frac{3}{8}\frac{e^2}{\pi\,\epsilon_0 a}
$$

as required.

3.19 Two Spherical Charge Distributions

The force exerted by distribution 1 on the other is $\mathbf{F} = -\nabla V_E$. This force can be calculated by integrating the electric stress tensor over any surface in vacuum which encloses one distribution but not the other. According to Gauss' law, $\mathbf{E}$ on such a surface is identical to the electric field produced by two point charges at the centers of the distributions with charges Q_1 and Q_2. Therefore, the force and potential energy are the same as well.

3.20 Two Electric Field Formulae

(a)

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) &= -\nabla \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{\rho}{4\pi\epsilon_0} \int_V d^3r' \, \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} \\
&= \frac{\rho}{4\pi\epsilon_0} \int_V d^3r' \, \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \\
&= \frac{\rho}{4\pi\epsilon_0} \int_S \frac{d\mathbf{S}'}{|\mathbf{r}-\mathbf{r}'|}.
\end{aligned}$$

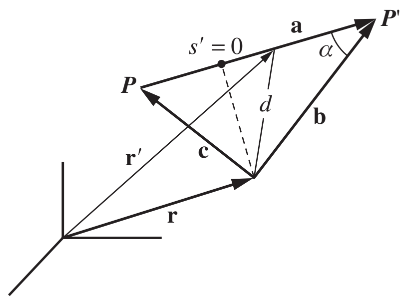
(b) For a general $\rho(\mathbf{r}')$,

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) &= -\nabla \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{1}{4\pi\epsilon_0} \int_V d^3r' \, \rho(\mathbf{r}')\nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|} \\
&= \frac{1}{4\pi\epsilon_0} \int_V d^3r' \, \rho(\mathbf{r}')\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} \\
&= \frac{1}{4\pi\epsilon_0} \int d^3r' \nabla' \left[\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \right] - \frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\nabla'\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{1}{4\pi\epsilon_0} \int d^3r' \frac{\nabla'\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.
\end{aligned}$$

The first integral in the penultimate line above vanishes because the charge distribution is localized and thus vanishes at infinity.

Source: O.D. Jefimenko, *Electricity and Magnetism* (Appleton-Century-Crofts, New York, 1966).

3.21 The Potential of a Charged Line Segment



We need to evaluate the integral

$$\varphi(\mathbf{r}) = \frac{\lambda}{4\pi\epsilon_0} \int_P^{P'} \frac{ds'}{|\mathbf{r} - \mathbf{r}'|}.$$

However, the point P corresponds to $s' = \mathbf{c}\cdot\mathbf{a}/a$ and the point P' corresponds to $s' = \mathbf{b}\cdot\mathbf{a}/a$. Moreover,

$$|\mathbf{r} - \mathbf{r}'| = d = \sqrt{s'^2 + b^2 \sin^2\alpha} = \sqrt{s'^2 + |\mathbf{b}\times\mathbf{a}|^2/a^2}.$$

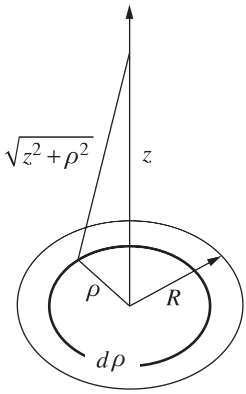
Therefore,

$$\varphi(\mathbf{r}) = \frac{\lambda}{4\pi\epsilon_0} \int_{\mathbf{c}\cdot\mathbf{a}/a}^{\mathbf{b}\cdot\mathbf{a}/a} \frac{ds'}{\sqrt{s'^2 + |\mathbf{b}\times\mathbf{a}|^2/a^2}} = \frac{\lambda}{4\pi\epsilon_0} \ln \frac{\left| \dfrac{\mathbf{b}\cdot\mathbf{a}}{a} + \sqrt{\left(\dfrac{\mathbf{b}\cdot\mathbf{a}}{a}\right)^2 + \dfrac{|\mathbf{b}\times\mathbf{a}|^2}{a^2}} \right|}{\left| \dfrac{\mathbf{c}\cdot\mathbf{a}}{a} + \sqrt{\left(\dfrac{\mathbf{c}\cdot\mathbf{a}}{a}\right)^2 + \dfrac{|\mathbf{b}\times\mathbf{a}|^2}{a^2}} \right|}.$$

Source: H.A. Haus and J.R. Melcher, *Electromagnetic Fields and Energy* (Prentice Hall, Englewood Cliffs, NJ, 1989)

3.22 A Variation on Coulomb's Law

(a) By translational symmetry, $\varphi(\mathbf{r}) = \varphi(z)$ and we can choose any observation distance z we wish. The calculation mimics the usual one done with an inverse-square potential; namely, we superpose the contribution from concentric annular rings with charge $dq = 2\pi\rho d\rho\sigma$:



Every point on each annulus lies a distance $s = \sqrt{\rho^2 + z^2}$ from the observation point on the z-axis. Therefore,

$$\varphi(z) = \frac{1}{4\pi\epsilon_0} \int dq f(\sqrt{\rho^2 + z^2}) = \frac{\sigma}{2\pi\epsilon_0} \int_0^\infty d\rho \rho f(\sqrt{\rho^2 + z^2}) = \frac{\sigma}{2\epsilon_0} \int_z^\infty ds s f(s).$$

(b) The electric field follows immediately because we are taking a derivative with respect to the upper limit of integration:

$$\mathbf{E} = -\nabla\varphi(z) = \hat{\mathbf{z}} \frac{\sigma}{2\epsilon_0} \frac{\partial}{\partial z} \int_\infty^z ds s f(s) = \hat{\mathbf{z}} \frac{\sigma}{2\epsilon_0} z f(z).$$

3.23 A Non-Uniform Charge Distribution on a Surface

(a)

$$Q = \int dS \sigma(\rho) = 2\pi \int_0^\infty d\rho \rho \sigma(\rho) = -qd \int_0^\infty d\rho \frac{\rho}{(\rho^2 + s^2)^{3/2}} = qd \left. \frac{1}{\sqrt{\rho^2 + s^2}} \right|_0^\infty = -q\frac{d}{s}.$$

(b) We add up the contribution from annular rings with charge $dQ = 2\pi\sigma(\rho)\rho d\rho$. All points $\mathbf{z} = z\hat{\mathbf{z}}$ lie a distance $\sqrt{z^2 + \rho^2}$ from points $\boldsymbol{\rho} = \rho\hat{\boldsymbol{\rho}}$ on an annulus of radius ρ. Therefore,

$$\begin{aligned}
\varphi(z) &= \frac{1}{4\pi\epsilon_0}\int dS\frac{\sigma(\rho)}{|\mathbf{z}-\boldsymbol{\rho}|} \\
&= \frac{-qd}{4\pi\epsilon_0}\int_0^\infty d\rho\frac{1}{(\rho^2+s^2)^{3/2}}\frac{\rho}{(z^2+\rho^2)^{1/2}} \\
&= \frac{qd}{4\pi\epsilon_0}\frac{1}{s}\frac{d}{ds}\int_0^\infty d\rho\frac{\rho}{(\rho^2+s^2)^{1/2}}\frac{1}{(z^2+\rho^2)^{1/2}} \\
&= \frac{q(d/s)}{4\pi\epsilon_0}\frac{d}{ds}\int_0^\infty d\rho\frac{\rho}{\sqrt{\rho^4+\rho^2(s^2+z^2)+s^2z^2}} \\
&= \frac{q(d/s)}{4\pi\epsilon_0}\frac{1}{2}\frac{d}{ds}\int_0^\infty \frac{du}{\sqrt{u^2+u(s^2+z^2)+s^2z^2}} \\
&= \frac{q(d/s)}{4\pi\epsilon_0}\frac{1}{2}\frac{d}{ds}\left[\ln\left\{2\sqrt{u^2+u(s^2+z^2)+s^2z^2}+2u+s^2+z^2\right\}\right]_0^\infty \\
&= -\frac{q(d/s)}{4\pi\epsilon_0}\frac{1}{2}\frac{d}{ds}\ln\left\{2\sqrt{s^2+z^2}+s^2+z^2\right\} \\
&= -\frac{q(d/s)}{4\pi\epsilon_0}\frac{1}{2}\frac{d}{ds}\ln(s+z)^2 \\
&= -\frac{1}{4\pi\epsilon_0}\frac{q(d/s)}{s+z}.
\end{aligned}$$

The last line is indeed the electrostatic potential produced by a point charge $Q = -qd/s$ on the axis at $z = -s$.

Source: E.J. Konopinski, *Electromagnetic Fields and Relativistic Particles* (McGraw-Hill, New York, 1981).

3.24 The Energy outside a Charged Volume

Since $\rho(\mathbf{r}) = 0$ outside V, we can write the total energy as

$$U_E = \frac{1}{2}\int d^3r\,\rho(\mathbf{r})\varphi(\mathbf{r}) = \frac{1}{2}\int_V d^3r\,\rho(\mathbf{r})\varphi(\mathbf{r}) = \frac{\epsilon_0}{2}\int_V d^3r\,\varphi(\mathbf{r})\nabla\cdot\mathbf{E}.$$

An integration by parts gives

$$U_E = \frac{\epsilon_0}{2}\int_V d^3r\,\nabla\cdot[\mathbf{E}\phi] + \frac{\epsilon_0}{2}\int_V d^3r\,\mathbf{E}\cdot\mathbf{E}.$$

We rewrite the first term above using the divergence theorem to get

$$U_E = \frac{\epsilon_0}{2}\int_S d\mathbf{S}\cdot\mathbf{E}\phi + \frac{\epsilon_0}{2}\int_V d^3r\,\mathbf{E}\cdot\mathbf{E}.$$

Finally, using Gauss' law and the constancy of φ on S,

$$U_E = \frac{1}{2}Q\varphi_0 + \frac{\epsilon_0}{2}\int\limits_V d^3r\,\mathbf{E}\cdot\mathbf{E}.$$

The second term in this expression is the total energy contained *inside* S. Therefore, the first term must be the total energy contained *outside* S.

Source: A.M. Portis, *Electromagnetic Fields* (Wiley, New York, 1978).

3.25 Overcharging

(a) We want the total energy of a spherical shell of radius R with charge Nq distributed uniformly over its surface. The electric field is zero inside the sphere and $\mathbf{E} = Nq\hat{\mathbf{r}}/4\pi\epsilon_0 r^2$ outside the sphere. The total energy of this configuration is, as suggested,

$$U = \frac{\epsilon_0}{2}\int d^3r|\mathbf{E}|^2 = \frac{\epsilon_0}{2}\left[\frac{Nq}{4\pi\epsilon_0}\right]^2\int d\Omega\int\limits_R^\infty \frac{dr}{r^2} = \frac{q^2N^2}{8\pi\epsilon_0 R}.$$

(b) U in part (a) overestimates the energy because it puts bits of charge arbitrarily close to each other on the sphere's surface. We must subtract from U the interaction energy of each micro-ion with the smeared-out charge within its immediate area $\pi a^2 = 4\pi R^2/N$. The latter is the surface area of the macro-ion per micro-ion. There are N micro-ions, so the correction term is

$$\Delta U \approx -N\times\frac{q^2}{4\pi\epsilon_0}\frac{1}{a} = -\frac{q^2}{4\pi\epsilon_0 R}\frac{1}{2}N^{3/2}.$$

(c) The electrostatic potential of the macro-ion at its own surface is $-|Q|/4\pi\epsilon_0 R$. Therefore the interaction potential energy of the N micro-ions adsorbed on the macro-ion surface is $-q|Q|N/4\pi\epsilon_0 R$. Taking out a common pre-factor, we determine N minimizing

$$V = \frac{q}{2}(N^2 - N^{3/2}) - |Q|N.$$

This gives

$$\frac{\partial V}{\partial N} = \frac{q}{2}\left(2N - \frac{3}{2}N^{1/2}\right) - |Q| = 0,$$

which we rearrange to

$$N^2 - \left(2\frac{|Q|}{q} + \frac{9}{16}\right)N + \frac{Q^2}{q} = 0.$$

The physical ($N > 0$) solution of this quadratic equation is

$$N = \frac{|Q|}{q} + \frac{9}{32} + \frac{9}{32}\sqrt{1 + \frac{64}{9}\frac{|Q|}{q}}.$$

Since $N > |Q|/q$, the micro-ions do not simply neutralize the charge of the macro-ion, they *overcharge* the macro-ion.

Chapter 4: Electric Multipoles

4.1 Dipole Moment Practice

In both cases, the total charge of the system is zero. Therefore, the dipole moment is unique and independent of the choice of origin. We choose the natural center of each as the origin.

(a) For a linear charge density $\lambda = \lambda(\phi)$ around a ring, the volume charge density is

$$\rho(\mathbf{r}) = \lambda(\phi)\frac{\delta(\theta - \pi/2)}{\sin\theta}\frac{\delta(r - R)}{r}.$$

To check this, we let $\lambda = Q/2\pi R$ and confirm that

$$\int d^3r\,\rho(\mathbf{r}) = \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\sin\theta \int_0^{\infty} drr^2 \left[\lambda\frac{\delta(\theta - \pi/2)}{\sin\theta}\frac{\delta(r - R)}{r}\right] = 2\pi R\lambda = Q.$$

Now, because $\hat{\mathbf{r}} = \hat{\mathbf{z}}\cos\theta + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{x}}\sin\theta\cos\phi$, the electric dipole moment of the ring is

$$\begin{aligned}
\mathbf{p} &= \int d^3r\,\mathbf{r}\,\rho(\mathbf{r}) \\
&= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\sin\theta \int_0^{\infty} drr^2 \left[\hat{\mathbf{z}}r\cos\theta + \hat{\mathbf{y}}r\sin\theta\sin\phi + \hat{\mathbf{x}}r\sin\theta\cos\phi\right] \\
&\qquad \times \left[\lambda_0\cos\phi\frac{\delta(\theta - \pi/2)}{\sin\theta}\frac{\delta(r - R)}{r}\right].
\end{aligned}$$

Only the $\hat{\mathbf{x}}$ integral is non-zero above. This result of the integration around the ring is

$$\mathbf{p} = \tfrac{1}{2}\lambda_0 R^2\hat{\mathbf{x}}.$$

(b) For a surface charge density $\sigma = \sigma(\theta)$ on a spherical shell, the volume charge density is

$$\rho(r) = \sigma(\theta)\delta(r - R).$$

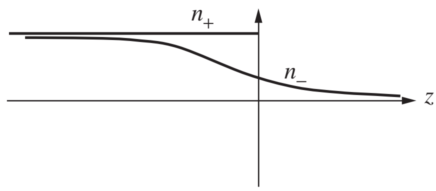
Hence,

$$\begin{aligned}
\mathbf{p} &= \int d^3r\mathbf{r}\,\rho(\mathbf{r}) \\
&= \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\sin\theta \int_0^{\infty} drr^2 \left[\hat{\mathbf{z}}r\cos\theta + \hat{\mathbf{y}}r\sin\theta\sin\phi + \hat{\mathbf{x}}r\sin\theta\cos\phi\right]\left[\sigma_0\cos\theta\delta(r - R)\right].
\end{aligned}$$

Only the $\hat{\mathbf{z}}$ integral is non-zero. Its value gives $\mathbf{p} = \frac{4}{3}\pi R^3\sigma_0\,\hat{\mathbf{z}}$ for the spherical shell.

4.2 Smolochowski's Model of a Metal Surface

(a) The form of $n_+(z)$ "smears out" the charge due to the positive nuclei but recognizes that there is a well-defined "last layer" of nuclei at $z = 0$. The form of $n_-(x)$ models the fact that the electron wave functions "spill out" into the vacuum beyond the last row of nuclei.



(b) There is only a z-component to the dipole moment $\mathbf{p}$ by symmetry. So, the dipole moment *per unit area* is

$$p_z = \int_{-\infty}^{\infty} dz\, z\,(n_+ - n_-) = \tfrac{1}{2}\bar{n}\,\frac{d}{d\kappa}\left\{\int_{-\infty}^{0} dz\, e^{\kappa z} + \int_{0}^{\infty} dz\, e^{-\kappa z}\right\} = -\frac{\bar{n}}{\kappa^2}.$$

(c) The total charge density is $\rho(z) = n_+(z) - n_-(z) = -\mathrm{sgn}(z)\frac{1}{2}\bar{n}\exp(-\kappa|z|)$. It must be that $\mathbf{E} = E(z)\hat{\mathbf{z}}$ by symmetry. Therefore, because $sgn(z) = d|z|/dz$, Gauss' law gives

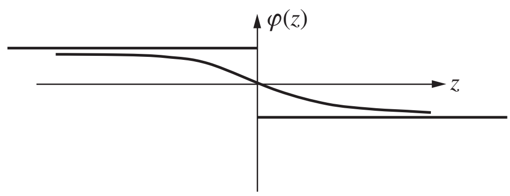
$$\frac{dE(z)}{dz} = \frac{\rho(z)}{\epsilon_0} = -\frac{\bar{n}}{2\epsilon_0}\frac{d|z|}{dz}\exp(-\kappa|z|).$$

This can be integrated by inspection to

$$E(z) = \frac{\bar{n}}{2\epsilon_0\kappa}\exp(-\kappa|z|).$$

The integration constant is zero because only a charged sheet produces an electric field at infinity. The electric field is finite at $z = 0$ so the potential must be continuous there. Hence, if we let $\varphi(0) = 0$,

$$\varphi(z) = -\int_0^z dz'\, E(z') = -\frac{\bar{n}}{2\epsilon_0\kappa}\int_0^z dz'\,\exp(-\kappa|z|) = \frac{\bar{n}}{2\epsilon_0\kappa^2}\mathrm{sgn}(z)\left\{\exp(-\kappa|z|) - 1\right\}.$$



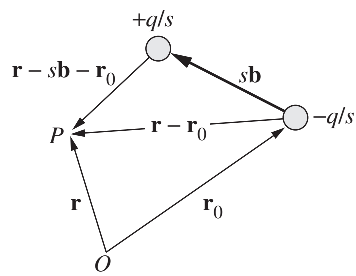
(d) This potential gives $\varphi(\infty) - \varphi(-\infty) = -\bar{n}/\kappa^2\epsilon_0 = p_z/\epsilon_0$. The right side is the change in potential which occurs across a double layer. Rather than a sudden jump, the change is spread out over the entire length of the system.

(e) The total energy per unit area is

$$U_E = \tfrac{1}{2}\epsilon_0 \int_{-\infty}^{\infty} dz\, E^2(z) = \frac{\bar{n}^2}{8\epsilon_0\kappa^2}\left\{\int_{-\infty}^{0} dz\, e^{2\kappa z} + \int_0^{\infty} dz\, e^{-2\kappa z}\right\} = \frac{\bar{n}^2}{8\epsilon_0\kappa^3}.$$

4.3 The Charge Density of a Point Electric Dipole

(a) Begin with two point charges arranged as shown below.



The charge density of this system is

$$\rho(\mathbf{r}) = \frac{q}{s}\left[\delta(\mathbf{r}-\mathbf{r}_0-s\mathbf{b}) - \delta(\mathbf{r}-\mathbf{r}_0)\right].$$

For the point electric dipole, we are interested in the limit as $s \to 0$. Therefore, we expand the argument of the delta function of the positive charge to get

$$\rho_D(\mathbf{r}) = \lim_{s\to 0}\frac{q}{s}\left[\delta(\mathbf{r}-\mathbf{r}_0) - s\mathbf{b}\cdot\nabla\delta(\mathbf{r}-\mathbf{r}_0) + \cdots - \delta(\mathbf{r}-\mathbf{r}_0)\right].$$

All the higher terms in the expansion are proportional to s, s^2, etc. and thus go to zero in the limit. Therefore, with $\mathbf{p} = q\mathbf{b}$, we get the advertised result,

$$\rho_D(\mathbf{r}) = -\mathbf{p}\cdot\nabla\delta(\mathbf{r}-\mathbf{r}_0).$$

(b) The suggested charged density is correct because the electrostatic potential it produces is

$$\begin{aligned}
\varphi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho_D(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{\mathbf{p}}{4\pi\epsilon_0}\cdot\int d^3r'\,\frac{\nabla'\delta(\mathbf{r}'-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}'|} \\
&= \frac{\mathbf{p}}{4\pi\epsilon_0}\cdot\nabla_0\int d^3r'\,\frac{\delta(\mathbf{r}'-\mathbf{r}_0)}{|\mathbf{r}-\mathbf{r}'|} \\
&= \frac{\mathbf{p}}{4\pi\epsilon_0}\cdot\nabla_0\frac{1}{|\mathbf{r}-\mathbf{r}_0|} \\
&= -\frac{\mathbf{p}}{4\pi\epsilon_0}\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_0|}.
\end{aligned}$$

4.4 Stress Tensor Proof of No Self-Force

Let S be any surface in vacuum which completely encloses the distribution $\rho(\mathbf{r})$ in question. The net force on $\rho(\mathbf{r})$ is

$$\mathbf{F} = \epsilon_0 \int_S dS \left[(\hat{\mathbf{n}} \cdot \mathbf{E})\mathbf{E} - \frac{1}{2}\hat{\mathbf{n}}(\mathbf{E} \cdot \mathbf{E}) \right].$$

Nothing changes if we expand S all the way out to infinity. If $\rho(\mathbf{r})$ has a net charge, the asymptotic electric field varies as $1/r^2$. Therefore, $dSE^2 \propto 1/r^2$ as $r \to \infty$ and the surface integral is zero. We get the same result if $\rho(\mathbf{r})$ does *not* have a net charge because the field goes to zero even faster as $r \to \infty$.

4.5 Point Charge Motion in an Electric Dipole Field

The electrostatic potential and electric field of the dipole are

$$\varphi(r, \theta) = \frac{p \cos\theta}{4\pi\epsilon_0 r^2} \qquad\qquad \mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}\left[2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right].$$

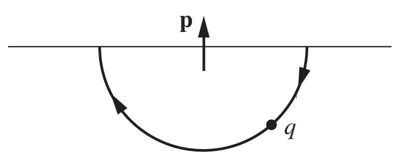
The initial condition is $\upsilon = 0$ when $r = R = \sqrt{x_0^2 + y_0^2}$ and $\theta = \pi/2$. Therefore, conservation of energy guarantees that

$$\tfrac{1}{2}m\upsilon^2 + \frac{qp\cos\theta}{4\pi\epsilon_0 R^2} = 0.$$

On the other hand, the motion will be circular if the radial force equals the centripetal acceleration, that is, if

$$\frac{m\upsilon^2}{R} = -qE_r = -\frac{2qp\cos\theta}{4\pi\epsilon_0 R^3}.$$

This equation is identical to the energy conservation equation so the motion is indeed semi-circular. A moment's reflection shows that the particle moves periodically back and forth along the arc shown below.

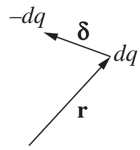


Source: R.S. Jones, *American Journal of Physics* **63**, 1042 (1995).

4.6 The Energy to Assemble a Point Dipole

We know that $dW = \varphi(\mathbf{r})dq$ is the work required to move charge dq quasistatically from infinity to the point $\mathbf{r}$. Therefore, the work required to bring charge dq to $\mathbf{r}$ and charge $-dq$ to $\mathbf{r} + \boldsymbol{\delta}$ is

$$dW = \varphi(\mathbf{r})dq - \varphi(\mathbf{r} + \boldsymbol{\delta})dq.$$



We will take the limit $\boldsymbol{\delta} \to 0$ presently so it is appropriate to perform a Taylor expansion to get

$$dW = \varphi(\mathbf{r})dq - [\varphi(\mathbf{r}) + \boldsymbol{\delta}\cdot\nabla\varphi(\mathbf{r})]dq = -\boldsymbol{\delta}q\cdot\nabla\varphi(\mathbf{r}).$$

From the figure, it is consistent to define $d\mathbf{p} = -\boldsymbol{\delta}dq$ in the limit when $dq \to \infty$ and $\boldsymbol{\delta} \to 0$ such that their product remains finite. Therefore, because $\mathbf{E} = -\nabla\varphi$, we get the desired result,

$$dW = -\mathbf{E}(\mathbf{r})\cdot d\mathbf{p}.$$

Source: A.M. Portis, *Electromagnetic Fields* (Wiley, New York, 1978).

4.7 Dipoles at the Vertices of Platonic Solids

The electric field of a point dipole is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{\mathbf{n}}(\hat{\mathbf{n}}\cdot\mathbf{p}) - \mathbf{p}}{|\mathbf{r}-\mathbf{r}_0|^3} - \frac{4\pi}{3}\mathbf{p}\,\delta(\mathbf{r}-\mathbf{r}_0)\right],$$

where $\hat{\mathbf{n}} = (\mathbf{r}-\mathbf{r}_0)/|\mathbf{r}-\mathbf{r}_0|$. The delta function has no effect since we are interested in the field $\mathbf{E}(0)$ at the center of each polyhedron. Also, $\hat{\mathbf{n}} = \hat{\mathbf{r}}_0$ at this observation point.

(a) The positions $\mathbf{r}_0$ of the dipoles for the octahedron on the far left can be taken to be $\pm a\hat{\mathbf{x}}$, $\pm a\hat{\mathbf{y}}$, and $\pm a\hat{\mathbf{z}}$. Therefore, $r_0 = a$ and $\hat{\mathbf{n}}$ takes the values $\pm a\hat{\mathbf{x}}$, $\pm a\hat{\mathbf{y}}$, and $\pm a\hat{\mathbf{z}}$ when we sum over dipoles. Hence, the total field at the origin is

$$\begin{aligned}\mathbf{E}(0) &= \frac{1}{4\pi\epsilon}\frac{1}{a^3}\left[-6\mathbf{p} + 3\hat{\mathbf{x}}p_x + 3(-\hat{\mathbf{x}})(-p_x) + 3\hat{\mathbf{y}}p_y + 3(-\hat{\mathbf{y}})(-p_y) + 3\hat{\mathbf{z}}p_z + 3(-\hat{\mathbf{z}})(-p_z)\right]\\ &= 0.\end{aligned}$$

(b) The positions $\mathbf{r}_0$ of the dipoles for the tetrahedron in the middle are $a(\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}})$, $a(-\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}})$, $a(-\hat{\mathbf{x}}+\hat{\mathbf{y}}-\hat{\mathbf{z}})$, and $a(\hat{\mathbf{x}}-\hat{\mathbf{y}}-\hat{\mathbf{z}})$. Therefore, $r_0 = \sqrt{3}a$ and $\hat{\mathbf{n}}$ takes the values $(\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}})/\sqrt{3}$, $(-\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}})/\sqrt{3}$, $(-\hat{\mathbf{x}}+\hat{\mathbf{y}}-\hat{\mathbf{z}})/\sqrt{3}$, and $(\hat{\mathbf{x}}-\hat{\mathbf{y}}-\hat{\mathbf{z}})\sqrt{3}$. Hence, the total field at the origin is

$$\begin{aligned}\mathbf{E}(0) &= \frac{1}{4\pi\epsilon_0}\frac{1}{3a^3}\left[-4\mathbf{p} + (\hat{\mathbf{x}}+\hat{\mathbf{y}}+\hat{\mathbf{z}})(p_x+p_y+p_z) + (-\hat{\mathbf{x}}-\hat{\mathbf{y}}+\hat{\mathbf{z}})(-p_x-p_y+p_z)\right]\\ &\quad+ \frac{1}{4\pi\epsilon_0}\frac{1}{3a^3}\left[(-\hat{\mathbf{x}}+\hat{\mathbf{y}}-\hat{\mathbf{z}})(-p_x+p_y-p_z) + (\hat{\mathbf{x}}-\hat{\mathbf{y}}-\hat{\mathbf{z}})(p_x-p_y-p_z)\right]\\ &= 0.\end{aligned}$$

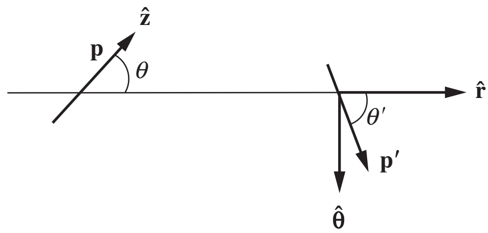
(c) The eight dipoles at the corners of the cube are the superposition of two tetrahedra with dipoles at their corners rotated by 90° with respect to one another. From part (b), each tetrahedron contributes zero to the electric field at the center. Hence, $\mathbf{E}(0) = 0$ for this case also.

Source: A.M. Portis, *Electromagnetic Fields* (Wiley, New York, 1978).

4.8 Two Coplanar Dipoles

Choose a polar coordinate system with $\mathbf{p} \parallel \hat{\mathbf{z}}$. The field produced by $\mathbf{p}$ in this system is

$$\mathbf{E} = \frac{p}{4\pi\epsilon_0 r^3}\left(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}\right).$$



At equilibrium, the potential energy $V = -\mathbf{p}' \cdot \mathbf{E}$ is a minimum. In the pictured coordinate system,

$$\mathbf{p}' = p'\cos\theta'\hat{\mathbf{r}} + p'\sin\theta'\hat{\boldsymbol{\theta}}.$$

Therefore,

$$V = -(2\cos\theta\cos\theta' + \sin\theta\sin\theta'),$$

and the minimum energy occurs at

$$\frac{\partial V}{\partial\theta'} = -(\sin\theta\cos\theta' - 2\cos\theta\sin\theta') = 0.$$

This gives the final result as

$$\tan\theta = 2\tan\theta'.$$

4.9 Potential of a Double Layer

(a) Begin with our fundamental formula for the potential due to a double layer:

$$\varphi(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\int_S dS\boldsymbol{\tau}(\mathbf{r}_S)\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_S|}.$$

Now $dS\boldsymbol{\tau} = dS\tau\hat{\mathbf{n}} = d\mathbf{S}\tau$. Therefore, working out the gradient,

$$\varphi(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\int_S d\mathbf{S}\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_S|}\tau(\mathbf{r}_S) = \frac{1}{4\pi\epsilon_0}\int_S d\mathbf{S}\cdot\frac{\mathbf{r}-\mathbf{r}_S}{|\mathbf{r}-\mathbf{r}_S|^3}\tau(\mathbf{r}_S).$$

On the other hand, the solid angle is defined as

$$\Omega(\mathbf{r}) = \int_S d\Omega = \int_S d\mathbf{S}\cdot\frac{\mathbf{r}_S-\mathbf{r}}{|\mathbf{r}_S-\mathbf{r}|^3}.$$

Combining the preceding equations completes the demonstration.

(b) Let $\mathbf{r}_L(\mathbf{r}_R)$ be a point infinitesimally close to the surface point $\mathbf{r}_S$ in region $L(R)$. The surface appears to have infinite extent when viewed at very close range, so

$$\varphi_R(\mathbf{r}_S) - \varphi_L(\mathbf{r}_S) = -\frac{1}{4\pi\epsilon_0}\tau(\mathbf{r}_S)\left[\Omega_R - \Omega_L\right] = \frac{\tau(\mathbf{r}_S)}{\epsilon_0}.$$

The square brackets contribute 4π because $\Omega_L = 2\pi$ and $\Omega_R = -2\pi$ are the solid angles subtended at $\mathbf{r}_L$ and $\mathbf{r}_R$ by an infinite plane.

4.10 A Spherical Double Layer

The outward normal to the sphere is $\hat{\mathbf{r}}$. Therefore, using the divergence, the potential at any point in space due to a surface dipole density $\boldsymbol{\tau} = \tau\hat{\mathbf{r}}$ is

$$\begin{aligned}
\varphi(\mathbf{r}) &= -\frac{1}{4\pi\epsilon_0}\int_S dS'\boldsymbol{\tau}\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{\tau}{4\pi\epsilon_0}\int_S dS'\hat{\mathbf{r}}\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{\tau}{4\pi\epsilon_0}\int d\mathbf{S}'\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{\tau}{4\pi\epsilon_0}\int_V d^3r'\,\nabla'\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} \\
&= +\frac{\tau}{4\pi\epsilon_0}\int_V d^3r'\,\nabla^2\frac{1}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{\tau}{4\pi\epsilon_0}\int_V d^3r'\,\delta(\mathbf{r}-\mathbf{r}').
\end{aligned}$$

Because $\mathbf{r}$ is the observation point and V is the volume enclosed by the spherical shell, the last integral above gives

$$\varphi(r) = \begin{cases} -\tau/\epsilon_0 & r < R, \\ 0 & r > R. \end{cases}$$

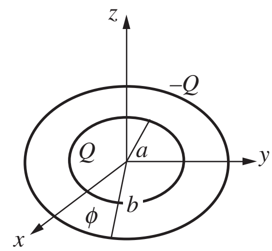
Notice that the matching condition is satisfied:

$$\varphi(r > R) - \varphi(r < R) = \tau/\epsilon_0.$$

4.11 The Distant Potential of Two Charged Rings

In cylindrical coordinates (s, ϕ, z), the charge density of the inner ring is

$$\rho(s, \phi, z) = \frac{Q}{2\pi s}\delta(s-a)\delta(z).$$



Since $x = s\cos\phi$, the x-component of the electric dipole moment vector is

$$p_x = \int_0^{2\pi} d\phi \int_0^{\infty} dss x\rho = \int_0^{2\pi} d\phi \cos\phi \int_0^{\infty} dss\delta(s-a) \int_{-\infty}^{\infty} dz\delta(z) = 0.$$

The y-component vanishes similarly because $y = s\sin\phi$. The z-component vanishes because of the factor $\delta(z)$. Hence, $\mathbf{p} = 0$.

The components of the primitive Cartesian quadrupole tensor are

$$Q_{ij} = \frac{1}{2}\int_0^{2\pi} d\phi \int_0^{\infty} dss \int_{-\infty}^{\infty} dz\, r_i r_j \rho(s).$$

All the off-diagonal elements are zero because of the ϕ-integration. The diagonal elements are

$$Q_{xx} = Q_{yy} = \frac{Qa^2}{8\pi} \qquad\qquad Q_{zz} = 0.$$

We conclude that the distant electric field of this ring is

$$\varphi_a(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0 r} + \frac{Q_{ij}}{4\pi\epsilon_0}\frac{3r_i r_j - r^2\delta_{ij}}{r^5} = \frac{Q}{4\pi\epsilon_0 r} + \frac{Qa^2}{8\pi\epsilon_0}\frac{x^2+y^2-2z^2}{r^5}.$$

The potential of the outer ring is similar except with opposite charge. The monopole terms cancel and the quadrupole terms add. Therefore, the asymptotic potential is a pure quadrupole:

$$\varphi(\mathbf{r}) = \frac{Q(a^2-b^2)}{8\pi\epsilon_0}\frac{x^2+y^2-2z^2}{r^5}.$$

4.12 The Potential Far from Two Neutral Disks

Each disk has no charge and no dipole moment. The latter is true because the charge density depends only on the radial distance from the disk center. Therefore, *with respect to its own symmetry axis*, each disk produces a quadrupole potential

$$\varphi = \frac{1}{4\pi\epsilon_0}\frac{\mathcal{Q}}{r^3}P_2(\cos\theta) = \frac{1}{4\pi\epsilon_0}\frac{\mathcal{Q}}{r^3}(3\cos^2\theta - 1),$$

where $\mathcal{Q}$ is a quadrupole moment, θ is the polar angle from its symmetry axis, and $r = \sqrt{x^2+y^2+z^2}$. We now restrict ourselves to the x-y plane, where $r = s$, and write θ for the polar angle for the horizontal disk and θ' for the polar angle of the tipped disk. This gives the total potential,

$$\varphi_{\rm tot}(x,y) = \frac{1}{4\pi\epsilon_0}\frac{\mathcal{Q}}{s^3}\left[3(\cos^2\theta + \cos^2\theta') - 2\right].$$

This formula will have the desired form (independent of all angles) if $\cos^2\theta + \cos^2\theta' = 1$, which will be true if $\theta' = \theta + \pi/2$. Hence, $\alpha = \pi/2$.

Source: J.A. Greenwood, *British Journal of Applied Physics* **17**, 1621 (1966).

4.13 Interaction Energy of Adsorbed Molecules

(a) The interaction energy between a point dipole $\mathbf{p}_1$ at $\mathbf{r}_1$ and a point dipole $\mathbf{p}_2$ at $\mathbf{r}_2$ is

$$U_{12} = \frac{1}{4\pi\epsilon_0}\left\{\frac{\mathbf{p}_1\cdot\mathbf{p}_2 - 3(\mathbf{p}_1\cdot\hat{\mathbf{n}})(\mathbf{p}_2\cdot\hat{\mathbf{n}})}{|\mathbf{r}_1 - \mathbf{r}_2|^3}\right\},$$

where $\hat{\mathbf{n}}$ is a unit vector that points from $\mathbf{r}_1$ to $\mathbf{r}_2$. The total energy per dipole is

$$U_T = \frac{1}{4\pi\epsilon_0}\frac{1}{2}\frac{p^2}{a^3}\left(U_{NN} + \frac{U_{NNN}}{2\sqrt{2}}\right),$$

where U_{NN} comes from the four nearest neighbors at a distance a, U_{NNN} comes from the four next-nearest neighbors at a distance $\sqrt{2}a$, and the factor $\frac{1}{2}$ corrects for double-counting in the total. By direct evaluation, we get

$$\begin{aligned} U_{NN} &= 2\left\{\cos 2\alpha - 3\cos^2\alpha\right\} + \left\{\cos(\pi - 2\alpha) - 3\cos^2(\tfrac{\pi}{2} + \alpha)\right\} \\ &\quad + \left\{\cos(\pi - 2\alpha) - 3\cos^2(\tfrac{\pi}{2} - \alpha)\right\} \\ &= -6 \\ U_{NNN} &= -4 - 6\left\{\cos(\tfrac{\pi}{4} - \alpha)\cos(\tfrac{3\pi}{4} + \alpha) + \cos(\tfrac{\pi}{4} + \alpha)\cos(\tfrac{3\pi}{4} - \alpha)\right\} = 2, \end{aligned}$$

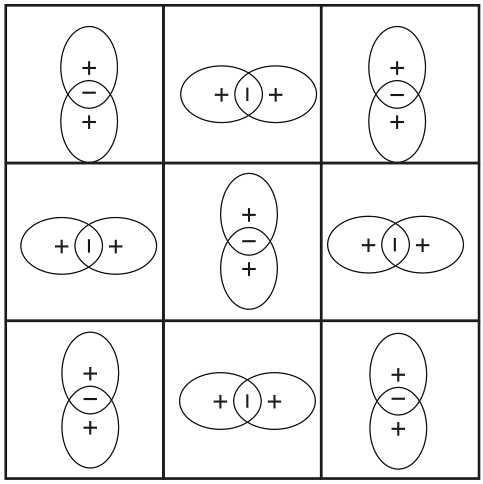
so

$$U_T = \frac{1}{4\pi\epsilon_0}\frac{1}{2}\frac{p^2}{a^3}\left(-6 + \frac{2}{2\sqrt{2}}\right) = \frac{1}{8\pi\epsilon_0}\frac{p^2}{a^3}\left\{\frac{1}{\sqrt{2}} - 6\right\}$$

as required. The energy is independent of the angle α!

Source: V.M. Rozenbaum and V.M. Ogenko, *Soviet Physics Solid State* **26**, 877 (1984).

(b) A point charge representation of each N_2 molecule is $+ - +$. A qualitative argument to find the preferred orientation focuses on maximizing the Coulomb attraction between molecules on the four nearest-neighbor sites. This suggests that the most favorable arrangement is the following.



Source: L. Mederos, E. Chacón, and P. Tarazona, *Physical Review B* **42**, 8571 (1990).

4.14 Practice with Cartesian Multipole Moments

(a) The total charge $Q = 0$. There is no dipole moment because the charge is distributed symmetrically about the origin. The components of the quadrupole moment tensor are

$$Q_{ij} = \tfrac{1}{2}\int d^3r\, r_i r_j \rho(\mathbf{r}).$$

Since

$$\begin{aligned}\rho(x,y) &= q\delta(z)\left\{\delta(x-a)\delta(y-a)+\delta(x+a)\delta(y+a)\right\}\\ &\quad - q\delta(z)\left\{\delta(x-a)\delta(y+a)+\delta(x-a)\delta(y+a)\right\},\end{aligned}$$

all four terms contribute equally to both Q_{xy} and Q_{yx}. In detail,

$$\mathbf{Q} = 2qa^2\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(b) The total charge is $q = 2\lambda\ell$. The dipole moment is zero because the charge is symmetrical around the origin. All $Q_{ij} = 0$ except

$$Q_{zz} = \tfrac{1}{2}\lambda\int_{-\ell}^{\ell} dz\, z^2 = \tfrac{1}{3}\lambda\ell^3.$$

(c) The charge density is $\rho(x,y,z)=\lambda\delta(z)\delta(r-R)$ in cylindrical coordinates. The total charge is $q=2\pi R\lambda$ trivially. The dipole moment is zero because the charge is symmetrically distributed. The charge lies entirely in the x-y plane so $Q_{xz}=Q_{yz}=Q_{zz}=0$. This leaves only

$$Q_{xy}=\tfrac{1}{2}\lambda\int_0^{2\pi}d\theta\sin\theta\cos\theta\int_0^{+\infty}dr\,r^3\,\delta(r-R)=0$$

$$Q_{xx}=Q_{yy}=\tfrac{1}{2}\lambda\int_0^{2\pi}d\theta\cos^2\theta\int_0^{\infty}drr^3\delta(r-R)=\tfrac{1}{2}\lambda\pi R^3.$$

4.15 The Many Faces of a Quadrupole

(a) The components of the primitive electric quadrupole moment tensor are

$$Q_{ij}=\tfrac{1}{2}\int d^3r\,r_ir_j\rho(\mathbf{r}).$$

Since

$$\begin{aligned}\rho(x,y) &= q\delta(z)\left\{\delta(x-a)\delta(y-a)+\delta(x+a)\delta(y+a)\right\}\\ &\quad - q\delta(z)\left\{\delta(x-a)\delta(y+a)+\delta(x-a)\delta(y+a)\right\},\end{aligned}$$

all four terms contribute equally to both Q_{xy} and Q_{yx}. Therefore,

$$\mathbf{Q}=2qa^2\begin{bmatrix}0&1&0\\1&0&0\\0&0&0\end{bmatrix}.$$

The potential produced by this quadrupole is

$$\varphi(\mathbf{r})=Q_{ij}\frac{3r_ir_j-\delta_{ij}r^2}{r^5}=12qa^2\frac{xy}{(x^2+y^2)^{5/2}}.$$

(b) The primitive quadrupole tensor is $Q=2qa^2(\hat{x}\hat{y}+\hat{y}\hat{x})$.

(c) Writing the Cartesian unit vectors in terms of the spherical polar unit vectors gives

$$\begin{aligned}\hat{x} &= \sin\theta\cos\phi\hat{r}+\cos\theta\cos\phi\hat{\theta}-\sin\phi\hat{\phi}\\ \hat{y} &= \sin\theta\sin\phi\hat{r}+\cos\theta\sin\phi\hat{\theta}+\cos\phi\hat{\phi}\\ \hat{z} &= \cos\theta\hat{r}-\sin\theta\hat{\theta}.\end{aligned}$$

Substituting these into part (b) and simplifying yields the nine matrix elements of Q in spherical polar coordinates:

$$Q_{rr}=4qa^2\sin^2\theta\cos\phi\sin\phi$$

$$Q_{\theta\theta} = 4qa^2 \cos^2\theta \cos\phi \sin\phi$$

$$Q_{\phi\phi} = -4qa^2 \sin\phi\cos\phi$$

$$Q_{r\theta} = Q_{\theta r} = 4qa^2 \sin\theta\cos\theta\sin\phi\cos\phi$$

$$Q_{r\phi} = Q_{\phi r} = 2qa^2 \sin\theta(\cos^2\phi - \sin^2\phi)$$

$$Q_{\theta\phi} = Q_{\phi\theta} = 2qa^2 \cos\theta(\cos^2\phi - \sin^2\phi).$$

(d) Since $\mathbf{r} = r\hat{r}$ in spherical polar coordinates, all terms involving r_θ and r_ϕ will vanish. Thus

$$\varphi_Q(\mathbf{r}) = Q_{ij}\frac{3r_ir_j - \delta_{ij}r^2}{r^5} = Q_{rr}\frac{2r^2}{r^5} - Q_{\theta\theta}\frac{r^2}{r^5} - Q_{\phi\phi}\frac{r^2}{r^5}.$$

(e) Substituting the appropriate results from part (c) into part (d) gives

$$\varphi_Q(\mathbf{r}) = 4qa^2\frac{r^2}{r^5}\sin\phi\cos\phi(2\sin^2\theta - \cos^2\theta + 1).$$

Since $x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$, the electric potential is

$$\varphi_Q(\mathbf{r}) = 12qa^2\frac{xy}{(x^2+y^2)^{5/2}},$$

the same as in part (a).

(f) The definition of the components of the quadrupole tensor in part (a) is valid in Cartesian coordinates only.

Source: Prof. R. Grigoriev, Georgia Institute of Technology (private communication).

4.16 Properties of a Point Electric Quadrupole

(a)

$$\begin{aligned}\varphi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int d^3r'\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi\epsilon_0}\int d^3r'\frac{1}{|\mathbf{r}-\mathbf{r}'|}Q_{ij}\partial_i'\partial_j'\delta(\mathbf{r}'-\mathbf{r}_0)\\ &= \frac{1}{4\pi\epsilon_0}\int d^3r'\delta(\mathbf{r}'-\mathbf{r}_0)Q_{ij}\partial_i'\partial_j'\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{4\pi\epsilon_0}Q_{ij}\partial_i\partial_j\int d^3r'\delta(\mathbf{r}'-\mathbf{r}_0)\frac{1}{|\mathbf{r}-\mathbf{r}'|}\\ &= \frac{1}{4\pi\epsilon_0}Q_{ij}\partial_i\partial_j\frac{1}{|\mathbf{r}-\mathbf{r}_0|}.\end{aligned}$$

This is a quadrupole potential so $\rho(\mathbf{r})$ is correct as stated.

(b) $\mathbf{F} = \int d^3r\rho(\mathbf{r})\mathbf{E}(\mathbf{r}) = Q_{ij}\int d^3r\,\mathbf{E}(\mathbf{r})\partial_i\partial_j\delta(\mathbf{r}-\mathbf{r}_0) = Q_{ij}\partial_i\partial_j\mathbf{E}(\mathbf{r}_0).$

(c) The torque is $\boldsymbol{\tau} = \int d^3r\, \mathbf{r} \times \rho(\mathbf{r})\mathbf{E}(\mathbf{r})$ so

$$\begin{aligned}
\tau_i &= \epsilon_{ijk} Q_{m\ell} \int d^3r\, r_j E_k \partial_m \partial_\ell \delta(\mathbf{r}-\mathbf{r}_0) \\
&= \epsilon_{ijk} Q_{m\ell} \int d^3r\, \delta(\mathbf{r}-\mathbf{r}_0) \partial_m \partial_\ell (r_j E_k) \\
&= \epsilon_{ijk} Q_{m\ell} \int d^3r\, \delta(\mathbf{r}-\mathbf{r}_0) \left\{\delta_{j\ell}\partial_m E_k + \delta_{mj}\partial_\ell E_k + r_j \partial_m \partial_\ell E_k\right\} \\
&= \epsilon_{ijk} Q_{mj} \partial_m E_k(\mathbf{r}_0) + \epsilon_{ijk} Q_{j\ell} \partial_\ell E_k(\mathbf{r}_0) + (\mathbf{r} \times \mathbf{F})_i,
\end{aligned}$$

where $\mathbf{F}$ is given by part (b). Finally, $Q_{mj} = Q_{jm}$ so the total torque can be written as

$$\mathbf{N} = 2(\mathbf{Q} \cdot \nabla) \times \mathbf{E} + \mathbf{r} \times \mathbf{F},$$

where $(\mathbf{Q} \cdot \nabla)_i = Q_{ij} \nabla_j$.

(d) $V_E = \int d^3r\, \varphi(\mathbf{r})\rho(\mathbf{r}) = Q_{ij} \int d^3r\, \delta(\mathbf{r}-\mathbf{r}_0)\partial_i \partial_j \varphi(\mathbf{r}) = -Q_{ij}\partial_i E_j(\mathbf{r}_0)$.

4.17 Interaction Energy of Nitrogen Molecules

The leading contribution to the interaction energy may be calculated by treating each molecule as a point quadrupole. The potential produced by molecule A is

$$\varphi_A(\mathbf{r}) = \frac{1}{2} Q^A_{ij} \nabla_i \nabla_j \frac{1}{|\mathbf{r}-\mathbf{r}_A|}.$$

The charge density associated with molecule B is

$$\rho_B(\mathbf{r}) = Q^B_{km} \nabla_k \nabla_m \delta(\mathbf{r}-\mathbf{r}_B).$$

Therefore, the interaction between the two is

$$\begin{aligned}
V_E &= \int d^3r \rho_B(\mathbf{r}) \varphi_A(\mathbf{r}) \\
&= \frac{1}{4\pi\epsilon_0} Q^A_{ij} Q^B_{k\ell} \nabla_i \nabla_j \nabla_k \nabla_\ell \frac{1}{|\mathbf{r}_A - \mathbf{r}_B|} \\
&= \frac{1}{8\pi\epsilon_0} Q^A_{ij} Q^B_{km} \int d^3r\, \nabla_k \nabla_m \delta(\mathbf{r}-\mathbf{r}_B) \nabla_i \nabla_j \frac{1}{|\mathbf{r}-\mathbf{r}_A|} \\
&= \frac{1}{8\pi\epsilon_0} Q^A_{ij} Q^B_{km} \int d^3r\, \delta(\mathbf{r}-\mathbf{r}_B) \nabla_k \nabla_m \nabla_i \nabla_j \frac{1}{|\mathbf{r}-\mathbf{r}_A|} \\
&= \frac{1}{8\pi\epsilon_0} Q^A_{ij} Q^B_{km} \nabla^B_k \nabla^B_m \nabla^B_i \nabla^B_j \frac{1}{|\mathbf{r}_B - \mathbf{r}_A|}.
\end{aligned}$$

This shows that the interaction energy varies as R^{-5} where $R = |\mathbf{r}_A - \mathbf{r}_B|$.

4.18 A Black Box of Charge

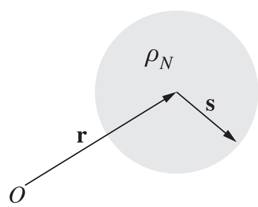
Place a point charge at the center of the box. This gives an $\ell = 0$ multipole and no others. Now take a point charge q and surround it by a spherical shell of uniformly distributed surface charge which integrates to $-q$. This point-plus-shell (PPS) object produces no electric field outside of itself. Therefore, placing any number of these PPS objects in the box away from the exact center produces a non-spherically symmetric charge distribution (with respect to the center of the box) with the desired property.

Source: Prof. Scott Tremaine, Institute for Advanced Study (private communication).

4.19 Foldy's Formula

(a) As indicated in the figure below, we choose an arbitrary origin O and locate the center of the neutron charge distribution at $\mathbf{r}$. In that case, the electrostatic interaction energy is

$$V_E(\mathbf{r}) = \int d^3s\, \rho_N(\mathbf{s})\varphi(\mathbf{r}+\mathbf{s}).$$



(b) When $\varphi(\mathbf{r})$ varies slowly over the size of $\rho_N(\mathbf{s})$, a Taylor series expansion is appropriate:

$$\varphi(\mathbf{r}+\mathbf{s}) = \varphi(\mathbf{r}) + s_k\partial_k\varphi(\mathbf{r}) + \frac{1}{2}s_j s_k\varphi(\mathbf{r}) + \cdots .$$

Substituting this above with $\rho_N(\mathbf{s}) = \rho_N(s)$ gives

$$\begin{aligned} V_E(\mathbf{r}) &= \left[\int d^3s\, \rho_N(s)\right]\varphi(\mathbf{r}) + \left[\int d^3s\, s_k\rho(s)\right]\mathbf{s}\cdot\nabla\varphi(\mathbf{r}) \\ &\quad + \frac{1}{2}\left[\int d^3s\, s_j s_k\rho_N(s)\right]\partial_j\partial_k\varphi(\mathbf{r}) + \cdots . \end{aligned}$$

The first bracketed integral is zero because the neutron has no charge. The second bracketed integral is zero because ρ_N is spherically symmetric. For the same reason, only the $j = k$ terms survive in the third bracketed integral. Hence,

$$V_E(\mathbf{r}) = \frac{1}{2}\left[\int d^3s\, s_k^2\rho_N(s)\right]\nabla_k^2\varphi(\mathbf{r}) = \frac{1}{2}\left[\int d^3s\, \frac{s^2}{3}\rho_N(s)\right]\nabla^2\varphi(\mathbf{r}).$$

The last equality follows because the spherical symmetry of $\rho_N(s)$ implies that the integrals with s_x^2, s_y^2, and s_z^2 are all equal to $1/3$ of the integral with $s^2 = s_x^2 + s_y^2 + s_z^2$. Finally, Poisson's equation for the electrostatic potential of the electron is

$$\epsilon_0 \nabla^2 \varphi = -e\delta(\mathbf{r} - \mathbf{r_0}).$$

Therefore, integrating $V_E(\mathbf{r})$ over all of space gives the desired formula.

Source: L.L. Foldy, *Reviews of Modern Physics* **30**, 471 (1958).

4.20 Practice with Spherical Multipoles

(a) The volume charge density of the shell is

$$\rho = \sigma_0 \delta(r - R) \sin\theta \cos\phi = \sigma_0 \sqrt{\frac{2\pi}{3}} \delta(r - R) \left[Y_{1,-1}(\theta, \phi) - Y_{1,1}(\theta, \phi) \right].$$

If $B = \sqrt{2\pi/3}$, the orthonormality of the spherical harmonics gives the exterior multipole moments as

$$\begin{aligned}
A_{\ell m} &= \frac{4\pi}{2\ell + 1} \int d^3 r \, \rho(\mathbf{r}) r^\ell Y_{\ell m}^*(\theta, \phi) \\
&= \frac{4\pi\sigma_0}{2\ell + 1} B \int_0^\infty dr \, r^{2+\ell} \delta(r - R) \int d\Omega \left\{ Y_{1,-1}(\Omega) - Y_{1,1}(\Omega) \right\} Y_{\ell m}^*(\Omega) \\
&= \frac{4\pi\sigma_0}{3} R^3 B \delta_{\ell,1} \left\{ \delta_{m,-1} - \delta_{m,1} \right\}.
\end{aligned}$$

(b) The potential outside the sphere is

$$\begin{aligned}
\varphi(r > R, \theta, \phi) &= \frac{1}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} \frac{Y_{\ell m}(\theta, \phi)}{r^{\ell+1}} \\
&= \frac{1}{4\pi\epsilon_0} \frac{4\pi\sigma_0}{3} \frac{R^3}{r^2} B \left\{ Y_{1,-1}(\theta, \phi) - Y_{1,1}(\theta, \phi) \right\} \\
&= \frac{\sigma_0}{3\epsilon_0} \frac{R^3}{r^2} \sin\theta \cos\phi \\
&= \frac{\sigma_0 R^3}{3\epsilon_0} \frac{x}{r^3}.
\end{aligned}$$

(c) The charge density is real so $\rho(\mathbf{r}) = \rho^*(\mathbf{r})$ and we can compute the interior spherical multipole moments from

$$\begin{aligned}
B_{\ell m} &= \frac{4\pi}{2\ell + 1} \int d^3 r \, \frac{\rho^*(\mathbf{r})}{r^{\ell+1}} Y_{\ell m}(\theta, \phi) \\
&= \frac{4\pi\sigma_0}{2\ell + 1} B \int_0^\infty dr \, r^{1-\ell} \delta(r - R) \int d\Omega \left\{ Y_{1,-1}^*(\Omega) - Y_{1,1}^*(\Omega) \right\} Y_{\ell m}^*(\Omega) \\
&= \frac{4\pi\sigma_0}{3} B \delta_{\ell,1} \left\{ \delta_{m,-1} - \delta_{m,1} \right\}.
\end{aligned}$$

(d) The potential inside the sphere is

$$\begin{aligned}\varphi(r<R,\theta,\phi) &= \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} B_{\ell m} r^\ell Y^*_{\ell m}(\theta,\phi) \\ &= \frac{1}{4\pi\epsilon_0}\frac{4\pi\sigma_0}{3} rB\left\{Y^*_{1,-1}(\theta,\phi) - Y^*_{1,1}(\theta,\phi)\right\} \\ &= \frac{\sigma_0}{3\epsilon_0} r\sin\theta\cos\phi \\ &= \frac{\sigma_0}{3\epsilon_0}x.\end{aligned}$$

(e) The potential is continuous at $r = R$ as it should be. The tangential components of the electric field are similarly continuous at $r = R$. As for the normal component of the electric field, direct calculation shows that this matching condition is also satisfied:

$$\begin{aligned}\left[\frac{\partial\varphi_<}{\partial r} - \frac{\partial\varphi_>}{\partial r}\right]_{r=R} &= \left[\frac{\sigma_0}{3\epsilon_0} - \frac{-2\sigma_0 R^3}{3\epsilon_0 r^3}\right]_{r=R}\sin\theta\cos\phi \\ &= \frac{\sigma_0}{\epsilon_0}\sin\theta\cos\phi \\ &= \frac{\sigma(\theta,\phi)}{\epsilon_0}.\end{aligned}$$

(f) A general electric dipole potential is

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}.$$

Comparing this with results of (b) shows that the shell carries an electric dipole moment $\mathbf{p} = \frac{1}{3}QR\hat{\mathbf{x}}$ where $Q = 4\pi R^2\sigma_0$ is the total charge of the shell.

4.21 Proof by Interior Multipole Expansion

Choose the origin at the center of a charge-free spherical sub-volume. All the source charge must be outside the sphere so an interior multipole expansion for points inside the sphere is

$$\varphi(r,\theta,\varphi) = \frac{1}{4\pi\epsilon_0}\sum_{\ell m} B_{\ell m} r^\ell Y^*_{\ell m}(\theta,\varphi)$$

where

$$B_{\ell m} = \frac{4\pi}{2\ell+1}\int d^3r\,\frac{\rho(\mathbf{r})}{r^{\ell+1}}Y_{\ell m}(\theta,\phi).$$

Using $Y_{00} = 1/ = \sqrt{4\pi}$ and the orthonormality of the spherical harmonics, the desired average is

$$\langle\varphi\rangle_S = \frac{1}{4\pi}\int d\Omega\,\varphi(R,\Omega) = \frac{\sqrt{4\pi}}{4\pi}\int d\Omega\,Y_{00}(\Omega)\frac{1}{4\pi\epsilon_0}\sum_{\ell m} B_{\ell m}R^\ell Y^*_{\ell m}(\theta,\varphi) = \frac{\sqrt{4\pi}}{4\pi}\frac{1}{4\pi\epsilon_0}B_{00}.$$

Rewriting this slightly gives the desired result:

$$\langle\varphi\rangle = \frac{\sqrt{4\pi}}{4\pi}\frac{1}{4\pi\epsilon_0}B_{00}Y_{00}\sqrt{4\pi} = \frac{1}{4\pi\epsilon_0}B_{00}Y_{00} = \varphi(0).$$

4.22 The Potential outside a Charged Disk

(a) The exterior moments are

$$A_{\ell m} = \frac{4\pi}{2\ell+1}\int d^3r\,\rho(\mathbf{r})r^\ell Y^*_{\ell m}(\theta,\phi).$$

Since $Q = \int d^3r\rho(\mathbf{r}) = 4\pi R^2\sigma$, we must have $\rho(r) = (\sigma/r)\delta(\cos\theta)\Theta(R-r)$ as the volume charge density of the disk. This gives

$$A_{\ell m} = \frac{4\pi\sigma}{2\ell+1}\int_{-1}^{1} d(\cos\theta)\delta(\cos\theta)\int_0^{2\pi} d\phi\,Y^*_{\ell m}(\theta,\phi)\int_0^\infty dr\,r^{\ell+1}\Theta(R-r).$$

But

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi\,Y^*_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta)\delta_{m,0}$$

so

$$A_{\ell m} = \sqrt{\frac{4\pi}{2\ell+1}}\frac{\sigma R^{\ell+2}}{\ell+2}2\pi P_\ell(0)\delta_{m,0}.$$

Substituting this into the exterior multipole expansion,

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m}\frac{Y_{\ell m}(\Omega)}{r^{\ell+1}},$$

and using $Y_{\ell 0}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}}P_\ell(\cos\theta)$ gives the desired result,

$$\varphi(r,\theta) = \frac{Q}{4\pi\epsilon_0 r}\sum_{\ell-0}^{\infty}\left(\frac{R}{r}\right)^\ell \frac{2}{\ell+2}P_\ell(0)P_\ell(\cos\theta) \qquad r > R. \tag{1}$$

(b) The potential on the z-axis of the disk is

$$\begin{aligned}\varphi(z) &= \frac{\sigma}{4\pi\epsilon_0}\int_0^{2\pi} d\varphi \int_0^R \frac{dr\,r}{\sqrt{r^2+z^2}} = \frac{\sigma}{2\epsilon_0}\left\{\sqrt{R^2+z^2}-|z|\right\} \\ &= \frac{q}{2\pi\epsilon_0}\frac{z}{R^2}\left\{\sqrt{1+R^2/z^2}-1\right\}. \end{aligned} \tag{2}$$

On the other hand,

$$\sqrt{1+t^2}-1 = \int_0^t ds \frac{s}{\sqrt{1+s^2}} = \sum_{\ell=0}^{\infty} P_\ell(0)\int_0^t ds s^{\ell+1} = \sum_{\ell=0}^{\infty}\frac{t^{\ell+2}}{\ell+2}P_\ell(0)$$

and $P_\ell(1) = 1$. Therefore, (2) is indeed the same as the multipole expansion (1) evaluated at $\theta = 0$.

4.23 Exterior Multipoles for Specified Potential on a Sphere

(a) The general form of an exterior, spherical multipole expansion is given by

$$\varphi(\mathbf{r}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m}\frac{Y_{\ell m}(\Omega)}{r^{\ell+1}} \qquad\qquad r > R,$$

where

$$A_{\ell m} = \frac{4\pi}{2\ell+1}\int \rho(\mathbf{r}')r'^{\ell}Y^{\star}_{\ell m}(\Omega')dV'.$$

On the surface of the sphere,

$$\varphi(R,\Omega) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_{\ell m}\frac{Y_{\ell m}(\Omega)}{R^{\ell+1}}$$

$$\int \varphi(R,\Omega')Y^{\star}_{\ell' m'}(\Omega')d\Omega' = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{A_{\ell m}}{R^{\ell+1}}\int Y_{\ell m}(\Omega')Y^{\star}_{\ell' m'}(\Omega')d\Omega'.$$

The orthonormality of the spherical harmonics gives the expansion coefficients as

$$A_{\ell m} = R^{\ell+1}\int \varphi(R,\Omega')Y^{\star}_{\ell' m'}(\Omega')d\Omega'.$$

Thus

$$\varphi(\mathbf{r}) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left(\frac{R}{r}\right)^{\ell+1} Y_{\ell m}(\Omega)\int \varphi(R,\Omega')Y^{\star}_{\ell' m'}(\Omega')d\Omega' \qquad\qquad r > R.$$

(b) By examining figure (b), it is clear that we need the potential to change signs every time ϕ is an integer multiple of $\pi/2$. Thus, $m = \pm 2$, which in turn implies that $\ell \geq 2$. For the asymptotic form of the potential we need only keep the lowest value of ℓ necessary. Examining figure (a), we can see that the potential must change signs every time θ is an integer multiple of $\pi/2$ as well. It is clear that $Y_{22} \propto \sin^2\theta$ does not satisfy this requirement. However, $Y_{32} \propto \sin^2\theta\cos\theta \propto \cos\theta - \cos 3\theta$ does. Thus, the potential must be a linear combination of $Y_{3\pm 2}$ and must equal $\pm V$ at $r = R$. That is,

$$\varphi(\mathbf{r}) = V\left(\frac{R}{r}\right)^4 2\sqrt{\frac{2\pi}{105}}\,(Y_{32} + Y_{3-2}) = V\left(\frac{R}{r}\right)^4 \sin^2\theta\cos\theta\cos 2\phi, \qquad r \to \infty.$$

4.24 A Hexagon of Point Charges

(a) Choose the origin of coordinates $\mathbf{r} = 0$ at the center of the hexagon. The charges q_α ($\alpha = 1, \ldots, 6$) are positioned at $\mathbf{r} = \mathbf{r}_\alpha$. Using the geometry of a hexagon, we label the (x, y) position of each charge beginning with the topmost and proceeding clockwise:

q	$-q$	q	$-q$	q	$-q$
$(0, a)$	$(x, a/2)$	$(x, -a/2)$	$(0, -a)$	$(-x, -a/2)$	$(-x, a/2)$

Then, by direct computation,

$$Q = \sum_\alpha q_\alpha = q - q + q - q + q - q = 0.$$

The dipole component $p_z = 0$ because $z = 0$ for all the charges. Otherwise,

$$p_x = \sum_\alpha q_\alpha x_\alpha = 0 - qx + qx + 0 - qx + qx = 0$$

$$p_y = \sum_\alpha q_\alpha y_\alpha = qa - qa/2 - qa/2 + qa - qa/2 - qa/2 = 0.$$

The quadrupole matrix is symmetric and $Q_{zz} = Q_{xz} = Q_{yz} = 0$ because $z = 0$ for all the charges. Otherwise,

$$Q_{xy} \propto \sum_\alpha q_\alpha x_\alpha y_\alpha = 0 - qxa/2 - qxa/2 + 0 + qxa/2 + qxa/2 = 0$$

$$Q_{xx} \propto \sum_\alpha q_\alpha x_\alpha^2 = 0 - qx^2 + qx^2 + 0 + qx^2 - qx^2 = 0$$

$$Q_{yy} \propto \sum_\alpha q_\alpha y_\alpha^2 = qa^2 - qa^2/4 + qa^2/4 - qa^2/4 - qa^2 + qa^2/4 - qa^2/4 = 0.$$

However, the next (octupole) moment has non-zero components. An example is

$$O_{yyy} \propto \sum_\alpha q_\alpha x_\alpha^3 = qa^3 - qa^3/8 - qa^3/8 + qa^3 - qa^3/8 - qa^3/8 = 3qa^3/2.$$

(b) The potential of a monopole, dipole, and quadrupole vary as $\varphi \propto r^{-1}$, $\varphi \propto r^{-2}$, and $\varphi \propto r^{-3}$. Therefore, the next term in the expansion must behave as

$$\varphi(r) \propto r^{-4}.$$

4.25 Analyze This Potential

(a) The charge distribution must have (i) zero net charge; (ii) no dipole moment; and (iii) a quadrupole potential with only the single term Bx^2/r^5. Consider the *traceless* multipole expansion,

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + \Theta_{ij}\frac{r_i r_j}{r^5} + \cdots\right],$$

and focus on the quadrupole term. The desired term is $\Theta_{xx}x^2/r^5$, so we must have $\Theta_{xx} \neq 0$. However, the traceless condition is

$$\Theta_{xx} + \Theta_{yy} + \Theta_{zz} = 0.$$

Therefore, we must have $\Theta_{yy} \neq 0$ or $\Theta_{zz} \neq 0$ or both. In other words, quadrupole terms like y^2/r^5 or z^2/r^5 or both must also be present if x^2/r^5 is present. We conclude that no charge distribution can produce the stated potential.

(b) We can also use the *primitive* Cartesian multipole expansion,

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{Q}{r} + \frac{\mathbf{p}\cdot\mathbf{r}}{r^3} + Q_{ij}\frac{3r_i r_j - r^2\delta_{ij}}{r^5} + \cdots\right].$$

We eliminate the terms with no factor of x^2/r^5 by requiring that $Q_{xy} = Q_{xz} = Q_{yz} = 0$. The desired factor x^2/r^5 appears in the remaining diagonal terms,

$$\varphi_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{Q_{xx}(2x^2 - y^2 - z^2) + Q_{yy}(2y^2 - x^2 - z^2) + Q_{zz}(2z^2 - x^2 - y^2)}{r^5}.$$

Rearranging this gives

$$\varphi_{\text{quad}} = \frac{1}{4\pi\epsilon_0}\frac{x^2(2Q_{xx} - Q_{yy} - Q_{zz}) + y^2(2Q_{yy} - Q_{zz} - Q_{xx}) + z^2(2Q_{zz} - Q_{xx} - Q_{yy})}{r^5}.$$

We want to eliminate the y^2/r^5 and z^2/r^5 terms. This means that

$$2Q_{yy} = Q_{xx} + Q_{zz} \qquad \text{and} \qquad 2Q_{zz} - Q_{yy} - Q_{zz}.$$

The only solution to these equations in $Q_{xx} = Q_{yy} = Q_{zz}$. However, this makes the coefficient of the x^2/r^5 term zero also. Therefore, once again, we conclude that there is no charge distribution with a quadrupole potential of the form Bx^3/r^5.

(c) If we permit charge to extend to any point in space, a charge distribution which produces the stated potential can always be found from the Poisson equation,

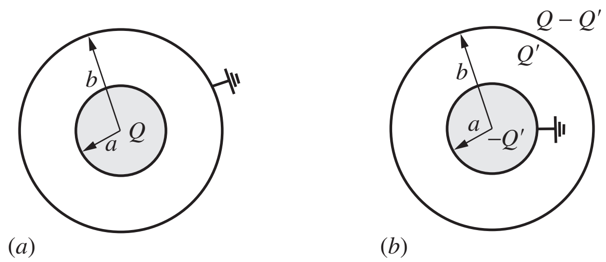
$$\rho(\mathbf{r}) = \epsilon_0\nabla^2\varphi(\mathbf{r}).$$

Chapter 5: Conductoring Matter

5.1 A Conductor with a Cavity

Earnshaw's theorem applies to the cavity because it is free of charge. The potential on the conductor surface which defines the cavity is everywhere a constant φ_0 because the potential is a constant at all points of a conductor. But Earnshaw's theorem says that the potential has no local minimum or local maximum inside the cavity. The only possibility is that $\varphi(\mathbf{r}) = \varphi_0$ at every point in the cavity. Therefore, $\mathbf{E} = 0$ everywhere in the cavity.

5.2 Two Spherical Capacitors



(a) The electric field between the conductors is determined by Gauss' law to be $\mathbf{E} = Q\hat{\mathbf{r}}/4\pi\epsilon_0 r^2$. Therefore,

$$\varphi(a) - \varphi(b) = \int_a^b d\mathbf{s} \cdot \mathbf{E} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right).$$

Therefore,

$$C = \frac{Q}{\Delta\varphi} = 4\pi\epsilon_0 \frac{ab}{b-a}.$$

(b) We will assume (and then verify) that the presence of Q on the shell draws up a charge $-Q'$ onto the ball from ground. This induces a charge Q' on the inner surface of the shell. This leaves a charge $Q - Q'$ on the outer surfaces of the shell so the total charge of the shell is Q. Both shells contribute a constant to the potential at the surface of the ball. Since the latter is grounded, we get

$$-\frac{Q'}{a} + \frac{Q}{b} = 0.$$

This shows that $Q' \neq 0$. Now, the potential infinitesimally outside the shell is

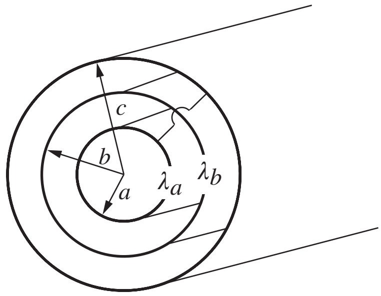
$$\varphi(b) = \frac{1}{4\pi\epsilon_0}\frac{Q - Q'}{b} = \frac{Q}{4\pi\epsilon_0}\frac{b-a}{b^2}.$$

Moreover $\varphi(a) = 0$, so

$$C = \frac{Q}{\Delta\varphi} = 4\pi\epsilon_0 \frac{b^2}{b-a}.$$

Source: J.H. Jeans, *The Mathematical Theory of Electricity and Magnetism* (University Press, Cambridge, 1925).

5.3 Concentric Cylindrical Shells



We ignore fringe effects and treat the cylinders as infinitely long. The inner and outer cylinders share charge, but it is enough to focus on the charge per unit length λ_a which happens to reside on the inner cylinder. By Gauss' law, the electric field between the inner and middle cylinders is $\mathbf{E} = \lambda_a \hat{\boldsymbol{\rho}}/2\pi\epsilon_0 \rho$. Therefore,

$$\varphi(b) - \varphi(a) = \frac{\lambda_a}{2\pi\epsilon_0} \int\limits_a^b \frac{d\rho}{\rho} = -\frac{\lambda_a}{2\pi\epsilon_0} \ln\frac{b}{a}.$$

Similarly,

$$\varphi(c) - \varphi(b) = \frac{\lambda_a + \lambda_b}{2\pi\epsilon_0} \ln\frac{c}{b}.$$

But $\varphi(a) = \varphi(c)$ so adding the previous two equations gives

$$\lambda_a \ln\frac{c}{a} = -\lambda_b \ln\frac{c}{b}.$$

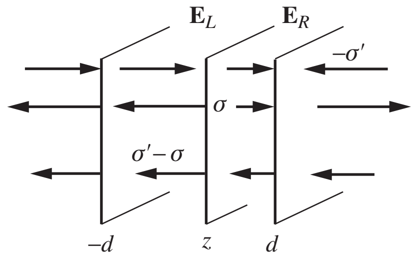
Therefore,the capacitance per unit length of this structure is

$$C = \frac{\lambda_b}{\varphi(b) - \varphi(a)} = \frac{\lambda_b}{-\frac{\lambda_a}{2\pi\epsilon_0}\ln(b/a)} = \frac{2\pi\epsilon_0}{\ln(b/a)}\frac{\ln(c/a)}{\ln(c/b)}.$$

Source: V.C.A. Ferraro, *Electromagnetic Theory* (Athlone Press, London, 1954).

5.4 A Charged Sheet between Grounded Planes

Neutralizing charge of opposite sign to σ is brought up from ground but we don't know how it is distributed between the two grounded planes. Put $-\sigma'$ on the $z = d$ plane and the remainder on the $z = -d$ plane. The figure below shows a representative field line for the uniform field produced by each of these three sources. Note that the sum of the fields from all three correctly gives zero when $z < -d$ and when $z > d$.



(a) Let $\mathbf{E}_L$ be the total field between $z = -d$ and the charged sheet and let $\mathbf{E}_R$ be the total field between the charged sheet and $z = d$. Clearly,

$$\mathbf{E}_L = \frac{1}{\epsilon_0}\left(\frac{\sigma'}{2} - \frac{\sigma}{2} - \frac{\sigma - \sigma'}{2}\right)\hat{\mathbf{z}} = \frac{\sigma' - \sigma}{\epsilon_0}\hat{\mathbf{z}}$$

$$\mathbf{E}_R = \frac{1}{\epsilon_0}\left(\frac{\sigma}{2} + \frac{\sigma'}{2} - \frac{\sigma - \sigma'}{2}\right)\hat{\mathbf{z}} = \frac{\sigma'}{\epsilon_0}\hat{\mathbf{z}}.$$

By definition of the electrostatic potential,

$$\varphi(-d) - \varphi(z) = \int_{-d}^{z} d\hat{\mathbf{z}} \cdot \mathbf{E}_L = \frac{\sigma' - \sigma}{\epsilon_0}(z + d) = -\varphi(z)$$

and

$$\varphi(z) - \varphi(d) = \int_{z}^{d} d\hat{\mathbf{z}} \cdot \mathbf{E}_R = \frac{\sigma'}{\epsilon_0}(d - z) = \varphi(z).$$

Equating the two expressions for $\varphi(z)$ gives

$$(\sigma - \sigma')(z + d) = \sigma'(d - z) \qquad \rightarrow \qquad \sigma' = \sigma\frac{z + d}{2d}.$$

(b) The force per unit area on the sheet of charge is

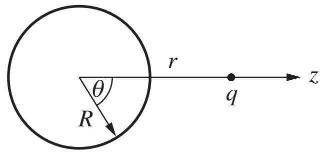
$$\mathbf{f} = \frac{1}{2}\sigma(\mathbf{E}_L + \mathbf{E}_R) = \frac{\sigma}{\epsilon_0}(2\sigma' - \sigma)\hat{\mathbf{z}} = \frac{\sigma^2 z}{2\epsilon d}\hat{\mathbf{z}}.$$

Source: P.C. Clemmow, *An Introduction to Electromagnetic Theory* (University Press, Cambridge, 1973).

5.5 The Charge Distribution Induced on a Neutral Sphere

(a) The total electrostatic energy is the sum of the self-energy of the sphere and the interaction energy between the sphere and the point charge. If $\mathbf{r}$ points from the sphere center to the position of q and $\mathbf{r}'$ points to the surface of the sphere,

$$U_E = U_{\rm I} + U_{\rm S} = \frac{q}{4\pi\epsilon_0}\int dS' \frac{\sigma(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{8\pi\epsilon_0}\int dS_1 \int dS_2 \frac{\sigma(\mathbf{r}_1)\sigma(\mathbf{r}_2)}{|\mathbf{r}_1 - \mathbf{r}_2|}.$$



We have $\sigma(\mathbf{r}') = \sum_\ell \sigma_\ell P_\ell(\cos\theta)$ with $\mathbf{r}' = (R,\theta)$. Therefore, since $R < r$,

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \frac{1}{r}\sum_m \left(\frac{R}{r}\right)^m P_m(\cos\theta).$$

$$\begin{aligned} U_\mathrm{I} &= \frac{q}{4\pi\epsilon_0}\frac{2\pi R^2}{r}\sum_{\ell,m}\sigma_\ell\left(\frac{R}{r}\right)^m\int_{-1}^{1} d(\cos\theta)P_\ell(\cos\theta)P_m(\cos\theta) \\ &= \frac{q}{4\pi\epsilon_0}\frac{2\pi R^2}{r}\sum_{\ell,m}\sigma_\ell\frac{2}{2\ell+1}\delta_{\ell m}\left(\frac{R}{r}\right)^m. \end{aligned}$$

Therefore,

$$U_\mathrm{I} = \frac{1}{\epsilon_0}\sum_{\ell=0}^{\infty}\frac{\sigma_\ell}{2\ell+1}\frac{qR^{\ell+2}}{r^{\ell+1}}. \tag{1}$$

To compute the self-energy, we have $r_1 = r_2 = R$ so we use

$$\frac{1}{|\mathbf{r}_1-\mathbf{r}_2|} = \frac{1}{R}\sum_{L=0}^{\infty}\sum_{M=-L}^{L}\frac{4\pi}{2L+1}Y^*_{LM}(\theta_1,\phi_1)Y_{LM}(\theta_2,\phi_2)$$

and the self-energy is

$$\begin{aligned} U_\mathrm{S} &= \frac{R^4}{8\pi\epsilon_0}\sum_{\ell,\ell',L}\frac{4\pi\sigma_\ell\sigma_{\ell'}}{2L+1}\int_{-1}^{1} d(\cos\theta_1)\int_{-1}^{1} d(\cos\theta_2)P_\ell(\cos\theta_1)P_{\ell'}(\cos\theta_2) \\ &\quad\times\frac{1}{R}\sum_{M=-L}^{L}\int_0^{2\pi} d\phi_1 Y^*_{LM}(\theta_1,\phi_1)\int_0^{2\pi} d\phi_2 Y_{LM}(\theta_2,\phi_2). \end{aligned}$$

However,

$$\frac{1}{2\pi}\int_0^{2\pi} d\phi Y_{LM}(\theta,\phi) = \sqrt{\frac{2L+1}{4\pi}}P_L(\cos\theta)\delta_{M,0}$$

and

$$\int_{-1}^{1} dx P_\ell(x)P_{\ell'}(x) = \frac{2}{2\ell+1}\delta_{\ell,\ell'}.$$

Therefore,

$$U_\mathrm{S} = \frac{2\pi R^3}{\epsilon_0}\sum_{\ell=0}^{\infty}\frac{\sigma_\ell^2}{2\ell+1}. \tag{2}$$

(b) To find $\sigma(\theta)$, we minimize $U_E(\sigma_\ell)$ with respect to each σ_ℓ. Adding (1) to (2) gives the total energy so

$$\frac{\partial U_E}{\partial \sigma_\ell} = \frac{1}{\epsilon_0}\sum_{\ell=0}^{\infty}\frac{1}{2\ell+1}\left[R^3\sigma_\ell\frac{4\pi}{2\ell+1}+\frac{qR^{\ell+2}}{r^{\ell+1}}\right] = 0.$$

This is solved by

$$\sigma_\ell = -\frac{(2\ell+1)qR^{\ell-1}}{4\pi r^{\ell+1}},$$

so the charge density induced on the sphere is

$$\sigma(\theta) = -\frac{q}{4\pi R^2}\sum_{\ell=0}^{\infty}(2\ell+1)\left(\frac{R}{r}\right)^{\ell+1}P_\ell(\cos\theta).$$

Source: C. Donolato, *American Journal of Physics* **71**, 1232 (2003).

5.6 Charge Transfer between Conducting Spheres

After connection, the balls have charges Q_1 and $Q - Q_1$. Since R is large, we assume that a point charge potential adequately describes the potential each contributes at the position of the other. Therefore,

$$4\pi\epsilon_0\varphi_1 = \frac{Q_1}{R_1} + \frac{Q-Q_1}{R}$$

$$4\pi\epsilon_0\varphi_2 = \frac{Q_1}{R} + \frac{Q-Q_1}{R_2}.$$

The two balls are connected together, so $\varphi_1 = \varphi_2$. This gives

$$Q_1\left[\frac{1}{R_1}+\frac{1}{R_2}-\frac{1}{R}\right] = Q\left[\frac{1}{R_2}-\frac{1}{R}\right]$$

or

$$Q_1 = \frac{Q(R-R_2)R_1}{(R_1+R_2)R - 2R_1R_2}.$$

This is correct, but does not yet take account of the fact that $R \gg R_1, R_2$. For that purpose, we write

$$Q_1 = \frac{QR_1}{R_1+R_2}\frac{\left(1-\dfrac{R_2}{R}\right)}{\left(1-\dfrac{2R_1R_2}{R_1+R_2}\dfrac{1}{R}\right)} \approx \frac{QR_1}{R_1+R_2}\left[\left(1-\frac{R_2}{R}\right)\left(1+\frac{2R_1R_2}{R_1+R_2}\frac{1}{R}\right)\right].$$

This gives the desired formula if we ignore the term of order R_1R_2/R^2.

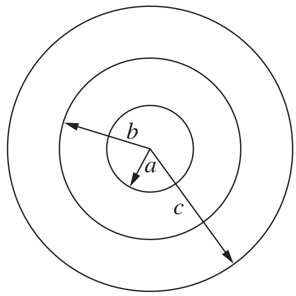
Source: V.C.A. Ferraro, *Electromagnetic Theory* (Athlone Press, London, 1954).

5.7 Concentric Spherical Shells

Before grounding, the potentials of the innermost and outermost shells are

$$\varphi_0(c) = \frac{1}{4\pi\epsilon_0}\frac{e_a + e_b + e_c}{c}$$

$$\varphi_0(a) = \frac{1}{4\pi\epsilon_0}\left[\frac{e_a}{a} + \frac{e_b}{b} + \frac{e_c}{c}\right].$$



When it is grounded, charge Q' flows from ground to the inner shell so that its total potential is zero, i.e.,

$$\varphi(a) = \varphi_0(a) + \frac{Q'}{4\pi\epsilon_0 a} = 0.$$

The charge added to the inner shell contributes to the potential at the position of the outer shell also. Therefore,

$$\varphi(c) - \varphi_0(c) = \frac{Q'}{4\pi\epsilon_0 c} = -\frac{a}{c}\varphi_0(a) = -\frac{1}{4\pi\epsilon_0}\frac{a}{c}\left[\frac{e_a}{a} + \frac{e_b}{b} + \frac{e_c}{c}\right].$$

Source: V.C.A. Ferraro, *Electromagnetic Theory* (Athlone Press, London, 1952).

5.8 Don't Believe Everything You Read in Journals

The electrostatic torque on any charge distribution is

$$\boldsymbol{\tau} = \int d^3r\, \{\mathbf{r} \times \rho(\mathbf{r})\mathbf{E}(\mathbf{r})\}.$$

For the distribution $\sigma(\mathbf{r}_S)$ at the surface of a spherical surface S of a conductor we must use the average of the interior and exterior electric fields. So, with respect to an origin at the center of the sphere,

$$\boldsymbol{\tau} = \frac{1}{2}\int_S dS\, \{\mathbf{r}_S \times \sigma(\mathbf{r}_S)\mathbf{E}_{\rm out}(\mathbf{r}_S)\} = \frac{1}{2\epsilon_0}\int_S dS\, \left\{\mathbf{r}_S \times \sigma^2(\mathbf{r}_S)\hat{\mathbf{n}}(\mathbf{r}_S)\right\} = 0.$$

We get zero because $\mathbf{n} = \mathbf{r}_S$ for a sphere.

Source: K. Hense, M. Tajmar, and K. Marhold, *Journal of Physics A* **37**, 8747 (2004).

5.9 A Dipole in a Cavity

(a) The surface density σ must be chosen to cancel the electric field outside the cavity produced by the point dipole outside the cavity. This will certainly be true if, outside the cavity, the potential due to σ is equal and opposite to the dipole potential produced by the dipole. If we choose $\hat{\mathbf{z}} \parallel \mathbf{p}$, the latter is

$$\varphi_{\mathbf{p}}(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\frac{p}{r^2}\cos\theta.$$

Now, we learned in Application 4.3 that a charge density $\sigma(\theta) = \sigma_0\cos\theta$ on the surface of spherical shell of volume $V = 4\pi R^3/3$ produces a potential

$$\varphi_\sigma = \begin{cases} \dfrac{\sigma_0 z}{3\epsilon_0} & r \leq R \\[2ex] \dfrac{V\sigma_0}{4\pi\epsilon_0}\dfrac{\cos\theta}{r^2} & r \geq R. \end{cases}$$

This shows that we will get the desired field cancellation outside the cavity if

$$\sigma(\mathbf{r}_S) = -\frac{\mathbf{p}\cdot\hat{\mathbf{r}}_S}{V}.$$

(b) The field produced by σ inside the cavity is constant. This produces no force on a point dipole because the general expression for the force on a dipole is $\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E}$.

5.10 Charge Induction by a Dipole

Consider a conductor with charge Q and a volume charge distribution $\rho(\mathbf{r})$. Together, they produce a potential φ_C on the conductor and a potential $\varphi(\mathbf{r})$ elsewhere in space. A comparison system is the same conductor with charge Q' and a volume charge distribution $\rho'(\mathbf{r})$ which together produce a potential φ'_C on the conductor and $\varphi'(\mathbf{r})$ otherwise. Green's reciprocity states that

$$Q\varphi'_C + \int d^3r\,\rho(\mathbf{r})\varphi'(\mathbf{r}) = Q'\varphi_C + \int d^3r\,\rho'(\mathbf{r})\varphi(\mathbf{r}).$$

For this problem, $\varphi_C = 0$ and we are told to choose $\rho'(\mathbf{r}) = 0$. Therefore,

$$Q\varphi'_C + \int d^3r\,\rho(\mathbf{r})\varphi'(\mathbf{r}) = 0. \tag{1}$$

For our problem, $\rho(\mathbf{r}) = -\mathbf{p}\cdot\nabla\delta(\mathbf{r}-\mathbf{r}_0)$ and $\varphi'_C = Q'/4\pi\epsilon_0 R$ with $\varphi(r) = Q'/4\pi\epsilon_0 r$ for points outside the sphere. Substituting these into (1) gives

$$\frac{Q}{R} - \int d^3r\,\frac{\mathbf{p}}{r}\cdot\nabla\delta(\mathbf{r}-\mathbf{r}_0) = 0,$$

or, after integrating by parts,

$$\frac{Q}{R} = -\int d^3r\, \delta(\mathbf{r} - \mathbf{r}_0)\mathbf{p} \cdot \nabla \frac{1}{r} = \int d^3r\, \delta(\mathbf{r} - \mathbf{r}_0) \frac{\mathbf{p} \cdot \hat{\mathbf{r}}}{r^2}.$$

Therefore,

$$Q = (\mathbf{p} \cdot \hat{\mathbf{r}}_0) \frac{R}{r_0^2}.$$

5.11 Charge Induction by a Potential Patch

Label the top plate as conductor 1, the bottom plate minus the finite square region as conductor 2, and the square region itself as conductor 3. We have $\varphi_1 = \varphi_2 = 0$ and $\varphi_3 = V$. Our task is to find $Q = Q_2 + Q_2$. The reciprocity theorem reads

$$\varphi_1 Q_1' + \varphi_2 Q_2' + \varphi_3 Q_3' = \varphi_1' Q_1 + \varphi_2' Q_2 + \varphi_3' Q_3,$$

so

$$V Q_3' = \varphi_1' Q_1 + \varphi_2' Q_2 + \varphi_3' Q_3.$$

We choose the primed, comparison system as a parallel-plate capacitor where $\varphi_1' = 0$ and $\varphi_2' = \varphi_3' = \Phi'$. In that case, the preceding equation reduces to

$$V Q_3' = \Phi' Q.$$

Now, if A is the area of the entire lower plate, and $Q' = Q_2' + Q_2'$, the definition of capacitance tells us that

$$Q' = \frac{\epsilon_0 A}{d} \Phi'.$$

Therefore,

$$Q = \frac{V Q_3'}{\Phi'} = \frac{V \dfrac{Q'}{A}(2a)^2}{\dfrac{Q'd}{\epsilon_0 A}} = \frac{4V\epsilon_0 a^2}{d}.$$

5.12 Charge Sharing among Three Metal Balls

There are many solutions; here is one. Move the ball on the far right to $+\infty$. Bring the other two into contact and bring them (together) nearby to $+Q$. By electrostatic induction, the leftmost ball acquires a charge $-q$. By neutrality, the ball it touches acquires a charge $+q$. Now move the $-q$ ball to the right nearly to $+\infty$. Then move the ball at ∞ to the left and touch it to the $-q$ ball. These two are nearly isolated. Therefore, by symmetry, each will have charge $-q/2$.

5.13 A Conducting Disk

(a) The text gives the capacitance of a conducting disk is $C = 8\epsilon_0 R$. Using this and the charge density computed in the text, we can write the latter in the form

$$\sigma(\rho) = \frac{2\epsilon_0 V}{\pi} \frac{1}{\sqrt{R^2 - \rho^2}}.$$

Since every ring with area $2\pi\rho d\rho$ lies a distance $\sqrt{z^2 + \rho^2}$ from a point z on the axis, and both sides of the disk contribute to the potential,

$$\varphi(z) = \frac{2}{4\pi\epsilon_0} \frac{2\epsilon_0 V}{\pi} \int_0^R \frac{2\pi\rho d\rho}{\sqrt{R^2 - \rho^2}} \frac{1}{\sqrt{z^2 + \rho^2}} = \frac{2V}{\pi} \frac{1}{2} \int_0^{R^2} \frac{dx}{\sqrt{(R^2 - x)(z^2 + x)}} = \frac{2V}{\pi} \tan^{-1} \frac{R}{|z|}.$$

(b) We treat the point charge as a tiny sphere. When the disk has charge Q and potential V, the sphere has charge $q = 0$ and potential $v = \varphi(d)$ from part (a). The comparison has the disk grounded ($V' = 0$) with a charge Q' we wish to determine. The tiny sphere has charge $q' = q_0$ and potential v'. The reciprocity theorem tells us that

$$QV' + qv' = Q'V + q'v$$

or

$$Q \cdot 0 + 0 \cdot v' = Q'V + q_0\varphi(z).$$

Therefore,

$$Q = -\frac{2q_0}{\pi} \tan^{-1} \frac{R}{d}.$$

Source: V.C.A. Ferraro, *Electromagnetic Theory* (Athlone Press, London, 1954).

5.14 The Capacitance of Spheres

(a) The self-capacitance of a sphere is $C = 4\pi\epsilon_0 R$ where R is the sphere radius. The radius of the Earth is $R = 6.4 \times 10^6$ m and $\epsilon_0 = 8.85 \times 10^{-12}$ F/m. Therefore,

$$C \approx 7 \times 10^{-4} \text{ F}.$$

The self-energy of a sphere is $Q^2/2C$. Therefore, the energy difference between a neutral sphere and a sphere with charge e is

$$U = \frac{e^2}{2C} = \frac{(1.6 \times 10^{-19} \text{ C})^2}{2 \times 7 \times 10^{-4} \text{ F}} = 1.8 \times 10^{-35} \text{ J} \approx 10^{-16} \text{ eV}.$$

(b) For the nanometer-sized sphere,

$$C = 4\pi \times 8.85 \times 10^{-12} \text{ F/m} \times 10 \times 10^{-9} \text{ m} \approx 10^{-18} \text{ F}$$

$$U = \frac{e^2}{2C} = \frac{(1.6 \times 10^{-19} \text{ C})^2}{2 \times 10^{-18} \text{ F}} \approx 0.1 \text{ eV}.$$

(c) If the sphere separation is large compared to their radii, each sphere feels the other as a point charge. Therefore,

$$\varphi_A = \frac{Q_A}{4\pi\epsilon_0 R_A} + \frac{Q_B}{4\pi\epsilon_0 R}$$

$$\varphi_B = \frac{Q_B}{4\pi\epsilon_0 R_B} + \frac{Q_A}{4\pi\epsilon_0 R},$$

so

$$P = \frac{1}{4\pi\epsilon_0}\begin{bmatrix} R_A^{-1} & R^{-1} \\ R^{-1} & R_B^{-1} \end{bmatrix}$$

and

$$C = 4\pi\epsilon_0 \begin{bmatrix} R_B^{-1} & -R^{-1} \\ -R^{-1} & R_A^{-1} \end{bmatrix} \frac{R^2 R_A R_B}{R^2 - R_A R_B} \approx \frac{4\pi\epsilon_0}{R}\begin{bmatrix} RR_A & -R_A R_B \\ -R_A R_B & R_B R \end{bmatrix}.$$

(d) The diagonal elements C_{AA} and C_{BB} approach the self-capacitances C_A and C_B in this limit.

5.15 Practice with Green's Reciprocity

(a) We have $q_i = C_{ij}\varphi_j$ and $\tilde{q}_i = C_{ij}\tilde{\varphi}_j$ because the same set of conductors is involved. Therefore,

$$\sum_{i=1}^{N} q_i\tilde{\varphi}_i = \sum_{i=1}^{N}\sum_{j=1}^{N} C_{ij}\varphi_j\tilde{\varphi}_i = \sum_{i=1}^{N}\sum_{j=1}^{N} C_{ji}\tilde{\varphi}_i\varphi_j = \sum_{j=1}^{N}\tilde{q}_j\varphi_j.$$

(b) This is a direct application of the theorem. We get

$$q'\cdot\phi + q\cdot 0 + q\cdot 0 = q\cdot\phi' + q_0\cdot\phi' + q_0\phi'$$

so

$$q' = (q + 2q_0)\frac{\phi'}{\phi}.$$

(c) By symmetry, the potentials are (ϕ'', ϕ_0, ϕ_0) when the charges are $(q'', 0, 0)$. We apply the reciprocity theorem twice, first using $(\phi, 0, 0)$ with (q, q_0, q_0). This gives

$$\phi'' q + 2\phi_0 q_0 = \phi q''.$$

Now use the result of part (b) where we have (ϕ', ϕ', ϕ') with (q', q', q'). This gives

$$(\phi'' + 2\phi_0)q' = q''\phi'.$$

We now have two equations for the two unknowns, ϕ'' and ϕ_0. Eliminating ϕ'' yields

$$\phi_0 = \frac{q''(\phi' - \phi q'/q)}{2q'(1 - q_0/q)} = \frac{q_0 q''}{(q_0 - q)(q + 2q_0)}\phi.$$

Substituting back gives

$$\phi'' = -\frac{(q_0 + q)q''}{(q_0 - q)(q + 2q_0)}\phi.$$

5.16 Maxwell Was Not Always Right

(a) Maxwell discusses four objects: (A) a non-conducting square; (B) a non-conducting rectangle; (C) a conducting rectangle; and (D) a conducting square. It is meaningful to discuss the electrostatic energy U_E for each of them. Capacitance is a meaningful concept for the two conductors, where $U_E = Q^2/2C$. Maxwell argues that $U_E(B) < U_E(A)$ and also that $U_E(C) < U_E(B)$. This shows hat $U_E(C) < U_E(A)$. He doesn't say so explicitly, but his line of argument also implies that $U_E(D) < U_E(A)$. Unfortunately, all these facts **do not** allow you to conclude that $U_E(C) < U_E(D)$. The latter is needed to conclude that $C_{\text{rect}} > C_{\text{sq}}$.

(b) It costs an energy $\delta U_E = (Q/C)\delta Q$ to add charge to a conductor. This number is bigger for a square than for a rectangle because, by the geometry, the added charge is closer to all the other charge on the square than for the rectangle. Hence, $\delta U_E(D) > \delta U_E(C)$ or $C_{\text{rect}} > C_{\text{sq}}$.

5.17 Two-Dimensional Electron Gas Capacitor

The electric field in the upper gap is $E_1 = \sigma_1/\epsilon_0$ directed downward. The electric field in the lower gap is $E_2 = \sigma_2/\epsilon_0$ directed upward. Since the electrostatic energy density is $u_E = \frac{1}{2}\epsilon_0 \mathbf{E}\cdot\mathbf{E}$, the total energy per unit area of the system is

$$u = \tfrac{1}{2}\epsilon_0(d - L)|E_1|^2 + \tfrac{1}{2}\epsilon_0 L|E_2|^2 + \ u_0.$$

Since σ_2 is the independent variable, the minimum energy requirement is

$$0 = \frac{d}{d\sigma_2}\left\{\frac{L\sigma_2^2}{2\epsilon_0} + \frac{\pi\hbar^2(\sigma_1 + \sigma_2)^2}{2me^2}\right\} = \frac{L\sigma_2}{\epsilon_0} + \frac{\pi\hbar^2}{me^2}(\sigma_1 + \sigma_2).$$

This gives $\sigma_2 = -\sigma_1 \dfrac{C_2}{C_2 + C_0}$ as required, with $C_2 = \dfrac{A\epsilon_0}{L}$ and $C_0 = \dfrac{Ame^2}{\pi\hbar^2}$. The classical limit is $\hbar \to 0$, so

$$C_0 \to \infty \text{ and } \sigma_2 \to 0.$$

This is the answer when the two-dimensional sheet is replaced by a perfect conductor, an example when the energy/area of the "sheet" $u_0 \to 0$.

Source: S. Luryi, *Applied Physics Letters* **52**, 501 (1988).

5.18 Two Pyramidal Conductors

(a) For a general two-conductor system, the symmetry of the P_{ij} implies that

$$U_E = \frac{1}{2}\left[Q_1^2 P_{11} + 2Q_1Q_2P_{12} + Q_2^2 P_{22}\right].$$

The charge transfer implies that $\delta Q_1 = \delta Q$ and $\delta Q_2 = -\delta Q$. Therefore,

$$\delta U_E = \frac{1}{2}\left[2Q_1\delta Q P_{11} + 2Q_2\delta Q P_{12} - 2Q_1\delta Q P_{12} - 2Q_2\delta Q P_{22}\right].$$

But $Q_1 = Q_2$ so $\delta U_E = Q\delta Q(P_{11} - P_{22})$. We conclude that $\delta U < 0$ requires $P_{22} > P_{11}$.

(b) The **P** matrix is the inverse of the symmetric capacitance matrix **C**. Therefore,

$$\begin{pmatrix} C_{11} & C_{12} \\ C_{12} & C_{22} \end{pmatrix}\begin{pmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This gives

$$C_{11}P_{12} + C_{12}P_{22} = 0$$
$$C_{12}P_{11} + C_{22}P_{12} = 0.$$

Eliminating P_{12} from these two gives

$$\frac{P_{22}}{P_{11}}\frac{C_{22}}{C_{11}} = 1.$$

Therefore, since $P_{22} > P_{11}$ by assumption, we must have $C_{22} < C_{11}$.

(c) When the two pyramids are widely separated, C_{11} and C_{22} are the self-capacitances of the bodies, which scale with their linear size. Therefore, pyramid 1 is larger than pyramid 2. This makes sense, since the charge can spread out when we transfer charge from a smaller conductor to a larger conductor.

5.19 Capacitance Matrix Practice

Let A, B, and C denote the regions to the left, above, and below the grounded plate, respectively. If up is positive, the electric fields in the three regions are

$$E_A = \frac{\varphi_2 - \varphi_1}{b} \qquad E_B = -\frac{\varphi_1}{b-y} \qquad \text{and} \qquad E_C = \frac{\varphi_2}{y}.$$

The charge per unit area on any plate is $\sigma = \epsilon_0 \mathbf{E}\cdot\hat{\mathbf{n}}$. Therefore, the charges on the upper and lower plates are

$$Q_1 = -\epsilon_0 d\frac{\varphi_1 - \varphi_2}{b}d(a-x) + \epsilon_0\frac{\varphi_1}{b-y}xd$$
$$Q_2 = \epsilon_0 d\frac{\varphi_1 - \varphi_2}{b}d(a-x) + \epsilon_0\frac{\varphi_2}{b-y}xd.$$

Therefore, the capacitance matrix is

$$\mathbf{C} = \epsilon_0 d \begin{bmatrix} \dfrac{x}{b-y} + \dfrac{a-x}{b} & -\dfrac{a-x}{b} \\[2ex] -\dfrac{a-x}{b} & \dfrac{a-x}{b} + \dfrac{x}{y} \end{bmatrix}.$$

5.20 Bounds on Parallel-Plate Capacitance

(a) We write out the right-hand side of the proposed identity:

$$\begin{aligned}
\int\limits_V d^3r\, |\delta\mathbf{E}|^2 + 2\int\limits_V d^3r\, \mathbf{E}_0 \cdot \delta\mathbf{E} &= \int\limits_V d^3r |\mathbf{E} - \mathbf{E}_0|^2 + 2\int\limits_V d^3r\, \mathbf{E}_0 \cdot (\mathbf{E} - \mathbf{E}_0) \\
&= \int\limits_V d^3r\, |\mathbf{E}|^2 + \int\limits_V d^3r\, |\mathbf{E}_0|^2 - 2\int\limits_V d^3r\, \mathbf{E} \cdot \mathbf{E}_0 \\
&\quad + 2\int\limits_V d^3r\, \mathbf{E}_0 \cdot (\mathbf{E} - \mathbf{E}_0) \\
&= \int\limits_V d^3r\, |\mathbf{E}|^2 - \int\limits_V d^3r\, |\mathbf{E}_0|^2.
\end{aligned}$$

(b) If Δ is the potential difference between the plates, the exact capacitance is defined in terms of the total energy U_E:

$$C = \frac{2U_E}{\Delta^2} = \frac{\epsilon_0}{\Delta^2} \int d^3r\, |\mathbf{E}|^2.$$

Since the integrand is positive definite we can restrict the integral to the volume V between the finite plates to get

$$C > \frac{\epsilon_0}{\Delta^2} \int\limits_V d^3r\, |\mathbf{E}|^2.$$

On the other hand, by the definition of $\mathbf{E}_0$,

$$C_0 = \frac{\epsilon_0}{\Delta^2} \int\limits_V d^3r\, |\mathbf{E}_0|^2.$$

Therefore, we need only prove that the right-hand side of the identity in part (a) is not negative. The first term is manifestly positive. The second term is zero. To see this, we write it in the form

$$\int\limits_V d^3r\, \nabla\varphi_0 \cdot \nabla\delta\varphi = \int\limits_V d^3r\, \nabla \cdot (\delta\varphi\nabla\varphi_0) - \int\limits_V d^3r\, \delta\varphi\nabla^2\varphi_0.$$

The last term on the right is zero because $\nabla^2\varphi_0 = 0$ in V. Gauss' law transforms the first term on the right side to

$$\int\limits_S d\mathbf{S} \cdot \delta\varphi \cdot \nabla\varphi_0.$$

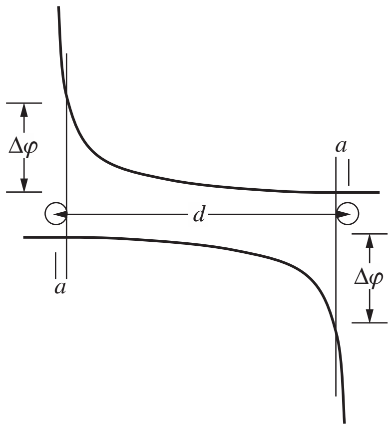
This is also zero because $\delta\varphi = 0$ on the plates and $\hat{\mathbf{n}} \cdot \nabla\varphi_0$ on the cylinder walls.

5.21 A Two-Wire Capacitor

(a) With respect to an origin at its center, the electrostatic potential produced by an infinitely long wire (we are advised to neglect end effects) with uniform charge per unit length $\lambda = Q/L$ is

$$\varphi(\rho) = -\frac{\lambda}{2\pi\epsilon_0}\ln\rho.$$

Therefore, when the wires have equal and opposite charge per unit length, the total electrostatic potential is the sum of the two curves drawn below.



(b) The potential difference between the wires is $2\Delta\varphi$, so the capacitance per unit length of this system is

$$\frac{C}{L} = \frac{\lambda}{2|\Delta\varphi|} = \frac{\lambda}{(\lambda/\pi\epsilon_0)[\ln(d-a) - \ln a]}.$$

Therefore, since $d \gg a$, the capacitance of the two-wire system is approximately

$$C \approx \frac{\pi\epsilon_0 L}{\ln(d/a)}.$$

5.22 An Off-Center Spherical Capacitor

(a) When Δ is small, we expect $C' = C + A\Delta + B\Delta^2 + \cdots$. But the $+\Delta$ capacitor is just the $-\Delta$ capacitor rotated by 180°. Therefore, we must have $A = 0$.

(b) Since the outer sphere does not move, the magnitude of the force on the inner sphere is

$$F = \frac{\partial U}{\partial \Delta} = \frac{1}{2}V^2\frac{\partial C'}{\partial \Delta}.$$

But, $F = 0$ when $\Delta = 0$, because the outer sphere produces zero field inside itself when the spheres are concentric. In other words,

$$0 = \left.\frac{\partial C'}{\partial \Delta}\right|_{\Delta=0}.$$

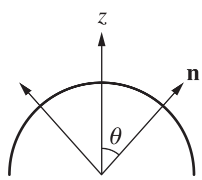
This is true only if $A = 0$.

5.23 The Force between Conducting Hemispheres

(a) For a sphere charged to a potential V, $Q = 4\pi\epsilon_0 RV$. Therefore,

$$\sigma = \frac{Q}{4\pi R^2} = \frac{\epsilon_0 V}{R}.$$

Moreover, the force per unit area which acts on any charged surface is $\mathbf{f} = \hat{\mathbf{n}}\sigma^2/2\epsilon_0$. Therefore, integrating over a hemispherical surface S and inserting a factor of $\cos\theta$ because only the z-component survives the integration (see figure below), we find



$$\mathbf{F} = \int_S dS\mathbf{f} = \hat{\mathbf{z}}\int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta \sin\theta\cos\theta \frac{\sigma^2}{2\epsilon_0}R^2 = \frac{1}{2}\pi\epsilon_0 V^2\hat{\mathbf{z}}.$$

(b) For a spherical capacitor, the charge resides on the outer surface of the inner sphere and on the inner surface of the outer sphere. Therefore, the force density acts outward radially on the inner sphere and inward radially on the outer sphere. Therefore, because $|Q| = 4\pi\epsilon_0 aV_a = 4\pi\epsilon_0 bV_b$, the result of part (a) gives a net force of repulsion with magnitude

$$F = \frac{1}{2}\pi\epsilon_0(V_a^2 - V_b^2) = \frac{Q^2}{32\pi\epsilon_0}\left[\frac{1}{a^2} - \frac{1}{b^2}\right].$$

5.24 Holding a Sphere Together

(a) Let $\sigma = Q/4\pi R^2$ be the uniform charge density of the shell. The shell charge produces zero electric field just inside the shell boundary and $\mathbf{E}_{\rm out} = (\sigma/\epsilon_0)\hat{\mathbf{r}}$ just outside the shell boundary. $\mathbf{E}'$ is the continuous field produced by the point charge. The force density which acts on any element of shell is

$$\mathbf{f} = \sigma\left[\mathbf{E}' + \frac{1}{2}\mathbf{E}_{\rm out}\right].$$

For this to be zero, we need $\mathbf{E}' = -\frac{1}{2}\mathbf{E}_{\rm out}$ or

$$\frac{Q'}{4\pi\epsilon_0 R^2} = -\frac{1}{2}\frac{\sigma}{\epsilon_0} = -\frac{1}{2}\frac{Q}{4\pi\epsilon_0 R^2}.$$

This gives $Q' = -Q/2$.

(b) There is no change. By Gauss' law, the electric field produced by the shell is the same in the two cases.

Source: D. Budker, D.P. DeMille, and D.F. Kimball, *Atomic Physics* (2004).

5.25 Force Equivalence

We adopt the Einstein summation convention so the inverse relation between **C** and **P** implies that

$$P_{ik}C_{km} = \delta_{im}.$$

This implies that

$$\delta(P_{ik}C_{km}) = \delta P_{ik}C_{km} + P_{ik}\delta C_{km} = 0.$$

Hence,

$$\delta P_{ik}C_{km}C^{-1}_{mj} = -P_{ik}\delta C_{km}C^{-1}_{mj}$$

so

$$\delta P_{ik}\delta_{kj} = -P_{ik}\delta C_{km}C^{-1}_{mj}.$$

Therefore, because $P_{ik} = P_{ki}$,

$$\delta P_{ij} = -P_{ik}\delta C_{km}C^{-1}_{mj} = -P_{ki}\delta C_{km}P_{mj}.$$

This is the key result we need. Now, using $\varphi_i = P_{ij}Q_j$ we see that

$$-Q_i\delta P_{ij}Q_j = Q_iP_{ki}\delta C_{km}P_{mj}Q_j = \varphi_k\delta C_{km}\varphi_m.$$

Dividing by $2\delta\mathbf{R}_p$ gives the desired final result:

$$\tfrac{1}{2}\varphi_k\frac{\delta C_{km}}{\delta\mathbf{R}_p}\varphi_m = -\tfrac{1}{2}Q_i\frac{\delta P_{ij}}{\delta\mathbf{R}_p}Q_j.$$

Chapter 6: Dielectric Matter

6.1 Polarization by Superposition

The Gauss' law electric field produced by a sphere with uniform charge density ρ centered at the origin is

$$\mathbf{E}(r) = \begin{cases} \dfrac{\rho}{3\epsilon_0}\mathbf{r} & r < R \\ \\ \dfrac{\rho}{3\epsilon_0}\dfrac{R^3}{r^3}\mathbf{r} & r > R. \end{cases}$$

An identical sphere, but with charge density $-\rho$ displaced from the origin by $\boldsymbol{\delta}$, produces the negative of this field except that $\mathbf{r} \to \mathbf{r} - \boldsymbol{\delta}$. Moreover, to lowest order in $\boldsymbol{\delta}$,

$$\begin{aligned} |\mathbf{r} - \boldsymbol{\delta}|^{-3} &= [(\mathbf{r} - \boldsymbol{\delta}) \cdot (\mathbf{r} - \boldsymbol{\delta})]^{-3/2} \\ &= \frac{1}{r^3}\left[1 - \frac{2\mathbf{r} \cdot \boldsymbol{\delta}}{r^2} + \frac{\delta^2}{r^2}\right]^{-3/2} \\ &\approx \frac{1}{r^3}\left[1 + \frac{3\mathbf{r} \cdot \boldsymbol{\delta}}{r^2}\right]. \end{aligned}$$

Hence, the total field produced by the superposition of the two spheres is

$$\mathbf{E}(r) = \begin{cases} \dfrac{\rho}{3\epsilon_0}[\mathbf{r} - (\mathbf{r} - \boldsymbol{\delta})] = \dfrac{\rho\boldsymbol{\delta}}{3\epsilon_0} & r < R, \\ \dfrac{\rho R^3}{3\epsilon_0}\left\{\dfrac{\mathbf{r}}{r^3} - \dfrac{\mathbf{r} - \boldsymbol{\delta}}{r^3}\left[1 + \dfrac{3\mathbf{r} \cdot \boldsymbol{\delta}}{r^2}\right]\right\} = \dfrac{\rho R^3}{3\epsilon_0}\left[\dfrac{\boldsymbol{\delta} - 3(\hat{\mathbf{r}} \cdot \boldsymbol{\delta})\hat{\mathbf{r}}}{r^3}\right] & r > R. \end{cases}$$

This may be compared with the field produced by a sphere with volume V and polarization $\mathbf{P}$:

$$\mathbf{E}(r) = \begin{cases} -\dfrac{\mathbf{P}}{3\epsilon_0} & r < R, \\ \\ \dfrac{V}{4\pi\epsilon_0}\left[\dfrac{3(\hat{\mathbf{r}} \cdot \mathbf{P})\hat{\mathbf{r}}}{r^3} - \dfrac{\mathbf{P}}{r^3}\right] & r > R. \end{cases}$$

The two are identical if we choose $\mathbf{P} = -\rho\boldsymbol{\delta}$.

6.2 How to Make a Uniformly Charged Sphere

The equation to be solved is

$$\nabla \cdot \mathbf{P} = \begin{cases} -\rho_{\rm P} & r < R, \\ 0 & r > R. \end{cases}$$

We solve this by analogy with the problem

$$\nabla \cdot \mathbf{E} = \begin{cases} \rho/\epsilon_0 & r < R, \\ 0 & r > R. \end{cases}$$

The Gauss' law solution for the latter problem is

$$\mathbf{E}(r) = \begin{cases} \dfrac{\rho}{3\epsilon_0}\mathbf{r} & r < R, \\ \\ \dfrac{\rho}{3\epsilon_0}\dfrac{R^3}{r^3}\mathbf{r} & r > R. \end{cases}$$

Therefore, the desired polarization is

$$\mathbf{P}(r) = \begin{cases} -\dfrac{\rho_\mathrm{P}}{3}\mathbf{r} & r < R, \\ \\ \dfrac{\rho_\mathrm{P}}{3}\dfrac{R^3}{r^3}\mathbf{r} & r > R. \end{cases}$$

Source: A.M. Portis, *Electromagnetic Fields* (Wiley, New York, 1978).

6.3 The Energy of a Polarized Ball

Choose $\mathbf{P} = P\hat{\mathbf{z}}$ so the surface polarization charge density is

$$\sigma_\mathrm{P}(\theta) = \mathbf{P} \cdot \hat{\mathbf{n}} = P\hat{\mathbf{z}} \cdot \mathbf{r} = P\cos\theta.$$

The volume polarization density is zero for this system. Therefore, the total energy is

$$U_E = \frac{1}{2}\int dS\, \sigma(\theta)\varphi(\mathbf{r}_S) = \frac{1}{8\pi\epsilon_0}\int dS \int dS' \frac{\sigma(\theta)\sigma(\theta')}{|\mathbf{r}_S - \mathbf{r'}_S|} = \frac{P^2}{8\pi\epsilon_0}\int dS \int dS' \frac{\cos(\theta)\cos(\theta')}{|\mathbf{r}_S - \mathbf{r'}_S|}.$$

Now, $\cos\theta = \sqrt{\frac{4\pi}{3}}Y_{10}(\theta,\phi)$ and, since $r_S = r'_S$, we can use

$$\frac{1}{|\mathbf{r}_S - \mathbf{r'}_S|} = \frac{1}{R}\sum_{L=0}^{\infty}\sum_{M=-L}^{L}\frac{4\pi}{2L+1}Y^*_{LM}(\theta,\phi)Y_{LM}(\theta',\phi').$$

Both integrals are now examples of the orthonormality relation for the spherical harmonics:

$$\int d\Omega\, Y^*_{\ell m}(\Omega)Y_{\ell' m'}(\Omega) = \delta_{\ell\ell}\delta_{mm'}.$$

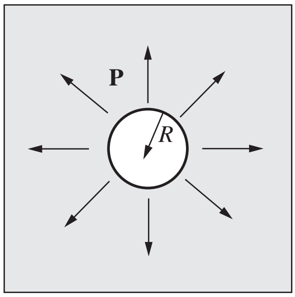
Hence,

$$U_E = \frac{2P^2R^3}{9\epsilon_0}.$$

Source: A.M. Portis, *Electromagnetic Fields* (Wiley, New York, 1978).

6.4 A Hole in Radially Polarized Matter

The polarization is uniform in magnitude but always points in the radial direction outside an origin-centered sphere of radius R as shown below.



The surface density of polarization charge is $\sigma = -\mathbf{P}\cdot\hat{\mathbf{r}} = -P$ at $r = R$. The volume density of polarization charge density is

$$\rho_P = -\nabla\cdot\mathbf{P} = -\nabla\cdot[P\hat{\mathbf{r}}].$$

Now,

$$\nabla\cdot\hat{\mathbf{r}} = \nabla\cdot\left(\frac{\mathbf{r}}{r}\right) = \frac{\nabla\cdot\mathbf{r}}{r} + \mathbf{r}\cdot\nabla\left(\frac{1}{r}\right) = \frac{3}{r} + \mathbf{r}\cdot\left(-\frac{\hat{\mathbf{r}}}{r^2}\right) = \frac{2}{r}.$$

Therefore,

$$\rho_P(r) = -\frac{2P}{r} \qquad r \geq R.$$

By Gauss' law, both σ_P and ρ_P produce purely radial electric fields *outside* the hole. Specifically,

$$\mathbf{E}_\rho(r) = \frac{Q(r)}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \qquad \text{with} \qquad Q(r) = -4\pi\int_0^r ds s^2\rho_P(s) = 4\pi P(R^2 - r^2),$$

and

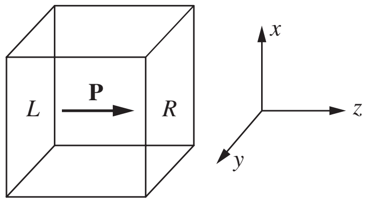
$$\mathbf{E}_\sigma(r) = -\frac{4\pi P R^2}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} = -\frac{PR^2}{\epsilon_0 r^2}\hat{\mathbf{r}}.$$

Therefore, the total electric field everywhere is

$$\mathbf{E}(r) = \mathbf{E}_\rho(r) + \mathbf{E}_\sigma(r) = \begin{cases} -\dfrac{P}{\epsilon_0}\hat{\mathbf{r}} & r \geq R, \\ 0 & r \leq R. \end{cases}$$

6.5 The Field at the Center of a Polarized Cube

The volume polarization charge density is zero for a uniformly polarized object. The surface polarization $\sigma_P = \mathbf{P}\cdot\hat{\mathbf{n}}$ is P on the right (R) face of the cube and $-P$ on the left (L) face of the cube.



Since we only have surface charge,

$$\mathbf{E}(\mathbf{r}) = \frac{P}{4\pi\epsilon_0}\left[\int_R dS'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3} - \int_L dS'\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right].$$

At the origin,

$$\mathbf{E}(0) = -\frac{P}{4\pi\epsilon_0}\left[\int_R dS'\frac{\mathbf{r}'}{r'^3} - \int_L dS'\frac{\mathbf{r}'}{r'^3}\right].$$

By symmetry, the x and y components of these integrals are zero. Therefore, if the origin of the primed system is at the center of the cube,

$$\begin{aligned} E_z(0) &= -\frac{P}{4\pi\epsilon_0}\left[\int_R dS'\frac{z'}{r'^3} - \int_L dS'\frac{z'}{r'^3}\right] = -\frac{2P}{4\pi\epsilon_0}\int_R dS'\frac{z'}{r'^3} \\ &= -\frac{2P}{4\pi\epsilon_0}\int_R d\mathbf{S}'\cdot\frac{\mathbf{r}'}{r'^3} = \frac{2P}{4\pi\epsilon_0}\int_R d\Omega'. \end{aligned}$$

The integral is the solid angle subtended by the right face at the center of the cube. By symmetry, this number must be $4\pi/6$. Therefore, the electric field at the center of the cube is

$$\mathbf{E}(0) = -\frac{\mathbf{P}}{3\epsilon_0}.$$

This is exactly the same as the electric field at the center of a uniformly polarized sphere found in Application 6.1!

Source: Prof. H.B. Biritz, Georgia Institute of Technology (private communication).

6.6 Practice with Poisson's Formula

A body with volume V and uniform charge density ρ_0 produces an electric field $\mathbf{E}_0(r)$. If we replace ρ_0 in the body by a uniform polarization $\mathbf{P}_0$, Poisson's relation asserts that the electrostatic potential produced by the polarized body is

$$\varphi(r) = \frac{\mathbf{P}_0 \cdot \mathbf{E}_0(r)}{\rho_0}.$$

Let the plane $z = 0$ bisect the slab with uniform charge density ρ. From Gauss' law in integral form, the electric field everywhere due to the slab is

$$\mathbf{E}(z) = \begin{cases} \operatorname{sgn}(z)(\rho t/\epsilon_0)\hat{\mathbf{z}} & |z| > t, \\ (\rho z/\epsilon_0)\,\hat{\mathbf{z}} & |z| < t. \end{cases}$$

As far as electrostatics is concerned, the slab with uniform polarization $\mathbf{P}$ is equivalent to a plane at $z = t$ with uniform charge/area $\sigma = \mathbf{P} \cdot \hat{\mathbf{z}}$ and a plane at $z = -t$ with uniform charge/area $\sigma = -\mathbf{P} \cdot \hat{\mathbf{z}}$. The potential of a sheet of charge at $z = 0$ with uniform charge/area σ is $\varphi_\sigma(z) = -(\sigma/2\epsilon_0)|z|$. Therefore, the potential of the polarized sheet is

$$\varphi(z) = \frac{\mathbf{P} \cdot \hat{\mathbf{z}}}{2\epsilon_0} \{|z + t| - |z - t|\}.$$

By checking the four intervals $z < -t$, $-t \le z < 0$, $0 \le z < t$, and $t \le z$ separately, it is easy to confirm that $\varphi(z)$ and $\mathbf{E}(z)$ do indeed satisfy the Poisson relation.

6.7 Isotropic Polarization

$$\begin{aligned} \varphi(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int d^3r'\, \mathbf{P}(\mathbf{r}') \cdot \nabla' \frac{1}{|\mathbf{r} - \mathbf{r}'|} = -\frac{1}{4\pi\epsilon_0} \int d^3r'\, \mathbf{P}(r') \cdot \nabla \frac{1}{|\mathbf{r} - \mathbf{r}'|} \\ &= -\int_0^R dr'\, \mathbf{P}(r') \cdot \nabla \left[\int \frac{dS'}{4\pi\epsilon_0} \frac{1}{|\mathbf{r} - \mathbf{r}'|} \right]. \end{aligned}$$

The quantity in square brackets is the potential $\varphi'(r)$ of a sphere of radius r' with uniform surface charge density $\sigma = 1$. This means that the charge of that sphere is $Q' = 4\pi r'^2$ and

$$\varphi'(r) = \begin{cases} \dfrac{r'}{\epsilon_0} & r < r', \\[2ex] \dfrac{r'^2}{\epsilon_0 r} & r > r'. \end{cases}$$

(a) Substituting $\varphi'(r)$ for $r > r'$ above gives the potential of the polarized sphere for $r > R$:

$$\varphi(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \int_0^R dr'\, r'^2 \mathbf{P}(r') \cdot \nabla \frac{1}{r} = -\frac{1}{4\pi\epsilon_0} \int d^3r' \mathbf{P}(r') \cdot \nabla \frac{1}{r}.$$

This is exactly the potential of a point dipole at the origin,

$$\varphi(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0} \mathbf{p} \cdot \nabla \frac{1}{r},$$

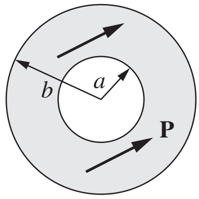
with electric dipole moment

$$\mathbf{p} = \int d^3r'\, \mathbf{P}(r').$$

(b) $\varphi'(r)$ is independent of r when $r < r'$ so the potential of the polarized sphere is zero when $r < R$.

6.8 E and D for an Annular Dielectric

(a) We treat the geometry shown below as the superposition of a ball with radius b and uniform polarization $\mathbf{P}$ and a concentric ball with radius a and uniform polarization $-\mathbf{P}$.



From the text, the field produced by an origin-centered polarized ball with volume V is

$$\mathbf{E}(\mathbf{r}) = \begin{cases} -\dfrac{\mathbf{P}}{3\epsilon_0} & r < R, \\[2ex] \dfrac{V}{4\pi\epsilon_0}\left\{\dfrac{3(\mathbf{r}\cdot\mathbf{P})\mathbf{r}}{r^5} - \dfrac{\mathbf{P}}{r^3}\right\} & r > R. \end{cases}$$

Therefore, the field in question is

$$\mathbf{E}(\mathbf{r}) = \begin{cases} 0 & r < a, \\[2ex] -\dfrac{\mathbf{P}}{3\epsilon_0} - \dfrac{a^3}{3\epsilon_0}\left\{\dfrac{3(\mathbf{r}\cdot\mathbf{P})\mathbf{r}}{r^5} - \dfrac{\mathbf{P}}{r^3}\right\} & a < r < b, \\[2ex] \dfrac{b^3 - a^3}{3\epsilon_0}\left\{\dfrac{3(\mathbf{r}\cdot\mathbf{P})\mathbf{r}}{r^5} - \dfrac{\mathbf{P}}{r^3}\right\} & r > b. \end{cases}$$

(b) By symmetry, we must have $\mathbf{D}(\mathbf{r}) = D(r)\hat{\mathbf{r}}$. Therefore, the choice of a spherical Gaussian surface of radius r gives

$$\int_S d\mathbf{S}\cdot\mathbf{D} = D(r)4\pi r^2 = Q_{c,encl} = 0.$$

Therefore, $\mathbf{D} = 0$ everywhere.

Source: A.M. Portis, *Electromagnetic Fields* (Wiley, New York, 1978).

6.9 The Correct Way to Define E

In the presence of a charge q, nearby conductors or dielectrics are polarized and create a field $\mathbf{E}_{\text{ind}}$ at the position of the charge. Therefore, if $\mathbf{E}$ is the field of interest, the force measured when q is placed at a point is

$$\mathbf{F}_q = q(\mathbf{E} + \mathbf{E}_{\text{ind}}).$$

The linearity of electrostatics guarantees that the induced field changes sign when the charge that polarizes the conductor/dielectric changes sign.Therefore, the force measured when $-q$ sits at the point in question is

$$\mathbf{F}_{-q} = -q(\mathbf{E} - \mathbf{E}_{\rm ind}).$$

Therefore, the electric field we seek is

$$\mathbf{E} = \frac{\mathbf{F}_q - \mathbf{F}_{-q}}{2q}.$$

Source: W.M. Saslow, *Electricity, Magnetism, and Light* (Academic, Amsterdam, 2002).

6.10 Charge and Polarizable Matter Coincident

(a) We will compute the polarization from

$$\mathbf{P} = \mathbf{D} - \epsilon_0 \mathbf{E} = \mathbf{D} - \frac{\mathbf{D}}{\kappa} = \frac{\kappa - 1}{\kappa}\mathbf{D}.$$

Gauss' law in integral form applies to a volume V enclosed by a surface S:

$$\int_S d\mathbf{S} \cdot \mathbf{D} = \int_V d^3 r \, \rho_{\rm c}.$$

This problem has spherical symmetry so $\mathbf{D}(\mathbf{r}) = D(r)\hat{\mathbf{r}}$. Choosing a Gaussian sphere of radius r gives

$$D(r) 4\pi r^2 = \rho_{\rm c} \frac{4\pi}{3} r^3.$$

Hence,

$$\mathbf{D}(r) = \frac{\rho_{\rm c}}{3}\mathbf{r} \qquad \Rightarrow \qquad \mathbf{P} = \rho_{\rm c}\frac{\kappa - 1}{3\kappa}\mathbf{r}.$$

(b) The volume density of polarization charge is

$$\rho_{\rm P} = -\nabla \cdot \mathbf{P} = -\rho_{\rm c}\frac{\kappa - 1}{3\kappa}\nabla \cdot \mathbf{r} = \rho_{\rm c}\frac{1-\kappa}{\kappa}.$$

The surface density of polarization charge is

$$\sigma_{\rm P} = \mathbf{P} \cdot \hat{\mathbf{r}} = \rho_{\rm c}\frac{\kappa - 1}{3\kappa}R.$$

Therefore, the total polarization charge is

$$Q_P = \rho_{\rm P} V + \sigma_{\rm P} A = \rho_{\rm c}\frac{1-\kappa}{\kappa} \times \frac{4\pi R^3}{3} + \rho_{\rm c}\frac{\kappa - 1}{3\kappa} R \times 4\pi R^2 = 0.$$

This is the expected value because the dielectric is neutral. The free charge $\rho_{\rm c}$ is extraneous to the dielectric matter.

6.11 Cavity Field

The matching conditions at a dielectric interface are the continuity of $\mathbf{E}_\parallel$ and the continuity of $\mathbf{D}_\perp = [\epsilon\mathbf{E}]_\perp$. When $h \ll l$, it is sufficient to enforce these conditions on the large flat surface with $\hat{\mathbf{n}}$ as its normal. All we need is the decomposition

$$\mathbf{E} = \mathbf{E}_\perp + \mathbf{E}_\parallel = (\mathbf{E}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + [\mathbf{E} - (\mathbf{E}\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}]\,.$$

Applying the matching conditions gives

$$\mathbf{E}_{\rm cav} = \frac{\epsilon}{\epsilon_0}(\mathbf{E}_0\cdot\hat{\mathbf{n}})\hat{\mathbf{n}} + [\mathbf{E}_0 - (\mathbf{E}_0\cdot\hat{\mathbf{n}})\hat{\mathbf{n}}]\,.$$

6.12 Making External Fields Identical

Both spheres produce a dipole field outside of themselves. The dipole moment of the dielectric sphere is

$$\mathbf{p} = 4\pi a^3\epsilon_0\frac{\kappa-1}{\kappa+2}\mathbf{E}_0.$$

The dipole moment of the conducting sphere is the $\kappa\to\infty$ limit: $\mathbf{p}' = 4\pi b^3\epsilon_0\mathbf{E}_0$. The fields will be identical for $r > a$ if $\mathbf{p} = \mathbf{p}'$ or

$$b = a\left[\frac{\kappa-1}{\kappa+2}\right]^{1/3}.$$

6.13 The Capacitance Matrix for a Spherical Sandwich

Let φ_2 and φ_1 be the potentials of the shells. For $r \geq R_2$, the system acts like a point charge, so

$$\varphi_2 = \frac{q_1+q_2}{4\pi\epsilon_0 R_2}.$$

From Gauss' law, the electric field between the spheres is $\mathbf{E}(r) = \hat{\mathbf{r}}\,E(r) = \hat{\mathbf{r}}\dfrac{q_1}{4\pi\epsilon_0\kappa r^2}$. Therefore,

$$\varphi_1 - \varphi_2 = \int_{R_1}^{R_2} d\mathbf{s}\cdot\mathbf{E}(r) = \frac{q_1}{4\pi\epsilon_0\kappa}\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

From the previous two equations, we deduce that

$$q_1 = 4\pi\epsilon_0\kappa\frac{R_1R_2}{R_2-R_1}(\varphi_1-\varphi_2)$$

$$q_2 = 4\pi\epsilon_0\kappa\frac{R_1R_2}{R_1-R_2}\varphi_1 + 4\pi\epsilon_0 R_2\left(1+\kappa\frac{R_1}{R_2-R_1}\right)\varphi_2.$$

By definition, the elements of the capacitance matrix satisfy

$$q_1 = C_{11}\varphi_1 + C_{12}\varphi_2$$

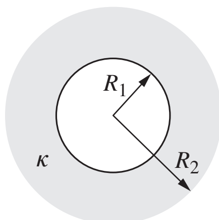
$$q_2 = C_{21}\varphi_1 + C_{22}\varphi_2.$$

Therefore,

$$C_{11} = -C_{12} = -C_{21} = 4\pi\epsilon_0\kappa\frac{R_1R_2}{R_2 - R_1}$$

$$C_{22} = 4\pi\epsilon_0 R_2\left(1 + \kappa\frac{R_1}{R_2 - R_1}\right).$$

6.14 A Spherical Conductor Embedded in a Dielectric



(a) Gauss' law in integral form is $\int_S d\mathbf{S}\cdot\mathbf{D} = Q_{f,encl}$. For this problem with spherical symmetry where $\mathbf{D} = \kappa\epsilon_0\mathbf{E}$, we find immediately that

$$\mathbf{E}(r) = \begin{cases} 0 & r < R_1, \\ \dfrac{Q}{4\pi\epsilon}\dfrac{\hat{\mathbf{r}}}{r^2} & R_1 < r < R_2, \\ \dfrac{Q}{4\pi\epsilon_0}\dfrac{\hat{\mathbf{r}}}{r^2} & r > R_2. \end{cases}$$

There is no free charge anywhere except on the conductor surface so the bulk polarization charge is zero everywhere:

$$\rho_{\mathrm{P}} = -\nabla\cdot\mathbf{P} = -(\kappa - 1)\nabla\cdot\mathbf{E} = 0.$$

The surface density of polarization charge is

$$\begin{aligned} \sigma_{\mathrm{P}}(R_1) &= -\mathbf{P}\cdot\hat{\mathbf{r}}|_{r=R_1} = -(\kappa-1)\mathbf{E}(R_1)\cdot\hat{\mathbf{r}} = -\frac{\kappa-1}{\kappa}\frac{Q}{4\pi\epsilon_0 R_1} \\ \sigma_{\mathrm{P}}(R_2) &= \mathbf{P}\cdot\hat{\mathbf{r}}|_{r=R_2} = (\kappa-1)\mathbf{E}(R_2)\cdot\hat{\mathbf{r}} = \frac{\kappa-1}{\kappa}\frac{Q}{4\pi\epsilon_0 R_2}. \end{aligned}$$

The dielectric is neutral so the total polarization charge vanishes, as it must:

$$\int_{r=R_1} dS\sigma_{\rm P}(R_1) + \int_{r=R_2} dS\sigma_{\rm P}(R_2) = 0.$$

(b) There is no bulk polarization charge as in part (a). Besides $\mathbf{E}_0$, which imposes azimuthal symmetry on the problem, the only electric fields in the problem are produced by surface polarization charge densities at $r = R_1$ and $r = R_2$. Therefore, the potential must have the form of an exterior azimuthal multipole expansion for $r > R_2$ (supplemented by $\mathbf{E}_0$) and the form of a sum of interior and exterior azimuthal multipole expansions for $R_1 < r < R_2$.

$$\varphi_{\rm out}(r,\theta) = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} \frac{A_\ell}{r^{\ell+1}} P_\ell(\cos\theta) \qquad r > R_2,$$

$$\varphi_{\rm in}(r,\theta) = \sum_{\ell=0}^{\infty} \left[B_\ell r^\ell + \frac{C_\ell}{r^{\ell+1}} \right] P_\ell(\cos\theta) \qquad R_1 < r < R_2.$$

From the fact that $\varphi_{\rm in}(R_1,\theta) = 0$, we conclude that

$$\varphi_{\rm in}(r,\theta) = \sum_{\ell=0}^{\infty} B_\ell \left[r^\ell - \frac{R_1^{2\ell+1}}{r^{\ell+1}} \right] P_\ell(\cos\theta) \qquad R_1 < r < R_2.$$

We also have the two matching conditions at $r = R_2$. One is $\varphi_{\rm out} = \varphi_{\rm in}$, which tells us that

$$B_1 \left[R_2^3 - R_1^3 \right] = -E_0 R_2^3 + A_1$$

and

$$A_\ell = B_\ell [R_2^{2\ell+1} + R_1^{2\ell+1}] \qquad \ell \neq 1. \tag{1}$$

The other matching condition at $r = R_2$ is

$$\kappa \frac{\partial \varphi_{\rm in}}{\partial r} = \frac{\partial \varphi_{\rm out}}{\partial r}.$$

This tells us that

$$\kappa B_1 [R_2^3 + 2R_1^3] = -E_0 R_2^3 - 2A_1$$

and

$$A_\ell = -\kappa B_\ell \left[\frac{\ell}{\ell+1} R_2^{2\ell+1} + R_1^{2\ell+1} \right] \qquad \ell \neq 1. \tag{2}$$

Equations (1) and (2) are not compatible unless $A_\ell = B_\ell = 0$ for $\ell \neq 0$. Therefore,

$$\varphi_{\rm in}(r,\theta) = B_1 \left[r - \frac{R_1^3}{r^2} \right] \cos\theta$$

$$\varphi_{\rm out}(r,\theta) = \left[-E_0 r + \frac{A_1}{r^2} \right] \cos\theta,$$

where

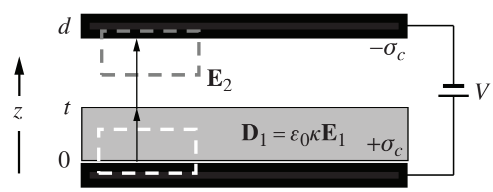
$$A_1 = \frac{-E_0 R_2^3 (R_1^3 - R_2^3)}{2(R_2^3 - R_1^3) + \kappa(R_1^3 + R_2^3)}$$

$$B_1 = \frac{-3E_0}{2[1-(R_1/R_2)^3] + \kappa[1+2(R_1/R_2)^3]}.$$

The charge density on the conductor surface is $\sigma(\theta) = -\epsilon_0(\partial\varphi_{\rm in}/\partial r)|_{r=R_1} \propto \cos\theta$. This integrates to zero; no charge is drawn up from ground. The external field polarizes both the neutral metal and the neutral dielectric, but there is no impetus to attract charge from ground.

6.15 A Parallel-Plate Capacitor with an Air Gap

The figure below shows the capacitor with the dielectric slab inserted. The potential difference is maintained at V and a charge per unit area $\pm\sigma_f$ develops on the inner surface of the conducting plates. A polarization charge develops on the surfaces of the dielectric, but we will not need this information to solve the problem.



(a) We use the dark dashed Gaussian surface to find the electric field E_2 in the air:

$$E_2 = \frac{\sigma_f}{\epsilon_0}. \tag{1}$$

We use the white dashed Gaussian surface to find the **D**-field in the dielectric:

$$D_1 = \sigma_f = \kappa\epsilon_0 E_1.$$

Finally, since the potential difference is maintained at V,

$$V = \int_0^t E_1 dz + \int_t^d E_2 dz = E_1 t + (d-t)E_2 = \frac{\sigma_f}{\kappa\epsilon_0}t + \frac{\sigma_f}{\epsilon_0}(d-t). \tag{2}$$

The capacitance C is defined, so

$$Q = \sigma_f A = CV.$$

Therefore, using (2) to find σ_f, we find

$$C = \frac{\epsilon_0 \kappa A}{\kappa d - (\kappa - 1)t}.$$

(b) Electric breakdown occurs when the electric field exceeds a critical value. Therefore, the criterion that V be the breakdown voltage is

$$E_2 = \frac{V_0}{d}.$$

Using this and (1) with the results of (a) gives the desired result,

$$V = \frac{V_0}{\kappa d}\left[t + \kappa(d-t)\right] = V_0\left[1 - \frac{t}{d}\left(1 - \frac{1}{\kappa}\right)\right].$$

Source: O.D. Jefimenko, *Electricity and Magnetism* (Appleton-Century-Crofts, New York, 1966).

6.16 Helmholtz Theorem for $\mathbf{D}(\mathbf{r})$

The Maxwell equations for dielectric matter are

$$\nabla \cdot \mathbf{D} = \rho_f \qquad\qquad \nabla \times \mathbf{E} = 0.$$

To exploit the Helmholtz theorem, we use $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ to write these in the form

$$\nabla \cdot \mathbf{D} = \rho_f \qquad\qquad \nabla \times \mathbf{D} = \nabla \times \mathbf{P}.$$

The Helmholtz theorem gives

$$\begin{aligned}
\mathbf{D}(\mathbf{r}) &= -\frac{1}{4\pi}\nabla \int d^3r' \, \frac{\nabla' \cdot \mathbf{D}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi}\nabla \times \int d^3r' \, \frac{\nabla' \times \mathbf{D}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} \\
&= -\frac{1}{4\pi}\nabla \int d^3r' \, \frac{\rho_f(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi}\nabla \times \int d^3r' \, \frac{\nabla' \times \mathbf{P}(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}.
\end{aligned}$$

For simple dielectric matter, $\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$. Therefore, $\nabla \times \mathbf{P} = \epsilon_0 \chi_e \nabla \times \mathbf{E} = 0$. Consequently,

$$\mathbf{D}(\mathbf{r}) = -\frac{1}{4\pi}\nabla \int d^3r' \, \frac{\rho_f(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{4\pi}\int d^3r' \, \rho_f(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}.$$

6.17 Electrostatics of a Doped Semiconductor

(a) Gauss' law is $\nabla \cdot \mathbf{D} = \rho_f$. Therefore, the electric field produced by the ions satisfies $(d/dz)\epsilon E_+ = eN_D$ for $0 < z \leq d$ and $(d/dz)\epsilon E_+ = 0$ outside this region. The latter means that the field is constant outside the doping region. Moreover, $E_+(d/2) = 0$ by symmetry, so

$$E_+(z) = \frac{eN_D}{\epsilon}\left(z - \frac{d}{2}\right) \qquad\qquad 0 \leq z \leq d.$$

$E_+(z)$ is continuous everywhere because the charge density is not singular. The resulting field is plotted as the solid line in the left figure below.

(b) The free charge in the doping layer has volume density eN_D. Each ion polarizes the dielectric medium so the total charge in the doping layer has volume density eN_D/κ. Therefore, by conservation of charge, the surface charge density (composed of ionized electrons and positive surface polarization charge) must be $\sigma = -eN_D d/\kappa$. This layer of charge produces a field

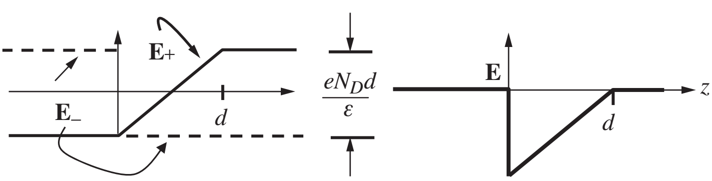
$$E_0(z) = \text{sgn}(z)\frac{\sigma}{2\epsilon_0} = -\text{sgn}(z)\frac{eN_D d}{2\epsilon}.$$

This is plotted as the dashed line in the left figure below.

(c) The total field $E = E_+ + E_-$ is plotted in the right figure below. The system of semiconductor plus dopant atoms has net zero charge. Therefore, from Gauss' law for the total field,

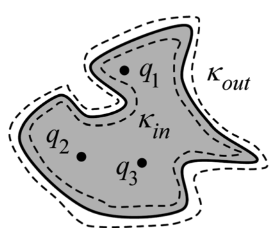
$$\int_{-\infty}^{\infty} dz \frac{dE}{dz} = \frac{1}{\epsilon_0}\int_{-\infty}^{\infty} dz\rho = 0.$$

On the other hand, the integral on the far left is $E(\infty) - E(-\infty)$. This is consistent with our graph where $E(\infty) = E(-\infty) = 0$.



6.18 Surface Polarization Charge

Let S be the dividing surface between the dielectrics. The dashed surfaces in the diagram below are S_{out} and S_{in}. Each lies entirely in the κ_{out} or κ_{in} material, respectively.



If $Q_f = \sum q_k$, Gauss' law tells us that

$$\epsilon_0 \int\limits_{S_{\rm out}} d\mathbf{S} \cdot \mathbf{E}_{\rm out} = Q_f + Q_{pol}$$

and

$$\epsilon_0 \int\limits_{S_{\rm in}} d\mathbf{S} \cdot \mathbf{E}_{\rm in} = Q_f.$$

Therefore,

$$\epsilon_0 \left[\int\limits_{S_{\rm out}} d\mathbf{S} \cdot \mathbf{E}_{\rm out} - \int\limits_{S_{\rm in}} d\mathbf{S} \cdot \mathbf{E}_{\rm in} \right] = Q_{pol}.$$

But $\mathbf{D} = \epsilon\mathbf{E}$, so we can change the integration range in both cases to the surface S when we write

$$\int\limits_{S} d\mathbf{S} \cdot \mathbf{D} \left[\frac{\epsilon_0}{\epsilon_{\rm out}} - \frac{\epsilon_0}{\epsilon_{\rm in}} \right] = Q_{pol}.$$

The surface integral of $\mathbf{D}$ over S is Q_f. Therefore,

$$Q_{\rm pol} = Q \left[\frac{1}{\kappa_{\rm out}} - \frac{1}{\kappa_{\rm in}} \right].$$

Source: T.P. Doerr and Y.-K. Yu, *American Journal of Physics* **72**, 190 (2004).

6.19 An Elastic Dielectric

(a) The energy is

$$U(q,d) = \frac{1}{2}k(d-d_0)^2 + \frac{q^2}{2C(d)} = \frac{1}{2}k(d-d_0)^2 + \frac{q^2}{2}\frac{d}{\epsilon A}.$$

The equilibrium is reached when $\partial U/\partial d = 0$, i.e., for

$$d(q) = d_0 - \frac{q^2}{2k\epsilon A}.$$

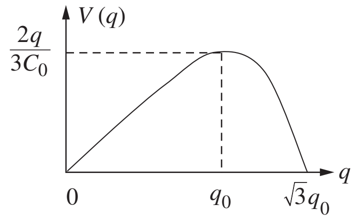
(b) At equilibrium the potential difference is

$$V(q) = \frac{q}{C(q)} = \frac{qd(q)}{\epsilon A} = \frac{q}{C_0}\left(1 - \frac{q^2}{3q_0^2}\right), \quad C_0 = \frac{\epsilon A}{d_0}, \quad 3q_0^2 = 2k\epsilon d_0 A.$$

Therefore, the capacitance is

$$C_d(q) = dq/dV = \frac{C_0}{1 - q^2/q_0^2},$$

which diverges at $q = q_0$.



6.20 A Dielectric Inclusion

Let $V_{\rm in}$ be the volume with permittivity $\epsilon_{\rm in}$ and $V_{\rm out}$ be the complementary volume with permittivity $\epsilon_{\rm out}$. The dipole moment of the entire system is

$$\mathbf{p} = \int_{V_{\rm in}} d^3r\,\mathbf{P}_{\rm in} + \int_{V_{\rm out}} d^3r\,\mathbf{P}_{\rm out} = \epsilon_0\chi_{\rm in}\int_{V_{\rm in}} d^3r\,\mathbf{E}_{\rm in} + \epsilon_0\chi_{\rm out}\int_{V_{\rm out}} d^3r\,\mathbf{E}_{\rm out}.$$

But, because $\mathbf{E}_{\rm in} = -\nabla\varphi_{\rm in}$, $\mathbf{E}_{\rm out} = -\nabla\varphi_{\rm out}$, and $d\mathbf{S}$ points outward from the body,

$$\mathbf{p} = -\epsilon_0\chi_{\rm in}\int_{V_{\rm in}} d^3r\,\nabla\varphi_{\rm in} - \epsilon_0\chi_{\rm out}\int_{V_{\rm out}} d^3r\,\nabla\varphi_{\rm out} = -\epsilon_0\chi_{\rm in}\int_S d\mathbf{S}\,\varphi_{\rm in} + \epsilon_0\chi_{\rm out}\int_S d^3r\,d\mathbf{S}\,\varphi_{\rm out}.$$

Finally, $\varphi_{\rm in} = \varphi_{\rm out}$ on the boundary, and $\epsilon = \epsilon_0(1+\chi)$. Therefore,

$$\mathbf{p} = \epsilon_0(\chi_{\rm out} - \chi_{\rm in})\int_S d\mathbf{S}\,\varphi = (\epsilon_{\rm out} - \epsilon_{\rm in})\int_S d\mathbf{S}\,\varphi.$$

6.21 A Classical Meson

(a) If we orient $\mathbf{p}$ along the z-axis, the dipole makes a contribution $(p/4\pi\epsilon_0 r^2)\cos\theta$ to the interior potential, $\varphi_{\rm in}$. Our experience with matching conditions and interior and exterior multipole expansions tells us that the potential produced by the medium must vary as $\cos\theta$ also. Therefore, the most general potential for this problem is

$$\varphi(r,\theta) = \begin{cases} \left[Ar + \dfrac{p}{4\pi\epsilon_0 r^2}\right]\cos\theta & r \le R, \\[2ex] \dfrac{B}{r^2}\cos\theta & r \ge R. \end{cases}$$

There is no free charge at the cavity boundary, so the matching conditions are

$$\varphi_{\rm in}(R,\theta) = \varphi_{\rm out}(R,\theta) \qquad\qquad \left.\frac{\partial\varphi_{\rm in}}{\partial r}\right|_{r=R} = \kappa\left.\frac{\partial\varphi_{\rm out}}{\partial r}\right|_{r=R}.$$

Solving these for A and B gives

$$A = -\frac{2p}{4\pi\epsilon_0 a^3}\frac{\kappa-1}{2\kappa+1} \qquad \text{and} \qquad B = \frac{p}{4\pi\epsilon_0}\frac{3}{2\kappa+1}.$$

The corresponding electric field is

$$\mathbf{E}(r,\theta) = \begin{cases} \dfrac{p}{4\pi\epsilon_0}\dfrac{3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}}{r^3} - A\hat{\mathbf{z}} & r < R, \\ \\ B\dfrac{3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}}{r^3} & r > R. \end{cases}$$

Finally, $\mathbf{D}_{\rm in} = \epsilon_0\mathbf{E}_{\rm in}$ and $\mathbf{D}_{\rm out} = \kappa\epsilon_0\mathbf{E}_{\rm out}$.

(b) We confirm immediately that $\mathbf{D}_{\rm out} = 0$ when $\kappa = 0$. Otherwise,

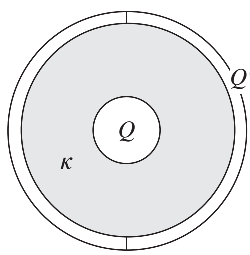
$$U_{\rm in} = \frac{1}{2}\int\limits_{a\le r\le R} d^3r\, \mathbf{E}_{\rm in}\cdot\mathbf{D}_{\rm in} = \frac{1}{2}\int d\Omega \int\limits_a^R dr r^2\epsilon_0|\mathbf{E}_{\rm in}|^2.$$

The various contributions to $U_{\rm in}$ behave like

$$\frac{1}{R^3}, \qquad \frac{1}{a^3}, \qquad \frac{R^3}{a^6}, \qquad \frac{\ln R}{a^3}, \qquad \text{and} \qquad \frac{\ln a}{a^3}.$$

Only the first of these is independent of the cutoff and competes with the surface energy ($\propto R^2$) to determine the size of the cavity.

6.22 An Application of the Dielectric Stress Tensor



There is a radial inward force per unit area $f_{\rm in}$ which acts on the inner surface of the shell and a radial outward force per unit area $f_{\rm out}$ which acts on the outer surface of the shell. The hemispheres will stay together if $f_{\rm in} \ge f_{\rm out}$.

The stress tensor formalism gives the force exerted on a sub-volume enclosed by a surface S as

$$\mathbf{F} = \int\limits_S dS\,\mathbf{f} = \int\limits_S dS[(\hat{\mathbf{n}}\cdot\mathbf{D})\mathbf{E} - \tfrac{1}{2}\hat{\mathbf{n}}(\mathbf{E}\cdot\mathbf{D})].$$

To find $f_{\rm in}$, we use the Gauss' law fields in the dielectric just inside the inner surface of the shell:

$$\mathbf{D}_{\rm in} = \epsilon_0 \kappa \mathbf{E}_{\rm in} = \frac{Q}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}.$$

If the shell has radius R, this gives

$$f_{\rm in} = \frac{D_{\rm in}^2}{2\epsilon_0\kappa} = \frac{Q^2/\kappa}{32\epsilon_0 R^4\pi^2}.$$

To find $f_{\rm out}$, we use the Gauss' law fields in the vacuum just outside the outer surface of the shell:

$$\mathbf{D}_{\rm out} = \epsilon_0 \mathbf{E}_{\rm out} = \frac{Q+Q'}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}.$$

This gives

$$f_{\rm out} = \frac{D_{\rm out}^2}{2\epsilon_0} = \frac{(Q+Q')^2}{32\epsilon_0 R^4\pi^2}.$$

Therefore, the shell will not separate if $Q^2/\kappa \geq (Q+Q')^2$ or

$$0 \geq Q^2(1-\frac{1}{\kappa}) + Q'^2 + 2QQ'.$$

Since $\kappa > 1$, this shows that Q and Q' must have opposite sign. If we put $Q \to -Q'$ and let $x = Q'/Q$, the no-separation condition reads

$$y = x^2 - 2x + 1 - \frac{1}{\kappa} \leq 0. \qquad (1)$$

This function is positive at $x = 0$ and has positive curvature. Therefore (1) is satisfied for values of x that lie between the zeroes of $y(x)$. From the quadratic equation, these occur at

$$x = \frac{Q'}{Q} = 1 \pm \frac{1}{\sqrt{\kappa}}.$$

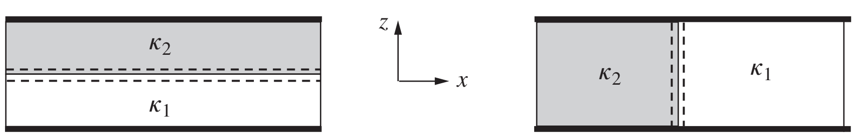
Source: V.C.A. Ferraro, *Electromagnetic Theory* (Athlone Press, London, 1954).

6.23 Two Dielectric Interfaces

The dashed lines in each figure below show a surface S which snugly encloses an interface. The force on the interface is

$$\mathbf{F} = \int_S dS\left[(\hat{\mathbf{n}}\cdot\mathbf{E})\mathbf{D} - \frac{1}{2}\hat{\mathbf{n}}(\mathbf{E}\cdot\mathbf{D})\right].$$

Let the z-axis point upward so the electric field in each medium is $\mathbf{E}_1 = E_1\hat{\mathbf{z}}$ and $\mathbf{E}_2 = E_2\hat{\mathbf{z}}$.



Horizontal Interface: The force per unit area $\mathbf{f} = f\hat{\mathbf{z}}$ is

$$f = \frac{1}{2}(E_2 D_2 - E_1 D_1) = \frac{1}{2}\epsilon_0(\kappa_2 E_2^2 - \kappa_1 E_1^2).$$

But $\mathbf{D} = D\hat{\mathbf{z}}$ is continuous at the interface, so

$$D = \epsilon_0 \kappa_1 E_1 = \epsilon_0 \kappa_2 E_2.$$

Therefore,

$$\mathbf{f} = \hat{\mathbf{z}}\frac{D^2}{2\epsilon_0}\left(\frac{1}{\kappa_2} - \frac{1}{\kappa_1}\right).$$

If the thickness of each dielectric layer is $d/2$, we have $E_0 = V/d$ and

$$E_1\frac{d}{2} + E_2\frac{d}{2} = V = \frac{D}{\epsilon_0\kappa_1}\frac{d}{2} + \frac{D}{\epsilon_0\kappa_2}\frac{d}{2} \qquad \Rightarrow \qquad D = 2\epsilon_0 E_0\left[\frac{1}{\kappa_1} + \frac{1}{\kappa_2}\right]^{-1}.$$

Vertical Interface: The force per unit area $\mathbf{f} = f\hat{\mathbf{x}}$ is

$$f = E_2 D_2 - E_1 D_1.$$

But $E_1 = E_2 = E_0 = V/d$, so

$$\mathbf{f} = \hat{\mathbf{x}}\epsilon_0 E_0^2(\kappa_2 - \kappa_1).$$

6.24 The Force on an Isolated Dielectric

We set $\rho_f = 0$ because this piece requires no comment. Otherwise, we use

$$(\partial_k P_k)E_i = \partial_k(P_k E_i) + (\partial_k E_i)P_k$$

to eliminate the $\nabla \cdot \mathbf{P}$ contribution to get

$$F_i = -\int d^3r\, \partial_k(P_k E_i) + \int d^3r\, (\mathbf{P}\cdot\nabla)E_i + \frac{1}{2}\int dS\,[\hat{\mathbf{n}}\cdot\mathbf{P}(\mathbf{r}_S)]\,[\mathbf{E}_{\rm in} + \mathbf{E}_{\rm out}]_i.$$

Transforming the first term on the right into a surface integral gives

$$F_i = -\int dS\,[\hat{\mathbf{n}}\cdot\mathbf{P}]\,[\mathbf{E}_{\rm in}]_i + \int d^3r\,(\mathbf{P}\cdot\nabla)E_i + \frac{1}{2}\int dS\,[\hat{\mathbf{n}}\cdot\mathbf{P}(\mathbf{r}_S)]\,[\mathbf{E}_{\rm in} + \mathbf{E}_{\rm out}]_i$$

or

$$\mathbf{F} = \int d^3r\,(\mathbf{P}\cdot\nabla)\mathbf{E} + \frac{1}{2}\int dS\,[\hat{\mathbf{n}}\cdot\mathbf{P}(\mathbf{r}_S)]\,[\mathbf{E}_{\rm out} - \mathbf{E}_{\rm in}].$$

Now, the matching conditions at the surface of a polarized dielectric are

$$\hat{\mathbf{n}}\cdot(\mathbf{E}_{\rm out} - \mathbf{E}_{\rm in}) = \frac{\sigma_{\rm P}}{\epsilon_0} = \frac{\hat{\mathbf{n}}\cdot\mathbf{P}}{\epsilon_0} \qquad \text{and} \qquad \hat{\mathbf{n}}\times(\mathbf{E}_{\rm out} - \mathbf{E}_{\rm in}) = 0.$$

Therefore,

$$\mathbf{E}_{\text{out}} - \mathbf{E}_{\text{in}} = \frac{\sigma_{\text{P}}}{\epsilon_0}\hat{\mathbf{n}} = \frac{\hat{\mathbf{n}} \cdot \mathbf{P}}{\epsilon_0}\hat{\mathbf{n}}.$$

Using $d\mathbf{S} = dS\hat{\mathbf{n}}$ and restoring the free charge gives the final result:

$$\mathbf{F} = \int d^3r\,[\rho_f(\mathbf{r}) + \mathbf{P}(\mathbf{r}) \cdot \nabla]\mathbf{E}(\mathbf{r}) + \frac{1}{2\epsilon_0}\int d\mathbf{S}\;[\hat{\mathbf{n}}(\mathbf{r}_S) \cdot \mathbf{P}(\mathbf{r}_S)]^2\,.$$

6.25 Minimizing the Total Energy Functional

Using the hint, we seek a minimum of the functional

$$F[\mathbf{D}\,] = \frac{1}{2}\int_V d^3r\,\frac{|\mathbf{D}\,|^2}{\epsilon} - \int_V d^3r\,\varphi(\mathbf{r})(\nabla \cdot \mathbf{D} - \rho_f).$$

The factor of $\frac{1}{2}$ and the minus sign are inserted for convenience. Operationally, we compute $\delta F = F\,[\mathbf{D} + \delta\mathbf{D}] - F\,[\mathbf{D}]$ and look for the conditions that make $\delta F = 0$ to first order in $\delta\mathbf{D}$. This extremum is a minimum if $\delta F > 0$ to second order in $\delta\mathbf{D}$.

The first step is to integrate by parts to get

$$F[\mathbf{D}\,] = \frac{1}{2}\int_V d^3r\,\frac{|\mathbf{D}\,|^2}{\epsilon} + \int_V d^3r\,[\mathbf{D}\,\cdot \nabla\varphi + \rho_f\varphi] \quad - \int_S d\mathbf{S} \cdot \mathbf{D}\,\varphi.$$

A direct calculation of δF to first order in $\delta\mathbf{D}$ gives

$$\delta F = \int_V d^3r\,\left[\frac{\mathbf{D}}{\epsilon} + \nabla\varphi\right] \cdot \delta\mathbf{D} - \int_S d\mathbf{S} \cdot \delta\mathbf{D}\,\varphi.$$

Finally, since the variation $\delta\mathbf{D}$ is arbitrary, δF vanishes if $\mathbf{D}\,(\mathbf{r}) = -\epsilon\nabla\varphi(\mathbf{r})$ and $\hat{\mathbf{n}}\cdot\delta\mathbf{D}|_S = 0$. The second of these is true if we specify the normal component of $\mathbf{D}$ on the boundary surface. The first implies that $\nabla \times \mathbf{D} = 0$. Together with the divergence constraint, this guarantees that $\mathbf{D}$ and $\mathbf{E} = -\nabla\varphi$ satisfy Maxwell's electrostatic equations. The second-order term in the variation of $F[\mathbf{D}]$ is $\frac{1}{2}\int d^3r\,|\delta\mathbf{D}\,|^2/\epsilon$. This is a positive quantity, so the extremum we have found does indeed correspond to a minimum of the total electrostatic energy.

Chapter 7: Laplace's Equation

7.1 Two Electrostatic Theorems

We will use these facts about spherical harmonics:

$$\begin{aligned}
Y_{00}(\Omega) &= \frac{1}{\sqrt{4\pi}} \\
Y_{10}(\Omega) &= \sqrt{\frac{3}{4\pi}}\cos\theta = \sqrt{\frac{3}{4\pi}}\frac{z}{r} \\
Y_{1\,\pm 1}(\Omega) &= \mp\sqrt{\frac{3}{8\pi}}\sin\theta\exp(\pm i\phi) = \mp\sqrt{\frac{3}{8\pi}}\frac{x \pm iy}{r} \\
\delta_{\ell\ell'}\delta_{mm'} &= \int d\Omega Y_{\ell m}(\Omega)Y^*_{\ell' m'}(\Omega).
\end{aligned}$$

We will also use the fact that the potential satisfies Laplace's equation in and on S, so

$$\varphi(r,\Omega) = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} A_\ell r^\ell Y_{\ell m}(\Omega) = A_{00}Y_{00} + \sqrt{\frac{3}{4\pi}}zA_{10} + \cdots .$$

(a)

$$\begin{aligned}
\int dS\varphi(\mathbf{r}) &= R^2\sum_{ellm} A_\ell R^\ell \int d\Omega Y_{\ell m}(\Omega)Y_{00}(\Omega)\sqrt{4\pi} \\
&= \sqrt{4\pi}R^2 A_0 \\
&= 4\pi R^2\varphi(0).
\end{aligned}$$

(b)

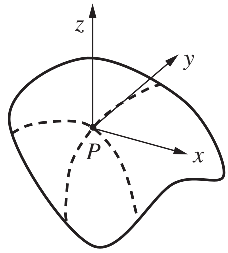
$$\int dSz\varphi(\mathbf{r}) = R^2\sum_{ellm} A_\ell R^{\ell+2}\int d\Omega Y_{\ell m}(\Omega)Y_{10}(\Omega)\sqrt{\frac{4\pi}{3}} = A_{10}R^4\sqrt{\frac{4\pi}{3}}.$$

But $\left.\dfrac{\partial\varphi}{\partial z}\right|_{\mathbf{r=0}} = \sqrt{\dfrac{3}{4\pi}}A_{10}$. Therefore,

$$\int_S dSz\varphi(\mathbf{r}) = \frac{4\pi}{3}R^4\left.\frac{\partial\varphi}{\partial z}\right|_{\mathbf{r}=0}.$$

7.2 Green's Formula

Orient the equipotential surface so that, at the point of interest P, the z-axis is normal and the x- and y-axes point along the directions of the principal radii.



Method I: Then, if $z = z(x, y)$ is the equation of the equipotential surface, we have $\partial z/\partial x|_P = \partial z/\partial y|_P = 0$ and $\kappa = \frac{1}{2}\left[\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2\right]_P$. Laplace's equation is

$$0 = \nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = \frac{\partial}{\partial x}\left(\frac{\partial \varphi}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial z}\frac{\partial z}{\partial y}\right) + \frac{\partial^2 \varphi}{\partial z^2}.$$

But $\partial/\partial z \equiv \partial/\partial n$ and $\partial z/\partial x|_P = \partial z/\partial y|_P = 0$ by construction, so this is

$$0 = \frac{\partial \varphi}{\partial n}\left(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) + \frac{\partial^2 \varphi}{\partial n^2} = 2\kappa \frac{\partial \varphi}{\partial n} + \frac{\partial^2 \varphi}{\partial n^2}$$

by the definition of the curvature given above. This was Green's method of proof.

Method II: The coordinate-free definition of mean curvature is $2\kappa = \nabla \cdot \hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is the unit normal to the surface. For our problem, the gradient $\nabla \hat{\mathbf{n}} = \hat{\mathbf{n}}\partial\varphi/\partial n$ because the surface is an equipotential. Therefore,

$$0 = \nabla^2 \varphi = \nabla \cdot (\nabla \varphi) = \nabla \cdot \left(\frac{\partial \varphi}{\partial n}\hat{\mathbf{n}}\right) = (\hat{\mathbf{n}} \cdot \nabla)\frac{\partial \varphi}{\partial n} + \frac{\partial \varphi}{\partial n}\nabla \cdot \hat{\mathbf{n}} = \frac{\partial^2 \varphi}{\partial n^2} + 2\kappa \frac{\partial \varphi}{\partial n}.$$

7.3 Poisson's Formula for a Sphere

The general solution of Laplace's equation inside the sphere is

$$\varphi(r) = \varphi(r, \Omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_\ell r^\ell Y_{\ell m}(\Omega).$$

The boundary values are $\varphi(R, \Omega) = \bar{\varphi}(\Omega)$ so, using the orthogonality of the spherical harmonics, we get

$$\varphi(r, \Omega) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} (r/R)^\ell Y_{\ell m}(\Omega) \int d\Omega' \bar{\varphi}(\Omega') Y^*_{\ell m}(\Omega').$$

We do the sum on m using the addition theorem for spherical harmonics, so

$$\varphi(r,\Omega) = \frac{R}{4\pi}\int d\Omega' \bar{\varphi}(\Omega') \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}}(2\ell+1)P_\ell(\hat{r}\cdot\hat{y}_S),$$

where $y_S \equiv (y_S, \Omega')$. Now let

$$I = \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}} P_\ell(\hat{r}\cdot\hat{y}_S) = \frac{1}{|r-y_S|} = \frac{1}{[r^2+R^2-2rR\cos(\hat{r}\cdot\hat{y}_S)]^{1/2}}$$

so

$$r\frac{\partial I}{\partial r} - R\frac{\partial I}{\partial R} = \sum_{\ell=0}^{\infty} \frac{r^\ell}{R^{\ell+1}}(2\ell+1)P_\ell(\hat{r}\cdot\hat{y}_S) = \frac{R^2-r^2}{|r-y_S|^3}.$$

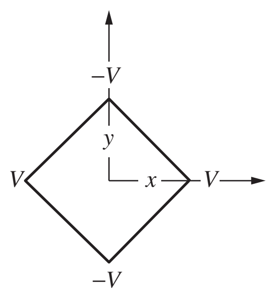
Therefore,

$$\varphi(r,\Omega) = \frac{R(R^2-r^2)}{4\pi}\int d\Omega' \frac{\bar{\varphi}(\Omega')}{|r-y_S|^3}$$

as required because $dy_S = R^2 d\Omega'$.

7.4 The Potential inside an Ohmic Duct

The geometry is



The corner values imply that the potential is invariant when $x \to -x$ or when $y \to -y$. This tells us that $B = C = F = 0$, so $\varphi(x,y) = A + Dx^2 + Ey^2$. Moreover, the potential inside the duct satisfies Laplace's equation. Therefore,

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} = 2D + 2E = 0.$$

This gives $D = -E$ and our trial solution becomes

$$\varphi(x,y) = A + D(x^2-y^2).$$

We get $V = A + Da^2$ from $\varphi(a,0) = V$. We get $-V = A - Da^2$ from $\varphi(0,a) = -V$. Combining these two gives $A = 0$ and $D = V/a^2$. Therefore, the potential inside the duct is

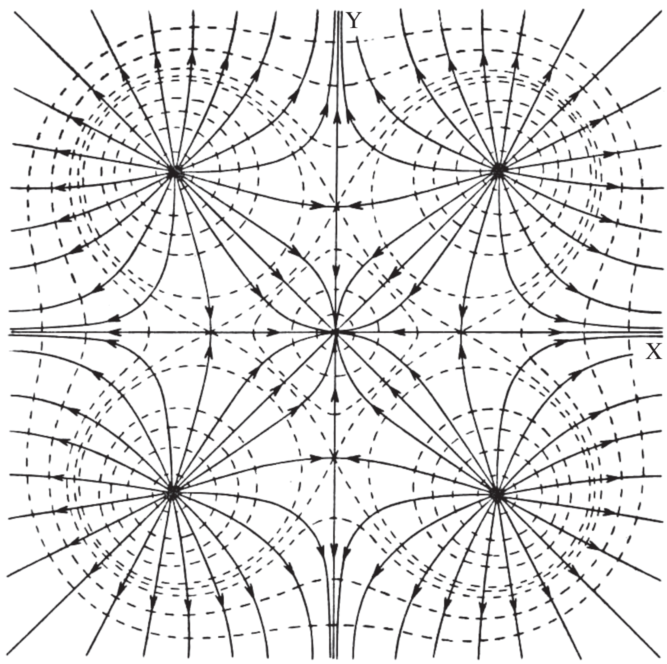
$$\varphi(x,y) = \frac{V}{a^2}(x^2 - y^2).$$

7.5 The Near-Origin Potential of Four Point Charges

(a) The charge distribution is invariant when $x \to -x$ or $y \to -y$ or $z \to -z$. This means that all the terms proportional to x, y, z, xy, xz, and yz are absent. Symmetry also implies that $H = I$. Otherwise, $A > 0$ trivially and $J < 0$ because the potential must decrease away from the origin on the $\pm z$-axes. Finally, $0 = \nabla^2\varphi = 4H - 2|J|$ so $H > 0$. We conclude that

$$\varphi(x,y,z) = A + H(x^2 + y^2) - |J|z^2 + \cdots .$$

(b) We get a point charge field very near each charge and far away from all of them. Otherwise, part (a) implies that **E** points radially inward near the origin in the $z = 0$ plane. Therefore,



Source: E. Durand, *Electrostatique* (Masson, Paris, 1964).

7.6 The Microchannel Plate

The general separated-variable solution in Cartesian coordinates has the form

$$\varphi(x,y) = (A_0 + B_0 x)(C_0 + D_0 y) + \sum_{\alpha,\beta}(A_\alpha e^{\alpha x} + B_\alpha e^{-\alpha x})(C_\beta e^{\beta y} + D_\beta e^{-\beta y}),$$

where $\alpha^2 + \beta^2 = 0$. On the other hand, the boundary conditions imply that $\varphi(x, y + b) = \varphi(x, y) + 2$. We can achieve this by writing $\varphi(x, y) = 2y/b + f(x, y)$ where $f(x, y) = f(x, y + b)$ is periodic. In that case,

$$\varphi(x,y) = \frac{2y}{b} + \sum_{n=1}^{\infty} \{A_n \sinh(2\pi nx/b) + B_n \cosh(2\pi nx/b)\} \{C_n \sin(2\pi ny/b) + D_n \cos(2\pi ny/b)\}$$

where $y = 0$ corresponds to the midpoint of the $\varphi = 0$ electrode. The coefficients are found by imposing the Dirichlet boundary conditions on the plates. Thus, multiplying the $x = 0$ condition by $\sin(2\pi my/b)$ and integrating gives

$$\begin{aligned}\int_{-b/2}^{b/2} dy\, [\varphi(0,y) - 2y/b] \sin(2\pi my/b) = \sum_{n=1}^{\infty} B_n \Bigg\{ & C_n \int_{-b/2}^{b/2} dy \sin(2\pi ny/b) \sin(2\pi my/b) \\ & + D_n \int_{-b/2}^{b/2} dy \cos(2\pi ny/b) \sin(2\pi my/b) \Bigg\}. \end{aligned} \tag{1}$$

Making use of, e.g., $\int_{-b/2}^{b/2} dy \sin(2\pi ny/b) \sin(2\pi my/b) = \frac{1}{2} b\, \delta_{mn}$, gives

$$B_n C_n = (2/\pi n) \cos \pi m = (-)^m (2/\pi n)$$

because the integrand of the second integral on the right side of (1) is odd. Similarly, multiplying the $x = 0$ condition by $\cos(2\pi my/b)$ and integrating similarly gives

$$\begin{aligned}\int_{-b/2}^{b/2} dy\, [\varphi(0,y) - 2y/b] \cos(2\pi my/b) = \sum_{n=1}^{\infty} B_n \Bigg\{ & C_n \int_{-b/2}^{b/2} dy \sin(2\pi ny/b) \cos(2\pi my/b) \\ & + D_n \int_{-b/2}^{b/2} dy \cos(2\pi ny/b) \cos(2\pi my/b) \Bigg\}. \end{aligned} \tag{2}$$

The integral on the left and the first integral on the right side of (2) are zero because their integrands are odd. This gives $B_n D_n = 0$. Combining this with the above gives $D_n = 0$.

We now impose the boundary condition at $x = d$, multiply by $\sin(2\pi my/b)$ and integrate. This gives

$$\int_{-b/2}^{0} dy\, [\varphi(d,y) - 2y/d] \sin(2\pi my/b) + \int_{0}^{b/2} dy\, [\varphi(d,y) - 2y/d] \sin(2\pi my/b)$$

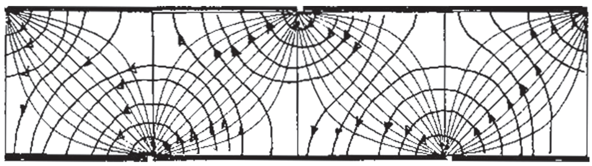
$$= \sum_{n=1}^{\infty} C_n \int_{-b/2}^{b/2} dy \sin(2\pi ny/b) \sin(2\pi my/b) \{A_n \sinh(2\pi d/b) + B_n \cosh(2\pi d/b)\}.$$

The integrals are the same as before and we get

$$A_n C_n = \frac{2\pi/n - C_n B_n \cosh(2\pi d/b)}{\sinh(2\pi d/b)} = \frac{2\pi}{n \sinh(2\pi d/b)} \{1 - (-)^n \cosh(2\pi d/b)\}.$$

Putting all of this together gives the final result for the potential:

$$\varphi(x,y) \;=\; \frac{2y}{d}+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2\pi ny/b)}{n}\left\{\frac{1-(-)^n\cosh(2\pi/b)}{\sinh(2\pi/b)}\sinh\frac{2\pi nx}{b}+(-)^n\cosh\frac{2\pi nx}{b}\right\}.$$



Source: V.K. Zworykin, G.A. Morton, E.G. Ramberg, J. Hillier, and A.W. Vance, *Electron Optics and the Electron Microscope* (Wiley, New York, 1945).

7.7 A Potential Patch by Separation of Variables

(a) Separation of variables in Cartesian coordinates gives a general solution of the form

$$\varphi(x,y,z)=\sum_{\alpha,\beta,\gamma}X_\alpha(x)Y_\beta(y)Z_\gamma(z)$$

where

$$X_\alpha(x)=\begin{cases} A_0+B_0x & \alpha=0,\\ A_\alpha e^{i\alpha x}+B_\alpha e^{-i\alpha x} & \alpha\neq 0\end{cases}$$

and similarly for $Y_\beta(y)$ and $Z_\gamma(z)$, subject to the constraint that $\alpha^2+\beta^2+\gamma^2=0$. The potential is an even function of x and y and must be bounded when either of these variables goes to infinity. This suggests an expansion of the sort

$$\varphi(x,y,z)=V\int_{-\infty}^{\infty}d\alpha\int_{-\infty}^{\infty}d\beta\,A(\alpha)B(\beta)\cos(\alpha x)\cos(\beta y)Z_{\alpha,\beta}(z).$$

The choice

$$Z_{\alpha,\beta}(z)=\frac{e^{\sqrt{\alpha^2+\beta^2}\,z}-e^{\sqrt{\alpha^2+\beta^2}\,(2d-z)}}{1-e^{2d\sqrt{\alpha^2+\beta^2}}}$$

guarantees that $\varphi(x,y,z=d)=0$ and $Z_{\alpha,\beta}(z=0)=1$. The boundary condition at $z=0$ is satisfied if we demand that

$$\int_{-\infty}^{\infty}d\alpha\,A(\alpha)\cos(\alpha x)=\begin{cases}0 & |x|>a,\\ 1 & |x|<a,\end{cases}$$

$$\int_{-\infty}^{\infty}d\beta\,B(\alpha)\cos(\beta y)=\begin{cases}0 & |y|>a,\\ 1 & |y|<a.\end{cases}$$

To extract, say, $A(\alpha)$, multiply both sides of the first equation just above by $\cos(\beta x)$ and integrate:

$$\int_{-\infty}^{\infty} d\alpha A(\alpha) \int_{-\infty}^{\infty} dx\, \cos(\alpha x)\cos(\beta x) = \int_{-a}^{a} dx\, \cos(\beta x) = \frac{2\sin(\beta a)}{\beta}.$$

The orthogonality integral we need,

$$\int_{\infty}^{\infty} dx\, \cos(\alpha x)\cos(\beta x) = \pi\delta(\alpha - \beta),$$

is derived by taking the real and imaginary parts of the identity

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \exp[ik(\alpha-\beta)] = \delta(\alpha-\beta).$$

The final result is $A(p) = B(p) = \dfrac{1}{\pi}\dfrac{\sin pa}{p}$. Therefore,

$$\varphi(x,y,z) = \frac{V}{\pi^2}\int_{-\infty}^{\infty} d\alpha\, \frac{\sin\alpha a}{\alpha}\cos\alpha x \int_{-\infty}^{\infty} d\beta\, \frac{\sin\beta a}{\beta}\cos\beta y \left[\frac{e^{\sqrt{\alpha^2+\beta^2}\,z} - e^{\sqrt{\alpha^2+\beta^2}\,(2d-z)}}{1-e^{2d\sqrt{\alpha^2+\beta^2}}}\right].$$

(b) The induced charge on the lower plate is

$$Q = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, \sigma(x,y) = -\epsilon_0 \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \frac{\partial\varphi}{\partial z}\bigg|_{z=0}.$$

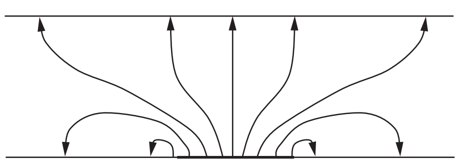
But

$$\int_{-\infty}^{\infty} dx \cos\alpha x \int_{-\infty}^{\infty} dy \cos\beta y = 4\pi^2\delta(\alpha)\delta(\beta),$$

so

$$\begin{aligned} Q &= -\,4V\epsilon_0 a^2 \int_{-\infty}^{\infty} d\alpha\, \delta(\alpha) \int_{-\infty}^{\infty} d\beta\, \delta(\beta)\, \frac{\sin\alpha a}{\alpha a}\frac{\sin\beta a}{\beta a}\sqrt{\alpha^2+\beta^2}\,\frac{1+e^{2d\sqrt{\alpha^2+\beta^2}}}{1-e^{2d\sqrt{\alpha^2+\beta^2}}} \\ &= -\frac{4V\epsilon_0 a^2}{d}. \end{aligned}$$

(c)



Source: Prof. M.J. Cohen, University of Pennsylvania (private communication).

7.8 A Conducting Slot

The potential $\varphi(x,z)$ does not depend on y and $\varphi(x,z) \to 0$ as $z \to \infty$. Because the potential must reflect the symmetry of the slot with respect to reflection through $x = a/2$, we conclude that

$$\varphi(x,z) = \sum_{\gamma>0} \left[A_\gamma \exp(i\gamma x) + B_\gamma \exp(-i\gamma x)\right] \exp(-\gamma z).$$

The Dirichlet boundary conditions on the walls and base of the slot determine the expansion coefficients and further restrict the allowed values of the separation constant γ. Thus, $\varphi(0,z) = 0$ fixes $B_\gamma = -A_\gamma$ and $\varphi(a,z) = 0$ requires that $\gamma = n\pi/a$ where n is a positive integer. The final boundary condition at $z = 0$ is

$$\varphi_0 = \sum_{n=1}^{\infty} A_n \sin(n\pi x/a) \qquad\qquad 0 \le x \le a.$$

We determine the coefficients of this Fourier series by multiplying both sides of the foregoing by $\sin(m\pi x/a)$, integrating over the indicated interval, and using the orthogonality integral

$$\int_0^a dx \sin\frac{n\pi x}{a} \sin\frac{m\pi x}{a} = \frac{a}{2}\delta_{m,n}.$$

The result is $A_m = (2\varphi_0/m\pi)(1 - \cos m\pi)$. Therefore, the electrostatic potential in the slot is

$$\varphi(x,z) = \frac{4\varphi_0}{\pi} \sum_{m=1,3,5,\ldots} \frac{1}{m}\sin(m\pi x/a)\, \exp(-m\pi z/a).$$

The most significant feature of this potential is its behavior when $z \gg a$. The $m = 1$ term dominates in that case and

$$\varphi(x,z) \approx \frac{4\varphi_0}{\pi} \sin\left(\pi x/a\right) \exp\left(-\pi z/a\right).$$

This shows that the influence of the source charge at $y = 0$ penetrates up the slot no farther than a distance of the order of a itself. The transverse variations of the potential vary on the same spatial scale. It could hardly be otherwise—the slot width is the only characteristic length in the problem.

7.9 A Two-Dimentional Potential Problem in Cartesian Coordinates

By symmetry, $\varphi(x,z) = \varphi(x,-z)$. This tells us that the potential cannot contain a linear term in z. There also cannot be a term linear in x because the solution domain extends to $\pm\infty$ in the x-direction. Hence, if we insist on a Fourier representation in the x-direction, the general form of the potential between the plates must have the form

$$\varphi(x,z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk A(k) e^{ikx} \cosh kz.$$

To find the function $A(k)$, we evaluate this expression at $z = \pm d$, use the Fourier inversion theorem, and regularize the integrals using

$$\int_0^{\pm\infty} dx e^{-ikx} = \lim_{\delta\to 0}\int_0^{\pm\infty} dx e^{-ikx} e^{\mp\delta x}.$$

The result is

$$A(k)\cosh kd = \int_{-\infty}^{\infty} dx\,\varphi(x,d) e^{-ikx} = \varphi_0\left\{\int_0^{\infty} dx\, e^{-ikx} - \int_{-\infty}^{0} dx\, e^{-ikx}\right\} = \frac{2\varphi_0}{ik}.$$

Therefore,

$$\varphi(x,z) = \frac{\varphi_0}{\pi i}\int_{-\infty}^{\infty}\frac{dk}{k}\frac{\cosh kz}{\cosh kd}e^{ikx} = \frac{2\varphi_0}{\pi}\int_0^{\infty}\frac{dk}{k}\frac{\cosh kz}{\cosh kd}\sin kx.$$

7.10 An Electrostatic Analog of the Helmholtz Coil

The general solution to Laplace's equation inside the shell is

$$\varphi(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell\left(\frac{r}{R}\right)^{\ell} P_\ell(\cos\theta) \qquad\qquad r \le R.$$

We would get a uniform electric field $\mathbf{E} = -(A_1/R)\hat{\mathbf{z}}$ everywhere inside the sphere if all the A_k were zero except for $k = 1$. By symmetry, only odd ℓ contribute to the sum. Hence, we should choose θ_0 so $A_3 = 0$ because this term in the sum varies most rapidly near the origin. The orthogonality integral for the Legendre polynomials is

$$\int_0^{\pi} d\theta \sin\theta P_n(\cos\theta) P_m(\cos\theta) = \frac{2}{2n+1}\delta_{nm}.$$

Therefore, if $V(\theta) = \varphi(R,\theta)$, the expansion coefficients above are

$$A_\ell = \frac{2\ell+1}{2}\int_0^{\pi} d\theta \sin\theta P_\ell(\cos\theta) V(\theta).$$

In particular,

$$\begin{aligned} A_3 &= \frac{7}{2}\int_0^{\pi} d\theta \sin\theta P_3(\cos\theta) V(\theta) \\ &= 7V\int_0^{\pi} d\theta \sin\theta P_3(\cos\theta) \\ &= 7V\int_{\cos\theta_0}^{0} dx P_3(x). \end{aligned}$$

Now $P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$ so the condition $A_3 = 0$ becomes

$$5\cos^4\theta_0 - 6\cos^2\theta_0 + 1 = 0.$$

The solutions to this quadratic equation are $\cos^2\theta_0 = 1$ and $\cos^2\theta_0 = 1/5$. Only the second of these makes physical sense so we conclude that

$$\theta_0 \approx 63^\circ.$$

Source: C.E. Baum, *IEEE Transaction on Electromegnetic Compatibility* **30**, 9 (1988).

7.11 Make a Field inside a Sphere

Integrating each component of $\mathbf{E} = -\nabla\varphi$ gives us

$$\hat{x}: \qquad \varphi = \frac{\mathrm{V}_0}{R^3}x^2y + f(y,z)$$

$$\hat{y}: \qquad \varphi = \frac{\mathrm{V}_0}{R^3}\left(x^2y - \tfrac{1}{3}y^3\right) + g(x,z)$$

$$\hat{z}: \qquad \varphi = \frac{V_0}{R}z + h(x,y).$$

Therefore, when $x^2 + y^2 + z^2 \le R^2$,

$$\varphi(x,y,z) = \frac{V_0}{R^3}\left(x^2y - \tfrac{1}{3}y^3\right) + \frac{V_0}{R}z + const. \tag{1}$$

This potential satisfies Laplace's equation. Therefore, no volume charge is present inside the sphere. On the other hand, in spherical coordinates, we know that solutions of Laplace's equation take the form

$$\varphi(r,\theta) = \begin{cases} \displaystyle\sum_{\ell,m} c_\ell \left(\frac{r}{R}\right)^\ell Y_{\ell,m}(\theta,\phi) & r \le R, \\ \displaystyle\sum_{\ell,m}^{\infty} c_\ell \left(\frac{R}{r}\right)^{\ell+1} Y_{\ell,m}(\theta,\phi) & r \ge R. \end{cases}$$

The cubic terms in (1) can come only from a linear combination of $\ell = 3$ terms. The linear term is an $\ell = 1$ term. Therefore, because $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$, we get

$$\varphi(r,\theta,\phi) = \begin{cases} V_0\dfrac{r}{R}\cos\theta + V_0\left(\dfrac{r}{R}\right)^3\sin^3\theta\left(\cos^2\phi\sin\phi - \tfrac{1}{3}\sin^3\phi\right) & r \le R, \\ V_0\left(\dfrac{R}{r}\right)^2\cos\theta + V_0\left(\dfrac{R}{r}\right)^4\sin^3\theta\left(\cos^2\phi\sin\phi - \tfrac{1}{3}\sin^3\phi\right) & r \ge R. \end{cases}$$

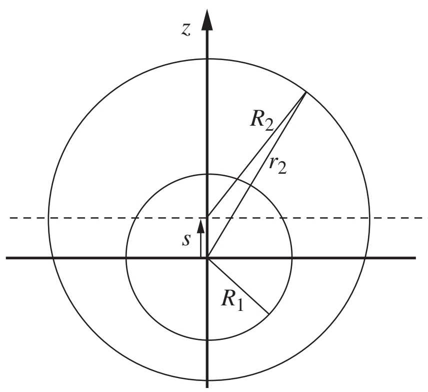
The charge density follows at once from the matching condition

$$\sigma(\theta,\phi) = \epsilon_0 \left.\frac{\partial\varphi}{\partial r}\right|_{r=R^-} - \epsilon_0 \left.\frac{\partial\varphi}{\partial r}\right|_{r=R^+}$$

$$= \epsilon_0 \frac{V_0}{R}\left[3\cos\theta + 7\sin^3\theta\left(\cos^2\phi\sin\phi - \tfrac{1}{3}\sin^3\phi\right)\right].$$

Source: Prof. M.J. Cohen, University of Pennsylvania (private communication).

7.12 The Capacitance of an Off-Center Capacitor



(a) Let (r_2, θ) denote a point on the outer shell with respect to the origin of the inner shell. By the law of cosines, $R^2 = r_2^2 + s^2 - 2r_2 s\cos\theta$. Therefore, to first order in s, the boundary of the outer shell is

$$r_2 = R_2 + s\cos\theta. \tag{1}$$

If the shells were exactly concentric, the potential between them would have the form $\varphi(r) = a + b/r$. Therefore, in light of (1) and the general solution of Laplace's equation in polar coordinates, we expect the potential in the space between the displaced shells to take the form

$$\varphi(r,\theta) = a + \frac{b}{r} + s\left(cr + \frac{d}{r^2}\right)\cos\theta + O(s^2). \tag{2}$$

To order s, the boundary conditions at the shell surfaces are

$$V_1 = \varphi(R_1,\theta) = a + \frac{b}{R_1} + s\left(cR_1 + \frac{d}{R_1^2}\right)\cos\theta \tag{3}$$

$$V_2 = \varphi(r_2,\theta) = a + \frac{b}{R_2 + s\cos\theta} + s\left(c[R_2 + s\cos\theta] + \frac{d}{[R_2 + s\cos\theta]^2}\right)\cos\theta$$
$$= a + \frac{b}{R_2} + s\left(cR_2 + \frac{d}{R_2^2} - \frac{b}{R_2^2}\right)\cos\theta. \qquad (4)$$

V_1 and V_2 are constants so the coefficients of $\cos\theta$ must vanish in (3) and (4). This fixes $d = -cR_1^3$ and $b = c(R_2^3 - R_1^3)$. Moreover, subtracting (4) from (3) gives

$$b = (V_1 - V_2)R_1R_2/(R_2 - R_1).$$

Therefore,

$$c = (V_1 - V_2)\frac{R_1R_2}{(R_2^3 - R_1^3)(R_2 - R_1)} \qquad d = -(V_1 - V_2)\frac{R_1^4R_2}{(R_2^3 - R_1^3)(R_2 - R_1)}.$$

Using (2), we conclude that the charge density on the surface of the inner shell is

$$\sigma(\theta) = -\epsilon_0 \left.\frac{\partial\varphi}{\partial r}\right|_{r=R_1} = \epsilon_0\frac{R_1R_2(V_2 - V_1)}{R_2 - R_1}\left[\frac{1}{R_1^2} - \frac{3s}{R_2^3 - R_1^3}\cos\theta\right].$$

The angular term in $\sigma(\theta)$ integrates to zero. Therefore, the total charge on the inner shell and the capacitance (to first order in s) are identical to the zero-order case of a concentric capacitor:

$$C_0 = \frac{Q}{V_1 - V_2} = 4\pi\epsilon_0\frac{R_1R_2}{R_2 - R_1}.$$

(b) By symmetry, there is only a z-component to the force on inner shell. Explicitly,

$$\mathbf{F} = \int dS\,\frac{\sigma^2}{2\epsilon_0}\hat{\mathbf{n}} = \hat{\mathbf{z}}2\pi R_1^2\int_0^\pi d\theta\sin\theta\frac{\sigma^2(\theta)}{2\epsilon_0}\cos\theta = -\frac{Q^2}{4\pi\epsilon_0}\frac{s\hat{\mathbf{z}}}{R_2^3 - R_1^3}.$$

(c) The force in part (b) can be computed from a variation of the capacitor energy. Therefore, if we imagine the charge fixed,

$$F_z = -\frac{dU_E}{ds} = -\frac{d}{ds}\frac{Q^2}{2C} = \frac{Q^2}{2C^2}\frac{dC}{ds} = -\frac{Q^2}{4\pi\epsilon_0}\frac{s}{R_2^3 - R_1^3}.$$

Integrating this gives

$$\frac{1}{C} = \frac{1}{4\pi\epsilon_0}\frac{s^2}{R_2^3 - R_1^3} + \frac{1}{C_0}.$$

Therefore, to second order in s,

$$C = C_0\left[1 - \frac{C_0}{4\pi\epsilon_0}\frac{s^2}{R_2^3 - R_1^3}\right].$$

7.13 The Plane-Cone Capacitor

(a) The geometry of the capacitor is invariant to rotations around the cone axis and to rescaling the radial variable from r to λr (where λ is a constant). These facts imply that the potential cannot depend on either variable.

(b) Since $\varphi(\mathbf{r}) = \varphi(\theta)$, Laplace's equation reads

$$\nabla^2 \varphi = \frac{1}{r^2 \sin\theta} \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\varphi}{\partial\theta}\right) = 0.$$

Hence,

$$\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\varphi}{\partial\theta}\right) = 0 \quad \Rightarrow \quad \sin\theta \frac{\partial\varphi}{\partial\theta} = K = const.$$

This gives the potential as

$$\varphi(\theta) = A + \int \frac{K}{\sin\theta} = A + B \ln\tan(\theta/2).$$

The boundary conditions are

$$\begin{aligned} \varphi(\pi/2) = 0 &= A + B\ln\tan(\pi/4) = A \\ \varpi(\pi/4) &= V = B\ln\tan(\pi/8). \end{aligned}$$

Therefore,

$$\varphi(\theta) = V \frac{\ln\tan(\theta/2)}{\ln\tan(\pi/8)}.$$

7.14 A Conducting Sphere at a Dielectric Boundary

Let the polar z-axis pass through the center of the sphere perpendicular to the dielectric interface.

(a) The general solution of Laplace's equation outside the sphere is

$$\varphi(r,\theta) = \sum_{\ell=0}^{\infty} \frac{A_\ell}{r^{\ell+1}} P_\ell(\cos\theta).$$

At the sphere boundary, we must have $\varphi(R,\theta) = const.$ This tells us that $A_\ell = 0$ for all $\ell \neq 0$ so

$$\varphi(r,\theta) = \frac{A_0}{r} \quad \Rightarrow \quad \mathbf{E} = \frac{A_0}{r^2}\hat{\mathbf{r}}.$$

Therefore, wherever the dielectric constant is κ_i $(p = 1, 2)$,

$$\mathbf{D}_i(r) = \epsilon_0 \kappa_i \frac{A_0}{r^2}\hat{\mathbf{r}}.$$

The constant A_0 follows from $\nabla \cdot \mathbf{D} = \rho_c$. Using a spherical Gaussian surface,

$$\int_S d\mathbf{S} \cdot \mathbf{D} = \epsilon_0 A_0 2\pi \left[\kappa_1 \int_0^{\pi/2} d\theta \sin\theta + \kappa_2 \int_{\pi/2}^{\pi} d\theta \sin\theta \right] = 2\pi\epsilon_0 A_0(\kappa_1 + \kappa_2) = Q.$$

We conclude that

$$\varphi(r) = \frac{Q}{2\pi\epsilon_0(\kappa_1 + \kappa_2)} \frac{1}{r}.$$

(b) The free charge on the surface of the sphere follows from Gauss' law as

$$\sigma_c = \mathbf{D}(R) \cdot \hat{\mathbf{r}} = \begin{cases} \dfrac{\kappa_1}{\kappa_1 + \kappa_2} \dfrac{Q}{2\pi R^2} & \text{in region } \kappa_1, \\[2ex] \dfrac{\kappa_2}{\kappa_1 + \kappa_2} \dfrac{Q}{2\pi R^2} & \text{in region } \kappa_2. \end{cases}$$

There is polarization charge at the sphere boundary. Its value is $\sigma_P = (1 - \kappa)\sigma_c/\kappa$. This charge is compensated by polarization charge at infinity. There is no polarization charge at the κ_1/κ_2 interface because $\mathbf{E}$ and hence $\mathbf{P}$ are everywhere radial. This means that $\mathbf{P} \cdot \hat{\mathbf{n}} = 0$ at the interface.

7.15 The Force on an Inserted Conductor

The grounded, inserted sphere draws up charge to its surface. The potential φ_σ due to this surface distribution must exactly cancel $\varphi_{\rm ext}$ everywhere inside the sphere. Moreover, φ_σ satisfies Laplace's equation and is continuous at $r = R$. Therefore,

$$\varphi_\sigma(r, \theta) = \begin{cases} -\sum\limits_{n=1}^{\infty} \alpha_n \left(\dfrac{r}{R}\right)^n P_n(\cos\theta) & r \le R, \\[2ex] -\sum\limits_{n=1}^{\infty} \alpha_n \left(\dfrac{R}{r}\right)^{n+1} P_n(\cos\theta) & r \ge R. \end{cases}$$

We get the force on the sphere by integrating the induced surface charge density over the sphere surface S:

$$\mathbf{F} = \int_S dS \frac{\sigma^2}{2\epsilon_0} \hat{\mathbf{n}}.$$

The unit normal is $\hat{\mathbf{n}} = \hat{\mathbf{r}} = \cos\theta\hat{\mathbf{z}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \sin\theta\cos\phi\hat{\mathbf{x}}$ and the surface charge density is

$$\sigma(\theta) = -\,\epsilon_0 \frac{\partial(\varphi_\sigma + \varphi_{\rm ext})}{\partial r}\bigg|_{r=R^+} = -\epsilon_0 \sum_{n=1}^{\infty} (2n+1) \frac{\alpha_n}{R} P_n(\cos\theta).$$

Only the $\hat{\mathbf{z}}$ contribution survives the ϕ integration in $dS = R^2 \sin\theta d\theta d\phi$. Therefore, if $x = \cos\theta$,

$$F_z = \pi\epsilon_0 \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(2n+1)(2k+1)\alpha_n\alpha_k \int_{-1}^{1} dx P_k(x) x P_n(x).$$

Using the hint,

$$F_z = \pi\epsilon_0 \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(2n+1)(2k+1)\alpha_n\alpha_k \frac{1}{2n+1}\int_{-1}^{1} dx P_k(x)\left[(n+1)P_{n+1}(x) + nP_{n-1}(x)\right].$$

On the other hand,

$$\int_{-1}^{1} dx P_n(x)P_k(x) = \frac{2}{2n+1}\delta_{nk}.$$

Therefore,

$$F_z = 2\pi\epsilon_0 \sum_{n=1}^{\infty}\frac{(2n+1)(2n+3)}{2n+1}\alpha_n\alpha_{n+1}\frac{n+1}{2n+3} + 2\pi\epsilon_0\sum_{n=1}^{\infty}\frac{(2n+1)(2n-1)}{2n+1}\alpha_n\alpha_{n-1}\frac{n}{2n-1}.$$

The $n = 1$ term in the second sum does not contribute because $\alpha_0 = 0$. Therefore, we start the sum at $n = 2$ and define $m = n - 1$. This makes the second sum identical to the first sum, so

$$F_z = 4\pi\epsilon_0 \sum_{n=1}^{\infty}(n+1)\alpha_n\alpha_{n+1}.$$

7.16 A Segmented Cylinder

Inside the cylinder, the general solution to Laplace's equation in polar coordinates that satisfies $\varphi(\rho,\phi) = \varphi(\rho,-\phi)$ and is finite at the origin is

$$\varphi(\rho,\phi) = \sum_{m=0}^{\infty} A_m \left(\frac{\rho}{R}\right)^m \cos m\phi.$$

Therefore, using the boundary conditions and the integral

$$\int_0^{2\pi} d\phi \cos m\phi \cos n\phi = \pi\delta_{mn} \qquad (m \neq 0),$$

we find

$$\int_0^{2\pi} d\phi\, \varphi(R,\varphi)\cos n\phi = \int_{-\alpha}^{\alpha} d\phi \cos n\phi = \sum_{m=0}^{\infty} A_m \int_0^{2\pi} d\phi \cos m\phi \cos n\phi.$$

We conclude that $A_m = \dfrac{2\sin m\alpha}{m\pi}$ when $m > 0$ and $A_0 = \alpha/\pi$, so

$$\varphi(\rho,\phi) = \frac{\alpha}{\pi} + \frac{2}{\pi}\sum_{m=0}^{\infty}\frac{1}{m}\left(\frac{\rho}{R}\right)^m \sin m\alpha \cos m\phi.$$

7.17 An Incomplete Cylinder

The general solution of Laplace's equation inside and outside a cylinder of radius R is

$$\varphi_{\rm in}(r,\theta) = A_0 + \sum_n A_n (r/R)^n G_n(\theta)$$

$$\varphi_{\rm out}(r,\theta) = A_0 - \frac{\lambda}{2\pi\epsilon_0}\ln(r/R) + \sum_n A_n (R/r)^n G_n(\theta),$$

where $G_n(\theta)$ stands for a linear combination of $\exp(\pm in\theta)$ functions. The logarithmic term is present because the cylinder looks like a line charge when viewed from a great distance. The difference in induced surface charge density between the inner and outer surfaces of the shell is

$$\sigma_{\rm out} - \sigma_{\rm in} = -\epsilon_0 \frac{\partial}{\partial r}\left[\varphi_{\rm out} + \varphi_{\rm in}\right]_{r=R} = \frac{\lambda}{2\pi R}.$$

Therefore, the difference in the total charge per unit length between the inside and outside is

$$Q_{\rm out} - Q_{\rm in} = R\int_0^{2\pi/p} d\theta(\sigma_{\rm out} - \sigma_{\rm in}) = \frac{\lambda}{p}.$$

But $Q_{\rm tot} = Q_{\rm out} + Q_{\rm in} = \lambda$ so we can solve for

$$Q_{\rm out} = \frac{1}{2}\lambda\left[1 + \frac{1}{p}\right].$$

This gives the fraction of charge on the inner surface of the shell as

$$\frac{Q_{\rm tot} - Q_{\rm in}}{Q_{\rm tot}} = \frac{1}{2}\left[1 - \frac{1}{p}\right].$$

Source: Prof. M.J. Cohen, University of Pennsylvania (private communication).

7.18 The Two-Cylinder Electron Lens

Our strategy is to find $\varphi(\rho,z)$ separately for $z<0$ and $z>0$ and match the solutions together at $z=0$. The boundary conditions are

$$\varphi(R,z) = \begin{cases} V_L & z<0 \\ V_R & z>0. \end{cases}$$

Rotational symmetry demands that only the zero-order Bessel functions $J_0(k\rho)$ and $N_0(k\rho)$ can be involved. The latter is not regular at the origin and the potential must be bounded at

$|z| \to \infty$ inside either tube. Therefore, if we write the zeroes of $J_0(x)$ in the form $x_m = k_m R$,

$$\varphi(\rho, z) = \begin{cases} V_L + \sum_{n=1}^{\infty} A_n J_0(k_n \rho) e^{k_n z} & z < 0, \\ V_R + \sum_{n=1}^{\infty} B_n J_0(k_n \rho) e^{-k_n z} & z > 0. \end{cases}$$

The cylinders possess a reflection symmetry through the plane $z = 0$, which implies that $E_z(\rho, z) = E_z(\rho, -z)$ where $E_z = -\partial\varphi/\partial z$. This gives $A_n = -B_n$. The same conclusion follows from the fact that $\varphi(\rho, z)$ varies smoothly from V_L at $z \to -\infty$ to $\frac{1}{2}(V_L + V_R)$ at $z = 0$ to V_R at $z \to +\infty$. Finally, the potential must be continuous at $z = 0$ for $\rho < R$. This yields

$$V_L - V_R = 2 \sum_{n=1}^{\infty} B_n J_0(k_n \rho).$$

Now multiply both sides of this equation by $d\rho\, J_0(k_m \rho)\rho$ and integrate from 0 to R. Using the integrals given in the statement of the problem, we find

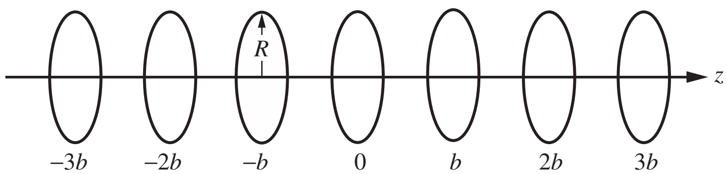
$$B_n = \frac{V_L - V_R}{k_n R J_1(x_n)}.$$

If $\text{sgn}(z) = z/|z|$, the potential inside the cylinders is

$$\varphi(\rho, z) = \frac{1}{2}(V_R + V_L) + \text{sgn}(z) \left[\frac{1}{2}(V_L + V_R) - (V_L - V_R) \sum_{n=1}^{\infty} \frac{J_0(k_n \rho)}{k_n R J_1(k_n R)} \exp(-k_n |z|) \right].$$

This expression is discontinuous at $z = 0$ when $\rho = R$ but is perfectly continuous everywhere within the cylinder (where the particle trajectories are confined).

7.19 A Periodic Array of Charged Rings



In cylindrical coordinates, the general solution for $\varphi(\rho, z)$ involves a multiplicative factor of

$$Z_k(z) = \begin{cases} s_k \exp(kz) + t_k \exp(-kz) & k \neq 0, \\ s_0 + t_0 z & k = 0. \end{cases}$$

The potential must be bounded as $|z| \to \infty$, so we are obliged to set $t_0 = 0$ and choose the separation constant $k = i\kappa$ to be purely imaginary. Combining this with the fact that $\varphi(\rho, z)$ is independent of the azimuthal angle ϕ tells us that the radial solution is a linear combination of $\ln \rho$ and the modified Bessel functions $I_0(\kappa\rho)$ and $K_0(\kappa\rho)$. $I_0(\kappa\rho)$ is finite at the origin and diverges exponentially as $\rho \to \infty$. $K_0(\kappa\rho)$ diverges as $\rho \to 0$ but goes to zero exponentially as $\rho \to \infty$. This suggests we construct a matching surface at $\rho = R$ and retain only the I_0 functions when $\rho > R$ and only the K_0 functions when $\rho > R$. We retain the $\ln \rho$ piece of the radial solution when $\rho > R$ because the potential of the rings approaches the potential of a charged line when $\rho \to \infty$. Finally, the potential is periodic $[\varphi(z+b) = \varphi(z)]$ and even $[\varphi(z) = \varphi(-z)]$ so $\cos \kappa z$ with $\kappa = 2\pi nz/b$ is the only possible z-dependence. Combining all of this information together leads us to write

$$\varphi(\rho, z) = \begin{cases} \displaystyle\sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n}{b} z\right) I_0\left(\frac{2\pi n}{b}\rho\right) K_0\left(\frac{2\pi n}{b} R\right) & \rho \le R, \\ \displaystyle\sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi n}{b} z\right) K_0\left(\frac{2\pi n}{b}\rho\right) I_0\left(\frac{2\pi n}{b} R\right) + A_0 \ln(\rho/R) & \rho \ge R. \end{cases}$$

Notice that we have "built in" the continuity of the potential at $\rho = R$ by inserting the constants $K_0(2\pi nR/b)$ and $I_0(2\pi nR/b)$ into the $\rho \le R$ and $\rho \ge R$ sums, respectively. To determine the expansion coefficients, we use the matching condition

$$\left[-\frac{\partial \varphi_{out}}{\partial \rho} + \frac{\partial \varphi_{in}}{\partial \rho}\right]_{\rho=R} = \frac{\sigma(z)}{\epsilon_0}. \tag{1}$$

Exploiting Example 1.6, we write the surface charge density of the rings on the $\rho = R$ cylindrical surface in the form

$$\frac{\sigma(z)}{\epsilon_0} = \frac{Q}{2\pi\epsilon_0 R} \sum_{m=-\infty}^{\infty} \delta(z - mb) = \frac{Q}{2\pi\epsilon_0 R}\left[\frac{1}{b} + \frac{2}{b}\sum_{n=1}^{\infty} \cos\left(\frac{2\pi nz}{b}\right)\right]. \tag{2}$$

With $y = 2\pi nR/b$, the left side of the matching condition (1) is

$$\sum_{n=1}^{\infty} A_n \cos\left(\frac{2\pi nz}{b}\right) \frac{y}{R}\left[I_0'(y)K_0(y) - I_0(y)K_0'(y)\right] - \frac{A_0}{R}. \tag{3}$$

Using the hint and the linear independence of the cosine functions, we impose the matching condition by equating similar terms in (2) and (3). The final result which completes the solution is

$$A_0 = -\frac{Q}{2\pi\epsilon_0 b} \qquad \text{and} \qquad A_n = \frac{Q}{\pi\epsilon_0 b}.$$

The sums in $\varphi(\rho, z)$ converge (although A_n does not depend on n) due to the exponential behavior of the modified Bessel functions when their arguments get large.

Our solution has the property that

$$\varphi \to -\frac{Q}{2\pi\epsilon_0 b}\ln(\rho/R)$$

as $\rho \to \infty$. This is exactly the result we expect because the "apparent" line charge has a charge/length $\lambda = Q/b$. Notice that this observation would have told us to retain the logarithm term (to match the A_0 term) if we had not realized at the beginning it should be there. Always check asymptotic and limiting cases!

Source: Prof. M.J. Cohen, University of Pennsylvania (private communication).

7.20 Axially Symmetric Potentials

The proposed solution satisfies $V(0,z) = V(z)$. Uniqueness guarantees that the proposed solution is the only solution if it satisfies Laplace's equation. The latter is

$$\frac{1}{r}\frac{d}{d\rho}\rho\frac{dV}{d\rho} + \frac{d^2V}{dz^2} = 0.$$

Now,

$$\frac{1}{\rho}\frac{d}{d\rho}\rho\frac{dV}{d\rho} = \frac{i}{\pi\rho}\int_0^\pi d\zeta\,\cos\zeta\,V'(z+i\rho\cos\zeta) - \frac{1}{\pi}\int_0^\pi d\zeta\,\cos^2\zeta\,V''(z+i\rho\cos\zeta)$$

and

$$\frac{d^2V}{dz^2} = \frac{1}{\pi}\int_0^\pi d\zeta\,V''(z+i\rho\cos\zeta) = \frac{1}{\pi}\int_0^\pi d\zeta\,\sin^2\zeta\,V''(z+i\rho\cos\zeta) + \frac{1}{\pi}\int_0^\pi d\zeta\,\cos^2\zeta\,V''(z+i\rho\cos\zeta).$$

This leaves us with

$$\nabla^2 V = \frac{i}{\pi\rho}\int_0^\pi d\zeta\,\cos\zeta\,V'(z+i\rho\cos\zeta) + \frac{1}{\pi}\int_0^\pi d\zeta\,\sin^2\zeta\,V''(z+i\rho\cos\zeta).$$

Finally, integrate the first term by parts to get

$$\nabla^2 V = \frac{i}{\pi\rho}V'\sin\zeta\Big|_0^\pi - \frac{1}{\pi}\int_0^\pi d\zeta\,\sin^2\zeta\,V''(z+i\rho\cos\zeta) + \frac{1}{\pi}\int_0^\pi d\zeta\,\sin^2\zeta\,V''(z+i\rho\cos\zeta) = 0.$$

This proves the assertion.

7.21 Circular-Plate Capacitor

(a) Laplace's equation must be satisfied everywhere except on the plates. Since $J_0(k\rho)$ is part of the proposed solution, the most general solution for the z-dependence is

$$Z_k(z) = s_k \exp(kz) + t_k \exp(-kz).$$

To avoid divergences at $z = \pm\infty$, we divide the space into three regions and write

$$f(k,z) = \begin{cases} a_k \exp(-kz) & z > L, \\ b_k \exp(kz) + c_k \exp(-kz) & -L < z < L, \\ d_k \exp(kz) & z < -L. \end{cases}$$

Using the given integral, inspection shows that the boundary conditions at the plate are satisfied in all three regions if

$$a_k = \exp(kL) - \exp(-kL)$$

$$b_k = \exp(-kL) \qquad c_k = -\exp(-kL)$$

$$d_k = \exp(-kL) - \exp(kL).$$

All of this can be combined to write the proposed solution as

$$\varphi(\rho,z) = \int_0^\infty dk \, \frac{2V}{1-e^{-2kL}} \frac{\sin(ka)}{\pi k} \left[\exp(-k|z-L|) - \exp(-k|z+L|)\right] J_0(k\rho).$$

(b) Using the results of (a), the discontinuity in the normal (z) component of the electric field at $z = L$ is

$$\left.\frac{\partial\varphi}{\partial z}\right|_{z=L^-} - \left.\frac{\partial\varphi}{\partial z}\right|_{z=L^+} = 2\int_0^\infty dk\, kA(k) J_0(k\rho).$$

When $\rho < a$, this quantity is the difference in the charge density induced on the top and bottom of the $z = L$ plate. But when $\rho > a$, we are out in the vacuum, and this quantity should be zero, which it is not. Therefore, the proposed solution is fallacious.

Source: B.D. Hughes, *Journal of Physics A* **17**, 1385 (1985).

7.22 A Dielectric Wedge in Polar Coordinates

(a) Put the junction at the origin of a two-dimensional polar coordinate system. The potential cannot depend on the radial coordinate ρ because the geometry is invariant to a change of scale where $\rho \to \lambda\rho$.

(b) Since there is no ρ-dependence, the most general solution to Laplace's equation in the two regions between the plates is

$$\varphi_1(\phi) = a + b\phi \qquad \varphi_2(\phi) = a' + b'\phi.$$

Continuity of the potential at $\phi = 0$ tells us that $a = a'$. Continuity of the normal component of $\mathbf{D}$ at $\phi - 0$ tells us that $\kappa_1 b = \kappa_2 b'$. Moreover,

$$\varphi_1(\phi_1) = a + b\phi_1 = V_1 \qquad \text{and} \qquad \varphi_2(-\phi_2) = a - \frac{\kappa_1}{\kappa_2} b\phi_2 = V_2.$$

Therefore,

$$a = \frac{\kappa_2 \phi_1 V_2 - \kappa_1 \phi_2 V_1}{\kappa_2 \phi_1 - \kappa_1 \phi_2} \qquad\qquad b = \kappa_2 \frac{V_1 - V_2}{\kappa_2 \phi_1 - \kappa_1 \phi_2}.$$

7.23 Contact Potential

(a) Imagine a radial expansion of the space $z \geq 0$ by a real scale factor, so $\rho \to \lambda\rho$. This has no physical effect whatsoever on the $z = 0$ boundary conditions because the potential is constant for $\phi = 0$ and also for $\phi = \pi$. Therefore, the potential for $z > 0$ cannot be affected either. But if $\varphi(\rho, \phi)$ depended in any way on ρ, factors of λ would appear in the solution, which contradicts the previous sentence. Hence, $\varphi(\rho, \phi) = \varphi(\phi)$.

(b) The general solution of Laplace's equation in plane polar coordinates is

$$\varphi(\rho, \phi) = (A_0 + B_0 \ln \rho)(a_0 + b_0 \phi) + \sum_{\alpha \neq 0} [A_\alpha \rho^\alpha + B_\alpha \rho^{-\alpha}][a_\alpha \sin \alpha\phi + b_\alpha \cos \alpha\phi].$$

Since there is no ρ-dependence, the only possibility is

$$\varphi(\phi) = A + B\phi.$$

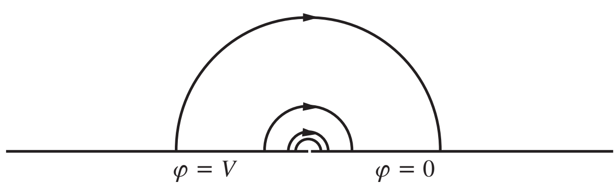
The boundary conditions force $A = 0$ and $B = V/\pi$, so

$$\varphi(\phi) = \frac{V}{\pi}\phi.$$

(c) The electric field is

$$\mathbf{E} = -\nabla\varphi = -\frac{1}{\rho}\frac{\partial\varphi}{\partial\phi}\hat{\boldsymbol{\phi}} = -\frac{V}{\pi}\frac{\hat{\boldsymbol{\phi}}}{\rho}.$$

The field lines are half-circles as shown below. The field intensity (and hence the density of field lines) increases as the origin is approached.



7.24 A Complex Potential

The given function is analytic for $|w| < R$ and $|w| > R$. This suggests that $|w| = R$ is the boundary of interest. Both the real and imaginary parts of this function satisfy Laplace's equation. We investigate the latter in light of the factor i in the first term. Therefore, our trial potential is

$$\varphi(w) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{2} \text{Im} \ln \left[\frac{R + iw}{R - iw} \right].$$

Now, since $w = x + iy$,

$$\begin{aligned}
\ln \left[\frac{R + iw}{R - iw} \right] &= \ln \left[\frac{(R + iw)(R - iw)^*}{(R - iw)(R - iw)^*} \right] \\
&= \ln \left[\frac{(R + iw)(R + iw^*)}{(R - iw)(R + iw^*)} \right] \\
&= \ln \left[\frac{R^2 + iR(w + w^*) - |z|^2}{R^2 + iR(w^* - w) + |z|^2} \right] \\
&= \ln \left[\frac{R^2 + i2Rx - x^2 - y^2}{R^2 + 2Ry + x^2 + y^2} \right].
\end{aligned}$$

Only the numerator of the bracketed quantity has an imaginary part. Therefore,

$$\text{Im} \ln \left[\frac{R + iw}{R - iw} \right] = \text{Im} \ln \left[R^2 - x^2 - y^2 + i2Rx \right].$$

We now recall that

$$\ln w = \ln |w| + i \arg(w) = \ln |w| + i \tan^{-1} \left[\frac{\text{Im} w}{\text{Re} w} \right].$$

Therefore,

$$\text{Im} \ln \left[\frac{R + iw}{R - iw} \right] = \tan^{-1} \left[\frac{2Rx}{R^2 - x^2 - y^2} \right].$$

Writing this in polar coordinates (ρ, ϕ) gives

$$\text{Im} \ln \left[\frac{R + iw}{R - iw} \right] = \tan^{-1} \left[\frac{2R\rho}{R^2 - \rho^2} \cos \phi \right],$$

and we conclude that the potential in question is

$$\varphi(\rho, \phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{\pi} \tan^{-1} \left[\frac{2R\rho}{R^2 - \rho^2} \cos \phi \right]. \tag{1}$$

The argument of the inverse tangent diverges when $\rho \to R$. Therefore, the inverse tangent itself approaches $\phi/2$ when $-\pi/2 < \phi < \pi/2$ and approaches $-\pi/2$ when in the interval $\pi/2 < \phi < 3\pi/2$. Consequently, (1) is the electrostatic potential inside the circle $\rho = R$, with $\varphi(R, \phi) = V_1$ when $-\pi/2 < \phi < \pi/2$ and $\varphi(R, \phi) = V_2$ when $\pi/2 < \phi < 3\pi/2$. A physical realization is a metal cylinder of radius R cut in half lengthwise. The two halves are separated by an infinitesimal distance, with one half held at potential V_1 and the other half held at potential V_2.

7.25 A Cylinder in a Uniform Field by Conformal Mapping

(a) The circle $w = a\exp(i\theta)$ and the part of the x-axis outside the circle map onto the u-axis as follows:

$$\begin{aligned} w = a\exp(i\theta), \quad 0 < \theta < \pi \quad \Rightarrow \quad g &= a(\exp(i\theta) + (\exp(-i\theta)) \\ &= 2a\cos\theta \Rightarrow \begin{cases} -2a < u < 2a, \\ v = 0, \end{cases} \end{aligned}$$

$$w = x, \quad -\infty < x < -a \quad \Rightarrow \quad g = w + \frac{a^2}{w} \Rightarrow \begin{cases} -\infty < u < -2a, \\ v = 0, \end{cases}$$

$$w = x, \quad a < x < \infty \quad \Rightarrow \quad g = w + \frac{a^2}{w} \Rightarrow \begin{cases} 2a < u < \infty, \\ v = 0. \end{cases}$$

(b) If the potential of the cylinder is zero, symmetry demands that the potential must be zero everyone on the x-axis outside the cylinder. In the g-plane, we define a complex potential

$$f(g) = \varphi(u, v) + i\psi(u, v) \tag{1}$$

and insist that $\nabla^2\varphi = 0$ with the boundary condition $\varphi(u, v = 0) = 0$. We also need $-\nabla\varphi = E_0\hat{\mathbf{v}}$. By inspection, the solution is $\varphi(u, v) = -E_0 v$. The Cauchy-Riemann relations tell us that

$$\frac{\partial\varphi}{\partial v} = -\frac{\partial\psi}{\partial u}.$$

Therefore, $\psi(u, v) = E_0 u$. Hence, using (1),

$$f(g) = -E_0 v + iE_0 u = iE_0(u + iv) = iE_0 g.$$

(c) Returning to the w-plane, the complex potential is

$$f(x, y) = iE_0 g = iE_0\left(w + \frac{a^2}{g}\right) = iE_0\left(x + iy + \frac{a^2}{x + iy}\right) = iE_0\left[x + iy + \frac{a^2(x - iy)}{x^2 + y^2}\right].$$

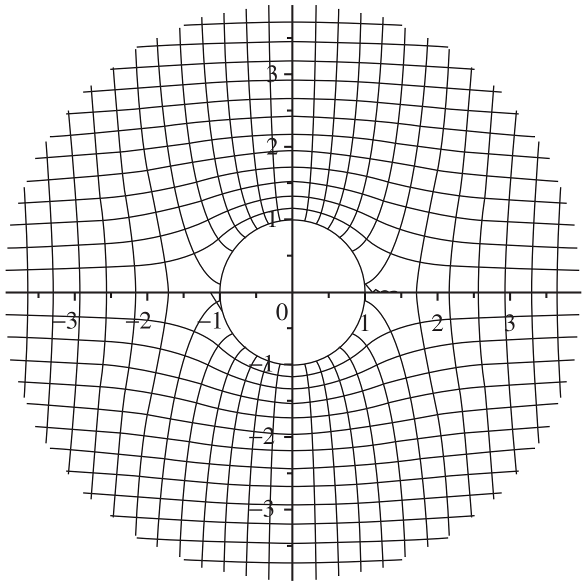
Therefore, the physical potential is

$$\varphi(x, y) = \text{Re}f = -E_0 y\left[1 - \frac{a^2}{x^2 + y^2}\right].$$

The corresponding electric field is

$$\mathbf{E} = -\nabla\varphi = \frac{2E_0 a^2 xy}{(x^2+y^2)^2}\,\hat{\mathbf{x}} + \left[E_0 + \frac{E_0 a^2 (y^2 - x^2)}{(x^2+y^2)^2}\right]\hat{\mathbf{y}}.$$

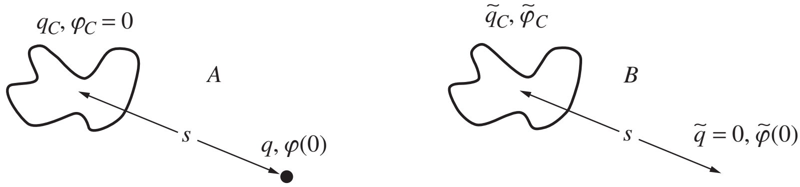
The corresponding field lines and equipotentials are shown below:



Chapter 8: Poisson's Equation

8.1 The Image Force and Its Limits

The diagram below shows the real system, called A, and the comparison system, B, which replaces the point charge q by no charge at all ($\tilde{q} = \tilde{\varphi} = 0$) and replaces the grounded conductor by a conductor with charge $\tilde{q_C}$ that produces a potential $\tilde{\varphi}_C$.



If the position of q is the origin, the reciprocity theorem for this situation reads

$$q\tilde{\varphi}(0) + q_C\tilde{\varphi}_C = \tilde{q}\varphi(0) + \tilde{q}_C\varphi_C.$$

Because the conductor is far from the body, we may write $\tilde{\varphi}(0) \approx \tilde{q}_C/4\pi\epsilon_0 s$. Moreover, because $\tilde{q} = 0$, the charge and potential of the conductor in B are related by the self-capacitance of the conductor: $\tilde{q}_C = \tilde{C}\tilde{\varphi}_C$. Therefore, the equation above simplifies to

$$q\frac{\tilde{q}_C}{4\pi\epsilon_0 s} + q_C\frac{\tilde{q}_C}{\tilde{C}} = 0 + 0.$$

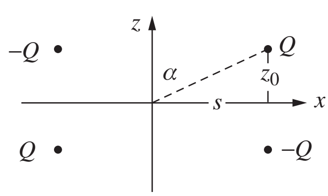
Hence,

$$q_C = -\frac{q\tilde{C}}{4\pi\epsilon_0 s}.$$

The self-capacitance depends only on the size and shape of the conductor. Therefore, in the limit of interest, the Coulomb force between the conductor and q varies as

$$F \propto \frac{qq_C}{s^2} \propto \frac{1}{s^3}.$$

8.2 Point Charge near a Corner



The diagram above shows an image system for the potential in the volume $x > 0$ and $z > 0$ because it grounds both the $z = 0$ and $x = 0$ planes. Moreover, all the field lines which leave Q end on one of the conducting surfaces because both are infinite in extent. This permits us to focus on the charge $Q_{\rm ind}$ induced on the $z = 0$ plane for $x > 0$; the charge on the $x = 0$ plane for $z > 0$ must be $-Q - Q_{\rm ind}$.

The text tells us that the image solution for a charge q at the point $(0, 0, z_0)$ above the (grounded) $z = 0$ plane corresponds to a charge induced on that plane equal to

$$\sigma(x,y) = -\frac{qz_0}{2\pi}\frac{1}{(x^2+y^2+z_0^2)^{3/2}}.$$

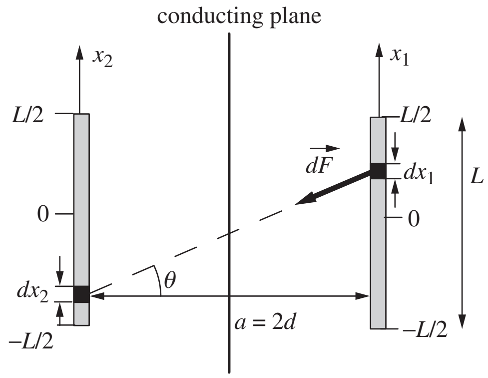
For our problem, we have a charge Q at $(s, 0, z_0)$ and a charge $-Q$ at $(-s, 0, z_0)$. Taking account of their images, both contribute to the charge density induced on the $z = 0$ plane. The total charge induced does not depend on the choice of origin for either charge. Therefore, the charge induced on the horizontal plate is

$$\begin{aligned}
Q_{\rm ind} &= -\frac{Qz_0}{2\pi}\int_{-s}^{\infty} dx \int_{-\infty}^{\infty}\frac{dy}{(y^2+x^2+z_0^2)^{3/2}} + \frac{Qz_0}{2\pi}\int_{s}^{\infty} dx \int_{-\infty}^{\infty}\frac{dy}{(y^2+x^2+z_0^2)^{3/2}} \\
&= -\frac{Qz_0}{2\pi}\int_{-s}^{\infty}\frac{2dx}{x^2+z_0^2} + \frac{Qz_0}{2\pi}\int_{s}^{\infty}\frac{2dx}{x^2+z_0^2} \\
&= -\frac{Qz_0}{2\pi}\frac{2}{z_0}\left\{\left[\tan^{-1}\left(\frac{x}{z_0}\right)\right]_{-s}^{\infty} - \left[\tan^{-1}\left(\frac{x}{z_0}\right)\right]_{s}^{\infty}\right\} \\
&= -\frac{2Q}{\pi}\tan\left(\frac{s}{z_0}\right) \\
&= -\frac{2Q}{\pi}\alpha.
\end{aligned}$$

The charge induced on the vertical plate is $Q(2\alpha/\pi - 1)$.

8.3 Rod and Plane

The total electric field is produced by the rod and its oppositely charged image rod located at the mirror position. Hence, the problem is to evaluate the force between two identical rods with opposite charges at distance $a = 2d$ from each other.



(a) If $k = 1/4\pi\epsilon_0$, the force between two small segments of the rods is attractive and has magnitude

$$dF = k\frac{\lambda dx_1 \lambda dx_2}{a^2 + (x_1 - x_2)^2}.$$

We only need the component of the force normal to the plane. This is

$$dF_\perp = dF \cdot \cos\theta = k\frac{\lambda^2 dx_1 dx_2}{a^2 + (x_1 - x_2)^2} \cdot \frac{a}{\sqrt{a^2 + (x_1 - x_2)^2}} = ka\lambda^2 \frac{dx_1 dx_2}{\left(a^2 + (x_1 - x_2)^2\right)^{3/2}}.$$

Consequently, the net force is

$$\begin{aligned} F &= ka\lambda^2 \int_{-L/2}^{L/2}\int_{-L/2}^{L/2} \frac{dx_1 dx_2}{\left[a^2 + (x_1 - x_2)^2\right]^{3/2}} \\ &= \frac{k\lambda^2}{a}\int_{-L/2}^{L/2}\left[\frac{L/2 - x_2}{\sqrt{(L/2 - x_2)^2 + a^2}} - \frac{-L/2 - x_2}{\sqrt{(-L/2 - x_2)^2 + a^2}}\right] dx_2 \\ &= \frac{2k\lambda^2}{a}\left(\sqrt{L^2 + a^2} - a\right). \end{aligned}$$

Since $a = 2d$, the final force is

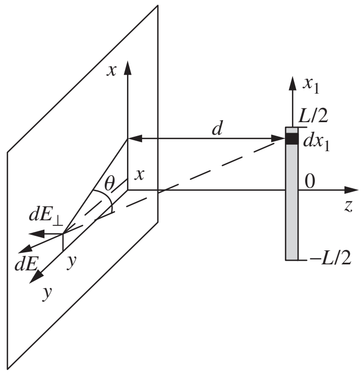
$$F = \frac{\lambda^2}{2\pi\epsilon_0}\left[\sqrt{1 + \left(\frac{L}{2d}\right)^2} - 1\right].$$

(b) In the limit when $d \gg L$, we expand the square root above and use $Q = \lambda L$ to get

$$F = \frac{1}{4\pi\epsilon_0}\frac{Q^2}{4d^2}.$$

This is the force between two point charges, as expected.

(c) The induced charge density is $\sigma(x, y) = 2\epsilon_0 E_\perp(x, y)$, where $E_\perp(x, y)$ is the normal component of the electric field created by the rod at the surface point (x, y). The factor of 2 accounts for the fact that the image rod contributes equally to the total electric field at (x, y). Using the angle θ as defined in the figure below (note difference from previous figure),



$$dE_\perp = dE \sin\theta = k \frac{\lambda dx_1 d}{\left[(x_1 - x)^2 + y^2 + d^2\right]^{3/2}}.$$

Therefore,

$$\begin{aligned} E_\perp &= k\lambda d \int_{-L/2}^{L/2} \frac{dx_1}{\left[(x_1 - x)^2 + y^2 + d^2\right]^{3/2}} \\ &= \frac{k\lambda d}{y^2 + d^2} \left[\frac{L/2 - x}{\sqrt{(L/2 - x)^2 + y^2 + d^2}} + \frac{L/2 + x}{\sqrt{(L/2 + x)^2 + y^2 + d^2}} \right]. \end{aligned}$$

Hence,

$$\sigma(x, y) = 2E_\perp \varepsilon_0 = \frac{\lambda d}{2\pi\,(y^2 + d^2)} \left[\frac{L/2 - x}{\sqrt{(L/2 - x)^2 + y^2 + d^2}} + \frac{L/2 + x}{\sqrt{(L/2 + x)^2 + y^2 + d^2}} \right].$$

(d) The plane has infinite capacity to draw charge up from ground. Therefore, the total induced charge is equal, but opposite, to the total charge of the rod. In other words, $Q_{\text{ind}} = -\lambda L$.

Source: Dr. A. Scherbakov, Georgia Institute of Technology (private communication).

8.4 A Dielectric Slab Intervenes

From Section 8.3.2, we recall that

$$\frac{1}{\sqrt{\rho^2 + z^2}} = \int_0^\infty dk\, J_0(k\rho) \exp(-k|z|).$$

Then, in Region I ($z < a$), a general solution of Poisson's equation that respects the cylindrical symmetry of the problem is the sum of the potential of the point charge,

$$V(\rho, z) = \frac{q}{4\pi\epsilon_0} = \frac{q}{4\pi\epsilon_0}\frac{1}{\sqrt{\rho^2 + z^2}} = \frac{q}{4\pi\epsilon_0}\int_0^\infty dk\, J_0(k\rho)\exp(-k|z|),$$

plus a general solution of Laplace's equation that does not diverge as $z \to -\infty$, namely,

$$\varphi_1 = \frac{q}{4\pi\epsilon_0}\left[\int_0^\infty dk\, J_0(k\rho)\exp(-k|z|) + \int_0^\infty dk\, A(k)J_0(k\rho)\exp(kz)\right].$$

In region II ($a \le z \le b$), a general solution of Laplace's equation is

$$\varphi_2 = \frac{q}{4\pi\epsilon_0}\left[\int_0^\infty dk\, B(k)J_0(k\rho)\exp(-kz) + \int_0^\infty dk\, C(k)J_0(k\rho)\exp(kz)\right].$$

In region III ($z > b$), a general solution of Laplace's equation is

$$\varphi_3 = \frac{q}{4\pi\epsilon_0}\int_0^\infty dk\, D(k)J_0(k\rho)\exp(-kz).$$

The linear independence of the solutions indexed by different values of k implies that it is sufficient to require the integrands to satisfy the matching conditions: the continuity of φ and $\kappa\partial\varphi/\partial z$ at $z = a$ and $z = b$. The result is four equations in four unknowns:

$$\begin{aligned}
e^{-ka} + Ae^{ka} &= Be^{-ka} + Ce^{ka} \\
De^{-kb} &= Be^{-kb} + Ce^{kb} \\
-ke^{-ka} + kAe^{ka} &= \kappa\left[-kBe^{-ka} + kCe^{ka}\right] \\
-kDe^{-kb} &= \kappa\left[-kBe^{-kb} + kCe^{kb}\right].
\end{aligned}$$

Solving for D gives

$$D(k) = \frac{4\kappa}{(\kappa+1)^2 - (\kappa-1)^2\exp\left[2k(a-b)\right]}.$$

Since $c = b - a$ and $1 - \beta^2 = 4\kappa/(\kappa+1)^2$, the potential in Region III is

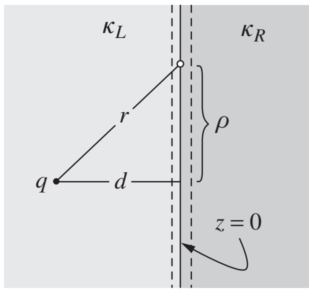
$$\varphi_3(\rho, z) = \frac{q(1-\beta^2)}{4\pi\epsilon_0}\int_0^\infty dk\frac{J_0(k\rho)\exp(-kz)}{1 - \beta^2\exp(-2kc)}.$$

The infinite-sum form of the potential follows immediately from the integral quoted above from the text and

$$\frac{1}{1-\beta^2\exp(-2kc)}=\sum_{n=0}^{\infty}\beta^{2n}\exp(-2knc).$$

Source: W.R. Smythe, *Static and Dynamic Electricity* (McGraw-Hill, New York, 1939), Section 5.304.

8.5 The Force Exerted by a Charge on a Dielectric Interface



(a) The stress tensor formalism calls for a surface which encloses the volume upon which the desired force acts. The "volume" here is the $z=0$ plane and the dashed lines in the diagram indicate the appropriate enclosing surface: a sandwich S composed of a plane at $z=\epsilon$ and a plane at $z=-\epsilon$, both in the limit when $\epsilon\to 0$. The force on the interface is

$$\mathbf{F}=\int_S dS\;[(\hat{\mathbf{n}}\cdot\mathbf{D})\mathbf{E}-\hat{\mathbf{n}}(\mathbf{E}\cdot\mathbf{D})].$$

By symmetry, the force is in the z-direction. Moreover, the outward normal is $\hat{\mathbf{n}}=\hat{\mathbf{z}}$ for the $z=\epsilon$ surface and $\hat{\mathbf{n}}=-\hat{\mathbf{z}}$ for the $z=-\epsilon$ surface. Therefore, using the notation we use to evaluate an integrated quantity between its limits of integration,

$$F_z=\int dS\left[D_zE_z-\frac{1}{2}\mathbf{D}_{\|}\cdot\mathbf{E}_{\|}-\frac{1}{2}D_zE_z\right]_L^R=\frac{1}{2}\int dS\;[D_zE_z-\mathbf{D}_{\|}\cdot\mathbf{E}_{\|}]\big|_L^R.$$

Now, D_z and $\mathbf{E}_{\|}$ are continuous at $z=0$. Therefore, we use $\mathbf{D}=\epsilon\mathbf{E}$ to write

$$F_z=\frac{1}{2}\int dS\left[\frac{D_z^2}{\epsilon}-\epsilon\mathbf{E}_{\|}\cdot\mathbf{E}_{\|}\right]_L^R,$$

which evaluates to

$$F_z=\frac{1}{2}\int_{z=0} dS\,D_z^2\left(\frac{1}{\epsilon_R}-\frac{1}{\epsilon_L}\right)-\frac{1}{2}\int_{z=0} dS\,\mathbf{E}_{\|}\cdot\mathbf{E}_{\|}(\epsilon_R-\epsilon_L).$$

Using the image-theory discussion of this problem in the text and the factors d/r and ρ/r to project out the normal and tangential components of the fields (see diagram above), we find that

$$D_z(z=0) = \frac{q_L}{4\pi r^2}\frac{d}{r} \qquad \text{and} \qquad E_\parallel(z=0) = \frac{1}{\epsilon_L}\frac{q_L}{4\pi r^2}\frac{\rho}{r} \qquad \text{where} \qquad q_L = \frac{2\kappa_R}{\kappa_L+\kappa_R}q.$$

The integrals we need are

$$\int\limits_{z=0} dS\, D_z^2 = \left(\frac{q_L}{4\pi}\right)^2 2\pi \int\limits_0^\infty d\rho \frac{d^2\rho}{(\rho^2+d^2)^3} = \left(\frac{q_L}{4\pi}\right)^2 2\pi \frac{1}{4d^2}$$

and

$$\int\limits_{z=0} dS\, E_\parallel^2 = \left(\frac{q_L}{4\pi\epsilon_R}\right)^2 2\pi \int\limits_0^\infty d\rho \frac{\rho^3}{(\rho^2+d^2)^3} = \left(\frac{q_L}{4\pi\epsilon_R}\right)^2 2\pi \frac{1}{4d^2}.$$

Therefore,

$$\begin{aligned} F_z &= \frac{1}{2}\left(\frac{q_L}{4\pi}\right)^2 \frac{\pi}{2d^2}\left[\left(\frac{1}{\epsilon_R}-\frac{1}{\epsilon_L}\right) - \frac{\epsilon_R-\epsilon_L}{\epsilon_R^2}\right] \\ &= \frac{q^2}{16\pi d^2}\frac{\kappa_R^2}{(\kappa_L+\kappa_R)^2}\left[\frac{(\kappa_L-\kappa_R)(\kappa_R+\kappa_L)}{\kappa_R^2}\frac{1}{\epsilon_0\kappa_L}\right]. \end{aligned}$$

In other words,

$$F_z = \frac{1}{4\pi\epsilon_L}\frac{q^2}{4d^2}\frac{\kappa_L-\kappa_R}{\kappa_L+\kappa_R}\hat{\mathbf{z}},$$

which is indeed the opposite of $\mathbf{F}_q$ quoted in the statement of the problem.

(b) By symmetry, the force is in the z-direction. Then, from Example 6.2, we use

$$\sigma(\rho) = \frac{1}{2\pi\kappa_L}\frac{\kappa_L-\kappa_R}{\kappa_L+\kappa_R}\frac{qd}{r^3}$$

and

$$\epsilon_0\hat{\mathbf{z}}\cdot\frac{\mathbf{E}_R+\mathbf{E}_L}{2} = \frac{1}{4\pi\kappa_L}\frac{qd}{r^3},$$

where $r=\sqrt{\rho^2+d^2}$. Therefore,

$$F_z = \frac{2\pi}{\epsilon_0}\int\limits_0^\infty d\rho\rho\left[\frac{1}{2\pi\kappa_L}\frac{\kappa_L-\kappa_R}{\kappa_L+\kappa_R}\frac{qd}{(\rho^2+d^2)^{3/2}}\right]\left[\frac{1}{4\pi\kappa_L}\frac{q^2d}{(\rho^2+d^2)^{3/2}}\right].$$

The integral is the same as the D_z^2 integral in part (a) and we find

$$F_z = F_z = \frac{1}{4\pi\epsilon_L}\frac{q^2}{4d^2}\frac{\kappa_L-\kappa_R}{\kappa_L+\kappa_R}\frac{1}{\kappa_L}.$$

This differs from the correct answer by a factor of $1/\kappa_L$.

8.6 Image Energy and Real Energy

Let $\varphi(\mathbf{r})$ be the potential produced at $\mathbf{r}$ by the conductor held at potential φ_C. If q_C is the charge on the conductor, the total electrostatic energy of the real system is

$$U_A = \frac{1}{2}\int d^3r\,\rho(\mathbf{r})\varphi(\mathbf{r}) = \frac{1}{2}q_C\varphi_C + \frac{1}{2}q\varphi(\mathbf{r}_q).$$

By construction, $\varphi(\mathbf{r}_q)$ is the same as the potential produced by the image point charges. Therefore,

$$U_A = \frac{1}{2}\int d^3r\,\rho(\mathbf{r})\varphi(\mathbf{r}) = \frac{1}{2}\sum_{k=1}^{N}\frac{qq_C}{|\mathbf{r}_q - \mathbf{r}_k|} = \frac{1}{2}q_C\varphi_C + \frac{1}{2}U_B.$$

Specifically, $U_A = U_B/2$ when the conductor is grounded.

8.7 Images in Spheres I

The sphere cannot be neutral because there is an attractive force between a point charge and any isolated neutral object. According to image theory, a grounded sphere acquires a charge $q' = -qR/s$ where s is the distance between q and the center of the sphere. The image itself lies a distance $d = R^2/s$ from the center. Now, suppose we add a second image charge $q'' = Q - q'$ at the center of the sphere. By Gauss' law, the total charge on the sphere is now Q and the sphere boundary is still an equipotential. This is the situation we want.

We have $s = 2R$, so $q' = -q/2$, $s = R/2$, and $q'' = Q + q/2$. We need to choose Q so that the force between q and the two images is zero:

$$F = \frac{q}{4\pi\epsilon_0}\left\{\frac{q'}{(s-d)^2} + \frac{q''}{s^2}\right\} = \frac{q}{4\pi\epsilon_0}\left\{\frac{-q/2}{(2R - R/2)^2} + \frac{Q + q/2}{(2R)^2}\right\} = 0.$$

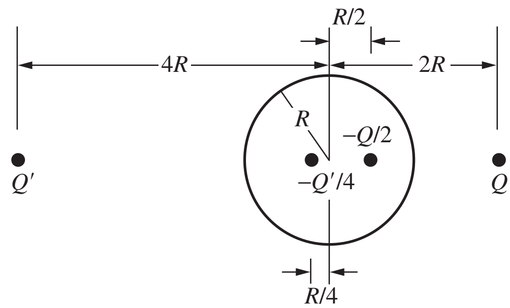
This gives $Q = \frac{7}{18}q$. When we move q so that $s = 3R$, the force formula on the left is still correct with $q' = -q/3$, $d = R/3$, and $q'' = Q + q/3$. Therefore, the force is

$$F = \frac{q^2}{4\pi\epsilon_0}\left\{\frac{-1/3}{(3R - R/3)^2} + \frac{7/18 + 1/3}{(3R)^2}\right\} = \frac{q^2}{4\pi\epsilon_0 R^2}\left\{-\frac{3}{64} + \frac{13}{162}\right\} = \frac{1}{4\pi\epsilon_0}\frac{173}{5184}\frac{q^2}{R^2}.$$

Source: Prof. M.J. Cohen, University of Pennsylvania (private communication).

8.8 Images in Spheres II

Let the sphere center be the origin. The image of Q at $2R$ is $Q_0 = -Q/2$ at $R/2$. The image of Q' at $-4R$ is $Q_0' = -Q'/4$ at $-R/4$.



The force on Q' is the force due to the other three collinear charges. It is

$$
\begin{aligned}
F &= \frac{Q'}{4\pi\epsilon_0}\left[\frac{-Q'/4}{(4R-R/4)^2}+\frac{Q}{(4R+2R)^2}+\frac{-Q/2}{(4R+R/2)^2}\right] \\
&= \frac{Q'}{4\pi\epsilon_0 R^2}\left[-\frac{4Q'}{15\cdot 15}+\frac{Q}{36}-\frac{2Q}{9\cdot 9}\right].
\end{aligned}
$$

This is negative (Q' repelled from sphere) if $Q' < (25/144)Q$.

Source: Prof. M.J. Cohen, University of Pennsylvania (private communication).

8.9 Debye's Model for the Work Function

(a) The text gives the potential outside an isolated conducting sphere with radius R and charge Q in the presence of a point charge q at a distance $r > R$ from the center of the sphere. From this, we may immediately infer that the force which the sphere exerts on q is

$$\mathbf{F} = \frac{q}{4\pi\epsilon_0}\left[\frac{Q}{r^2}+\frac{qR}{r^3}-\frac{QRr}{(r^2-R^2)^2}\right]\hat{\mathbf{r}}.$$

The Debye model proposes that $W = -\int_{R+d}^{\infty} d\mathbf{r}\cdot\mathbf{F}$. The integrals are elementary and we find

$$W = -\frac{q}{4\pi\epsilon_0}\left\{\left[-\frac{Q}{r}\right]_{R+d}^{\infty}+\left[-\frac{qR}{2r^2}\right]_{R+d}^{\infty}-qR\left[-\frac{1}{2}\frac{1}{r^2-R^2}\right]_{R+d}^{\infty}\right\}.$$

Evaluating this expression with $q=-e$ and $Q=e$ gives the proposed formula,

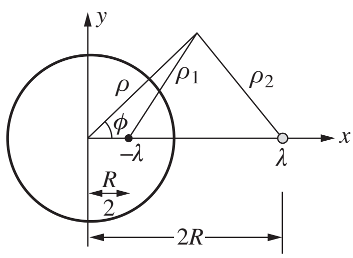
$$
\begin{aligned}
W &= \frac{e}{4\pi\epsilon_0}\left[\frac{e}{R+d}-\frac{eR}{2(R+d)^2}-\frac{eR}{2}\frac{1}{R^2-(R+d)^2}\right] \\
&= \frac{e^2}{8\pi\epsilon_0}\left[\frac{2}{R+d}-\frac{R}{(R+d)^2}+\frac{R}{(R+d)^2-R^2}\right].
\end{aligned}
$$

(b) The limit $R \to \infty$ naively gives $W = 0$. To do better, we let $x = d/R$ and take the limit as $x \to 0$. The result is

$$W = \frac{e^2}{8\pi\epsilon_0 d}\left[\frac{1}{x+2} + \frac{x(1+2x)}{(1+x)^2}\right] \quad \to \quad \frac{e^2}{16\pi\epsilon_0 d} \quad \text{as} \quad x \to 0.$$

8.10 Force between a Line Charge and a Conducting Cylinder

(a) We need the potential to compute the charge density. The text shows that the cylinder is an equipotential if the line charge λ at distance $b = 2R$ from the center is accompanied by an image line charge with strength $-\lambda$ placed at a distance $R^2/b = R/2$ from the center. We are interested in computing $\varphi(\rho, \phi)$ with ρ and ϕ defined in the diagram below.



Adding the contributions from the two line charges gives

$$\varphi(\rho,\phi) = -\frac{\lambda}{2\pi\epsilon_0}\ln\rho_2 + \frac{\lambda}{2\pi\epsilon_0}\ln\rho_1 = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{\rho_1^2}{\rho_2^2} = \frac{\lambda}{4\pi\epsilon_0}\ln\frac{\rho^2 + R^2/4 - \rho R\cos\phi}{\rho^2 + 4R^2 - 4\rho R\cos\phi}.$$

Consequently,

$$\begin{aligned}
\sigma(\phi) &= -\epsilon_0 \left.\frac{\partial\varphi}{\partial r}\right|_{\rho=R} \\
&= -\frac{\lambda}{4\pi}\left[\frac{2\rho - R\cos\phi}{\rho^2 + R^2/4 - \rho R\cos\phi} - \frac{2\rho - 4R\cos\phi}{\rho^2 + 4R^2 - 4\rho R\cos\phi}\right]_{\rho=R} \\
&= -\frac{\lambda}{2\pi R}\left(\frac{3}{5 - 4\cos\phi}\right).
\end{aligned}$$

(b) If $\hat{\mathbf{n}}$ is the outward normal to the conductor, the net force on the cylinder may be computed from

$$\mathbf{F} = \frac{1}{2\epsilon_0}\int dS\sigma^2\hat{\mathbf{n}}.$$

For the cylinder, $\hat{\mathbf{n}} = \hat{\boldsymbol{\rho}} = \hat{\mathbf{x}}\cos\phi + \hat{\mathbf{y}}\sin\phi$. By symmetry, the force per unit length $\mathbf{f}$ is along $\hat{\mathbf{x}}$. Hence,

$$\mathbf{f} = \frac{9\lambda^2}{4\pi^2\epsilon_0 R}\int_0^\pi d\phi \frac{\cos\phi}{(5-4\cos\phi)^2}\hat{\mathbf{x}}.$$

Using *Tables of Integrals, Series, and Products* by Gradshteyn and Ryzhik (1980),

$$\int_0^\pi d\phi \frac{\cos\phi}{(5-4\cos\phi)^2} = \frac{1}{9}\left[\frac{5\sin\phi}{5-4\cos\phi}\Bigg|_0^\pi + \int_0^\pi d\phi\frac{4}{5-4\cos\phi}\right].$$

The integrated term is zero and

$$\frac{4}{9}\int_0^\pi d\phi \frac{1}{5-4\cos\phi} = \frac{8}{27}\left[\tan^{-1}\{3\tan(\phi/2)\}\right]_0^\pi = \frac{4\pi}{27}.$$

Hence, the force per unit length that acts on the cylinder is

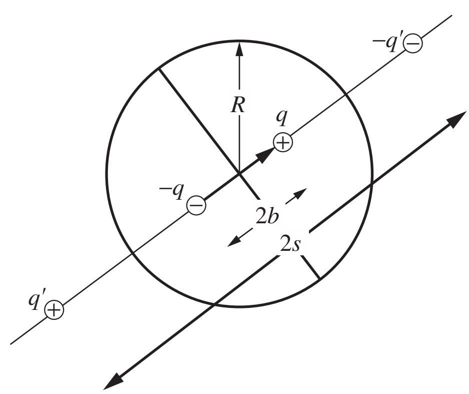
$$\mathbf{f} = \frac{9\lambda^2}{4\pi^2\epsilon_0 R}\frac{4\pi}{27}\hat{\mathbf{x}} = \frac{4}{3}\frac{\lambda^2}{4\pi\epsilon_0 R}\hat{\mathbf{x}}.$$

(c) We can also compute the force on the cylinder as the *negative* of the force that acts on the line charge λ. The distance between this line and its image is $3R/2$. Therefore,

$$\mathbf{f} = \frac{\lambda^2}{2\pi\epsilon_0(3R/2)}\hat{\mathbf{x}} = \frac{4}{3}\frac{\lambda^2}{4\pi\epsilon_0 R}\hat{\mathbf{x}}.$$

8.11 Point Dipole in a Grounded Shell

We treat the point dipole as finite, with an electric moment with magnitude $p = 2qb$. We keep the shell an equipotential using two image charges, each a distance $s = R^2/b$ from the origin (but on opposite sides of the shell) and each with charge magnitude $q' = (R/b)q$.



$\mathbf{E}_{\text{in}}$ is the field produced by q' and $-q'$ when $q \to \infty$ and $b \to 0$ but $p = 2qb$ remains finite. In that limit, q' and $-q'$ go off to infinity and $\mathbf{E}_{\text{in}}$ differs negligibly from the field at the midpoint between q' and $-q'$. That field, in turn, is identical to the field at the $\rho = 0$ point of the surface of the grounded, conducting plane for which q' and $-q'$ are the image system. This we compute using the stated charge density when $s^2 = R^2/b \to \infty$. When b and q are finite, the magnitude of $\mathbf{E}_{\rm in}$ is

$$E_{\rm in} = \frac{1}{\epsilon_0}\sigma(0) = \frac{q'}{2\pi\epsilon_0 z_0^2} = \frac{Rq/b}{2\pi\epsilon_0 (R^2/b)^2} = \frac{qb}{2\pi\epsilon_0 R^3}.$$

The field due to q' and $-q'$ points in the direction of $\mathbf{p}$, so

$$\mathbf{E}_{\rm in} = \frac{\mathbf{p}}{4\pi\epsilon_0 R^3}.$$

8.12 Inversion in a Cylinder

(a) The problem states that

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left\{\rho\frac{\partial\Phi}{\partial\rho}\right\} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0 \qquad (\rho < R).$$

Multiply this Laplace equation above by ρ^2, let $u = R^2/\rho$, and use the fact that

$$du = -(R^2/\rho^2)d\rho = -(u/\rho)d\rho.$$

This gives

$$u\frac{\partial}{\partial u}\left\{u\frac{\partial\Phi}{\partial u}\right\} + \frac{\partial^2\Phi}{\partial\phi^2} = 0 = \frac{1}{u}\frac{\partial}{\partial u}\left\{u\frac{\partial\Phi}{\partial u}\right\} + \frac{1}{u^2}\frac{\partial^2\Phi}{\partial\phi^2}.$$

Therefore, $\Psi(\rho,\phi) = \Phi(u,\phi)$ satisfies Laplace's equation when $u < R$, i.e., when $\rho > R$.

(b) Center the cylinder on the origin. Choose Φ as the potential of a line charge placed at a distance $s > R$ from the origin. This function satisfies Laplace's equation inside the cylinder. On the other hand, $\Psi(\rho,\phi) = \Phi(R^2/\rho,\phi)$ satisfies Laplace's equation outside the cylinder. Finally,

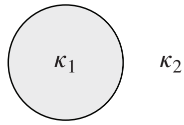
$$\Psi(\rho = R,\phi) = \Phi(\rho = R,\phi).$$

Therefore, outside the cylinder, the potential

$$\varphi(\rho,\phi) = \Phi(\rho,\phi) - \Psi(\rho,\phi)$$

vanishes on the cylinder and satisfies Poisson's equation outside the cylinder (with the line charge as its source). Since $\rho = R^2/u$, the same argument shows that $\varphi(\rho,\phi) = \Psi(\rho,\phi) - \Phi(\rho,\phi)$ satisfies the line charge Poisson equation inside the cylinder and vanishes on the surface of the cylinder.

(c) The geometry is



Again, choose Φ as the potential of a line charge placed at a distance $s > R$ from the origin. As suggested, we try a linear combinations of Φ and Ψ outside the cylinder. Inside the cylinder, we can try only a multiple of Φ because the potential must satisfy Laplace's equation there:

$$\varphi_{\rm out}(\rho > R, \phi) = \Phi(\rho, \phi) + S\Psi(\rho, \phi) = \Phi(\rho, \phi) + S\Phi(R^2/\rho, \phi)$$

$$\varphi_{\rm in}(\rho < R, \phi) = T\Phi(\rho, \phi).$$

The matching conditions at $\rho = R$ are

$$\varphi_{\rm in}(R, \phi) = \varphi_{\rm out}(R, \phi) \qquad\qquad \kappa_1 \left.\frac{\partial \varphi_{\rm in}}{\partial \rho}\right|_{\rho=R} = \kappa_2 \left.\frac{\partial \varphi_{\rm in}}{\partial \rho}\right|_{\rho=R}.$$

The continuity of the potential (left equation above) gives $1+S = T$. The continuity of the normal component of $\mathbf{D}$ (right equation above) gives $\kappa_1 T = (1-S)\kappa_2$. Combining these completes the solution:

$$S = \frac{\kappa_2 - \kappa_1}{\kappa_2 + \kappa_1} \qquad\qquad T = \frac{2\kappa_2}{\kappa_2 + \kappa_1}.$$

An entirely similar argument applies if the line charge lies inside the κ_1 cylinder.

(d) $\Psi(\rho, \phi)$ is the potential of a line charge located at a distance R^2/ρ from the origin on the line which connects the origin to the line charge represented by $\Phi(\rho, \phi)$. But if the line source represented by $\Psi(\rho, \phi)$ lies inside the cylinder, we only used this function in the space outside the sphere and vice versa. Therefore, $\Psi(\rho, \phi)$ has exactly the characteristics of an image potential.

Source: L.G. Chambers, *An Introduction to the Mathematics of Electricity and Magnetism* (Chapman and Hall, London, 1973).

8.13 Symmetry of the Dirichlet Green Function

Green's second identity is

$$\int\limits_V d^3r\, [f\nabla^2 g - g\nabla^2 f] = \int\limits_S dS\, \hat{\mathbf{n}} \cdot [f\nabla g - g\nabla f].$$

We choose $f(\mathbf{r} = G_D(\mathbf{r}', \mathbf{r})$ and $g(\mathbf{r} = G_D(\mathbf{r}'', \mathbf{r})$ where

$$\epsilon_0 \nabla^2 G_D(\mathbf{r}', \mathbf{r}) = -\delta(\mathbf{r}' - \mathbf{r}) \qquad\qquad G_D(\mathbf{r}', \mathbf{r}_S) = 0$$

and

$$\epsilon_0 \nabla^2 G_D(\mathbf{r}'', \mathbf{r}) = -\delta(\mathbf{r}'' - \mathbf{r}) \qquad\qquad G_D(\mathbf{r}'', \mathbf{r}_S) = 0.$$

Substituting these into the left side of Green's identity gives

$$\begin{aligned}
&\int_V d^3r \left[G_D(\mathbf{r}', \mathbf{r})\nabla^2 G_D(\mathbf{r}'', \mathbf{r}) - G_D(\mathbf{r}'', \mathbf{r})\nabla^2 G_D(\mathbf{r}', \mathbf{r})\right] \\
&\quad = -\frac{1}{\epsilon_0}\int_V d^3r\, G_D(\mathbf{r}', \mathbf{r})\delta(\mathbf{r}'' - \mathbf{r}) + \frac{1}{\epsilon_0}\int_V d^3r\, G_D(\mathbf{r}'', \mathbf{r})\delta(\mathbf{r}' - \mathbf{r}) \\
&\quad = -G_D(\mathbf{r}', \mathbf{r}'') + G_D(\mathbf{r}'', \mathbf{r}').
\end{aligned}$$

With our choices for f and g, the right side of Green's identity gives

$$\int_S dS\hat{\mathbf{n}} \cdot [G_D(\mathbf{r}', \mathbf{r}_S)\nabla G_D(\mathbf{r}'', \mathbf{r}) - G_D(\mathbf{r}'', \mathbf{r}_S)\nabla G_D(\mathbf{r}', \mathbf{r})] = 0,$$

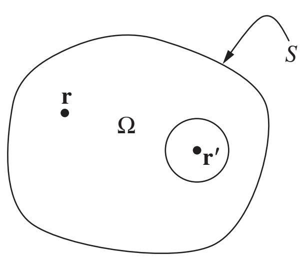
because $G_D(\mathbf{r}', \mathbf{r}_S) = G_D(\mathbf{r}'', \mathbf{r}_S) = 0$. This proves the desired result.

8.14 Green Function Inequalities

(a) Let S be the surface of the closed volume V. $G_D(\mathbf{r}, \mathbf{r}')$ is the potential at $\mathbf{r} \in V$ due to a unit positive point charge at $\mathbf{r}' \in V$ when the boundary S of V is grounded. This means that $\Lambda(\mathbf{r}, \mathbf{r}')$ is the potential due to *negative* charge that is drawn up from ground and resides on S. This potential must be negative so $\Lambda < 0$. Therefore,

$$G_D(\mathbf{r}, \mathbf{r}') < \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r} - \mathbf{r}'|}.$$

(b) Let Ω be the volume V minus an infinitesimally small sphere centered on the point $\mathbf{r}'$ where the unit point charge resides. This means that $\nabla^2 G_D(\mathbf{r}, \mathbf{r}') = 0$ everywhere in Ω. Earnshaw's theorem says that G_D cannot have either a local maximum or a local minimum in Ω. But G_D is zero on the surface of V and large and positive on the surface of the infinitesimal sphere. Hence, G_D cannot be negative anywhere in Ω (or V).



Source: G. Barton, *Elements of Green's Functions and Propagation* (Clarendon, Oxford, 1989).

8.15 The Potential of a Voltage Patch

By symmetry, $\varphi(x,y,z) = \varphi(x,y,-z)$. Therefore, we restrict our attention to the $z > 0$ half-space where $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$. There is no charge in this volume so the magic rule gives the solution as

$$\varphi(x,y,z>0) = -\epsilon_0 \int\limits_{z=0} dS' \varphi(\mathbf{r}') \frac{\partial}{\partial n'} G_D(\mathbf{r},\mathbf{r}') = \epsilon_0 \varphi_0 \int\limits_{S_0} dS' \left. \frac{\partial}{\partial z'} G_D(\mathbf{r},\mathbf{r}') \right|_{z'=0}. \tag{1}$$

From the method of images, the Dirichlet Green function for the volume $z > 0$ is

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right].$$

Therefore,

$$\left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial z'}\right|_{z'=0} = \frac{1}{4\pi\epsilon_0}\frac{2z}{[(x-x')^2+(y-y')^2+z^2]^{3/2}}. \tag{2}$$

Substituting (2) into (1) gives

$$\varphi(x,y,z>0) = \frac{\varphi_0 z}{2\pi}\int\limits_{S_0}\frac{d^2r'}{|\mathbf{r}-\mathbf{r}'|^3}.$$

The symmetry $\varphi(x,y,z) = \varphi(x,y,-z)$ produces the suggested result.

8.16 The Charge Induced by Induced Charge

The unit normal to the solution volume $z > 0$ is $\hat{\mathbf{n}} = -\hat{\mathbf{z}}$. Moreover, the only free charge in that volume is confined to S_0. Therefore, the magic rule gives the potential in the solution volume as

$$\varphi(x,y,z\geq 0) = \int\limits_{S_0} dS' G_D(\mathbf{r},\mathbf{r}')\sigma_0(\mathbf{r}'_S) + \epsilon_0 V \int\limits_{z'=0} dS' \left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial z'}\right|_{z'=0}.$$

The first term above is the potential induced by the added conductor. Therefore, the charge induced on the $z = 0$ surface by that conductor is the $z = 0$ value of

$$\sigma(x,y) = -\epsilon_0 \frac{\partial}{\partial z}\int\limits_{S_0} dS' G_D(\mathbf{r},\mathbf{r}')\sigma_0(\mathbf{r}'_S)\Bigg|_{z=0}. \tag{1}$$

From the method of images, the Dirichlet Green function for this geometry is

$$G_D(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\epsilon_0}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}} - \frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z+z')^2}}\right].$$

Therefore,

$$\left.\frac{\partial G_D(\mathbf{r},\mathbf{r}')}{\partial z}\right|_{z=0} = \frac{1}{4\pi\epsilon_0}\frac{2z'}{[(x-x')^2+(y-y')^2+z'^2]^{3/2}} = \frac{1}{2\pi\epsilon_0}\frac{z'}{|\mathbf{r}_S-\mathbf{r}'|^3}. \tag{2}$$

Substituting (2) into (1) gives the final result,

$$\sigma(x,y) = -\frac{1}{2\pi}\int_{S_0} dS' \, \frac{\sigma_0(\mathbf{r'}_S)z'}{|\mathbf{r}_S-\mathbf{r}'|^3}.$$

8.17 Free-Space Green Functions by Eigenfunction Expansion

In N dimensions, the normalized eigenfunctions of $\nabla^2\psi = -\lambda\psi$ are plane waves,

$$\psi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{(2\pi)^{N/2}}\exp(i\mathbf{k}\cdot\mathbf{r}),$$

where $\mathbf{k}$ and $\mathbf{r}$ are N-dimensional vectors. The eigenvalue is $\lambda = \mathbf{k}\cdot\mathbf{k} = k^2$ so

$$G_0^{(N)}(\mathbf{r},\mathbf{r}') = \frac{1}{\epsilon_0}\int \frac{d^N k}{(2\pi)^N}\frac{\exp[i\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')]}{k^2}.$$

In three dimensions,

$$\begin{aligned} G_0^{(3)}(\mathbf{r},\mathbf{r}') &= \frac{1}{(2\pi)^2\epsilon_0}\int_0^\infty dk \int_{-1}^1 d(\cos\theta)\, e^{ik|\mathbf{r}-\mathbf{r}'|\cos\theta} \\ &= \frac{1}{2\pi^2\epsilon_0|\mathbf{r}-\mathbf{r}'|}\int_0^\infty dx\frac{\sin x}{x} = \frac{1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \end{aligned}$$

In two dimensions, we need a few tabulated integrals of Bessel functions:

$$G_0^{(2)}(\mathbf{r},\mathbf{r}') = \frac{1}{\epsilon_0}\int_0^{2\pi}\frac{d\phi}{(2\pi)^2}\int_0^\infty \frac{dk}{k}e^{ik|\mathbf{r}-\mathbf{r}'|\cos\phi} = \frac{1}{2\pi\epsilon_0}\int_0^\infty \frac{dk}{k}J_0(k|\mathbf{r}-\mathbf{r}'|).$$

The last integral does not exist so we use a limiting process with the Bessel functions $J_0(x)$ and $K_0(x)$:

$$\begin{aligned} G_0^{(2)}(\mathbf{r},\mathbf{r}') &= \frac{1}{2\pi\epsilon_0}\lim_{\eta\to 0}\int_0^\infty dk\, k\frac{J_0(k|\mathbf{r}-\mathbf{r}'|)}{k^2+\eta^2} = \frac{1}{2\pi\epsilon_0}\lim_{\eta\to 0}K_0(\eta|\mathbf{r}-\mathbf{r}'|) \\ &= -\frac{1}{2\pi\epsilon_0}\ln|\mathbf{r}-\mathbf{r}'| + \textit{const}. \end{aligned}$$

The constant is proportional to $\ln\eta$. We drop it because its divergence has no physical consequences. In one dimension,

$$G_0^{(1)}(x,x') = \frac{1}{2\pi\epsilon_0}\int_{-\infty}^\infty \frac{dk}{k^2}e^{ik(x-x')} = -\frac{1}{2\epsilon_0}|x-x'|.$$

This follows because $d^2 G_0^{(1)}/dx^2 = -\delta(x-x')/\epsilon_0$ by inspection and

$$\frac{d}{dx}|x-x'| \;=\; sgn(x) = -1 + 2\int_{-\infty}^{x-x'} dy\,\delta(y).$$

8.18 Free-Space Green Function in Polar Coordinates

In polar coordinates, we want to solve

$$\nabla^2 G = \frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right) + \frac{1}{\rho}\frac{\partial^2 G}{\partial\phi^2} = -\frac{1}{\epsilon_0}\delta(\rho-\rho')\delta(\phi-\phi')$$

because the source is a line charge with unit charge/length. The completeness relation

$$\frac{1}{2\pi}\sum_{m=-\infty}^{\infty}\exp[im(\phi-\phi')] = \delta(\phi-\phi')$$

suggests the ansatz

$$G_0^{(2)}(r,r') = \frac{1}{2\pi\epsilon_0}\sum_{m=-\infty}^{\infty} G_m(\rho,\rho')\,e^{im(\phi-\phi')}.$$

Substituting this above yields

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G_m}{\partial\rho}\right) - \frac{m^2}{\rho}G_m = -\delta(\rho-\rho').$$

The solution to the homogeneous equation is

$$\begin{aligned} G_m &= A\rho^{-mp} + B\rho^{mp} && m \neq 0 \\ G_m &= C\ln\rho + D && m = 0. \end{aligned}$$

Continuity at $\rho = \rho'$ gives

$$G_m(\rho,\rho') = A_m(1-\delta_{m,0})\frac{\rho_<^{|m|}}{\rho_>^{|m|}} + B_m\delta_{m,0}\ln\rho_>.$$

The jump condition

$$\rho\,\frac{\partial G_m}{\partial\rho}\bigg|_{\rho=\rho'-\epsilon}^{\rho=\rho'+\epsilon} = -1$$

determines the coefficients A and B to be $A_m = 1/2|m|$ and $B_m = -1$, so

$$G_0^{(2)}(r,r') = -\frac{1}{2\pi\epsilon_0}\ln\rho_> + \frac{1}{2\pi\epsilon_0}\sum_{m=1}^{\infty}\frac{1}{m}\frac{\rho_<^m}{\rho_>^m}\cos m(\phi-\phi').$$

8.19 Using a Cube to Simulate a Point Charge

(a) A completeness relation for the one-dimensional interval $-a \le x \le a$ is formed from solutions of Laplace's equation with homogeneous boundary conditions, namely,

$$\psi_m(x) = \sqrt{\frac{1}{a}}\sin\left[\frac{m\pi x}{a}\right] \qquad \text{and} \qquad \sqrt{\frac{1}{a}}\cos\left[\frac{(2m-1)\pi x}{2a}\right],$$

where $m = 1, 2, 3, \ldots$ Both are needed because the sine functions are odd in x and the cosine functions are even in x. We use these to make the delta function

$$\delta(x-x')\delta(y-y') = \frac{1}{a^2}\sum_{m=1}^{\infty}\sum_{m=1}^{\infty}\psi_n(x)\,\psi_n^*(x)\,\psi_m(y)\,\psi_m^*(y).$$

This motivates us to make an ansatz for the Green function:

$$G(x,x') = \frac{1}{\epsilon_0 a^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\psi_m(x)\psi_m^*(x)\,\psi_n(y)\psi_n^*(y)g(z,z'|m,n).$$

Substituting this into $\epsilon_0\nabla^2 G(x,x') = -\delta(x-x')\delta(y-y')\delta(z-z')$ gives the ordinary differential equation

$$\left(\frac{\partial^2}{\partial z^2} - \kappa^2\right)g(z,z'|\kappa) = -\delta(z-z'),$$

where

$$\kappa^2 = \frac{\pi^2}{4a^2}(m^2+n^2) \qquad \text{and} \qquad g(\pm a, z'|\kappa) = 0.$$

The continuous solution that satisfies both boundary conditions is

$$g(z,z|\kappa) = C\sinh\kappa(a+z_<)\sinh\kappa(a-z_>).$$

The jump condition,

$$\left.\frac{dg(z,z'|\kappa)}{dz}\right|_{z=z'-\epsilon}^{z=z'+\epsilon} = -1,$$

fixes

$$C = 1/\kappa\sinh(2\kappa a).$$

Therefore, the final Green function is

$$G(x,x') = \frac{1}{\epsilon_0 a^2}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\psi_m(x)\psi_m^*(x)\,\psi_n(y)\psi_n^*(y)\frac{\sinh\kappa(a+z_<)\sinh\kappa(a-z_>)}{\kappa\sinh(2\kappa a)}.$$

(b) Put a charge $-Q$ at the center of the box. When the box is grounded, the charge density induced on the inside walls of the box must exactly annul the field of $-Q$ at every point outside box. That is, it is the field outside the box that would be produced by a point charge $+Q$ at the center of the box. We will compute the charge density induced on the plane $z = a$. Since $\sinh 2\kappa a = 2\sinh\kappa a\cosh\kappa a$,

$$\sigma(x,y) = -Q\epsilon_0 \left.\frac{\partial G(x,y,z,0,0,0)}{\partial z}\right|_{z=a} = \frac{Q}{2a^2}\sum_{m=1,3,\ldots}\sum_{n=1,3,\ldots}\cos\frac{n\pi x}{2a}\cos\frac{m\pi y}{2a}\frac{1}{\cosh\kappa a}.$$

Notice that only the cosine terms survive the sum because the position of the point charge is $x' = 0$. We conclude that

$$\sigma(0,0) = \frac{Q}{2a^2}\sum_{m=1,3,\ldots}\sum_{n=1,3,\ldots}\frac{1}{\cosh\left[(\pi/2)\sqrt{m^2+n^2}\right]} \approx 0.1233\frac{Q}{a^2}.$$

(c) Example 8.3 gives the charge density on the $z = a$ face of a box which occupies $0 \le x, y, z \le a$ as

$$\sigma(x,y) = \frac{8Q}{\pi a^2}\sum_{n,\ell,m \text{ odd}}(-1)^{\frac{1}{2}(n+\ell+m-3)}\frac{m}{n^2+\ell^2+m^2}\sin\left[\frac{n\pi x}{a}\right]\sin\left[\frac{\ell\pi y}{a}\right].$$

We evaluate the foregoing at the center of the face ($x = y = a/2$) and then let $a \to 2a$ because the edge length of the box in the present problem is $2a$. The final result for the charge density at the center agrees with part (b):

$$\begin{aligned}\sigma &= \frac{2Q}{\pi a^2}\sum_{n,\ell,m \text{ odd}}(-1)^{\frac{1}{2}(n+\ell+m-3)}\frac{m}{n^2+\ell^2+m^2}\sin\left[\frac{n\pi}{2}\right]\sin\left[\frac{\ell\pi}{2}\right] \\ &= \frac{2Q^2}{\pi a^2}\sum_{n,\ell,m \text{ odd}}(-1)^{\frac{1}{2}(m-1)}\frac{m}{n^2+\ell^2+m^2} \\ &= 0.1233\frac{Q}{a^2}.\end{aligned}$$

Source: Prof. M.J. Cohen, University of Pennsylvania (private communication).

8.20 Green Function for a Sphere by Direct Integration

(a) Since

$$\epsilon_0\nabla^2 G(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') = -\frac{\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')}{r^2\sin\theta},$$

we use the given completeness relation to make the ansatz

$$\epsilon_0 G(\mathbf{r},\mathbf{r}') = \sum_{l=0}^{\infty}\sum_{m=-\ell}^{\ell} G_{\ell m}(r,r')Y_{\ell m}(\hat{\mathbf{r}})Y^*_{\ell m}(\hat{\mathbf{r}}')$$

for the Green function. From, say, the wave mechanics of the hydrogen atom, we know that the angular part of the Laplacian in spherical coordinates makes up the total angular momentum operator. Specifically,

$$\nabla^2 G = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) - \frac{\hat{L}^2}{r^2}G,$$

where

$$\hat{L}^2 Y_{\ell m} = \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}.$$

Using the three preceding formulae in the differential equation for G isolates the differential equation satisfied by $G_{\ell m}(r, r')$:

$$\left[-\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \ell(\ell+1)\right] G_\ell(r, r') = \delta(r - r').$$

Notice that we have changed notation to $G_\ell(r, r')$ because the equation above does not depend on m. When $r \neq r'$, our experience with Laplace's equation in spherical coordinates leads us to write

$$G_\ell(r, r') = \begin{cases} B_\ell r^\ell & 0 \leq r < r' \\ A'_\ell r^{-(\ell+1)} + A_\ell r^\ell & r' < r \leq R. \end{cases}$$

because the Green function must be regular at the origin. The three coefficients are determined from (a) the boundary condition $G_\ell(R, r') = 0$; (b) the continuity of $G_\ell(r, r')$ at $r = r'$; and (c) the jump condition

$$\lim_{\epsilon\to 0}\left(r^2\frac{\partial G_\ell}{\partial r}\right)_{r=r'-\epsilon}^{r=r'+\epsilon} = -1.$$

A few lines of algebra give

$$G_\ell(r, r') = \frac{1}{2\ell+1}\left\{\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r_<^\ell\, 2r_>^\ell}{R^{2\ell+1}}\right\}$$

so

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-\ell}^{\ell}\frac{1}{2\ell+1}\left\{\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r_<^\ell r_>^\ell}{R^{2\ell+1}}\right\} Y_{\ell m}(\hat{\mathbf{r}})Y_{\ell m}^*(\hat{\mathbf{r}}').$$

The stated formula,

$$G(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi\epsilon_0}\sum_{\ell=0}^{\infty}\left\{\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{r_<^\ell r_>^\ell}{R^{2\ell+1}}\right\} P_\ell(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'),$$

is correct because

$$P_\ell(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}') = \frac{4\pi}{2\ell+1}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\hat{\mathbf{r}})Y_{\ell m}^*(\hat{\mathbf{r}}').$$

(b) Define a vector $\mathbf{Q} = (R^2/r')\hat{\mathbf{r}}'$ so $\hat{\mathbf{r}}\cdot\hat{\mathbf{Q}} = \hat{\mathbf{r}}\cdot\hat{\mathbf{r}}'$. Now use

$$\frac{1}{|\mathbf{r} - \mathbf{r}'|} = \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}}')$$

twice: once as it stands and once with $\mathbf{r}'$ replaced by $\mathbf{Q}$. This shows that the interior Green function we derived above can be written in the form

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\epsilon_0}\left\{\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{R/r'}{|\mathbf{r}-\mathbf{Q}|}\right\} = \frac{1}{4\pi\epsilon_0}\left\{\frac{1}{|\mathbf{r}-\mathbf{r}'|} - \frac{R/r'}{|\mathbf{r}-R^2\mathbf{r}'/r'^2|}\right\}.$$

This is the image formula for the potential derived in the text.

8.21 The Charge Induced on a Conducting Tube

Begin with the exterior Green function for a tube of radius R derived in the text:

$$\begin{aligned} G(r,r') &= \frac{1}{2\pi^2\epsilon_0}\sum_{m=-\infty}^{\infty}\int_0^\infty dk\frac{K_m(k\rho_>)}{K_m(kR)}\left\{K_m(kR)I_m(k\rho_<) - K_m(k\rho_<)I_m(kR)\right\} \\ &\qquad \times\, e^{im(\phi-\phi')}\cos k(z-z'). \end{aligned}$$

(a) To find the charge density induced on the tube surface, let $z' = \phi' = 0$ and $\rho' = s$ be the coordinates of q. Using the Wronskian for Bessel functions,

$$\begin{aligned} \sigma(\phi,z) &= -q\epsilon_0\left.\frac{\partial G(\rho,\phi,z|s)}{\partial\rho}\right|_{\rho=R} \\ &= -\frac{q}{2\pi^2}\sum_{m=-\infty}^{\infty}e^{im\phi}\int_0^\infty dk\cos(kz)\frac{K_m(ks)}{k_m(kR)}\frac{d}{d\rho}\left[K_m(kR)I_m(k\rho)\right. \\ &\quad \left.-\ K_m(k\rho)I_m(kR)\right]_{\rho=R} \\ &= -\frac{q}{2\pi^2 R}\sum_{m=-\infty}^{\infty}e^{im\phi}\int_0^\infty dk\cos(kz)\frac{K_m(ks)}{K_m(kR)}. \end{aligned}$$

(b) The total induced charge is

$$Q = R\int_0^{2\pi}d\phi\int_{-\infty}^{\infty}dz\,\sigma(\phi,z) = -\frac{q}{\pi}\int_0^\infty dk\frac{K_0(ks)}{K_0(kR)}\int_{-\infty}^{\infty}dz\,\cos(kz) = -q$$

because $\lim_{x\to 0} K_0(x) = -\ln x$ and $\int_0^\infty dk\,\delta(k) = \frac{1}{2}$.

(c) By symmetry, the angle-averaged linear charge density is an even function of z:

$$\lambda(z) = R\int_0^{2\pi}d\phi\,\sigma(\phi,z) = -\frac{q}{\pi}\int_0^\infty dk\cos kz\frac{K_0(ks)}{K_0(kR)}.$$

Therefore, it is sufficient to focus on $z > 0$. When $z\to\infty$, the integral is dominated by wave vectors in the immediate vicinity of $k = 0$ because $\cos kz$ oscillates wildly otherwise. Moreover, $K_0(x)\to\exp(-x)/\sqrt{2\pi x}$ as $x\to\infty$ and $K_0(x)\to -\ln(x/2)$ as $x\to 0$. Therefore, when $z\to\infty$, $\lambda(z)\approx -(q/\pi)A(z)$, where (using a convergence factor $\epsilon > 0$)

$$A(z) = \int_0^\infty dke^{-k\epsilon}\cos kz\frac{\ln(ks)}{\ln(kR)} = \int_0^\infty dke^{-k\epsilon}\cos kz\left[1+\frac{\ln(s/R)}{\ln(kR)}\right].$$

The factor $\exp(-k\epsilon)$ cuts off the integral for values of $k \gg 1/\epsilon$. Therefore, it has no effect on an integral dominated by values of k near zero. Below, we put $\epsilon \to 0$ (after the integral is performed) unless a non-zero value is needed to guarantee a finite result. With this understanding, the first term in the square brackets gives no contribution. Hence,

$$\begin{aligned} A(z) &= \int_0^\infty dk f(k) \frac{\ln(s/R)}{\ln(kR)} \\ &= -\frac{\ln(s/R)}{\ln(z/R)} \int_0^\infty \frac{dk\, f(k)}{1 - \ln(kz)/\ln(z/R)} \\ &\approx -\frac{\ln(s/R)}{\ln(z/R)} \int_0^\infty dk\, f(k) \left[1 + \frac{\ln kz}{\ln(z/R)}\right], \end{aligned}$$

where $f(k) = \exp(-k\epsilon)\cos kz$. Again, the first term gives zero so, for large z,

$$A(z) \sim \frac{\ln(s/R)}{z \ln^2(z/R)} \int_0^\infty dy\, e^{-k\epsilon} \cos y \ln y.$$

The integral is finite so we get the desired result.

8.22 Green Function for a Dented Beer Can

(a) The ansatz is based on two completeness relations. One is for the particle-in-a-box eigenfunctions for a grounded, one-dimensional "box" defined by $0 \le z \le h$. The other is the same except the box is defined by $0 \le \phi \le 2\pi/p$. The first of these is

$$\frac{2}{h} \sum_{n=1}^\infty \sin\left(\frac{n\pi z}{h}\right) \sin\left(\frac{n\pi z'}{h}\right) = \delta(z - z').$$

The other is the same except that $h \to 2\pi/p$:

$$\frac{p}{\pi} \sum_{m=1}^\infty \sin\left(\frac{mp\phi}{2}\right) \sin\left(\frac{mp\phi'}{2}\right) = \delta(\phi - \phi').$$

Consequently, all the requested boundary conditions are satisfied by the ansatz

$$G_D(\mathbf{r}, \mathbf{r}') = \frac{2p}{\pi h} \sum_{n=1}^\infty \sum_{m=1}^\infty \sin\left(\frac{n\pi z}{h}\right) \sin\left(\frac{n\pi z'}{h}\right) \sin\left(\frac{mp\phi}{2}\right) \sin\left(\frac{mp\phi'}{2}\right) g_{mn}(\rho, \rho').$$

Substituting this into

$$\epsilon_0 \nabla^2 G(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') = -\frac{1}{\rho}\delta(\rho - \rho')\delta(\phi - \phi')\delta(\phi - \phi')\delta(z - z')$$

and defining $\kappa = n\pi/h$ and $\alpha = mp/2$ gives the desired one-dimensional equation:

$$\epsilon_0 \left\{ \frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d}{d\rho}\right) - \left(\kappa^2 + \frac{\alpha^2}{\rho^2}\right) \right\} g_\alpha(\rho, \rho'|\kappa) = -\frac{1}{\rho}\delta(\rho - \rho').$$

(b) The equation above defines the modified Bessel functions. We need a solution that is regular at the origin, vanishes at $\rho = R$, and is continuous when $\rho = \rho'$. If $\rho_<$ denotes the smaller of ρ and ρ' and $\rho_>$ is the larger of ρ and ρ', a linear combination of $I_\alpha(\kappa\rho)$ and $K_\alpha(\kappa\rho)$ that satisfies these requirements is

$$g_\alpha(\rho, \rho'|\kappa) = AI_\alpha(\kappa\rho_<)\left[K_\alpha(\kappa R)I_\alpha(\kappa\rho_>) - I_\alpha(\kappa R)K_\alpha(\kappa\rho_>)\right].$$

The coefficient A is determined by the jump condition

$$\epsilon_0\ \rho' \frac{dg}{d\rho}\bigg|_{\rho=\rho'-\delta}^{\rho=\rho'+\delta} = -1.$$

Writing this out gives

$$A\epsilon_0\kappa\rho' I_\alpha(\kappa R)\left\{I'_\alpha(\kappa\rho')K_\alpha(\kappa\rho') - I_\alpha(\kappa\rho')K'_\alpha(\kappa\rho')\right\} = -1.$$

The Wronskian in the brackets is given in the text as $1/\kappa\rho'$. Therefore, $A = -1/\epsilon_0 I_\alpha(\kappa R)$ and the final Green function is

$$\begin{aligned} G_D(\mathbf{r}, \mathbf{r}') \quad &= \quad \frac{1}{\epsilon_0}\frac{2p}{\pi h}\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\left(\frac{n\pi z}{h}\right)\sin\left(\frac{n\pi z'}{h}\right)\sin\left(\frac{mp\phi}{2}\right)\sin\left(\frac{mp\phi'}{2}\right)\frac{I_{mp/2}(n\pi\rho_</h)}{I_{mp/2}(n\pi R/h)} \\ &\times \left[I_{mp/2}(n\pi R/h)K_{mp/2}(n\pi\rho_>/h) - K_{mp/2}(n\pi R/h)I_{mp/2}(n\pi\rho_>/h)\right]. \end{aligned}$$

Source: Prof. M.J. Cohen, University of Pennsylvania (private communication).

8.23 Weyl's Formula

A two-dimensional delta function in the x and y directions is

$$\int \frac{d^2k_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}_\perp - \mathbf{r}'_\perp)} = \delta(\mathbf{r}_\perp - \mathbf{r}'_\perp).$$

We use this to make the ansatz

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{\epsilon_0}\int \frac{d^2k_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}_\perp - \mathbf{r}'_\perp)} G_0(z, z'|k_\perp).$$

Substituting this into $\nabla^2 G_0(\mathbf{r}, \mathbf{r}' = -\delta(\mathbf{r} - \mathbf{r}')/\epsilon_0$ gives

$$\left[-\frac{\partial^2}{\partial z^2} + k_\perp^2\right] G(z, z'|k_\perp) = \frac{1}{\epsilon_0}\delta(z - z').$$

When $z \neq z'$,

$$G_0(z, z'|k_\perp) = \begin{cases} Ae^{-k_\perp z} & z > z' \\ \\ Be^{+k_\perp z} & z < z' \end{cases}$$

on account of the Dirichlet boundary condition $G_0(z \to \pm\infty, z'|k_\perp) = 0$ appropriate for free space. We get the constants from continuity,

$$\lim_{\epsilon\to 0} G_0(z, z'|k_\perp)\big|_{z=z'-\epsilon} = \lim_{\epsilon\to 0} G_0(z, z'|k_\perp)\big|_{z=z'+\epsilon},$$

and the jump condition

$$\lim_{\epsilon\to 0} \epsilon_0 \, \frac{\partial}{\partial z} G_0(z,z'|k_\perp)\Big|_{z=z'-\epsilon}^{z=z'+\epsilon} = -1.$$

The result is

$$G_0(z,z'|k_\perp) = \frac{1}{2\epsilon_0 k_\perp} e^{-k_\perp|z-z'|}.$$

Therefore, as advertised,

$$G_0(\mathbf{r},\mathbf{r}') = \frac{1}{2\epsilon_0}\int \frac{d^2k_\perp}{(2\pi)^2} e^{i\mathbf{k}_\perp\cdot(\mathbf{r}_\perp-\mathbf{r}'_\perp)} \frac{1}{k_\perp} e^{-k_\perp|z-z'|}.$$

8.24 Electrostatics of a Cosmic String

(a) $G_0(\boldsymbol{\rho},\boldsymbol{\rho}')$ is the potential at $\boldsymbol{\rho}$ due to a unit line charge at $\boldsymbol{\rho}'$. For a unit line source at infinity, Gauss' law gives the electric field as $\mathbf{E} = -\hat{\boldsymbol{\rho}}/2\pi\epsilon_0\rho$. Therefore, the Green function of interest is

$$G(\boldsymbol{\rho},\boldsymbol{\rho}') = -\frac{1}{2\pi\epsilon_0}\ln|r-r'|.$$

(b) The periodicity condition is $G_0^p(\rho,\phi,\boldsymbol{\rho}') = G_0^p(\rho,\phi+2\pi/p,\boldsymbol{\rho}')$ so we will need to have $\delta(\phi-\phi') = \delta(\phi+2\pi/p-\phi')$. This suggests a Fourier series in the interval $(0,2\pi/p)$. The basis functions are $\exp(imp\phi)$ rather than the usual $\exp(im\phi)$ so, by Fourier's integral theorem,

$$\delta(\phi) = \sum_{m=-\infty}^{\infty} f_m e^{imp\phi} = \sum_{m=-\infty}^{\infty}\left[\frac{p}{2\pi}\int_0^{2\pi/p} d\varphi\,\delta(\varphi)e^{-imp\varphi}\right]e^{imp\phi} = \frac{p}{2\pi}\sum_{m=-\infty}^{\infty} e^{imp\phi}.$$

(c) The result of part (b) suggests the ansatz

$$G_0^p(\rho,\phi|\rho',\phi') = \frac{p}{2\pi}\sum_{m=-\infty}^{\infty} e^{imp(\phi-\phi')}G_m(\rho,\rho').$$

Substituting this into $\epsilon_0\nabla^2 G(\boldsymbol{\rho},\boldsymbol{\rho}') = -\delta(\phi-\phi')\delta(\rho-\rho')/\rho$ gives

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G_m}{\partial\rho}\right) - \frac{m^2p^2}{\rho}G_m = -\delta(\rho-\rho').$$

The general solution when $\rho\neq\rho'$ is

$$G_m = A\rho^{-mp} + B\rho^{mp} \qquad m\neq 0$$

$$G_m = C\ln\rho + D \qquad m = 0.$$

Imposing continuity and the jump condition

$$\lim_{\epsilon\to 0}\epsilon_0\rho\;\partial G_m/\partial\rho|^{\rho=\rho'+\epsilon}_{\rho=\rho'-\epsilon}=-1$$

gives

$$\epsilon_0 G_m(\rho,\rho')=\frac{1-\delta_{m,0}}{2p|m|}\left(\frac{\rho_<}{\rho_>}\right)^{p|m|}-\delta_{m,0}\ln\rho_>\,.$$

Consequently,

$$G_0^p(\rho,\phi|\rho',\phi')=\frac{1}{2\pi\epsilon_0}\sum_{m=1}^{\infty}\frac{1}{m}\cos\left[mp(\phi-\phi')\right]\left(\frac{\rho_<}{\rho_>}\right)^{mp}-\frac{p}{2\pi\epsilon_0}\ln\rho_>\,.$$

(d) To perform the sum, note first that G_0^p is obtained from G_0^1 by the replacement $\rho\to\rho^p$ and $\phi\to\phi p$ and similarly for the primed variables. Moreover, for $p=1$, G_0^p must be identical to the result of part (a):

$$G_0^1(\boldsymbol{\rho},\boldsymbol{\rho}')=-\frac{1}{2\pi\epsilon_0}\ln|\boldsymbol{\rho}-\boldsymbol{\rho}'|\;=-\frac{1}{2\pi\epsilon_0}\ln\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos(\phi-\phi')}.$$

Combining these facts gives

$$G_0^p(\boldsymbol{\rho},\boldsymbol{\rho}')=-\frac{1}{2\pi\epsilon_0}\ln\sqrt{\rho^{2p}+\rho'^{2p}-2\rho^p\rho'^p\cos p(\phi-\phi')}.$$

(e) The required force is

$$\mathbf{F}=-q^2\nabla\left[G_0^p(\boldsymbol{\rho},\boldsymbol{\rho}')-G_0^1(\boldsymbol{\rho},\boldsymbol{\rho}')\right]_{\boldsymbol{\rho}=\boldsymbol{\rho}'}\,,$$

where the second term is present to ensure that there is no force when $p=1$. The result is

$$\mathbf{F}=\frac{q^2}{4\pi\epsilon_0}\lim_{\rho\to\rho'}\left[p\frac{\rho^{p-1}}{\rho^p-\rho'^p}-\frac{1}{\rho-\rho'}\right]\hat{\boldsymbol{\rho}}$$

or, using l'Hospital's rule,

$$\begin{aligned}\mathbf{F}&=\frac{q^2}{4\pi\epsilon_0}\lim_{\rho\to\rho'}\left[\frac{p\rho^{p-1}(\rho-\rho')-\rho^p+\rho'^p}{(\rho^p-\rho'^p)(\rho-\rho')}\right]\hat{\boldsymbol{\rho}}\\&=\frac{q^2}{4\pi\epsilon_0}\lim_{\rho\to\rho'}\left[\frac{p\rho^{p-1}-p\rho'^{p-1}}{\rho^p-\rho'^p}\right]\hat{\boldsymbol{\rho}}\\&=\frac{q^2}{4\pi\epsilon_0}\lim_{\rho\to\rho'}\left[\frac{p(p-1)\rho^{p-2}}{p\rho^{p-1}}\right]\hat{\boldsymbol{\rho}}\\&=(p-1)\frac{q^2\hat{\boldsymbol{\rho}}}{4\pi\epsilon_0\rho}.\end{aligned}$$

8.25 Practice with Complex Potentials

Write the potential in the form

$$f(z) = -\frac{\lambda}{2\pi\epsilon_0}\ln\tan\frac{\pi z}{a} = -\frac{\lambda}{2\pi\epsilon_0}\left[\ln\sin\frac{\pi z}{a} - \ln\cos\frac{\pi z}{a}\right].$$

Let $z = na + az_0/\pi$ where $n = 0, \pm 1, \pm 2, \ldots$ and $|z_0| \ll 1$, i.e., a complex number in the immediate vicinity of $x = na$. This gives $\ln\sin(\pi z/a) \approx \ln z_0$ (plus a non-essential constant) so the first term in the brackets approaches the potential of a *positive* line charge very near $x = na$. Now let $z = (n + \frac{1}{2})a + az_0/\pi$. This gives $\ln\cos(\pi z/a) \approx \ln z_0$ so the second term in the brackets approaches the potential of a *negative* line charge very near $x = (n + \frac{1}{2})a$. The physical potential is

$$\varphi(x, y) = Re f(z) = -\frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{\cosh(2\pi y/a) - \cos(2\pi x/a)}{\cosh(2\pi y/a) + \cos(2\pi x/a)}\right].$$

The asymptotic behavior of this potential is

$$\lim_{|y|\to\infty}\varphi(x, y) \sim e^{-2\pi|y|/a}\cos(2\pi x/a).$$

Chapter 9: Steady Current

9.1 A Power Theorem

The rate at which an electric field $\mathbf{E}$ does work on a current density $\mathbf{j}$ confined to V is

$$\mathcal{P} = \int_V d^3r\, \mathbf{j} \cdot \mathbf{E}.$$

A static electric field satisfies $\mathbf{E} = -\nabla\varphi$. Therefore,

$$\mathcal{P} = -\int_V d^3r\, \mathbf{j} \cdot \nabla\varphi = \int_V d^3r\, \varphi\nabla \cdot \mathbf{j} - \int_V d^3r \nabla \cdot (\mathbf{j}\varphi).$$

A steady current satisfies $\nabla \cdot \mathbf{j} = 0$. Therefore, using Gauss' theorem,

$$\mathcal{P} = \int_S d\mathbf{S} \cdot \mathbf{j}\varphi = 0.$$

This integral is zero because confinement of the particles implies that $d\mathbf{S} \cdot \mathbf{j} = 0$.

9.2 A Salt-Water Tank

Let σ be the conductivity of the water. The current density is $\mathbf{j} = \sigma\mathbf{E}$, where $\mathbf{E} = -\nabla\varphi$ and the electrostatic potential obeys Laplace's equation. The boundary condition at $z = h$ is

$$\varphi(x, y) = \begin{cases} V & 0 < x < L/2, \\ 0 & L/2 < x < L. \end{cases}$$

The boundary condition at $x = 0$, $x = L$, and $z = 0$ is

$$\hat{\mathbf{n}} \cdot \mathbf{j} = \sigma \frac{\partial\varphi}{\partial n} = 0.$$

Solutions of Laplace's equation pair sines and cosines with sinh and cosh functions. Therefore, a moment's reflection shows that

$$\varphi(x, y, z) = \sum_{n=0} A_n \cos\left(\frac{n\pi x}{L}\right) \cosh\left(\frac{n\pi z}{h}\right).$$

To get A_n, multiply by $\cos(m\pi x/L)$ and integrate from $x = 0$ to $x = L$. This gives

$$V \int_0^{L/2} dx \cos(m\pi x/L) = \sum_0^{\infty} A_n \cosh(n\pi) \int_0^L dx \cos(n\pi x/L) \cos(m\pi x/L).$$

The integral on the far right gives $(L/2)\delta_{mn}$ so

$$A_n = \begin{cases} 0 & n \text{ even,} \\ \dfrac{2V}{n\pi}\dfrac{\sin(m\pi/2)}{\cosh(m\pi)} & n \text{ odd.} \end{cases}$$

Source: Prof. T.M. O'Neil, University of California San Diego (public communication).

9.3 Radial Hall Effect

(a) From Ampère's law in integral form, the azimuthal magnetic field at radius r is

$$B(r) = \frac{\mu_0}{r}\int_0^r dsj(s)s.$$

The Lorentz force on a charge q is $F = q(E + v \times B)$, so $B(r)$ must be opposed by a radial electric field

$$E(r) = \frac{v\mu_0}{r}\int_0^r dsj(s)s.$$

(b) If $\rho(r) = \rho_+ + \rho_c(r)$ is the source of $E(r)$, Gauss' law in integral form says that

$$E(r) = \frac{1}{r\epsilon_0}\int_0^r ds\rho(s)s.$$

But $j = \rho_c v$, so equating the two expressions for $E(r)$ gives

$$v^2\mu_0\int_0^r ds\rho_c(s)s = \frac{1}{\epsilon_0}\int_0^r ds[\rho_c(s) + \rho_+]s.$$

This gives

$$\int_0^r ds\rho_c(s)s = -\frac{\rho_+}{1 - v^2/c^2}\int_0^r ds\, s,$$

which must be true for every value of r. Therefore,

$$\rho_c(r) = \rho_c = -\frac{\rho_+}{1 - v^2/c^2}.$$

Remark: By conservation of charge, the wire must still be charge-neutral. Therefore, a constant density of mobile charge that is slightly greater (in magnitude) than the immobile positive charge density can only be achieved if all the mobile charge carriers squeeze radially inward a tiny bit. This leaves a very thin layer adjacent to the surface swept free of mobile charges.

(c) The atomic weight of Cu is 63 and its density is 8.8 g/cm^3. Then, if each atom contributes one valence electron, the positive ion density is

$$\rho_+ = 8.8\frac{\text{gm}}{\text{cm}^3} \times \frac{1\,\text{mol}}{63\,\text{gm}} \times \frac{6\cdot 10^{23}\,\text{atom}}{\text{mole}} \times \frac{1\,\text{electron}}{1\,\text{atom}} \times \frac{1.6\cdot 10^{-19}\,\text{C}}{1\,\text{electron}} \times \frac{10^6\,\text{cm}^3}{\text{m}^3}.$$

We suppose that $\rho_c \approx \rho_+$ to estimate the mobile charge velocity from

$$v = \frac{j}{\rho_+} = \frac{I}{A\rho_+} = \frac{1\,\text{A}}{10^{-4}\,\text{m}^2 \times 1.3\times 10^{10}\,\text{C/m}^3} = 8\times 10^{-7}\ \text{m/s}.$$

This gives $v^2/c^2 \sim 10^{-30}$ so $\rho_c = \rho_+$ is an excellent approximation. The radial electric field is

$$E(r) = \frac{v^2\rho_c\mu_0}{r}\int_0^r ds\, s = \frac{1}{2}v^2\mu_0\rho_c r$$

so the potential difference in question is

$$\Delta\varphi = \int_0^R dr E(r) = \tfrac{1}{4}v^2\mu_0\rho_c R^2.$$

9.4 Acceleration EMF

(a) The total acceleration of the electrons is $a + \dot{v}$ and the Drude drag force depends on the relative velocity v. Therefore, with no external electric field,

$$m(a+\dot{v}) + mv/\tau = 0.$$

A steady solution has $\dot{v} = 0$ so $v = -a\tau$. With the Drude conductivity $\sigma = ne^2\tau/m$, this gives a current density

$$j = -nev = ne\tau a = \sigma ma/e.$$

Therefore, a current

$$I = A\sigma ma/e$$

flows through a wire with cross section A.

(b) We have

$$\mathcal{E} = \oint d\boldsymbol{\ell}\cdot\mathbf{E}',$$

where $\mathbf{E}'$ is an effective field which produces a force $\mathbf{F} = q\mathbf{E}'$ so the electron motion described by $m\dot{\mathbf{v}} = \mathbf{F}$ has the same effect as the true, non-electrodynamic current flow. We have $m\dot{\mathbf{v}} = -m\mathbf{a}$ from part (a). Therefore, because $q = -e$,

$$\mathbf{E}' = \frac{m\mathbf{a}}{e}.$$

(c) The linear acceleration is $a = r\dot{\Omega}$ and the ring area is $S = \pi r^2$ so

$$I = \frac{\mathrm{E}}{R} = \frac{1}{R}\frac{mr\dot{\Omega}}{e}2\pi r = \frac{2mS\dot{\Omega}}{eR}.$$

Let the angular motion of the ring be $\theta(t) = \theta_0 \cos\omega t$ with $\theta_0 = 2\pi/360$ radians.

$$\begin{aligned} I &= \frac{2m\pi r^2\omega^2\theta_0 A\sigma\Omega}{e2\pi r} = \frac{mr\omega^2\theta_0 A\sigma}{e} \\ &= \frac{(9\times10^{-31}\mathrm{kg})(0.01\,\mathrm{m})(500\,\mathrm{sec}^{-1})^2(2\pi/360)(10^{-6}\mathrm{m}^2)(6\times10^7\,\mathrm{ohm}^{-1}\mathrm{m}^{-1})}{1.6\times10^{-19}\,C} \\ &= 15\ \mathrm{nA}. \end{aligned}$$

9.5 Membrane Boundary Conditions

At an interface S where the conductivity changes abruptly from σ_1 to σ_2, the matching condition is

$$\sigma_1\hat{\mathbf{n}}_1\cdot\mathbf{E}_1|_S = \sigma_2\hat{\mathbf{n}}\cdot\mathbf{E}_2|_S\,.$$

The electrostatic condition $\nabla\times\mathbf{E} = 0$ implies that $\mathbf{E} = -\nabla\varphi$. Therefore, $\nabla\cdot\mathbf{j} = 0$ implies that each region of our problem is characterized by

$$E_z - -\frac{d\varphi}{dz} \qquad\qquad \frac{d^2\varphi}{dz^2} = 0.$$

Accordingly,

$$\varphi(z) = \begin{cases} A_1 z + B_1 & z < 0, \\ A_2 z + B_2 & 0 < z < \delta, \\ A_3 z + B_3 & z > \delta. \end{cases}$$

From the continuity of $\varphi(z)$ and $\sigma\varphi'(z)$ at each interface we get the conditions

$$\begin{aligned} B_1 &= B_2 \\ A_2\delta + B_2 &= A_3\delta + B_3 \\ \sigma A_1 &= \sigma' A_2 \\ \sigma' A_2 &= \sigma A_3\,. \end{aligned}$$

Using these identities, direct evaluation gives

$$\varphi(\delta^+) - \varphi(0^-) = A_3\delta + B_3 - B_1 = A_2\delta = \delta\frac{\sigma}{\sigma'}A_3 = \delta\frac{\sigma}{\sigma'}\left.\frac{d\varphi}{dz}\right|_{z=0^-}.$$

Similarly,

$$\left.\frac{d\varphi}{dz}\right|_{z=\delta^+} = A_3 = A_1 = \left.\frac{d\varphi}{dz}\right|_{z=0^-}.$$

Source: R.W.P. King and T.T. Wu, *Physical Review E* **58**, 2363 (1998).

9.6 Current Flow to a Bump

Let S be the hemispherical surface. Ohm's law is $\mathbf{j} = \sigma\mathbf{E}$ so

$$I = \int_S d\mathbf{A}\cdot\mathbf{j} = \sigma\int_S d\mathbf{A}\cdot\mathbf{E} = 2\pi\sigma R^2\int_0^{\pi/2} d\theta\sin\theta E_r(R).$$

The radial electric field at the surface of the bump is related to its surface charge density by $E_r(R) = \Sigma/\epsilon_0$. When $d \gg R$, the field near the surface of the bump is nearly the same as the field near a sphere in a uniform field $\mathbf{E}_0$ if the latter is perpendicular to the flat plate. Therefore,

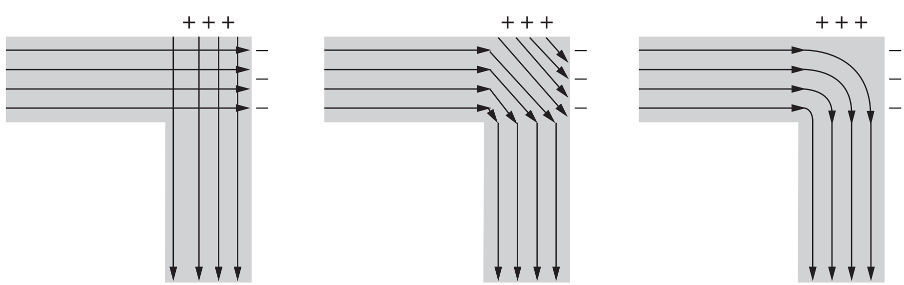
$$I = 6E_0\pi\sigma R^2\int_0^{\pi/2} d\theta\sin\theta\cos\theta = 3\pi\sigma R^2 E_0 = 3\pi\sigma R^2 V_0/d.$$

$E_0 \simeq V_0/d$ is approximately true when $d \gg R$.

9.7 The Charge at a Bend in a Wire

The figure below indicates the physical origin of the surface charges that appear at the bend. The leftmost panel is the situation when the lines of current density $j = I/A = \sigma E$ are unaware of the bend and thus terminate over an area A of wire surface. The charges induced on the wire surface produce the indicated electric field inside the wire (which "bends" the field line pattern) and are the source of a normal electric field $E_n = \sigma/\epsilon = Q/A\epsilon_0$ outside the wire. Combining the two equations in this paragraph gives the desired estimate:

$$Q = \sigma A = \epsilon_0 EA = \epsilon_0\frac{I}{A\sigma}A = \epsilon_0 I/\sigma.$$



Source: A. Butoni and J.-M. Levy-Léblond, *Electricité et Magnétisme* (Librairie Vuibert, Paris, 1999).

9.8 Spherical Child-Langmuir Problem

(a) We generalize the text's derivation of the Child-Langmuir law to a spherical geometry and assume that $\varphi(a) = 0$, $\varphi(b) = V$. The current density at distance r from the center of the system is

$$j = \frac{I}{4\pi r^2} = \rho(r)v(r).$$

where $\rho(r)$ is charge density and $v(r)$ is the velocity of the electrons. Because we assume $v(a) = 0$ the kinetic energy of any electron is $\frac{1}{2}mv^2(r) = e\varphi(r)$. Hence

$$\rho(r) = \frac{I}{4\pi r^2}\sqrt{\frac{m}{2e}}\frac{1}{\sqrt{\varphi}}.$$

Poisson's equation, $\nabla^2\varphi = -\rho/\epsilon_0$, for this situation reads

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\varphi}{\partial r}\right) = \frac{I}{4\pi\epsilon_0 r^2}\sqrt{\frac{m}{2e}}\frac{1}{\sqrt{\varphi}}, \tag{1}$$

and we will require $\varphi(a) = 0$ and $\partial\varphi/\partial r|_{r=a} = 0$.

The change of variable

$$r = a\exp(t) \qquad \text{and} \qquad \varphi(r) = y(t)\left(\frac{I}{4\pi\epsilon_0}\sqrt{\frac{m}{2e}}\right)^{2/3}$$

simplifies (1) to

$$\frac{\partial^2 y}{\partial t^2} + \frac{\partial y}{\partial t} = \frac{1}{\sqrt{y}}, \tag{2}$$

with the "initial" conditions

$$y(0) = 0 \qquad \text{and} \qquad \partial y/\partial t|_{t=0} = 0.$$

The differential equation for $y(t)$ cannot be solved in closed form. Nevertheless, given a solution which satisfies (2) and the initial conditions, the potential on the outer sphere of radius b is

$$V = y\left(\ln[b/a]\right)\left(\frac{I}{4\pi\epsilon_0}\sqrt{\frac{m}{2e}}\right)^{2/3}.$$

Solving for I in the preceding gives the advertised result,

$$I = \sqrt{\frac{2e}{m}}4\pi\epsilon_0\left[\frac{V}{y(x)}\right]^{3/2}, \qquad x = \ln(b/a).$$

(b) With $t = \ln(b/a)$, we search for an asymptotic ($t \gg 1$) solution to (2) of the form $y(t) = Ct^n$. Substituting this guess into the differential equation gives

$$y(t) = \frac{3}{2}^{2/3} t^{2/3},$$

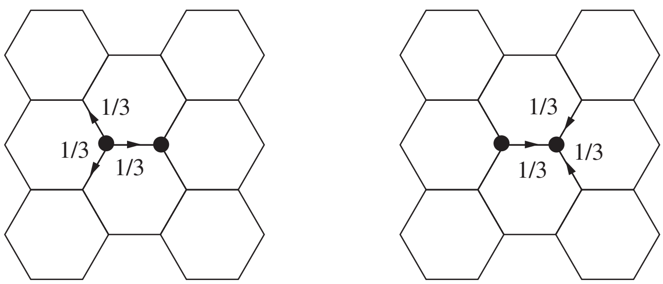
because we may disregard the term proportional to $t^{-4/3}$ in the limit of interest. Hence, when $b \gg a$, the maximum current takes the value

$$I \approx \sqrt{\frac{2e}{m}} \frac{8\pi\epsilon_0 V^{3/2}}{3\ln(b/a)}.$$

Source: I. Langmuir, *Physical Review* **2**, 409 (1913).

9.9 A Honeycomb Resistor Network

Assume segment AB is not missing and let a thin wire carry a current I from infinity to the point A. By symmetry, a current $I/3$ flows through the segment AB. Now, remove the first wire and use a second wire to carry a current I from point B to infinity. By symmetry, a current $I/3$ again flows through the segment AB. Finally, connect the first wire back to point A so a current I flows both from infinity to point A and from point B to infinity. In that case, the total current which flows through AB is $I/3 + I/3 = 2I/3$. This means that the total current which flows through the rest of the circuit is $I - 2I/3 = I/3$, which is one-half the current which flows through the segment AB. Since the rest of the circuit is connected in parallel to r at AB, the resistance of the infinite hexagonal network without segment AB is $R = 2r$.



Source: Dr. A. Scherbakov, Georgia Institute of Technology (private communication).

9.10 Refraction of Current Density

The matching conditions for ohmic matter are

$\hat{\mathbf{n}} \cdot [\mathbf{j}_1 - \mathbf{j}_2] = 0$, hence $j_{1\perp} = j_{2\perp}$

$\hat{\mathbf{n}} \times [\mathbf{E}_1 - \mathbf{E}_2] = 0$, hence $j_{1\|}/\sigma_1 = j_{2\|}/\sigma_2$.

Combining these matching conditions we obtain

$$j_{1\parallel}/j_{1\perp} = (\sigma_1/\sigma_2) \cdot \left(j_{1\parallel}/j_{1\perp}\right)$$

or

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\sigma_1}{\sigma_2}.$$

9.11 Resistance to Ground

Let a current I be expelled radially through the surface of a hemisphere S of radius r. By the definition of current density,

$$I = \int_S d\mathbf{S} \cdot \mathbf{j} = 2\pi r^2 j.$$

If this takes place in ohmic matter with conductivity σ,

$$j = \sigma \mathbf{E} = -\sigma \frac{\partial \varphi}{\partial r}.$$

Therefore,

$$\varphi(r) = \frac{I}{2\pi r \sigma},$$

and the voltage across the σ_2 layer next to the sphere is

$$\varphi(a) - \varphi(b) = \frac{I}{2\pi\sigma_2}\left(\frac{1}{a} - \frac{1}{b}\right).$$

The resistance of this layer is

$$R_2 = \frac{1}{2\pi\sigma_2}\left(\frac{1}{a} - \frac{1}{b}\right).$$

Similarly, the resistance through the Earth is

$$R_1 = \frac{1}{2\pi\sigma_E}\left(\frac{1}{b} - \frac{1}{\infty}\right).$$

The overall resistance of the earthing device is equivalent to the resistances R_1 and R_2 in series. Hence,

$$R = \frac{1}{2\pi\sigma_2}\left(\frac{1}{a} - \frac{1}{b}\right) + \frac{1}{2\pi\sigma_E}\frac{1}{b}.$$

Source: V.V. Batygin and I.N. Toptygin, *Problems in Electrodynamics* (Academic, London, 1978).

9.12 A Separation-Independent Resistance

The text shows that $RC = \epsilon/\sigma$ relates the desired resistance R to the capacitance of a two-conductor capacitor with the same geometry. Moreover, when $a_1, a_2 \ll d$, each sphere contributes a point charge potential at the position of the other. Therefore,

$$\left[\begin{array}{c}\varphi_1 \\ \\ \varphi_2\end{array}\right] \approx \left[\begin{array}{cc}\dfrac{1}{4\pi\epsilon a_1} & \dfrac{1}{4\pi\epsilon d} \\ \dfrac{1}{4\pi\epsilon d} & \dfrac{1}{4\pi\epsilon a_2}\end{array}\right]\left[\begin{array}{c}Q_1 \\ \\ Q_2\end{array}\right] = \left[\begin{array}{cc}P_{11} & P_{12} \\ P_{12} & P_{22}\end{array}\right]\left[\begin{array}{c}Q_1 \\ Q_2\end{array}\right].$$

We know from our previous work with conductors that the P_{ij} are the coefficients of potential and that the capacitance of a two-conductor capacitor is

$$C = \frac{1}{P_{11} + P_{22} - 2P_{12}}.$$

Therefore, the resistance between the conductors for the problem at hand is

$$R \approx \frac{1}{4\pi\sigma}\left[\frac{1}{a_1} + \frac{1}{a_2} + \frac{2}{d}\right].$$

This expression becomes independent of d when d is large enough.

9.13 Inhomogeneous Conductivity

Inside the strip, the electric field is

$$\mathbf{E} = \mathbf{j}/\sigma = \frac{j}{\sigma_0}(1 + a\cos kx)\hat{\mathbf{x}} \qquad |y| < L.$$

Outside the strip, the potential $\varphi(x, y)$ satisfies $\nabla^2\varphi = 0$. The tangential component of the electric field is continuous, so we impose the boundary condition

$$\left.\frac{d\varphi}{dx}\right|_{y=\pm L} = -\frac{j}{\sigma_0}(1 + a\cos kx).$$

A convenient strategy is to write the potential as the sum of two functions,

$$\varphi(x, y) = \varphi_1(x, y) + \varphi_2(x, y),$$

each being a particular solution of Laplace's equation with the boundary conditions

$$\left.\frac{d\varphi_1}{dx}\right|_{y=\pm L} = -\frac{j}{\sigma_0}$$

and

$$\left.\frac{d\varphi_2}{dx}\right|_{y=\pm L} = -\frac{ja}{\sigma_0}\cos kx.$$

By inspection,

$$\varphi_1(x,y) = -\frac{j}{\sigma_0}x.$$

To find $\varphi_2(x,y)$ we separate variables in the usual way to sine and cosine functions in one direction and real exponentials in the other. A solution of this form that satisfies the boundary condition above is

$$\varphi_2(x,y) = -\frac{ja}{\sigma_0 k}\sin kx \exp\left[\pm k(y \pm L)\right],$$

where the $\pm$ sign is chosen for $y < -L$ and $y > L$ so $\varphi_2 \to 0$ as $y \to \pm\infty$. We conclude that

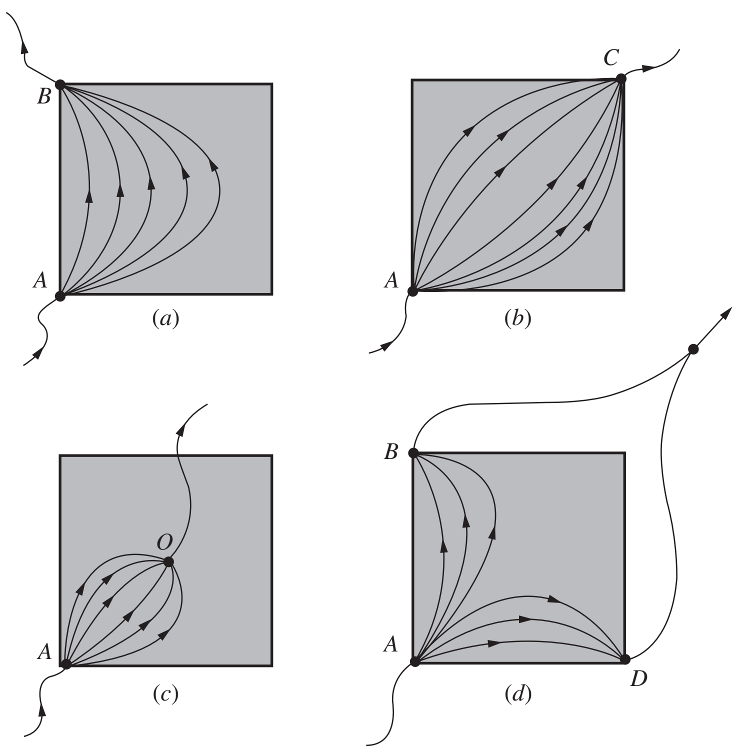
$$\varphi(x,y) = -\frac{j}{\sigma_0}x - \frac{ja}{\sigma_0 k}\sin kx \exp\left[\pm k(y \pm L)\right].$$

The corresopnding electric field outside the strip is

$$\mathbf{E} = -\nabla\varphi = \frac{j}{\sigma_0}\left\{1 + a\cos kx \exp\left[\pm k(y \pm L)\right]\right\}\hat{\mathbf{x}} \pm \frac{ja}{\sigma_0}\sin kx \exp\left[\pm k(y \pm L)\right]\hat{\mathbf{y}}.$$

Source: V.B. Gil'denburg and M.A. Miller, *Collection of Problems in Electrodynamics*, 2nd edition (FizMatLit, Moscow, 2001).

9.14 A Variable Resistor



If the current is I and copper has conductivity σ, the resistance between a point 1 and a point 2 is

$$R = \frac{1}{I\sigma}\int_1^2 d\boldsymbol{\ell}\cdot\mathbf{j}.$$

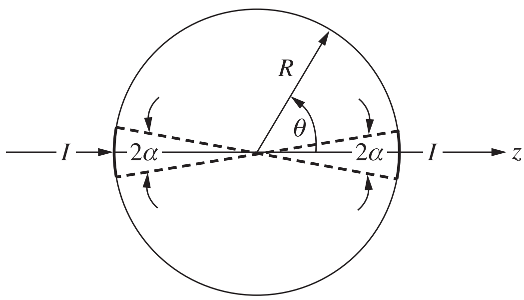
Any integration path between the two points can be used because $\nabla\times\mathbf{E} = 0$ and $\mathbf{j} = \sigma\mathbf{E}$. Unfortunately, we cannot simply compare the shortest path lengths (or even the shortest lines of $\mathbf{j}$) in each case to determine the relative resistances because the magnitude of $\mathbf{j}$ is not constant. On the other hand, it seems clear that $R_a < R_b$ and $R_d < R_a$ because the former is the latter taken in parallel. R_c is surely a small resistance, but it is difficult to compare it to R_d quantitatively. Therefore, the best we can do is

$$R_d, R_c < R_a < R_b.$$

Source: A. Butoni and J.-M. Levy-Léblond, *Electricité et Magnétisme* (Librairie Vuibert, Paris, 1999).

9.15 The Resistance of an Ohmic Sphere

The geometry of the problem is as follows.



The potential satisfies $\nabla^2\varphi = 0$ inside the sphere. The general solution with azimuthal symmetry which is regular everywhere is

$$\varphi(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta). \tag{1}$$

The boundary conditions involve the radial component of the electric field. This is

$$E_r = -\frac{\partial\varphi}{\partial r} = -\sum_{\ell=0}^{\infty} A_\ell \ell r^{\ell-1} P_\ell(\cos\theta). \tag{2}$$

To find the A_ℓ, evaluate (2) at $r = R$, multiply the far left and far right terms by $\sin\theta P_m(\cos\theta)$ and integrate over $x = \cos\theta$. Using the orthogonality of the Legendre polynomials,

$$\int_{-1}^{1} dx\, P_\ell(x)P_m(x) = \frac{2}{2m+1}\delta_{m\ell},$$

we find

$$A_m = -\frac{2m+1}{2m}\frac{1}{R^{m-1}}\int_{-1}^{1} dx\, E_r(R,x)P_m(x).$$

Because $\mathbf{j} = \sigma\mathbf{E}$, $E_r(R) = 0$ everywhere except on the electrodes. On the electrodes, the boundary condition is

$$E_r(R) = \frac{j_r}{\sigma} = \pm\frac{I}{\sigma\pi(R\alpha)^2},$$

where the plus (minus) sign applies to the electrode where the current leaves (enters) the sphere. Consequently,

$$\int_{-1}^{1} dx\, E_r(R,x)P_m(x) = \frac{I}{\sigma\pi(R\alpha)^2}\left[\int_{\cos\alpha}^{1} dx\, P_m(x) - \int_{-1}^{\pi-\cos\alpha} dx\, P_m(x)\right].$$

Using the hint given and the parity of Legendre polynomials, $P_m(-x) = (-)^m P_m(x)$, we find

$$\int_{-1}^{1} dx\, E_r(R,x)P_m(x) = \begin{cases} 0 & m \text{ even,} \\ \dfrac{I}{\sigma\pi(R\alpha)^2}\dfrac{2}{2m+1}\left[P_{m-1}(\cos\alpha) - P_{m+1}(\cos\alpha)\right] & m \text{ odd.} \end{cases}$$

Using this to evaluate A_m and substituting back into (1) gives the potential at any point inside the sphere as

$$\varphi(r,\theta) = \frac{I}{\sigma\pi(R\alpha)^2}\sum_{k=0}^{\infty}\frac{1}{2k+1}\frac{r^{2k+1}}{R^{2k}}\left[P_{2k+2}(\cos\alpha) - P_{2k}(\cos\alpha)\right]P_{2k+1}(\cos\theta).$$

The potential difference between the electrodes is

$$\begin{aligned} V &= \varphi(R,\theta=\pi) - \varphi(R,\theta=0) \\ &= \frac{I}{\sigma\pi(R\alpha)^2}\sum_{k=0}^{\infty}\frac{R}{2k+1}\left[P_{2k+2}(\cos\alpha) - P_{2k}(\cos\alpha)\right]\left[P_{2k+1}(-1) - P_{2k+1}(1)\right]. \end{aligned}$$

The last quantity in square brackets is equal to -2. Therefore, because $V = IR$ defines the resistance,

$$R = \frac{2}{\pi R\alpha^2\sigma}\sum_{k=0}^{\infty}\frac{1}{2k+1}\left[P_{2k}(\cos\alpha) - P_{2k+2}(\cos\alpha)\right].$$

The $k = 0$ term causes R to diverge when $\alpha = 0$ because we are trying to force a finite amount of current through a point of infinitesimal size.

Source: E. Weber, *Electromagnetic Fields* (Wiley, New York, 1950).

9.16 Space-Charge-Limited Current in Matter

This is a one-dimensional problem where $v = \tilde{\mu}E$ and $E = -\partial\varphi/\partial x$. Therefore, Poisson's equation takes the form

$$\frac{d^2\varphi}{dx^2} = -\frac{\rho}{\epsilon} = -\frac{j}{v\epsilon} = \frac{j}{\tilde{\mu}\epsilon}\left(\frac{\partial\varphi}{\partial x}\right)^{-1}.$$

This is the same as

$$\frac{d^2\varphi}{dx^2}\frac{d\varphi}{dx} = \frac{j}{\tilde{\mu}\epsilon},$$

or

$$\frac{d}{dx}\left(\frac{d\varphi}{dx}\right) = \frac{2j}{\tilde{\mu}\epsilon}.$$

Integrating this using the conditions $\varphi = 0$ and $d\varphi/dx = 0$ at $x = 0$ gives

$$\left(\frac{d\varphi}{dx}\right) = \frac{2jx}{\tilde{\mu}\epsilon}.$$

Taking the square root and integrating again gives

$$\varphi(x) = \frac{2}{3}\sqrt{\frac{2j}{\tilde{\mu}\epsilon}}x^{3/2}.$$

Now, $\varphi = V$ at $x = L$. Therefore

$$V = \frac{2}{3}\sqrt{\frac{2j}{\tilde{\mu}\epsilon}}L^{3/2},$$

or

$$j = \frac{9}{8}\tilde{\mu}\epsilon\frac{V^2}{L^3}.$$

This is often called the *Mott-Gurney* law.

Source: N.F. Mott and R.W. Gurney, *Electronic Processes in Ionic Crystals*, 2nd edition (Clarendon, Oxford, 1948).

9.17 van der Pauw's Formula

(a) The potential of a line source has the form $\varphi(\rho) = A\ln\rho$. To find the constant A we insist that

$$I = \int_S d\mathbf{S}\cdot\mathbf{j} = \int_S dSj_\rho,$$

where S is the surface of a half-cylinder of radius a which encloses the contact at A and j_ρ is the radial component of the current density in cylindrical coordinates. Because $j_\rho = \sigma E_\rho = -\sigma\partial\varphi/\partial\rho$,

$$I = -\pi a d\sigma A \,\frac{d}{d\rho} \ln \rho \bigg|_{\rho=a} = -\pi d\sigma A.$$

Therefore, $\varphi(\rho) = -(I/\pi d\sigma)\ln\rho$. The radial distance from A to C is $a+b$, so

$$\varphi_{AC} = -\frac{I}{\pi d\sigma}\ln(a+b).$$

(b) With our definitions,

$$\begin{aligned} V_D - V_C &= \varphi_{AD} - \varphi_{BD} - \varphi_{AC} + \varphi_{BC} \\ &= -\frac{I}{\pi d\sigma}\left[\ln(a+b+c) - \ln(b+c) - \ln(a+b) + \ln b\right]. \end{aligned}$$

Hence,

$$R_{AB,CD} = \frac{V_D - V_C}{I} = \frac{1}{\pi d\sigma}\ln\frac{(b+c)(a+b)}{b(a+b+c)}.$$

Similarly,

$$\begin{aligned} V_A - V_D &= \varphi_{BA} - \varphi_{CA} - \varphi_{BD} + \varphi_{CD} \\ &= -\frac{I}{\pi d\sigma}\left[\ln a - \ln(a+b) - \ln(b+c) + \ln c\right], \end{aligned}$$

so

$$R_{BC,DA} = \frac{V_A - V_D}{I} = \frac{1}{\pi d\sigma}\ln\frac{(b+c)(a+b)}{ca}.$$

Now, because $b(a+b+c) = (b+c)(a+b) - ac$, we have

$$\exp[-\pi d\sigma R_{AB,CD}] = \frac{(a+b)(b+c) - ca}{(a+b)(b+c)}$$

and

$$\exp[-\pi d\sigma R_{BC,DA}] = \frac{ca}{(a+b)(b+c)}.$$

Therefore,

$$\exp(-\pi d\sigma R_{AB,CD}) + \exp(-\pi d\sigma R_{BC,DA}) = 1.$$

Source: L.J. van der Pauw, *Phillips Research Reports* **13**, 1 (1958).

9.18 Rayleigh-Carson reciprocity

Following the discussion of current sources in the text, the Poisson equations satisfied in the two situations are

$$\sigma\nabla^2\varphi_A(\mathbf{r}) = -I_A\left[\delta(\mathbf{r}-\mathbf{r}_1) - \delta(\mathbf{r}-\mathbf{r}_2)\right]$$

and

$$\sigma\nabla^2\varphi_B(\mathbf{r}) = -I_B\left[\delta(\mathbf{r}-\mathbf{r}_3) - \delta(\mathbf{r}-\mathbf{r}_4)\right].$$

Moreover, $\mathbf{j} = \sigma\mathbf{E} = -\sigma\nabla\varphi$. Therefore

$$\begin{aligned}
M &= \int d^3r\,\left[\nabla\varphi_A\cdot\mathbf{j}_B - \varphi_A\nabla\cdot\mathbf{j}_B - \nabla\varphi_B\cdot\mathbf{j}_A - \varphi_B\nabla\cdot\mathbf{j}_A\right] \\
&= \int d^3r\,\left[-\nabla\varphi_A\sigma\cdot\nabla\varphi_B - \varphi_A\sigma\nabla^2\varphi_B + \nabla\varphi_B\sigma\cdot\nabla\varphi_A + \varphi_B\sigma\nabla^2\varphi_A\right] \\
&= \int d^3r\,\left(\varphi_A\sigma I_B\left[\delta(\mathbf{r}-\mathbf{r}_3) - \delta(\mathbf{r}-\mathbf{r}_4)\right] - \varphi_B\sigma I_A\left[\delta(\mathbf{r}-\mathbf{r}_1) - \delta(\mathbf{r}-\mathbf{r}_2)\right]\right) \\
&= \sigma I_B[\varphi_A(\mathbf{r}_3) - \varphi_A(\mathbf{r}_4)] - \sigma I_A[\varphi_B(\mathbf{r}_1) - \varphi_B(\mathbf{r}_2)] \\
&= \sigma(I_BV_A - I_AV_B).
\end{aligned}$$

On the other hand, evaluating M using the divergence theorem gives zero because there is no current density at infinity. Therefore, $V_A = V_B$ if $I_A = I_B$.

Source: H.H. Sample *et al.*, *Journal of Applied Physics* **61**, 1079 (1987).

9.19 The Electric Field of an Ohmic Tube

(a) The general separated-variable solution to Laplace's equation in two dimensions is

$$\varphi(\rho,\phi) = (A_0 + B_0\ln\rho)(x_0 + y_0\phi) + \sum_{\alpha\neq 0}[A_\alpha\rho^\alpha + B_\alpha\rho^{-\alpha}][x_\alpha\sin\alpha\phi + y_\alpha\cos\alpha\phi].$$

The foregoing must match continuously to $\varphi(a,\phi) = (V_0/2\pi)\phi$. This motivates us to write a Fourier series for a straight line on a finite interval:

$$\phi = 2\sum_{k=1}^{\infty}\frac{(-1)^{k-1}\sin k\phi}{k} \qquad\qquad -\pi < \phi < \pi.$$

By inspection, the unique solution inside and outside the cylinder is

$$\varphi(\rho,\phi) = \begin{cases} -\dfrac{V_0}{\pi}\displaystyle\sum_{k=1}^{\infty}\frac{(-\rho/a)^k\sin k\phi}{k} & \rho < a, \\[2ex] -\dfrac{V_0}{\pi}\displaystyle\sum_{k=1}^{\infty}\frac{(-a/\rho)^k\sin k\phi}{k} & \rho > a. \end{cases}$$

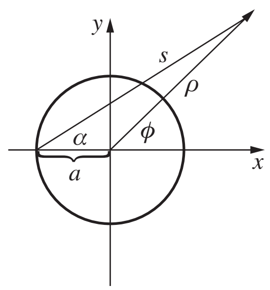
(b) Using the hint, we find immediately that

$$\varphi(\rho,\phi) = \begin{cases} \dfrac{V_0}{\pi}\tan^{-1}\left[\dfrac{\rho\sin\phi}{a+\rho\cos\phi}\right] & \rho < a, \\ \dfrac{V_0}{\pi}\tan^{-1}\left[\dfrac{a\sin\phi}{\rho+a\cos\phi}\right] & \rho > a. \end{cases} \tag{1}$$

From the diagram below, we see that $\tan\alpha = \rho\sin\phi/(a+\rho\cos\phi)$. Therefore,

$$\varphi(\rho < a, \phi) = \frac{V_0}{\pi}\alpha,$$

which shows that the equipotentials inside the cylinder are the suggested straight lines.



(c) Far from the origin of coordinates,

$$\varphi(\rho > a, \phi) \approx \frac{V_0}{\pi}\tan^{-1}\left[\frac{a\sin\phi}{\rho}\right] \approx \frac{V_0}{\pi}\frac{a\sin\phi}{\rho}. \tag{2}$$

This is a dipole potential in two dimensions. To see this, recall that the potential of a line with charge per unit length λ at the origin is

$$\varphi(\rho) = -\frac{\lambda}{2\pi\epsilon_0}\ln\rho.$$

Superposing this with a line with charge per unit length $-\lambda$ located on the x-axis at a small distance ρ_0 from the origin gives

$$\varphi_{\rm tot} = -\frac{\lambda}{2\pi\epsilon_0}\ln\rho + \frac{\lambda}{2\pi\epsilon_0}\ln\sqrt{\rho^2 - 2\rho\rho_0\cos\phi + \rho_0^2} \approx \frac{\lambda\rho_0}{2\pi\epsilon_0}\frac{\cos\phi}{\rho}.$$

We can now let $\lambda \to \infty$ and $\rho_0 \to 0$ in such a way that $\mathbf{d} = \lambda\rho_0\hat{\mathbf{x}}$ is a finite dipole moment per unit length. In that case,

$$\varphi_{\rm tot} = \frac{\mathbf{d}\cdot\hat{\boldsymbol{\rho}}}{2\pi\epsilon_0\rho} = \frac{d}{2\pi\epsilon_0}\frac{\cos\phi}{\rho}.$$

Comparing this with (2) shows that the dipole moment of the ohmic cylinder is oriented along $\hat{\mathbf{y}}$ rather than $\hat{\mathbf{x}}$, as we (arbitrarily) assumed for this calculation.

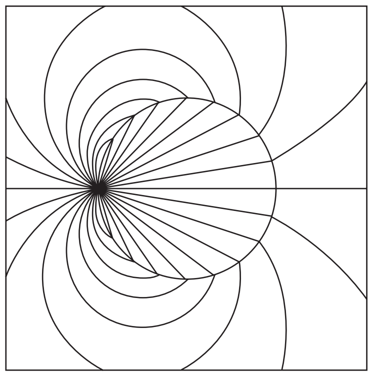
(d) Using (1), it is straightforward to find the radial component of the electric field. Surprisingly, perhaps,

$$E_\rho = -\frac{\partial \varphi}{\partial \rho} = \begin{cases} -\dfrac{V_0}{\pi} \dfrac{a \sin\phi}{a^2 + 2a\rho\cos\phi + \rho^2} & \rho < a, \\ \dfrac{V_0}{\pi} \dfrac{a \sin\phi}{a^2 + 2a\rho\cos\phi + \rho^2} & \rho > a. \end{cases} \tag{3}$$

Using $\sigma_{\text{out}} = \epsilon_0 \hat{\boldsymbol{\rho}} \cdot \mathbf{E}_S$ and $\sigma_{\text{in}} = -\epsilon_0 \hat{\boldsymbol{\rho}} \cdot \mathbf{E}_S$ gives

$$\sigma_{\text{out}} = \sigma_{\text{in}} = \frac{\epsilon_0 V_0}{2\pi a} \frac{\sin\phi}{1 + \cos\phi} = \frac{\epsilon_0 V_0}{2\pi a} \tan\alpha.$$

The equipotentials inside and outside the cylinder look like the following.



(d) The angular component of the electric field inside the cylinder is

$$E_\phi(\rho < a) = -\frac{1}{\rho}\frac{\partial \varphi}{\partial \varphi} = -\frac{V_0}{\pi} \frac{\rho + a\cos\phi}{a^2 + 2a\rho\cos\phi + \rho^2}.$$

From the figure above, the denominator of the last term is s^2. Using E_ρ from (3),

$$\mathbf{E}(\rho < a) = -\frac{V_0}{\pi s} \frac{a \sin\phi}{s} \hat{\boldsymbol{\rho}} - \frac{V_0}{\pi s} \frac{\rho + a\cos\phi}{s} \hat{\boldsymbol{\phi}}.$$

Substituting

$$\hat{\boldsymbol{\rho}} = \cos\phi \hat{\mathbf{x}} + \sin\phi \hat{\mathbf{y}} \qquad \hat{\boldsymbol{\phi}} = -\sin\phi \hat{\mathbf{x}} + \cos\phi \hat{\mathbf{y}} \tag{4}$$

into the preceding equation gives

$$\mathbf{E}(\rho < a) = -\frac{V_0}{\pi s} \left[-\frac{\rho \sin\phi}{s} \hat{\mathbf{x}} + \frac{a + \rho\cos\phi}{s} \hat{\mathbf{y}} \right].$$

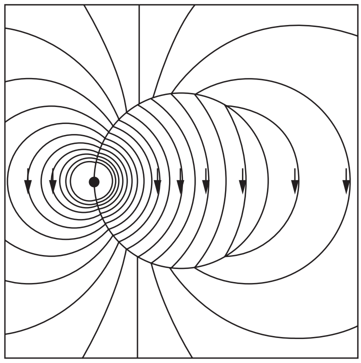
On the other hand, mimicking (4),

$$\hat{\boldsymbol{\alpha}} = -\sin\alpha \hat{\mathbf{x}} + \cos\alpha \hat{\mathbf{y}} = -\frac{\rho \sin\phi}{s} \hat{\mathbf{x}} + \frac{a + \rho\cos\phi}{s} \hat{\mathbf{y}}.$$

Therefore, as suggested,

$$\mathbf{E}(\rho < a) = -\frac{V_0}{\pi s}\hat{\boldsymbol{\alpha}}.$$

The electric field line pattern inside and outside the cylinder looks like the following.



Source: M.A. Heald, *American Journal of Physics* **52**, 522 (1984).

9.20 Current Density in a Curved Segment of Wire

The potential difference $\Delta\varphi$ is a constant, independent of r, between any two cross sectional cuts of the wire. Therefore, if θ parameterizes a traversal through the bend,

$$\Delta\varphi = \mathbf{E} \cdot d\boldsymbol{\ell} = Erd\theta$$

must be independent of r. Hence, $E \propto 1/r$ and $j = \sigma E \propto 1/r$ also. For a very narrow wire, $r \approx$ *const.* so $j \approx$ *const.*, which is the usual answer.

Source: F.B. Pidduck, *Lectures on the Mathematical Theory of Electricity* (Clarendon, Oxford, 1937).

9.21 The Annulus and the Trapezoid

(a) In polar coordinates, the potential $\varphi(r, \phi)$ satisfies the Laplace equation:

$$\nabla^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\phi^2} = 0.$$

Because the edges CD and FA are maintained at a constant potential difference, we see that $\varphi(r, \phi) = \varphi(\phi)$ and

$$\frac{\partial^2\varphi}{\partial\phi^2} = 0.$$

If $\varphi(0) = 0$ and $\varphi(\pi) = V$, the unique solution is

$$\varphi(\phi) = V\frac{\phi}{\pi}.$$

The associated current density is

$$\mathbf{j} = -\sigma\nabla\varphi = -\sigma\frac{1}{r}\frac{\partial\varphi}{\partial\phi}\hat{\boldsymbol{\phi}} = -\sigma\frac{1}{r}\frac{V}{\pi}\hat{\boldsymbol{\phi}},$$

so the total current through any line $\phi = const.$ is

$$I = -t\sigma\frac{V}{\pi}\int_a^b \frac{dr}{r} = -t\sigma\frac{V}{\pi}\ln\left(\frac{b}{a}\right).$$

Hence, the resistance of the annulus is

$$R = \frac{V}{|I|} = \frac{\pi}{t\sigma\ln(b/a)}.$$

(b) Using the result of part (a), the resistance to an azimuthal flow of current through an annulus with radius r, width dr, and length $L = \pi r$ is

$$dR(r) = \frac{\pi}{t\sigma\ln[(r+dr)/r]} = \frac{\pi r}{t\sigma dr}.$$

The resistances from different sub-annuli are in parallel with one another. Therefore,

$$\frac{1}{R} = \int\frac{1}{dR(r)} = \int_a^b \frac{t\sigma}{\pi}\frac{dr}{r} = \frac{t\sigma}{\pi}\ln(b/a).$$

This reproduces our previous result because the lines of current density in each annulus with infinitesimal thickness are the same as the lines of current density obtained for the complete annulus by solving Laplace's equation.

(c) Because the edges ABC and DEF are maintained at a constant potential we see that $\varphi(r,\phi) \equiv \varphi(r)$. Therefore, Laplace's equation reads

$$\nabla^2\varphi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\varphi}{\partial r}\right) = 0.$$

If $\varphi(a) = 0$ and $\varphi(b) = V$, the unique solution is

$$\varphi(r) = \frac{\ln(r/a)}{\ln(b/a)}.$$

The associated current density is

$$\mathbf{j} = -\sigma\nabla\varphi = -\sigma\frac{\partial\varphi}{\partial r}\hat{\mathbf{r}} = -\frac{\sigma V}{\ln(b/a)}\frac{1}{r}\hat{\mathbf{r}},$$

so the total current through any line $r = const.$ is

$$I = j \cdot \pi r t = -\frac{t\sigma\pi V}{\ln(b/a)}.$$

The resistance in this case is

$$R = |V/I| = \frac{\ln(b/a)}{\pi t\sigma}.$$

Using the result just above, the resistance to a radially flowing current through an annulus of radius r and width dr is

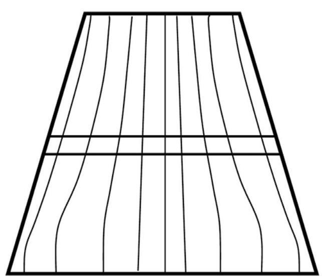
$$dR(r) = \frac{\ln[(r+dr)/r]}{\pi t\sigma} = \frac{dr}{\pi t\sigma r}.$$

The resistance from different annuli are in series for this geometry. Therefore,

$$R = \int dR(r) = \frac{1}{\pi t\sigma}\int_a^b \frac{dr}{r} = \frac{1}{\pi\sigma t}\ln\left(\frac{b}{a}\right).$$

This reproduces our previous result because, in both cases, the radial lines of current density are everywhere parallel to edges where no current exits perpendicular to the edges where the current enters and exits.

(d) The diagram below shows the lines of current density that would result from the Laplace's equation solution. Note that they are perpendicular to the edges where current enters and exits. If this solution were used for the tiny trapezoid indicated, the lines would be perpendicular to the horizontal edges of the tiny trapezoid also, which is not what the true lines of current density do at these points in space. Summing these tiny trapezoids overestimates the amount of current flowing normal to any horizontal line (because the real current density has a horizontal component almost everywhere) and thus *underestimates* the resistance, compared to the exact solution.



Source: L.G. Chambers, *An Introduction to the Mathematics of Electricity and Magnetism* (Chapman and Hall, London, 1973).

9.22 Joule Heating of a Shell

In spherical coordinates, Laplace's equation on the shell is

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\varphi}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2\varphi}{\partial\phi^2}=0.$$

The imposed potential suggests we seek a solution of the form $\varphi = U(\theta)\cos n\phi$. This guess reduces the equation above to

$$\sin\theta\frac{d}{d\theta}\left(\sin\theta\frac{dU}{d\theta}\right)-n^2U=0,$$

subject to the condition $U = V$ at $\theta = \alpha$, $U = -V$ at $\theta = \pi - \alpha$. The suggested substitution further simplifies the equation to

$$\frac{d^2U}{dy^2}-n^2U=0.$$

The general solution is

$$U = A\exp(ny)+B\exp(-ny) = A\tan^n(\theta/2)+B\cot^n(\theta/2),$$

where A and B are determined from

$$V = A\tan^n(\alpha/2)+B\cot^n(\alpha/2)$$

and

$$-V = A\tan^n((\pi-\alpha)/2)+B\cot^n((\pi-\alpha)/2) = A\cot^n(\alpha/2)+B\tan^n(\alpha/2).$$

Specifically,

$$A=\frac{V}{\tan^n(\alpha/2)-\cot^n(\alpha/2)}\quad\text{and}\quad B=-A=\frac{V}{\cot^n(\alpha/2)-\tan^n(\alpha/2)}.$$

Finally, the electric field

$$\mathbf{E}=-\nabla\varphi=-\frac{1}{R}\left[\left(\frac{\partial U}{\partial\theta}\cos n\phi\right)\hat{\boldsymbol{\theta}}-\left(\frac{U}{\sin\theta}n\sin n\phi\right)\hat{\boldsymbol{\phi}}\right]$$

has components

$$E_\theta=-\frac{n\cos n\phi}{R\sin\theta}\left(A\tan^n(\theta/2)-B\cot^n(\theta/2)\right)$$

and

$$E_\phi=\frac{n\sin n\phi}{R\sin\theta}\left(A\tan^n(\theta/2)+B\cot^n(\theta/2)\right).$$

The associated surface current density is

$$\mathbf{K}=\sigma\mathbf{E}=\sigma(E_\theta\hat{\boldsymbol{\theta}}+E_\phi\hat{\boldsymbol{\phi}}).$$

Therefore, the rate of Joule heating generated between the two rings is

$$\begin{aligned}
\mathcal{R} &= \int dS\, \mathbf{K}\cdot\mathbf{E} \\
&= R^2\sigma \int_{\alpha}^{\pi-\alpha} \sin\theta d\theta \int_0^{2\pi} \left(E_\theta^2 + E_\phi^2\right) d\phi \\
&= 2n^2\sigma\pi \int_{\alpha}^{\pi-\alpha} \frac{1}{\sin\theta} \left[A^2 \tan^{2n}(\theta/2) + B^2 \cot^{2n}(\theta/2)\right] d\theta \\
&= 2n^2\sigma\pi A^2 \frac{2\left[\cot^{2n}(\alpha/2) - \tan^{2n}(\alpha/2)\right]}{2n} \\
&= 2n\pi\sigma V^2 \frac{\cot^n(\alpha/2) + \tan^n(\alpha/2)}{\cot^n(\alpha/2) - \tan^n(\alpha/2)} \\
&= \frac{2n\pi\sigma V^2}{\cos\alpha}.
\end{aligned}$$

Source: L.G. Chambers, *An Introduction to the Mathematics of Electricity and Magnetism* (Chapman and Hall, London, 1973).

9.23 The Resistance of a Shell

(a) The potential is confined to a spherical surface and has azimuthal symmetery. Therefore, φ is a function of θ only and Laplace's equation simplifies to

$$\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial\varphi}{\partial\theta}\right) = 0.$$

The boundary conditions are $\varphi = 0$ at $\theta = \alpha_1$ and $\varphi = V$ at $\theta = \pi - \alpha_2$. The suggested substitution further simplifies Laplace's equation to

$$\frac{d^2\varphi}{dy^2} = 0.$$

The general solution is

$$\varphi = A + By = A + B\ln\left[\tan(\theta/2)\right],$$

where A and B are determined from

$$0 = A + B\ln\left[\tan(\alpha_1/2)\right] \qquad \text{and} \qquad V = A + B\ln\left[\tan((\pi - \alpha_2)/2)\right] = A - B\ln\left[\tan(\alpha_2/2)\right].$$

Solving these gives

$$B = \frac{V}{\ln\left[\cot(\alpha_1/2)\cdot\cot(\alpha_2/2)\right]},$$

and we will have no need for the constant A. The electric field is

$$\mathbf{E}(\theta) = \nabla\varphi(\theta) = -\frac{1}{a}\left(\frac{\partial\varphi}{\partial\theta}\right)\hat{\boldsymbol{\theta}} = -\frac{B}{a\sin\theta}\hat{\boldsymbol{\theta}}$$

and the associated surface current density is

$$\mathbf{K}(\theta) = \sigma\mathbf{E} = -\frac{B\sigma}{a\sin\theta}\hat{\boldsymbol{\theta}}.$$

The total current which flows past the sphere's equator, say, is

$$I = |K(\pi/2)|\cdot 2\pi a = 2\pi\sigma B = 2\pi\sigma V/\left\{\ln\left[\cot(\alpha_1/2)\cdot\cot(\alpha_2/2)\right]\right\}.$$

Therefore, the resistance between the terminals is

$$R = \frac{V}{I} = \frac{1}{2\pi\sigma}\ln\left[\cot(\alpha_1/2)\cdot\cot(\alpha_2/2)\right].$$

(b) Consider a thin ring on the surface of the sphere defined by an angle θ. The radius of the ring is $r = a\sin\theta$ and its width is $ad\theta$. Therefore, the resistance of the ring to the current flow is

$$dR = \frac{1}{\sigma}\frac{ad\theta}{2\pi r} = \frac{1}{2\pi\sigma}\frac{d\theta}{\sin\theta}.$$

Integrating this over the ohmic portion of the sphere gives the total resistance between the poles as

$$\begin{aligned} R &= \frac{1}{2\pi\sigma}\int_{\alpha_1}^{\pi-\alpha_2}\frac{d\theta}{\sin\theta} \\ &= \frac{1}{2\pi\sigma}\left[\ln\left(\tan(\pi-\alpha_2)/2\right) - \ln(\tan(\alpha_1/2))\right] \\ &= \frac{1}{2\pi\sigma}\ln\left[\cot(\alpha_1/2)\cdot\cot(\alpha_2/2)\right]. \end{aligned}$$

Source: L.G. Chambers, *An Introduction to the Mathematics of Electricity and Magnetism* (Chapman and Hall, London, 1973).

9.24 The Resistance of the Atmosphere

The resistance of the atmosphere is $R = V/I$ where I is the total current that flows through the atmosphere and V is the potential difference between the Earth's surface and the upper atmosphere. The first of these is straightforward because the current density near the surface of the Earth is

$$j_0 = \sigma_0 E_0 = -3 \times 10^{-12}\ \text{A/m}^2.$$

Therefore, the net steady current which flows from the atmosphere to the Earth is

$$I = |j_0|\, 4\pi r_0^2 \approx 1500\ \text{A}.$$

To find V, we need $E(r)$ and hence $j(r)$. The latter obeys the steady-current condition,

$$\nabla \cdot \mathbf{j} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 j\right) = 0.$$

Integrating this differential equation gives

$$j(r) = j_0 \frac{r_0^2}{r^2}.$$

Hence, the electric field at a distance r from the Earth's center is

$$E(r) = \frac{j(r)}{\sigma(r)} = \frac{j_0 r_0^2}{r^2\left[\sigma_0 + A\left(r - r_0\right)^2\right]}.$$

The potential difference follows immediately as

$$V = -\int_{r_0}^{r_0+H} E(r)dr = -j_0 r_0^2 \int_{r_0}^{r_0+H} \frac{dr}{r^2\left[\sigma_0 + A\left(r - r_0\right)^2\right]}.$$

The integral can be done, with the result that

$$\begin{aligned} V \;=\; & -\frac{j_0 r_0^2}{\sqrt{\sigma_0}(\sigma_0 + A r_0^2)^2}\left[\sqrt{A}(A r_0^2 - \sigma_0)\tan^{-1}\left(\frac{\sqrt{A}(r - r_0)}{\sqrt{\sigma_0}}\right)\right. \\ & \left. - \frac{\sigma_0 + A r_0^2}{r} - A r_0 \ln\left(\frac{\sigma_0 + A(r - r_0)^2}{r^2}\right)\right]_{r_0}^{r_0+H}. \end{aligned}$$

Substituting the given numerical values yields

$$V \approx 370\ \text{kV}.$$

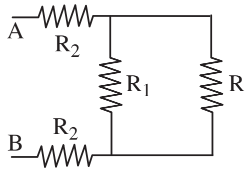
Therefore, our estimate for the resistance of the Earth's atmosphere is

$$R = \frac{V}{I} \approx 250\ \Omega.$$

Source: A.N. Matveev, *Electricity and Magnetism* (Mir, Moscow, 1986).

9.25 Ohmic Loss in an Infinite Circuit

Let R be the equivalent resistance between terminals A and B. Because the three-resistor motif is repeated indefinitely, nothing changes by the addition of one additinal motif. Hence, the circuit shown in the diagram below is equivalent to the original circuit.



As a result, the equivalent resistance of the circuit is

$$R = 2R_2 + \frac{R_1 R}{R_1 + R}.$$

The foregoing rearranges into the quadratic equation

$$R^2 - 2R_2 R - 2R_2 R_1 = 0,$$

which has the positive solution

$$R = R_2 + \sqrt{R_2^2 + 2R_2 R_1}.$$

Now, assume that the voltage between the terminals is V_0. The current flow between the terminals is $I = V_0/R$, so the voltage drop across the first resistor R_1 is

$$V_1 = V_0 - 2R_2 I = V_0 \left(1 - 2R_2/R\right).$$

Similarly, the voltage drop across the second resistor R_1 is

$$V_2 = V_1 \left(1 - 2R_2/R\right)$$

and the voltage across the nth resistor R_1 is

$$V_n = V_0 \left(1 - 2R_2/R\right)^n.$$

Using this information, the rate at which heat is produced by the nth resistor R_1 is

$$\mathcal{R}_n(1) = V_n^2/R_1 = V_0^2 \left(1 - 2R_2/R\right)^{2n}/R_1,$$

and the rate of Joule heating by all the R_1 resistors together is

$$\begin{aligned} \mathcal{R}(1) &= \frac{V_0^2}{R_1} \sum_{n=1}^{\infty} \left(1 - 2R_2/R\right)^{2n} = \frac{V_0^2}{R_1} \left(\frac{1}{1 - \left(1 - 2R_2/R\right)^2} - 1 \right) \\ &= \frac{V_0^2 \left(R_1 + R_2 - \sqrt{R_2^2 + 2R_2 R_1} \right)}{2R_1 \sqrt{R_2^2 + 2R_2 R_1}}. \end{aligned}$$

On the other hand, the rate at which heat is produced by the entire circuit is

$$\mathcal{R} = V_0^2/R = \frac{V_0^2}{R_2 + \sqrt{R_2^2 + 2R_2R_1}}.$$

Finally, by the definition of α,

$$\alpha = \frac{\mathcal{R}(1)}{\mathcal{R}} = \frac{\left(R_1 + R_2 - \sqrt{R_2^2 + 2R_2R_1}\right)\left(R_2 + \sqrt{R_2^2 + 2R_2R_1}\right)}{2R_1\sqrt{R_2^2 + 2R_2R_1}} = \frac{1}{2}\left(1 - \frac{R_2}{\sqrt{R_2^2 + 2R_2R_1}}\right).$$

This expression shows that

$$\alpha < \frac{1}{2}.$$

To complete the problem, we set $x = R_1/R_2$, write

$$\alpha = \frac{1}{2}\left(1 - \frac{1}{\sqrt{1 + 2x}}\right),$$

and solve for x. The result is

$$x = \frac{R_1}{R_2} = \frac{2\alpha(1-\alpha)}{(1-2\alpha)^2}.$$

Source: Dr. A. Scherbakov, Georgia Institute of Technology (private communication).

Chapter 10: Magnetostatics

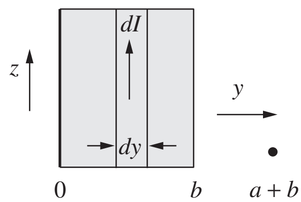
10.1 In-Plane Field of a Current Strip

This problem amounts to superposing the fields from a collection of long, straight wires. The surface current density is $\mathbf{K} = (I/b)\hat{\mathbf{z}}$. Therefore, an infinitely long filament at y with width dy carries a current $dI = K dy$. Treating this as a wire gives a contribution to the magnetic field of

$$d\mathbf{B} = -\frac{\mu_0 K dy}{2\pi(a+b-y)}\hat{\mathbf{x}}.$$

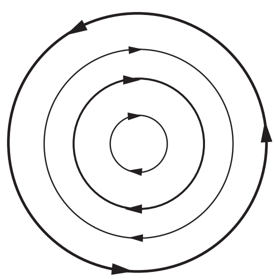
Therefore, the total field at the observation point is

$$\mathbf{B} = -\frac{\mu_0 I}{2\pi b}\int_0^b \frac{dy}{a+b-y}\hat{\mathbf{x}} = -\frac{\mu_0 I}{2\pi b}\ln\left(\frac{a+b}{b}\right)\hat{\mathbf{x}}.$$



10.2 Current Flow in a Disk

The field has only a tangential component in the immediate vicinity of the top and bottom of the disk. That field changes direction when the observation point passes through the disk itself. From the direction of the field, we must have a circulating distribution of current in the body of the disk as shown below. However, the field far away looks like that produced by a loop with current circulating in the opposite direction. We get a consistent picture if we assign that current to the perimeter of the disk. Thus,



Source: C.L. Pekeris and K. Frankowski, *Physical Review A* **36**, 5118 (1987).

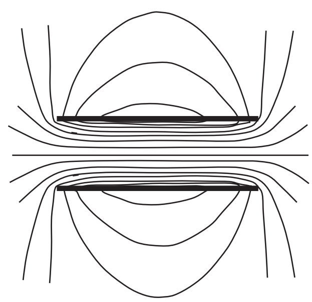
10.3 Finite-Length Solenoid I

(a) Place two identical, semi-infinite solenoids end-to-end so their windings turn in the same direction. The magnetic field through their common plane is longitudinal and the magnetic flux through this plane is the same as the flux through any cross-sectional plane far from their common plane. Now separate the two slightly. By symmetry, the field line pattern is symmetric with respect to the midplane between the two open ends. The longitudinal component of every field line which exits one solenoid (and thus contributes to the magnetic flux of interest) has a counterpart field line which enters the other solenoid with the same longitudinal component of the field. The radial components have opposite signs for the two solenoids. Therefore, adding the two fields together reproduces the field inside an infinite solenoid. This proves the assertion.

(b) All the field lines form closed loops. Moreover, from part (a), only half the field lines exit the solenoid at each end. The other half must pass through the walls and form closed loops as shown below. Indeed, the field lines which exit through the open ends meet up to form closed loops also. We emphasize that there is no reason for field lines *not* to pass through the current sheet. The requirement is that the matching conditions be satisfied:

$$\hat{\boldsymbol{\rho}} \cdot (\mathbf{B}_{\rm in} - \mathbf{B}_{\rm out}) = 0 \qquad \text{and} \qquad \hat{\boldsymbol{\rho}} \times (\mathbf{B}_{\rm in} - \mathbf{B}_{\rm out}) = \mu_0 \mathbf{K}.$$

The field lines very near the wall but very far from the ends are very nearly parallel to the walls. In that case, the matching rules force these lines to execute a sharp "hairpin" turn when they pass through the walls.



Source: I.E. Irodov, *Basic Laws of Electromagnetism* (Mir, Moscow, 1986).

10.4 Helmholtz and Gradient Coils

(a) The field on the z-axis of a current ring is $\mathbf{B}(z) = B(z)\hat{\mathbf{z}}$ where

$$B(z) = \frac{\mu_0 I}{2} \frac{R^2}{(R^2 + z^2)^{3/2}} = \frac{\mu_0 I R^2}{2} f(z) = K f(z).$$

For two coils that carry current in the same direction at a distance z from one of them,

$$\begin{aligned} B(z) &= K[f(z) + f(2b - z)] & B(b) &= 2Kf(b) \\ B'(z) &= K[f'(z) - f'(2b - z)] & B'(b) &= 0 \\ B''(z) &= K[f''(z) + f''(2b - z)] & B''(b) &= 2Kf''(b) \\ B'''(z) &= K[f'''(z) - f'''(2b - z)] & B'''(b) &= 0, \end{aligned}$$

where

$$f'(z) = -\frac{3z}{(R^2+z^2)^{5/2}} \qquad\qquad f''(z) = \frac{12z^2 - 3R^2}{(R^2+z^2)^{7/2}}.$$

Thus, $f''(b) = 0$ when $2b = R$, which is the condition for a Helmholtz coil.

(b) For the gradient coil,

$$\begin{aligned} B(z) &= K[f(z) - f(2b-z)] & B(b) &= 0 \\ B'(z) &= K[f'(z) + f'(2b-z)] & B'(b) &= 2Kf'(b) \\ B''(z) &= K[f''(z) - f''(2b-z)] & B''(b) &= 0 \\ B'''(z) &= K[f'''(z) + f'''(2b-z)] & B'''(b) &= 2Kf'''(b). \end{aligned}$$

Therefore, near the midpoint, $z = b$,

$$B(z) = B(b) + B'(b)(z-b) = 2Kf'(b)(z-b) = \mu_0 I R^2 \frac{3b}{(R^2+b^2)^{5/2}}(b-z) + \cdots.$$

10.5 A Step off the Symmetry Axis

By symmetry, the magnetic field of a current ring has no $\hat{\boldsymbol{\phi}}$ component. Therefore, $\nabla\cdot\mathbf{B} = 0$ reads

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho B_\rho) + \frac{\partial B_z}{\partial z} = 0.$$

Substituting into this $B_\rho = f(z)\rho$ gives

$$f(z) = -\frac{1}{2}\frac{\partial B_z}{\partial z} = \frac{3}{4}\mu_0 I \frac{R^2 z}{(R^2+z^2)^{5/2}}.$$

To find $B_z(\rho, z)$, use the fact that $\nabla\times\mathbf{B} = 0$ near the symmetry axis. Therefore,

$$\frac{\partial B_z}{\partial\rho} = \frac{\partial B_\rho}{\partial z} = \frac{3}{4}\mu_0 I R^2 \rho \frac{R^2 - 4z^2}{(R^2+z^2)^{7/2}}.$$

Integration with respect to ρ gives

$$B_z(\rho, z) = B_z(0, z) + \frac{3}{8}\mu_0 I R^2 \rho^2 \frac{R^2 - 4z^2}{(R^2+z^2)^{7/2}}.$$

10.6 Two Approaches to the Field of a Current Sheet

(a)

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int dS' \frac{\mathbf{K}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \frac{\mu_0 K}{4\pi}\int_{-\infty}^{\infty} dy' \int_{-\infty}^{\infty} dz' \frac{x\hat{\mathbf{y}} - (y-y')\hat{\mathbf{x}}}{[x^2 + (y-y')^2 + (z-z')^2]^{3/2}}.$$

There is no $\hat{\mathbf{x}}$ contribution because the integrand is an odd function of $y-y'$. Therefore,

$$\begin{aligned}\mathbf{B}(\mathbf{r}) &= \frac{\mu_0 K}{4\pi}\int\limits_{-\infty}^{\infty} dy' \int\limits_{-\infty}^{\infty} dz' \frac{x\hat{\mathbf{y}}}{[x^2+(y-y')^2+(z-z')^2]^{3/2}} \\ &= \frac{\mu_0 K}{2\pi}\int\limits_{-\infty}^{\infty} dy' \frac{x\hat{\mathbf{y}}}{x^2+(y-y')^2} \\ &= \frac{\mu_0 K}{2}\hat{\mathbf{y}}.\end{aligned}$$

(b) The magnetic field at a distance R from a wire with current I is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi R}\hat{\boldsymbol{\phi}}.$$

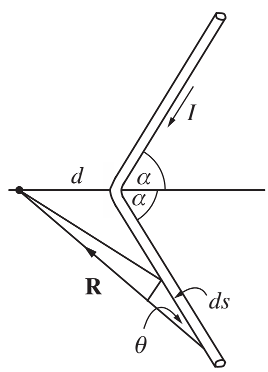
The right-hand rule tells us that the contributions to the $\hat{\mathbf{x}}$-component of $\mathbf{B}(x,y)$ from wires located at $y' = y \pm s$ cancel for all values of s. The remaining component of $\mathbf{B}$ points in the $\text{sgn}(x)\hat{\mathbf{y}}$-direction. Then, because $I = K dy$ and $R = x^2 + y^2$, the substitution $y = x\tan\theta$ gives

$$|B| = \frac{\mu_0}{2\pi}\int_{-\infty}^{\infty} \frac{dy\, K}{\sqrt{x^2+y^2}} \frac{x}{\sqrt{x^2+y^2}} = \frac{\mu_0 K}{2\pi}\int_{-\pi/2}^{\pi/2} d\theta = \tfrac{1}{2}\mu_0 K.$$

Source: W. Hauser, *Introduction to the Principles of Electromagnetism* (Addison-Westey, Reading, MA, 1971).

10.7 The Geometry of Biot and Savart

Adding some angles and labels to the figure gives



If $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, the Bio-Savart law for this wire is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\oint \frac{d\mathbf{s}\times\mathbf{R}}{R^3}.$$

From the geometry, we collect three bits information:

$$d\mathbf{s} \times \mathbf{R} = dsR\sin(\pi - \theta) = dsR\sin\theta$$

$$ds\sin\theta = Rd\theta$$

$$\frac{\sin\theta}{d} = \frac{\sin\alpha}{R}.$$

Since both legs of the wire contribute equally to a field which points into the paper at the indicated observation point (black dot), we get the field magnitude as

$$|\mathbf{B}| = \frac{\mu_0 I}{2\pi}\int_0^\alpha \frac{d\theta}{R} = \frac{\mu_0 I}{2\pi d}\frac{1}{\sin\alpha}\int_0^\alpha d\theta \sin\theta = \frac{\mu_0 I}{2\pi d}\frac{1-\cos\alpha}{\sin\alpha} = \frac{\mu_0 I}{2\pi d}\tan\tfrac{1}{2}\alpha.$$

10.8 The Magnetic Field of Planar Circuits

(a) The Biot-Savart law is

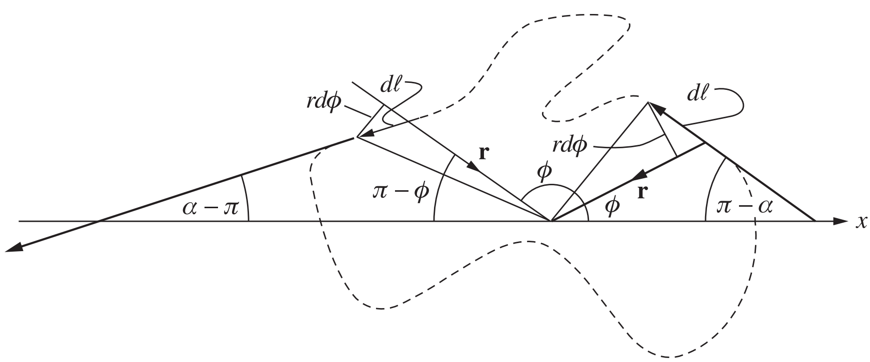
$$\mathbf{B}(\mathbf{x}) = \frac{\mu_0}{4\pi}\oint \frac{d\boldsymbol{\ell}\times(\mathbf{x}-\mathbf{y})}{|\mathbf{x}-\mathbf{y}|^3} = \frac{\mu_0 I}{4\pi}\oint \frac{d\boldsymbol{\ell}\times\hat{\mathbf{r}}}{r^2},$$

where $d\boldsymbol{\ell}$ is a differential element of the circuit at the point $\mathbf{y}$ and $\mathbf{r} = \mathbf{x} - \mathbf{y}$. When the observation point is in the plane of a planar current loop, the field direction is given by the right-hand rule. The figure below shows that the angle between $d\boldsymbol{\ell}$ and $\mathbf{r}$ is $\pi - (\alpha - \phi)$ whether this angle is acute or obtuse. Therefore,

$$|d\boldsymbol{\ell}\times\hat{\mathbf{r}}| = d\ell\sin[\pi - (\alpha - \phi)] = d\ell\sin(\alpha - \phi).$$

On the other hand, the diagram also shows that $r\phi = d\ell\sin(\alpha - \phi)$. Therefore,

$$B(P) = \frac{\mu_0 I}{4\pi}\oint \frac{d\phi}{r}.$$



(b) If P is at the center of an ellipse, we use the result of part (a) where $r(\phi)$ is the equation of an ellipse in polar coordinates:

$$r(\phi) = \frac{ab}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}}.$$

This gives

$$B = \frac{\mu_0 I}{\pi a} E(k),$$

where $k = \sqrt{1 - a^2/b^2}$ and

$$E(k) = \int_0^{\pi/2} d\phi \sqrt{1 - k^2 \sin^2 \phi}$$

is the complete elliptic integral of the second kind. When $a = b$, this simplifies to

$$B = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{d\phi}{r(\phi)} = \frac{\mu_0 I}{4\pi a} \int_0^{2\pi} d\phi = \frac{\mu_0 I}{2a},$$

which is the familiar result for the field at the center of a circular current loop. When $a \to \infty$ with b fixed we get

$$B = \frac{\mu_0 I}{4\pi} \int_0^{2\pi} \frac{d\phi}{r(\phi)} = \frac{\mu_0 I}{4\pi b} \int_0^{2\pi} d\phi |\sin \phi| = \frac{\mu_0 I}{\pi b},$$

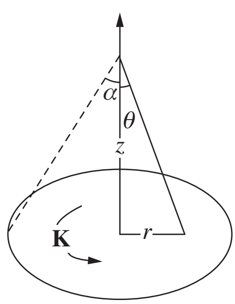
which is the field at the midpoint between two wires separated by a distance $2b$ which carry equal and opposite currents.

(c) The Biot-Savart law reads

$$\mathbf{B}(z) = \frac{\mu_0}{4\pi} \int dS' \frac{K\hat{\boldsymbol{\phi}} \times (z\hat{\mathbf{z}} - r'\hat{\mathbf{r}})}{(z^2 + r'^2)^{3/2}}.$$

By symmetry, $\mathbf{B} = B(z)\hat{\mathbf{z}}$. Therefore,

$$B(z) = \frac{1}{2}\mu_0 K \int dr' \frac{r^2}{z^2 + r'^2}.$$



Now, $\sin\theta = r'/\sqrt{r'^2+z^2}$, $\cos\theta = z/\sqrt{r'^2+z^2}$, and $r' = z\tan\theta$. Therefore, $dr' = (z/\cos^2\theta)d\theta$ and

$$B(\alpha) = \tfrac{1}{2}\mu_0 K \int_0^\alpha d\theta \left\{ \frac{1}{\cos\theta} + \cos\theta \right\} = \tfrac{1}{2}\mu_0 K \left\{ \ln(\sec\alpha + \tan\alpha) - \sin\alpha \right\}.$$

Source: J.A. Miranda, *American Journal of Physics* **68**, 254 (2000).

10.9 Invert the Biot-Savart Law

(a)

$$\begin{aligned}
\mathbf{B}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int dS' \frac{\mathbf{K}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} \\
&= \frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} dz' K(z') \int_{-\infty}^{\infty} dy' \frac{(z-z')\hat{\mathbf{x}} - (x-x_0)\hat{\mathbf{z}}}{[(x-x_0)^2+(y-y')^2+(z-z')^2]^{3/2}}.
\end{aligned}$$

Performing the y' integration gives

$$\begin{aligned}
B_x(x,z) &= \frac{\mu_0}{2\pi}\int_{-\infty}^{\infty} dz' K(z') \frac{(z-z')}{(x-x_0)^2+(z-z')^2} \\
B_y(x,z) &= 0 \\
B_z(x,z) &= -\frac{\mu_0}{2\pi}\int_{-\infty}^{\infty} dz' K(z') \frac{(x-x_0)}{(x-x_0)^2+(z-z')^2}.
\end{aligned}$$

These are indeed convolution integrals of the form

$$A(z) = \int_{-\infty}^{\infty} dz' B(z-z')C(z').$$

(b) Define the Fourier transform pairs,

$$B_z(x,z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \hat{B}_z(x,k)\exp(ikz) \qquad \text{and} \qquad \hat{B}_z(x,k) = \int_{-\infty}^{\infty} dz\, B_z(x,z)\exp(-ikz),$$

and similarly for $K(z)$ and

$$\Lambda(x,z) = \frac{x_0 - x}{(x_0-x)^2+z^2}.$$

In that case, the convolution theorem tells us that

$$\hat{B}_z(k) = \frac{\mu_0}{2\pi}\hat{K}(k)\hat{\Lambda}(k).$$

Solving this for $\hat{K}$ and Fourier transforming back to configuration space gives

$$K(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, \frac{2\pi}{\mu_0}\frac{\hat{B}_z(k)}{\hat{\Lambda}(k)}\exp(ikz). \tag{1}$$

It remains to compute the Fourier transform,

$$\begin{aligned}
\hat{\Lambda}(x,k) &= \int_{-\infty}^{\infty} dz \Lambda(x,z)\exp(-ikz) \\
&= \int_{-\infty}^{\infty} dz \frac{x_0 - x}{z^2 + (x_0^2 - x)^2}\exp(-ikz) \\
&= \frac{1}{2i}\int_{-\infty}^{\infty} dz \left[\frac{1}{z - i(x_0 - x)} - \frac{1}{z + i(x_0 - x)}\right]\exp(-ikz) \\
&= \pi\exp[-|k|(x_0 - x)].
\end{aligned}$$

The last line comes from using the residue theorem for $x < x_0$ and closing the contour in the lower (upper) half-plane when k is positive (negative). Collecting our results and substituting in (1) gives the advertised result:

$$K(z) = \frac{1}{\pi\mu_0}\int_{-\infty}^{\infty} dk\, \exp[|k|(x_0 - x)]\int_{-\infty}^{\infty} dz'\, B_z(x,z')\exp[ik(z - z')] \qquad x < x_0$$

(c) Since $B_y = 0$ and $x < x_0$ excludes the source current, the two components of the magnetic field are constrained by

$$\nabla \cdot \mathbf{B} = 0 \quad \Rightarrow \quad \frac{\partial B_x}{\partial x} = -\frac{\partial B_z}{\partial z}$$

and

$$\nabla \times \mathbf{B} = 0 \quad \Rightarrow \quad \frac{\partial B_x}{\partial z} = -\frac{\partial B_z}{\partial x}.$$

In other words, $B_x(x,z)$ adds no new information. Furthermore, our result says that $K(z)$ is determined entirely by, say, $B_z(0,z)$. This implies that the translationally invariant magnetic field $\mathbf{B}(x,z)$ is determined entirely by $B_z(0,z)$ also. This is analogous to the result proved in the text that an azimuthally invariant magnetic field $\mathbf{B}(\rho,z)$ is determined entirely by $B_z(0,z)$.

Source: D. Jette, *Medical Physics* **30**, 264 (2003).

10.10 Symmetry and Ampère's Law

(a) We work in cylindrical coordinates where $x = \rho\cos\phi$ and $y = \rho\sin\phi$. Then, because $\hat{\mathbf{x}} = \cos\phi\hat{\boldsymbol{\rho}} - \sin\phi\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{y}} = \sin\phi\hat{\boldsymbol{\rho}} + \cos\phi\hat{\boldsymbol{\phi}}$,

$$\begin{aligned}\mathbf{B} &= B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}} = (B_x\cos\phi + B_y\sin\phi)\hat{\boldsymbol{\rho}} + (B_y\cos\phi - B_x\sin\phi)\hat{\boldsymbol{\phi}} + B_z\hat{\mathbf{z}} \\ &= B_\rho\hat{\boldsymbol{\rho}} + B_\phi\hat{\boldsymbol{\phi}} + B_z\hat{\mathbf{z}}.\end{aligned}$$

The magnetic field is a pseudovector so, under reflection through $x = 0$,

$$\mathbf{B} \to \mathbf{B}' = B'_x\hat{\mathbf{x}} + B'_y\hat{\mathbf{y}} + B'_z\hat{\mathbf{z}} = B_x\hat{\mathbf{x}} - B_y\hat{\mathbf{y}} - B_z\hat{\mathbf{z}}.$$

Moreover, this reflection sends $\phi \to \pi - \phi$, so $\sin\phi \to \sin\phi$ and $\cos\phi \to -\cos\phi$. Combining these two bits of information tells us that $B'_\rho = -B_\rho$, $B'_\phi = B_\phi$, and $B'_z = -B_z$, as required.

(b) A π rotation around the z-axis sends $\mathbf{B}$ to $\tilde{\mathbf{B}}$ where

$$\tilde{\mathbf{B}} = \tilde{B}_x\hat{\mathbf{x}} + \tilde{B}_y\hat{\mathbf{y}} + \tilde{B}_z\hat{\mathbf{z}} = -B_x\hat{\mathbf{x}} - B_y\hat{\mathbf{y}} + B_z\hat{\mathbf{z}}.$$

On the other hand, this rotation sends $\phi \to \phi + \pi$, so $\sin\phi \to -\sin\phi$ and $\cos\phi \to -\cos\phi$. Consequently, $\tilde{B}_\rho = B_\rho$, $\tilde{B}_\phi = B_\phi$, and $\tilde{B}_z = B_z$. This result is consistent with the results of part (a) only if $B_\rho = 0$ and $B_z = 0$. Finally, rotational invariance around the z-axis ensures that $\mathbf{B}(\rho,\phi,z) = \mathbf{B}(\rho,z)$. Therefore, we conclude that $\mathbf{B} = B_\phi(\rho,z)\hat{\boldsymbol{\phi}}$.

(c) Since $\mathbf{B}(\mathbf{r}) = B(\rho,z)\hat{\boldsymbol{\phi}}$, we use Ampère's law and circular circuits parallel to the $z = 0$ plane centered on the z-axis. This gives immediately

$$\mathbf{B}(\rho,z) = \begin{cases} -\dfrac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\phi}} & z > 0, \\ 0 & z < 0. \end{cases}$$

(d) The magnetic field matching conditions are

$$\hat{\mathbf{n}}_2 \cdot [\mathbf{B}_1 - \mathbf{B}_2] = 0 \qquad \hat{\mathbf{n}}_2 \times [\mathbf{B}_1 - \mathbf{B}_2] = \mu_0\mathbf{K}(\mathbf{r}_S).$$

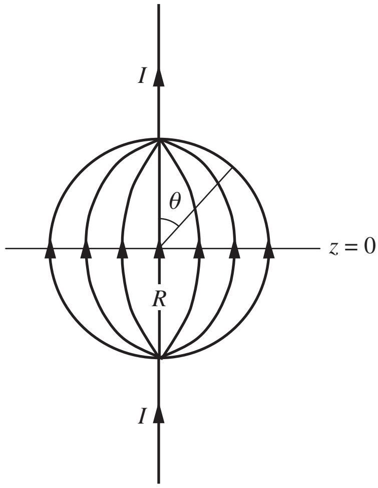
We choose $\hat{\mathbf{n}}_2 = -\hat{\mathbf{z}}$ so the normal component equation is automatically satisfied at the $z = 0$ surface. The other matching equation reads

$$-\hat{\mathbf{z}} \times \frac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\phi}} = \frac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\rho}} = \mu_0\mathbf{K}.$$

This is correct because the total current I flows radially outward through every circle with perimeter $2\pi\rho$.

10.11 Current Flow over a Sphere

The geometry of the problem is the following.



(a) The amount of current which flows past a curve C on a surface is

$$I = \int_C d\mathbf{s} \cdot \mathbf{K} \times \hat{\mathbf{n}}.$$

This must be true for every circle which is the intersection of the sphere with a horizontal plane. For such circles, $d\mathbf{s} = R \sin\theta d\phi \hat{\boldsymbol{\phi}}$. Moreover, $\mathbf{K} = -K(\theta)\hat{\boldsymbol{\theta}}$ and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. Hence,

$$I = 2\pi R \sin\theta K(\theta) \qquad \Rightarrow \qquad \mathbf{K} = -\frac{I}{2\pi R \sin\theta}\hat{\boldsymbol{\theta}}.$$

(b) The source current is invariant to rotations around the z-axis. Therefore, in cylindrical coordinates,

$$\mathbf{j} = j_\rho(\rho, z)\hat{\boldsymbol{\rho}} + j_z(\rho, z)\hat{\mathbf{z}}.$$

The magnetic field satisfies $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ and $\mathbf{B}$ cannot depend on ϕ. Therefore, the only components of the curl that may be non-zero are

$$\nabla \times \mathbf{B} = -\frac{\partial B_\phi}{\partial z}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho B_\phi)\hat{\mathbf{z}}.$$

This tells us that $\mathbf{B} = B_\phi(\rho, z)\hat{\boldsymbol{\phi}}$ and that we should use Ampèrian circuits which are horizontal circles coaxial with the z-axis. When $|z| > R$, this gives the infinite-wire result that $\mathbf{B} = (\mu_0 I/2\pi\rho)\hat{\boldsymbol{\phi}}$. When $|z| < R$, we get zero when $\rho < R\sin\theta$ (inside the sphere). When $\rho > R\sin\theta$ (outside the sphere),

$$\oint_C d\mathbf{s} \cdot \mathbf{B} = 2\pi\rho B_\phi = \mu_0 I.$$

Therefore,

$$\mathbf{B} = \begin{cases} 0 & \text{inside the sphere,} \\ \dfrac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\phi}}. & \text{outside the sphere.} \end{cases}$$

(c) The magnetic field matching conditions are

$$\hat{\mathbf{n}}_2 \cdot [\mathbf{B}_1 - \mathbf{B}_2] = 0 \qquad \hat{\mathbf{n}}_2 \times [\mathbf{B}_1 - \mathbf{B}_2] = \mu_0 \mathbf{K}(\mathbf{r}_S).$$

We choose $\hat{\mathbf{n}}_2 = \hat{\mathbf{r}}$ so the normal-component equation is automatically satisfied at the surface of the sphere. The other matching equation reads

$$\hat{\mathbf{r}} \times (\mathbf{B}_{\text{out}} - \mathbf{B}_{\text{in}})|_S = \mu_0 \mathbf{K}.$$

Since $\rho = R\sin\theta$ at the surface of the sphere, the left side is

$$\hat{\mathbf{r}} \times \frac{\mu_0 I}{2\pi R\sin\theta}\hat{\boldsymbol{\phi}} = -\frac{\mu_0 I}{2\pi R\sin\theta}\hat{\boldsymbol{\theta}}.$$

This agrees with the current density found in part (a).

Source: P.C. Clemmow, *An Introduction to Electromagnetic Theory* (University Press, Cambridge, 1973).

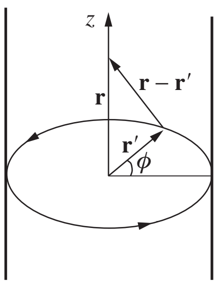
10.12 Finite-Length Solenoid II

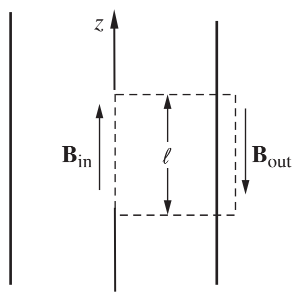
(a) The Biot-Savart law for a current ring is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int d\mathbf{s}' \times \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}.$$

Moreover, $d\mathbf{s} \times (\mathbf{r} - \mathbf{r}') = Rd\phi\hat{\boldsymbol{\phi}} \times (z\hat{\mathbf{z}} - R\hat{\boldsymbol{\rho}}) = Rzd\phi\hat{\boldsymbol{\rho}} + R^2 d\phi\hat{\mathbf{z}}$. Only the z-component survives the ϕ integration so

$$\mathbf{B}(z) = \frac{\mu_0 I R^2}{2(R^2+z^2)^{3/2}}\hat{\mathbf{z}}.$$





Since $n = N/L$, we get the field at the midpoint of the solenoid by symmetrically superposing the field from rings:

$$\mathbf{B}(z) = \frac{\mu_0 I R^2}{2}\int_{-L/2}^{L/2} \frac{ndz}{(R^2+z^2)^{3/2}}\hat{\mathbf{z}} = \frac{\mu_0 InL}{\sqrt{4R^2+L^2}}\hat{\mathbf{z}}.$$

(b) The magnetic field inside a finite solenoid very near the wall and far from the ends is very nearly parallel to the wall, with only a tiny component normal to the wall. The field just outside will be nearly parallel to the wall also, because the normal component of **B** is continuous passing through the wall. Indeed, the outside field near the wall is nearly anti-parallel to the inside field near the wall because $\nabla \cdot \mathbf{B}$ implies that field lines form closed loops. This motivates us to use Ampère's law in integral form and the square-loop path shown dashed in the figure above. There is no contribution from the paths normal to the wall so

$$\oint d\mathbf{s} \cdot \mathbf{B} = B_{\text{in}}\ell + B_{\text{out}}\ell = \mu_0 I n \ell.$$

Since $L \gg R$, we conclude that

$$\mathbf{B}_{\text{out}} = -\mu_0 n I \left[1 - \frac{L}{\sqrt{4R^2 + L^2}} \right] \hat{\mathbf{z}} \approx -\frac{2\mu_0 n I R^2}{L^2} \hat{\mathbf{z}}.$$

10.13 How the Biot-Savart Law Differs from Ampère's Law

(a) The Biot-Savart law for a line source is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I_0}{4\pi} \int \frac{d\boldsymbol{\ell} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}.$$

For observation points in the $z = 0$ plane, $\mathbf{r} - \mathbf{r}' = \rho\hat{\boldsymbol{\rho}} - z'\hat{\mathbf{z}}$, so

$$I_0 dz \hat{\mathbf{z}} \times (\rho\hat{\boldsymbol{\rho}} - z'\hat{\mathbf{z}}) = I_0 \rho dz' \hat{\boldsymbol{\phi}},$$

and the magnitude of the field at any point on C is

$$B = \frac{\mu_0 I_0 \rho}{4\pi} \int_{z_1}^{z_2} \frac{dz'}{(z'^2 + \rho^2)^{3/2}} = \frac{\mu_0 I_0}{4\pi\rho} \frac{z'}{\sqrt{z'^2 + \rho^2}} \bigg|_{z_1}^{z_2} = \frac{\mu_0 I_0}{4\pi\rho} \left\{ \cos\theta_2 - \cos\theta_1 \right\}.$$

(b) The streamlines of $\mathbf{j}_1$ are radially *in* from infinity to the point $\mathbf{r}_1$ and the streamlines of $\mathbf{j}_2$ are radially *out* from the point $\mathbf{r}_2$ to infinity. A closed circuit results if these two, respectively, deliver/extract total current I_0 from/to the original segment; in other words, if

$$\int_{S_1} d\mathbf{S} \cdot \mathbf{j}_1 = -I_0 \qquad \text{and} \qquad \int_{S_2} d\mathbf{S} \cdot \mathbf{j}_1 = I_0,$$

where $S_1(S_2)$ is an infinitesimal sphere which surrounds $\mathbf{r}_2$ ($\mathbf{r}_1$). We confirm this using the divergence theorem and the fact (gleaned by direct analogy with the electric field of a point charge) that $\nabla \cdot \mathbf{j}_1 = -I_0 \delta(\mathbf{r} - \mathbf{r}_1)$ and $\nabla \cdot \mathbf{j}_2 = I_0 \delta(\mathbf{r} - \mathbf{r}_2)$.

(c) The net current through the circle C in the $z = 0$ plane due to $\mathbf{j}_1(\mathbf{r})$ is

$$I_1 = \int d\mathbf{S} \cdot \mathbf{j}_1 = \frac{I_0}{4\pi} \int_0^{2\pi} d\phi \int_0^r d\rho \rho \frac{\cos\theta}{\rho^2 + z_1^2} = \frac{I_0}{2} \int_0^{\theta_1} d\theta \sin\theta = \frac{I_0}{2}(1 - \cos\theta_1).$$

Similarly, the net current through this circle due to $\mathbf{j}_2(\mathbf{r})$ is $I_2 = \frac{I_0}{2}(\cos\theta_2 - 1)$. Therefore, applying Ampère's law to this circle gives

$$\oint_C d\mathbf{s}\cdot\mathbf{B} = 2\pi\rho B = \mu_0\frac{I_0}{2}(\cos\theta_2 - \cos\theta_1).$$

Therefore, in agreement with part (a),

$$\mathbf{B} = \frac{\mu_0 I_0}{4\pi\rho}\{\cos\theta_2 - \cos\theta_1\}\,\hat{\boldsymbol{\phi}}.$$

(d) Neither $\mathbf{j}_1$ nor $\mathbf{j}_2$ produces a Biot-Savart field because both have zero curl and both fall off fast enough to make the surface term vanish in this Biot-Savart equivalent formula derived in the text:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int d^3r'\,\frac{\nabla'\times\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \frac{\mu_0}{4\pi}\int d\mathbf{S}\cdot\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

Source: N. Fukushima, *Report of Ionosphere and Space Research* **30**, 113 (1976).

10.14 Find Surface Current from the Field inside a Sphere

We will use the magnetic scalar potential. Since the desired magnetic field is a polynomial of degree one, the fact that $B_< = -\nabla\psi_<$ inside the sphere tells us that $\psi_<$ is a polynomial of degree two. Moreover, $\nabla^2\psi_< = 0$. Therefore, we may conclude that

$$\psi_<(x,y,z) = -\frac{B_0}{2a}(x^2-y^2) = -\frac{B_0}{2a}r^2\sin^2\theta(\cos^2\phi - \sin^2\phi) = -\frac{B_0}{2a}r^2\sin^2\theta\cos 2\phi.$$

Outside the sphere, we have $\mathbf{B}_> = -\nabla\psi_>$ and $\nabla^2\psi_> = 0$. One matching condition is that the normal (radial) component of the magnetic field is continuous. Therefore, the angular dependence of $\psi_>$ must be the same as $\psi_>$. Accordingly,

$$\psi_>(r,\theta,\phi) = \frac{A}{r^3}\sin^2\theta\cos 2\phi.$$

We find the coefficient A by imposing this matching condition explicitly at $r = a$. This gives

$$B_0 = \frac{3A}{a^4} \;\Rightarrow\; \psi_>(r,\theta,\phi) = \frac{B_0 a^4}{3r^3}\sin^2\theta\cos 2\phi.$$

The current density we want follows from the other condition at the matching surface S:

$$\hat{\mathbf{r}}\times(\mathbf{B}_> - \mathbf{B}_<)|_S = \mu_0\mathbf{K}.$$

Writing this out in detail gives

$$\begin{aligned}\mu_0\mathbf{K} &= -\hat{\mathbf{r}}\times\nabla(\psi_> - \psi_<)|_{r=a}\\ &= -\frac{1}{a}\left[\hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta} - \hat{\boldsymbol{\theta}}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right](\psi_> - \psi_<)|_{r=a}\\ &= -\frac{5}{6}B_0\left[\hat{\boldsymbol{\phi}}\frac{\partial}{\partial\theta} - \hat{\boldsymbol{\theta}}\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right]\sin^2\theta\cos 2\phi.\end{aligned}$$

Therefore,

$$\mathbf{K} = -\frac{5B_0}{6\mu_0}\left[\sin 2\theta\cos 2\phi\,\hat{\boldsymbol{\phi}} + 2\sin\theta\sin 2\phi\,\hat{\boldsymbol{\theta}}\right].$$

Source: Prof. M.J. Cohen, University of Pennsylvania (private communication).

10.15 A Spinning Spherical Shell of Charge

If $\sigma = Q/4\pi R^2$ is the surface charge density, the magnetic field is produced by the surface current density

$$\mathbf{K} = \sigma\boldsymbol{\omega}\times\mathbf{r} = \omega\sigma R\sin\theta\hat{\boldsymbol{\phi}}.$$

The magnetic scalar potential satisfies Laplace's equation inside and outside the sphere. The text showed that the $l = 0$ term is absent from the expansion

$$\psi(r,\theta) = \sum_{\ell=1}^{\infty} P_\ell(\cos\theta)\times\begin{cases} A_\ell r^\ell & r < R,\\ \dfrac{B_\ell}{r^{\ell(\ell+1)}} & r > R.\end{cases} \tag{1}$$

One matching condition is continuity of the normal component of $\mathbf{B}$ at $r = R$. This gives

$$\sum_{\ell=1}^{\infty} A_\ell \ell R^{\ell-1}P_\ell(\cos\theta) = -\sum_{\ell=1}^{\infty} B_\ell(\ell+1)\frac{P_\ell(\cos\theta)}{R^{\ell+2}}.$$

Hence,

$$B_\ell = -\frac{\ell}{\ell+1}R^{2\ell+1}A_\ell. \tag{2}$$

The other matching condition is

$$\hat{\mathbf{r}}\times[\mathbf{B}_{\text{out}} - \mathbf{B}_{\text{in}}]_{r=R} = \mu_0\mathbf{K},$$

or

$$\frac{1}{R}\left[\frac{\partial\psi_{\text{in}}}{\partial\theta} - \frac{\partial\psi_{\text{out}}}{\partial\theta}\right]_{r=R} = \mu_0\omega\sigma R\sin\theta.$$

Therefore, using (1) and (2),

$$\frac{1}{R}\sum_{\ell=1}^{\infty} A_\ell R^\ell \left(\frac{2\ell+1}{\ell+1}\right)\frac{dP_\ell(\cos\theta)}{\partial\theta} = \mu_0\omega\sigma R\sin\theta.$$

From Appendix C.1.1., $dP_\ell/d\theta = P_\ell^1$ and $\sin\theta = -P_1^1$. Therefore,

$$\frac{3}{2}A_1 = -\mu_0\omega\sigma R \qquad \text{and} \qquad A_{\ell\neq 1} = 0.$$

We conclude that

$$\psi(r,\theta) = \begin{cases} -\dfrac{2}{3}\mu_0\omega\sigma R z & r < R, \\[2ex] \dfrac{\mu_0\omega\sigma R^4}{3r^2}\cos\theta & r > R. \end{cases}$$

Inside the sphere, the magnetic field $\mathbf{B} = (2/3)\mu_0\sigma R\boldsymbol{\omega}$ is uniform. Outside the sphere, the field is purely dipolar with magnetic moment $\mathbf{m} = (4\pi/3)\sigma R^4\boldsymbol{\omega}$.

Source: W. Hauser, *Introduction to the Principles of Electromagnetism* (Addison-Wesley, Reading, MA, 1971).

10.16 The Distant Field of a Helical Coil

The magnetic scalar potential outside the coil is a linear combination of solutions to Laplace's equation in cylindrical coordinates:

$$\psi(\rho,\phi,z) = \sum_\alpha\sum_k R_\alpha^k(\rho)G_\alpha(\phi)Z_k(z).$$

The source current has a fundamental periodicity ℓ in the z-direction. This suggests that $Z_k(z)$ is a Fourier series of terms like $\sin(kz)$ and $\cos(kz)$, where $k = 2\pi mz/\ell$ and m is an integer greater than zero (the $k = 0$ term is the solution for a straight wire along the z-axis whereas the current is almost entirely along $\hat{\boldsymbol{\phi}}$ for this problem). The radial partners for this set of functions are the modified Bessel functions $K_\alpha(k\rho)$ and $I_\alpha(k\rho)$. The second of these diverges as $z \to \infty$, while

$$K_\alpha(k\rho) \to \sqrt{\frac{\pi}{2k\rho}}\exp(-k\rho) \qquad \text{as} \qquad \rho\to\infty.$$

The most slowly decaying term has $m = 1$. Therefore, far from the coil,

$$\psi(\rho \gg R) \sim \exp(-2\pi\rho/\ell).$$

The magnetic field $\mathbf{B} = -\nabla\psi$ decays exponentially in the same way. Moreover, the exponential function goes to zero when $\ell \to 0$. This makes sense because an ideal solenoid (no pitch) has no magnetic field outside of itself.

10.17 The Distant Field of Helmholtz Coils

(a) The two coils of a Helmholtz pair of radius R lie on a spherical surface of radius $a = \sqrt{5/4}R$. If we orient the rings parallel to the x-y plane, the surface current density may be written

$$\mathbf{K} = K\hat{\boldsymbol{\phi}} = \frac{I}{a}\left[\delta(\theta-\theta_0)+\delta(\theta+\theta_0-\pi)\right]\hat{\boldsymbol{\phi}},$$

where $\cos\theta_0 = 1/\sqrt{5}$. We will find the magnetic field from $\mathbf{B} = -\nabla\psi$, where $\nabla^2\psi = 0$. Using separation of variables in spherical coordinates and the absence of monopole magnetic fields (no $\ell = 0$ term below), the magnetic scalar potential must have the form

$$\psi(r,\theta) = \begin{cases} \sum\limits_{\ell=1}^{\infty} A_\ell \left(\dfrac{r}{a}\right)^{\ell} P_\ell(\cos\theta) & r < a, \\ \\ \sum\limits_{\ell=1}^{\infty} B_\ell \left(\dfrac{a}{r}\right)^{-(\ell+1)} P_\ell(\cos\theta) & r > a. \end{cases}$$

The matching conditions at $r = a$ are $B_r(in) = B_r(out)$ and $B_\theta(out) - B_\theta(in) = \mu_0 K$. The first matching condition gives

$$\psi(r,\theta) = \begin{cases} \sum\limits_{\ell=1}^{\infty} A_\ell \left(\dfrac{r}{a}\right)^{\ell} P_\ell(\cos\theta) & r < a, \\ \\ -\sum\limits_{\ell=1}^{\infty} A_\ell \dfrac{\ell}{\ell+1} \left(\dfrac{a}{r}\right)^{(\ell+1)} P_\ell(\cos\theta) & r > a. \end{cases}$$

The second matching condition gives

$$\sum_{\ell=1}^{\infty} A_\ell \frac{2\ell+1}{\ell+1}\frac{d}{d\theta}P_\ell(\cos\theta) = \mu_0 I\left[\delta(\theta-\theta_0)+\delta(\theta+\theta_0-\pi)\right].$$

We now use three facts about the associated Legendre polynomials,

$$\frac{d}{d\theta}P_\ell(\cos\theta) = -P_\ell^1(\cos\theta)$$

$$P_\ell^1(\cos\theta) = (-1)^{\ell+1}P_\ell^1(-\cos\theta)$$

$$\int_0^\pi d\theta \sin\theta P_\ell^m(\cos\theta)P_{\ell'}^m(\cos\theta) = \frac{2}{2\ell+1}\frac{(\ell+m)!}{(\ell-m)!}\delta_{\ell\ell'},$$

to get

$$A_\ell = -\frac{\mu_0 I}{2\ell}\sin\theta_0\left[1+(-1)^{\ell+1}\right]P_\ell^1(\cos\theta_0).$$

This gives $A_\ell = 0$ when ℓ is even. Otherwise, because $\sin\theta_0 = 2/\sqrt{5}$ and $\cos\theta_0 = 1/\sqrt{5}$,

$$A_1 = -\mu_0 I \sin\theta_0 P_\ell^1(\cos\theta_0) = -\mu_0 I \sin^2\theta_0 = -\tfrac{4}{5}\mu_0 I$$

$$A_3 = -\tfrac{1}{3}\mu_0 I \sin\theta_0 P_3^1(\cos\theta_0) = -\tfrac{1}{2}\mu_0 I \sin^2\theta_0 \left[5\cos^2\theta_0 - 1\right] = 0$$

$$A_5 = -\tfrac{1}{5}\mu_0 I \sin\theta_0 P_5^1(\cos\theta_0) = \tfrac{9}{5}\mu_0 I \sin^2\theta_0 \cos^2\theta_0 = \tfrac{36}{125}\mu_0 I,$$

where we have used the recurrence formula

$$nP_{n+1}^1(x) - (2n+1)xP_n^1(x) + (n+1)P_{n-1}^1(x) = 0.$$

We conclude that, far from the origin,

$$\psi(r,\theta) = \frac{c_1}{r^2}P_1(\cos\theta) + \frac{c_2}{r^6}P_5(\cos\theta) + \cdots .$$

This gives the advertised form for every component of the magnetic field because

$$\mathbf{B}(r,\theta) = -\nabla\psi = -\frac{\partial\psi}{\partial r}\hat{\mathbf{r}} - \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}}.$$

(b) The leading (dipole) term of the scalar potential at long distance is

$$\psi_1(r,\theta) = \frac{4}{10}\mu_0 I\frac{a^2}{r^2}\cos\theta = \frac{1}{2}\mu_0 I\frac{R^2}{r^2}\cos\theta.$$

This term can be canceled by a second, coaxial set of Helmholtz coils with radius $R' > R$ and current I' as long as $I'R'^2 = -IR^2$.

Source: E.M. Purcell, *American Journal of Physics* **57**, 18 (1988).

10.18 Solid Angles for Magnetic Fields

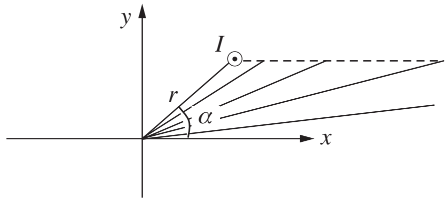
Put the origin at the observation point so the current flows parallel to the z-axis as shown below. The dashed line is an edge view of a semi-infinite plane that begins at the wire and extends to infinity. This plane can serve as the "cut". It is also the boundary of a giant square loop that closes the circuit. The magnetic scalar potential is

$$\psi(\mathbf{r}) = -\frac{\mu_0 I}{4\pi}\Omega = -\frac{\mu_0 I}{2\pi}\alpha$$

because $\Omega = \int_0^\pi d\theta \sin\theta \int_0^\alpha d\phi = 2\alpha$. The magnetic field is

$$\mathbf{B} = -\nabla\psi = -\frac{1}{r}\frac{\partial\psi}{\partial\alpha}\hat{\boldsymbol{\alpha}} = \frac{\mu_0 I}{2\pi r}\hat{\boldsymbol{\alpha}}$$

because α is the polar angle in cylindrical coordinates.



10.19 A Matching Condition for A

The text pointed out that the analytic connection between the k^{th} Cartesian component of Coulomb gauge vector potential $\mathbf{A}(\mathbf{r})$ and the current density $\mathbf{j}(\mathbf{r})$ is the same as the analytic connection between the electrostatic scalar potential $\varphi(\mathbf{r})$ and the charge density $\rho(\mathbf{r})$. Therefore, because

$$\hat{\mathbf{n}}_1 \cdot (\mathbf{E}_2 - \mathbf{E}_1)|_S = \frac{\sigma}{\epsilon_0} \qquad \Rightarrow \qquad \hat{\mathbf{n}}_1 \cdot (\nabla\varphi_1 - \nabla\varphi_2)|_S = \frac{\sigma}{\epsilon_0},$$

we conclude that

$$\hat{\mathbf{n}}_1 \cdot (\nabla\mathbf{A}_1 - \nabla\mathbf{A}_2)|_S = \mu_0\mathbf{K} \qquad \Rightarrow \qquad \left.\left(\frac{\partial\mathbf{A}_1}{\partial n_1} - \frac{\partial\mathbf{A}_2}{\partial n_1}\right)\right|_S = \mu_0\mathbf{K}.$$

10.20 Magnetic Potentials

In cylindrical coordinates,

$$\psi = \frac{C}{2}\ln\rho^2 = C\ln\rho.$$

This scalar potential produces the magnetic field

$$\mathbf{B} = -\nabla\psi = -\frac{C}{\rho}\hat{\boldsymbol{\rho}}.$$

This is the magnetic field of a wire carrying current in the z-direction. The vector potential for such a situation points in the z-direction also. Thus, we want

$$\mathbf{B} = \nabla\times\mathbf{A} = \nabla\times(A\hat{\mathbf{z}}) = -\hat{\mathbf{z}}\times\nabla A$$

to produce the magnetic field above. This tells us that $\mathbf{A} = C\ln\rho\,\hat{\mathbf{z}}$. To get a vector potential which lies in the x-y plane, we need a change of gauge: we choose a gauge function χ so $\nabla\chi$ cancels the vector potential we have and replaces it with a vector potential which lies in the x-y plane. A simple choice which does the trick is $\chi = -Cz\ln\rho$ because

$$\mathbf{A}' = \mathbf{A} + \nabla\chi = C\ln\rho\,\hat{\mathbf{z}} - \nabla[Cz\ln\rho] = -\frac{Cz}{\rho}\hat{\boldsymbol{\rho}} = -\frac{Cz}{\rho}(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}).$$

We also check that

$$\mathbf{B} = \nabla \times \mathbf{A}' = -\nabla \times \left[\frac{Cz}{\rho}\hat{\boldsymbol{\rho}}\right] = -\frac{\partial}{\partial z}\left[\frac{Cz}{\rho}\right]\hat{\boldsymbol{\phi}} = -\frac{C}{\rho}\hat{\boldsymbol{\phi}}.$$

Source: V.C.A. Ferraro, *Electromagnetic Theory* (Athlone Press, London, 1954).

10.21 Consequences of Gauge Choices

(a)

$$\begin{aligned}\nabla \cdot \mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{j}(\mathbf{r}')\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|} \\ &= -\frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{j}(\mathbf{r}')\cdot\nabla'\frac{1}{|\mathbf{r}-\mathbf{r}'|} \\ &= -\frac{\mu_0}{4\pi}\int d^3r'\,\nabla'\cdot\left[\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] + \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\nabla'\cdot\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.\end{aligned}$$

The second term vanishes for steady currents because $\nabla\cdot\mathbf{j} = 0$. The divergence theorem converts the first term to a surface integral at infinity where we presume that $\mathbf{j}(\mathbf{r}) \to 0$.

(b) From the Helmholtz theorem, the conventional Biot-Savart formula

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

is valid when $\nabla\cdot\mathbf{B} = 0$ and $\nabla\times\mathbf{B} = \mu_0\mathbf{j}$. Therefore, the proposed formula is valid when $\nabla\cdot\mathbf{A} = 0$ and $\nabla\times\mathbf{A} = \mathbf{B}$. This is the Coulomb gauge.

10.22 The Magnetic Field of Charge in Uniform Motion

(a) The relevant current density is $\mathbf{j}(\mathbf{r}) = \boldsymbol{v}\rho(\mathbf{r})$ so

$$\begin{aligned}\mathbf{A}(\mathbf{r}) &= \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\mu_0\boldsymbol{v}}{4\pi}\int d^3r'\,\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \frac{\boldsymbol{v}}{c^2}\varphi(\mathbf{r}) \\ \mathbf{B} &= \nabla\times\mathbf{A} = \frac{1}{c^2}\nabla\times(\boldsymbol{v}\varphi) = -\frac{1}{c^2}\boldsymbol{v}\times\nabla\varphi = \frac{1}{c^2}\boldsymbol{v}\times\mathbf{E}.\end{aligned}$$

(b) From Gauss' law, the electric field of a line charge (charge/length λ) coincident with the z-axis is

$$\mathbf{E}(\rho) = \frac{\lambda}{2\pi\epsilon_0\rho}\hat{\boldsymbol{\rho}}.$$

We get a flowing current $I = \rho v$ if $\boldsymbol{v} = v\hat{\mathbf{z}}$, so, from (a), the magnetic field is

$$\mathbf{B}(\rho) = \frac{\mu_0 \lambda v}{2\pi\rho}\hat{\mathbf{z}} \times \hat{\boldsymbol{\rho}} = \frac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\phi}}.$$

This agrees with Ampère's law. Similarly, Gauss' law gives the electric field of a sheet with uniform charge per area σ coincident with $x = 0$ as

$$\mathbf{E}(x) = \hat{\mathbf{x}}\frac{\sigma}{2\epsilon_0}\mathrm{sgn}(x).$$

If the surface current density is $\mathbf{K} = \sigma\boldsymbol{v}$ where $\boldsymbol{v} = v\hat{\mathbf{z}}$, part (a) gives the Ampère's law result

$$\mathbf{B}(x) = \hat{\mathbf{y}}\frac{K}{2\epsilon_0}\mathrm{sgn}(x).$$

10.23 A Geometry of Aharonov and Bohm

(a) By symmetry, $\mathbf{A} = A\,\hat{\boldsymbol{\phi}}$. We evaluate the flux integral in the statement of the problem for a circle of radius ρ. Since $\Phi = \int d\mathbf{S} \cdot \mathbf{B}$, this gives $\pi B\rho^2 = 2\pi\rho A(\rho)$ for $\rho < R$ and $\pi R^2 B = 2\pi\rho A(\rho)$ for $\rho > R$. So,

$$\mathbf{A} = \begin{cases} \dfrac{B\rho}{2}\hat{\boldsymbol{\phi}} & \rho \leq R, \\[2ex] \dfrac{BR^2}{2\rho}\hat{\boldsymbol{\phi}} & \rho \geq R. \end{cases}$$

We are in the Coulomb gauge because $\nabla \cdot \mathbf{A} = 0$.

(b)

$$\mathbf{A}' = \mathbf{A} + \nabla\left(-\frac{\Phi}{2\pi}\phi\right) = \mathbf{A} - \frac{\Phi}{2\pi\rho}\hat{\boldsymbol{\phi}} = \mathbf{A} - \frac{BR^2}{2\rho}\hat{\boldsymbol{\phi}}.$$

Using $\mathbf{A}$ from part (a) shows that

$$\mathbf{A}' = \begin{cases} \left(\dfrac{B\rho}{2} - \dfrac{BR^2}{2\rho}\right)\hat{\boldsymbol{\phi}} & \rho \leq R, \\[2ex] 0 & \rho \geq R. \end{cases}$$

This vector pot!ential is zero outside the solenoid, as advertised.

(c)

$$\mathbf{B}' = \nabla \times \mathbf{A}' = \mathbf{B} - \frac{BR^2}{2}\nabla \times \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = \mathbf{B} - \frac{\Phi}{2\pi}\nabla \times \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right). \qquad (1)$$

When $\rho \neq 0$,

$$\nabla \times \left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right) = \frac{\hat{\mathbf{z}}}{\rho}\frac{\partial}{\partial\rho}\left(\rho \times \frac{1}{\rho}\right) = 0.$$

Moreover, if S is a surface that cuts perpendicularly through the solenoid,

$$\int_S d\mathbf{S}\cdot\nabla\times\left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right)=\oint d\mathbf{s}\cdot\frac{\hat{\boldsymbol{\phi}}}{\rho}=\int_0^{2\pi}d\phi=2\pi.$$

Consequently,

$$\nabla\times\left(\frac{\hat{\boldsymbol{\phi}}}{\rho}\right)=\frac{\delta(\rho)}{\rho}\hat{\mathbf{z}}.$$

Using (1), we conclude finally that

$$\mathbf{B}'=\mathbf{B}-\frac{\Phi}{2\pi}\frac{\delta(\rho)}{\rho}.$$

10.24 Lamb's Formula

(a) If $\mathcal{B}$ is a constant vector,

$$(\nabla\times\mathbf{A})_i=\epsilon_{ijk}\partial_j\tfrac{1}{2}\epsilon_{kst}\mathcal{B}_s r_t=\tfrac{1}{2}\mathcal{B}_s\epsilon_{kij}\epsilon_{kst}\delta_{jt}=\tfrac{1}{2}\mathcal{B}_s(\delta_{is}\delta_{jt}-\delta_{it}\delta_{js})\delta_{jt}=\tfrac{1}{2}(\mathcal{B}_i\delta_{jj}-\mathcal{B}_j\delta_{ij})=\mathcal{B}_i.$$

(b) The source current is $\mathbf{j}(\mathbf{r})=(e/2m)\rho(r)\mathcal{B}\times\mathbf{r}$ so

$$\mathbf{A}_{\rm ind}(\mathbf{r})=\frac{\mu_0}{4\pi}\frac{e\mathcal{B}}{2m}\times\int d^3r'\,\frac{\mathbf{r}'\rho(r')}{|\mathbf{r}-\mathbf{r}'|}.$$

Let $\mathbf{r}$ point along $\hat{\mathbf{z}}$ and $\mathbf{r}'=r'\hat{\mathbf{r}}'$ where $\hat{\mathbf{r}}'=\hat{\mathbf{z}}\cos\theta'+\hat{\mathbf{y}}\sin\theta'\sin\phi'+\hat{\mathbf{x}}\sin\theta'\cos\phi'$. In that case,

$$\int d^3r'\,\frac{\mathbf{r}'\rho(r')}{|\mathbf{r}-\mathbf{r}'|}$$

$$=\sum_{\ell=0}^{\infty}\int_0^{2\pi}d\phi'\int_0^{\pi}d\theta'\sin\theta' P_\ell(\cos\theta')\hat{\mathbf{r}}'\left\{\frac{1}{r}\int_0^r dr'\,\rho(r')r'^3\left(\frac{r'}{r}\right)^\ell+\int_r^\infty dr'\,\rho(r')r'^2\left(\frac{r}{r'}\right)^\ell\right\}.\tag{10.1}$$

The ϕ' integration eliminates the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ components. Moreover, $P_1(x)=x$ and

$$\int_{-1}^{1}dxP_\ell(x)P_m(x)=\frac{2\delta_{\ell m}}{2\ell+1}.$$

Hence, only the $\ell=1$ term in the sum survives the integration in (1) and

$$\begin{aligned}\int d^3r'\,\frac{\mathbf{r}'\rho(r')}{|\mathbf{r}-\mathbf{r}'|}&=2\pi\times\frac{2}{3}\hat{\mathbf{z}}\left\{\frac{1}{r^2}\int_0^r dr'\,\rho(r')r'^4+r\int_r^\infty dr'\,\rho(r')r'\right\}\\&=\frac{r}{3}\left\{\frac{1}{r^3}\int_{r'<r}d^3r'\,\rho(r')r'^2+\int_{r'>r}d^3r'\,\frac{\rho(r')}{r'}\right\}\hat{\mathbf{z}}.\end{aligned}$$

We conclude that

$$\mathbf{A}_{\rm ind}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{e\mathcal{B}\times\mathbf{r}}{6m}\left\{\frac{1}{r^3}\int\limits_{r'<r} d^3r'\,\rho(r')r'^2 + \int\limits_{r'>r} d^3r'\,\frac{\rho(r')}{r'}\right\}.$$

(c) When r is very small,

$$\mathbf{A}_{\rm ind}(\mathbf{r}) \approx \frac{\mu_0}{4\pi}\frac{e\mathcal{B}\times\mathbf{r}}{6m}\left\{\frac{4\pi}{5}\rho(0)r^2 + \int d^3r'\,\frac{\rho(r')}{r'}\right\},$$

where the integral that remains is over all of space. In that case, the latter is equal to $4\pi\epsilon_0\varphi(0)$, where $\varphi(0)$ is the electrostatic potential of the atom evaluated at the origin. Moreover, the first term inside the curly brackets is higher order in r. Therefore,

$$\mathbf{A}_{\rm ind}(\mathbf{r}) \approx \frac{e\mathcal{B}\times\mathbf{r}}{6mc^2}\varphi(0).$$

Because $\mathcal{B}$ is a constant vector, $\nabla\times(\mathcal{B}\times\mathbf{r}) = 2\mathcal{B}$, and we confirm that

$$\mathbf{B}_{\rm ind}(0) = \nabla\times\mathbf{A}_{\rm ind}(\mathbf{r}) = \frac{e\varphi(0)}{3mc^2}\mathcal{B}.$$

Source: W. Lamb, *Physical Review* **60**, 817 (1941).

10.25 Toroidal and Poloidal Magnetic Fields

(a) Use the fact that $\nabla\cdot\nabla\times\mathbf{Q} = 0$ for any vector $\mathbf{Q}$. This gives $\nabla\cdot\mathbf{P} = \nabla\cdot\nabla\times\mathbf{L}\gamma = 0$ immediately. Similarly, $\mathbf{T} = \mathbf{L}\psi = -i\mathbf{r}\times\nabla\psi = i\nabla\times(\psi\mathbf{r})$ so $\nabla\cdot\mathbf{T} = 0$ as well.

(b) Suppose $\mathbf{B}$ is toroidal so $\mathbf{B} = \mathbf{L}\psi$. This implies that $\mathbf{j}$ is poloidal because $\mu_0\mathbf{j} = \nabla\times\mathbf{B} = \mu_0\nabla\times\mathbf{L}\xi$. Conversely, suppose $\mathbf{B}$ is poloidal so $\mathbf{B} = \nabla\times\mathbf{L}\psi$. In that case, $\mathbf{j}$ is toroidal because

$$\mu_0\mathbf{j} = \nabla\times\mathbf{B} = \nabla\times(\nabla\times\mathbf{L}\psi) = \nabla(\nabla\cdot\mathbf{L}\psi) - \nabla^2\mathbf{L}\psi = -\nabla^2\mathbf{L}\psi = -\mathbf{L}\nabla^2\psi.$$

(c) The magnetic field of a toroidal solenoid was found in the text to be

$$\mathbf{B}(\rho,z) = \begin{cases} \dfrac{\mu_0 NI}{2\pi\rho}\hat{\boldsymbol{\phi}} & \text{for points inside the torus,} \\[2ex] 0 & \text{for points not outside the torus.} \end{cases}$$

Therefore, if C is a constant, we need to show that a function $\psi(\mathbf{r})$ exists such that

$$\mathbf{B} = \frac{C}{\rho}\hat{\boldsymbol{\phi}} = \mathbf{L}\psi = i\nabla\psi\times\mathbf{r}.$$

Now, $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}}$. Comparing this with the equation above and switching from cylindrical to polar coordinates tells us that

$$-i\frac{\partial \psi}{\partial \theta} = \frac{C}{\rho} = \frac{C}{r\sin\theta}.$$

Integration gives $\psi(r,\theta) = \dfrac{i}{r}\ln\tan\dfrac{\theta}{2}$, which proves the assertion.

(d) We have $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$ in V. The Helmholtz theorem would give $\mathbf{B} \equiv 0$ if V were all of space. When V is finite, the double-curl identity tells us that

$$0 = \nabla \times (\nabla \times \mathbf{B}) - \nabla(\nabla \cdot \mathbf{B}) = -\nabla^2 \mathbf{B}.$$

Therefore, $\nabla^2 \mathbf{B} = 0$ in V.

(d) We have

$$\begin{aligned} \mathbf{B} &= \nabla \times \mathbf{A} \\ &= \nabla \times \mathbf{L}\psi + \nabla \times (\nabla \times \mathbf{L}\gamma) \\ &= \nabla \times \mathbf{L}\psi + \nabla(\nabla \cdot \mathbf{L}\gamma) - \nabla^2 \mathbf{L}\gamma \\ &= \nabla \times \mathbf{L}\psi - \nabla^2 \mathbf{L}\gamma. \end{aligned}$$

Now take the Laplacian of both sides. We get $\nabla^2 \mathbf{B} = 0$ if $\psi(x)$ and $\gamma(x)$ both satisfy Laplace's equation, i.e., $\nabla^2\psi = 0$ and $\nabla^2\gamma = 0$. In that case, $\nabla^2 \mathbf{L}\gamma = \mathbf{L}\nabla^2\gamma = 0$ so we are left with $\mathbf{B} = \nabla \times \mathbf{L}\psi$, which implies that the vector potential $\mathbf{A} = \mathbf{L}\psi$ in V.

Chapter 11: Magnetic Multipoles

11.1 Magnetic Dipole Moment Practice

We will find the current using $\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j}$. First,

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{\mu_0 A_0}{4\pi}\left[\hat{\mathbf{r}}\frac{2\cos\theta}{r^2} + \hat{\boldsymbol{\theta}}\frac{\lambda \sin\theta}{r}\right]\exp(-\lambda r).$$

Therefore,

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B} = \hat{\boldsymbol{\phi}} 4\pi A_0 \sin\theta \left\{\frac{2}{r^3} - \frac{\lambda^2}{r}\right\}\exp(-\lambda r).$$

The associated magnetic moment is

$$\mathbf{m} = \tfrac{1}{2}\int d^3r\, \mathbf{r} \times \mathbf{j} = -\frac{A_0}{8\pi}\int d^3r\, \hat{\boldsymbol{\theta}}\, r\sin\theta \left\{\frac{2}{r^3} - \frac{\lambda^2}{r}\right\}\exp(-\lambda r).$$

But $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}\cos\theta\cos\phi + \hat{\mathbf{y}}\cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$. This shows that only the $\hat{\mathbf{z}}$-component survives the integration. Hence,

$$\begin{aligned}
\mathbf{m} &= \hat{\mathbf{z}}\frac{A_0}{4}\int_0^\pi d\theta\, \sin^2\theta \int_0^\infty dr r^3 \exp(-\lambda r)\left\{\frac{2}{r^3} - \frac{\lambda^2}{r}\right\} \\
&= \hat{\mathbf{z}}\frac{\pi A_0}{8}\left\{2 - \lambda^2 \frac{d^2}{d\lambda^2}\right\}\int_0^\infty dr \exp(-\lambda r) \\
&= 0.
\end{aligned}$$

11.2 Origin Independence of Magnetic Multipole Moments

(a) If we shift the origin by a vector $\mathbf{d}$, the new magnetic moment is

$$\mathbf{m}' = \int d^3r\, (\mathbf{r} - \mathbf{d}) \times \mathbf{j} = \mathbf{m} - \mathbf{d} \times \int d^3r\, \mathbf{j} = \mathbf{m}.$$

The last equality above is true by conservation of charge. In the language of current loops,

$$\int d^3r\, \mathbf{j} = I \oint d\mathbf{s} = 0.$$

(b) Similarly, $m'^{(2)}_{ij} = \dfrac{1}{3}\displaystyle\int d^3r\, [(\mathbf{r} - \mathbf{d}) \times \mathbf{j}]_i (\mathbf{r} - \mathbf{d})_j$. Writing out the four terms gives

$$m'^{(2)}_{ij} = \frac{1}{3}\int d^3r\, (\mathbf{r}\times\mathbf{j})_i r_j - \frac{1}{3}\int d^3r\, (\mathbf{r}\times\mathbf{j})_i d_j - \frac{1}{3}\int d^3r\, (\mathbf{d}\times\mathbf{j})_i r_j + \frac{1}{3}\int d^3r\, (\mathbf{d}\times\mathbf{j})_i d_j.$$

The last integral above is zero because $\int d^3r\, j_k = 0$ as in part (a). What remains is

$$m'^{(2)}_{ij} = m^{(2)}_{ij} - \frac{3}{2}m_i d_j - \frac{1}{3}\epsilon_{ist} d_s \int d^3r\, j_t r_j. \tag{11.1}$$

But our derivation of the dipole vector potential exploited the identity

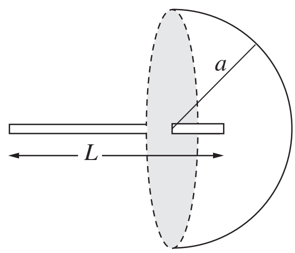
$$\int d^3r\, j_k r_l = -\frac{1}{2}\epsilon_{k\ell i}\int d^3r\, (\mathbf{r}\times\mathbf{j})_i = -\epsilon_{k\ell i} m_i.$$

Comparison with (1) shows that $m'^{(2)}_{ij} = m^{(2)}_{ij}$ only when $\mathbf{m} \equiv 0$.

11.3 The Field outside a Finite Solenoid

The volume V is bounded by a surface S composed of a hemisphere and a disk as shown below. Therefore,

$$0 = \int_V d^3r\, \nabla\cdot\mathbf{B} = \int_S d\mathbf{S}\cdot\mathbf{B} = \int_{\text{disk}} d\mathbf{S}\cdot\mathbf{B} + \int_{\text{hemisphere}} d\mathbf{S}\cdot\mathbf{B}.$$



The flux integral over the hemisphere is zero because $dS \sim r^2$. The general multipole expansion guarantees that the field strength falls off as $B \sim 1/r^3$ in the limit when $a \to \infty$. This means that the total flux through the disk must also be zero. The contribution to the flux integral from the part of the disk which lies inside the solenoid is $\Phi^{\text{in}}_{\text{disk}} = \mu_0 nIA$. Therefore, the contribution from the part of the disk which lies outside the solenoid must be equal and opposite. This tells us that the field outside points oppositely to the field inside. Moreover, only the portion of the disk near the solenoid contributes because the field becomes dipolar far away on the disk and fails to contribute (like the hemisphere contribution). By "near", we mean out to a distance of the order of L (there in no other length in the problem). Therefore, apart from the sign,

$$\Phi^{\text{out}}_{\text{disk}} \approx B_{\text{out}} L^2 = \Phi^{\text{in}}_{\text{disk}} = \mu_0 nIA.$$

This gives our estimate for the magnitude of the outside field as

$$B_{\text{out}} \approx \frac{\mu_0 nIA}{L^2}.$$

Source: J. Farley and R.H. Price, *American Journal of Physics* **69**, 751 (2001).

11.4 The Magnetic Moment of a Rotating Charged Disk

Orient the disk to lie in the x-y plane with its center at the origin and let $\boldsymbol{\rho}$ be the radial vector (in polar coordinates) in that plane. If each element of surface charge $dq = \sigma dS$ moves with velocity $\mathbf{v} = \boldsymbol{\omega} \times \boldsymbol{\rho}$, the magnetic dipole moment of the disk is

$$\mathbf{m} = \frac{1}{2}\int \boldsymbol{\rho} \times dq\mathbf{v} = \frac{1}{2}\sigma \int dS\boldsymbol{\rho} \times (\boldsymbol{\omega} \times \boldsymbol{\rho}). \tag{11.2}$$

(a) The rotation axis is normal to the plane. In other words, $\boldsymbol{\omega} = \omega\hat{\mathbf{z}}$ is perpendicular to $\boldsymbol{\rho}$. Therefore,

$$\mathbf{m} = \frac{1}{2}\sigma \int dS \left[\boldsymbol{\omega}\rho^2 - \boldsymbol{\rho}(\boldsymbol{\omega}\cdot\boldsymbol{\rho})\right] = \frac{1}{2}\sigma \int_0^{2\pi} d\phi \int_0^R d\rho\rho^3 \boldsymbol{\omega} = \frac{\pi}{4}\sigma R^4 \boldsymbol{\omega}.$$

(b) We can choose $\boldsymbol{\omega} = \omega\hat{\mathbf{x}}$ since the rotation axis lies along a diameter. Then, because $\boldsymbol{\rho} = \rho\cos\phi\hat{\mathbf{x}} + \rho\sin\phi\hat{\mathbf{y}}$, (11.2) becomes

$$\mathbf{m} = \frac{1}{2}\sigma\omega \int dS \,\left[\rho^2 \sin^2\phi\hat{\mathbf{x}} - \rho^2 \sin\phi\cos\phi\hat{\mathbf{y}}\right] = \frac{\pi}{8}\sigma R^4 \boldsymbol{\omega}.$$

11.5 The Field inside a Semi-Infinite Solenoid

The field inside an infinite solenoid is $\mathbf{B} = \mu_0 nI\hat{\mathbf{z}}$. If we apply this to the semi-infinite situation,

$$\mathbf{B}_{\rm in}(\rho, z) = \mu_0 nI\Theta(-z)\Theta(R-\rho)\hat{\mathbf{z}} = \mu_0 m\,\Theta(-z)\frac{\Theta(R-\rho)}{A}\hat{\mathbf{z}}.$$

If the delta function identity suggested in the hint is correct,

$$\lim_{A\to 0} \mathbf{B}_{\rm in} = \mu_0 \mathbf{m}\Theta(-z)\delta(x)\delta(y).$$

In this case,

$$\lim_{A\to 0} \nabla\cdot\mathbf{B}_{\rm in} = \frac{\partial}{\partial z}\mu_0 m\Theta(-z)\delta(x)\delta(y) = -\mu_0 m\delta(x)\delta(y)d\delta(z) = -\mu_0 m\delta(\mathbf{r}).$$

To prove the identity, we note first that $\lim\limits_{A\to 0}\Theta(R-\rho)/A = \infty$. Moreover,

$$\lim_{A\to 0}\int_0^{2\pi} d\phi \int_0^{\infty} d\rho\,\rho\frac{\Theta(R-\rho)}{A} = 1.$$

11.6 A Spinning Spherical Shell of Charge

(a) The surface current density is $\mathbf{K} = \sigma\mathbf{v}$ where $\sigma = Q/4\pi R^2$ is the surface charge density and $\mathbf{v} = \boldsymbol{\omega}\times\mathbf{r}$ is the velocity of a point on the surface at the point $\mathbf{r}$. The corresponding volume current density requires a delta function to define the surface:

$$\mathbf{j} = \frac{Q}{4\pi R^2}(\boldsymbol{\omega}\times\mathbf{r})\delta(r-R).$$

On the other hand,

$$\mathbf{j} = \nabla\times[\mathbf{M}\Theta(R-r)] = \mathbf{M}\times\nabla\Theta(r-R) = \mathbf{M}\times\hat{\mathbf{r}}\delta(r-R) = \mathbf{M}\times\frac{\mathbf{r}}{R}\delta(r-R).$$

Comparing these two formula shows that $\mathbf{M} = (Q/4\pi R)\boldsymbol{\omega}$.

(b) Using one of the expressions for $\mathbf{j}$ just above and the definition of magnetic moment,

$$\begin{aligned}\mathbf{m} &= \frac{1}{2}\int d^3r\,\mathbf{r}\times\mathbf{j} = \frac{1}{2}\int d^3r\,\mathbf{r}\times(\mathbf{M}\times\hat{\mathbf{r}})\delta(r-R)\\ &= \frac{1}{2}\int d^3r\,\left[\mathbf{M}(\mathbf{r}\cdot\hat{\mathbf{r}}) - \hat{\mathbf{r}}(\mathbf{M}\cdot\mathbf{r})\right]\delta(r-R).\end{aligned}$$

To do the second part of the last integral, choose $\mathbf{M} = M\hat{\mathbf{z}}$ and note that only the $\cos\theta\hat{\mathbf{z}}$ part of $\hat{\mathbf{r}}$ survives the integration over ϕ. Hence,

$$\begin{aligned}\mathbf{m} &= \frac{1}{2}\int d^3r\,\mathbf{M}r\delta(r-R) - \frac{1}{2}\hat{\mathbf{z}}M\int\limits_0^\infty drr^2\int_0^{2\pi}d\phi\int\limits_{-1}^{1}d(\cos\theta)\cos^2\theta Mr\delta(r-R)\\ &= \frac{1}{2}\left[4\pi R^3M - 2\pi MR^3\times\frac{2}{3}\right]\hat{\mathbf{z}}\\ &= \frac{4}{3}\pi R^3\mathbf{M} = \frac{1}{3}QR^2\boldsymbol{\omega}.\end{aligned}$$

(c) To get the vector potential, we integrate by parts:

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \mathbf{M}\times\frac{\mu_0}{4\pi}\int d^3r'\,\frac{\nabla'\Theta(r'-R)}{|\mathbf{r}-\mathbf{r}'|} = \mu_0\mathbf{M}\times\frac{1}{4\pi}\int\limits_{r'<R}d^3r'\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|}.$$

The integral is the electric field of a ball of radius R with uniform charge density $\rho(\mathbf{r}') = \epsilon_0$. This we compute straightforwardly using Gauss' law. Therefore,

$$\mathbf{A}(\mathbf{r}) = \begin{cases}\dfrac{\mu_0}{4\pi}\dfrac{\mathbf{m}\times\mathbf{r}}{r^3} & r > R,\\[2ex] \dfrac{\mu_0}{4\pi}\dfrac{\mathbf{m}\times\mathbf{r}}{R^3} & r < R.\end{cases}$$

(d) Outside the sphere, we get a pure dipole field with magnetic dipole moment $\mathbf{m}$. Inside the sphere, we get a uniform field:

$$\mathbf{B}(r < R) = \frac{\mu_0}{4\pi R^3}\nabla \times (\mathbf{m} \times \mathbf{r}) = \frac{\mu_0}{2\pi R^3}\mathbf{m} = \frac{\mu_0 Q}{6\pi R}\boldsymbol{\omega}.$$

11.7 Magnetic Moment of a Planar Spiral

One turn of the wire at radius r produces a magnetic moment $m(r) = I\pi r^2$ in the direction dictated by the right-hand rule. For the spiral, the number of turns in the interval between r and $r + dr$ is

$$dN = \frac{N}{b-a}dr.$$

Therefore, the magnetic moment of the complete spiral is

$$m = \int_a^b m(r)dN = \frac{\pi i N}{b-a}\int_a^b drr^2 = \frac{\pi I N}{b-a}\frac{b^3-a^3}{3}.$$

Source: I.E. Irodov, *Basic Laws of Electromagnetism* (Mir, Moscow, 1986).

11.8 A Hidden Delta Function

Let V be an infinitesimally small spherical volume centered on the origin. The problem will be solved if we can show that

$$\int_V d^3r\, \nabla\psi_0(\mathbf{r}) = \frac{1}{3}\mu_0\mathbf{m}.$$

Using a special case of the divergence theorem (Section 0.14) and the fact that $d\mathbf{S} = dS\hat{\mathbf{r}}$ for a spherical surface,

$$\int_V d^3r\, \nabla\psi_0 = \int_S d\mathbf{S}\,\psi = \frac{\mu_0}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\hat{\mathbf{r}}(\mathbf{m}\cdot\hat{\mathbf{r}}).$$

Now choose $\mathbf{m} = m\hat{\mathbf{z}}$ and recall that $\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$. These give

$$\begin{aligned}\int_V d^3r\, \nabla\psi_0 &= \frac{\mu}{4\pi}\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\, m\cos\theta\,\{\sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}\}\\ &= \frac{\mu}{4\pi}\mathbf{m}\int_0^{2\pi} d\phi \int_{-1}^{1} d(\cos\theta)\cos^2\theta\\ &= \frac{1}{3}\mu_0\mathbf{m}.\end{aligned}$$

11.9 Magnetic Dipole and Quadrupole Moments for $\psi(\mathbf{r})$

(a) We get a Cartesian expansion using

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|}=\frac{1}{r}-\mathbf{r}'\cdot\nabla\frac{1}{r}+\frac{1}{2}(\mathbf{r}'\cdot\nabla)^2\frac{1}{r}-\cdots.$$

Inserting this above gives

$$\begin{aligned}I &= \frac{\mu_0}{4\pi}\left[\frac{1}{r}\int d^3r'\,\mathbf{r}'\cdot\nabla'\times\mathbf{j}\,(\mathbf{r}')-\int d^3r'\,\mathbf{r}'\cdot\nabla'\times\mathbf{j}\,(\mathbf{r}')\mathbf{r}'\cdot\nabla\frac{1}{r}\right.\\ &\qquad\left.+\int d^3r'\,\mathbf{r}'\cdot\nabla'\times\mathbf{j}\,(\mathbf{r}')\frac{1}{2}(\mathbf{r}'\cdot\nabla)^2\frac{1}{r}\right].\end{aligned}$$

The first integral in the square brackets vanishes because the identity

$$\nabla\cdot(\mathbf{r}\times\mathbf{j})=\mathbf{j}\cdot(\nabla\times\mathbf{r})-\mathbf{r}\cdot\nabla\times\mathbf{j}=-\mathbf{r}\cdot\nabla\times\mathbf{j} \tag{11.3}$$

and Gauss' law produce a surface integral at infinity which is zero for a localized current distribution. The two terms which remain are exactly

$$I=-2\mathbf{m}\cdot\nabla\frac{1}{r}+m_{ij}^{(2)}\nabla_i\nabla_j\frac{1}{r}.$$

Now, $\nabla_i(1/r)=f(\theta,\phi)/r^2$ and $\nabla_i\nabla_j(1/r)=g(\theta,\phi)/r^3$. Therefore, since $I=-r\partial\psi/\partial r$, we get the advertised expansion for $\psi(\mathbf{r})$ immediately.

(b) Using (11.3) and integrating by parts,

$$\begin{aligned}\mathbf{m} &= \frac{1}{2}\int d^3r\,(\mathbf{r}\cdot\nabla\times\mathbf{j})\mathbf{r}\\ &= -\frac{1}{2}\int d^3r\,\nabla\cdot(\mathbf{r}\times\mathbf{j})\mathbf{r}\\ &= \frac{1}{2}\int d^3r\,(\mathbf{r}\times\mathbf{j})\cdot\nabla\mathbf{r}\\ &= \frac{1}{2}\int d^3r\,(\mathbf{r}\times\mathbf{j}).\end{aligned}$$

(c) Again using (11.3) and integrating by parts,

$$\begin{aligned}m_{ij}^{(2)} &= \frac{1}{2}\int d^3r\,(\mathbf{r}\cdot\nabla\times\mathbf{j})r_ir_j\\ &= -\frac{1}{2}\int d^3r\,\nabla\cdot(\mathbf{r}\times\mathbf{j})\,r_ir_j\\ &= \frac{1}{2}\int d^3r\,(\mathbf{r}\times\mathbf{j})\cdot\nabla(r_ir_j).\end{aligned} \tag{11.4}$$

(d) Because $\mathbf{r}\times\mathbf{j}$ is perpendicular to $\mathbf{r}$,

$$\mathrm{Tr}\;\mathbf{m}^{(2)}=\sum_{i=1}^{3}m_{ii}^{(2)}=\frac{1}{2}\int d^3r\,(\mathbf{r}\times\mathbf{j})\cdot\nabla(r^2)=\int d^3r\,(\mathbf{r}\times\mathbf{j})\cdot\mathbf{r}=0.$$

(e) Using the result of part (c),

$$\begin{aligned} m_{ij}^{(2)} &= \frac{1}{2}\int d^3r\,(\mathbf{r}\times\mathbf{j})_k\nabla_k(r_ir_j) \\ &= \frac{1}{2}\int d^3r\,(\mathbf{r}\times\mathbf{j})_k(\delta_{ki}r_j+r_i\delta_{kj}) \\ &= \frac{1}{2}\int d^3r\,[(\mathbf{r}\times\mathbf{j})_ir_j+r_i(\mathbf{r}\times\mathbf{j})_j]. \end{aligned}$$

(f) The key point is that $m_{ij}^{(2)}=\frac{1}{2}(M_{ij}^{(2)}+M_{ji}^{(2)})$. Therefore,

$$\begin{aligned} m_{ij}^{(2)}\nabla_i\nabla_j\frac{1}{r} &= \frac{1}{2}\left[M_{ij}^{(2)}\nabla_i\nabla_j\frac{1}{r}+M_{ji}^{(2)}\nabla_i\nabla_j\frac{1}{r}\right] \\ &= \frac{1}{2}\left[M_{ij}^{(2)}\nabla_i\nabla_j\frac{1}{r}+M_{ij}^{(2)}\nabla_j\nabla_i\frac{1}{r}\right]=M_{ij}^{(2)}\nabla_i\nabla_j\frac{1}{r}. \end{aligned}$$

Source: C.G. Gray, *American Journal of Physics* **46**, 582 (1978); *ibid.* **48**, 984 (1980).

11.10 Biot-Savart at the Origin

When the observation point $\mathbf{r}$ lies close to the center of a current-free, origin-centered sphere, we get an interior multiple expansion of the vector potential simply by exchanging $\mathbf{r}$ and $\mathbf{r}'$ in our basic exterior Cartesian expansion. The first two terms are

$$A_k(\mathbf{r})=\frac{\mu_0}{4\pi}\left[\int d^3r'\,\frac{j_k(\mathbf{r}')}{r'}+\mathbf{r}\cdot\int d^3r'\,\frac{j_k(\mathbf{r}')\mathbf{r}'}{r'^3}+\cdots\right].$$

The first term is a constant which does not contribute to the magnetic field $\mathbf{B}=\nabla\times\mathbf{A}$. We rewrite the second term using

$$(\mathbf{r}'\times\mathbf{j})\times\mathbf{r}=(\mathbf{r}\cdot\mathbf{r}')\mathbf{j}-(\mathbf{r}\cdot\mathbf{j})\mathbf{r}'.$$

This gives

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\int d^3r'\,\frac{[\mathbf{r}'\times\mathbf{j}(\mathbf{r}')]}{r'^3}\times\mathbf{r}+\frac{\mu_0}{4\pi}\mathbf{r}\cdot\int d^3r'\,\mathbf{j}(\mathbf{r}')\frac{\mathbf{r}'}{r'^3}.$$

We rewrite the second term using an identity proved in the text, namely,

$$\int d^3r'\,j_kr'_\ell=-\frac{1}{2}\epsilon_{k\ell i}\int d^3r'\,(\mathbf{r}'\times\mathbf{j})_i.$$

The result is

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\left[\int d^3r'\,\frac{[\mathbf{r}'\times\mathbf{j}(\mathbf{r}')]}{r'^3}\times\mathbf{r}-\frac{1}{2}\int d^3r'\,\frac{[\mathbf{r}'\times\mathbf{j}(\mathbf{r}')]}{r'^3}\times\mathbf{r}\right].$$

This has the desired form,

$$\mathbf{A}(\mathbf{r})=\frac{\mu_0}{4\pi}\mathbf{G}\times\mathbf{r},$$

if

$$\mathbf{G} = \frac{1}{2}\int d^3r' \, \frac{[\mathbf{r}' \times \mathbf{j}(\mathbf{r}')]}{r'^3}.$$

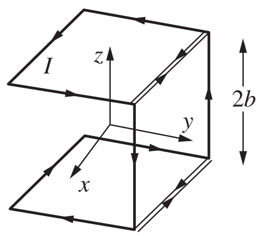
The corresponding magnetic field is the constant vector

$$\mathbf{B} = \frac{\mu_0}{2\pi}\mathbf{G} = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{r}' \times \mathbf{j}(\mathbf{r}')}{r'^3}.$$

This is exactly the usual Biot-Savart formula evaluated at the origin.

11.11 Purcell's Loop

(a) The figure below shows that Purcell's loop is equivalent to the sum of three square loops of current. This is so because no magnetic field is produced by the two pairs of oppositely directed and spatially coincident lines of current. Each square loop carries a magnetic dipole moment with magnitude $I(2b)^2$. The moments contributed by the top and bottom loops are oppositely directed and thus cancel. Therefore, using the right-hand rule, the magnetic dipole moment of Purcell's loop is $\mathbf{m} = 4Ib^2\hat{\mathbf{y}}$.



(b) By analogy with the electric case, the top and bottom square loops cancel for the dipole moment but add for the quadrupole moment. Thus, we expect a non-negligible magnetic quadrupole moment for the Purcell loop.

Source: E.M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1985).

11.12 Dipole Field from Monopole Field

In the text, we computed the magnetic field of a dipole from its vector potential as

$$\mathbf{B}(\mathbf{r}) = \nabla \times \left\{\frac{\mu_0}{4\pi}\frac{\mathbf{m} \times \mathbf{r}}{r^3}\right\} = \frac{\mu_0}{4\pi}\left[\mathbf{m}\left(\nabla \cdot \frac{\mathbf{r}}{r^3}\right) - (\mathbf{m} \cdot \nabla)\frac{\mathbf{r}}{r^3}\right].$$

The first term in square brackets is proportional to $\delta(\mathbf{r})$ and thus does not contribute away from the origin. The term that remains is exactly the desired result:

$$\mathbf{B} = -(\mathbf{m} \cdot \nabla)\frac{\mathbf{B}_{\rm mono}}{g} = \frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{m} \cdot \mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right].$$

11.13 The Spherical Magnetic Dipole Moment

We need to compute

$$\mathbf{m} = \frac{3}{2\mu_0} \int\limits_{\text{sphere}} d^3 r\, \nabla \times \mathbf{A} = \frac{3}{2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{1}{i\ell} M_{\ell m} \int\limits_{\text{sphere}} d^3 r\, \nabla \times \mathbf{L} \frac{Y_{\ell m}(\Omega)}{r^{\ell+1}}.$$

This is not difficult if we exploit an identity quoted in Section 11.4.4, namely,

$$(i\ell \nabla + \nabla \times \mathbf{L}) \frac{Y_{\ell m}(\Omega)}{r^{\ell+1}} = 0.$$

Therefore,

$$\mathbf{m} = -\frac{3}{2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} M_{\ell m} \int\limits_{\text{sphere}} d^3 r\, \nabla \frac{Y_{\ell m}(\Omega)}{r^{\ell+1}}.$$

We convert the volume integral to a surface integral using a corollary of the divergence theorem to get

$$\mathbf{m} = -\frac{3}{2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} M_{\ell m} \int\limits_{\text{sphere}} d\mathbf{S}\, \frac{Y_{\ell m}(\Omega)}{r^{\ell+1}}.$$

If we let the integration sphere have radius R,

$$\mathbf{m} = -\frac{3}{2} \sum_{\ell=1}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} M_{\ell m} R^{1-\ell} \int d\Omega\, \hat{\mathbf{r}}\, Y_{\ell m}(\Omega).$$

Now,

$$\hat{\mathbf{r}} = \hat{\mathbf{x}} \sin\theta \cos\phi + \hat{\mathbf{y}} \sin\theta \sin\phi + \hat{\mathbf{z}} \cos\theta.$$

11.14 No Magnetic Dipole Moment

Method I: Use the identity proved in Example 11.1:

$$\mathbf{m} = \frac{3}{2\mu_0} \int\limits_{\text{sphere}} d^3 r\, \mathbf{B}(\mathbf{r}).$$

Then, using a corollary of the divergence theorem,

$$\begin{aligned}
\mathbf{m} &= \frac{3}{2\mu_0} \int\limits_{\text{sphere}} d^3 r\, \nabla \times \mathbf{A} \\
&= \frac{3}{2\mu_0} \int\limits_{\text{sphere}} d\mathbf{S} \times \mathbf{A} \\
&= \frac{3}{2\mu_0} \int\limits_{\text{sphere}} dS\, \hat{\mathbf{r}} \times (f\mathbf{r}) \\
&= 0.
\end{aligned}$$

Method II: From $\nabla\times\mathbf{B}=\mu_0\mathbf{j}$ and the definition of the magnetic moment,

$$\mathbf{m}=\frac{1}{2}\int d^3r\,\mathbf{r}\times\mathbf{j}=\frac{1}{2\mu_0}\int d^3r\,\mathbf{r}\times[\nabla\times(\nabla\times\mathbf{A})].$$

Also,

$$\nabla\times\mathbf{A}=\nabla\times(f\mathbf{r})=f(\nabla\times\mathbf{r})-\mathbf{r}\times\nabla f=-\mathbf{r}\times\nabla f,$$

so

$$\nabla\times(\nabla\times\mathbf{A})=-\nabla\times(\mathbf{r}\times\nabla f).$$

But Section 11.6.3 of the text defines $\mathbf{L}=-i\mathbf{r}\times\nabla$ and quotes the identity

$$\nabla\times\mathbf{L}=-i\mathbf{r}\nabla^2+i\nabla(1+\mathbf{r}\cdot\nabla).$$

Therefore,

$$-\nabla\times(\mathbf{r}\times\nabla f)=-\mathbf{r}\nabla^2 f+\nabla f+\nabla\frac{\partial f}{\partial r},$$

and we see that

$$\mathbf{m}=\frac{1}{2\mu_0}\int d^3r\,\mathbf{r}\times[\nabla\times(\nabla\times\mathbf{A})]=\frac{1}{2\mu_0}\int d^3r\,\mathbf{r}\times\left\{-\mathbf{r}\nabla^2 f+\nabla f+\nabla\frac{\partial f}{\partial r}\right\}.$$

The first term in the curly brackets does not contribute because $\mathbf{r}\times\mathbf{r}=0$. The other two pieces of the integrand are

$$\mathbf{r}\times\nabla f=f(\nabla\times\mathbf{r})-\nabla\times(\mathbf{r}f)=-\nabla\times(\mathbf{r}f)$$

and a similar term with f replaced by $\partial f/\partial r$. However, using a corollary of the divergence theorem, the integral over all space can be done using a sphere. Therefore, because $d\mathbf{S}=dS\hat{\mathbf{r}}$,

$$\int d^3r\,\nabla\times(\mathbf{r}f)=\int d\mathbf{S}\times(\mathbf{r}f)=0$$

and similarly for the $\partial f/\partial r$ term. Thus all three contributions to the curly brackets produce zero and there is no magnetic dipole moment.

11.15 A Spherical Superconductor

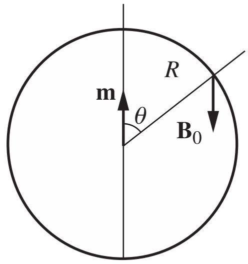
(a) The net magnetic field is

$$\mathbf{B}=\frac{\mu_0}{4\pi}\left(\frac{3\,(\mathbf{m}\cdot\mathbf{r})}{r^5}\mathbf{r}-\frac{\mathbf{m}}{r^3}\right)+\mathbf{B}_0.$$

The matching condition is that the normal component of $\mathbf{B}$ is continuous. Since $\mathbf{B}=0$ inside the superconductor, the requirement is

$$\hat{\mathbf{r}}\cdot\mathbf{B}\big|_{r=R}=0. \tag{11.5}$$

A moment's reflection shows that this is possible only if $\mathbf{m}=m\hat{\mathbf{z}}$ is *anti-parallel* to $\mathbf{B}$ as shown.



Accordingly, (1) reads

$$\frac{\mu_0}{4\pi}\left[\frac{3m\cos\theta}{R^3}-\frac{m\cos\theta}{R^3}\right]-B_0\cos\theta=0,$$

so

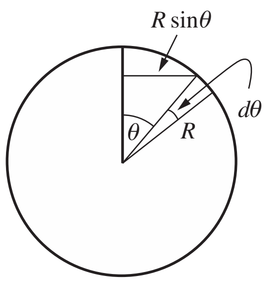
$$\mathbf{m}=-\frac{2\pi}{\mu_0}R^3\mathbf{B}_0.$$

(b) The other matching condition we need is $\hat{\mathbf{r}}\times[\mathbf{B}_{\text{out}}-\mathbf{B}_{\text{in}}]=\mu_0\mathbf{K}$. Hence,

$$\mu_0\mathbf{K}=-\hat{\mathbf{r}}\times\mathbf{B}_{\text{in}}=\left[\frac{\mu_0}{4\pi}\frac{m}{R^3}+B_0\right](\hat{\mathbf{r}}\times\hat{\mathbf{z}})=-\left[\frac{\mu_0}{4\pi}\frac{m}{R^3}+B_0\right]\sin\theta\hat{\boldsymbol{\phi}}=-\frac{3}{2}B_0\sin\theta\hat{\boldsymbol{\phi}}.$$

(c) We compute the magnetic moment by summing the magnetic moments from a collection of current rings. The current in the ring at angle θ is

$$dI=Rd\theta K(\theta)=\frac{3R}{2\mu_0}B_0\sin\theta d\theta.$$



The radius of such a ring is $R\sin\theta$ so its magnetic dipole moment is

$$dm=area\times dI=\pi(R\sin\theta)^2\frac{3R}{2\mu_0}B_0\sin\theta d\theta=\frac{3\pi R^3}{2\mu_0}B_0\sin^3\theta d\theta.$$

Therefore, in agreement with part (a), the net dipole moment of the sphere is

$$\begin{aligned} m &= \int dm=\frac{3\pi R^3}{2\mu_0}B_0\int_0^\pi d\theta\,\sin^3\theta=\frac{3\pi R^3}{2\mu_0}B_0\int_{-1}^{1}\left(1-t^2\right)dt \\ &= \frac{3\pi R^3}{2\mu_0}B_0\frac{4}{3}=\frac{2\pi R^3}{\mu_0}B_0. \end{aligned}$$

11.16 Azimuthal Moments for Concentric Current Rings

Consider an origin-centered ring parallel to the x-y plane with radius R and current I flowing counter-clockwise as viewed from the positive z-axis. The text computed the azimuthal multipoles for this system when the plane of the ring is $z = a$ and $\tan\theta = R/a$:

$$M_\ell^+ = \frac{2\pi IR^2}{\ell+1}\left(\frac{R}{\sin\theta}\right)^{\ell-1} P_\ell'(\cos\theta).$$

The Helmholtz coil has $a = R/2$ and a second identical current-carrying ring located at $z = -R/2$. The azimuthal moments associated with the second ring are

$$M_\ell^- = \frac{2\pi IR^2}{\ell+1}\left(\frac{R}{\sin(\pi-\theta)}\right)^{\ell-1} P_\ell'(\cos(\pi-\theta)) = (-1)^{\ell+1} M_\ell^+.$$

The second equality above is true because $\sin(\pi-\theta) = \sin\theta$ and $P_\ell(-x) = (-1)^\ell P_\ell(x)$. The derivative of the Legendre function introduces one more minus sign. This shows that the total moment $M_\ell = M_\ell^+ + M_\ell^-$ is *zero* when ℓ is even and

$$M_\ell = \frac{4\pi IR^2}{\ell+1}\left(\frac{R}{\sin\theta}\right)^{\ell-1} P_\ell'(\cos\theta)$$

when ℓ is odd.

For our Helmholtz geometry, $\cos\theta = 1/\sqrt{5}$ and $\sin\theta = 2/\sqrt{5}$, so

$$M_\ell = \frac{4\pi IR^{\ell+1}}{\ell+1}\left(\frac{\sqrt{5}}{2}\right)^{\ell-1} P_\ell'(1/\sqrt{5}), \qquad \ell \text{ odd}.$$

Moreover,

$$P_1(x) = x$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x)$$

$$P_5(x) = \tfrac{1}{8}(63x^5 - 70x^3 + 15x)$$

$$P_7(x) = \tfrac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x),$$

which implies that

$$P_1'(x) = 1$$

$$P_3'(x) = \tfrac{1}{2}(15x^2 - 3)$$

$$P_5'(x) = \tfrac{1}{8}(315x^4 - 210x^2 + 15)$$

$$P_7'(x) = \tfrac{1}{16}(3003x^6 - 3465x^4 + 945x^2 - 35).$$

Therefore, there are only three non-zero moments between $\ell = 1$ and $\ell = 8$:

$$M_1 = 2\pi R^2 I \qquad M_5 = -\frac{15}{8}\pi R^6 I \qquad M_7 = \frac{77}{32}\pi R^8 I.$$

Source: D.G. Smith, *American Journal of Physics* **48**, 739 (1980).

11.17 Dipole Field from Biot-Savart

The Biot-Savart law is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{j}(\mathbf{r}') \times (\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \, \mathbf{j}(\mathbf{r}') \times \nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|}.$$

Into this, we insert the $r \gg r'$ expansion,

$$\nabla' \frac{1}{|\mathbf{r}-\mathbf{r}'|} = \nabla' \left[\frac{1}{r} + \mathbf{r}' \cdot \nabla \frac{1}{r} + \frac{1}{2}(\mathbf{r}' \cdot \nabla)^2 \frac{1}{r} + \cdots \right].$$

The first term is clearly zero. The second term is zero also because

$$\nabla'(\mathbf{r}' \cdot \nabla)\frac{1}{r} = \frac{\mathbf{r}}{r^3}.$$

Therefore, dropping all higher-order terms,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{8\pi}\int d^3r' \, \mathbf{j}(\mathbf{r}') \times \nabla'(\mathbf{r}' \cdot \nabla)^2 \frac{1}{r},$$

or

$$B_i(\mathbf{r}) = \frac{\mu_0}{8\pi}\int d^3r' \, \epsilon_{i\ell m} j_\ell \partial'_m (r'_p r'_s) \partial_p \partial_p \frac{1}{r}.$$

However,

$$\epsilon_{i\ell m} j_\ell \partial'_m (r'_p r'_s) \partial_p \partial_p \frac{1}{r} = 2\epsilon_{i\ell p} j_\ell r'_s \partial_p \partial_s \frac{1}{r},$$

and we proved in the text that

$$\int d^3r' \, j_\ell(\mathbf{r}') r'_s = \epsilon_{\ell k s} m_k,$$

where m_k is the kth Cartesian component of the magnetic dipole moment vector $\mathbf{m}$. Therefore,

$$B_i = \frac{\mu_0}{4\pi}\epsilon_{\ell p i}\epsilon_{\ell k s} m_k \partial_p \partial_s \frac{1}{r} = \frac{\mu_0}{4\pi}\left[m_k \partial_k \partial_i \frac{1}{r} - m_i \partial_p \partial_p \frac{1}{r} \right].$$

The last term is proportional to a delta function at the origin, which we drop. Consequently,

$$B_i = \frac{\mu_0}{4\pi} m_k \partial_k \left(-\frac{r_i}{r^3} \right) = \frac{\mu_0}{4\pi} m_k \left(\frac{3 r_k r_i}{r^5} - \frac{\delta_{ik}}{r^3} \right),$$

which is indeed the dipole field,

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right].$$

11.18 Octupoles from Dipoles

(a) Ignoring the pre-factor $\mu_0/4\pi$, the magnetic scalar potential for this system is

$$\psi(\mathbf{r}) = \sum_\alpha \frac{\mathbf{m}_\alpha\cdot(\mathbf{r}-\mathbf{r}_\alpha)}{|\mathbf{r}-\mathbf{r}_\alpha|^3}.$$

At large distance,

$$\psi(\mathbf{r}) \approx \sum_\alpha \frac{\mathbf{m}_\alpha\cdot(\mathbf{r}-\mathbf{r}_\alpha)}{r^3}\left(1+3\frac{\mathbf{r}\cdot\mathbf{r}_\alpha}{r^2}\right),$$

or

$$\psi(\mathbf{r}) \approx \frac{\hat{\mathbf{r}}}{r^2}\cdot\sum_\alpha \mathbf{m}_\alpha + \frac{1}{r^3}\sum_\alpha\left[3(\hat{\mathbf{r}}\cdot\mathbf{r}_\alpha)(\mathbf{m}_\alpha\cdot\hat{\mathbf{r}}) - \mathbf{m}_\alpha\cdot\mathbf{r}_\alpha\right] + O(1/r^4).$$

The $1/r^4$ term is the octupole, so we must eliminate the dipole and quadrupole terms. Therefore, the conditions needed are

$$\sum_\alpha \mathbf{m}_\alpha = 0 \qquad \text{and} \qquad \sum_\alpha\left[3(\hat{\mathbf{r}}\cdot\mathbf{r}_\alpha)(\mathbf{m}_\alpha\cdot\hat{\mathbf{r}}) - \mathbf{m}_\alpha\cdot\mathbf{r}_\alpha\right] = 0. \tag{1}$$

The condition on the left is straightforward: the total dipole moment must be zero. Writing out the scalar products, the condition on the right becomes

$$\sum_\alpha \hat{r}_i\left[3r_{\alpha i}m_{\alpha j} - \mathbf{m}_\alpha\cdot\mathbf{r}_\alpha\delta_{ij}\right]\hat{r}_j = 0.$$

This a quadratic form. Using the hint, the conditions are that $A_{ij}+A_{ji}=0$, where

$$A_{ij} = \sum_\alpha\left[3r_{\alpha i}m_{\alpha j} - \mathbf{m}_\alpha\cdot\mathbf{r}_\alpha\delta_{ij}\right].$$

We begin with $A_{xx}=A_{yy}=A_{zz}=0$. These constraints give

$$\sum_\alpha\left[2m_{\alpha x}x_\alpha - m_{\alpha y}y_\alpha - m_{\alpha z}z_\alpha\right] = 0$$

$$\sum_\alpha\left[2m_{\alpha y}y_\alpha - m_{\alpha z}z_\alpha - m_{\alpha x}x_\alpha\right] = 0$$

$$\sum_\alpha\left[2m_{\alpha z}z_\alpha - m_{\alpha x}x_\alpha - m_{\alpha y}y_\alpha\right] = 0.$$

In other words,

$$\sum_\alpha m_{\alpha x} x_\alpha = \sum_\alpha m_{\alpha y} y_\alpha = \sum_\alpha m_{\alpha z} z_\alpha. \tag{2}$$

We also must impose $A_{xy} = -A_{yx}$, $A_{xz} = -A_{zx}$, and $A_{yz} = -A_{zy}$. These give, respectively,

$$\begin{aligned} \sum_\alpha (m_{\alpha x} y_\alpha + m_{\alpha y} x_\alpha) &= 0 \\ \sum_\alpha (m_{\alpha x} z_\alpha + m_{\alpha z} x_\alpha) &= 0 \\ \sum_\alpha (m_{\alpha y} z_\alpha + m_{\alpha z} y_\alpha) &= 0. \end{aligned} \tag{3}$$

(b) There is no loss of generality if we put $\mathbf{m}_3$ at the origin of coordinates and point $\mathbf{m}_1$ in the x-direction. Similarly, we are free to point $\mathbf{m}_2$ in the x-y plane. Therefore, using the left side of (1), $\mathbf{m}_3$ points in the x-y plane also:

$$\begin{aligned} \mathbf{m}_1 &= (m_{1x}, 0, 0) \\ \mathbf{m}_2 &= (m_{2x}, m_{2y}, 0) \\ \mathbf{m}_3 &= (-m_{1x} - m_{2x}, -m_{2y}, 0) \\ \\ \mathbf{r}_1 &= (x_1, y_1, z_1) \\ \mathbf{r}_2 &= (x_2, y_2, z_2) \\ \mathbf{r}_3 &= (0, 0, 0). \end{aligned} \tag{4}$$

At this point, we have nine parameters and five conditions represented by (2) and (3) constrained by (4), namely,

$$\begin{aligned} m_{2y} y_2 &= 0 \\ m_{1x} x_1 + m_{2x} x_2 &= 0 \\ m_{1x} y_1 + m_{2x} y_2 + m_{2y} x_2 &= 0 \\ m_{1x} z_1 + m_{2x} z_2 &= 0 \\ m_{2y} z_2 &= 0. \end{aligned} \tag{5}$$

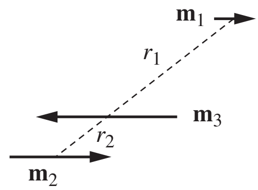
We can satisfy the first equation in (5) using $m_{2y} = 0$ or $y_2 = 0$. Suppose we choose $m_{2y} = 0$. In that case,

$$\mathbf{m}_1 = m_{1x}\hat{\mathbf{x}}, \ \mathbf{m}_2 = m_{2x}\hat{\mathbf{x}}, \ \mathbf{m}_3 = -(\mathbf{m}_1 + \mathbf{m}_2),$$

and

$$m_{1x}\mathbf{r}_1 + m_{2x}\mathbf{r}_2 = 0.$$

In other words, all three moments lie on a common line and all three moments are parallel or anti-parallel to one another. The ratio $m_1/m_2 = r_2/r_1$. An example is shown below.



(c) The second choice is $y_2 = 0$. In that case,

$$z_1 = z_2 = 0 \qquad \text{and} \qquad m_{1x}\mathbf{r}_1 + \mathbf{m}_2 x_2 = 0.$$

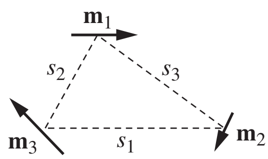
In other words, the moments are located at the corners of a triangle in the x-y plane with $\mathbf{m}_2$ anti-parallel to $\mathbf{r}_1$. Moreover, because $\mathbf{r}_2$ is opposite $\mathbf{m}_1$ and $\mathbf{r}_1$ is opposite $\mathbf{m}_2$,

$$m_1 : m_2 : m_3 = r_2 : r_1 : |\mathbf{r}_1 + \mathbf{r}_2|,$$

or

$$m_1 : m_1 : m_2 = s_1 : s_2 : s_3.$$

The choice $z_2 = 0$ in the last equation of (5) reproduces this solution. An example is shown below.



Source: H.J. Butterweck, *Archiv für Electrotechnik* **74**, 203 (1991).

11.19 Magnetic Multipoles from Electric Multipoles

Let z be the symmetry axis of the charge distribution $\rho(r,\theta)$. The exterior electrostatic potential produced by such a distribution is

$$\varphi(r,\theta) = \frac{1}{4\pi\epsilon_0}\sum_{L=0}^{\infty}\frac{A_L}{r^{L+1}}P_L(\cos\theta),$$

where

$$A_L = \int d^3r\, r^L P_L(\cos\theta)\rho(r,\theta).$$

Therefore, a distribution which produces a pure multipole field of order ℓ must have the form

$$\rho(r,\theta) = f(r)P_\ell(\cos\theta).$$

The current density produced when this distribution is rotated rigidly about $\hat{\boldsymbol{\omega}} = \omega\hat{\mathbf{z}}$ is

$$\mathbf{j}(r,\theta) = \hat{\boldsymbol{\omega}} \times \mathbf{r}\rho(r,\theta) = \omega r(\hat{\mathbf{z}} \times \hat{\mathbf{r}})\rho(r,\theta) = \omega r f(r)\sin\theta P_\ell(\cos\theta)\hat{\boldsymbol{\phi}}.$$

Now, the text showed that the exterior magnetic scalar potential produced by an azimuthally symmetric current density $\mathbf{j}(r,\theta)$ is

$$\psi(r,\theta) = \frac{\mu_0}{4\pi}\sum_{\ell=1}^{\infty} M_\ell \frac{P_\ell(\cos\theta)}{r^{\ell+1}},$$

where

$$M_\ell = \frac{i}{\ell+1}\int d^3r\, r^\ell P_\ell(\cos\theta)\mathbf{L}\cdot\mathbf{j}.$$

Because $\mathbf{L} = -i\mathbf{r}\times\nabla$ is Hermitian, we can also write

$$M_\ell = -\frac{i}{\ell+1}\int d^3r\, r^\ell\, [\mathbf{L}P_\ell(\cos\theta)]\cdot\mathbf{j}.$$

Finally, $\mathbf{L}P_\ell \propto \mathbf{L}Y_{\ell 0} \propto \mathbf{X}_{\ell 0}$ and the vector spherical harmonics $\mathbf{X}_{\ell m}$ are orthogonal to a solid angle integration. Therefore, it is sufficient to show that our $\mathbf{j}$ is proportional to a linear combination of $\mathbf{L}P_{\ell+1}$ and $\mathbf{L}P_{\ell-1}$.

Using the representation of the gradient in spherical coordinates, we find that

$$\mathbf{L}P_{\ell\pm1}(\theta) = -i\mathbf{r}\times\nabla P_{\ell\pm1}(\cos\theta) = -i\frac{\partial P_{\ell\pm1}(\cos\theta)}{\partial\theta}\hat{\boldsymbol{\phi}} = -i\sin\theta P'_{\ell\pm1}(\cos\theta)\hat{\boldsymbol{\phi}}.$$

This establishes the desired result because the Legendre polynomials satisfy the recurrence relation

$$P'_{\ell+1}(x) - P'_{\ell-1}(x) = (2\ell+1)P_\ell(x).$$

Source: S. Datta, *American Journal of Physics* **60**, 47 (1992).

11.20 A Seven-Wire Circuit

The two square loops are identical except that they carry current I in opposite directions. The loop centered at $x = a/2$ produces a magnetic moment $\mathbf{m}_+ = Ia^2\hat{\mathbf{z}}$. The loop centered at $x = -a/s$ produces a magnetic moment $\mathbf{m}_- = -Ia^2\hat{\mathbf{z}}$. These cancel in the far field and we expect a magnetic quadrupole field. We get the asymptotic vector potential by superposing the vector potentials from these two slightly displaced magnetic dipoles:

$$
\begin{aligned}
\mathbf{A}(x,y,z) &= \frac{\mu_0}{4\pi}\left[\frac{\mathbf{m}_+ \times (\mathbf{r}-(a/2)\hat{\mathbf{x}})}{|\mathbf{r}-(a/2)\hat{\mathbf{x}}|^3} - \frac{\mathbf{m}_- \times (\mathbf{r}+(a/2)\hat{\mathbf{x}})}{|\mathbf{r}+(a/2)\hat{\mathbf{x}}|^3}\right] \\
&= \frac{\mu_0}{4\pi} I a^2 \left[\frac{(x-a/2)\hat{\mathbf{y}} - y\hat{\mathbf{x}}}{[(x-a/2)^2+y^2+z^2]^{3/2}} - \frac{(x+a/2)\hat{\mathbf{y}} - y\hat{\mathbf{x}}}{[(x+a/2)^2+y^2+z^2]^{3/2}}\right] \\
&= \frac{\mu_0}{4\pi} I a^2 \left[\frac{(x-a/2)\hat{\mathbf{y}} - y\mathbf{x}}{[r^2 - xa + (a/2)^2]^{3/2}} - \frac{(x+a/2)\hat{\mathbf{y}} - y\mathbf{x}}{[r^2 + xa + (a/2)^2]^{3/2}}\right] \\
&= \frac{\mu_0 I a^2}{4\pi r^3}\left\{[(x-a/2)\hat{\mathbf{y}} - y\hat{\mathbf{x}}]\left[1 + \frac{3}{2}\frac{xa}{r^2} + \cdots\right]\right. \\
&\qquad \left. - \; [(x+a/2)\hat{\mathbf{y}} - y\hat{\mathbf{x}}]\left[1 - \frac{3}{2}\frac{xa}{r^2} + \cdots\right]\right\} \\
&= \frac{\mu_0 I a^2}{4\pi r^3}\left[-a\hat{\mathbf{y}} + 3\frac{xa}{r^2}(x\hat{\mathbf{y}} - y\mathbf{x})\right] + O\left(\frac{1}{r^4}\right) \\
&= \frac{\mu_0 I}{4\pi}\frac{a^3}{r^3}\left[\left(\frac{3x^2}{r^2} - 1\right)\hat{\mathbf{y}} - \frac{3xy}{r^2}\hat{\mathbf{y}}\right].
\end{aligned}
$$

Chapter 12: Magnetic Force and Energy

12.1 Bleakney's Theorem

The equation of motion for m is

$$m\frac{d^2\mathbf{r}}{dt^2} = q(\mathbf{E} + \frac{d\mathbf{r}}{dt} \times \mathbf{B}). \tag{1}$$

The equation of motion for M in a field $k\mathbf{B}(\mathbf{r})$ is

$$M\frac{d^2\mathbf{r}}{dt^2} = q(\mathbf{E} + \frac{d\mathbf{r}}{dt} \times k\mathbf{B}).$$

To avoid changing $\mathbf{E}$, we need only define a new time variable τ using

$$\frac{M}{t^2} = \frac{m}{\tau^2}.$$

This transforms the M equation of motion into

$$m\frac{d^2\mathbf{r}}{d\tau^2} = q(\mathbf{E} + \sqrt{\frac{m}{M}}\frac{d\mathbf{r}}{d\tau} \times k\mathbf{B}).$$

The choice $k = \sqrt{M/m}$ reduces this to

$$m\frac{d^2\mathbf{r}}{d\tau^2} = q(\mathbf{E} + \frac{d\mathbf{r}}{d\tau} \times \mathbf{B}).$$

This equation predicts the same trajectory as (1). They differ only by the speed at which the particle traverses the trajectory.

Source: W. Bleakney, *American Journal of Physics* **4**, 12 (1936).

12.2 A Hall Thruster

(a) We get uniform electron drift in the z-direction if the electric force density $-en_e\mathbf{E}$ exactly cancels the Lorentz magnetic force density $-en_e\mathbf{v}\times\mathbf{B}$. Imposing this condition, $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$, implies that

$$\mathbf{E} \times \mathbf{B} = \mathbf{B} \times (\mathbf{v} \times \mathbf{B}) = \mathbf{v}B^2 - \mathbf{B}(\mathbf{B} \cdot \mathbf{v}).$$

This, in turn, implies the suggested result,

$$\mathbf{v} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}.$$

(b) Using the equality of the forces in part (a) and $n_i = n_e$, the electric force on the ions is

$$\mathbf{F}_i = en_i\mathbf{E} = en_e\mathbf{E} = -en_e\mathbf{v} \times \mathbf{B} = \mathbf{j}_{\text{Hall}} \times \mathbf{B}.$$

By Newton's third law, the reaction thrust on the shells is $\mathbf{T} = -\mathbf{F}_i = \mathbf{B} \times \mathbf{j}_{\text{Hall}}$ *if* the ions are ejected from V before the magnetic Lorentz force on the ions begins to act. This will be the case because a xenon ion is much more massive than an electron.

Source: D.M. Goebel and I. Katz, *Fundamentals of Electric Propulsion* (Wiley, Hoboken, NJ, 2008).

12.3 Charged Particle Motion near a Straight, Current-Carrying Wire

(a) The magnetic field points in the z-direction at the initial position of the particle. Since the initial velocity is in the y-direction, the initial force is in the x-direction. Subsequent forces are confined to the x-y plane, so the particle trajectory is in this plane also.

(b) The relevant magnetic field is

$$\mathbf{B} = -\frac{\mu_0 I}{2\pi x}\hat{\mathbf{z}}.$$

Therefore, with $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ and $\beta = \mu_0 Iq/2\pi m$, Newton's equation of motion is

$$\mathbf{F} = m\beta\frac{v_x}{x}\hat{\mathbf{y}} - m\beta\frac{v_y}{x}\hat{\mathbf{x}} = m\frac{dv_x}{dt}\hat{\mathbf{x}} + m\frac{dv_y}{dt}\hat{\mathbf{y}}$$

or

$$\frac{dv_y}{dt} = \beta\frac{v_x}{x} \qquad \text{and} \qquad \frac{dv_x}{dt} = -\beta\frac{v_y}{x}. \tag{1}$$

The leftmost of these integrates immediately to

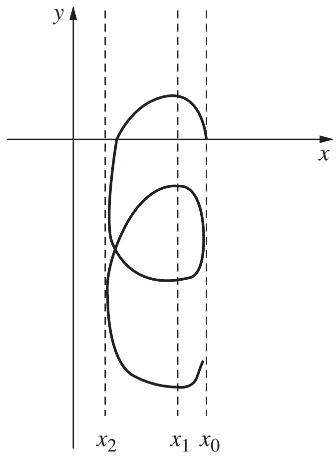
$$v_y - v_0 = \beta \ln(x/x_0). \tag{2}$$

The speed of the particle is v_0 and the magnetic field cannot change this because it does no work. Therefore, we can use (2) to make a table:

$$\begin{aligned} &v_y = v_0 && \text{when} \quad x = x_0 \\ &v_y = 0 && \text{when} \quad x = x_1 = x_0 \exp(-v_0/\beta) \\ &v_y = -v_0 && \text{when} \quad x = x_2 = x_0 \exp(-2v_0/\beta). \end{aligned}$$

This shows that the particle never leaves the proposed interval.

(c) To conserve the speed, we must have $v_x = 0$ at $x = x_2$ as well as at $x - x_0$. Using this information, and the fact that $v_x \propto \dot{v}_y$ and $v_y \propto -\dot{v}_x$, we can sketch the trajectory:



(d) Combining the right side of (1) with $dv_x/dt = (dv_x/dx)v_x$ gives

$$v_x dv_x = -\frac{\beta}{x}\left[v_0 + \beta \ln(x/x_0)\right] dx.$$

This integrates to

$$v_x^2 = -\beta \ln(x/x_0)\left[2v_0 + \beta \ln(x/x_0)\right].$$

Therefore,

$$\frac{dy}{dx} = \frac{v_y}{v_x} = \frac{v_0 + \beta \ln(x/x_0)}{\sqrt{-\beta \ln(x/x_0)[2v_0 + \beta \ln(x/x_0)]}}.$$

Source: J. Neuberger and J. Gruenbaum, *European Journal of Physics* **3**, 22 (1982).

12.4 Anti-Parallel Currents Do Not Always Repel

The fact that R is very large means that the entire voltage drop occurs over the resistor. Therefore, the wires are equipotentials with net charges $\pm Q$ to maintain the potential difference V between them. In other words, they form a two-wire capacitor with capacitance $C = Q/V$. The potential produced by one wire with charge per unit length λ is $\varphi(\rho) = -(\lambda/2\pi\epsilon_0)\ln\rho$. Therefore,

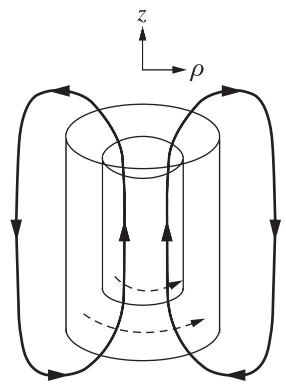
$$C = \frac{\lambda L}{2|\Delta\varphi|} = \frac{\lambda L}{(\lambda/\pi\epsilon_0)[\ln(d-a) - \ln a]} \approx \frac{\pi\epsilon_0 L}{\ln(d/a)}.$$

Because $V = IR$, the net force between the wires is

$$F = F_{\rm mag} - F_{\rm el} = \frac{\mu_0}{2\pi}\frac{I^2 L}{d} - \frac{1}{2\pi\epsilon_0}\frac{Q^2 L}{d} = F_{\rm mag}\left[1 - \left(\frac{R}{R_0}\right)^2\right].$$

Source: I.E. Irodov, *Basic Laws of Electromagnetism* (Mir, Moscow, 1986).

12.5 The Mechanical Stability of Concentric Solenoids



The dashed lines in the diagram indicate the surface current density $\mathbf{K} = K\hat{\boldsymbol{\theta}}$ of each solenoid. The solid line is a representative field line of the outer solenoid. The latter is continuous near the walls of the inner solenoid and takes the value $\mathbf{B}$ on those walls. Therefore, the force density on the inner solenoid is

$$\mathbf{f} = \mathbf{K} \times \mathbf{B} = K\hat{\boldsymbol{\theta}} \times (B_z\hat{\mathbf{z}} + B_\rho\hat{\boldsymbol{\rho}}) = KB_z\hat{\boldsymbol{\rho}} - KB_\rho\hat{\mathbf{z}}.$$

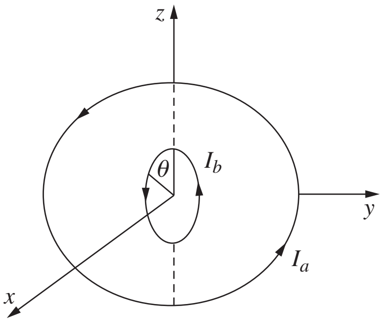
When the two solenoids are concentric, the radial component of the force cancels when integrated around the inner solenoid. The z-component cancels also if the mid-planes of the two solenoids are coincident because $B_\rho > 0$ above the mid-plane and $B_\rho < 0$ below the mid-plane. However, the radial component increases in magnitude as we approach the open ends of the solenoid. Therefore, the z-components fail to cancel if the inner solenoid is displaced either up or down from the mid-plane. However, the force tends to return the inner solenoid to the mid-plane in both cases. In other words, the system is stable with respect to this perturbation.

The currents in the two solenoids are parallel and thus attract. This is the radial force which integrated to zero above when the two solenoids are concentric. However, any radial motion will bring one pair of parallel currents closer together than any other pair of parallel currents. This produces an unstable situation because the force increases as the distance between two parallel currents decreases.

Source: Y. Iwasa, *Case Studies in Superconducting Magnets* (Springer, New York, 1994).

12.6 The Torque between Nested Current Rings

We locate the ring I_a in the x-y plane and the ring I_b in the x-z plane as shown below. The magnetic field on the axis of I_a points in the z-direction. The magnetic field of I_b points in the y-direction.



The vector torque which acts on I_b is

$$\mathbf{N} = I_b \oint \mathbf{r} \times (d\boldsymbol{\ell} \times \mathbf{B}),$$

where $\mathbf{r} = b\cos\theta\hat{\mathbf{z}} + b\sin\theta\hat{\mathbf{z}}$, $d\boldsymbol{\ell} = bd\theta\hat{\boldsymbol{\theta}} = bd\theta(\cos\theta\hat{\mathbf{x}} - \sin\theta\hat{\mathbf{z}})$, and $\mathbf{B} = B_\rho\hat{\boldsymbol{\rho}} + B_z\hat{\mathbf{z}}$ is the magnetic field due to I_a in cylindrical coordinates. Since parallel currents attract and anti-parallel currents repel, the only component of the torque which survives is N_x. Focusing on this component, substituting $\mathbf{r}$, $d\boldsymbol{\ell}$, and $\mathbf{B}$ into the torque formula gives

$$N_x = I_b b^2 \int_0^{2\pi} d\theta \cos\theta (B_z \cos\theta + B_\rho \sin\theta).$$

Our task now is to write the components of $\mathbf{B}(\rho, z)$ near the origin, where $\rho = b\sin\theta$ and $z = b\cos\theta$. In the text, we used the technique of "going off the axis" to find the exact magnetic scalar potential of a current loop. In polar coordinates, the first two terms of this expansion for $r < a$ were

$$\psi(r,\theta) = -\frac{1}{2}\mu_0 I_a \left[\frac{r}{a}P_0(0)P_1(\cos\theta) + \left(\frac{r}{a}\right)^3 P_2(0)P_3(\cos\theta)\right].$$

Using $P_0 = 1$, $P_1 = \cos\theta$, $P_2 = (1/2)(3\cos^2\theta - 1)$, and $P_3 = (1/2)(5\cos^3\theta - 3\cos\theta)$, together with $z = r\cos\theta$ and $\rho = r\sin\theta$, we get

$$\psi(\rho, z) = -\frac{1}{2}\mu_0 I_a \left[\frac{z}{a} - \frac{1}{2}\frac{z^3}{a^3} + \frac{3}{4}\frac{z\rho^2}{a^3}\right].$$

Therefore,

$$B_z = -\frac{\partial\psi}{\partial z} = \frac{\mu_0 I_a}{2a}\left[1 - \frac{3}{2}\frac{z^2}{a^2} + \frac{3}{4}\frac{\rho^2}{a^2}\right] = \frac{\mu_0 I_a}{2a}\left[1 - \frac{3}{2}\frac{b^2\cos^2\theta}{a^2} + \frac{3}{4}\frac{b^2\sin^2\theta}{a^2}\right]$$

and

$$B_\rho = -\frac{\partial\psi}{\partial\rho} = \frac{3\mu_0 I_a}{4}\frac{z\rho}{a^3} = \frac{3\mu_0 I_a}{4}\frac{b^2}{a^3}\sin\theta\cos\theta.$$

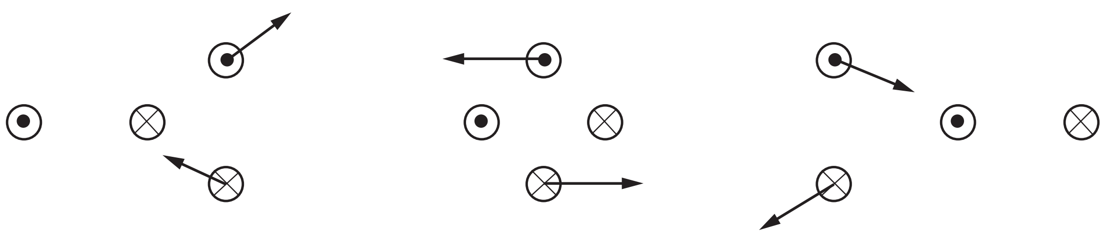
Substituting these fields into the torque formula and collecting terms gives the advertised result:

$$N_x = \frac{1}{2}\mu_0 I_a I_b \frac{b^2}{a} \int_0^{2\pi} d\theta \left[\cos^2\theta \left(1 - \frac{3b^2}{2a^2}\right) + \frac{15b^2}{16a^2}\sin^2 2\theta\right] = \frac{\pi}{2}\mu_0 I_a I_b \frac{b^2}{a}\left[1 - \left(\frac{3b}{4a}\right)^2\right].$$

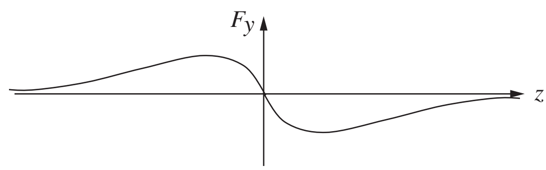
Source: J. Jeans, *The Mathematical Theory of Electricity and Magnetism* (University Press, Cambridge, 1908).

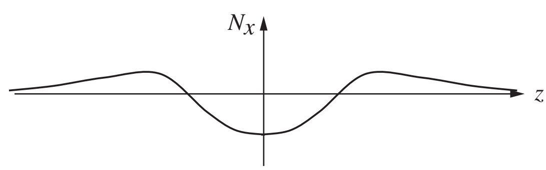
12.7 Force and Torque

The text shows that the force between two current elements $I_1 d\mathbf{s}_1$ and $I_2 d\mathbf{s}_2$ is proportional to $I_1 I_2 d\mathbf{s}_1 \cdot \mathbf{s}_2$. Therefore, the vertical legs of the vertical loop feel no force at all from the horizontal loop. Moreover, only the two x-legs of the horizontal loop contribute to the force on the (x-legs of the) vertical loop. The figure below shows only these x-legs (oriented perpendicular to the paper for both loops) at three different relative distances. Parallel currents attract, anti-parallel currents, and the force decreases as the distance between parallel wires decreases. With this information, the solid arrows below indicate the net force on the two x-legs of the vertical loop.



Qualitatively, the forces on the two x-legs of the vertical loop produce the net force and the net torque on the vertical loop sketched below.





12.8 A General Formula for Magnetostatic Torque

Using the Biot-Savart law and expanding the triple product gives the torque on $\mathbf{j}(\mathbf{r})$ produced by the magnetic field $\mathbf{B}'(\mathbf{r})$ produced by $\mathbf{j}'(\mathbf{r}')$ as

$$\begin{aligned}
\mathbf{N} &= \frac{\mu_0}{4\pi}\int d^3r\,\mathbf{r}\times[\mathbf{j}(\mathbf{r})\times\mathbf{B}'(\mathbf{r})]\\
&= \int d^3r\,\mathbf{r}\times\left[\mathbf{j}(\mathbf{r})\times\frac{\mu_0}{4\pi}\int d^3r'\,\mathbf{j}'(\mathbf{r}')\times\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\right]\\
&= \frac{\mu_0}{4\pi}\int d^3r\int d^3r'\,\mathbf{r}\times\frac{\mathbf{j}'(\mathbf{r}')\,[\mathbf{j}(\mathbf{r})\cdot(\mathbf{r}-\mathbf{r}')]-[\mathbf{j}'(\mathbf{r}')\cdot\mathbf{j}(\mathbf{r})]\,(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\\
&= \frac{\mu_0}{4\pi}\int d^3r\int d^3r'\,\frac{[\mathbf{r}\times\mathbf{j}'(\mathbf{r}')][\mathbf{j}(\mathbf{r})\cdot(\mathbf{r}-\mathbf{r}')]-[\mathbf{r}\times(\mathbf{r}-\mathbf{r}')][\mathbf{j}(\mathbf{r})\cdot\mathbf{j}'(\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|^3}\\
&= \frac{\mu_0}{4\pi}\int d^3r\int d^3r'\left\{[\mathbf{r}\times\mathbf{j}'(\mathbf{r}')]\mathbf{j}(\mathbf{r})\cdot\nabla\frac{-1}{|\mathbf{r}-\mathbf{r}'|}+\frac{\mathbf{r}\times\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\mathbf{j}(\mathbf{r})\cdot\mathbf{j}'(\mathbf{r}')\right\}.
\end{aligned}$$

Integrating the first term in curly brackets by parts and using $\nabla\cdot\mathbf{j}=0$ gives the desired result.

Source: P.C. Clemmow, *An Introduction to Electromagnetic Theory* (University Press, Cambridge, 1973).

12.9 Force-Free Magnetic Fields

(a) Let $\nabla\times\mathbf{B}_1 = \alpha_1(\mathbf{r})\mathbf{B}_1$ and $\nabla\times\mathbf{B}_2 = \alpha_2(\mathbf{r})\mathbf{B}_2$. Their sum is force-free only if $\alpha_1(\mathbf{r})=\alpha_2(\mathbf{r})=\alpha(\mathbf{r})$, so

$$\nabla\times(\mathbf{B}_1+\mathbf{B}_2)=\alpha_1(\mathbf{r})\mathbf{B}_1+\alpha_2(\mathbf{r})\mathbf{B}_2=\alpha(\mathbf{r})(\mathbf{B}_1+\mathbf{B}_2).$$

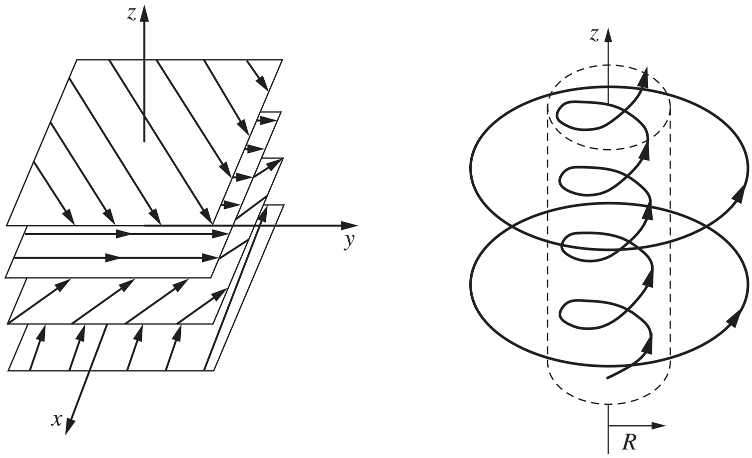
(b) Inserting the assumed form into $\nabla\times\mathbf{B}=\alpha\mathbf{B}$ gives the two equations

$$-\frac{dB_y}{dz}=\alpha B_x \qquad \text{and} \qquad \frac{dB_x}{dz}=\alpha B_y.$$

Taking the z-derivative of the right equation and using the left equation gives

$$\frac{d^2B_x}{dz^2}=\alpha\frac{dB_y}{dz}=-\alpha^2B_x.$$

This is solved by $\mathbf{B}(z)=\hat{\mathbf{x}}\,B_0\sin\alpha z+\hat{\mathbf{y}}\,B_0\cos\alpha z$ if we impose the condition $B(0)=B_0\hat{\mathbf{y}}$. As shown on the left side of the diagram below, the field is constant in each plane $z=$ *const.* and the direction of the field rotates as z increases.



(c) Inserting the assumed form into $\nabla \times \mathbf{B} = \alpha \mathbf{B}$ gives the two equations

$$\frac{1}{\rho}\frac{d}{d\rho}(\rho B_\phi) = \alpha B_z \qquad \text{and} \qquad -\frac{dB_z}{d\rho} = \alpha B_\phi.$$

Combining these two as in part (b) gives Bessel's equation for B_z:

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{dB_z}{d\rho}\right) + \alpha^2 B_z = 0.$$

We get B_ϕ by differentiation. Therefore, since $J_1(x) = J_0'(x)$, the solution which is regular at the origin is

$$\mathbf{B}(\rho) = AJ_1(\alpha\rho)\hat{\boldsymbol{\phi}} + AJ_0(\alpha\rho)\hat{\mathbf{z}}.$$

The right side of the diagram above shows that the field lines "spiral" up the z-axis.

(d) One matching condition is the continuity of $\hat{\mathbf{n}} \cdot \mathbf{B}$ on the $\rho = R$ cylinder. This will be true if $\mathbf{B}^{\text{out}}$ has no $\hat{\boldsymbol{\rho}}$ component. We want no current generated at $\rho = R$ so the tangential matching condition is

$$\hat{\boldsymbol{\rho}} \times \left[\mathbf{B}^{\text{out}}(R) - \mathbf{B}^{\text{in}}(R)\right] = 0.$$

If we impose $B_z(R) = 0$ as an additional condition, a non-trivial ($A \neq 0$) solution requires that αR be one of the zeroes of $J_0(x)$. In that case,

$$B_\phi^{\text{out}}(R) = B_\phi^{\text{in}}(R) = AJ_1(\alpha R).$$

Finally, we want no current for $\rho > R$. This implies that $\mathbf{B}^{\text{out}}$ can be derived from a magnetic scalar potential (which must satisfy Laplace's equation) as

$$B_\phi^{\text{out}} = -\frac{1}{\rho}\frac{d\psi}{d\phi}.$$

The simplest possible choice is the linear function $\psi(\phi) = -RAJ_1(\alpha R)\phi$. This gives the desired field as

$$\mathbf{B}^{\text{out}}(\rho) = \frac{RAJ_1(\alpha R)}{\rho}\hat{\boldsymbol{\phi}}.$$

Source: G.E. Marsh, *Force-Free Magnetic Fields* (World Scientific, Singapore, 1996).

12.10 Nuclear Magnetic Resonance

The total magnetic field is $\mathbf{B} = (B_1\cos\omega t, -B_1\sin\omega t, B_0)$. Therefore, the components of the torque equation $\dot{\mathbf{m}} = \gamma\mathbf{m}\times\mathbf{B}$ are

$$\frac{dm_x}{dt} = \gamma(m_y B_0 + m_z B_1\sin\omega t)$$

$$\frac{dm_y}{dt} = \gamma(m_z B_1\cos\omega t - m_x B_0)$$

$$\frac{dm_z}{dt} = -\gamma B_1(m_x\sin\omega t + m_y\cos\omega t).$$

The transverse components of $\mathbf{m}$ in a coordinate system that rotates with $\mathbf{B}_1$ are

$$M_x(t) = m_x\cos\omega t - m_y\sin\omega t \qquad \text{and} \qquad M_y(t) = m_x\sin\omega t + m_y\cos\omega t.$$

The time derivatives of these components are

$$\dot{M}_x = \dot{m}_x\cos\omega t - \dot{m}_y\sin\omega t - \omega M_y \qquad \text{and} \qquad \dot{M}_y = \dot{m}_x\sin\omega t + \dot{m}_y\cos\omega t + \omega M_x.$$

Therefore, substituting from the first set of equations into the last set of equations gives

$$\dot{M}_x = (\Omega_L - \omega)M_y$$

$$\dot{M}_y = -(\Omega_L - \omega)M_x + \gamma B_1 m_z$$

$$\dot{m}_z = -\gamma B_1 M_y.$$

When $\omega = \Omega_L$, these simplify to

$$\dot{M}_x = 0$$

$$\dot{M}_y = \gamma B_1 m_z$$

$$\dot{m}_z = -\gamma B_1 M_y.$$

In the rotating frame, we observe a moment that precesses around $\mathbf{B}_1$ at the Larmor frequency, γB_1. In the lab frame, this motion is superimposed on a rotation around the $\mathbf{B}_0$ axis at Ω_L.

Source: G. Scharf, *From Electrostatics to Optics* (Springer, Berlin, 1994).

12.11 Two Dipoles in a Uniform Field

If **R** points from the center of one dipole to the other, the potential energy for this situation is

$$\hat{V}_B = -\mathbf{m}_1 \cdot \mathbf{B} - \mathbf{m}_2 \cdot \mathbf{B} + \frac{\mu_0}{4\pi}\left[\frac{\mathbf{m}_1 \cdot \mathbf{m}_2}{R^3} - \frac{3(\mathbf{m}_1 \cdot \mathbf{R})(\mathbf{m}_2 \cdot \mathbf{R})}{R^5}\right].$$

Since $\cos(\pi - \theta) = -\cos\theta$, this is

$$\hat{V}_B = (m_1 \cos\alpha + m_2 \cos\beta)B + \frac{\mu_0}{4\pi}\left[\frac{m_1 m_2}{R^3}\cos(\beta - \alpha) - \frac{3m_1 m_2}{R^3}\cos\alpha\cos\beta\right].$$

The angles are supposed to be small. Therefore, expanding to second order gives

$$\hat{V}_B = A\alpha^2 + B\beta^2 + 2C\alpha\beta + const.$$

where

$$A = \frac{\mu_0}{4\pi}\frac{m_1 m_2}{R^3} - \frac{1}{2}m_1 B \qquad B = \frac{\mu_0}{4\pi}\frac{m_1 m_2}{R^3} - \frac{1}{2}m_2 B \qquad C = \frac{\mu_0}{8\pi}\frac{m_1 m_2}{R^3}.$$

We can shift the zero to eliminate the constant. Therefore, if $\alpha = \beta = 0$ is to be a point of stable mechanical equilibrium, we must have $\hat{V}_B > 0$. The latter is a quadratic form, which is positive if and only if

$$A > 0 \qquad B > 0 \qquad AB > C^2.$$

If we let $p = 4\pi R^3 B/\mu_0$, these conditions imply that

$$m_1 > p/2 \qquad m_2 > p/2 \qquad (2m_1 - p)(2m_2 - p) > m_1 m_2. \tag{1}$$

The last of these we may rewrite as the condition

$$F(p) = p^2 - 2(m_1 + m_2)p + 3m_1 m_2 > 0.$$

Now, the zeroes of $F(p)$ are

$$p_+ = m_1 + m_2 + \sqrt{m_1^2 - m_1 m_2 + m_2^2} \qquad p_- = m_1 + m_2 + \sqrt{m_1^2 - m_1 m_2 + m_2^2}.$$

Since $p_+ > p_-$, we get $F(p) > 0$ if either

$$p < p_- \qquad \text{or} \qquad p > p_+.$$

However, we know from the left side of (1) that $p < m_1 + m_2$. In addition, $m_1 + m_2 < p_+$. Therefore, $p < p_+$. This means that the criterion for stability is $p < p_-$ or

$$B < \frac{\mu_0}{4\pi R^3}\left[m_1 + m_2 - \sqrt{m_1^2 - m_1 m_2 + m_2^2}\right].$$

Source: V.C.A. Ferraro, *Electromagnetic Theory* (Athlone Press, London, 1954).

12.12 Three Point Dipoles

Let $\overrightarrow{CA}$ be the vector that points from $\mathbf{m}_\mathrm{C}$ to $\mathbf{m}_\mathrm{A}$. The magnetic field at $\mathbf{m}_\mathrm{A}$ is

$$\mathbf{B}_\mathrm{A} = \frac{\mu_0}{4\pi}\left[-\frac{\mathbf{m}_\mathrm{B}}{a^3} + \frac{3(\mathbf{m}_\mathrm{B}\cdot\overrightarrow{BA})\,\overrightarrow{BA}}{a^3} - \frac{\mathbf{m}_\mathrm{C}}{a^3} + \frac{3(\mathbf{m}_\mathrm{C}\cdot\overrightarrow{CA})\,\overrightarrow{CA}}{a^3}\right].$$

Therefore, the potential energy of $\mathbf{m}_\mathrm{A}$ is

$$\begin{aligned}
\hat{V}_\mathrm{A} &= -\mathbf{m}_\mathrm{A}\cdot\mathbf{B}_\mathrm{A} \\
&= \frac{\mu_0}{4\pi}\left[\frac{\mathbf{m}_\mathrm{A}\cdot\mathbf{m}_\mathrm{B}}{a^3} + \frac{\mathbf{m}_\mathrm{A}\cdot\mathbf{m}_\mathrm{C}}{a^3} - \frac{3(\mathbf{m}_\mathrm{B}\cdot\overrightarrow{BA})(\mathbf{m}_\mathrm{A}\cdot\overrightarrow{BA})}{a^3} - \frac{3(\mathbf{m}_\mathrm{C}\cdot\overrightarrow{CA})(\mathbf{m}_\mathrm{A}\cdot\overrightarrow{CA})}{a^3}\right] \\
&= \frac{\mu_0 m^2}{4\pi a^3}\left[\cos\left(\theta - \frac{\pi}{3}\right) + \cos\left(\theta + \frac{\pi}{3}\right) - 3\cos\left(\frac{\pi}{6}\right)\cos\left(\theta - \frac{\pi}{6}\right) - 3\cos\left(\frac{\pi}{6}\right)\cos\left(\theta + \frac{\pi}{6}\right)\right] \\
&= -\frac{7\mu_0 m^2\cos\theta}{8\pi a^3}.
\end{aligned}$$

The potential energy is smallest at $\theta = 0$, when $\mathbf{m}_\mathrm{A}$ points straight up. For small oscillations, we write $\cos\theta \approx 1 - \frac{1}{2}\theta^2$ and write the torque equation,

$$I\ddot{\theta} = -\frac{\partial\hat{V}_\mathrm{A}}{\partial\theta} = -\frac{7\mu_0 m^2}{8\pi a^3}\theta.$$

Therefore the period of small oscillations is

$$T = 2\pi\sqrt{\frac{8\pi a^3 I}{7m^2\mu_0}} = \frac{4\pi a}{m}\sqrt{\frac{2\pi I a}{7\mu_0}}.$$

Source: B.H. Chirgwin, C. Plumpton, and C.W. Kilmister, *Elementary Electromagnetic Theory*, Volume 2 (Pergamon, Oxford, 1972).

12.13 A Dipole in the Field of Two Dipoles

The orientation is determined by minimizing the potential energy $\hat{V}_B = -\mathbf{M}\cdot\mathbf{B}$ where $\mathbf{B}$ is the magnetic field produced by the two fixed dipoles at the position of $\mathbf{M}$. In other words, $\mathbf{M}$ is parallel to $\mathbf{B}$ in stable equilibrium. To proceed, define vectors $\mathbf{r}_1 = a\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ and

$\mathbf{r}_2 = -a\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ which point from the two fixed dipoles to the point $(0, y, z)$. Both vectors have magnitude $r = \sqrt{a^2 + y^2 + z^2}$. Then, by direct computation,

$$\begin{aligned}
\mathbf{B}(0, y, z) &= \frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{r}_1)\mathbf{r}_1}{r^5} - \frac{\mathbf{m}}{r^3} + \frac{3(\mathbf{m}\cdot\mathbf{r}_2)\mathbf{r}_2}{r^5} - \frac{\mathbf{m}}{r^3}\right] \\
&= \frac{\mu_0}{4\pi}\left[\frac{6mz}{r^5}(y\hat{\mathbf{y}} + z\hat{\mathbf{z}}) - \frac{2m}{r^3}\hat{\mathbf{z}}\right] \\
&= \frac{\mu_0}{4\pi}\frac{2m}{r^5}\left[3yz\hat{\mathbf{y}} + \left(2z^2 - a^2 - y^2\right)\hat{\mathbf{z}}\right].
\end{aligned}$$

We conclude that, in stable equilibrium, the direction of $\mathbf{M}$ is

$$\hat{\mathbf{M}} = \frac{3yz\hat{\mathbf{y}} + (2z^2 - a^2 - y^2)\hat{\mathbf{z}}}{\sqrt{(2yz)^2 + (2z^2 - a^2 - y^2)^2}}.$$

Source: V.C.A. Ferraro, *Electromagnetic Theory* (Athlone Press, London, 1954).

12.14 Superconductor Meets Solenoid

The dipole moment of the sphere is induced, so the only valid expression for the force on it (treated as a point dipole on the z-axis) is

$$\mathbf{F} = m_k \nabla B_k = -\frac{2\pi}{\mu_0}R^3 B_k \nabla B_k = -\frac{2\pi}{\mu_0}R^3 B\frac{dB}{dz}\hat{\mathbf{z}}.$$

The last equality follows because $\mathbf{B} = B\hat{\mathbf{z}}$ on the symmetry axis of the solenoid. The work done on the sphere as it moves from infinity to deep inside the solenoid is

$$W = \int d\mathbf{s}\cdot\mathbf{F} = \int_{-\infty}^{0} dz F_z = -\frac{2\pi}{\mu_0}R^3 \int_{-\infty}^{0} B\frac{dB}{dz}dz = -\frac{2\pi}{\mu_0}R^3 \int_{0}^{B_S} BdB = -\frac{\pi B_S^2}{\mu_0}R^3.$$

Note: an answer different from this by a factor of 2 results if one incorrectly uses the potential energy function $V_B = -\mathbf{m}\cdot\mathbf{B}$ appropriate for a permanent moment. We get the minimum speed by equating W to the initial kinetic energy $\frac{1}{2}Mv_0^2$. Therefore,

$$v_{\min} = B_S\sqrt{\frac{2\pi R^3}{M\mu_0}}.$$

Source: S.M. Kozel, E.I. Rashba, and S.A. Slavatinsky, *Collection of Problems in Physics* (Nauka, Moscow, 1987).

12.15 The Levitron

(a) Above the base and near the z-axis, we write

$$\psi(\rho, z) = a(z) + b(z)\rho + c(z)\rho^2 + d(z)\rho^3 + \cdots .$$

The magnetic scalar potential satisfies

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{\partial^2\psi}{\partial z^2} = 0.$$

Substituting the first equation into the second equation gives

$$\frac{b}{\rho} + 4c + 9d\rho + a'' + b''\rho + c''\rho^2 + \cdots = 0.$$

The coefficient of each power of ρ must be zero. Therefore, $b = d = 0$ and $c = -a''/4$. Moreover, $a(z) = \psi(0, z) = \psi_0(z)$, using the notation defined by the problem statement. Hence, to second order in ρ,

$$\psi(\rho, z) = \psi_0(z) - \frac{1}{4}\psi_2(z)\rho^2 + \cdots = \psi_0(z) - \frac{1}{4}\psi_2(z)(x^2 + y^2) + \cdots .$$

(b) We need $|\mathbf{B}|$ to construct the potential energy E. The magnetic field is

$$\mathbf{B} = -\nabla\psi = -\psi_1(z)\hat{\mathbf{z}} + \frac{1}{4}\psi_3(z)(x^2 + y^2)\hat{\mathbf{z}} + \frac{1}{2}\psi_2(z)x\hat{\mathbf{x}} + \frac{1}{2}\psi_2(z)y\hat{\mathbf{y}}.$$

Therefore,

$$|\mathbf{B}|^2 = \left(\frac{1}{4}\psi_3\rho^2 - \phi_1\right)^2 + \frac{1}{4}\psi_2^2\rho^2 \approx \psi_1^2\left[1 - \frac{1}{2}\frac{\psi_3}{\psi_1}\rho^2 + \frac{1}{4}\left(\frac{\psi_2}{\psi_1}\right)^2\rho^2\right],$$

and we conclude that

$$B(\rho, z) = |\mathbf{B}(\rho, z)| = |\psi_1|\left[1 + \frac{\rho^2}{8}\left(\frac{\psi_2^2}{\psi_1^2} - 2\frac{\psi_3}{\psi_2}\right)\right] = \psi_1 \,\mathrm{sgn}(\psi_1)\left[1 + \frac{\rho^2}{8}\left(\frac{\psi_2^2}{\psi_1^2} - 2\frac{\psi_3}{\psi_2}\right)\right].$$

The equilibrium condition is $\nabla E = 0$. The z derivative gives

$$Mg = m_B\frac{\partial B}{\partial z}.$$

Since $Mg > 0$ and ρ is small, this amounts to the condition

$$m_B\psi_2 \,\mathrm{sgn}(\psi_1) > 0 \qquad \text{(equilibrium)}.$$

The x and y pieces of the gradient say that equilibrium requires $\rho = 0$. The vertical stability condition is

$$0 < \frac{\partial^2 E}{\partial z^2} = -m_B \frac{\partial^2 B}{\partial z^2},$$

or

$$m_B \psi_3 \,\mathrm{sgn}(\psi_1) < 0 \qquad \text{(vertical stability)}.$$

Finally, the horizontal stability condition amounts to

$$0 < \frac{\partial^2 E}{\partial \rho^2} = -m_B \mathrm{sgn}(\psi_1)\psi_1 \left(\frac{\psi_2^2}{\psi_1^2} - 2\frac{\psi_3}{\psi_1} \right),$$

or

$$m_B \,\mathrm{sgn}(\psi_1) \left(2\psi_3 - \frac{\psi_2^2}{\psi_1} \right) > 0 \qquad \text{(horizontal stability)}.$$

Assuming first that $\psi_1 > 0$ and then that $\psi_1 < 0$, it is straightforward to check that all three conditions cannot be satisfied simultaneously unless $m_B < 0$.

(c) We now assume that $m_B < 0$ and, once again, study $\psi_1 > 0$ and $\psi_1 < 0$ separately. The equilibrium condition tells us that ψ_1 and ψ_2 must have opposite signs. The vertical stability condition tells us that ψ_1 and ψ_3 must have the same signs. The horizontal stability condition tells us that

$$\psi_2^2 > 2\psi_3\psi_1.$$

Source: M.V. Berry, *Proceedings of the Royal Society of London A* **452**, 1207 (1996).

12.16 Magnetic Trap I

The magnetic field from a wire at the origin is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\theta}} = \frac{\mu_0 I}{2\pi\rho^2}(-y\hat{\mathbf{x}} + x\hat{\mathbf{y}}),$$

so, if $\alpha = \mu_0 I/2\pi$, the total magnetic field of the system is

$$B = B_0\hat{\mathbf{z}} + \alpha\left[\frac{-y\hat{\mathbf{x}} + (x-1)\hat{\mathbf{y}}}{(x-1)^2 + y^2} + \frac{-y\hat{\mathbf{x}} + (x+1)\hat{\mathbf{y}}}{(x+1)^2 + y^2}\right].$$

For an anti-parallel dipole with magnetic moment $\mathbf{m}_0$, the potential energy is $\hat{U}_B = -\mathbf{m}_0 \cdot \mathbf{B} = m_0|B|$. Near the origin, we can use the approximation

$$\mathbf{B} = B_0\hat{\mathbf{z}} + 2\alpha(x\hat{\mathbf{y}} - y\hat{\mathbf{x}}).$$

Therefore,

$$|B| = \sqrt{B_0^2 + 4\alpha^2\rho^2} \approx B_0 + 2\alpha\rho^2/B_0.$$

The small oscillation frequency ω arises from a potential energy function $U = \frac{1}{2}M\omega^2\rho^2$. Therefore,

$$\omega = \sqrt{\frac{4\alpha^2 m_0}{MB_0}} = \frac{\mu_0 I}{\pi}\sqrt{\frac{m_0}{MB_0}}.$$

12.17 Magnetic Trap II

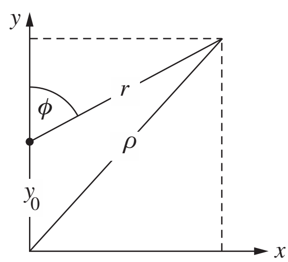
(a) If $\lambda = \mu_0 I/2\pi$, the total magnetic field in Cartesian coordinates is

$$\mathbf{B} = \left(B_0 - \frac{\lambda y}{x^2+y^2}\right)\hat{\mathbf{x}} + \frac{\lambda x}{x^2+y^2}\hat{\mathbf{y}} + B'\hat{\mathbf{z}}.$$

By inspection, the smallest field magnitude occurs when the x and y components of $\mathbf{B}$ are zero. Thus, we get $|\mathbf{B}| = B'$ along the line parallel to the wire defined by

$$x_0 = 0 \qquad \text{and} \qquad y_0 = \frac{\lambda}{B_0} = \frac{\mu_0 I}{2\pi B_0}.$$

(b) The potential energy of interaction between $\mathbf{m}$ and $\mathbf{B}$ when they are anti-parallel is $\hat{V} = -\mathbf{m}\cdot\mathbf{B} = m|\mathbf{B}|$. Therefore, to find the frequency of small (radial) oscillations away from the minimum of energy, we must expand $|\mathbf{B}|$ to quadratic order in the radial distance r from the line.



Using $\rho^2 = x^2 + y^2$ and, from the diagram, $x = r\sin\phi$, $y = y_0 + r\cos\phi$, and $\rho^2 = r^2 + y_0^2 + 2ry_0\cos\phi$, we find

$$\begin{aligned}
|\mathbf{B}|^2 &= \left(B_0 - \frac{\lambda y}{\rho^2}\right)^2 + \frac{\lambda^2 x^2}{\rho^2} + B'^2 \\
&= \frac{B_0^2}{\rho^2}\left[\rho^2 + y_0^2 - 2y_0 y\right] + B'^2 \\
&= \frac{B_0^2}{\rho^2} r^2 + B'^2.
\end{aligned}$$

We are interested in very small values of r, where $\rho \approx y_0$. Therefore,

$$\hat{V} = m|\mathbf{B}| \approx mB'^2\left[1 + \frac{B_0^4}{B'^2}\frac{r^2}{\lambda^2}\right] \approx mB' + \frac{1}{2}\frac{mB_0^4}{B'\lambda^2}r^2.$$

Setting this equal to $\hat{V}_0 + \frac{1}{2}M\omega^2 r^2$ gives the frequency of small radial oscillations as

$$\omega = \sqrt{\frac{m}{MB'}\frac{2\pi B_0^2}{\mu_0 I}}.$$

(c) If $B' = 0$, we get $|\mathbf{B}| \approx (B_0/y_0)r$ and there is no harmonic motion at all.

Source: J. Forágh and C. Zimmerman, *Reviews of Modern Physics* **79**, 235 (2007).

12.18 Roget's Spiral

(a) At constant current, an increase in L generates a force $\mathbf{F} = -\frac{\partial \hat{U}_B}{\partial L}\hat{\mathbf{z}}$. The magnetic field inside the solenoid has magnitude $B = \mu_0 IN/L$. There is no field outside the solenoid if we neglect stray fields. Therefore, the magnetic potential energy is

$$\hat{U}_B = -\frac{1}{2\mu_0}\int d^3r\, B^2 = -\frac{\pi R^2 \mu_0 N^2 I^2}{2L}.$$

In equilibrium, the magnetic force

$$\mathbf{F} = -\frac{\pi R^2 \mu_0 N^2 I^2}{2L^2}\hat{\mathbf{z}}$$

balances the weight mg. Therefore,

$$I = \sqrt{\frac{2mg}{\pi\mu_0}}\frac{L}{NR}.$$

(b) The attractive force between parallel currents in adjacent rings causes the coil to contract without any motion of its center of mass.

Source: O.D. Jefimenko, *Electricity and Magnetism* (Appleton-Century-Crofts, New York, 1966).

12.19 Equivalence of Force Formulae

U_B must be expressed as a function of the flux variables. $\hat{U}_B$ must be expressed as a function of the current variables. To do this, we use

$$\Phi_k = M_{k\ell} I_\ell \qquad \text{and} \qquad I_k = M^{-1}_{k\ell}\Phi_\ell. \tag{1}$$

Therefore,

$$U_B = \frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k = \frac{1}{2}\sum_{k=1}^{N} \Phi_k M^{-1}_{k\ell}\Phi_\ell$$

$$\hat{U}_B = -\frac{1}{2}\sum_{k=1}^{N} I_k \Phi_k = -\frac{1}{2}\sum_{k=1}^{N} I_k M_{k\ell} I_\ell.$$

Substituting these expressions into the force formulae in the statement of the problem shows that the proposition will be proved if we can show that

$$\Phi_k \nabla M^{-1}_{k\ell} \Phi_\ell = -I_k \nabla M_{k\ell} I_\ell. \tag{2}$$

We begin with $\mathbf{M}\mathbf{M}^{-1} = \mathbf{I}$ written in component form:

$$M_{k\ell} M^{-1}_{\ell p} = \delta_{kp}.$$

Using this,

$$(\nabla M_{k\ell}) M^{-1}_{\ell p} + M_{k\ell}(\nabla M^{-1}_{\ell p}) = 0.$$

Multiplying on the right by M_{ps} and summing over p gives

$$(\nabla M_{k\ell}) M^{-1}_{\ell p} M_{ps} = -M_{k\ell}(\nabla M^{-1}_{\ell p}) M_{ps}.$$

Using the definition of the inverse,

$$(\nabla M_{k\ell})\delta_{\ell s} = -M_{k\ell}(\nabla M^{-1}_{\ell p}) M_{ps}.$$

The left side of this equation is ∇M_{ks}. Therefore, using (1) and the fact that $M_{k\ell} = M_{\ell k}$,

$$\begin{aligned} -I_k \nabla M_{ks} I_s &= -I_k \left[-M_{k\ell}(\nabla M^{-1}_{\ell p}) M_{ps} \right] I_s \\ &= I_k M_{k\ell} \nabla M^{-1}_{\ell p} \Phi_p \\ &= M_{\ell k} I_k \nabla M^{-1}_{\ell p} \Phi_p \\ &= \Phi_\ell \nabla M^{-1}_{\ell p} \Phi_p. \end{aligned}$$

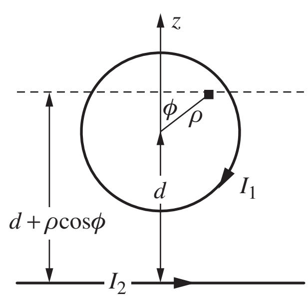
This is (2), as required.

12.20 The Force between a Current Loop and a Wire

(a) Let Φ_1 be the flux through the loop produced by the wire. The force on the loop is

$$\mathbf{F} = -\frac{\partial \hat{V}_B}{\partial d} \hat{\mathbf{z}},$$

where $\hat{V}_B$ is the interaction potential energy $\hat{V}_B = -I_1 \Phi_1$. The black square in the diagram below is an area element $dS = \rho d\rho d\phi$ at a distance $d + \rho\cos\phi$ from the wire.



The magnetic field at that element points out of the paper in the plane of the loop with magnitude

$$B_2 = \frac{\mu_0 I_2}{2\pi} \frac{1}{d + \rho \cos \phi}.$$

By the right-hand rule, $d\mathbf{S}$ points into the paper. Therefore,

$$\begin{aligned} \Phi_1 &= \int d\mathbf{S} \cdot \mathbf{B}_2 = -\frac{\mu_0 I_2}{2\pi} \int_0^R \int_0^{2\pi} \frac{\rho d\rho d\phi}{d + \rho \cos \phi} = -\mu_0 I_2 \int_0^R \frac{\rho d\rho}{\sqrt{d^2 - \rho^2}} \\ &= -\mu_0 I_2 \left[\sqrt{d^2 - R^2} - d \right]. \end{aligned}$$

This gives the force on the loop as

$$\mathbf{F} = \mu_0 I_1 I_2 \frac{\partial}{\partial d} \left[\sqrt{d^2 - R^2} - d \right] \hat{\mathbf{z}} = \mu_0 I_1 I_2 \left[\frac{d}{\sqrt{d^2 - R^2}} - 1 \right] \hat{\mathbf{z}}.$$

(b) In the limit $d \gg R$, we use $(1 - R^2/d^2)^{-1/2} \approx 1 + R^2/2d^2$ to get the repulsive force on the loop as

$$\mathbf{F} = \frac{\mu_0 I_1 I_2 R^2}{2d^2} \hat{\mathbf{z}}.$$

The magnetic moment of the loop is $m_1 = I_1 \pi R^2$ and points into the paper. Therefore, since $\mathbf{B}_2$ points out of the paper, the force on the loop should be

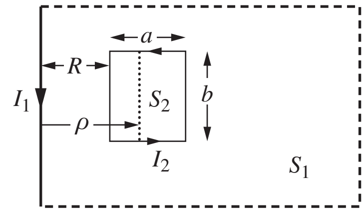
$$\mathbf{F} = \nabla(\mathbf{m}_1 \cdot \mathbf{B}_2) = -(I_1 \pi R^2) \frac{\mu_0 I_2}{2\pi} \frac{\partial}{\partial d} \left(\frac{1}{d} \right) \hat{\mathbf{z}} = \frac{\mu_0 I_1 I_2 R^2}{2d^2} \hat{\mathbf{z}}.$$

Source: M.H. Nayfeh and M.K. Brussel, *Electricity and Magnetism* (Wiley, New York, 1985).

12.21 Toroidal Inductance

(a) The inductance L_{21} of the coil with respect to the wire is N times the inductance of the rectangular loop (shown below) with respect to the wire. The latter is the ratio of the flux through the loop, Φ_{21}, to the current through the wire, I_1. The right-hand rule connects the direction of I_2 to the direction of $d\mathbf{S}$ in the flux integral. Therefore, since the field of the wire, $\mathbf{B}_1 = \hat{\boldsymbol{\phi}} \mu_0 I_1 / 2\pi\rho$, is normal to the plane S enclosed by the loop,

$$L_{21} = N \frac{\Phi_{21}}{I_1} = \frac{N}{I_1} \int_{S_2} d\mathbf{S} \cdot \mathbf{B}_1 = \frac{N}{I_1} \int_{S_2} dS B_1 = \frac{\mu_0 N b}{2\pi} \int_R^{R+a} \frac{d\rho}{\rho} = \frac{\mu_0 N b}{2\pi} \ln \frac{R + a}{a}.$$



(b) The text computed the field produced by a toroidal solenoid. The field lies entirely within the solenoid with magnitude $B_2 = \mu_0 N I_2 / 2\pi\rho$. The direction of $\mathbf{B}_2$ is normal to every element $d\mathbf{S}$ of the surface S_1 enclosed by the infinitely large rectangular loop shown in the figure which has I_1 as one of its legs. With this choice,

$$L_{12} = \frac{\Phi_{12}}{I_2} = \frac{1}{I_2}\int_{S_1} d\mathbf{S}\cdot\mathbf{B}_2 = \frac{1}{I_2}\int_{S_1} dSB_2 = \frac{\mu_0 N b}{2\pi}\int_R^{R+a}\frac{d\rho}{\rho} = \frac{\mu_0 N b}{2\pi}\ln\frac{R+a}{a}.$$

Source: O.D. Jefimenko, *Electricity and Magnetism* (Appleton-Century-Crofts, New York, 1966).

12.22 Force between Square Current Loops

(a) By symmetry, the force is along the symmetry axis. The loop currents are fixed, so if M is the mutual inductance between the loops, the force between them is $\mathbf{F} = -\nabla \hat{U}_B$. More precisely, let I_1 lie in the $z=0$ plane so I_2 lies in the $z=c$ plane. Then, because an increases of c is in the $+\hat{\mathbf{z}}$-direction, the force on I_2 is

$$\mathbf{F} = -\frac{\partial \hat{U}_B}{\partial c}\hat{\mathbf{z}} = I_1 I_2 \frac{\partial M}{\partial c}\hat{\mathbf{z}}. \quad (1)$$

We calculate M from Neumann's formula:

$$M = \frac{\mu_0}{4\pi}\oint\oint\frac{d\mathbf{s}_1\cdot d\mathbf{s}_2}{|\mathbf{r}_1-\mathbf{r}_2|}.$$

Each of the four sides of loop I_1 gives a non-zero contribution to this integral from the two sides of loop I_2 which are parallel to it. However, by symmetry, there are only two distinct terms among these eight contributions. Specifically, if the variables $-a/2 \le x_1 \le a/2$ and $-a/2 \le x_2 \le a/2$ label distances measured from the midpoint of a segment as shown in the diagram,

$$M = \frac{\mu_0}{\pi}\int_{-a/2}^{a/2} dx_1 \int_{-a/2}^{a/2} dx_2 \left[\frac{1}{\sqrt{(x_1-x_2)^2+c^2}} - \frac{1}{\sqrt{(x_1-x_2)^2+a^2+c^2}}\right].$$

We do the x_2 integral using

$$\int\frac{dx}{\sqrt{Ax^2+Bx+C}} = \frac{1}{\sqrt{A}}\sinh^{-1}\frac{2Ax+B}{\sqrt{4AC-B^2}}.$$

Since $\sinh^{-1}(-x) = -\sinh^{-1}(x)$, we find

$$M = \frac{\mu_0}{\pi}\int_{-a/2}^{a/2} dx_1 \left[\sinh^{-1}\frac{2x_2-2x_1}{2c}\Big|_{-a/2}^{a/2} - \sinh^{-1}\frac{2x_2-2x_1}{2\sqrt{c^2+a^2}}\Big|_{-a/2}^{a/2}\right]$$

$$= \frac{\mu_0}{\pi}\int_{-a/2}^{a/2} dx_1 \left[\sinh^{-1}\frac{2x_1+a}{2c} - \sinh^{-1}\frac{2x_1-a}{2c} + \sinh^{-1}\frac{2x_1-a}{2\sqrt{c^2+a^2}} - \sinh^{-1}\frac{2x_1+a}{2\sqrt{c^2+a^2}}\right].$$

We do the x_1 integral using

$$\int dx \sinh^{-1}\frac{x}{p} = x\sinh^{-1}\frac{x}{p} - \sqrt{x^2+p^2}.$$

The final result is

$$M = \frac{2\mu_0}{\pi}\left[a\sinh^{-1}\left(\frac{a}{c}\right) - a\sinh^{-1}\left(\frac{a}{\sqrt{a^2+c^2}}\right) - 2\sqrt{a^2+c^2} + \sqrt{2a^2+c^2} + c\right]. \tag{2}$$

(b) Using (1) and (2), the force exerted on I_2 is

$$\mathbf{F} = \frac{2\mu_0 I_1 I_2}{\pi}\left[\frac{c\sqrt{2a^2+c^2}}{a^2+c^2} + 1 - \frac{a^2+2c^2}{c\sqrt{a^2+c^2}}\right]\hat{\mathbf{z}}.$$

(c) If $\delta = a^2/c^2$, the force in part (b) is

$$\mathbf{F} = \frac{2\mu_0 I_1 I_2}{\pi}\left[(1+2\delta)^{1/2}(1+\delta)^{-1} + 1 - (2+\delta)(1+\delta)^{-1/2}\right]\hat{\mathbf{z}}.$$

Expanding and keeping all terms to second order in $\delta \ll 1$ gives the attractive force

$$\mathbf{F} = -\frac{3\mu_0 I_1 I_2}{2\pi}\frac{a^4}{c^4}\hat{\mathbf{z}} = \frac{\partial}{\partial c}\left(\frac{\mu_0}{2\pi}\frac{I_1 a^2 I_2 a^2}{c^3}\right)\hat{\mathbf{z}}. \tag{3}$$

To interpret this formula, we use the fact that the interaction between the loops should be of dipole-dipole type when when $c \gg a$. Using the right-hand rule, the moments in question are $\mathbf{m}_1 = I_1 a^2\hat{\mathbf{z}}$ and $\mathbf{m}_2 = I_2 a^2\hat{\mathbf{z}}$, each located at the center of the corresponding loop. If $\hat{\mathbf{n}}$ is a unit vector which points from the center of one loop to the center of the other loop, the interaction potential energy is

$$\hat{V}_B = \frac{\mu_0}{4\pi}\frac{\mathbf{m}_1\cdot\mathbf{m}_2 - 3(\hat{\mathbf{n}}\cdot\mathbf{m}_1)(\hat{\mathbf{n}}\cdot\mathbf{m}_2)}{c^3} = -\frac{\mu_0}{2\pi}\frac{m_1 m_2}{c^3}.$$

Using this, the force $\mathbf{F} = -\nabla\hat{V}_B$ (calculated as described at the beginning of the solution) agrees with (3).

Source: V.C.A. Ferraro, *Electromagnetic Theory* (Athlone Press, London, 1954).

12.23 The History of Mutual Inductance

The total mutual inductance between circuit 1 and circuit 2 is

$$M_{12} = \frac{\mu_0}{4\pi}\left\{\left(\frac{1+k}{2}\right)\oint_{C_1}\oint_{C_2}\frac{d\mathbf{s}_1\cdot d\mathbf{s}_2}{|\mathbf{r}_1-\mathbf{r}_2|}+\left(\frac{1-k}{2}\right)\oint_{C_1}\oint_{C_2}\frac{d\mathbf{s}_1\cdot(\mathbf{r}_1-\mathbf{r}_2)(\mathbf{r}_1-\mathbf{r}_2)\cdot d\mathbf{s}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}\right\}.$$

Focus on the second integral. Use Stokes' theorem, a bit of vector calculus, and Stokes' theorem again to write

$$\begin{aligned}
\oint_{C_1} d\mathbf{s}_1\cdot\frac{(\mathbf{r}_1-\mathbf{r}_2)(\mathbf{r}_1-\mathbf{r}_2)\cdot d\mathbf{s}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3} &= \int_{S_1} d\mathbf{S}_1\cdot\nabla_1\times\left[(\mathbf{r}_1-\mathbf{r}_2)\frac{(\mathbf{r}_1-\mathbf{r}_2)\cdot d\mathbf{s}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}\right]\\
&= \int_{S_1} d\mathbf{S}_1\cdot d\mathbf{s}_2\times\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}\\
&= \int_{S_1} d\mathbf{S}_1\cdot\nabla_2\times\frac{d\mathbf{s}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}\\
&= \oint_{C_1}\frac{d\mathbf{s}_1\cdot d\mathbf{s}_2}{|\mathbf{r}_1-\mathbf{r}_2|^3}.
\end{aligned}$$

Substituting this above recovers Neumann's formula:

$$M_{12} = \frac{\mu_0}{4\pi}\oint_{C_1}\oint_{C_2}\frac{d\mathbf{s}_1\cdot d\mathbf{s}_2}{|\mathbf{r}_1-\mathbf{r}_2|}.$$

Source: E.T. Whittaker, *A History of the Theories of Aether and Electricity* (Philosophical Library, New York, 1951).

12.24 An Inductance Inequality

Let $f = I_1/I_2$ and write the total magnetic energy as

$$U_B = \frac{1}{2}I_2^2\left(L_1 f^2 + 2Mf + L_2\right).$$

We will find the minimum value of U_B and insist that it be positive. The minimum is determined by

$$\frac{dU_B}{df} = \frac{1}{2}I_2^2\left(2L_1 f + 2M\right) = 0,$$

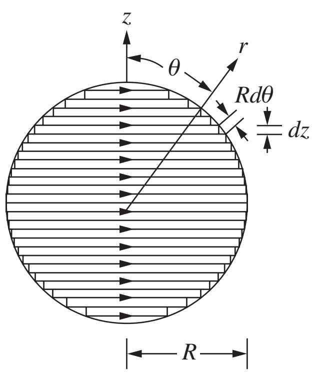
or $f = -M/L_1$. Therefore,

$$U_B(\min) = \frac{1}{2}I_2^2\left(\frac{M^2}{L_1} - 2\frac{M^2}{L_1} + L_2\right) = \frac{1}{2}I_2^2\left(-\frac{M^2}{L_1} + L_2\right) \geq 0.$$

We conclude that $M^2 \leq L_1 L_2$, as advertised.

12.25 The Self-Inductance of a Spherical Coil

The diagram shows the geometry of the winding.



It will be simplest to compute the flux of the winding through itself. The density of the winding is $N/2R$. Therefore, the number of turns in a linear distance dz is $(N/2R)$. Because $dz = -\sin\theta R d\theta$, the number of turns along the arc length $Rd\theta$ is $(N/2R)\sin\theta$ and the surface current density is

$$\mathbf{K} = \frac{NI}{2R}\sin\theta\hat{\boldsymbol{\phi}}.$$

Inside and outside the coil, we can write $\mathbf{B} = -\nabla\psi$ where ψ satisfies Laplace's equation. The matching conditions for the magnetic field,

$$\hat{\mathbf{r}} \cdot \mathbf{B}^{\text{out}} = \hat{\mathbf{r}} \cdot \mathbf{B}^{\text{in}} \qquad \text{and} \qquad \hat{\mathbf{r}} \times (\mathbf{B}^{\text{out}} - \mathbf{B}^{\text{in}}) = \mu_0 \mathbf{K},$$

imply the matching conditions for the potential,

$$\left[\frac{\partial\psi^{\text{in}}}{\partial r} - \frac{\partial\psi^{\text{out}}}{\partial r}\right]_{r=R} = 0 \qquad \text{and} \qquad \left[\frac{\partial\psi^{\text{in}}}{\partial\theta} - \frac{\partial\psi^{\text{out}}}{\partial\theta}\right]_{r=R} = \frac{1}{2}\mu_0 NI\sin\theta.$$

The second of these tells us that we need only the solutions of Laplace's equation which are proportional to $\cos\theta$, namely

$$\psi^{\text{in}}(r,\theta) = C\frac{r}{R}\cos\theta \qquad \text{and} \qquad \psi^{\text{out}}(r,\theta) = A\left(\frac{R}{r}\right)^2\cos\theta.$$

We find immediately that

$$A = \frac{1}{6}\mu_0 NI \qquad \text{and} \qquad C = -\frac{1}{3}\mu_0 NI.$$

In particular, the magnetic field inside the sphere is uniform:

$$\mathbf{B}(r < R) = \frac{1}{3R}\mu_0 NI\hat{\mathbf{z}}.$$

The magnetic flux through through an arc length $Rd\theta$ is

$$d\Phi = \frac{\text{flux}}{\text{turn}} \times \frac{\text{turn}}{\text{arc length}} \times \text{arc length} = \pi(R\sin\theta)^2 B \times \frac{N}{2R}\sin\theta \times Rd\theta.$$

Therefore, the self-inductance of the winding is

$$L = \frac{1}{I}\int d\Phi = \frac{\mu_0}{6}\pi R N^2 \int_0^\pi d\theta \, \sin^3\theta = \frac{2}{9}\mu_0 \pi R N^2.$$

Chapter 13: Magnetic Matter

13.1 The Magnetic Field of an Ideal Solenoid

Fill the solenoid with a uniform distribution of magnetization $\mathbf{M} = M\hat{\mathbf{z}}$. The surface normal $\mathbf{n}$ is everywhere perpendicular to $\mathbf{M}$. Therefore, there is no effective magnetic charge anywhere and $\mathbf{H} = 0$ everywhere. Then, because $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M})$, we get $\mathbf{B} = \mu_0 M\hat{\mathbf{z}}$ inside the solenoid and $\mathbf{B} = 0$ outside the solenoid. On the other hand, the effective surface current associated with the magnetization is $\mathbf{K} = \mathbf{M} \times \mathbf{n}$, which reproduces the imposed azimuthal current. Hence, $M = K$ and we reproduce the Biot-Savart result.

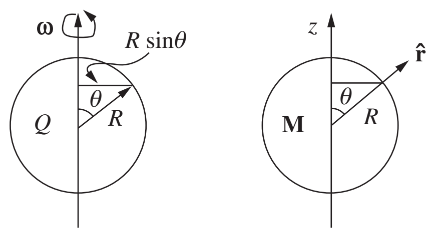
13.2 Equal and Opposite Magnetization

(a) There is no free current. The magnetization in each region is uniform so the bulk magnetization current density $\mathbf{j}_{\mathrm{M}} = \nabla \times \mathbf{M} = 0$. The magnetization is normal to the $z = 0$ interface so the surface magnetization current density $\mathbf{K} = \mathbf{M} \times \hat{\mathbf{n}} = 0$. There is no source current of any kind, so $\mathbf{B} = 0$ everywhere.

(b) There is no bulk magnetic charge $\rho^* = -\nabla \cdot \mathbf{M}$ but there is a surface charge density $\sigma^* = \mathbf{M} \cdot \hat{\mathbf{n}}$. There is a contribution $\sigma = M$ at $z = 0$ due to the $z > 0$ region. An identical contribution comes from the $z < 0$ region. Therefore, since an outward-pointing electric field $E = \sigma/2\epsilon_0$ is created by a planar surface density of electric charge σ, we get an outward-pointing field $H = M$ in this case. Since $\mathbf{M}$ points inward to the same interface, we conclude that $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = 0$ everywhere.

13.3 Equivalent Currents

(a) The statement will be true if the two spheres produce exactly the same distributions of current. Schematically, the two spheres are as follows.



The magnetized sphere produces no volume current density because $\mathbf{j}_{\mathrm{M}} = \nabla \times \mathbf{M}$. However, it does produce an azimuthal surface current density

$$\mathbf{K} = \mathbf{M} \times \hat{\mathbf{n}} = \mathbf{M} \times \hat{\mathbf{r}} = M \sin\theta\hat{\boldsymbol{\phi}}.$$

If $\boldsymbol{\omega}$ is the angular velocity, the velocity of a point on the surface of the rotating sphere is $\boldsymbol{v} = \boldsymbol{\omega} \times \mathbf{r}$. Therefore, the rotating sphere produces a surface current density

$$\mathbf{K} = \sigma\boldsymbol{v} = \frac{Q}{4\pi R^2}\omega R \sin\theta\hat{\boldsymbol{\phi}}.$$

The two are the same with the choice $M = \omega Q/4\pi R$.

(b) The magnetic moment of any object with magnetization $\mathbf{M}$ is

$$\mathbf{m} = \int_V d^3r\, \mathbf{M}(\mathbf{r}).$$

For our sphere,

$$\mathbf{m} = MV\hat{\mathbf{z}} = \frac{\omega Q}{4\pi R}\frac{4\pi R^3}{3} = \frac{1}{3}\omega Q R^2 \hat{\mathbf{z}}.$$

13.4 The Helmholtz Theorem for M

(a) The Helmholtz theorem says that

$$\mathbf{M}(\mathbf{r}) = -\nabla \int \frac{d^3r'}{4\pi}\frac{\nabla' \cdot \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \nabla \times \int \frac{d^3r'}{4\pi}\frac{\nabla' \times \mathbf{M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

But $\rho^*(\mathbf{r}) = -\nabla \cdot \mathbf{M}(\mathbf{r})$ is the fictitious magnetic charge density which enters the magnetic scalar potential

$$\psi_{\rm M}(\mathbf{r}) = \frac{1}{4\pi}\int d^3r' \,\frac{\rho^*(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}$$

and $\mathbf{j}_{\rm M}(\mathbf{r}) = \nabla \times \mathbf{M}(\mathbf{r})$ is the effective current density of magnetized matter that enters the magnetic vector potential

$$\mathbf{A}_{\rm M}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int d^3r' \,\frac{\mathbf{j}_{\rm M}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

Hence, because $\mathbf{H}_{\rm M} = -\nabla\psi_{\rm M}$ and $\mathbf{B}_{\rm M} = \nabla \times \mathbf{A}_{\rm M}$, the Helmholtz representation of $\mathbf{B}$ has the anticipated form,

$$\mathbf{M} = -\mathbf{H}_{\rm M} + \mathbf{B}_{\rm M}/\mu_0.$$

(b) The magnetization of the stripes satisfies $\nabla \times \mathbf{H}_{\rm M} = 0 = \nabla \cdot \mathbf{H}_{\rm M}$ because $\nabla \cdot \mathbf{M} = 0$. Therefore, by the Helmholtz theorem, $\mathbf{H}_{\rm M} = 0$ everywhere and $\mathbf{B} = \mu_0 \mathbf{M}$ everywhere. The magnetization current density has a surface piece $\mathbf{K}_{\rm M} = \mathbf{M} \times \hat{\mathbf{n}}$ which is solenoidal around each uniform block of magnetization.

13.5 The Virtues of Magnetic Charge

(a) The text establishes that $\mathbf{m} = \int d^3r\, \mathbf{M}$. On the other hand, using the proposed formula, the $k^{\rm th}$ component of the magnetic dipole moment of the sample is

$$m_k = -\int d^3r\, r_k \,\nabla \cdot \mathbf{M} = -\int d^3r\, \nabla \cdot (\mathbf{M}\, r_k) + \int d^3r\, (\mathbf{M} \cdot \nabla) r_k = \int d^3r\, M_k.$$

(b) By definition, the interaction energy between two current distributions is

$$\hat{V}_B = -\frac{\mu_0}{4\pi}\int d^3r \int d^3r' \frac{\mathbf{j}_1(\mathbf{r})\cdot\mathbf{j}_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}.$$

Using the definition of the vector potential in the Coulomb gauge, this is

$$\hat{V}_B = -\int d^3r\,\mathbf{j}_1\cdot\mathbf{A}_2 = -\int d^3r\,\mathbf{A}_2\cdot\nabla\times\mathbf{M}_1 = \int d^3r\,\nabla\cdot(\mathbf{A}_2\times\mathbf{M}_1) - \int d^3r\,\mathbf{M}_1\cdot\nabla\times\mathbf{A}_2.$$

Finally, using the divergence theorem and the fact that $\mathbf{M}_1$ is zero on the integration surface at infinity, we conclude that

$$\hat{V}_B = -\int d^3r\,\mathbf{M}_1\cdot\mathbf{B}_2.$$

Precisely the same steps beginning with $\hat{V}_B = -\int d^3r\,\mathbf{j}_2\cdot\mathbf{A}_1$ establish the reciprocity relation.

(c) It is simplest to begin with the proposed formula and show that it is equivalent to the expression derived in part (b). Then, because $\mathbf{B}_2 = \mu_0\mathbf{H}_2$ in the part of space where $\mathbf{M}_1 \neq 0$,

$$\begin{aligned}
\hat{V}_B &= \frac{\mu_0}{4\pi}\int d^3r \int d^3r'\,\frac{\nabla\cdot\mathbf{M}_1(\mathbf{r})\nabla'\cdot\mathbf{M}_2(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\
&= \frac{\mu_0}{4\pi}\int d^3r'\,\nabla'\cdot\mathbf{M}_2(\mathbf{r}')\int d^3r\,\left\{\nabla\cdot\left[\frac{\mathbf{M}_1(\mathbf{r})}{|\mathbf{r}-\mathbf{r}'|}\right] - \mathbf{M}_1(\mathbf{r})\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right\} \\
&= \int d^3r\,\mathbf{M}_1(\mathbf{r})\cdot\nabla\,\frac{\mu_0}{4\pi}\int d^3r'\,\frac{\rho_2^*(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\int d^3r\,\mathbf{M}_1(\mathbf{r})\cdot\mu_0\mathbf{H}_2(\mathbf{r}) \\
&= -\int d^3r\,\mathbf{M}_1(\mathbf{r})\cdot\mathbf{B}_2(\mathbf{r}).
\end{aligned}$$

13.6 Atom Optics with Magnetic Recording Tape

(a) For an infinitely wide tape, the magnetic field due to $\mathbf{M}(x)$ is equivalent to the field produced by the surface magnetization current densities

$$\mathbf{K}_\pm(x) = \mathbf{M}\times\hat{\mathbf{n}}_\pm = \pm M\cos kx\,\hat{\mathbf{z}},$$

where the upper (lower) sign refers the upper (lower) surface of the tape. Let $\mathbf{B}_+$ be the field produced by the upper surface ($y = 0$) and let $\mathbf{B}_-$ be the field produced by the lower surface ($y = -t$).

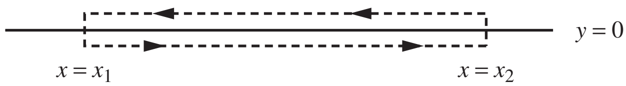
Away from $y = 0$, the field $\mathbf{B}_+$ produced by the upper surface current at $y = 0$ is derivable from a magnetic scalar potential. That is, $\mathbf{B}_+(x,y) = -\nabla\psi_+(x,y)$ is the magnetic field produced by $\mathbf{K}_+(x)$. $\psi_+(x,y)$ is periodic in x (because the source current is periodic in x) and satisfies Laplace's equation away from $y = 0$. Moreover, $\psi_+(x,y) \to 0$ as $y \to \infty$. Separation of variables in Cartesian coordinates then gives $\psi_+(x, y > 0) = A\sin(kx + \phi)e^{-ky}$. Hence,

$$\mathbf{B}_+(x, y > 0) = -\nabla\psi = Ake^{-ky}\left[\hat{\mathbf{y}}\sin(kx+\phi) - \hat{\mathbf{x}}\cos(kx+\phi)\right].$$

From our discussion of the symmetry of a current sheet, we know that $B_y(x,-y) = B_y(x,y)$ because the normal component of $\mathbf{B}$ must be continuous when we cross through the surface of the source current. Therefore,

$$\mathbf{B}_+(x,y) = Ake^{-k|y|}\left[\hat{\mathbf{y}}\sin(kx+\phi) - \hat{\mathbf{x}}sgn(y)\cos(kx+\phi)\right].$$

To evaluate the constants A and ϕ, we use the Ampèrian loop sketched below.



Using the definition of surface current density, this gives

$$\oint \mathbf{B}_+ \cdot d\mathbf{s} = 2kA\int_{x_1}^{x_2} dx\cos(kx+\phi) = \mu_0 I_{enclosed} = \mu_0\int_{x_1}^{x_2} dx\, M\;\cos kx.$$

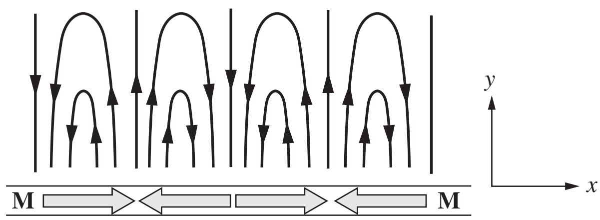
We conclude that

$$\mathbf{B}_+(x, y > 0) = \tfrac{1}{2}\mu_0\, M\,\left[\hat{\mathbf{y}}\sin kx - \hat{\mathbf{x}}\cos kx\right]e^{-ky}.$$

The magnetic field $\mathbf{B}_-(x,y)$ due to the current $\mathbf{K}_-(x)$ on the lower surface of the tape is identical except that it has the opposite sign and the origin is shifted to $y = -t$. The total magnetic field is therefore

$$\mathbf{B}(x, y > 0) = \mathbf{B}_+ + \mathbf{B}_- = \tfrac{1}{2}\mu_0\, M(1-e^{-kt})\;\left[\hat{\mathbf{y}}\sin kx - \hat{\mathbf{x}}\cos kx\right]e^{-ky}.$$

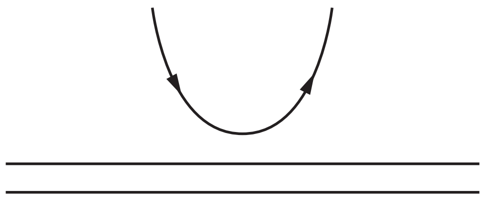
(b) The field line pattern is as follows.



(c) The interaction potential energy is $\hat{V}_B = -\mathbf{m}\cdot\mathbf{B}$. For permanent anti-alignment between the moment and the field, this gives

$$\hat{V}_B = m|\mathbf{B}| = \tfrac{1}{2}\mu\mu_0\, M(1-e^{-kt})e^{-ky}.$$

This potential is purely repulsive as the atom approaches from above. Moreover, $\hat{V}_B = \hat{V}_B(y)$. Therefore, the magnetic tape acts like a flat mirror and reflects the atom as sketched below.



Source: E.A. Hinds and I.G. Hughes, *Journal of Physics D: Applied Physics* **32**, R119 (1999).

13.7 Bitter's Iron Magnet

(a) By symmetry, it is sufficient to let $\hat{\mathbf{r}}$, $\hat{\mathbf{m}}$, and $\hat{\mathbf{z}}$ be coplanar. The z-component of the dipole field is

$$\begin{aligned} B_z(0) &= \frac{\mu_0}{4\pi}\frac{3(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}})(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}\cdot\hat{\mathbf{z}}}{r^3} \\ &= \frac{\mu_0 m}{4\pi}\frac{3\cos\theta\cos\alpha - \cos(\theta+\alpha)}{r^3} \\ &= \frac{\mu_0 m}{4\pi}\frac{2\cos\theta\cos\alpha + \sin\theta\sin\alpha}{r^3}. \end{aligned}$$

The minimum corresponds to $\partial B_z(0)/\partial\alpha = 0$ or $2\tan\alpha = \tan\theta$ as required.

(b) From the geometry, it is easy to see that $\tan\alpha = r d\theta/dr$ so $\frac{1}{2}\tan\theta = r d\theta/dr$. This can be written in the form

$$\frac{dr}{r} = 2\frac{\cos\theta}{\sin\theta}d\theta = 2\frac{d(\sin\theta)}{\sin\theta} \quad\Rightarrow\quad \ln r = \ln\sin^2\theta + const.$$

Hence, the spin directions which yield the maximal field obey $r = K\sin^2\theta$. This is exactly the equation of the field lines for a z-oriented point dipole at the origin.

(c) We want to substitute $2\tan\alpha = \tan\theta$ into

$$B_z(0) = \frac{\mu_0 m}{4\pi}\frac{2\cos\theta\cos\alpha + \sin\theta\sin\alpha}{r^3}.$$

Using

$$\cos\alpha = \frac{1}{\sqrt{1+\tan^2\alpha}} \qquad\qquad \sin\alpha = \frac{1}{\sqrt{1+\cot^2\alpha}}$$

we get

$$B_z(0) = \frac{\mu_0 m}{4\pi r^3}\frac{4\cos\theta + \sin\theta\tan\theta}{\sqrt{4+\tan^2\theta}} = \frac{\mu_0 m}{4\pi r^3}\frac{3\cos^2\theta + 1}{\cos\theta\sqrt{4+\tan^2\theta}} = \frac{\mu_0 m}{4\pi r^3}\sqrt{1+3\cos^2\theta}$$

as the magnetic field per spin. So, if there are N spins/volume, the total magnetic field at the origin is

$$B_z(0) = \frac{\mu_0 m N}{4\pi}\int d^3r\,\frac{\sqrt{1+3\cos^2\theta}}{r^3}.$$

For a spherical shell, $d^3r = 2\pi r^2\sin\theta d\theta$ so

$$B_z(0) = \frac{\mu_0 m N}{2}\int_{r_1}^{r_2}\frac{dr}{r}\int_{-1}^{1}dx\sqrt{1+3x^2} \approx \frac{5}{3}\mu_0 m N\ln\frac{r_2}{r_1}.$$

Source: F. Bitter, *Review of Scientific Instruments* **7**, 479 (1936).

13.8 Einstein Errs!

(a) We showed in Chapter 9 that the currents which flow on the surfaces of the can have densities

$$\mathbf{K}_\pm = \pm\frac{I}{2\pi\rho}\hat{\boldsymbol{\rho}} \qquad \text{and} \qquad K_W = -\frac{I}{2\pi R}\hat{\mathbf{z}}.$$

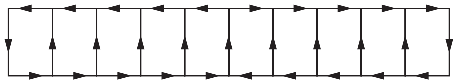
The force of levitation is

$$\mathbf{F} = \int_{\text{top cap}} dS\,\mathbf{K}_+ \times \mathbf{B}_\text{M} = \int_0^{2\pi} d\phi\int_0^R d\rho\rho\frac{I}{2\pi\rho}\hat{\boldsymbol{\rho}}\times\mu_0 M\hat{\boldsymbol{\phi}} = IR\mu_0 M\hat{\mathbf{z}}.$$

(b) The volume magnetization current density is

$$\mathbf{j} = \nabla\times\mathbf{M} = \nabla\times(M\hat{\boldsymbol{\phi}}) = \frac{M}{\rho}\hat{\mathbf{z}}.$$

The surface magnetization current density is $\mathbf{K} = \mathbf{M}\times\hat{\mathbf{n}}$. There are three surfaces to this finite-thickness end cap. On the top and bottom of the coin, $\mathbf{K} = M\hat{\boldsymbol{\phi}}\times\pm\hat{\mathbf{z}} = \pm M\hat{\boldsymbol{\rho}}$. On the edge, $\mathbf{K} = M\hat{\boldsymbol{\phi}}\times\hat{\boldsymbol{\rho}} = -M\hat{\mathbf{z}}$. All three are sketched below in side view.



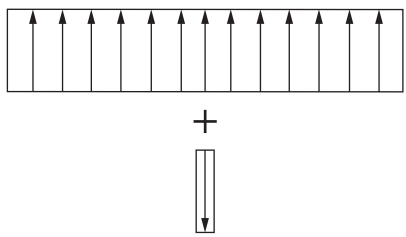
(c) For a finite thickness end-cap, the magnetic field of the can-of-current must fall to zero smoothly from $\mathbf{B}_{\text{in}}$ at the bottom surface of the end-cap to zero at the top surface of the end-cap. This shows that the magnetization current density on the top surface never feels a force from $\mathbf{B}_{\text{in}}$. Moreover, the force density is purely radial, both on the volume magnetization current and on the surface magnetization current on the side wall of the end-cap. Both integrate to zero. This leaves only the magnetization current density $\mathbf{K}_b$ on the bottom surface of the top end-cap. As anticipated, the force exerted on this current density cancels the force computed in part (a):

$$\mathbf{F} = \int_{\text{top cap}} dS\,\mathbf{K}_b \times \mathbf{B}_{\text{in}} = 2\pi \int_0^R d\rho\rho[-M\hat{\boldsymbol{\rho}}] \times \frac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\phi}} = -IR\mu_0 M\hat{\mathbf{z}}.$$

Source: A. Einstein, *Archives des Sciences Physiques et Naturelles* **30**, 323 (1910).

13.9 A Hole Drilled through a Permanent Magnet

We treat the drilled-out magnetic slab as a pristine magnetic slab superposed with a narrow cylinder with opposite magnetization:



The magnetic charge picture gives the **H** field of the pristine slab as isomorphic to the **E** field of a capacitor: $\mathbf{H}_{\text{out}} = 0$ and $\mathbf{H}_{\text{in}} = -M\hat{\mathbf{z}}$. Therefore, the pristine slab produces $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) = 0$ everywhere. The **H** field produced by the narrow rod is equivalent to the **E** field produced by a charge $q^\star = M\pi R^2$ at its bottom and a second charge $-q^\star$ at is top, i.e., the field of a finite dipole. Therefore, with respect to an origin at the center of the rod, the **H** field at every point in space is

$$\mathbf{H}(\mathbf{r}) = \frac{MR^2}{4}\left[\frac{\mathbf{r} + (t/2)\hat{\mathbf{z}}}{|\mathbf{r} + (t/2)\hat{\mathbf{z}}|^3} - \frac{\mathbf{r} - (t/2)\hat{\mathbf{z}}}{|\mathbf{r} - (t/2)\hat{\mathbf{z}}|^3}\right].$$

The corresponding magnetic field is

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \mu_0\mathbf{H}(\mathbf{r}) - \mu_0 M\hat{\mathbf{z}} & \text{inside the cylindrical hole} \\ \\ \mu_0\mathbf{H}(\mathbf{r}) & \text{outside the cylindrical hole and outside the slab.} \end{cases}$$

Source: E.B. Moullin, *The Principles of Electromagnetism* (Clarendon, Oxford, 1950).

13.10 The Demagnetization Factor for an Ellipsoid

To find N_{zz}, we begin with the equation for the surface of such an ellipsoid,

$$\frac{z^2}{a^2} + \frac{\rho^2}{b^2} = 1. \tag{1}$$

Next, since $\mathbf{M}$ points along $\hat{\mathbf{z}}$, we can think about a disk of area A and thickness dz with magnetic moment per unit volume (magnetization) $M = dm/dV$. Ampère's theorem permits us to parameterize the magnetic moment as $dm = dIA$, where dI is an effective current that circulates around the perimeter of the disk. Hence,

$$Mdz = \frac{dm}{dV}dz = \frac{dm}{A} = dI.$$

We conclude that the vector potential—and thus the magnetic field—produced by the shaded slice in the figure is identical to the vector potential produced by a ring with radius ρ that carries a current $dI = Mdz$. Since $\mathbf{B}_{\rm M}$ is uniform, it is sufficient to use the Biot-Savart result for a current ring to evaluate B_z at the center of the ellipsoid. Summing this field over all slices gives

$$B_z(0) = \frac{\mu_0 M}{2} \int\limits_{-a}^{a} dz \, \frac{\rho^2}{(\rho^2 + z^2)^{3/2}}. \tag{2}$$

The integral (2) takes a standard form if we use (1) to eliminate ρ, and the definition $\epsilon^2 = 1 - b^2/a^2$ of the eccentricity of an ellipsoid. The result is

$$B_z(0) = \mu_0 M(1 - N_{zz}),$$

where

$$N_{zz} = \frac{1-\epsilon^2}{\epsilon^2}\left[\frac{1}{2\epsilon}\ln\left(\frac{1+\epsilon}{1-\epsilon}\right) - 1\right] \qquad (\epsilon < 1). \tag{3}$$

A Taylor series expansion of the logarithm confirms that $N_{zz} \to 1/3$ in the spherical limit ($\epsilon \to 0$). The demagnetization factors for needle-shaped samples and disk-shaped samples can be derived from (3) and compared with more direct calculations of $\mathbf{H}_{\rm M}$ for these geometries.

Source: C. Birch, *European Journal of Physics* **6**, 180 (1985).

13.11 Lunar Magnetism

We have $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$, where $\mathbf{H} = -\nabla\psi$ and ψ satisfies the Poisson-like equation

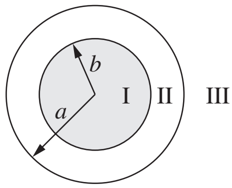
$$\nabla^2\psi = \nabla \cdot \mathbf{M}.$$

In addition, at the boundary between regions, it is necessary to satisfy the matching conditions

$$\psi_1(\mathbf{r}_S) = \psi_2(\mathbf{r}_S)$$

and
$$\left[\frac{\partial\psi_1}{\partial n_1} - \frac{\partial\psi_2}{\partial n_1}\right]_S = [\mathbf{M}_1 - \mathbf{M}_2]_S \cdot \hat{\mathbf{n}}_1.$$

We call the core, crust, and exterior of the Moon regions I, II, and III, respectively, as shown below.



The impressed magnetization $\mathbf{M}$ of the core is stated to be proportional to a dipole field $\mathbf{B}_{\rm d}$ centered at the origin. If we align the magnetic moment $\mathbf{m}$ with the z-axis,

$$\mathbf{B}_{\rm d}(r,\theta) = \frac{\mu_0 m}{4\pi}\frac{3\cos\theta\hat{\mathbf{r}} - \hat{\mathbf{z}}}{r^3} = \frac{\mu_0 m}{4\pi}\frac{2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}}{r^3}.$$

Since $\nabla\cdot\mathbf{B} = 0$, we know that $\nabla\cdot\mathbf{M} = 0$ and the magnetic scalar potential above satisfies Laplace's equation everywhere. Specifically,

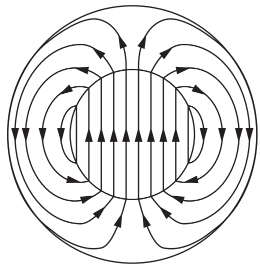
$$\begin{aligned}
\psi_{\rm I} &= D\left(\frac{r}{b}\right)\cos\theta \\
\psi_{\rm II} &= \left[B\left(\frac{a}{r}\right)^2 + C\left(\frac{r}{a}\right)\right]\cos\theta \\
\psi_{\rm III} &= A\left(\frac{a}{r}\right)^2\cos\theta.
\end{aligned}$$

Applying the matching conditions, noting that $\hat{\mathbf{n}} = \hat{\mathbf{r}}$ and that the only non-zero magnetization is
$$\mathbf{M}_{\rm II} = M\frac{2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}}{r^3},$$
gives

$$\begin{aligned}
A &= B + C \\
D &= B\frac{a^2}{b^2} + C\frac{b}{a} \\
\frac{2M}{a^3} &= -\frac{2B}{a} + \frac{C}{a} + \frac{2A}{a} \\
-\frac{2M}{b^3} &= \frac{D}{b} + 2B\frac{a^2}{b^3} - \frac{C}{a}.
\end{aligned}$$

It is straightforward to check that this system is solved by
$$A = 0 \qquad C = -B = \frac{2M}{3a^2} \qquad D = B\left[\frac{a^2}{b^2} - \frac{b}{a}\right],$$

which confirms that $\mathbf{H} = \mathbf{B} = 0$ in region III outside the Moon. We can sketch $\mathbf{B}$ inside the Moon using the fact that $\mathbf{B}_{\rm I} = \mathbf{H}_{\rm I} = -\nabla\psi_{\rm I}$ is constant, the lines of $\mathbf{B}$ must form closed loops, and $\mathbf{B}$ must be tangent to the sphere at $r = b$ because its radial component is continuous there.



Source: S.K. Runcorn, *Physics of the Earth and Planetary Interiors* **10**, 327 (1975).

13.12 A Dipole in a Magnetizable Sphere

We will use the magnetic scalar potential and write $\mathbf{H} = -\nabla\psi(\mathbf{r})$. The potential satisfies Laplace's equation $\nabla^2\psi = 0$ everywhere except at the origin where, due to the presence of the dipole with moment $\mathbf{m} = m\hat{\mathbf{z}}$,

$$\lim_{r\to 0}\psi(r,\theta) = \frac{\mu_0}{4\pi}\frac{m\cos\theta}{r^2}. \tag{1}$$

There is no free current, so the matching conditions at the $r = R$ boundary are

$$\psi_{\rm in}(\mathbf{r}_S) = \psi_{\rm out}(\mathbf{r}_S) \qquad \text{and} \qquad \mu\left.\frac{\partial\psi_{\rm in}}{\partial r}\right|_{r=R} = \mu_0\left.\frac{\partial\psi_{\rm out}}{\partial r}\right|_{r=R}.$$

Given (1), the matching conditions will be satisfied only if the potential varies everywhere as $\cos\theta$. Therefore, since the contributions to the potential other than (1) satisfy Laplace's equation:

$$\psi(r,\theta) = \begin{cases} \dfrac{\mu_0}{4\pi}\left[Ar + \dfrac{m}{r^2}\right]\cos\theta & r < R, \\[2ex] \dfrac{\mu_0}{4\pi}\dfrac{M\cos\theta}{r^2} & r > R. \end{cases}$$

Direct application of the matching conditions gives

$$M = \frac{3\mu}{\mu + 2\mu_0}m \qquad \text{and} \qquad A = \frac{2(\mu - \mu_0)}{\mu + 2\mu_0}\frac{m}{R^3}.$$

Therefore, with $\mathbf{M} = M\hat{\mathbf{z}}$,

$$\mathbf{H}(\mathbf{r}) = \begin{cases} \dfrac{\mu_0}{4\pi}\dfrac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} - \dfrac{\mu_0\mathbf{M}}{4\pi} & r < R, \\[2ex] \dfrac{\mu_0}{4\pi}\dfrac{3(\mathbf{M}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} & r > R. \end{cases}$$

13.13 Magnetic Shielding

We use a magnetic scalar potential where $\mathbf{H} = -\nabla\psi$. There is no free current, and the problem is two-dimensional, so

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(r\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} = 0.$$

By standard separation of variables, the general solution is a superposition of terms of the form

$$\psi(\rho,\theta) = (A_n \cos n\phi + B_n \sin n\phi)(C_n\rho^n + D_n\rho^{-n}).$$

Inside the shell, the solution must be finite and reflect the symmetry of the external field. Since $B_{\rm ext} = \mu_0 H_{\rm ext}$ and $\psi_{\rm ext} = -H_{\rm ext}x = -H_{\rm ext}\rho\cos\phi$,

$$\psi_{\rm in} = A\rho\cos\phi.$$

Within the shell, we have the slightly more general potential

$$\psi_{\rm shell} = (C\rho + D\rho^{-1})\cos\phi.$$

Outside the shell, the field must reduce to $\mathbf{B}_{\rm ext}$ as $\rho \to \infty$. Therefore,

$$\psi_{\rm out} = -H_{\rm ext}\rho\cos\phi + E\rho^{-1}\cos\phi.$$

The matching conditions are continuity for the normal component of $\mathbf{B}$ and continuity for the tangential component of $\mathbf{H}$. The latter is equivalent to the continuity of ψ itself. Applying these at $\rho = a$ gives

$$\left(\frac{\partial\psi_{\rm in}}{\partial\rho}\right)_{\rho=a} = \kappa\left(\frac{\partial\psi_{\rm shell}}{\partial\rho}\right)_{\rho=a} \qquad \text{and} \qquad \psi_{\rm in}\big|_{\rho=a} = \psi_{\rm shell}\big|_{\rho=a}$$

or

$$A = \kappa\left(C - \frac{D}{a^2}\right) \qquad \text{and} \qquad Aa = Ca + \frac{D}{a}.$$

The matching conditions at $\rho = b$ are

$$\left(\frac{\partial\psi_{\rm out}}{\partial\rho}\right)_{\rho=a} = \kappa\left(\frac{\partial\psi_{\rm shell}}{\partial\rho}\right)_{\rho=b} \qquad \text{and} \qquad \psi_{\rm out}\big|_{\rho=a} = \psi_{\rm shell}\big|_{\rho=b}$$

or

$$-H_{\rm ext} = \frac{E}{b^2} = \kappa\left(C - \frac{D}{b^2}\right) \qquad \text{and} \qquad -H_{\rm ext} + \frac{E}{b} = Cb + \frac{D}{b}.$$

From the matching conditions at $\rho = a$, we deduce that

$$\frac{C}{D} = \frac{\kappa+1}{\kappa-1}\frac{1}{a^2} \quad\Rightarrow\quad \frac{A}{D} = \frac{\kappa}{\kappa+1}\frac{2}{a^2} \quad\Rightarrow\quad \frac{A}{C} = \frac{2\kappa}{\kappa+1}. \tag{1}$$

Eliminating E from the matching conditions at $\rho = b$ gives

$$D = \frac{2H_{\rm ext} + (\kappa+1)C}{\kappa-1}b^2.$$

Substituting this into the expression for C/D in (1) gives

$$C\left\{1 - \frac{b^2}{a^2}\left(\frac{\kappa+1}{\kappa-1}\right)^2\right\} = 2H_{\rm ext}\frac{b^2}{a^2}\frac{\kappa+1}{(\kappa-1)^2}.$$

Using this to eliminate C from the expression for A/C in (1) gives

$$A = \frac{4\kappa b^2}{(\kappa-1)^2a^2 - (\kappa+1)^2b^2}H_{\rm ext}.$$

This gives the advertised result because

$$\mathbf{B}_{\rm in} = -\mu_0 A\hat{\mathbf{z}} = \frac{4\kappa b^2}{(\kappa+1)^2b^2 - (\kappa-1)^2a^2}\mathbf{B}_{\rm ext}.$$

13.14 The Force on a Current-Carrying Magnetizable Wire

(a) $\mathbf{B} = \mu_0(\mathbf{H}+\mathbf{M}) = \mu\mathbf{H}$ so $\mathbf{M} = (\mu/\mu+_0 -1)\mathbf{H}$. We find $\mathbf{H}$ from $\nabla\times\mathbf{H} = \mathbf{j}_0$. Ampère's law gives the particular solution

$$\mathbf{H}''_{\rm in} = \frac{j_0\rho}{2}\hat{\boldsymbol{\phi}}$$

$$\mathbf{H}''_{\rm out} = \frac{a^2 j_0}{2\rho}\hat{\boldsymbol{\phi}}$$

inside and outside the wire. The total field $\mathbf{H} = \mathbf{H}' + \mathbf{H}''$ where $\mathbf{H}'$ solves the homogeneous equation $\nabla\times\mathbf{H}' = 0$. Now $\nabla\cdot\mathbf{B} = 0$ so $\nabla\cdot\mathbf{H} = \nabla\cdot\mathbf{H}' = 0$ except at the wire surface. With $\mathbf{H}' = -\nabla\psi$ we solve $\nabla^2\psi = 0$ subject to the boundary condition $\mathbf{H}' = H_0\hat{\mathbf{x}}$ when $\rho\to\infty$ and the matching conditions

$$\hat{\boldsymbol{\phi}}\cdot\mathbf{H}'_{\rm in} = \hat{\boldsymbol{\phi}}\cdot\mathbf{H}'_{\rm out} \quad\Rightarrow \frac{\partial\psi_{\rm in}}{\partial\phi}\bigg|_{\rho=a} = \frac{\partial\psi_{\rm out}}{\partial\phi}\bigg|_{\rho=a}$$

$$\hat{\mathbf{r}}\cdot\mathbf{B}_{\rm in} = \hat{\mathbf{r}}\cdot\mathbf{B}_{\rm out} \quad\Rightarrow \mu\,\frac{\partial\psi_{\rm in}}{\partial r}\bigg|_{\rho=a} = \mu_0\,\frac{\partial\psi_{\rm out}}{\partial r}\bigg|_{\rho=a}.$$

The solution is

$$\mathbf{H}'_{\rm in} = \hat{\mathbf{x}}\frac{2\mu_0 H_0}{\mu+\mu_0}$$

$$\mathbf{H}'_{\rm out} = \hat{\mathbf{x}}H_0 + \hat{\boldsymbol{\rho}}\frac{\mu-\mu_0}{\mu+\mu_0}H_0\cos\phi + \hat{\boldsymbol{\phi}}\frac{\mu-\mu_0}{\mu+\mu_0}H_0\cos\phi.$$

Therefore,

$$\mathbf{M} = (\mu/\mu_0 - 1)\left(\frac{2\mu_0 H_0}{\mu+\mu_0}\hat{\mathbf{x}} + \frac{1}{2}\rho j_0\hat{\boldsymbol{\phi}}\right).$$

This gives a volume magnetization current density

$$\mathbf{j}_{\rm M} = \nabla\times\mathbf{M} = (\mu/\mu_0 - 1)j_0\hat{\mathbf{z}}$$

and (using $\hat{\mathbf{x}} = \hat{\boldsymbol{\rho}}\cos\phi - \hat{\boldsymbol{\phi}}\sin\phi$) a surface magnetization current density

$$\mathbf{K}_{\rm M} = \mathbf{M}\times\hat{\mathbf{n}} = \mathbf{M}\times\hat{\boldsymbol{\rho}} = (\mu/\mu_0 - 1)\left(\frac{2\mu_0 H_0}{\mu+\mu_0}\sin\phi - \tfrac{1}{2}\rho j_0\right)\hat{\mathbf{z}}.$$

(b) The total current density is $\mathbf{j} = \mathbf{j}_0 + \mathbf{j}_{\rm M} = (\mu/\mu_0)j_0\,\hat{\mathbf{z}}$. The volume force density is

$$\mathbf{f}_V = \mathbf{j}\times\mu\mathbf{H}_{\rm in} = \mu\frac{\mu j_0}{\mu_0}\hat{\mathbf{z}}\times\left(\frac{2\mu_0 H_0}{\mu+\mu_0}\hat{\mathbf{x}} + \tfrac{1}{2}\rho j_0\,\hat{\boldsymbol{\phi}}\right) = \frac{2\mu^2 H_0 j_0}{\mu+\mu_0}\hat{\mathbf{y}} - \frac{\mu^2 j_0\rho}{2\mu_0}\hat{\boldsymbol{\rho}}$$

because

$$\hat{\boldsymbol{\rho}} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\phi}} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}.$$

The volume integral of the second term is zero so the total volume force is

$$\mathbf{F}_V = \int_0^{2\pi} d\phi\int_0^a d\rho\,\rho\,\mathbf{f}_V = \frac{2\mu^2 H_0 I_0}{\mu+\mu_0}\hat{\mathbf{y}}.$$

The surface contribution to the force is $\mathbf{F}_S = a\int_0^{2\pi} d\phi\,\mathbf{K}_{\rm M}\times\bar{\mathbf{B}}$, where $\bar{\mathbf{B}} = \frac{1}{2}[\mathbf{B}_{\rm in}(a) + \mathbf{B}_{\rm out}(a)]$. Using the results above, this is

$$\bar{\mathbf{B}} = \tfrac{1}{2}[\mu\mathbf{H}_{\rm in}(a) + \mu_0\mathbf{H}_{\rm out}(a)] = \tfrac{1}{2}(\mu+\mu_0)\left[\tfrac{1}{2}\rho j_0 - \frac{2\mu_0\mathbf{H}_0}{\mu+\mu_0}\sin\phi\right]\hat{\boldsymbol{\phi}} + \frac{2\mu\mu_0 H_0}{\mu+\mu_0}\cos\phi\,\hat{\boldsymbol{\rho}}.$$

Many of the terms which could contribute to $\mathbf{F}_S$ integrate to zero. Those that do not give

$$\mathbf{F}_S = -I_0 H_0\frac{(\mu-\mu_0)(2\mu+\mu_0)}{\mu+\mu_0}\hat{\mathbf{y}}.$$

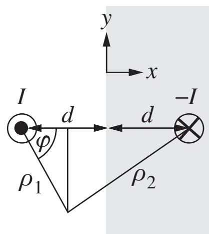
Because $\mathbf{B}_0 = \mu_0\mathbf{H}_0$, the total force on the wire is

$$\mathbf{F} = \mathbf{F}_V + \mathbf{F}_S = I_0 H_0\frac{2\mu^2 - (\mu-\mu_0)(2\mu+\mu_0)}{\mu+\mu_0}\hat{\mathbf{y}} = I_0 B_0\,\hat{\mathbf{y}}.$$

Source: F.N.H. Robinson, *Macroscopic Electromagnetism* (Pergamon, Oxford, 1973).

13.15 Active Magnetic Shielding

(a) The text shows that the boundary condition $B_x(x=0)=0$ is satisfied at the conductor surface by placing an image line current flowing in the $-z$-direction at $x=d$.



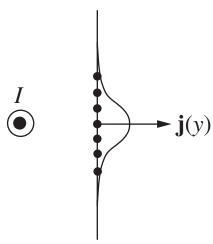
Therefore, in the $x<0$ half-space, the magnetic field is

$$\mathbf{B} = \frac{\mu_0 I}{2\pi\rho_1}\hat{\boldsymbol{\phi}} - \frac{\mu_0 I}{2\pi\rho_2}\hat{\boldsymbol{\phi}}.$$

Because $\mathbf{B}=0$ inside the conductor, the surface current density obeys the matching condition

$$\mu_0\mathbf{K} = \mathbf{B}(x=0)\times\hat{\mathbf{x}} = -2B_y(x=0)\hat{\mathbf{z}} = -\frac{\mu_0 I}{\pi}\frac{d}{y^2+d^2}\hat{\mathbf{z}}.$$

(b) If we remove the conductor, but want to shield $x>0$ from the effect of the wire in the $x<0$ space, we need only synthesize the surface current density $\mathbf{K}$ in part (a) because this guaranteed that $\mathbf{B}(x>0)=0$. If $N=7$, an arrangement like the one shown below will do, where the current in each wire is chosen equal to value $j(y)$ at the y-position of the wire. The larger N is, the better the shielding will be.



Source: P. Mansfield and B. Chapman, *Journal of Physics E* **19**, 540 (1986).

13.16 The Role of Interface Magnetization Current

The direct solution exploits the cylindrical symmetry of the problem and Ampère's law in integral form:

$$\oint d\mathbf{s} \cdot \mathbf{H} = I_f.$$

This gives $\mathbf{H} = (I_f/2\pi\rho)\hat{\boldsymbol{\phi}}$. Therefore, since $\mathbf{B} = \mu\mathbf{H}$ in linear matter,

$$\mathbf{B}(\rho) = \begin{cases} \dfrac{\mu_1 I_f}{2\pi\rho}\hat{\boldsymbol{\phi}} & \rho < R, \\ \\ \dfrac{\mu_1 I_f}{2\pi\rho}\hat{\boldsymbol{\phi}} & \rho > R. \end{cases} \tag{1}$$

For linear matter, the magnetization current is proportional to the free current: $\mathbf{j}_{\mathrm{M}} = \chi_m \mathbf{j}_f$. This gives a total current density

$$\mathbf{j} = \mathbf{j}_f + \mathbf{j}_{\mathrm{M}} = (1 + \chi_m)\mathbf{j}_f = (\mu/\mu_0)\mathbf{j}_f.$$

This means that the field produced by a wire embedded in a medium with permeability μ_1 *at every point in space* is

$$\mathbf{B}_I = \frac{\mu_1 I_f}{2\pi\rho}\hat{\boldsymbol{\phi}}. \tag{2}$$

This is not inconsistent with (1) because the total field is the sum of $\mathbf{B}_I$ and the field $\mathbf{B}_{\mathrm{M}}$ produced by all the (induced) magnetization current density in the system. The bulk piece of this current density, $\mathbf{j}_{\mathrm{M}}$, is zero for all linear matter. Using the continuity of the normal (radial) component of $\mathbf{H}$, the surface piece is

$$\mathbf{K} = \mathbf{M}_1 \times \hat{\mathbf{n}}_1 + \mathbf{M}_2 \times \hat{\mathbf{n}}_2 = (\mathbf{M}_1 - \mathbf{M}_2) \times \hat{\boldsymbol{\rho}} = [\chi_1 \mathbf{H}_1(R) - \chi_2 \mathbf{H}_2(R)] \times \hat{\boldsymbol{\phi}} = (\chi_2 - \chi_1)H(R)\hat{\boldsymbol{\phi}}.$$

The field $\mathbf{B}_{\mathrm{M}} = B_{\mathrm{M}}\hat{\boldsymbol{\phi}}$ due to this current is calculable by Ampère's law in the form

$$2\pi\rho B_{\mathrm{M}} = \oint d\mathbf{s} \cdot \mathbf{B}_{\mathrm{M}} = \mu_0 I_{\mathrm{enclosed}} = \begin{cases} 0 & \rho < a, \\ \\ \mu_0 2\pi R K & \rho > R. \end{cases}$$

This shows that (2) is the correct total field when $\rho < R$. This agrees with (1). Otherwise,

$$\begin{aligned} \mathbf{B}(\rho > R) &= \mathbf{B}_I + \mathbf{B}_{\mathrm{M}} \\ &= \frac{\mu_1 I_f}{2\pi\rho}\hat{\boldsymbol{\phi}} + \mu_0 \frac{R}{\rho}(\chi_2 - \chi_1)\frac{I}{2\pi R}\hat{\boldsymbol{\phi}} \\ &= \frac{I_f}{2\pi\rho}\left\{\mu_1 + \mu_0\left[\left(\frac{\mu_2}{\mu_0} - 1\right) - \left(\frac{\mu_1}{\mu_0} - 1\right)\right]\right\}\hat{\boldsymbol{\phi}} \\ &= \frac{\mu_2 I_f}{2\pi\rho}\hat{\boldsymbol{\phi}}. \end{aligned}$$

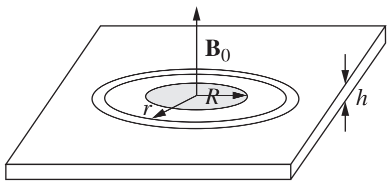
This agrees with (1) also.

Source: L. Egyes, *The Classical Electromagnetic Field* (Dover, New York, 1972).

13.17 Magnetic Film and Magnetic Disk

(a) By symmetry, the field inside the film must be in the same direction as $\mathbf{B}_0$. But $\mathbf{B}\cdot\hat{\mathbf{n}}$ is continuous everywhere. Therefore, $\mathbf{B}_{\rm in} = \mathbf{B}_0$.

(b) To sum the dipole fields produced by the matter for $r > R$, we let $d\mathbf{m} = \mathbf{M}_0 dV$, where $dV = 2\pi r dr h$ is the annular volume shown below.



The field due to the annulus at the center of the disk (where $d\mathbf{m}\cdot\hat{\mathbf{r}} = 0$ because $h \ll R$) is

$$d\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{3(d\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - d\mathbf{m}}{r^3} = -\frac{\mu_0}{4\pi}\frac{d\mathbf{m}}{r^3}.$$

Therefore, the field at the center of the disk is

$$\mathbf{B}(0) = \mathbf{B}_0 - \int d\mathbf{B} = \mathbf{B} + \frac{\mu_0 \mathbf{M}_0 h}{2}\int_R^\infty \frac{dr}{r^2} = \mathbf{B}_0 + \frac{\mu_0 \mathbf{M}_0}{2}\frac{h}{R}.$$

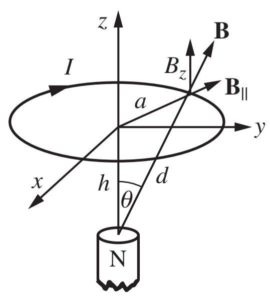
Now we need $\mathbf{M}_0$. For linear matter, $\mathbf{M} = \chi_m \mathbf{H}$ and $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M})$. Therefore,

$$\mathbf{M}_0 \frac{\mu - \mu_0}{\mu\mu_0}\mathbf{B}_0$$

and

$$\mathbf{B}(0) = \mathbf{B}_0 + \frac{\mu - \mu_0}{2\mu}\frac{h}{R}\mathbf{B}_0.$$

13.18 A Current Loop Levitated by a Bar Magnet



The stated conditions permit us to ignore the south pole of the bar magnet and treat the north pole as a point magnetic charge that produces a radial field (emanating from the pole) with magnitude $B = \mu_0 g/4\pi r^2$. The pole strength $g = \pi r^2 M$ where M is the magnetization.

The weight of the loop is $W = mg = 2\pi a\rho g$. This must be balanced against the magnetic force. The latter is the sum of the forces $d\mathbf{F} = Id\boldsymbol{\ell} \times \mathbf{B}$ on the line elements $d\boldsymbol{\ell}$ of the ring. The contributions from B_z cancel when integrated around the ring. The other component of the magnetic field contributes an upward force of equal strength from every element with magnitude

$$dF = Id\ell B_{\|} = Id\ell B\sin\theta = Id\ell\frac{\mu_0 g}{4\pi d^2}\left(\frac{a}{d}\right).$$

Therefore,

$$F = I(2\pi a)\frac{\mu_0\pi r^2 M}{4\pi d^2}\frac{a}{d} = mg = 2\pi a\rho g$$

and

$$M = \frac{4d^3\rho g}{I\mu_0 ar^2} = \frac{4(h^2+a^2)^{3/2}\rho g}{I\mu_0 ar^2}.$$

13.19 A Real Electromagnet

(a) The magnetic field on the z-axis of a coaxial ring at $z=0$ with radius R and current I is

$$\mathbf{B}_{\rm ring}(z) = \frac{1}{2}\mu_0 I\frac{R^2}{(R^2+z^2)^{3/2}}\hat{\mathbf{z}}.$$

Summing the field from a uniform distribution of such rings in the interval $-L/2 \leq z \leq L/2$ gives

$$\mathbf{B}_0(z) = \frac{1}{2}\mu_0\frac{NI}{L}\int\limits_{-L/2}^{L/2}\frac{dz'}{[R^2+(z-z')^2]^{3/2}}\hat{\mathbf{z}} = \frac{\mu_0 NI}{L}f(z)\hat{\mathbf{z}},$$

where

$$f(z) = \frac{1}{2}\left[\frac{z+L/2}{\sqrt{(z+L/2)^2+R^2}} - \frac{z-L/2}{\sqrt{(z-L/2)^2+R^2}}\right].$$

The ratio

$$\frac{B_0(\pm L/2)}{B_0(0)} = \frac{f(L/2)}{f(0)} = \frac{\sqrt{(L/2)^2+R^2}}{\sqrt{L^2+R^2}} \approx \frac{L/2}{L} = \frac{1}{2}$$

when $L \gg R$.

(b) Because $\mathbf{M} = \chi\mathbf{H}$, the field of interest is

$$\mathbf{B}(z) = \mu_0[\mathbf{H}(z)+\mathbf{M}(z)] = \mu_0(1+\chi_m)\mathbf{H}(z). \tag{1}$$

In this expression, $\mathbf{H}(z) = \mathbf{H}_f(z)+\mathbf{H}^*(z)$, where $\mathbf{H}_f(z) = \mathbf{B}_0(z)/\mu_0$ is the field produced by the free current of the solenoid coils and $\mathbf{H}^*(z)$ is the field produced by the magnetization of the rod. If we ignore the volume magnetic charge, the latter is the field from two disks of radius R at $z = \pm L/2$ with surface magnetic charge density $\pm\sigma^* = \pm\mathbf{M}|_S\cdot\hat{\mathbf{z}}$. A sensible approximation to compute the latter is

$$\mathbf{M}(\mathbf{r}) = M(z)\hat{\mathbf{z}} = \chi_m H(z)\hat{\mathbf{z}},$$

in which case

$$\sigma^* = \chi_m H(L/2). \tag{2}$$

To find $H^*(z)$, we recall that the electrostatic field on the symmetry axis of a disk with radius R and uniform surface charge density σ located at $z = 0$ is

$$\varphi(z) = \frac{1}{4\pi\epsilon_0}\int_0^R d\rho \frac{2\pi\rho\sigma}{\sqrt{R^2+z^2}} = \frac{\sigma}{2\epsilon_0}\left[\sqrt{R^2+z^2} - |z|\right].$$

The associated electric field is

$$\mathbf{E}(z) = -\nabla\varphi = \frac{\sigma}{2\epsilon_0}\frac{z}{|z|}\left[1 - \frac{|z|}{\sqrt{R^2+z^2}}\right]\hat{\mathbf{z}}.$$

Hence, for a magnetic disk at $z = 0$,

$$H_0(z) = \frac{\sigma^*}{2}\frac{z}{|z|}\left[1 - \frac{|z|}{\sqrt{R^2+z^2}}\right]\hat{\mathbf{z}}.$$

Our problem has a disk with charge σ^* at $z = L/2$ and a disk with charge $-\sigma^*$ at $z = -L/2$. A brief calculation shows that the superposition of the fields produced by these disks gives

$$\mathbf{H}^*(z) = \begin{cases} \sigma^* f(z)\hat{\mathbf{z}} & \text{outside,} \\ \sigma^*[f(z)-1]\hat{\mathbf{z}} & \text{inside.} \end{cases}$$

We conclude that the total auxiliary field inside the solenoid is

$$\mathbf{H}_{\text{in}}(z) = \mathbf{H}_f(z) + \mathbf{H}^*_{\text{in}}(z) = \frac{NI}{L}f(z)\hat{\mathbf{z}} + \sigma^*[f(z)-1]\hat{\mathbf{z}} = \frac{1}{\mu_0}\mathbf{B}_0(z) + \sigma^*[f(z)-1]\hat{\mathbf{z}}. \tag{3}$$

To find σ^*, we substitute (2) into (3) and evaluate the latter at $z = L/2$ to get

$$H_{\text{in}}(L/2) = \frac{NI}{L}f(L/2) + \chi_m H_{\text{in}}(L/2)[f(L/2)-1].$$

Because $f(L/2) \approx 1/2$ when $L \gg R$,

$$\sigma^* = \chi_m H_{\text{in}}(L/2) = \frac{NI}{L}\frac{\chi_m f(L/2)}{1 - \chi_m[f(L/2)-1]} \approx \frac{NI}{L}\frac{\chi_m}{\chi_m + 2}.$$

Using (3) to evaluate (1), our approximate expression for the magnetic field inside the solenoid is

$$\mathbf{B}(z) = (1+\chi_m)\left\{B_0(z) + \mu_0\sigma^*[f(z)-1]\right\}\hat{\mathbf{z}}.$$

(c) A bit of algebra shows that

$$\frac{B(z)}{B_0(z)} = \frac{\chi_m + 1}{\chi_m + 2}\left[2\chi_m + 2 - \frac{\chi_m}{f(z)}\right].$$

Because $f(0) \approx 1$ when $L \gg R$,

$$\frac{B(0)}{B_0(0)} \approx \chi_m + 1 \to \chi_m.$$

Similarly, because $f(L/2) \approx 1/2$,

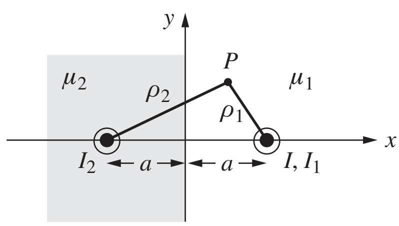
$$\frac{B(L/2)}{B_0(L/2)} \approx 2\frac{\chi_m + 1}{\chi_m + 2} \to 2.$$

Thus, the "freshman physics" amplification of the magnetic field occurs only far away from the ends of the solenoid. Near the ends, the effect of the demagnetization field is to limit the amplification to a factor of 2, even when $\chi_m \to \infty$.

Source: Prof. B.P. Tonner, University of Central Florida (private communication).

13.20 Vector Potential Approach to Image Currents

(a) The vector potential **A** is continuous at a boundary and the tangential component of **H** is continuous at a boundary. Mimicking electrostatics, we treat the field in medium 1 as produced by the current I plus a parallel image current I_2 in medium 2 at a perpendicular distance a from the interface. We treat the field in medium 2 as produced by an image current I_1 at the position of I.



The magnetic field produced by I at a point where the permeability is μ^* is $\mathbf{B} = \dfrac{\mu^* I}{2\pi\rho_1}\hat{\boldsymbol{\phi}}$. Because $\mathbf{B} = \nabla \times \mathbf{A}$, the curl information given says that the corresponding vector potential is

$$\mathbf{A} = -\hat{\mathbf{z}}\frac{\mu^* I}{2\pi}\ln\rho_1.$$

Accordingly, with $\mathbf{A} = A\hat{\mathbf{z}}$,

$$A = \begin{cases} -\dfrac{\mu_2 I_1}{2\pi}\ln\rho_1 & x < 0, \\ \\ -\dfrac{\mu_1 I}{2\pi}\ln\rho_1 - \dfrac{\mu_1 I_2}{2\pi}\ln\rho_2 & x > 0. \end{cases}$$

On the boundary, $\rho_1 = \rho_2$, so the matching condition for $\mathbf{A}$ is

$$\mu_2 I_1 = -\mu_1 I + \mu_1 I_2.$$

Since $\mathbf{B} = \mu\mathbf{H}$ and the tangential components involve the same trigonometric factors for every current, the matching condition for $\mathbf{H}$ gives

$$-I_1 = I + I_2.$$

Combining these two gives the image currents as

$$I_2 = \frac{\mu_2 - \mu_1}{\mu_2 + \mu_1} I \qquad \text{and} \qquad I_1 = \frac{2\mu_1}{\mu_2 + \mu_1} I.$$

(b) The force per unit length exerted on I may be computed from the magnetic field due to I_2. This has magnitude

$$F = IB_2 = \frac{I\mu_1 I_2}{2a}.$$

The force is attractive if $I_2 > 0$ and repulsive if $I_2 < 0$.

13.21 The London Equations for a Superconductor

(a) The curl of the London constitutive law is

$$\nabla \times \mathbf{j} = -\frac{1}{\mu_0 \delta^2} \nabla \times \mathbf{A} = -\frac{1}{\mu_0 \delta^2} \mathbf{B}, \tag{1}$$

and the curl of the Ampère-Maxwell law is

$$\nabla \times \nabla \times \mathbf{B} = \mu_0 \nabla \times \mathbf{j} + \frac{1}{c^2} \nabla \times \frac{\partial \mathbf{E}}{\partial t}. \tag{2}$$

Because $\nabla \cdot \mathbf{B} = 0$, we have

$$\nabla \times \nabla \times \mathbf{B} = \nabla(\nabla \cdot \mathbf{B}) - \nabla^2 \mathbf{B} = -\nabla^2 \mathbf{B}.$$

Therefore, inserting (1) into (2) and using Faraday's law, $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$, gives

$$\nabla^2 \mathbf{B} - \frac{1}{c^2} \frac{\partial^2 \mathbf{B}}{\partial t^2} = \frac{\mathbf{B}}{\delta^2}.$$

(b) From translational invariance and the boundary condition $\mathbf{B}(z = \pm d) = B_0\hat{\mathbf{x}}$, we deduce that $\mathbf{B} = B(z)\hat{\mathbf{x}}$. The external field is static, so the equation to be solved is

$$\frac{d^2 B}{dz^2} = \frac{B}{\delta^2}.$$

The solution is the linear combination

$$B(z) = a\exp(z/\delta) + b\exp(-z/\delta),$$

subject the boundary conditions

$$B(d) = a\exp(d/\delta) + b\exp(-d/\delta) = B_0 \tag{3}$$

and

$$B(-d) = a\exp(-d/\delta) + b\exp(d/\delta) = B_0. \tag{4}$$

It follows from (3) and (4) that $a = b$. Hence,

$$\mathbf{B} = \frac{\cosh(z/\delta)}{\cosh(d/\delta)}\mathbf{B}_0.$$

(c) The current density is

$$\mathbf{j} = \frac{1}{\mu_0}\nabla \times \mathbf{B} = \frac{B_0}{\mu_0 \delta}\frac{\sinh(z/\delta)}{\cos(d/\delta)}\hat{\mathbf{y}}.$$

13.22 Supercurrent on a Sphere

$\mathbf{B} = 0$ inside the sphere because the material is a superconductor. Therefore, the surface current density at the sphere surface is

$$\mu_0\mathbf{K} = \hat{\mathbf{r}} \times [\mathbf{B}_{\text{out}} - \mathbf{B}_{\text{in}}]_S = \hat{\mathbf{r}} \times \mathbf{B}_{\text{out}}|_S.$$

We let $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$ and use a magnetic scalar potential approach. The latter satisfies $\nabla^2\psi = 0$ away from the sphere surface. Now, the external field contributes $\psi_{\text{ext}} = -B_0 r\cos\theta$ everywhere and the current induced on the sphere surface contributes a general solution of Laplace's equation valid for $R > r$. Therefore, outside the sphere,

$$\psi_{\text{out}}(r,\theta) = \sum_{n=1}^{\infty} A_n \left(\frac{r}{R}\right)^{n+1} P_n(\cos\theta) - B_0 r\cos\theta.$$

The normal component of the magnetic field is always continuous. Therefore,

$$0 = -\left.\frac{\partial\psi_{\text{out}}}{\partial r}\right|_S = \sum_{n=1}^{\infty} A_n \frac{n+1}{R} P_n(\cos\theta) + B_0\cos\theta.$$

This shows that $A_1 = -RB_0/2$ and all other $A_n = 0$. We conclude that

$$\psi_{\rm out} = -B_0 \left[\frac{R^3}{2r^2} + r\right] \cos\theta,$$

so

$$\mu_0 \mathbf{K} = -\hat{\mathbf{r}} \times \nabla\psi_{\rm ext}\big|_{r=R} = -\frac{3}{2}B_0 \sin\theta\hat{\boldsymbol{\phi}}.$$

This agrees with the stated answer because $\hat{\mathbf{r}} \times \hat{\mathbf{z}} = -\sin\theta\hat{\boldsymbol{\phi}}$.

13.23 A Cylindrical Refrigerator Magnet

The first term is the capacitor-like attractive force between the end of the cylinder and its infinitesimally close image derived in the text. The far end of the cylinder behaves like a point magnetic charge $Q^* = MA$ at a distance L from the permeable surface and induces an image magnetic charge $-Q^*$ at a distance L into the permeable matter. Therefore, there are three Coulomb correction terms: attraction between the far end of the cylinder and its image, repulsion between the far end of the cylinder and the image of the near end, and repulsion between the near end of the cylinder and the image of the far end. The sum of these forces is

$$\Delta F = \frac{\mu_0 (Q^*)^2}{4\pi}\left[\frac{1}{(2L)^2} - \frac{1}{L^2} - \frac{1}{L^2}\right] = -\frac{7}{16\pi}\mu_0 M^2 \frac{A^2}{L^2}.$$

13.24 Magnetic Total Energy

By definition, the total energy is the work required to assemble the configuration. Opposite poles repel, so more work is required to assemble configuration 2 than configuration 1. Hence, its total energy is larger.

Source: J.R. Pierce, *Journal of Applied Physics* **24**, 1247 (1953).

13.25 Inductance in a Magnetic Medium

In vacuum, the defining equations are

$$\nabla \times \mathbf{H}_0 = \mathbf{j}_f \qquad \text{and} \qquad \nabla \cdot \mathbf{H}_0.$$

The magnetic flux through the loop is

$$\Phi_0 = \int d\mathbf{S} \cdot \mathbf{B}_0 = \mu_0 \int d\mathbf{S} \cdot \mathbf{H}_0.$$

Therefore, the self-inductance of the loop is

$$L_0 = \Phi_0 / I_f.$$

There is no change to any of the defining equations if all we do is let $\mu_0 \to \mu$. Therefore, $\mathbf{H} = \mathbf{H}_0$. On the other hand, the flux through the loop is

$$\Phi = \int d\mathbf{S} \cdot \mathbf{B} = \mu \int d\mathbf{S} \cdot \mathbf{H}_0 = \kappa_m \Phi_0.$$

Therefore,

$$L = \Phi / I_f = \kappa_m \Phi_0 / I_f = \kappa_m L_0.$$

Source: R.K. Wangsness, *Electromagnetic Fields* (Wiley, New York, 1986).

Chapter 14: Dynamic and Quasistatic Fields

14.1 A Polarized Slab in Motion

The field of the polarized medium is equivalent to the field produced by two uniformly charged sheets. One at $z = d$ has $\sigma_+ = P$; the other at $z = 0$ has $\sigma_- = -P$. Both move with velocity $v\hat{\mathbf{x}}$. This is equivalent to a surface current at $z = d$ with density $\mathbf{K}_+ = \sigma_+\boldsymbol{v} = Pv\hat{\mathbf{x}}$, and a surface current at $z = 0$ with density $\mathbf{K}_- = \sigma_-\boldsymbol{v} = -Pv\hat{\mathbf{x}}$. On the other hand, if there were a magnetization $\mathbf{M} = \mathbf{P} \times \boldsymbol{v} = Pv\hat{\mathbf{y}}$, we would expect a surface magnetization current density $\mathbf{K} = \mathbf{M} \times \hat{\mathbf{n}}$. In agreement with the first calculation, this gives $\mathbf{K} = Pv\hat{\mathbf{x}}$ on $z = d$ and $\mathbf{K} = -Pv\hat{\mathbf{x}}$ on $z = 0$.

14.2 Broken Wire?

The field in the gap is capacitor-like if $b \ll a$ and points along the wire, as does the current density j. Moreover,

$$E_{\text{gap}} = \frac{\sigma}{\epsilon_0} \qquad \text{where } \sigma = \int dt j = \int dt \frac{I}{\pi a^2}.$$

By definition, the displacement current density is

$$j_d = \epsilon_0 \frac{\partial E_{\text{gap}}}{\partial t} = \frac{d\sigma}{dt} = j = \frac{I(t)}{\pi a^2}.$$

Therefore, the displacement current flowing in the gap is

$$I_d = \int dS j_d = \pi a^2 j_d = I(t).$$

14.3 Charge Accumulation at a Line

(a) By charge conservation, the charge per unit length $\lambda(t)$ at $x = 0$ satisfies

$$\frac{d\lambda}{dt} = K.$$

In the quasi-electrostatic approximation, the electric field is given by the usual static formula,

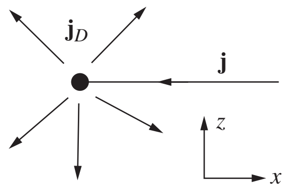
$$\mathbf{E}(\boldsymbol{\rho}, t) = \frac{\lambda(t)}{2\pi\epsilon_0} \frac{\hat{\boldsymbol{\rho}}}{\rho} \qquad \boldsymbol{\rho} = x\hat{\mathbf{x}} + z\hat{\mathbf{z}}.$$

Taking account of the displacement current, the magnetic field satisfies

$$\nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J} = \mu_0 \mathbf{j} + \mu_0 \mathbf{j}_d = \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t},$$

where, as illustrated in the figure below,

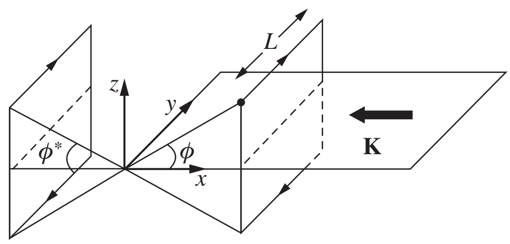
$$\mathbf{J} = -K\theta(x)\delta(z)\hat{\mathbf{x}} + \frac{K}{2\pi} \frac{\hat{\boldsymbol{\rho}}}{\rho}.$$



Now, it follows from the symmetry of the problem that

$$\mathbf{B} = B(x,z)\hat{\mathbf{y}} \qquad \text{and} \qquad B(x,z) = -B(x,-z).$$

Remembering both contributions to $\mathbf{J}$, we now apply Ampère's law in integral form using the circuits shown below.



Beginning with the $x > 0$ circuit for the observation point (x,z) shown as a black dot,

$$\oint d\mathbf{s}\cdot\mathbf{B} = 2BL = \mu_0 KL - \frac{\mu_0 KL}{2\pi}\int_{-z}^{z}\frac{dz'}{\sqrt{x^2+z'^2}}\frac{x}{\sqrt{x^2+z'^2}} = \mu_0 KL\left(1-\phi/\pi\right).$$

Therefore, for $0 \le \phi \le \pi/2$,

$$\mathbf{B}(x>0,z) = \operatorname{sgn}(z)\frac{\mu_0 K}{2}\left(1-\frac{\phi}{\pi}\right)\hat{\mathbf{y}},$$

where $\tan\phi = z/x$ as shown in the figure. Note that this reproduces the infinite sheet result when $x\to\infty$ so $\phi\to 0$.

We use the circuit to the left for $x < 0$. Here, only the displacement current contributes to the magnetic field. The integral is the same as above except the overall minus sign is absent and the angle $\phi \to \phi^*$. This gives

$$2BL = \mu_0 KL\frac{\phi^*}{\pi} = \mu_0 KL\frac{\pi-\phi}{\pi} = \mu_0 KL\left(1-\phi/\pi\right).$$

Therefore, if $0\le\phi\le\pi$, we get a single formula for the magnetic field everywhere:

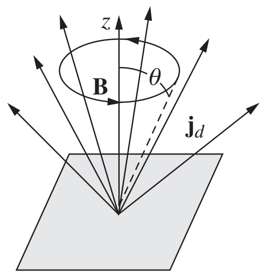
$$\mathbf{B}(x,z) = \operatorname{sgn}(z)\frac{\mu_0 K}{2}\left(1-\frac{\phi}{\pi}\right)\hat{\mathbf{y}}.$$

(b) Since $\mathbf{J}$ is time-independent, we found a time-independent $\mathbf{B}$. This means that $\partial\mathbf{B}/\partial t = 0$ and the solution we have found satisfies the full Maxwell equations and not merely their quasi-static approximation to them.

14.4 Charge Accumulation in a Plane

(a) The displacement current is

$$\mathbf{j}_d(\mathbf{r}) = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} = \frac{dq/dt}{4\pi} \frac{\hat{\mathbf{r}}}{r^2} = \frac{I}{4\pi} \frac{\hat{\mathbf{r}}}{r^2}.$$



(b) By symmetry, the magnetic field can only have the form $\mathbf{B} = B(r, \theta)\mathbf{e}_\phi$. The field lines of $\mathbf{B}$ are circles centered on the z-axis as shown in the figure. Ampère's law in integral form shows that $\mathbf{K}$ does not produce any magnetic field at all. The total $\mathbf{B}$ comes from the displacement current. Then, using a spherical capping surface to evaluate the surface integral of $\mathbf{j}_d \cdot d\mathbf{S} = j_d dS$ in spherical coordinates,

$$2\pi r \sin\theta B = 2\pi r^2 \mu_0 j_d \int_0^\theta d\theta' \sin\theta' = 2\pi \frac{\mu_0 I}{4\pi}(1 - \cos\theta) \quad \Rightarrow \quad B(r, \theta) = \frac{\mu_0 I}{4\pi r} \tan(\theta/2).$$

14.5 Rogowski Coil

For an Ampèrian loop that lies inside the torus and is everywhere perpendicular to the cross section,

$$\oint d\mathbf{s} \cdot \mathbf{B} = \mu_0 I(t),$$

if $I(t)$ flows through the hole of the torus. On the other hand, the magnetic flux through one turn of the torus is

$$\Phi_B = \int d\mathbf{S} \cdot \mathbf{B} = A \int d\hat{\mathbf{s}} \cdot \mathbf{B},$$

where $d\hat{\mathbf{s}}$ is a unit vector perpendicular to the cross section. Therefore, if one turn advances us along the torus by distance ds, the total flux through the entire torus is

$$\Phi_B = nA \oint d\mathbf{s} \cdot \mathbf{B} = nA\mu_0 I(t).$$

The induced EMF is

$$\mathcal{E} = \frac{d\Phi_B}{dt} = nA\mu_0 \dot{I}(t).$$

No EMF arises if $I(t)$ does not flow through the hole of the torus because the Ampère's law calculation above then gives zero for the magnetic flux.

14.6 Magnetic Field of an AC Capacitor

The electric field directed from one plate to the other follows immediately from the static formula:

$$E(t) = \frac{V(t)}{d}.$$

By symmetry, the magnetic field lines form closed circles concentric with the symmetry axis of the plates. Using circuits like this of radius ρ to evaluate the Ampère-Maxwell law in integral form gives

$$\oint d\mathbf{s} \cdot \mathbf{B} = 2\pi\rho B(\rho) = \int dS \left[\mu_0 \sigma E + \frac{1}{c^2}\frac{\partial E}{\partial t}\right] = \frac{\pi\rho^2 V_0}{d}\left[\mu_0\sigma \sin\omega t + \frac{\omega}{c^2}\cos\omega t\right].$$

Hence,

$$\mathbf{B}(\rho) = \frac{\rho V_0}{2d}\left[\mu_0\sigma\sin\omega t + \frac{\omega}{c^2}\cos\omega t\right]\hat{\boldsymbol{\phi}}.$$

14.7 A Resistive Ring Comes to Rest

When $a \ll x_0$, the magnetic field is nearly constant over the area of the ring. In that case, Faraday's law gives the magnitude of the EMF generated in the ring by its motion as

$$\mathcal{E} = \frac{d}{dt}\int d\mathbf{S}\cdot\mathbf{B} = \pi a^2 \frac{dB}{dt} = \frac{\pi a^2 B_0}{x_0}\frac{dx}{dt} = \frac{\pi a^2 B_0}{x_0} v.$$

Ohm's law says that $\mathcal{E} = IR$, so we can equate the rate at which energy is dissipated in the ring with the rate at which the ring loses kinetic energy:

$$\mathcal{P} = \mathcal{E} I = \frac{\mathcal{E}^2}{R} = -\frac{d}{dt}\frac{1}{2}Mv^2 = Mv\dot{v}.$$

This gives the equation of motion

$$\dot{v} = -\frac{\gamma}{M}v \qquad \text{with } \gamma = \left(\frac{\pi a^2 B_0}{x_0}\right)^2.$$

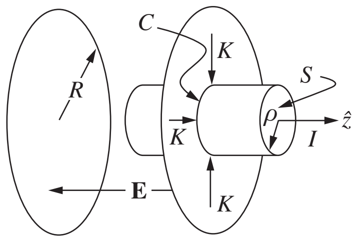
The solution is immediate: $v(t) = v_0 \exp(-\gamma t/M)$. Therefore, the total distance traveled from the origin is

$$x = \int_0^\infty dt v(t) = \frac{mv_0}{\gamma}.$$

Source: MIT Physics General Exam I, Spring 2001.

14.8 A Discharging Capacitor

(a) By symmetry, $\mathbf{K}_L(\rho,t) = -\mathbf{K}_R(\rho,t) = K(\rho,t)\hat{\boldsymbol{\rho}}$. To find $K(\rho,t)$, we integrate the continuity equation over the rod-shaped volume with radius ρ shown below.



From Chapter 5, the surface charge density is $\sigma(\rho) = Q/4\pi R\sqrt{R^2-\rho^2}$ on each side of a circular conducting plate. Therefore,

$$Q(\rho,t) = 2Q(t)\int_0^{2\pi} d\phi \int_0^{\rho} d\rho' \frac{\rho}{4\pi R\sqrt{R^2-\rho'^2}} = Q(t)\left[1-\sqrt{1-\rho^2/R^2}\right].$$

The surface current brings charge into the rod-shaped volume and the wire carries charge out of the rod-shaped volume. Therefore,

$$\begin{aligned}\int_V d^3r\,\nabla\cdot\mathbf{j} + \int_V d^3r\,\frac{\partial\rho}{\partial t} = 0 \quad &= \quad \oint_C d\boldsymbol{\ell}\cdot\mathbf{K} - \int_S d\mathbf{S}\cdot\mathbf{j} + \frac{d}{dt}\int_S dS\sigma \\ &= \quad K2\pi\rho - I + \frac{dQ}{dt}\left[1-\sqrt{1-\rho^2/R^2}\right].\end{aligned}$$

Since $dQ/dt = I$, we conclude that

$$K(\rho,t) = I(t)\frac{\sqrt{1-\rho^2/R^2}}{2\pi\rho}. \tag{1}$$

(b) The Ampère-Maxwell law in integral form is

$$\oint_C d\boldsymbol{\ell}\cdot\mathbf{B} = \int_S d\mathbf{S}\cdot\left[\mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right].$$

We will always choose C as a circle of radius ρ concentric with the wire. We are treating the wire as filamentary (rather than ohmic), so the only source term outside the plates is the wire current. In that case,

$$\mathbf{B}_{\rm out}(\rho,t) = \frac{\mu_0 I(t)}{2\pi\rho}\hat{\boldsymbol{\phi}}. \tag{2}$$

Between the plates, the only source term is the displacement current. When $d \ll R$, the field between the plates is $E(\rho) = 2\sigma(\rho)/\epsilon_0$ Therefore, since $d\mathbf{S} = dS\hat{\mathbf{z}}$,

$$B_\phi(\rho,t)2\pi\rho = \frac{1}{c^2}\frac{d}{dt}\int_S d\mathbf{S}\cdot\mathbf{E} = \frac{1}{\epsilon_0 c^2}\frac{d}{dt}\int_S dS\sigma(\rho) = \mu_0\dot{Q}(\rho,t),$$

so

$$\mathbf{B}_{\rm in}(\rho,t) = \frac{\mu_0 I(t)}{2\pi\rho}\left[1-\sqrt{1-\rho^2/R^2}\right]\hat{\boldsymbol{\phi}}. \tag{3}$$

(c) The general matching condition is $[\mathbf{B}_2 - \mathbf{B}_1] = \mu_0 \mathbf{K} \times \hat{\mathbf{n}}_1$. For the right plate, this reads

$$\mathbf{B}_{\text{out}} - \mathbf{B}_{\text{in}} = \mu_0 \mathbf{K}_R \times \hat{\mathbf{z}} = -\mu_0 K(\rho, t)\hat{\boldsymbol{\rho}} \times \hat{\mathbf{z}} = \mu_0 K(\rho, t)\hat{\boldsymbol{\phi}}. \tag{4}$$

On the other hand, from (2) and (3),

$$\mathbf{B}_{\text{out}} - \mathbf{B}_{\text{in}} = \frac{\mu_0 I(t)}{2\pi\rho}\sqrt{1 - \rho^2/R^2}\,\hat{\boldsymbol{\phi}}.$$

This reproduces (4) when we use (1). The calculation for the left plate is identical.

14.9 What Do the Voltmeters Read?

The magnetic field is increasing into the paper. Therefore, a counter-clockwise current is induced in the inner circuit that includes R_1 and R_2. By Faraday's law and Ohm's law, the magnitude of that current is

$$I(R_1 + R_2) = \dot{B}\pi r^2.$$

Again by Faraday's law, the small current that flows through the outer circuit that includes both voltmeters is also counterclockwise. This is the direction where V_1 reads a positive voltage and the voltage drop across this meter is the same as the voltage drop across R_1. Therefore,

$$V_1 = \frac{R_1}{R_1 + R_2}\dot{B}\pi r^2.$$

The current flows through V_2 in the opposite direction, and the voltage drop across this meter is the same as the voltage drop across R_2. Therefore,

$$V_2 = -\frac{R_2}{R_1 + R_2}\dot{B}\pi r^2.$$

Source: R.H. Romer, *American Journal of Physics* **50**, 1089 (1982).

14.10 A Sliding Circuit

There is both a "flux" and a "motional" contribution to the EMF so we use

$$\mathcal{E} = -\frac{d}{dt}\int_S d\mathbf{S} \cdot \mathbf{B} + \oint_C d\mathbf{s} \cdot (\boldsymbol{v}_{\text{d}} \times \mathbf{B}),$$

and pick C as the stationary circuit PP′Q′QP indicated in the figure. This choice makes the wire coincident with three of the four legs of C. The solid arrows in the right panel indicate the presumed direction of current flow. The drift velocity of the electrons in the wire is along the wire and $\boldsymbol{v}_{\text{d}} = \boldsymbol{v}$ for electrons in the magnet. This assumes that the latter are simply dragged along by the ions of the magnet. The motional EMF associated with C comes entirely from the segment PP′ where $\boldsymbol{v}_{\text{d}}$, $\mathbf{B}$, and $d\mathbf{s}$ are mutually orthogonal. We find

$$\mathcal{E}_{\text{motional}} = -Bhv.$$

As the surface S that bounds C, we choose the union of the two rectangular and two triangular flat surfaces outlined by dashed lines in the figure. The flux of $\mathbf{B}$ is non-zero through the large rectangular surface only. Therefore, if x stands for the distance PE, the fact that $\mathbf{B}$ and $d\mathbf{S}$ are anti-parallel gives

$$\mathcal{E}_{\rm flux} = Bh\frac{dx}{dt} = Bhv.$$

The total EMF is $\mathcal{E} = \mathcal{E}_{\rm motional} + \mathcal{E}_{\rm flux} = 0$. Once the segment QQ′ enters the field of the magnet, the flux contribution disappears and only the motional EMF remains.

Source: G.C. Scorgie, *European Journal of Physics* **16**, 36 (1995).

14.11 Townsend-Donaldson Effect

The magnetic field inside an ideal solenoid is $B = \mu_0 IN/L$. Its self-inductance is $L = \Phi_B/I = NBA/I$ where $A = \pi R^2$ is the cross sectional area. The voltage drop across the solenoid is $V = E_z\ell = LdI/dt = NBA(\dot{I}/I)$. Faraday's law applied to one loop of the solenoid gives $E_\theta \cdot 2\pi R = BA$. Therefore,

$$\frac{E_\theta}{E_z} = \frac{\ell/N}{2\pi R}\frac{\dot{B}}{B}\frac{I}{\dot{I}} = \frac{\ell/N}{2\pi R}.$$

Source: J.S. Townsend and R.H. Donaldson, *Philosophical Magazine* **5**, 178 (1928).

14.12 A Magnetic Monopole Detector

The charge is the time integral of the Faraday current induced in the ring:

$$Q = \int dtI = \frac{1}{R}\int dt\mathcal{E} = \frac{1}{R}\int dt\frac{d\Phi_B}{dt} = \frac{1}{R}\int d\Phi_B = \frac{\Phi_B(\text{final}) - \Phi_B(\text{initial})}{R}.$$

The monopole creates a magnetic field which satisfies the flux integral

$$\int_S d\mathbf{S}\cdot\mathbf{B} = \mu_0 g,$$

where S is any surface that completely encloses the monopole. Let the ring lie in the $z = 0$ plane and choose this plane as half the surface S. The other half of S is a hemisphere of infinite radius. When the monopole reaches $z = +\Delta$, half the flux integral above comes from the $z = 0$ plane and half passes through the hemisphere at $z > 0$. When the monopole reaches $z = -\Delta$, half the flux integral above comes from the $z = 0$ plane and half passes through the hemisphere at $z < 0$. The two $z = 0$ contributions have opposite signs. Therefore, since the monopole actually passes through the ring, the change in flux through the ring is the sum of these two contributions. Hence,

$$Q = \frac{\mu_0 g}{R}.$$

14.13 Corbino Disk

(a) The radial current density associated with the radial current I_0 is $j(r) = I_0/2\pi rt$. The corresponding carrier speed is $v = j/ne = I_0/2\pi rtne$. This motion is perpendicular to the external magnetic field so the "motional" EMF at radius r is

$$\mathcal{E} = \oint d\boldsymbol{\ell} \cdot (\mathbf{v} \times \mathbf{B}) = 2\pi r \times I_0/2\pi rtne \times B = I_0 B/tne.$$

The resistance of a volume composed of an area tdr that sweeps around in a circle of length $2\pi r$ is $2\pi r/\sigma t dr$. Therefore, the circular current is

$$I = \frac{\mathcal{E}}{R} = \mathcal{E} \int_{R_1}^{R_2} dr \frac{\sigma t}{2\pi r} dr = \frac{\sigma B}{2\pi ne} I_0 \ln \frac{R_2}{R_1}.$$

(b) Consider a circle of radius $R_1 < r < R_2$ in the plane of the disk. The potential difference between any two points on the circle is

$$\varphi(A) - \varphi(B) = \int_A^B d\boldsymbol{\ell} \cdot \mathbf{E},$$

because the Ohm's law electric field $E = j/\sigma$ is radial and therefore perpendicular to a path that follows a circular arc between A and B.

Remark: There is no azimuthal electric field. The circular current is driven by the magnetic part of the Lorentz force, which does no work. Since $IR_{AB} = \varphi_A - \varphi_B + E_{AB}$, we get $E_{AB} - IR_{AB} = 0$ for this situation. That is, each bit of circular arc is a "battery" that produces a current through its own resistance.

14.14 A Falling Ring and the Lorentz Force

(a) The changing magnetic field produces a changing magnetic flux through the ring as it falls. The induced electric field produces a torque on the distributed charge which induces rotation.

(b) The flux rule is

$$\frac{d}{dt} \int_S d\mathbf{S} \cdot \mathbf{B} = - \oint_C d\boldsymbol{\ell} \cdot \mathbf{E}.$$

The ring is horizontal and the induced electric field is azimuthal. Therefore, an elementary calculation gives the instantaneous electric field at the position of the ring as

$$\mathbf{E} = -\frac{1}{2} R \frac{dB_z}{dt} \hat{\boldsymbol{\phi}}.$$

The Coulomb force $q\mathbf{E}$ exerts a net torque on the ring equal to

$$\mathbf{N} = \oint \mathbf{r} \times dq\mathbf{E} = -QER\hat{\mathbf{z}} = -\frac{QR^2}{2} \frac{dB_z}{dt} \hat{\mathbf{z}}.$$

On the other hand the torque is also equal to $\mathbf{N} = Id\omega/dt$, where $I = MR^2$ is the moment of inertia of the ring. Therefore,

$$-\frac{QR^2}{2}\frac{dB_z}{dt} = MR^2\frac{d\omega}{dt}$$

or

$$-\int_0^\Omega d\omega = \frac{Q}{2M}\int_{B_z(h)}^{B_z(0)} dB_z,$$

where Ω is the angular velocity of the ring when it hits the ground. Integrating gives

$$\Omega = \frac{Q}{2M}[B_z(h) - B_z(0)].$$

By conservation of energy the change in potential energy is equal to the sum of the translational and rotational kinetic energies, i.e.,

$$Mgh = \frac{1}{2}Mv_{\text{CM}}^2 + \frac{1}{2}I\Omega^2.$$

Inserting Ω and I from just above gives the required center-of-mass speed.

(c) At a moment when the ring has angular velocity ω, its rotational motion produces a current $i = Q\omega/(2\pi)$. Therefore, the ring possesses a magnetic dipole moment with magnitude

$$m = \pi R^2 \times \frac{Q\omega}{2\pi} = \frac{1}{2}QR^2\omega.$$

The magnetic force on this dipole, $\mathbf{F} = m\nabla B_z$, opposes gravity as the ring falls. The work done by this force over the course of the fall is

$$W = \int d\mathbf{r}\cdot\mathbf{F} = -\int m\frac{dB_z}{dz}dz = -\frac{1}{2}QR^2\int_{B_z(h)}^{B_z(0)} \omega(B_z)dB_z.$$

Using the result from part (b) that

$$\omega(z) = \frac{Q}{2M}[B_z(z) - B_z(0)],$$

we find that

$$W = \frac{Q^2R^2}{4M}\int_{B_z(h)}^{B_z(0)} [B_z(0) - B_z(z)]dB_z = -\frac{Q^2R^2}{8m}[B_z(h) - B_z(0)]^2.$$

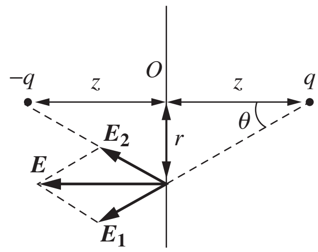
This is indeed equal to the change in the rotational energy of the ring:

$$\frac{1}{2}MR^2\Omega^2 = \frac{Q^2R^2}{8M}[B_z(0) - B_z(h)]^2.$$

Source: Dr. A. Scherbakov, Georgia Institute of Technology (private communication).

14.15 Ohmic Dissipation by a Moving Charge

(a) The conducting plane is an equipotential with $V = 0$. By the method of images, the electric field to the right of the plane may be represented as a superposition of the field produced by q at $(0, 0, z)$ and its "image" $-q$ located at $(0, 0, -z)$. The field components parallel to the surface cancel from these two sources and the field components normal to the surface add.



Therefore, the electric field on the surface of the plane at distance r from the origin is

$$\mathbf{E} = -\hat{\mathbf{z}}\frac{1}{4\pi\epsilon_0}\frac{2qz}{\left(z^2 + r^2\right)^{3/2}}.$$

The corresponding induced surface charge density is

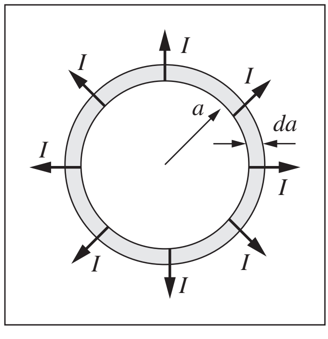
$$\sigma(r) = \epsilon_0\hat{\mathbf{z}} \cdot \mathbf{E}(r) = -\frac{q}{2\pi}\frac{z}{\left(z^2 + r^2\right)^{3/2}}.$$

(b) The amount of induced charge located within distance a from the origin is

$$Q(a) = \int_0^a 2\pi r\sigma(r)dr = -qz\int_0^a \frac{rdr}{\left(z^2 + r^2\right)^{3/2}} = qz\left.\frac{1}{\sqrt{z^2 + r^2}}\right|_0^a = q\left(\frac{z}{\sqrt{z^2 + a^2}} - 1\right).$$

Referring to the diagram below, the rate of increase of $Q(a)$ is the current flowing into the disk of radius a:

$$I = \frac{dQ\,(a)}{dt} = \frac{\partial Q}{\partial z} \cdot \frac{\partial z}{\partial t} = \frac{qa^2v}{\left(z^2 + a^2\right)^{3/2}}.$$



Therefore, the power dissipated within a ring of radius a and thickness da is $dP(a) = I^2 dR$ where dR is defined in the problem statement. Hence,

$$dP(a) = \frac{q^2 a^4 v^2}{(z^2+a^2)^3} \cdot R_S \frac{da}{2\pi a},$$

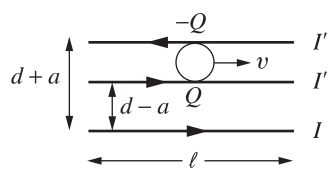
and the instantaneous power dissipated in the plane is

$$\begin{aligned} P &= \frac{q^2 v^2 R_S}{2\pi} \int_0^\infty \frac{a^3 da}{(z^2+a^2)^3} = \frac{q^2 v^2 R_S}{2\pi} \int_{z^2}^\infty \frac{\frac{1}{2}(t-z^2)\, dt}{t^3} \\ &= \frac{q^2 v^2 R_S}{4\pi} \left(-\frac{1}{t} + \frac{z^2}{2t^2} \right) \Bigg|_{z^2}^{\infty} = \frac{q^2 v^2 R_S}{8\pi z^2} \end{aligned}$$

Source: Dr. A. Scherbakov, Georgia Institute of Technology (private communication).

14.16 An Unusual Attractive Force

(a) The wire produces a magnetic field $B_0 = \mu_0 I/2\pi r$ which points out of the paper near the sphere. The Lorentz force acts like an effective electric field $\mathbf{E}_0 = \mathbf{v} \times \mathbf{B}$ which polarizes the sphere (positive charge Q closest to the wire; negative charge $-Q$ farthest from the wire). The moving positive charge at distance $d-a$ from the wire behaves like a current I' flowing parallel to I. The moving negative charge at a distance $d+a$ from the wire behaves like an identical current I' flowing anti-parallel to I. The parallel current is closer to the wire, so the net force is attractive.



(b) The force exerted on a length ℓ of wire is $F = I\ell B$, where

$$B = \frac{\mu_0 I'}{2\pi} \left[\frac{1}{d-a} - \frac{1}{d+a} \right] \approx \frac{\mu_0 I'}{2\pi} \frac{2a}{d^2}.$$

An estimate of the effective current produced by a moving charge Q is $I' = Qv/\ell$. Therefore,

$$F \approx \frac{\mu_0 I}{2\pi} \frac{2a}{d^2} Qv, \tag{1}$$

and it remains only to estimate Q. We do this by equating the estimate $p \approx Qa$ for the dipole moment with the estimate (based on the hint)

$$p \approx \epsilon_0 a^3 E_0 \approx \epsilon_0 a^3 v B_0 \approx \epsilon_0 a^3 v \frac{\mu_0 I}{2\pi d}.$$

Since $\epsilon_0 \mu_0 = 1/c^2$, the result is $Q = \dfrac{1}{c^2}\dfrac{a^2 v I}{2\pi d}$. Inserting this into (1) gives the suggested result,

$$F \propto \frac{v^2}{c^2}\frac{a^3}{d^3}\mu_0 I^2.$$

14.17 Quasi-Electrostatic Fields

Since $\mathbf{E} = -\nabla\varphi$ and $\mathbf{B} = \nabla \times \mathbf{A}$, we get $\nabla \times \mathbf{E} = \nabla \cdot \mathbf{B} = 0$ immediately. In addition,

$$\nabla \cdot \mathbf{E} = -\frac{1}{4\pi\epsilon_0}\int d^3r' \rho(\mathbf{r}', t)\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\epsilon_0}\int d^3r'\, \rho(\mathbf{r}', t)\delta(\mathbf{r} - \mathbf{r}') = \frac{\rho(\mathbf{r}, t)}{\epsilon_0}.$$

Finally, making use of the continuity equation, $\nabla \cdot \mathbf{j} + \partial\rho/\partial t$, to get the penultimate line,

$$\begin{aligned}
\nabla \times \mathbf{B}(\mathbf{r}, t) &= \nabla \times \nabla \times \frac{\mu_0}{4\pi}\int d^3r'\, \frac{\mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} \\
&= \nabla\frac{\mu_0}{4\pi}\int d^3r'\, \nabla \cdot \frac{\mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} - \frac{\mu_0}{4\pi}\int d^3r'\, \mathbf{j}(\mathbf{r}', t)\nabla^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} \\
&= \nabla\frac{\mu_0}{4\pi}\int d^3r' \left\{\frac{1}{|\mathbf{r} - \mathbf{r}'|}\nabla \cdot \mathbf{j}(\mathbf{r}', t) + \mathbf{j}(\mathbf{r}', t) \cdot \nabla\frac{1}{|\mathbf{r} - \mathbf{r}'|}\right\} + \mu_0 \mathbf{j}(\mathbf{r}, t) \\
&= -\nabla\frac{\mu_0}{4\pi}\int d^3r'\, \mathbf{j}(\mathbf{r}', t) \cdot \nabla'\frac{1}{|\mathbf{r} - \mathbf{r}'|} + \mu_0 \mathbf{j}(\mathbf{r}, t) \\
&= \nabla\frac{\mu_0}{4\pi}\int d^3r'\, \frac{\nabla' \cdot \mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} + \mu_0 \mathbf{j}(\mathbf{r}, t) \\
&= -\frac{1}{c^2}\frac{\partial}{\partial t}\nabla\frac{1}{4\pi\epsilon_0}\int d^3r'\, \frac{\rho(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|} + \mu_0 \mathbf{j} \\
&= \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0 \mathbf{j}.
\end{aligned}$$

14.18 Casimir's Circuit

When $a \ll b \ll L$, we may approximate the magnetic field produced by each wire using the infinite-wire formula $\mathbf{B}(\rho) = \hat{\boldsymbol{\phi}}\mu_0 I/2\pi\rho$. The magnetic flux $\Phi_{\rm B}$ through the rectangular area bounded by I_1 and I_2 is

$$\Phi_{\rm B} = \frac{\mu_0 L}{2\pi}\left\{(I_1 - I_2)\ln\frac{b}{a} - I_3 \ln 2\right\}.$$

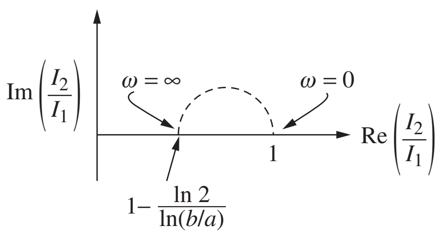
Since each $I_k(t)$ varies as $\exp(-i\omega t)$ and $r = 1/\sigma\pi a^2$ is the wire resistance per unit length, the flux rule $\mathcal{E} = -d\Phi_B/dt$ applied to this rectangle gives

$$\mathcal{E} = (I_1 - I_2)rL = -\frac{d}{dt}\Phi_B = i\omega\frac{\mu_0 L}{2\pi}\left\{(I_1 - I_2)\ln\frac{b}{a} - I_3\ln 2\right\}.$$

We have neglected the voltage drop across the horizontal busses that connect the wires because $L \gg b$. By symmetry, $I_1 = I_3$. Therefore, since $2\pi r/\omega\mu_0 = \delta^2/a^2$, we get the final result,

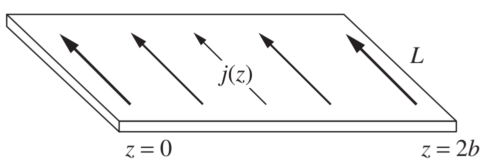
$$\frac{I_2}{I_1} = \frac{I_2}{I_3} = 1 - \frac{\ln 2}{\ln(b/a) + i(\delta^2/a^2)}. \tag{1}$$

The figure below is a plot of $\mathrm{Im}(I_2/I_1)$ versus $\mathrm{Re}(I_2/I_1)$ according to (1). Two points stand out. First, although the current in each wire is the same when $\omega = 0$, I_2 steadily decreases compared to its neighbors as ω increases. Second, the ratio I_2/I_1 always has a positive imaginary part, so $I_2(t)$ always lags in time behind $I_1(t)$ and $I_3(t)$. Both are a consequence of transient, Faraday-induced currents that circulate in the three closed circuits defined by the three resistive wires.



These results may be compared with the time-harmonic current density in a thin, rectangular metal slab in the small-skin-depth regime. The results of the text imply that the electric field near the $z = 0$ edge of such a slab is

$$\mathbf{E}_{\|}(z) = \mathbf{E}_{\|}(0)\exp\left\{(i-1)z/\delta(\omega)\right\}.$$



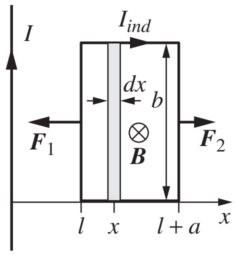
A similar formula applies near the $z = 2b$ edge. The symmetric minimum in the current density sketched in the figure above is a continuous version of the fact that $I_2 < I_1 = I_3$ in the three-wire problem. The prediction of (2) that the current flow away from the edges lags behind the current flow at the edges is a continuous version of the lag of $I_2(t)$ compared to $I_1(t)$ and $I_3(t)$.

Source: H.B.G. Casimir and J. Ubbink, *Philips Technical Review* **28**, 271 (1967).

14.19 Inductive Impulse

Let $B(x)$ be the magnitude of the field produced by the straight wire at a perpendicular distance x from itself. By Ampère's law,

$$B(x) = \frac{\mu_0 I}{2\pi x}.$$



The flux of B through the frame is

$$\Phi = \int_l l + adx\, B\,(x)\, bdx = \frac{\mu_0 Ib}{2\pi} \int_l^{l+a} \frac{dx}{x} = \frac{\mu_0 Ib}{2\pi} \ln\left(1 + \frac{a}{l}\right).$$

Using Faraday's law, the magnitude of the induced EMF in the frame is

$$\mathcal{E} = \frac{d\Phi}{dt} = \frac{\mu_0 b}{2\pi} \ln\left(1 + \frac{a}{l}\right) \frac{dI}{dt}.$$

Therefore, the current induced in the frame is

$$I_{ind} = \frac{\mathcal{E}}{R} = \frac{\mu_0 b}{2\pi R} \ln\left(1 + \frac{a}{l}\right) \frac{dI}{dt}.$$

The direction of the current is clockwise, as dictated by Lenz' law. Hence, the net force on the frame is toward the straight wire, with magnitude

$$F = F_1 - F_2 = \frac{\mu_0 bI \cdot I_{ind}}{2\pi l} - \frac{\mu_0 bI \cdot I_{ind}}{2\pi\,(l+a)} = \frac{\mu_0 baI \cdot I_{ind}}{2\pi l\,(l+a)} = \frac{\mu_0^2 b^2 a}{4\pi^2 l\,(l+a)\,R} \ln\left(1 + \frac{a}{l}\right) I \frac{dI}{dt}.$$

The impulse imparted by the force during the switch-off is

$$J = \int F(t)dt = \frac{\mu_0^2 b^2 a}{4\pi^2 l\,(l+a)\,R} \ln\left(1 + \frac{a}{l}\right) \int_{I_0}^{0} I \cdot dI = -\frac{\mu_0^2 b^2 a I_0^2}{8\pi^2 l\,(l+a)\,R} \ln\left(1 + \frac{a}{l}\right).$$

Hence, the velocity imparted to the frame is

$$v = \frac{J}{m} = \frac{\mu_0^2 b^2 a I_0^2}{8\pi^2 ml\,(l+a)\,R} \ln\left(1 + \frac{a}{l}\right).$$

The frame moves toward the wire.

14.20 AC Resistance of an Ohmic Wire

The DC resistance of a wire with cross sectional area A is $R = L/\sigma A$. The high-frequency limit of quasi-magnetostatics is the limit of small skin depth when

$$\delta(\omega) = \sqrt{\frac{2}{\mu_0 \omega \sigma}}.$$

Since all the current flows within a distance δ of the surface of the wire, the effective area of the wire is reduced from πa^2 to $2\pi a \delta$. Therefore,

$$R(\omega) = \frac{L}{\sigma 2\pi a \delta} = \frac{L}{2\pi a \sigma}\sqrt{\frac{\mu_0 \omega \sigma}{2}} \propto \frac{L}{a}\sqrt{\frac{\mu_0 \omega}{\sigma}}.$$

14.21 A Rotating Magnet

The neglect of electromagnetic waves means we may neglect the displacement current. This is a problem with zero charge density so the relevant Maxwell equations are the formulae of quasi-magnetostatics:

$$\nabla \cdot \mathbf{E} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$
$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j}.$$

The magnetic field is

$$\mathbf{B} = \nabla \times \mathbf{A} = \nabla \times \left[\frac{\mu_0}{4\pi}\frac{\mathbf{m}(t) \times \hat{\mathbf{r}}}{r^2}\right].$$

We get the electric field from Faraday's law, namely,

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}\nabla \times \mathbf{A} = -\nabla \times \frac{\partial \mathbf{A}}{\partial t}.$$

Therefore,

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0}{4\pi}\frac{\dot{\mathbf{m}} \times \hat{\mathbf{r}}}{r^2} = \frac{\mu_0}{4\pi}\frac{(\boldsymbol{\Omega} \times \mathbf{m}) \times \hat{\mathbf{r}}}{r^2}.$$

Source: A. Kovetz, *Electromagnetic Theory* (University Press, Oxford, 2000).

14.22 Magnetic Metal Slab

(a) In the quasi-magnetostatic limit, the magnetic field satisfies the diffusion equation

$$\nabla^2 \mathbf{B}(\mathbf{r}.t) = \mu\sigma \frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t}.$$

The driving field is time-harmonic, so the steady-state field in the medium will be time-harmonic also. This means we need to solve

$$\nabla^2 \mathbf{B}(\mathbf{r}) = -i\omega\mu\sigma \mathbf{B}(\mathbf{r}) = k^2 \mathbf{B}(\mathbf{r}).$$

The boundary conditions and symmetry tell us that $\mathbf{B}(\mathbf{r}) = \hat{\mathbf{z}}B(y)$ with $B(y) = B(-y)$. From these facts, we conclude that

$$B(y) = A\cosh ky.$$

Ohmic material cannot support a singular surface current density $\mathbf{K}$. Therefore, the tangential component of $\mathbf{H}$ is continuous and $B_0/\mu_0 = (A/\mu)\cosh kd$. Therefore, if $\kappa = \mu/\mu_0$, the field inside the slab is

$$\mathbf{B}(y,t) = \hat{\mathbf{z}}\frac{\kappa B_0 \cosh ky}{\cosh kd}e^{-i\omega t}.$$

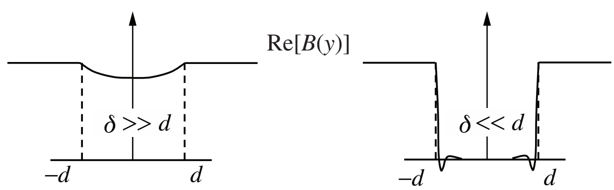
(b) From the definition in (a),

$$k = \frac{1-i}{\delta} \qquad \text{where} \qquad \delta = \sqrt{\frac{2}{\mu\sigma\omega}}.$$

Therefore,

$$\begin{aligned}
&\text{Re } B_z(y,0) \\
&= \kappa B_0 \text{Re}\left\{\frac{\cosh(y/\delta)\cos(y/\delta) - i\sinh(y/\delta)\sin(y/\delta)}{\cosh(d/\delta)\cos(d/\delta) - i\sinh(d/\delta)\sin(d/\delta)}\right\} \\
&= \kappa B_0 \frac{\cosh(y/\delta)\cosh(d/\delta)\cos(y/\delta)\cos(d/\delta) + \sinh(y/\delta)\sinh(d/\delta)\sin(y/\delta)\sin(d/\delta)}{\cosh^2(d/\delta)\cos^2(d/\delta) + \sinh^2(d/\delta)\sin^2(d/\delta)}.
\end{aligned}$$

The graph of this function is as follows.



Source: T.P. Orlando and K.A. Delin, *Foundations of Applied Superconductivity* (Addison-Westey, Reading, MA, 1991).

14.23 Azimuthal Eddy Currents in a Wire

(a) The first approximation to the eddy-current density inside the tube is $\mathbf{j} = \sigma\mathbf{E}$ where $\mathbf{E}$ is the electric field induced by Faraday's law. By symmetry, this field is in the $\hat{\boldsymbol{\phi}}$ direction. Therefore, using the integral form and closed circles of radius ρ,

$$\oint d\mathbf{s}\cdot\mathbf{E} = 2\pi\rho E_\phi(\rho) = -\frac{d}{dt}\Phi_B(\rho) = -\pi\rho^2\dot{B}(t).$$

The result is

$$\mathbf{j}(\rho) = \hat{\boldsymbol{\phi}}\frac{1}{2}\sigma B_0\rho\omega\sin\omega t.$$

(b) The eddy-current density calculated in (a) produces a magnetic field which supplements the external magnetic field. Repeating the calculation in part (a) using this additional magnetic field produces a correction to the current density calculated in part (a). This is precisely a self-inductive effect. Now, the total azimuthal $\mathbf{j}(\rho)$ may be regarded as a superposition of nested solenoids. The "solenoid" at radius ρ produces a magnetic field

$$d\mathbf{B}'(\rho') = \begin{cases} \mu_0 j(\rho) d\rho \hat{\mathbf{z}} & \rho' < \rho, \\ 0 & \rho' > \rho. \end{cases}$$

Therefore, the Faraday-induced supplement to the external magnetic field is

$$\mathbf{B}'(\rho) = \int_\rho^R d\rho' d\mathbf{B}' = \frac{1}{2}\mu_0 \sigma B_0 \omega \sin \omega t \left(\frac{R^2 - \rho^2}{2} \right) \hat{\mathbf{z}}.$$

The associated correction to the magnetic flux through the tube is

$$\Phi'(\rho) = 2\pi \int_0^\rho d\rho' \rho' \frac{1}{2}\mu_0 \sigma B_0 \omega \sin \omega t \left(\frac{R^2 - \rho'^2}{2} \right).$$

Repeating the calculation in part (a) gives the correction to the eddy-current density:

$$\mathbf{j}'(\rho) = -\frac{1}{4}\mu_0 B_0 \rho \sigma^2 \omega^2 \cos \omega t \left(\frac{R^2}{2} - \frac{\rho^2}{4} \right) \hat{\boldsymbol{\phi}}.$$

(c) The self-inductance can be neglected when $j'/j \sim \mu_0 \sigma \omega R^2 \sim \omega \tau_M \ll 1$.

14.24 Eddy-Current Levitation

(a) Because $a \ll b$, the magnetic field of the loop is nearly constant over the volume of the sphere and may be approximated by the on-axis field of a current loop at a height z above the plane of the loop. This was calculated in Chapter 10 as

$$\mathbf{B}_0 = \frac{1}{2} \frac{\mu_0 I_0 b^2}{(b^2 + z^2)^{3/2}} \exp(-i\omega t) \hat{\mathbf{z}} = \mu_0 \mathbf{H}_0.$$

Because $\delta \ll a$, the ohmic sphere behaves no differently than a perfectly conducting sphere in a uniform magnetic field. Therefore, we can use the results of Chapter 13 and conclude that the eddy currents are characterized by a magnetic moment $\mathbf{m} = -2\pi a^3 \mathbf{H}_0$. By symmetry, the instantaneous force on the dipole is in the z-direction:

$$\mathbf{F}(t) = m_k(t) \nabla B_k(t) = -2\pi a^3 \frac{B_0(t)}{\mu_0} \frac{\partial B_0(t)}{\partial z}.$$

The time-averaged force is

$$\langle \mathbf{F} \rangle = \frac{3}{4}\pi a^3 b^4 \frac{z}{(b^2+z^2)^4}\mu_0 I_0^2 \hat{\mathbf{z}}.$$

Setting this upward force equal to the downward force mg gives the value of I_0 required for levitation.

(b) Using a^3 as the characteristic volume of the sample and b as the characteristic size of the source, the text suggests that, when $\delta \ll a$,

$$\langle \mathbf{F} \rangle \sim \frac{a^3}{\mu_0 b} B_0^2 \sim a^3 b^4 \frac{1/b}{(b^2+z^2)^3}\mu_0 I_0^2 \hat{\mathbf{z}}.$$

This does not quite agree with the foregoing until we put $z = b$. This is reasonable because only one length scale was used in the text to characterize the source.

Source: G. Wouch, *American Journal of Physics* **46**, 464 (1978).

14.25 Dipole down the Tube

(a) When the dipole is at z_0, the vector potential at a point $\mathbf{r} = a\hat{\boldsymbol{\rho}} + z'\hat{\mathbf{z}}$ on the ring is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times(\mathbf{r}-z_0\hat{\mathbf{z}})}{|\mathbf{r}-z_0\hat{\mathbf{z}}|^3} = \frac{\mu_0}{4\pi}\frac{ma}{[a^2+(z_0-z')^2]^{3/2}}\hat{\boldsymbol{\phi}} = \frac{\mu_0 ma}{4\pi r_0^3}\hat{\boldsymbol{\phi}}.$$

The associated magnetic flux through the ring is

$$\Phi_{\rm B} = \int d\mathbf{S}\cdot\mathbf{B} = \oint d\mathbf{s}\cdot\mathbf{A} = a\int_0^{2\pi} d\phi\, A_\phi = \frac{\mu_0 ma^2}{2[a^2+(z_0-z')^2]^{3/2}} = \frac{\mu_0 ma^2}{2r_0^3}.$$

(b) Using the convective derivative and the quasi-static approximation, Faraday's law reads

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\partial\Phi}{\partial t} - (\mathbf{v}\cdot\nabla)\Phi = v\frac{\partial\Phi}{dz'}.$$

(c) The resistance of a wire with conductivity σ, length L, and cross sectional area A is $R = L/\sigma A$. Here, the current is circumferential so $L = 2\pi a$ and $A = dz't$. Therefore, setting $G = 1/R$ so Ohm's law reads $I = \mathcal{E}G$, we have

$$dI = \mathcal{E}dG = \mathcal{E}\frac{\sigma t}{2\pi a}dz' = \frac{\sigma vt}{2\pi a}\frac{\partial\Phi_{\rm B}}{\partial z'}dz' = \frac{3\mu_0 mav\sigma t}{4\pi}\frac{(z_0-z')}{r_0^5}dz'.$$

(d) The total power dissipated by Joule heating is

$$\begin{aligned}
\int \mathcal{E} dI &= \int_{-\infty}^{\infty} dz' \left[\frac{3\mu_0 m a^2 v}{2} \frac{(z_0 - z')}{r_0^5}\right] \left[\frac{3\mu_0 m a v \sigma t}{4\pi} \frac{(z_0 - z')}{r_0^5}\right] \\
&= \frac{9\mu_0^2 m^2 a^3 v^2 \sigma t}{8\pi} \int_{-\infty}^{\infty} dz' \frac{(z_0 - z')^2}{r_0^{10}} \\
&= \frac{9\mu_0^2 m^2 a^3 v^2 \sigma t}{8\pi} \int_{-\infty}^{\infty} dz' \frac{(z_0 - z')^2}{[a^2 + (z_0 - z')^2]^5} \\
&= \frac{9\mu_0^2 m^2 a^3 v^2 \sigma t}{8\pi} \frac{1}{a^7} \int_{-\pi/2}^{\pi/2} d\theta \sin^2\theta \cos^6\theta \\
&= \frac{9\mu_0^2 m^2 a^3 v^2 \sigma t}{8\pi} \frac{1}{a^7} \frac{5\pi}{138} \\
&= \frac{45}{1024} \frac{\mu_0^2 m^2 v^2 \sigma t}{a^4}.
\end{aligned}$$

Hence, the drag force is

$$F = \frac{45}{1024} \frac{\mu_0^2 m^2 v \sigma t}{a^4}.$$

(e) We get the terminal velocity by setting the drag force equal to the weight w of the magnet. This gives

$$v_T = \frac{1024}{45} \frac{w a^4}{\mu_0^2 m^2 \sigma t}.$$

Chapter 15: General Electromagnetic Fields

15.1 Continuous Creation

(a) Operate with $(1/c^2)\partial/\partial t$ on the modified Gauss' law and add this to the divergence of the modified Ampère-Maxwell law. The result is

$$\frac{1}{c^2}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{E}) = \mu_0\left(\frac{\partial\rho}{\partial t}+\nabla\cdot\mathbf{j}\right)+\frac{1}{c^2}\frac{\partial}{\partial t}(\nabla\cdot\mathbf{E}) - \lambda\left(\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\varphi}{\partial t}\right).$$

The modified continuity equation will be satisfied if we choose the potentials so

$$\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\varphi}{\partial t} = \frac{\mu_0}{\lambda}R.$$

The theory is *not* gauge invariant because the equation above chooses a gauge.

(b) The unaltered Maxwell equations, $\nabla\cdot\mathbf{B}=0$ and $\nabla\times\mathbf{E}=-\partial\mathbf{B}/\partial t$, guarantee that $\mathbf{B}=\nabla\times\mathbf{A}$ and $\mathbf{E}=-\nabla\varphi-\partial\mathbf{A}/\partial t$ are still true. Then, because φ_0 is a constant and $f(r,t)$ is a function of the radial variable only,

$$\mathbf{E} = -\frac{\partial\mathbf{A}}{\partial t} = -\mathbf{r}\dot{f}$$

$$\mathbf{B} = \nabla\times\mathbf{A} = f\,\nabla\times\mathbf{r} - \mathbf{r}\times\nabla f = 0.$$

From the modified Maxwell equations, the charge and current densities associated with these fields are

$$\rho = \epsilon_0\nabla\cdot\mathbf{E}+\epsilon_0\lambda\varphi = -3\epsilon_0\dot{f} - \epsilon_0 r\dot{f}' + \epsilon_0\lambda\varphi_0 \tag{15.1}$$

$$\mathbf{j} = \nabla\times\mathbf{B}/\mu_0 + \mathbf{A}\lambda/\mu_0 - \epsilon_0\frac{\partial\mathbf{E}}{\partial t} = \mathbf{r}\left(\epsilon_0\ddot{f}+f\lambda/\mu_0\right). \tag{15.2}$$

Substituting these into the modified continuity equation gives

$$\nabla\cdot\mathbf{j}+\frac{\partial\rho}{\partial t} = \frac{\lambda}{\mu_0}\left(3f+rf'\right) = R. \tag{15.3}$$

(c) A particular solution of (3) is $f_0 = R\mu_0/3\lambda$. The solution to the homogeneous equation $3f+rf'=0$ is

$$f = \frac{const.}{r^3}.$$

This is singular at the origin and so must be discarded.

(d) In light of part (c), we choose $f=f_0$ so (1) and (2) give

$$\mathbf{v} = \frac{\mathbf{j}}{\rho} = \frac{R}{3\epsilon_0\lambda\varphi_0}\mathbf{r}.$$

This means that matter moves out radially from every point in space. This is consistent with the modified continuity equation which says that matter is created at every point in space.

Source: R.A. Lyttleton and H. Bondi, *Proceedings of the Royal Society of London A* **252**, 313 (1959).

15.2 Lorenz Gauge Forever

(a) The Lorenz gauge constraint is

$$\nabla \cdot \mathbf{A}_L + \frac{1}{c^2}\frac{\partial \varphi_L}{\partial t} = 0.$$

If we insist that $\mathbf{A}'$ and φ' do the same, we must have

$$0 = \nabla \cdot \mathbf{A}_L - \nabla \cdot \nabla\Lambda + \frac{1}{c^2}\frac{\partial \varphi_L}{\partial t} + \frac{\partial^2 \Lambda}{\partial t^2} = -\nabla^2\Lambda + \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2}.$$

In other words, Λ must satisfy the homogeneous wave equation.

(b) The Lorenz gauge potentials satisfy

$$\begin{aligned}\nabla^2\varphi_L - \frac{1}{c^2}\frac{\partial^2\varphi_L}{\partial t^2} &= -\frac{\rho}{\epsilon_0}\\ \nabla^2\mathbf{A}_L - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} &= -\mu_0\mathbf{j}.\end{aligned}$$

The change

$$\varphi_L \to \varphi_L + \frac{\partial\Lambda}{\partial t} \qquad\qquad \mathbf{A}_L \to \mathbf{A}_L - \nabla\Lambda$$

transforms the equations of motion to

$$\begin{aligned}\nabla^2\varphi_L + \nabla^2\frac{\partial\Lambda}{\partial t} - \frac{1}{c^2}\frac{\partial^2\varphi_L}{\partial t^2} - \frac{1}{c^2}\frac{\partial^3\Lambda}{\partial t^3} &= -\frac{\rho}{\epsilon_0}\\ \nabla^2\mathbf{A}_L - \nabla^2\nabla\Lambda - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2\nabla\Lambda}{\partial t^2} &= -\mu_0\mathbf{j},\end{aligned}$$

or

$$\begin{aligned}\nabla^2\varphi_L - \frac{1}{c^2}\frac{\partial^2\varphi_L}{\partial t^2} + \frac{\partial}{\partial t}\left[\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}\right] &= -\frac{\rho}{\epsilon_0}\\ \nabla^2\mathbf{A}_L - \frac{1}{c^2}\frac{\partial^2\mathbf{A}_L}{\partial t^2} - \nabla\left[\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}\right] &= -\mu_0\mathbf{j}.\end{aligned}$$

We conclude that the Lorenz equations of motion for the potentials are invariant to gauge transformations where the gauge function $\Lambda(\mathbf{r}, t)$ satisfies the homogeneous wave equation.

15.3 Gauge Invariant Vector Potential

Let us apply the Helmholtz theorem to $\mathbf{A}_\perp$ and $\mathbf{A}_\parallel$ separately. This gives

$$\mathbf{A}_\perp(\mathbf{r},t) = -\nabla\int d^3r'\,\frac{\nabla'\cdot\mathbf{A}_\perp(\mathbf{r}',t)}{4\pi|\mathbf{r}-\mathbf{r}'|} + \nabla\times\int d^3r'\,\frac{\nabla'\times\mathbf{A}_\perp(\mathbf{r}',t)}{4\pi|\mathbf{r}-\mathbf{r}'|} = \nabla\times\int d^3r'\,\frac{\nabla'\times\mathbf{A}_\perp(\mathbf{r}',t)}{4\pi|\mathbf{r}-\mathbf{r}'|}$$

and

$$\mathbf{A}_{\|}(\mathbf{r},t) = -\nabla \int d^3r' \, \frac{\nabla' \cdot \mathbf{A}_{\|}(\mathbf{r}',t)}{4\pi|\mathbf{r}-\mathbf{r}'|} + \nabla \times \int d^3r' \, \frac{\nabla' \times \mathbf{A}_{\|}(\mathbf{r}',t)}{4\pi|\mathbf{r}-\mathbf{r}'|} = -\nabla \int d^3r' \, \frac{\nabla' \cdot \mathbf{A}_{\|}(\mathbf{r}',t)}{4\pi|\mathbf{r}-\mathbf{r}'|}.$$

A general change of gauge for the vector potential is $\mathbf{A} \to \mathbf{A} + \nabla\Lambda$. This has no effect on $\mathbf{A}_\perp$ because $\nabla \times \nabla\Lambda = 0$. By contrast, $\mathbf{A}_{\|}$ changes because there is no reason to suppose that $\nabla^2\Lambda = 0$.

15.4 Transverse Current Density in the Coulomb Gauge

The definition of the transverse current density given in the text is

$$\mathbf{j}_\perp(\mathbf{r},t) = \mathbf{j}(\mathbf{r},t) + \nabla \frac{1}{4\pi} \int d^3r' \, \frac{\nabla' \cdot \mathbf{j}(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}. \tag{1}$$

On the other hand, the text discussion of the Helmholtz theorem proves that

$$\frac{\nabla' \cdot \mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \nabla' \cdot \left[\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] + \mathbf{j}(\mathbf{r}') \cdot \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}. \tag{2}$$

Insert (2) into (1) and note that the total divergence term vanishes for a localized current distribution. Therefore,

$$j_{\perp,k}(\mathbf{r},t) = j_k(\mathbf{r},t) + \nabla_k \frac{1}{4\pi} \int d^3r' \, j_\ell(\mathbf{r}') \nabla_\ell \frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

or

$$j_{\perp,k}(\mathbf{r},t) = j_k(\mathbf{r},t) + \frac{1}{4\pi} \int d^3r' \, \left[\nabla_k \nabla_\ell \frac{1}{|\mathbf{r}-\mathbf{r}'|}\right] j_\ell(\mathbf{r}'). \tag{3}$$

On the other hand,

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}'|} = -4\pi\delta(\mathbf{r}-\mathbf{r}').$$

Using this, (3) can be written in the form

$$j_{\perp,k}(\mathbf{r},t) = \int d^3r' \, \left[-\frac{1}{4\pi}\left(\delta_{k\ell}\nabla^2 - \nabla_k\nabla_\ell\right)\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right] j_\ell(\mathbf{r}').$$

This completes the proof because the quantity in square brackets is the transverse delta function defined in the problem statement.

15.5 Poincaré Gauge

(a) When $\mathbf{E}$ is a constant vector,

$$-\nabla\varphi = -\nabla\left[-\mathbf{r}\cdot\mathbf{E}\right] = \nabla(r_k E_k) = \mathbf{E}.$$

Similarly, when $\mathbf{B}$ is a constant vector,

$$\nabla \times \mathbf{A} = -\frac{1}{2}\nabla \times (\mathbf{r} \times \mathbf{B}) = -\frac{1}{2}\left[(\mathbf{B}\cdot\nabla)\mathbf{r} - (\mathbf{r}\cdot\nabla)\mathbf{B} + \mathbf{r}(\nabla\cdot\mathbf{B}) - \mathbf{B}(\nabla\cdot\mathbf{r})\right].$$

But $\nabla\cdot\mathbf{r} = 3$, $\nabla\cdot\mathbf{B} = 0$, and $(\mathbf{B}\cdot\nabla)\mathbf{r} = \mathbf{B}$. Therefore,

$$\nabla \times \mathbf{A} = -\frac{1}{2}\left[\mathbf{B} - 3\mathbf{B}\right] = \mathbf{B}.$$

(b) By the chain rule,

$$\frac{d\mathbf{G}}{d\lambda} = \frac{d(\lambda r_k)}{d\lambda}\frac{d\mathbf{G}}{d(\lambda r_k)} = r_k \frac{1}{\lambda}\frac{d\mathbf{G}}{dr_k} = \frac{1}{\lambda}\mathbf{r}\cdot\nabla\mathbf{G}.$$

We will use this identity below with both $\mathbf{G} = \mathbf{B}$ and $\mathbf{G} = \mathbf{E}$.

$$\begin{aligned}\nabla \times \mathbf{A}(\mathbf{r},t) &= \int_0^1 d\lambda\,\lambda\,\nabla\times\{\mathbf{B}(\lambda\mathbf{r},t)\times\mathbf{r}\}\\ &= \int_0^1 d\lambda\,\lambda\,\{\mathbf{B}\nabla\cdot\mathbf{r} - \mathbf{r}\nabla\cdot\mathbf{B} + (\mathbf{r}\cdot\nabla)\mathbf{B} - (\mathbf{B}\cdot\nabla)\mathbf{r}\}.\end{aligned}$$

As above, $\nabla\cdot\mathbf{r} = 3$, $\nabla\cdot\mathbf{B} = 0$, and $(\mathbf{B}\cdot\nabla)\mathbf{r} = \mathbf{B}$. Therefore,

$$\nabla \times \mathbf{A} = \int_0^1 d\lambda\,\lambda\,\{2\mathbf{B}(\lambda\mathbf{r},t) + (\mathbf{r}\cdot\nabla)\mathbf{B}(\lambda\mathbf{r},t)\}.$$

Hence, using the identity proved above,

$$\nabla \times \mathbf{A}(\mathbf{r},t) = \int_0^1 d\lambda\,\lambda\left\{2\mathbf{B}(\lambda\mathbf{r},t) + \lambda\frac{d}{d\lambda}\mathbf{B}(\lambda\mathbf{r},t)\right\} = \int_0^1 d\lambda\frac{d}{d\lambda}\left\{\lambda^2\mathbf{B}(\lambda\mathbf{r},t)\right\} = \mathbf{B}(\mathbf{r},t).$$

Similarly,

$$\begin{aligned}\nabla\varphi(\mathbf{r},t) &= -\int_0^1 d\lambda\nabla\{\mathbf{r}\cdot\mathbf{E}(\lambda\mathbf{r},t)\}\\ &= -\int_0^1 d\lambda\,\{(\mathbf{r}\cdot\nabla)\mathbf{E} + (\mathbf{E}\cdot\nabla)\,\mathbf{r} + \mathbf{r}\times(\nabla\times\mathbf{E}) + \mathbf{E}\times(\nabla\times\mathbf{r})\}.\end{aligned}$$

But $\nabla\times\mathbf{r} = 0$ and $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ so

$$-\nabla\varphi(\mathbf{r},t) = \int_0^1 d\lambda\left\{\frac{d}{d\lambda}\{\lambda\mathbf{E}(\lambda\mathbf{r},t)\} - \mathbf{r}\times\frac{\partial\mathbf{B}}{\partial t}\right\} = \mathbf{E}(\mathbf{r},t) - \int_0^1 d\lambda\,\mathbf{r}\times\frac{\partial\mathbf{B}}{\partial t}.$$

This proves that $\mathbf{E} = -\nabla\varphi - \dfrac{\partial \mathbf{A}}{\partial t}$ because

$$\frac{\partial \mathbf{A}}{\partial t} = -\int_0^1 d\lambda\,\lambda\,\mathbf{r} \times \frac{\partial}{\partial t}\mathbf{B}(\lambda\mathbf{r}, t).$$

Source: W.E. Brittin, W.R. Smythe, and W.Wyss, *American Journal of Physics* **50**, 693 (1982).

15.6 First-Order Equations for Numerical Electrodynamics

(a) $\mathbf{B} = \nabla \times \mathbf{A}$ guarantees that $\nabla \cdot \mathbf{B} = 0$. Similarly, $\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t$ guarantees that $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$. Hence, if Gauss' law is maintained, it is sufficient to satisfy the Ampère-Maxwell law,

$$\nabla \times \mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

The two equations given do this because the final one fixes the Lorenz gauge.

(b) Using the given equations and the (implied) continuity equation,

$$\frac{\partial \mathcal{C}}{\partial t} = \frac{\partial}{\partial t}\left[\nabla \cdot \mathbf{E} - \rho/\epsilon_0\right] = \nabla \cdot \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{\epsilon_0}\frac{\partial \rho}{\partial t} = c^2\mu_0\nabla \cdot \mathbf{j} - \frac{1}{\epsilon_0}\frac{\partial \rho}{\partial t} = 0.$$

(c) Since $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$, we see that $\Gamma = \nabla \cdot \mathbf{A}$ and that the first two equations given reproduce the two equations in (a). All that remains is the time evolution of Γ. Using $\dot{\mathbf{A}}$ from the statement of the problem, this is

$$\frac{\partial \Gamma}{\partial t} = \nabla \cdot \frac{\partial \mathbf{A}}{\partial t} = -\nabla \cdot \mathbf{E} - \nabla^2\varphi.$$

This equation and the others form a closed set. The crucial step is to assume Gauss' law and then check that it is maintained. This gives the final evolution equation as

$$\frac{\partial \Gamma}{\partial t} = -\rho/\epsilon_0 - \nabla^2\varphi.$$

(d)

$$\begin{aligned}
\frac{\partial^2 \mathcal{C}}{\partial t^2} &= \frac{\partial}{\partial t}\nabla \cdot \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{\epsilon_0}\frac{\partial^2 \rho}{\partial t^2} \\
&= c^2\frac{\partial}{\partial t}\nabla \cdot \left[-\nabla^2\mathbf{A} + \nabla\Gamma - \mu_0\mathbf{j}\right] - \frac{1}{\epsilon_0}\frac{\partial^2 \rho}{\partial t^2} \\
&= -c^2\nabla \cdot \left[\nabla^2\frac{\partial \mathbf{A}}{\partial t} - \nabla\frac{\partial \Gamma}{\partial t}\right] - \frac{1}{\epsilon_0}\frac{\partial}{\partial t}\left[\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t}\right] \\
&= c^2\nabla \cdot \left[\nabla^2(\mathbf{E} + \nabla\varphi) - \nabla(\rho/\epsilon_0 + \nabla^2\varphi)\right] \\
&= c^2\nabla^2(\nabla \cdot \mathbf{E} - \rho/\epsilon_0) \\
&= c^2\nabla^2\mathcal{C}.
\end{aligned}$$

Source: A.M. Knapp, E.J. Walker, and T.W. Baumgarte, *Physical Review D* **65**, 064031 (2002).

15.7 Elementary Energy Conservation

As the bar moves, Faraday induction drives a current $I(t)$ through the expanding rectangular circuit formed by the bar and the U-shaped portion of the rails to its left. Our task is to prove that

$$\frac{1}{2}mv_0^2 = R\int_0^\infty dt I^2(t).$$

We determine $I(t)$ from the flux law,

$$I(t)R = \frac{d}{dt}\int d\mathbf{S}\cdot\mathbf{B} = B\frac{dA}{dt} = B\ell v(t),$$

where $v(t)$ is the speed of the moving bar. The magnitude of the drag force on the bar is $F(t) = I(t)\ell B$. Therefore,

$$m\dot{v} = -I\ell B = -\frac{\ell^2 B^2}{R}v,$$

from which we deduce that

$$v(t) = v_0 \exp(-\ell^2 B^2 t/mR).$$

Therefore, it remains only to check that

$$\frac{1}{2}mv_0^2 = R\left(\frac{\ell B}{R}\right)^2\int_0^\infty dt v^2(t) = \frac{\ell^2 B^2 v_0^2}{R}\int_0^\infty dt \exp(-2\ell^2 B^2/mR) = \frac{1}{2}mv_0^2.$$

Source: E.M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1985).

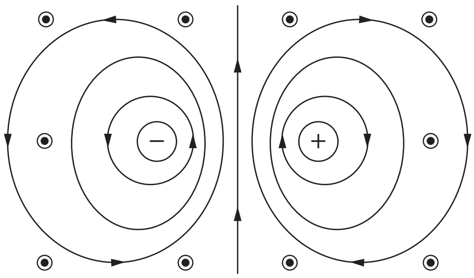
15.8 The Poynting Vector Field

(a) Let $\mathbf{E}_+$ and $\mathbf{E}_-$ be the electric fields produced by q and $-q$. The Poynting vector will have the suggested properties if it has zero divergence. By direct computation,

$$\nabla\cdot\mathbf{S} = \frac{1}{\mu_0}\nabla\cdot[(\mathbf{E}_+ + \mathbf{E}_-)\times\mathbf{B}] = \mathbf{B}\cdot\nabla\times(\mathbf{E}_+ + \mathbf{E}_-) - (\mathbf{E}_+ + \mathbf{E}_-)\cdot\nabla\times\mathbf{B}.$$

This is zero because $(\mathbf{E}_+ + \mathbf{E}_-)$ is a static field with zero curl and $\mathbf{B}$ is uniform with zero curl.

(b)



15.9 Poynting Vector Matching Condition

Write $\mathbf{E} = \mathbf{E}_{\|} + \mathbf{E}_{\perp}$ to partition the electric field into components parallel and perpendicular to the interface. If we do the same for $\mathbf{H}$ and the Poynting vector in matter, $\mathbf{S} = \mathbf{E} \times \mathbf{H}$,

$$\mathbf{S} = (\mathbf{E}_{\|} + \mathbf{E}_{\perp}) \times (\mathbf{H}_{\|} + \mathbf{H}_{\perp}).$$

We have $\mathbf{E}_{\perp} \times \mathbf{H}_{\perp} = 0$, but

$$\mathbf{S}_{\perp} = \mathbf{E}_{\|} \times \mathbf{H}_{\| } \qquad \text{and} \qquad \mathbf{S}_{\|} = \mathbf{E}_{\|} \times \mathbf{H}_{\perp} + \mathbf{E}_{\perp} \times \mathbf{H}_{\|}.$$

The tangential component of $\mathbf{E}$ is always continuous. In the absence of free surface current, the tangential component of $\mathbf{H}$ is also continuous. Therefore, $\mathbf{S}_{\perp}$ is continuous. This is a physical necessity if we regard this vector as an energy current density. By contrast, $\mathbf{S}_{\|}$ is generally *discontinuous* because $\mathbf{E}_{\perp}$ is generally discontinuous (if either free charge or polarization charge is present at the interface) and, because $\mathbf{B}_{\perp}$ is always continuous, $\mathbf{H}_{\perp}$ will be discontinuous at the boundary between magnetically dissimilar materials.

Source; F.N.H. Robinson, *SIAM Review* **36**, 633 (1994).

15.10 A Poynting Theorem Check

The Poynting theorem is

$$\nabla \cdot \mathbf{S} + \frac{\partial u_{\rm EM}}{\partial t} = -\mathbf{j} \cdot \mathbf{E}.$$

The magnetic field produced by the moving sheet has magnitude $\mu_0 \sigma v/2$ and points parallel to the sheet (but in opposite directions on opposite sides of the sheet). Both $\mathbf{E}$ and $\mathbf{B}$ are constant in space and time so the left side of the theorem is zero. The right side of the theorem is zero because $\mathbf{j} = 0$ at points away from the sheet.

15.11 A Charged Particle in a Static Electromagnetic Field

Let $\mathbf{j}_q$, $\mathbf{E}_q$, and $\mathbf{B}_q$ be the current density, electric field, and magnetic field produced by the moving particle. The total fields are $\mathbf{E}_0 + \mathbf{E}_q$ and $\mathbf{B}_0 + \mathbf{B}_q$, but only the *cross* terms make physically relevant contributions to Poynting's theorem. Moreover, the flux of the Poynting vector $\mathbf{S}$ through a surface at infinity is zero if there is no radiation. Therefore, the relevant statement of the theorem is

$$\frac{d}{dt}\int d^3r\left[\epsilon_0\mathbf{E}_0\cdot\mathbf{E}_q+\frac{1}{\mu_0}\mathbf{B}_0\cdot\mathbf{B}_q\right]+\int d^3r\,\mathbf{j}_q\cdot\mathbf{E}_0=0. \tag{1}$$

Now, $\mathbf{E}_0=-\nabla\varphi_0$ and $\rho(\mathbf{r})=q\delta(\mathbf{r}-\mathbf{r}_q)$ is the charge density of the charge. Therefore the electric energy is

$$\epsilon_0\int d^3r\mathbf{E}_0\cdot\mathbf{E}_q=-\epsilon_0\int d^3r\,\nabla\varphi_0\cdot\mathbf{E}_q=\int d^3r\,\varphi_0\nabla\cdot\mathbf{E}_q=\int d^3r\,\varphi_0\rho_q=q\varphi_0(\mathbf{r}_q).$$

On the other hand,

$$\int d^3r\,\mathbf{j}_q\cdot\mathbf{E}_0=\int d^3r\,q\mathbf{v}\delta(\mathbf{r}-\mathbf{r}_q)\cdot\mathbf{E}_0=q\mathbf{E}_0\cdot\mathbf{v}$$

is equal to the change in the particle's kinetic energy computed using Newton's second law and the fact that the Lorentz magnetic force does no work on a charged particle:

$$\frac{d}{dt}\left(\frac{1}{2}mv^2\right)=m\mathbf{v}\cdot\mathbf{a}=q\mathbf{E}_0\cdot\mathbf{v}.$$

We conclude from these calculations that

$$\frac{d}{dt}\left[\frac{1}{2}mv^2+q\varphi_0(\mathbf{r}_q)\right]=0$$

is equivalent to Poynting's theorem (1) if we ignore the magnetic energy. The hint from the problem statement is that the magnetic energy is related to the work done by $\mathbf{E}_q$ and $\mathbf{B}_q$ on the sources of $\mathbf{E}_0$ and $\mathbf{B}_0$. To check, we write

$$\frac{1}{\mu_0}\int d^3r\,\mathbf{B}_0\cdot\mathbf{B}_q=\frac{1}{\mu_0}\int d^3r\,\mathbf{j}_0\cdot\mathbf{A}_q,$$

where $\mathbf{j}_0$ is the source of $\mathbf{B}_0$ and $\mathbf{E}_q=-\nabla\varphi_q-\partial\mathbf{A}_q/\partial t$. Hence,

$$\frac{d}{dt}\frac{1}{\mu_0}\int d^3r\,\mathbf{B}_0\cdot\mathbf{B}_q=\frac{1}{\mu_0}\int d^3r\,\mathbf{j}_0\cdot\frac{\partial\mathbf{A}_q}{\partial t}=-\frac{1}{\mu_0}\int d^3r\,\mathbf{j}_0\cdot(\mathbf{E}_q+\nabla\varphi_q)\,.$$

Integrating by parts and using $\nabla\cdot\mathbf{j}_0=0$ gives

$$\frac{d}{dt}\frac{1}{\mu_0}\int d^3r\,\mathbf{B}_0\cdot\mathbf{B}_q=-\frac{1}{\mu_0}\int d^3r\,\mathbf{j}_0\cdot\mathbf{E}_q+\frac{1}{\mu_0}\int d^3r\,\varphi(\nabla\cdot\mathbf{j})=-\frac{1}{\mu_0}\int d^3r\,\mathbf{j}_0\cdot\mathbf{E}_q.$$

In other words the time rate of change of the magnetic energy is the negative of the work done by the moving particle on the sources of $\mathbf{E}_0$ and $\mathbf{B}_0$. This work is non-zero because the moving particle induces an EMF on the source charge and current. Collecting results, the Poynting theorem (1) says

$$\frac{d}{dt}\left[\frac{1}{2}mv^2 + q\varphi_0(\mathbf{r}_q)\right] = \frac{1}{\mu_0}\int d^3r\,\mathbf{j}_0\cdot\mathbf{E}_q,$$

and we get proper conservation of energy by canceling the right-hand side with $dW_{\rm ext}/dt$, which is the power supplied by an external source to maintain $\mathbf{E}_0$ and $\mathbf{B}_0$.

Source: J. Paton, *European Journal of Physics* **13**, 280 (1992).

15.12 Energy Flow in a Coaxial Cable

(a) Using Gauss' law and Ampère's law in integral form, it is straightforward to find the electric field and magnetic field at every point space. If z is the direction of the current flow in the outer cylinder,

$$\mathbf{E} = \begin{cases} 0 & \rho < a, \\ -\dfrac{\lambda}{2\pi\epsilon_0\rho}\hat{\boldsymbol{\rho}} & a < \rho < b, \\ 0 & \rho > b, \end{cases}$$

and

$$\mathbf{B} = \begin{cases} 0 & \rho < a, \\ -\dfrac{\mu_0 I}{2\pi\rho}\hat{\boldsymbol{\phi}} & a < \rho < b, \\ 0 & \rho > b. \end{cases}$$

The Poynting vector, which is non-zero only between the cylinders, is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{\lambda I}{4\pi^2\epsilon_0\rho^2}\hat{\mathbf{z}}.$$

Therefore, the power through a cross section is

$$\mathcal{P} = \int dA\,\hat{\mathbf{z}}\cdot\mathbf{S} = 2\pi\int_a^b d\rho\frac{\lambda I}{4\pi\epsilon_0\rho} = \frac{\lambda I}{2\pi\epsilon_0}\ln\frac{b}{a}.$$

(b) The potential difference between the cylinders is

$$V = \int_a^b d\boldsymbol{\ell}\cdot\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0}\int_a^b\frac{d\rho}{\rho} = \frac{\lambda}{2\pi\epsilon_0}\ln\frac{b}{a} = IR.$$

Comparing this to the answer in part (a) shows that we get the expected result,

$$\mathcal{P} = I^2R.$$

15.13 Energy Conservation for Quasi-Magnetostatic Fields

(a) The applied magnetic field is along y. This implies that the magnetic field everywhere has only a y-component because the normal component of $\mathbf{B}$ and the tangential components of $\mathbf{H}$ are continuous at $z=0$. In the quasi-magnetostatic limit, we solve the diffusion equation with an assumed time dependence $\exp(-i\omega t)$:

$$\nabla^2 B_y = \frac{d^2 B_y}{dz^2} = \mu_0\sigma\frac{\partial B_y}{\partial t} = -i\omega\mu_0\sigma B_y = -k^2 B_y.$$

Here, $k=\sqrt{i\mu_0\sigma\omega}=(1+i)/\delta$ where $\delta=\sqrt{2/\mu_0\sigma\omega}$ is the skin depth. The general solution of this equation is a linear combination of $\exp(ikz)$ and $\exp(-ikz)$. The boundary condition is satisfied and the magnetic field decays exponentially into the conductor as

$$\mathbf{B}_{\rm in}(z,t) = \hat{\mathbf{y}}B_0 e^{i(kz-\omega t)} = \hat{\mathbf{y}}B_0 e^{-z/\delta}e^{iz/\delta}e^{-i\omega t}.$$

We get the associated electric field from Ampère's law neglecting the displacement current:

$$\nabla\times\mathbf{B} = -\hat{\mathbf{x}}\frac{\partial B_y}{\partial z} = \mu_0\sigma\mathbf{E}.$$

This gives

$$\mathbf{E}_{\rm in}(z,t) = \hat{\mathbf{x}}\frac{1-i}{\mu_0\sigma\delta}B_y(z,t).$$

(b) We take the real part of the fields calculated in (a) to evaluate the Poynting theorem. Moreover, if $A(\mathbf{r},t)=a(\mathbf{r})e^{-i\omega t}$ and $B(\mathbf{r},t)=b(\mathbf{r})e^{-i\omega t}$, we have shown that the time average over one period of a quantity that is quadratic in the field is

$$\langle A(\mathbf{r},t)B(\mathbf{r},t)\rangle = \frac{1}{2}\mathrm{Re}\left[a^*(\mathbf{r})b(\mathbf{r})\right].$$

On the other hand, every term in the time average of the *time derivative* of a quantity which is quadratic in the fields. Therefore, the time-average is zero because

$$\int_0^{2\pi/\omega} dt\frac{\partial}{\partial t}\cos^2\omega t = \int_0^{2\pi/\omega} dt\frac{\partial}{\partial t}\sin^2\omega t = \int_0^{2\pi/\omega} dt\frac{\partial}{\partial t}\sin\omega t\cos\omega t = 0.$$

Consequently, the $\partial u_{\rm EM}/\partial t$ term does not contribute to the theorem. Otherwise,

$$\langle\mathbf{S}\rangle = \frac{1}{2\mu_0}\mathrm{Re}\left[\mathbf{E}\times\mathbf{B}^*\right] = \hat{\mathbf{z}}\frac{B_0^2}{2\mu_0^2\sigma\delta}$$

and

$$\int_0^\infty dz\langle\mathbf{j}\cdot\mathbf{E}^*\rangle = \frac{B_0^2}{\mu_0^2\sigma\delta^2}\int_0^\infty dz\exp(-2z/\delta) = \frac{B_0^2}{2\mu_0^2\sigma\delta}.$$

For this problem, the unit normal $-\hat{\mathbf{n}}=\hat{\mathbf{z}}$ and the Poynting theorem is confirmed for every x-y unit area.

15.14 Energy to Spin Up a Charged Cylinder

(a) We treat the cylinder as infinitely long because $L \gg R$. An elementary application of Gauss' law gives

$$\mathbf{E}(\mathbf{r}) = \begin{cases} 0 & \rho < R, \\ \dfrac{\lambda\hat{\boldsymbol{\rho}}}{2\pi\epsilon_0\rho} & \rho > R. \end{cases}$$

Similarly, the rotating charged cylinder is a solenoid with surface current density $\mathbf{K} = \sigma R\omega_0$. Therefore,

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \mu_0\sigma R\omega_0\hat{\mathbf{z}} & \rho < R, \\ 0 & \rho > R. \end{cases}$$

(b) If we ignore the displacement current, the magnetic field is the same as in part (a) with ω_0 replaced by $\omega(t)$. That is,

$$\mathbf{B}(\mathbf{r}, t) = \begin{cases} \mu_0\sigma R\omega(t)\hat{\mathbf{z}} & \rho < R, \\ 0 & \rho > R. \end{cases}$$

By symmetry, the Faraday's law electric field is in the $\hat{\boldsymbol{\phi}}$ direction. Therefore, we choose C as a circle of radius ρ concentric with the z-axis to get

$$\frac{d\Phi_B}{dt} = \int_S d\mathbf{S}\cdot\frac{\partial\mathbf{B}}{\partial t} = -\oint_C d\boldsymbol{\ell}\cdot\mathbf{E} = E(\rho, t)2\pi\rho.$$

The magnetic flux is $\Phi_B = \pi\rho^2 B$ if $\rho < R$ and $\Phi_B = \pi R^2 B$ if $\rho > R$. Therefore, the Faraday electric field is

$$\mathbf{E}_F(\mathbf{r}, t) = \begin{cases} -\dfrac{1}{2}\mu_0\sigma R\rho\dfrac{d\omega}{dt}\hat{\boldsymbol{\phi}} & \rho < R, \\ -\dfrac{1}{2}\mu_0\sigma\dfrac{R^3}{\rho}\dfrac{d\omega}{dt}\hat{\boldsymbol{\phi}} & \rho > R. \end{cases}$$

The total electric field is the sum of $\mathbf{E}_F(\mathbf{r}, t)$ and $\mathbf{E}(\mathbf{r})$ found in part (a).

(c) If $\mathbf{S} = (\mathbf{E}\times\mathbf{B})/\mu_0$ and $u_{\rm EM} = \frac{1}{2}\epsilon_0(\mathbf{E}\cdot\mathbf{E} + c^2\mathbf{B}\cdot\mathbf{B})$, the Poynting theorem for an infinite volume is

$$-\int d^3r\,\mathbf{j}\cdot\mathbf{E} = \int dA\,\hat{\mathbf{n}}\cdot\mathbf{S} + \frac{d}{dt}\int d^3r\,u_{\rm EM}. \tag{1}$$

The current density $\mathbf{j} = \sigma\omega R\delta(\rho - R)\hat{\boldsymbol{\phi}}$ is localized on the cylinder surface. The power supplied by the external agent is the work done against the back EMF:

$$\mathcal{P} = -\int d^3r\,\mathbf{j}\cdot\mathbf{E} = -2\pi\sigma\omega RL\int_0^\infty d\rho\rho\delta(\rho - R)E_\phi = \pi\mu_0\sigma^2R^4L\omega\frac{d\omega}{dt}. \tag{2}$$

This work goes to establish the magnetic field, because

$$\frac{dU_B}{dt} = \frac{d}{dt}\frac{1}{2\mu_0}\int d^3r\mathbf{B}\cdot\mathbf{B} = \frac{d}{dt}\left[\frac{1}{2\mu_0}\left(\mu_0\sigma R\omega\right)^2\pi R^2 L\right] = \pi\mu_0\sigma^2 R^4 L\omega\frac{d\omega}{dt}. \tag{3}$$

The equality of (2) and (3) implies that the surface term and dU_E/dt do not contribute to the Poynting theorem in (1). The surface term is zero because $\mathbf{S}=0$ at infinity, in our approximation which neglects the displacement current. The time rate of change of the electric energy, $\dot{U}_E \propto \ddot{\omega}(t)$, is also negligible because the spin-up occurs very slowly.

(d) The rate at which energy flows into the interior of the cylinder is given by the Poynting vector surface integral evaluated on a surface slightly smaller than the shell radius. This is

$$-\int dA\hat{\boldsymbol{\rho}}\cdot\mathbf{S} = -2\pi RL\frac{1}{\mu_0}\hat{\boldsymbol{\rho}}\cdot\left(-\frac{1}{2}\mu_0\sigma R^2\frac{d\omega}{dt}\hat{\boldsymbol{\phi}}\right)\times\left(\mu_0\sigma R\omega\hat{\mathbf{z}}\right) = \pi\mu_0\sigma^2 R^4 L\omega\frac{d\omega}{dt} = \frac{dU_B}{dt}.$$

As anticipated, the rate of inward energy flow is equal to the rate of magnetic energy increase.

15.15 A Momentum Flux Theorem

If S is the boundary of V, we are interested in the integral

$$I = \epsilon_0\int_S d\mathbf{S}\cdot(\mathbf{E}_0\times\mathbf{B}_0) = \epsilon_0\int_S d^3r\,\nabla\cdot(\mathbf{E}\times\mathbf{B}).$$

Using a vector identity familiar from the Poynting theorem,

$$I = \epsilon_0\int_V d^3r\,(\nabla\times\mathbf{E})\cdot\mathbf{B} - \epsilon_0\int_V(\nabla\times\mathbf{B})\cdot\mathbf{E}.$$

The electric field is static, so $\nabla\times\mathbf{E}=0$, and the source of $\mathbf{B}$ is steady, so $\nabla\times\mathbf{B}=\mu_0\mathbf{j}$. Therefore,

$$I = -\frac{1}{c^2}\int_V d^3r\,\mathbf{j}\cdot\mathbf{E}.$$

We make further progress using the fact that $\mathbf{E}=-\nabla\varphi$. Therefore,

$$I = \frac{1}{c^2}\int_V d^3r\,\mathbf{j}\cdot\nabla\varphi = -\frac{1}{c^2}\int_V d^3r\,\varphi\nabla\cdot\mathbf{j} + \frac{1}{c^2}\int_V d^3r\nabla\cdot(\mathbf{j}\varphi).$$

A steady current satisfies $\nabla\cdot\mathbf{j}=0$. Therefore, using Gauss' theorem,

$$I = \frac{1}{c^2}\int_S d\mathbf{S}\cdot\mathbf{j}\varphi = 0.$$

This integral is zero because confinement of the particles implies that $d\mathbf{S}\cdot\mathbf{j}=0$.

15.16 An Electromagnetic Inequality

$$U_{\rm EM}=\tfrac{1}{2}\epsilon_0\int d^3r\,\left[\mathbf{E}\cdot\mathbf{E}+c^2\mathbf{B}\cdot\mathbf{B}\right]=\tfrac{1}{2}\epsilon_0\int d^3r\,\left[(|\mathbf{E}|-c|\mathbf{B}|)^2+2c|\mathbf{E}||\mathbf{B}|\right].$$

Therefore,

$$U_{\rm EM}\geq\epsilon_0 c\int d^3r\,|\mathbf{E}||\mathbf{B}|.$$

On the other hand,

$$\int d^3r\,|\mathbf{E}||\mathbf{B}|\geq\int d^3r\,|\mathbf{E}\times\mathbf{B}|,$$

while

$$\int d^3r\,|\mathbf{E}\times\mathbf{B}|\geq\left|\int d^3r\,\mathbf{E}\times\mathbf{B}\right|=|\mathbf{P}_{\rm EM}|/\epsilon_0.$$

This proves the assertion. Equality holds when each of the three inequalities above is separately an equality. The first is an equality when $|\mathbf{E}|=c|\mathbf{B}|$. The second is an equality when $\mathbf{E}\cdot\mathbf{B}=0$. The third is an equality when $\mathbf{E}\times\mathbf{B}$ has a constant direction at every point in space. These are the conditions that define a propagating plane wave.

15.17 Potential Momentum

(a) Faraday's law in integral form is

$$\oint_C d\boldsymbol{\ell}\cdot\mathbf{E}=-\frac{d}{dt}\int_S d\mathbf{S}\cdot\mathbf{B}.$$

The magnetic field is zero outside the solenoid. Inside, the field is $\mathbf{B}=\mu_0 nI\hat{\mathbf{z}}$. If we choose the circuit C coincident with the wire, a direct application of Faraday's law gives the electric field as

$$\mathbf{E}(r)=-\frac{\mu_0 nR^2}{2r}\frac{dI}{dt}\hat{\boldsymbol{\phi}}.$$

Since the particle velocity is $\mathbf{v}=v\hat{\boldsymbol{\phi}}$, Newton's second law gives

$$\frac{d}{dt}mv=qE_\phi=-\frac{\mu_0 nqR^2}{2r}\frac{dI}{dt}.$$

Therefore, after integrating from $t=0$ to $t=\infty$,

$$m[v(\infty)-v(0)]=-\frac{\mu_0 nqR^2}{2r}\left[I(\infty)-I(0)\right].$$

But $v(0) = 0$, $I(0) = I$, and $I(\infty) = 0$. Hence

$$v(\infty) = \frac{\mu_0 n q R^2 I}{2rm}.$$

(b) For a single charged particle in the presence of an external field, a statement of conservation of linear momentum is

$$m\mathbf{v} + \mathbf{P}_{\rm EM} = const. \tag{1}$$

For our situation, $\mathbf{v}(0) = 0$ and $\mathbf{P}_{\rm EM}(\infty) = 0$ because $\mathbf{B}(\infty) = 0$. Initially, the charge is at rest and the field it produces satisfies $\nabla \times \mathbf{E} = 0$. In that case, the text proved that

$$\mathbf{P}_{\rm EM} = q\mathbf{A}, \tag{2}$$

where $\mathbf{A}$ is evaluated at the position of the charge. To find $\mathbf{A}$, we note that the magnetic flux through any surface S bounded by a curve C is

$$\Phi_B = \int_S d\mathbf{S} \cdot \mathbf{B} = \oint_C d\boldsymbol{\ell} \cdot \mathbf{A}.$$

By symmetry, $\mathbf{A} = A\hat{\boldsymbol{\phi}}$, so we choose C as the wire loop and find

$$\mathbf{A} = \frac{\mu_0 n I R^2}{2r}\hat{\boldsymbol{\phi}}. \tag{3}$$

Substituting (3) into (2) and using (1) gives

$$\mathbf{v}(\infty) = \frac{\mu_0 q n I R^2}{2mr}\hat{\boldsymbol{\phi}},$$

which is the same answer as found in (a).

(c) The moving particle cannot exert a force on itself. Therefore there can be no transfer of momentum between its kinetic momentum and any field momentum associated with its self-fields.

Source: E.J. Konopinski, *American Journal of Physics* **46**, 499 (1978).

15.18 $\mathbf{P}_{\rm EM}$ for an Electric Dipole in a Uniform Magnetic Field

The electric field of a point dipole at the origin is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{p}) - \mathbf{p}}{r^3} - \frac{4\pi}{3}\mathbf{p}\,\delta(\mathbf{r})\right].$$

The magnetic field of the rotating shell is produced by a surface current density $\mathbf{K} = \sigma\boldsymbol{\omega}\times\mathbf{r}$. However, $\mathbf{K} = \mathbf{M}\times\hat{\mathbf{r}}$ is the effective surface current produced by a sphere with uniform magnetization $\mathbf{M} = M\hat{\mathbf{z}}$. The text found the field of the latter to be

$$\mathbf{B}(\mathbf{r}) = \begin{cases} \dfrac{2}{3}\mu_0\mathbf{M} & r < R, \\[2ex] \dfrac{\mu_0 R^3}{3}\left[\dfrac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{M})-\mathbf{M}}{r^3}\right] & r > R. \end{cases}$$

Therefore, if $\mathbf{B}_0$ is the uniform field inside the sphere, $\mathbf{P}_{\rm EM} = \mathbf{P}^{\rm in}_{\rm EM} + \mathbf{P}^{\rm out}_{\rm EM}$, where

$$\mathbf{P}^{\rm in}_{\rm EM} = \epsilon_0 \int\limits_{r<R} d^3r \left[-\frac{1}{3\epsilon_0}\mathbf{p}\right]\times\mathbf{B}_0 = -\frac{1}{3}(\mathbf{p}\times\mathbf{B}_0)$$

and

$$\begin{aligned}\mathbf{P}^{\rm out}_{\rm EM} &= \epsilon_0 \int\limits_{r>R} d^3r \left\{\frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p})-\mathbf{p}}{r^3}\right]\times\frac{\mu_0 R^3}{3}\left[\frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{M})-\mathbf{M}}{r^3}\right]\right\} \\ &= \frac{\mu_0 R^3}{12\pi}\int d\Omega \int\limits_R^\infty \frac{dr}{r^6}\left[r^2\mathbf{p}\times\mathbf{M} - 3(\mathbf{r}\times\mathbf{M})(\mathbf{r}\cdot\mathbf{p}) - 3(\mathbf{p}\times\mathbf{r})(\mathbf{r}\cdot\mathbf{M})\right].\end{aligned}$$

The first integral above is straightforward because $\mathbf{M}$ and $\mathbf{p}$ are constant vectors. Apart from constants, the second and third integrals above both involve the integral

$$Z_{ij} = \int d\Omega \int\limits_R^\infty dr \frac{r_i r_j}{r^6}.$$

By symmetry, $Z_{ij} \propto \delta_{ij}$ and the integrals with x^2, y^2, and z^2 in the numerator must all be equal. Therefore, because $x^2 + y^2 + z^3 = r^2$,

$$Z_{ij} = \frac{4\pi}{3}\int\limits_R^\infty \frac{dr}{r^4} = \frac{4\pi}{9R^4}\delta_{ij}.$$

Using this information and $\mathbf{B}_0 = (2/3)\mu_0\mathbf{M}$,

$$\mathbf{P}^{\rm out}_{\rm EM} = -\frac{\mu_0 R^3}{12\pi}\frac{4\pi}{3R^3}(\mathbf{p}\times\mathbf{M}) = -\frac{1}{6}(\mathbf{p}\times\mathbf{B}_0).$$

Therefore, in agreement with the result found in the text,

$$\mathbf{P}_{\rm EM} = \mathbf{P}^{\rm in}_{\rm EM} + \mathbf{P}^{\rm out}_{\rm EM} = -\frac{1}{3}(\mathbf{p}\times\mathbf{B}_0) - \frac{1}{6}(\mathbf{p}\times\mathbf{B}_0) = -\frac{1}{2}(\mathbf{p}\times\mathbf{B}_0).$$

Source: Prof. K.T. McDonald, Princeton University, http://cosmology.princeton.edu/~mcdonald/examples/

15.19 $\mathbf{P}_{\rm EM}$ for Electric and Magnetic Dipoles

(a) For a static electric field, $\mathbf{E}(\mathbf{r}) = -\nabla\varphi$. Therefore,

$$\mathbf{P}_{\rm EM} = \epsilon_0 \int d^3r\, \mathbf{E}\times\mathbf{B} = -\epsilon_0 \int d^3r\, \nabla\varphi \times \mathbf{B} = -\epsilon_0 \int d^3r\, [\nabla\times(\mathbf{B}\varphi) - \varphi(\nabla\times\mathbf{B})] \,.$$

Using $\nabla\times\mathbf{B} = \mu_0\mathbf{j}$ and a corollary of Stokes' theorem,

$$\mathbf{P}_{\rm EM} = \frac{1}{c^2}\int d^3r\, \varphi\,\mathbf{j} - \epsilon_0 \int_S d\mathbf{S}\times\mathbf{B}\,\varphi.$$

The surface integral vanishes because the integrand goes to zero faster than $1/r^2$ for a static field. Therefore,

$$\mathbf{P}_{\rm EM} = \frac{1}{c^2}\int d^3r\, \varphi\,\mathbf{j}.$$

(b) The current density of a point magnetic dipole is $\mathbf{j} = -\mathbf{m}\times\nabla\delta(\mathbf{r})$. Therefore, using the formula derived in part (a),

$$\mathbf{P}_{\rm EM} = -\frac{1}{c^2}\int d^3r\, \varphi\mathbf{m}\times\nabla\delta(\mathbf{r}) = -\frac{1}{c^2}\int d^3r\, \delta(\mathbf{r})\nabla\varphi\times\mathbf{m} = \frac{1}{c^2}\mathbf{E}(0)\times\mathbf{m}.$$

(c) The static potential for a point electric dipole at $\mathbf{r} = \mathbf{r}_0$ is

$$\varphi(\mathbf{r}) = -\frac{1}{4\pi\epsilon_0}\mathbf{p}\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_0|}.$$

Therefore, using the formula derived in part (a) and the definition of the vector potential in the Coulomb gauge,

$$\begin{aligned}
\mathbf{P}_{\rm EM} &= \frac{1}{c^2}\int d^3r\, \varphi\mathbf{j} = -\frac{1}{4\pi\epsilon_0 c^2}\int d^3r\, \mathbf{p}\cdot\nabla\frac{1}{|\mathbf{r}-\mathbf{r}_0|}\mathbf{j} \\
&= \frac{\mu_0}{4\pi}\mathbf{p}\cdot\nabla_0\int d^3r\frac{\mathbf{j}\,(\mathbf{r})}{|\mathbf{r}-\mathbf{r}_0|} = (\mathbf{p}\cdot\nabla_0)\mathbf{A}(\mathbf{r}_0)|_{\mathbf{r}_0=0}\,.
\end{aligned}$$

Source: D.J. Griffiths, *American Journal of Physics* **60**, 979 (1992).

15.20 $\mathbf{P}_{\rm EM}$ in the Coulomb Gauge

The electromagnetic momentum is

$$\mathbf{P}_{\rm EM} = \epsilon_0 \int d^3r\, \mathbf{E}\times\mathbf{B}.$$

Using the suggested splitting for $\mathbf{E}$ gives

$$\mathbf{P}_{\mathrm{EM}} = \epsilon_0 \int d^3r\, \mathbf{E}_{\|} \times \mathbf{B} + \epsilon_0 \int d^3r\, \mathbf{E}_{\perp} \times \mathbf{B}.$$

The second term already has the desired form so we focus on the first term and write it as

$$\mathbf{P}^{\star} = \epsilon_0 \int d^3r\, \mathbf{E}_{\|} \times \nabla \times \mathbf{A}_{\perp}.$$

A standard vector identity applied to the integrand gives

$$\mathbf{E}_{\|} \times \nabla \times \mathbf{A}_{\perp} = \nabla(\mathbf{E}_{\|} \cdot \mathbf{A}_{\perp}) - \mathbf{A}_{\perp} \times (\nabla \times \mathbf{E}_{\|}) - (\mathbf{E}_{\|} \cdot \nabla)\mathbf{A}_{\perp} - (\mathbf{A}_{\perp} \cdot \nabla)\mathbf{E}_{\|}.$$

Now, $\nabla \times \mathbf{E}_{\|} = 0$ so

$$\mathbf{P}^{\star} = \epsilon_0 \int d^3r\, [\nabla(\mathbf{E}_{\|} \cdot \mathbf{A}_{\perp}) - (\mathbf{E}_{\|} \cdot \nabla)\mathbf{A}_{\perp} - (\mathbf{A}_{\perp} \cdot \nabla)\mathbf{E}_{\|}].$$

The first term can be written as a surface integral (see below). For the second and third terms we use the supplied identity and the fact that $\nabla \cdot \mathbf{A}_{\perp} = 0$ to write

$$-\int d^3r\, (\mathbf{A}_{\perp} \cdot \nabla)\mathbf{E}_{\|} = \int d^3r\, \mathbf{E}_{\|} \nabla \cdot \mathbf{A}_{\perp} - \int (d\mathbf{S} \cdot \mathbf{A}_{\perp})\mathbf{E}_{\|} = -\int (d\mathbf{S} \cdot \mathbf{A}_{\perp})\mathbf{E}_{\|}$$

and

$$-\int d^3r\, (\mathbf{E}_{\|} \cdot \nabla)\mathbf{A}_{\perp} = \int d^3r\, \mathbf{A}_{\perp} \nabla \cdot \mathbf{E}_{\|} - \int (d\mathbf{S} \cdot \mathbf{E}_{\|})\mathbf{A}_{\perp}.$$

Gauss' law is $\nabla \cdot \mathbf{E}_{\|} = \rho/\epsilon_0$ so we can collect our results and write

$$\mathbf{P}^{\star} = \int d^3r\, \rho \mathbf{A}_{\perp} + \epsilon_0 \int dS \left\{ (\mathbf{E}_{\|} \cdot \mathbf{A}_{\perp})\hat{\mathbf{n}} - \mathbf{E}_{\|}(\mathbf{A}_{\perp} \cdot \hat{\mathbf{n}}) - \mathbf{A}_{\perp}(\mathbf{E}_{\|} \cdot \hat{\mathbf{n}}) \right\}.$$

The surface integral (at infinity) vanishes. To see this, adopt the Coulomb gauge where $\nabla \cdot \mathbf{A} = 0$ together with $\nabla \cdot \mathbf{A}_{\perp} = 0$ imply that $\nabla \cdot \mathbf{A}_{\|} = 0$. Therefore, by Helmholtz' theorem, $\mathbf{A}_{\|} = 0$. Hence,

$$\mathbf{E}_{\|} = -\nabla\varphi \qquad \text{and} \qquad \mathbf{E}_{\perp} = -\frac{\partial \mathbf{A}_{\perp}}{\partial t}.$$

The scalar potential is the Coulomb potential so $\mathbf{E}_{\|}$ falls off no slower than $1/r^2$ as $r \to \infty$. The slowest $\mathbf{A}_{\perp}$ can fall off is $1/r$ as $r \to \infty$. Therefore, every term in the surface integrals decreases no more slowly than $1/r^3$ and the integral vanishes. Consequently, the total linear momentum has the suggested form,

$$\mathbf{P}_{\mathrm{EM}} = \int d^3r\, \rho \mathbf{A}_{\perp} + \epsilon_0 \int d^3r \mathbf{E}_{\perp} \times \mathbf{B}.$$

Finally, the charge density for a collection of point charges q_k at positions $\mathbf{r}_k$ is

$$\rho(\mathbf{r},t) = \sum_k q_k \delta(\mathbf{r} - \mathbf{r}_k).$$

Therefore,

$$\int d^3r\, \rho \mathbf{A}_\perp = \sum_k q_k \mathbf{A}_\perp(\mathbf{r}_k, t),$$

and we conclude that

$$\mathbf{P}_{\rm EM} = \sum_k q_k \mathbf{A}_\perp(\mathbf{r}_k, t) + \epsilon_0 \int d^3r \mathbf{E}_\perp \times \mathbf{B}.$$

Source: M.G.Calkin, *American Journal of Physics* **34**, 921 (1966).

15.21 Hidden Momentum in a Bar Magnet?

(a) For a permanent magnet, $\mathbf{B} = \mu_0(\mathbf{M} + \mathbf{H})$ and $\mathbf{H} = -\nabla\psi$, where ψ is the magnetic scalar potential. Therefore,

$$\mathbf{P}_{\rm EM} = \epsilon_0 \int d^3r\, \mathbf{E} \times \mathbf{B} = \frac{1}{c^2} \int d^3r\, \mathbf{E} \times (\mathbf{M} - \nabla\psi).$$

The $\nabla\psi$ term is zero because integration by parts produces a factor of $\nabla \times \mathbf{E}$ in the integrand. This is zero because the point charge is at rest. Therefore,

$$\mathbf{P}_{\rm EM} = \frac{1}{c^2} \int d^3r\, \mathbf{E} \times \mathbf{M}.$$

(b) The center-of-energy theorem surely demands $\mathbf{P}_{\rm tot} = 0$ for this situation. If so, some hidden momentum to cancel $\mathbf{P}_{\rm EM}$ is required. However, there are no “moving parts”. The magnetic moment due to spin is a relativistic effect, but its origin is quantum-mechanical, rather than classical.

15.22 $\mathbf{L}_{\rm EM}$ for a Charge in a Two-Dimensional Magnetic Field

$$\mathbf{L}_{\rm EM} = \epsilon_0 \int d^3r\, \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) = \epsilon_0 \int d^3r\, \left[\mathbf{E}(\mathbf{r} \cdot \mathbf{B}) - \mathbf{B}(\mathbf{r} \cdot \mathbf{E})\right].$$

We have $\mathbf{B} = B(x,y)\hat{\mathbf{z}}$ and

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0} \frac{x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}}{r^3}.$$

Therefore,

$$\begin{aligned}\mathbf{L}_{\rm EM} &= \frac{q}{4\pi}\int\limits_{-\infty}^{\infty}dx\int\limits_{-\infty}^{\infty}dy\int\limits_{-\infty}^{\infty}dz\,B(x,y)\left[\frac{z}{r^2}\hat{\mathbf{r}}-\frac{\hat{\mathbf{z}}}{r}\right]\\ &= \frac{q}{4\pi}\int\limits_{-\infty}^{\infty}dx\int\limits_{-\infty}^{\infty}dy\int\limits_{-\infty}^{\infty}dz\,B(x,y)\left[\frac{x\hat{\mathbf{x}}+y\hat{\mathbf{y}}-(x^2+y^2)\hat{\mathbf{z}}}{r^2}\right]\\ &= -\frac{q}{4\pi}\int\limits_{-\infty}^{\infty}dx\int\limits_{-\infty}^{\infty}dy\,B(x,y)(x^2+y^2)\int\limits_{-\infty}^{\infty}\frac{dz}{r^3}\hat{\mathbf{z}}.\end{aligned}$$

The integral $\int_{-\infty}^{\infty}\frac{dz}{r^3}=\frac{2}{x^2+y^2}$. Therefore,

$$\mathbf{L}_{\rm EM}=-\frac{q}{2\pi}\hat{z}\int dxdy\,B(x,y)=-\frac{q\,\Phi_B}{2\pi}\hat{\mathbf{z}}.$$

Source: H.J. Lipkin and M. Peshkin, *Physics Letters B* **118**, 385 (1982).

15.23 Transformation of Angular Momentum

(a) Using Gauss' law in integral form, and the fact that the cylinder has zero net charge, we find without trouble that

$$\mathbf{E}=\begin{cases}\dfrac{r\tilde{\rho}}{2\epsilon_0}\hat{\boldsymbol{\rho}} & 0<\rho<a,\\[2ex] 0 & \rho>a.\end{cases}$$

The electromagnetic linear momentum density is zero outside the cylinder and

$$\mathbf{g}_{\rm EM}(\rho<a)=\epsilon_0\mathbf{E}\times\mathbf{B}=-\frac{1}{2}r\tilde{\rho}B\hat{\boldsymbol{\phi}}.$$

Consequently, the total electromagnetic angular momentum per unit length of cylinder is

$$\mathcal{L}_{\rm EM}=\int d^2r\,\boldsymbol{\rho}\times\mathbf{g}_{\rm EM}=-\pi\rho B\int\limits_0^a drr^3\hat{\mathbf{z}}=-\frac{1}{4}\pi\tilde{\rho}a^4B\hat{\mathbf{z}}.$$

(b) The torque per unit length which acts on the cylinder is $\mathcal{N}=\int d^2r\,\boldsymbol{\rho}\times\tilde{\rho}(\mathbf{r})\mathcal{E}$, where $\mathcal{E}$ is the electric field induced by the time rate of change of $B(t)$ and $\rho(\mathbf{r})=\tilde{\rho}\Theta(a-r)+\sigma\delta(r-a)$. To find σ, we impose neutrality:

$$0 = \int d^2r\,\rho(\mathbf{r}) = \int_0^{2\pi} d\phi \int_0^{\infty} drr[\tilde{\rho}\Theta(a-r) + \sigma\delta(r-a)] = 2\pi(a^2\tilde{\rho}/2 + \sigma a).$$

Therefore, $\sigma = -\tilde{\rho}a/2$, and we are ready to exploit the integral form of Faraday's law:

$$\oint d\boldsymbol{\ell}\cdot\mathcal{E} = -\frac{d}{dt}\int d\mathbf{S}\cdot\mathbf{B}.$$

The result is

$$\mathcal{E} = -\frac{1}{2}r\frac{dB}{dt}\hat{\boldsymbol{\phi}}.$$

Therefore,

$$\begin{aligned}
\mathcal{N} &= \int d^2r\,\boldsymbol{\rho}\times\tilde{\rho}(\mathbf{r})\mathcal{E} \\
&= -\int d\phi\int drr^2\,|\,\mathcal{E}\,|\,[\tilde{\rho}\Theta(a-r)+\sigma\delta(r-a)]\,\hat{\mathbf{z}} \\
&= -\int d\phi\int drr^3\frac{1}{2}\frac{dB}{dt}|[\tilde{\rho}\Theta(a-r)-\frac{1}{2}\tilde{\rho}a\delta(r-a)]\,\hat{\mathbf{z}} \\
&= \frac{1}{4}\pi a^4\tilde{\rho}\frac{dB}{dt}\,\hat{\mathbf{z}}.
\end{aligned}$$

(c) The torque is related to the angular momentum by $\dfrac{d\mathcal{L}}{dt} = \mathcal{N}$. Therefore, the difference between the initial and final angular momentum is

$$\mathcal{L}_f - \mathcal{L}_i = \int_{t_i}^{t_f} dt\mathcal{N} = \frac{1}{4}\pi a^4\tilde{\rho}(B_f - B_i)\hat{\mathbf{z}}.$$

But $B_i = B$ and $\mathcal{L}_i = B_f = 0$ for our problem. Therefore, when the magnetic field has disappeared, the total mechanical angular momentum of the cylinder is

$$\mathcal{L}_f = -\frac{1}{4}\pi a^4\tilde{\rho}B\hat{\mathbf{z}}.$$

This agrees with $\mathcal{L}_{\rm EM}$ calculated in part (a).

15.24 $\mathbf{L}_{\rm EM}$ for Static Fields

(a)

$$\mathbf{L}_{\rm EM} = \epsilon_0\int d^3r\,\mathbf{r}\times(\mathbf{E}\times\mathbf{B}) = \epsilon_0\int d^3r\,\mathbf{r}\times[\mathbf{E}\times(\nabla\times\mathbf{A})]\,.$$

A vector identity transforms this to

$$\mathbf{L}_{\rm EM} = \epsilon_0 \int d^3r\, \mathbf{r} \times \left[\nabla(\mathbf{E}\cdot\mathbf{A}) - (\mathbf{A}\cdot\nabla)\mathbf{E} - (\mathbf{E}\cdot\nabla)\mathbf{A} - \mathbf{A}\times(\nabla\times\mathbf{E})\right].$$

$\nabla \times \mathbf{E} = 0$ for a static electric field. Therefore,

$$\mathbf{L}_{\rm EM} = \epsilon_0(\mathbf{a} - \mathbf{b} - \mathbf{c}), \tag{1}$$

where

$$\mathbf{a} = \int d^3r\, \mathbf{r}\times\nabla(\mathbf{E}\cdot\mathbf{A}) \qquad \mathbf{b} = \int d^3r\, \mathbf{r}\times(\mathbf{A}\cdot\nabla)\mathbf{E} \qquad \mathbf{c} = \int d^3r\, \mathbf{r}\times(\mathbf{E}\cdot\nabla)\mathbf{A}.$$

The external fields are static so the integrals of total derivatives below will always vanish after integration by parts. First, because $\epsilon_{ijk}\delta_{jk} = 0$,

$$a_i = \epsilon_{ijk}\int d^3r\, r_j \partial_k(E_m A_m) = \epsilon_{ijk}\int d^3r\, \left[\partial_k(r_j E_m A_m) - E_m A_m \delta_{jk}\right] = 0.$$

Next, because the Coulomb gauge specifies $\nabla\cdot\mathbf{A} = 0$,

$$\begin{aligned} b_i &= \epsilon_{ijk}\int dr\, r_j A_m \partial_m E_k = \epsilon_{ijk}\int d^3r\, \left[\partial_m(r_j A_m E_k) - \delta_{mj} A_m E_k - r_j E_k \partial_m A_m\right] \\ &= -\epsilon_{imk}\int dr A_m E_k. \end{aligned}$$

In vector form, $\mathbf{b} = \int d^3r\, \mathbf{E}\times\mathbf{A}$.

Finally,

$$c_i = \epsilon_{ijk}\int d^3r\, r_j E_m \partial_m A_k = \epsilon_{ijk}\int d^3r\, \left[\partial_m(r_j E_m A_k) - \delta_{mj} E_m A_k - r_j A_k \partial_m E_m\right].$$

Therefore,

$$\mathbf{c} = -\int d^3r\, \mathbf{E}\times\mathbf{A} - \int d^3r\, (\nabla\cdot\mathbf{E})(\mathbf{r}\times\mathbf{A}).$$

Since $\mathbf{a} = 0$, substituting $\mathbf{b}$ and $\mathbf{c}$ in (1) and using Gauss' law gives the advertised result,

$$\mathbf{L}_{\rm EM} = \epsilon_0\int d^3r\, (\nabla\cdot\mathbf{E})(\mathbf{r}\times\mathbf{A}) = \int d^3r\, \rho(r)\mathbf{r}\times\mathbf{A}(\mathbf{r}).$$

(b) The charge density of a point electric dipole at the origin is $\rho = -\nabla\cdot[\mathbf{p}\delta(\mathbf{r})]$. Therefore,

$$\begin{aligned} \mathbf{L}_k &= -\epsilon_{k\ell m}\int d^3r\, \partial_j[p_j\delta(\mathbf{r})]r_\ell A_m \\ &= -\epsilon_{k\ell m}\int d^3r\, \left\{\partial_j\left[p_j r_\ell \delta(\mathbf{r})\right]A_m - p_j\delta_{j\ell}A_m\delta(\mathbf{r})\right\} \\ &= \epsilon_{k\ell m}p_\ell\int d^3r\, A_m\delta(\mathbf{r}) \end{aligned}$$

because the charge density of the point dipole is zero away from the origin. Therefore,

$$\mathbf{L}_{\rm EM} = \mathbf{p} \times \mathbf{A}(0).$$

(c) The current density of the rotating object is $\mathbf{j} = \rho\mathbf{v} = \rho\boldsymbol{\omega} \times \mathbf{r}$. Therefore, the magnetic energy can be written

$$U_B = \frac{1}{2}\int d^3r\, \mathbf{A}\cdot\mathbf{j} = \frac{1}{2}\int d^3r\, \rho\mathbf{A}\cdot(\boldsymbol{\omega}\times\mathbf{r}) = \frac{1}{2}\boldsymbol{\omega}\cdot\int d^3r\, \rho\mathbf{r}\times\mathbf{A} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}_{\rm EM}.$$

Source: N. Gauthier, *American Journal of Physics* **74**, 232 (2006).

15.25 The Dipole Force on Atoms and Molecules

As a first step, we note that

$$(\mathbf{m}\times\nabla)\times\mathbf{B} = m_k\nabla B_k - \mathbf{m}(\nabla\cdot\mathbf{B}) = \nabla(\mathbf{m}\cdot\mathbf{B}) - B_k\nabla m_k - \mathbf{m}(\nabla\cdot\mathbf{B}).$$

But $\mathbf{m}$ is not a function of position and $\nabla\cdot\mathbf{B} = 0$ always. Therefore, we get the last term in the proposed force formula:

$$(\mathbf{m}\times\nabla)\times\mathbf{B} = \nabla(\mathbf{m}\cdot\mathbf{B}).$$

Otherwise, the motion of the atom implies that we must use the convective derivative

$$\frac{d\mathbf{B}}{dt} = \frac{\partial\mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{B}.$$

Therefore, using Faraday's law,

$$\frac{d}{dt}(\mathbf{p}\times\mathbf{B}) = \dot{\mathbf{p}}\times\mathbf{B} + \mathbf{p}\times\left[\frac{\partial\mathbf{B}}{\partial t} + (\mathbf{v}\cdot\nabla\mathbf{B})\right] = \dot{\mathbf{p}}\times\mathbf{B} - \mathbf{p}\times(\nabla\times\mathbf{E}) + \mathbf{p}\times(\mathbf{v}\cdot\nabla)\mathbf{B}.$$

Expanding the triple cross product gives

$$\frac{d}{dt}(\mathbf{p}\times\mathbf{B}) = \dot{\mathbf{p}}\times\mathbf{B} - [\nabla(\mathbf{p}\cdot\mathbf{E}) - (\mathbf{p}\cdot\nabla)\mathbf{E}] + \mathbf{p}\times(\mathbf{v}\cdot\nabla)\mathbf{B}.$$

On the other hand, because $\mathbf{p}$ and $\mathbf{v}$ are constant vectors,

$$\nabla[\mathbf{p}\cdot(\mathbf{v}\times\mathbf{B})] = \mathbf{p}\times\{\nabla\times(\mathbf{v}\times\mathbf{B})\} + (\mathbf{p}\cdot\nabla)(\mathbf{v}\times\mathbf{B}) = -\mathbf{p}\times(\mathbf{v}\cdot\nabla)\mathbf{B} + (\mathbf{p}\cdot\nabla)(\mathbf{v}\times\mathbf{B}).$$

Adding $\nabla(\mathbf{p}\cdot\mathbf{E})$ to the last two equations reproduces the remaining terms in the original force law. This proves the result.

Chapter 16: Waves in Vacuum

16.1 Wave Equation vs. Maxwell Equations

We want the magnetic field to satisfy $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = c^{-2}\partial \mathbf{E}/\partial t$. By the Helmholtz theorem, this will be true if we define $\mathbf{B}$ using

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \frac{1}{4\pi c^2} \int d^3 r' \frac{\partial \mathbf{E}(\mathbf{r}', t)}{\partial t} \frac{1}{|\mathbf{r} - \mathbf{r}'|}.$$

Since $\nabla \cdot \mathbf{E} = 0$ by assumption, it remains only to prove that $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$. Using the fact that $\mathbf{E}$ satisfies the wave equation, a direction computation gives

$$\begin{aligned}
\frac{\partial \mathbf{B}(\mathbf{r}, t)}{\partial t} &= \nabla \times \frac{1}{4\pi c^2} \int d^3 r' \frac{1}{|\mathbf{r} - \mathbf{r}'|} \frac{\partial^2 \mathbf{E}(\mathbf{r}', t)}{\partial t^2} \\
&= \nabla \times \frac{1}{4\pi} \int d^3 r' \, \frac{1}{|\mathbf{r} - \mathbf{r}'|} \nabla'^2 \mathbf{E}(\mathbf{r}', t) \\
&= \nabla \times \frac{1}{4\pi} \int d^3 r' \, \mathbf{E}(\mathbf{r}', t) \nabla'^2 \frac{1}{|\mathbf{r} - \mathbf{r}'|} \\
&= -\nabla \times \int d^3 r' \, \mathbf{E}(\mathbf{r}', t) \delta(\mathbf{r} - \mathbf{r}') \\
&= -\nabla \times \mathbf{E}(\mathbf{r}, t).
\end{aligned}$$

16.2 No Electromagnetic Bullets

We are told that

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \psi(x, y, z - ct) = 0.$$

On the other hand, holding x and y constant,

$$\left[\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right] \psi(x, y, z - ct) = 0.$$

Therefore, ψ satisfies Laplace's equation in two dimensions:

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right] \psi(x, y, z - ct) = 0.$$

Now, if $\psi(x, y, z - ct)$ is localized in the x and y directions simultaneously, the origin must be part of a closed area A in the x-y plane such that $\psi(x, y, z - ct)$ is zero everywhere on and outside the boundary curve of A. However, the unique solution of Laplace's equation which satisfies this boundary condition is $\psi(x, y, z - ct) = 0$ everywhere in A. Therefore, our assumption that $\psi(x, y, z - ct)$ is localized in all three dimensions cannot be correct.

Source: J.N. Brittingham, *Journal of Applied Physics* **54**, 1179 (1983).

16.3 An Evanescent Wave in Vacuum

(a) The electric field must satisfy the wave equation

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.$$

Substituting the trial solution shows that

$$h^2 - \kappa^2 = \frac{\omega^2}{c^2}.$$

(b) For time-harmonic fields, Faraday's law gives

$$\mathbf{B} = -\frac{i}{\omega}\nabla \times \mathbf{E} = \frac{iE_0}{\omega}\hat{\mathbf{y}} \times (\nabla \exp[i(hz - \omega t) - \kappa x]) = \frac{iE_0}{\omega}(ih\hat{\mathbf{x}} + \kappa\hat{\mathbf{z}})e^{i(hz-\omega t)}e^{-\kappa z}.$$

(c) Given the result in (a), the magnetic field will be very nearly circularly polarized if $h \approx \kappa$, that is, $h, \kappa \gg \omega/c$.

(d) When $\mathbf{E} = \mathbf{E}_0 \exp(-i\omega t)$ and $\mathbf{B} = \mathbf{B}_0 \exp(-i\omega t)$, the time-averaged Poynting vector is

$$\langle \mathbf{S} \rangle = \frac{1}{2\mu_0}\mathrm{Re}\left[\mathbf{E}_0^* \times \mathbf{B}_0\right] = \frac{E_0^2}{2\mu_0\omega}e^{-2\kappa x}\hat{\mathbf{z}}.$$

16.4 Plane Waves from Potentials

(a) A Coulomb gauge vector potential, $\mathbf{A}_C$, must satisfy $\nabla \cdot \mathbf{A}_C = 0$ and the homogeneous wave equation. The latter is automatically true of $\mathbf{A}(\mathbf{k} \cdot \mathbf{r} - ckt)$. To satisfy the Coulomb gauge constraint, it is sufficient that $\mathbf{k} \cdot \mathbf{A} = 0$. Since $\varphi_C = 0$, we set $\phi = \mathbf{k} \cdot \mathbf{r} - \omega t$ and compute

$$\mathbf{E} = -\frac{\partial \mathbf{A}_C}{\partial t} = -\mathbf{A}'(\phi)\frac{\partial \phi}{\partial t} = \omega \mathbf{A}'(\phi)$$

$$\mathbf{B} = \nabla \times \mathbf{A}_C = \mathbf{k} \times \mathbf{A}'(\phi).$$

Both fields are transverse to $\mathbf{k}$ because $\mathbf{A}(\phi)$—and hence $\mathbf{A}'(\phi)$—are transverse to $\mathbf{k}$.

(b) We impose the Coulomb gauge constraint to get

$$0 = \nabla \cdot \mathbf{A}_C = \nabla \cdot (u\mathbf{a}) = \mathbf{a} \cdot \nabla u.$$

Therefore, $\mathbf{a} \perp \nabla u$ is necessary. Otherwise, we require

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}_C = \mathbf{a}\left[\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\right] = 0.$$

In words, $u(\mathbf{r}, t)$ must be a solution of the homogeneous wave equation. By direct computation,

$$\mathbf{E} = -\frac{\partial \mathbf{A}_C}{\partial t} = -\frac{\partial u}{\partial t}\mathbf{a} \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A}_C = -\mathbf{a} \times \nabla u.$$

(c) The special case $u = u(\phi)$, where $\phi = \mathbf{k}\cdot\mathbf{r} - ckt$, simplifies the fields in part (b) to

$$\mathbf{E} = cku'(\phi)\mathbf{a} \qquad\qquad \mathbf{B} = -\mathbf{a}\times u'(\phi)\hat{\mathbf{k}} \quad\Rightarrow\quad c\mathbf{B} = \hat{\mathbf{k}}\times\mathbf{E}.$$

(d) The vector potential satisfies

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}_L = \mathbf{s}\left[\nabla^2 u - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}\right] = 0.$$

Therefore, u must be a solution of the homogeneous wave equation and there is no restriction on the constant vector $\mathbf{s}$. The Lorenz gauge constraint reads

$$\frac{\partial\varphi_L}{\partial t} = -c^2\mathbf{s}\cdot\nabla u. \tag{1}$$

The electromagnetic fields in this case are

$$\mathbf{E} = -\nabla\varphi_L - \frac{\partial\mathbf{A}_L}{\partial t} = c^2(\mathbf{s}\cdot\nabla)\nabla\int\limits_{-\infty}^{t} dt'u(\mathbf{r},t') - \frac{\partial u}{\partial t}\mathbf{s} \qquad\qquad \mathbf{B} = \nabla\times\mathbf{A}_L = -\mathbf{s}\times\nabla u.$$

(e) With $\mathbf{A}_L = u(\phi)\hat{\mathbf{s}}$ and $\phi = \mathbf{k}\cdot\mathbf{r} - ckt$, we can write down the solution to (1) immediately:

$$\varphi_L(\phi) = c(\mathbf{s}\cdot\hat{\mathbf{k}})u(\phi).$$

In that case,

$$\mathbf{E} = -\nabla\varphi_L - \frac{\partial\mathbf{A}_L}{\partial t} = ck[\mathbf{s} - (\mathbf{s}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}]u' \qquad\qquad \mathbf{B} = ck(\hat{\mathbf{k}}\times\mathbf{s})u'. \tag{2}$$

The vector in square brackets is perpendicular to $\mathbf{k}$. This makes the electric field the same as in part (c), because $\mathbf{a}\perp\mathbf{k}$ also. The magnetic fields are the same also because the fields in (2) satisfy $c\mathbf{B} = \hat{\mathbf{k}}\times\mathbf{E}$.

16.5 Two Counter-Propagating Plane Waves

(a)

$$\begin{aligned}\mathbf{E} &= E_0[\cos(kz-\omega t) + \cos(kz+\omega t)]\,\hat{\mathbf{x}} \\ &= E_0[\cos kz\cos\omega t + \sin kz\sin\omega t + \cos kz\cos\omega t - \sin kz\sin\omega t]\,\hat{\mathbf{x}} \\ &= 2E_0\cos kz\cos\omega t\,\hat{\mathbf{x}}.\end{aligned}$$

The electric field is a standing wave, so we cannot use $c\mathbf{B} = \hat{\mathbf{k}}\times\mathbf{E}$. However, if $\tilde{\mathbf{E}}(z) = 2E_0\cos kz\,\hat{\mathbf{x}}$, we have $\mathbf{E} = \mathrm{Re}\,\mathcal{E}$ and $\mathbf{B} = \mathrm{Re}\,\mathcal{B}$, where

$$\mathcal{E}(z,t) = \tilde{\mathbf{E}}(z)\exp(-i\omega t) \qquad \text{and} \qquad \mathcal{B}(z,t) = \tilde{\mathbf{B}}\exp(-i\omega t).$$

The latter two satisfy Faraday's law, $\nabla\times\mathcal{E} = -\partial\mathcal{B}/\partial t$, which gives

$$\nabla\times\tilde{\mathbf{E}}(z) = i\omega\tilde{\mathbf{B}}(z)$$

or

$$-2kE_0 \sin kz\hat{\mathbf{y}} = i\omega\tilde{\mathbf{B}}(z).$$

The magnetic field is 90° out of phase with the electric field in both space and time:

$$c\mathbf{B} = \text{Re}\left[\frac{2ic}{k}\sin kz \exp(-i\omega t)\right]\hat{\mathbf{y}} = 2E_0 \sin kz \sin \omega t\,\hat{\mathbf{y}}.$$

(b)

$$\langle u_E(z)\rangle = \frac{1}{2}\text{Re}\,[\frac{1}{2}\epsilon_0\tilde{\mathbf{E}}^* \cdot \tilde{\mathbf{E}}] = \frac{1}{4}\epsilon_0 4E_0^2 \cos^2 kz$$

$$\langle u_B(z)\rangle = \frac{1}{2}\text{Re}\,[\frac{1}{2}\epsilon_0 c\tilde{\mathbf{B}}^* \cdot c\tilde{\mathbf{B}}] = \frac{1}{4}\epsilon_0 4E_0^2 \sin^2 kz.$$

As for the Poynting vector, $\tilde{\mathbf{E}}$ is real and $\tilde{\mathbf{B}}$ is pure imaginary, so $\langle\mathbf{S}(z)\rangle \propto \text{Re}\,[\tilde{\mathbf{E}}^* \times \tilde{\mathbf{B}}] = 0$.

(c) The given electric field is

$$\mathbf{E}(z,t) = \text{Re}\,[E_0 e^{ikz}e^{-i\omega t}]\hat{\mathbf{x}} + \text{Re}\,[iE_0 e^{-ikz}e^{-i\omega t}]\hat{\mathbf{y}} = \text{Re}\,[\tilde{\mathbf{E}}(z)e^{-i\omega t}]$$

where

$$\tilde{\mathbf{E}}(z) = (e^{ikz}\hat{\mathbf{x}} + ie^{-ikz}\hat{\mathbf{y}})E_0 = \left(\frac{\hat{\mathbf{x}} + ie^{-2ikz}\hat{\mathbf{y}}}{\sqrt{2}}\right)\sqrt{2}E_0 e^{ikz} = \mathbf{e}(z)\sqrt{2}E_0 e^{ikz}.$$

In this form, we can easily read off the normalized, position-dependent polarization vector $\mathbf{e}(z)$:

$$\mathbf{e}(z=0) = \frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} \qquad \text{(right-hand circular)}$$

$$\mathbf{e}(z=\lambda/8) = \frac{\hat{\mathbf{x}} + ie^{-i\pi/2}\hat{\mathbf{y}}}{\sqrt{2}} = \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}}}{\sqrt{2}} \qquad \text{(linear)}$$

$$\mathbf{e}(z=\lambda/4) = \frac{\hat{\mathbf{x}} + ie^{-i\pi}\hat{\mathbf{y}}}{\sqrt{2}} = \frac{\hat{\mathbf{x}} - i\hat{\mathbf{y}}}{\sqrt{2}} \qquad \text{(left-hand circular)}$$

$$\mathbf{e}(z=3\lambda/8) = \frac{\hat{\mathbf{x}} + ie^{-i3\pi/2}\hat{\mathbf{y}}}{\sqrt{2}} = \frac{\hat{\mathbf{x}} - \hat{\mathbf{y}}}{\sqrt{2}} \qquad \text{(linear)}$$

$$\mathbf{e}(z=\lambda/2) = \frac{\hat{\mathbf{x}} + ie^{-i2\pi}\hat{\mathbf{y}}}{\sqrt{2}} = \frac{\hat{\mathbf{x}} + i\hat{\mathbf{y}}}{\sqrt{2}} \qquad \text{(right-hand circular).}$$

(d) In part (c) we have $\tilde{\mathbf{E}} = E_0[e^{ikz}\hat{\mathbf{x}} + ie^{-ikz}\hat{\mathbf{y}}]E_0$. On the other hand, $\hat{\mathbf{e}}_\pm = \dfrac{\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}}{\sqrt{2}}$, so

$$\hat{\mathbf{x}} = \frac{\hat{\mathbf{e}}_+ + \hat{\mathbf{e}}_-}{\sqrt{2}} \qquad \text{and} \qquad \hat{\mathbf{y}} = \frac{\hat{\mathbf{e}}_+ - \hat{\mathbf{e}}_-}{\sqrt{2}}.$$

Therefore,

$$\begin{aligned}\tilde{\mathbf{E}} &= \left[e^{ikz}\left(\frac{\hat{\mathbf{e}}_+ + \hat{\mathbf{e}}_-}{\sqrt{2}}\right) + ie^{-ikz}\left(\frac{\hat{\mathbf{e}}_+ - \hat{\mathbf{e}}_-}{\sqrt{2}}\right)\right]E_0 \\ &= \left[\hat{\mathbf{e}}_+\left(\frac{e^{ikz} + e^{-ikz}}{\sqrt{2}}\right) + \hat{\mathbf{e}}_-\left(\frac{e^{ikz} - e^{-ikz}}{\sqrt{2}}\right)\right]E_0 \\ &= \sqrt{2}E_0(\hat{\mathbf{e}}_+ \cos kz + i\hat{\mathbf{e}}_- \sin kz).\end{aligned}$$

Therefore,

$$\mathbf{E} = \text{Re}\left[\sqrt{2}E_0(\hat{\mathbf{e}}_+ \cos kz + i\hat{\mathbf{e}}_- \sin kz)e^{-i\omega t}\right] = \sqrt{2}E_0\left[\hat{\mathbf{e}}_+ \cos kz \cos \omega t + \hat{\mathbf{e}}_- \sin kz \cos \omega t\right].$$

(e) The part (c) electric field is $\mathcal{E} = E_0 e^{i(kz-\omega t)}\hat{\mathbf{x}} + iE_0 e^{-i(kz+\omega t)}\hat{\mathbf{y}} = \tilde{\mathbf{E}}e^{-i\omega t}$. This is the sum of two propagating waves so the magnetic field is

$$c\mathcal{B} = (\hat{\mathbf{z}}\times\hat{\mathbf{x}})E_0 e^{i(kz-\omega t)} - iE_0(\hat{\mathbf{z}}\times\hat{\mathbf{y}})e^{-i(kz+\omega t)} = E_0\left[e^{ikz}\hat{\mathbf{y}} + ie^{-ikz}\hat{\mathbf{x}}\right]e^{-i\omega t} = c\tilde{\mathbf{B}}e^{-i\omega t}.$$

The time-averaged Poynting vector is

$$\langle \mathbf{S}(z)\rangle = \frac{1}{2}\text{Re}\,[\tilde{\mathbf{E}}^*\times\tilde{\mathbf{B}}] = \frac{E_0^2}{2c}\text{Re}\left\{\left(e^{ikz}\hat{\mathbf{x}} + ie^{-ikz}\hat{\mathbf{y}}\right)^* \times \left(e^{ikz}\hat{\mathbf{y}} + ie^{-ikz}\hat{\mathbf{x}}\right)\right\} = \frac{E_0^2}{2c}\text{Re}\,[\hat{\mathbf{z}} - \hat{\mathbf{z}}] = 0.$$

Source: Prof. C. Caves, University of New Mexico (public communication).

16.6 Transverse Plane Waves with $\mathbf{E} \parallel \mathbf{B}$

Since $\mathbf{E} = \mathbf{E}(z,t)$ and $\nabla \cdot \mathbf{E} = 0$, we know that $\partial E_z/\partial z = 0$. Therefore, $E_z = const. = 0$. By the same argument, $B_z = 0$.

(a) If $\mathbf{S} \propto \mathbf{E}\times\mathbf{B} = 0$, our task is to show that

$$\frac{\partial u_{EM}}{\partial t} = \frac{\partial}{\partial t}\left\{E_x^2 + E_y^2 + c^2\left(B_x^2 + B_y^2\right)\right\} = 2\left[E_x\dot{E}_x + E_y\dot{E}_y + c^2(B_x\dot{B}_x + B_y\dot{B}_y)\right] = 0 \tag{1}$$

and

$$\frac{\partial u_{EM}}{\partial z} = \frac{\partial}{\partial z}\left\{E_x^2 + E_y^2 + c^2\left(B_x^2 + B_y^2\right)\right\} = 2\left[E_xE_x' + E_yE_y' + c^2(B_xB_x' + B_yB_y')\right] = 0. \tag{2}$$

We begin with the Maxwell equations. These tell us that

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad\Rightarrow\quad E_x' = -\dot{B}_y \qquad \text{and} \qquad E_y' = \dot{B}_x$$

$$\nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} \quad\Rightarrow\quad c^2B_x' = \dot{E}_y \qquad \text{and} \qquad c^2B_y' = -\dot{E}_x.$$

The Poynting vector has only a z-component so, from above, $S_z = E_x B_y - E_y B_x = 0$. Therefore, using the Maxwell equation information,

$$\begin{aligned} 0 = \frac{\partial S_z}{\partial t} = \frac{\partial}{\partial t}(E_x B_y - E_y B_x) &= \dot{E}_x B_y + E_x \dot{B}_y - \dot{E}_y B_x - E_y \dot{B}_x \\ &= -c^2 B_y' B_y - E_x E_x' - c^2 B_x' B_x - E_y E_y'. \end{aligned}$$

This confirms (2). Similarly,

$$\begin{aligned} 0 = \frac{\partial S_z}{\partial z} = \frac{\partial}{\partial z}(E_x B_y - E_y B_x) &= E_x' B_y + E_x B_y' - E_y' B_x - E_y B_x' \\ &= -\dot{B}_y B_y - \frac{1}{c^2} E_x \dot{E}_x - \dot{B}_x B_x - \frac{1}{c^2} E_y \dot{E}_y. \end{aligned}$$

This confirms (1).

(b) The results of part (a) imply that $E^2 + B^2 = const.$ The suggested parameterization gives the special case $E^2 + B^2 = \cos^2 \alpha + \sin^2 \alpha = 1$. This gives the Poynting condition as

$$0 = E_x B_y - E_y B_x = \sin\alpha \cos\alpha \sin(\gamma - \beta).$$

The solutions with $\sin\alpha = 0$ or $\cos\alpha = 0$ are trivial so $\gamma(z,t) = \beta(z,t) + m\pi$ where m is a non-negative integer. Substituting this back into the parameterization gives

$$\begin{aligned} E_x &= \cos\alpha \cos\beta & cB_x &= (-)^m \sin\alpha \cos\beta \\ E_y &= \cos\alpha \sin\beta & cB_y &= (-)^m \sin\alpha \sin\beta. \end{aligned}$$

The Maxwell equations, e.g., $E_x' = -\dot{B}_y$, demand that

$$c\alpha' = (-)^m \dot{\beta} \quad \text{and} \quad c\beta' = (-)^m \dot{\alpha}. \tag{3}$$

Combining these two shows that $\alpha(z,t)$ satisfies the wave equation

$$\frac{d^2\alpha}{dz^2} + \frac{1}{c^2} \frac{\partial^2 \alpha}{\partial t^2} = 0.$$

The general solution is a sum of left-going and right-going plane waves. Therefore, using (3),

$$\begin{aligned} \alpha(z,t) &= F(z + ct) + G(z - ct) \\ \beta(z,t) &= (-)^m \{F(z + ct) - G(z - ct)\}. \end{aligned}$$

This has the required form. The corresponding fields are

$$\begin{aligned} E_x &= \tfrac{1}{2}\{\cos 2F + \cos 2G\} & cB_x &= (-)^m \tfrac{1}{2}\{\sin 2F + \sin 2G\} \\ E_y &= (-)^m \tfrac{1}{2}\{\sin 2F - \sin 2G\} & cB_y &= \tfrac{1}{2}\{-\cos 2F + \cos 2G\}. \end{aligned}$$

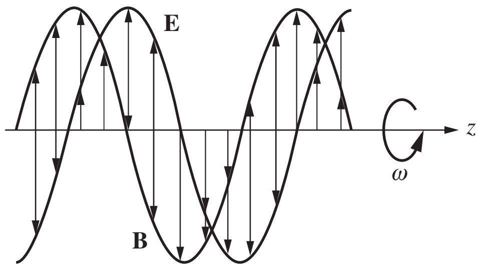
The factor of $(-)^m$ affects only the sense of rotation in the examples below so we drop it.

(c) When $F = \frac{1}{2}k(z + ct)$ and $G = \pm\frac{1}{2}k(z - ct)$ with $\omega = ck$,

$$E_x = \cos kz \cos \omega t \qquad cB_x = \begin{cases} \sin kz \cos \omega t \\ \cos kz \sin \omega t \end{cases}$$

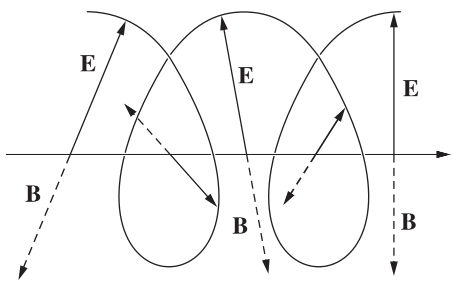
$$E_y = \begin{cases} \cos kz \sin \omega t \\ \sin kz \cos \omega t \end{cases} \qquad cB_y = \sin kz \sin \omega t,$$

where the upper/lower portion of the brackets corresponds to the plus/minus sign choice for G. For the plus sign, $E_y/E_x = B_y/B_x = \tan \omega t$ so $E(z,t)$ and $B(z,t)$ lie in the same plane, independent of z. The plane rotates with angular frequency ω. The field vectors are parallel but their magnitudes are 90° out of phase along the z-axis as shown below.



For the minus sign, $E_y/E_x = B_y/B_x = \tan kz$ at all times. This is a standing wave where the fields vary helically with z as shown below. An alternate description is a superposition of two circularly polarized waves that propagate in opposite directions. That is,

$$\mathbf{E}(z,t) \propto \mathrm{Re}\left\{(\hat{\mathbf{x}} + i\hat{\mathbf{y}})e^{i(kz-\omega t)} + (\hat{\mathbf{x}} - i\hat{\mathbf{y}})e^{i(kz+\omega t)}\right\}.$$

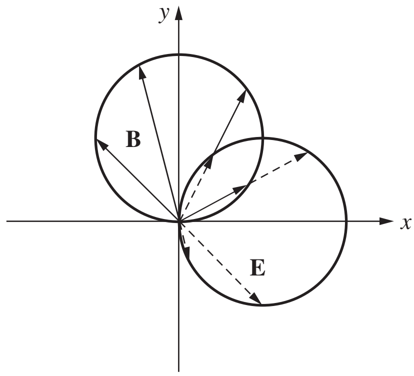


(d) When $F = \frac{1}{2}k(z + ct)$ and $G = 0$ we get

$$E_x = \tfrac{1}{2}\{1 + \cos(kz + \omega t)\} \quad cB_x = \tfrac{1}{2}\sin(kz + \omega t)$$

$$E_y = \tfrac{1}{2}\sin(kz + \omega t) \qquad cB_y = \tfrac{1}{2}\{1 - \cos(kz + \omega t)\}.$$

This is a superposition of a static field $E_x = B_y = \frac{1}{2}$ with a wave that propagates in the $-z$-direction. The Poynting vector for the propagating wave part is not zero but the *total* field has zero Poynting vector! In particular, in a plane of fixed z, the tips of the **E** and **B** vectors trace out identical circles oriented at 90° from one another as shown below.



Source: K. Shimoda, T. Kawai, and K. Uehara, *American Journal of Physics* **58**, 394 (1990).

16.7 Photon Spin for Plane Waves

(a) The angular momentum is $\mathbf{L} = \epsilon_0 \int d^3r\,\mathbf{r} \times (\mathbf{E} \times \mathbf{B}) = \epsilon_0 \int d^3r\,\mathbf{r} \times [\mathbf{E} \times (\nabla \times \mathbf{A})]$ so

$$\begin{aligned} L_i &= \epsilon_0 \int dr\, \{\epsilon_{ijk}\epsilon_{kmn}\epsilon_{npq} r_j E_m \partial_p A_q\} \\ &= \epsilon_0 \epsilon_{ijk} \int dr (\delta_{kp}\delta_{mq} - \delta_{kq}\delta_{mp}) \{\partial_p (r_j E_m A_q) - A_q E_m \delta_{pj} - A_q r_j \partial_p E_m\}. \end{aligned}$$

The total derivative term vanishes if $E_m A_q \to 0$ faster than $1/r^3$ so

$$L_i = \epsilon_0 \epsilon_{ijk} \int dr\, \{A_k E_j - A_m r_j \partial_k E_m + A_k r_j \nabla \cdot \mathbf{E}\}.$$

The last term is zero because $\nabla \cdot \mathbf{E} = 0$ in vacuum so

$$L_i = \epsilon_0 \int dr\, (\mathbf{E} \times \mathbf{A})_i - \epsilon_{ijk}\epsilon_0 \int dr\; \{\partial_k (A_m r_j E_m) - r_j E_m \partial_k A_m - A_m E_m \delta_{jk}\}.$$

The total derivative integrates to zero again and the last term in brackets is zero because $\epsilon_{ijk}\delta_{jk} = 0$. We conclude that

$$\mathbf{L}_{\mathrm{EM}} = \epsilon_0 \int d^3r\, \mathbf{E} \times \mathbf{A} + \epsilon_0 \int d^3r\, E_m (\mathbf{r} \times \nabla) A_m = \mathbf{L}_{\mathrm{spin}} + \mathbf{L}_{\mathrm{orbital}}.$$

(b) The decomposition is not gauge invariant because, if $\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla f$,

$$\mathbf{L}'_{\mathrm{spin}} = \mathbf{L}_{\mathrm{spin}} + \epsilon_0 \int d^3r\, \mathbf{E} \times \nabla f \neq \mathbf{L}_{\mathrm{spin}}.$$

(c) In the Coulomb gauge, we have $\nabla \cdot \mathbf{A} = 0$. When there are no sources, we can also choose $\varphi = 0$ so $\mathbf{E} = -\partial\mathbf{A}/\partial t$. In that case, the given electric field can be derived from

$$\mathbf{A}_{\pm} = -\frac{iE_0}{\omega}\frac{\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}}{\sqrt{2}}\exp[i(kz - \omega t)].$$

The time-averaged spin angular momentum is

$$< \mathbf{L}_{\text{spin}} > = \frac{\epsilon_0}{2}\text{Re}\int d^3r\,\mathbf{E}\times\mathbf{A}^* = \pm\hat{\mathbf{z}}\frac{\epsilon_0}{2\omega}\int d^3r\,E_0^2.$$

On the other hand, the time-averaged total energy is

$$< U_{\text{EM}} > = \frac{1}{2}\epsilon_0\int d^3r\mathbf{E}\cdot\mathbf{E}^* = \frac{1}{2}\epsilon_0\int d^3r\,E_0^2.$$

Therefore,

$$\hat{\mathbf{z}}\cdot < \mathbf{L}_{\text{spin}} > = \pm\frac{1}{\omega} < U_{\text{EM}} > .$$

If $< U_{\text{EM}} > = \hbar\omega$, we correctly get $\hat{\mathbf{z}}\cdot < \mathbf{L}_{\text{spin}} > = \pm\hbar$ for the "spin" of the photon.

16.8 When Interference Behaves Like Reflection

(a) The H and V beams are both propagating, monochromatic plane waves with electric field amplitude E_0. The time-averaged electromagnetic energy density for both is

$$\langle u_{\text{EM}}\rangle_{H,V} = \frac{1}{2}\text{Re}\left\{\frac{\epsilon_0}{2}(\mathbf{E}\cdot\mathbf{E}^* + c^2\mathbf{B}\cdot\mathbf{B}^*)\right\} = \frac{1}{2}\epsilon_0 E_0^2.$$

For the superposed beams, we note that $\mathbf{B}_H \cdot \mathbf{B}_V^* = 0$, so

$$\begin{aligned}
\langle u_{\text{EM}}\rangle &= \frac{1}{2}\text{Re}\left\{\frac{\epsilon_0}{2}[(\mathbf{E}_H + \mathbf{E}_V)\cdot(\mathbf{E}_H^* + \mathbf{E}_V^*) + c^2(\mathbf{B}_H + \mathbf{B}_H)\cdot(\mathbf{B}_H^* + \mathbf{B}_V^*)\right\} \\
&= \langle u_{\text{EM}}\rangle_H + \langle u_{\text{EM}}\rangle_V + \frac{1}{2}\epsilon_0\text{Re}\left\{\mathbf{E}_V\cdot\mathbf{E}_H^* + c^2\mathbf{B}_H\cdot\mathbf{B}_V^*\right\} \\
&= \epsilon_0 E_0^2 - \frac{1}{2}\epsilon_0 E_0^2\text{Re}\left\{e^{ikx}e^{-iky}\right\} \\
&= \epsilon_0 E_0^2\left[1 - \frac{1}{2}\cos[k(x-y)]\right].
\end{aligned}$$

This quantity is minimum when $x - y = m\lambda$ where m is an integer. The only solution in the overlap region is $m = 0$, so $x = y$. On that plane, the physical fields are

$$\mathbf{E}(x = y) = 0\hat{\mathbf{z}} \qquad \text{and} \qquad c\mathbf{B}(x = y) = E_0\cos(kx - \omega t)(\hat{\mathbf{x}} + \hat{\mathbf{y}}).$$

(b) The time-averaged Poynting vector for a plane wave propagating in the $\hat{\mathbf{k}}$ direction is

$$\langle \mathbf{S} \rangle = \frac{\langle u_{\rm EM} \rangle}{c} \hat{\mathbf{k}}.$$

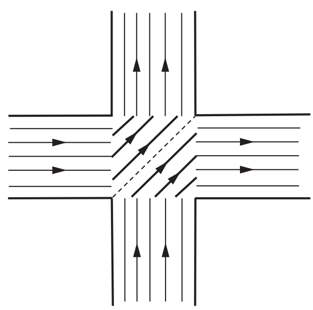
Therefore, for the horizontal and vertical beams,

$$\langle \mathbf{S} \rangle_V = \frac{\epsilon_0}{2c} E_0^2 \hat{\mathbf{y}} \qquad \text{and} \qquad \langle \mathbf{S} \rangle_H = \frac{\epsilon_0}{2c} E_0^2 \hat{\mathbf{x}}.$$

For the superposed beams,

$$\begin{aligned}
\langle \mathbf{S} \rangle &= \frac{1}{2\mu_0} \mathrm{Re}\left\{ (\mathbf{E}_H + \mathbf{E}_V) \times (\mathbf{E}_H + \mathbf{E}_H)^* \right\} \\
&= \langle \mathbf{S}_H \rangle + \langle \mathbf{S}_V \rangle + \frac{1}{2\mu_0} \mathrm{Re}\left\{ \mathbf{E}_H \times \mathbf{B}_V^* + \mathbf{E}_V \times \mathbf{B}_H^* \right\} \\
&= \frac{\epsilon_0}{2c} E_0^2 (\hat{\mathbf{x}} + \hat{\mathbf{y}}) - \frac{E_0^2}{2c\mu_0} \mathrm{Re}\left\{ e^{ik(x-y)} \hat{\mathbf{y}} + e^{ik(y-x)} \hat{\mathbf{x}} \right\} \\
&= \frac{\epsilon_0}{2c} E_0^2 \left[1 - \cos k(x-y) \right] (\hat{\mathbf{x}} + \hat{\mathbf{y}}).
\end{aligned}$$

The sketch below uses line thickness to indicate the intensity of $\langle \mathbf{S} \rangle$. The dashed line indicates that $\langle \mathbf{S}(x=y) \rangle = 0$.



(c) At the surface of a conductor, we must have $\mathbf{E}_{\parallel} = 0$ and $\mathbf{B}_{\perp} = 0$. For this problem, part (a) shows that $\mathbf{E}$ points along $\hat{\mathbf{z}}$ and goes to zero at $x = y$. Part (a) showed also that $\mathbf{B}$ is parallel to the $x = y$ plane everywhere in the overlap region. Hence, the boundary conditions for a perfect conductor are met at $x = y$.

Source: J.P. Dowling and J. Gea-Banacloche, *American Journal of Physics* **60**, 28 (1992).

16.9 Zeroes of the Transverse Field

(a) If $\psi = k(gx + kx^2 \sin\delta + iy\cos\delta + iz\sin\delta)e^{i(kz-\omega t)}$, direct calculation gives

$$\psi_{xx} = 2k^2 \sin\delta\, e^{i(kz-\omega t)}$$

$$\psi_{yy} = 0$$

$$\psi_{zz} = -2k^2 \sin\delta e^{i(kz-\omega t)} - k^2\psi \quad .$$

$$\psi_{tt} = -\omega^2\psi.$$

Consequently, $\nabla^2\psi - \dfrac{1}{c^2}\dfrac{\partial^2\psi}{\partial t^2} = 0$ if $\omega = ck$.

(b) We find E_z by integrating (over z) the requirement that $0 = \nabla\cdot\mathbf{E} = \partial_x\psi + i\partial_y\psi + \partial_z E_z$. This gives

$$E_x = k\left\{gx + kx^2\sin\delta + iy'\right\}e^{i(kz-\omega t)}$$

$$E_y = iE_x$$

$$E_z = i(g - \cos\delta + 2kx\sin\delta)e^{i(kz-\omega t)} + f(x,y,t).$$

A brief calculation shows that E_z satisfies the wave equation if the arbitrary function $f(x,y,t)$ does so. We will choose $f = 0$. Given the foregoing, we know that both the $\nabla\cdot\mathbf{B}$ and $\nabla\times\mathbf{B}$ Maxwell equations will be satisfied if $\mathbf{B} = -(i/\omega)\nabla\times\mathbf{E}$. In detail,

$$cB_x = -i\left\{k(gx + kx^2\sin\delta + iy') + \sin\delta\right\}e^{i(kz-\omega t)}$$

$$cB_y = \left\{k(gx + kx^2\sin\delta + iy') - \sin\delta\right\}e^{i(kz-\omega t)}$$

$$cB_z = (g + 2kx\sin\delta - \cos\delta)e^{i(kz-\omega t)}.$$

(c) If $g \gg 1$, we can drop the terms quadratic in x so that

$$E_x = k\{gx + iy'\}e^{i(kz-\omega t)} \qquad cB_x = -i\{k(gx + iy') + \sin\delta\}e^{i(kz-\omega t)}$$

$$E_y = ik\{gx + iy'\}e^{i(kz-\omega t)} \qquad cB_y = \{k(gx + iy') - \sin\delta\}e^{i(kz-\omega t)}.$$

Clearly, the zeroes of $E_\perp$ occur when $x = 0 = y' = y\cos\delta + z\sin\delta$. This is the line $y = -z\tan\delta$ in the $x = 0$ plane.

(d) For the magnetic field just above, it is simplest to let $\Omega = kz - \omega t$ and set Re B_x = Re $B_y = 0$. These give

$$(kgx + \sin\delta)\sin\Omega + ky'\cos\Omega = 0$$

$$(kgx - \sin\delta)\cos\Omega - ky'\sin\Omega = 0.$$

Substituting $gx = r\cos\theta$ and $y' = r\sin\theta$ into these two yields

$$kr\sin(\Omega + \theta) = -\sin\delta\sin\Omega \tag{1}$$

$$kr\cos(\Omega+\theta)=\sin\delta\cos\Omega. \tag{2}$$

Squaring and adding (1) and (2) gives

$$r=k^{-1}\sin\delta=const.$$

Therefore,

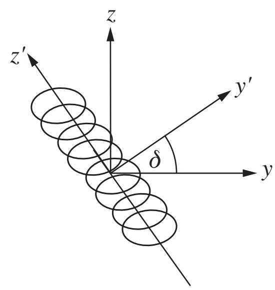
$$g^2x^2+y'^2=r^2=const.$$

This defines an ellipse centered on the z' axis.

Finally, multiply (1) by $\cos\Omega$ and add this to (2) multiplied by $\sin\Omega$. This gives

$$kr\sin(2\Omega+\theta)=0 \qquad \rightarrow \qquad \theta=2\omega t-2kz.$$

This shows that the polar angle which traces out the ellipse increases steadily as time goes on.



Source: J.F. Nye, *Proceedings of the Royal Society of London A* **387**, 105 (1983).

16.10 Superposition and Wave Intensity

The fields of a monochromatic plane wave propagating in the z-direction satisfy $c\mathbf{B}=\hat{\mathbf{z}}\times\mathbf{E}$ and the definition of the wave intensity is

$$I=\left|\frac{1}{T}\int_0^T dt\,\mathbf{S}\right|.$$

For our problem, the Poynting vector is

$$\mathbf{S}=\frac{1}{\mu_0}[\mathrm{Re}\mathbf{E}\times\mathrm{Re}\mathbf{B}]=\frac{1}{\mu_0}\left[|\mathrm{Re}\mathbf{E}_1|^2+|\mathrm{Re}\mathbf{E}_2|^2+2\mathrm{Re}\mathbf{E}_1\cdot\mathrm{Re}\mathbf{E}_2\right]\hat{\mathbf{z}}.$$

The first two terms produce I_1+I_2, so we focus on the last term. If $\phi_k=\omega_k(z/c-t)$,

$$\mathrm{Re}\mathbf{E}_k=A_k\cos(\phi_k+\delta_k)\hat{\mathbf{x}}+B_k\cos(\phi_k+\bar{\delta}_k)\hat{\mathbf{y}}.$$

Therefore,

$$\mathrm{Re}\mathbf{E}_1 \cdot \mathrm{Re}\mathbf{E}_2 = A_1 A_2 \cos(\phi_1 + \delta_1)\cos(\phi_2 + \delta_2) + B_1 B_2 \cos(\phi_1 + \bar{\delta}_1)\cos(\phi_2 + \bar{\delta}_2).$$

Both terms have the same structure in time, so we focus on the first one. If $\Delta_k = \delta_k + \omega_k z/c$,

$$\begin{aligned} X &= \frac{1}{T}\int_0^T dt\, \cos(\phi_1 + \delta_1)\cos(\phi_2 + \delta_2) \\ &= \frac{1}{T}\int_0^T dt\, \left[\cos[(\omega_1 + \omega_2)t + \Delta_1 + \Delta_2] + \cos[(\omega_1 - \omega_2)t + \Delta_1 - \Delta_2]\right]. \end{aligned}$$

Carrying out the integrals gives

$$\begin{aligned} X &= \frac{1}{T(\omega_1 + \omega_2)}\left[\sin[(\omega_1 + \omega_2)T + \Delta_1 + \Delta_2] - \sin(\Delta_1 + \Delta_2)\right] \\ &\quad + \frac{1}{T(\omega_1 - \omega_2)}\left[\sin[(\omega_1 - \omega_2)T + \Delta_1 - \Delta_2] - \sin(\Delta_1 - \Delta_2)\right]. \end{aligned}$$

The numerators in both terms are bounded by 2. Therefore, these integrals will be negligible (as we desire) if the averaging time satisfies

$$T(\omega_1 + \omega_2) \gg 1 \qquad \text{and} \qquad T(\omega_1 - \omega_2) \gg 1.$$

Source: G.S. Smith, *An Introduction to Classical Electromagnetic Radiation* (University Press, Cambridge, 1997).

16.11 Antipodes of the Poincaré Sphere

Let $\mathbf{E} = \mathcal{E}_1\hat{\mathbf{e}}_1 + \mathcal{E}_2\hat{\mathbf{e}}_2$ and $\mathbf{E}' = \mathcal{E}'_1\hat{\mathbf{e}}_1 + \mathcal{E}'_2\hat{\mathbf{e}}_2$. The condition for orthogonality is

$$\mathbf{E}\cdot\mathbf{E}'^* = \mathcal{E}_1\mathcal{E}'^*_1 + \mathcal{E}_2\mathcal{E}'^*_2 = 0. \tag{1}$$

Our task is to check that (1) is satisfied if the tips of $\mathbf{E}$ and $\mathbf{E}'$ are antipodes on the Poincaré sphere. For this purpose, we recall that the Cartesian coordinates of points on this sphere are (s_1, s_2, s_3), where the Stokes parameters are

$$s_1 = |\mathcal{E}_1|^2 - |\mathcal{E}_2|^2 \qquad s_2 = 2\mathrm{Re}\,[\mathcal{E}_1^*\mathcal{E}_2] \qquad s_3 = 2\mathrm{Im}\,[\mathcal{E}_1^*\mathcal{E}_2].$$

If $\mathbf{E}'$ is an antipode point, we must have

$$s'_1 = -s_1 \qquad s'_2 = -s_2 \qquad s'_3 = -s_3.$$

This will be true if

$$|\mathcal{E}_1|^2 - |\mathcal{E}_2|^2 = -|\mathcal{E}'_1|^2 + |\mathcal{E}'_2|^2$$

and

$$\mathcal{E}_1^*\mathcal{E}_2 = -\mathcal{E}'^*_1\mathcal{E}'_2. \tag{2}$$

The first of these is satisfied if

$$|\mathcal{E}'_1| = |\mathcal{E}_2| \qquad \text{and} \qquad |\mathcal{E}'_2| = |\mathcal{E}_1|.$$

Therefore, we write

$$\mathcal{E}_1 = ae^{i\delta_1} \qquad \mathcal{E}_2 = be^{i\delta_2}$$

$$\mathcal{E}'_1 = be^{i\delta'_1} \qquad \mathcal{E}'_2 = ae^{i\delta'_2}.$$

Substituting these into (2) gives

$$abe^{i(\delta_2-\delta_1)} = -abe^{i(\delta'_2-\delta'_1)}.$$

But this is the same as

$$abe^{i(\delta_1-\delta'_1)} + abe^{i(\delta_2-\delta'_2)} = 0,$$

which is the orthogonality condition (1).

16.12 Kepler's Law for Plane Wave Polarization

Let ψ be the angle between $\boldsymbol{\mathcal{E}}(t)$ and the $\hat{\mathbf{e}}_1$ axis. Then,

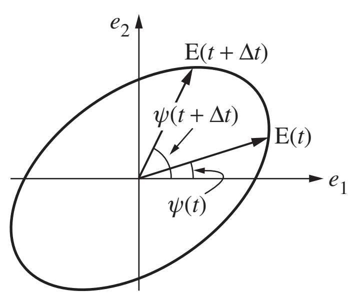
$$\tan\psi = \frac{a_2\cos(\omega t-\delta_2)}{a_1\cos(\omega t-\delta_1)}.$$

Taking the time derivative gives

$$\sec^2\psi\frac{d\psi}{dt} = \frac{\omega a_1 a_2\sin(\omega t-\delta_1)\cos(\omega t-\delta_2) - \omega a_2 a_1\sin(\omega t-\delta_2)\cos(\omega t-\delta_1)}{a_1^2\cos^2(\omega t-\delta_1)}$$

or

$$\frac{d\psi}{dt} = \frac{\omega a_1 a_2\sin(\delta_2-\delta_2)}{a_1^2\cos^2(\omega t-\delta_1+a_2^2\cos(\omega t-\delta_2)} = \frac{\omega a_1 a_2\sin(\delta_2-\delta_2)}{|\boldsymbol{\mathcal{E}}|^2}.$$



Ignoring the change in the magnitude of $\boldsymbol{\mathcal{E}}(t)$ over a time interval Δt, the area of the triangular sector swept out by $\boldsymbol{\mathcal{E}}$ in this much time is $dA = (1/2)|\boldsymbol{\mathcal{E}}|^2 d\psi$. Therefore,

$$\frac{dA}{dt} = \frac{1}{2}|\boldsymbol{\mathcal{E}}|^2 \frac{d\psi}{dt} = \frac{1}{2}\omega a_1 a_2 \sin(\delta_2 - \delta_1).$$

Source: H. Mott, *Polarization and Antennas in Radar* (Wiley, New York, 1986).

16.13 Elliptical Polarization

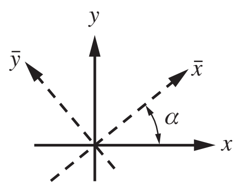
The text demonstrates that the locus of $|\mathbf{E}|$ is the ellipse

$$\left(\frac{E_x}{A}\right)^2 + \left(\frac{E_y}{B}\right)^2 - 2\frac{E_x E_y}{AB}\cos(\delta_1 - \delta_2) = \sin^2(\delta_1 - \delta_2).$$

Let the principal axis system $\bar{x}$-$\bar{y}$ be rotated from the x-y system by an angle α as shown below. In that case

$$E_x = \bar{E}_x \cos\alpha - \bar{E}_y \sin\alpha$$

$$E_y = \bar{E}_x \sin\alpha + \bar{E}_y \cos\alpha.$$



Substituting these into the ellipse equation above gives

$$\begin{aligned}
&\bar{E}_x^2 \left\{\frac{\cos^2\alpha}{A^2} + \frac{\sin^2\alpha}{B^2} - \frac{2\sin\alpha\cos\alpha}{AB}\cos(\delta_1 - \delta_2)\right\} \\
&\qquad + \bar{E}_y^2 \left\{\frac{\sin^2\alpha}{A^2} + \frac{\cos^2\alpha}{B^2} + \frac{2\sin\alpha\cos\alpha}{AB}\cos(\delta_1 - \delta_2)\right\} \\
&\qquad + 2\bar{E}_x\bar{E}_y \left\{-\frac{\sin\alpha\cos\alpha}{A^2} + \frac{\sin\alpha\cos\alpha}{B^2} - \frac{\cos^2\alpha - \sin^2\alpha}{AB}\cos(\delta_1 - \delta_2)\right\} \\
&\qquad = \sin^2(\delta_1 - \delta_2).
\end{aligned}$$

Since this is the principal axis system, the coefficient of $\bar{E}_x\bar{E}_y$ must be zero. This gives

$$\frac{1}{2}\sin 2\alpha \left\{\frac{A^2 - B^2}{AB}\right\} = \frac{1}{AB}\cos^2 2\alpha \cos(\delta_1 - \delta_2)$$

or

$$\tan 2\alpha = \frac{2AB}{A^2 - B^2}\cos(\delta_2 - \delta_1)$$

as required.

16.14 A Vector Potential Wave Packet

(a) The electric and magnetic fields are

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = -\frac{\partial \mathbf{A}}{\partial \zeta}\frac{\partial \zeta}{\partial t} = -ck\frac{\partial \mathbf{A}}{\partial \zeta} \qquad \mathbf{B} = \nabla \times \mathbf{A} = \nabla\zeta \times \frac{\partial \mathbf{A}}{\partial \zeta} = -\frac{\hat{\mathbf{z}}}{c} \times \mathbf{E}.$$

The chain rule gives the exact electric field as

$$\mathbf{E} = -(a\hat{\mathbf{x}} + ib\hat{\mathbf{y}})\left[A_0'(\zeta) + iA_0(\zeta)\right]\exp(i\zeta)ck.$$

The slow-variation assumption means that $A_0'(\zeta) \ll A_0(\zeta)$. Therefore,

$$\mathbf{E} = \text{Re}\ \{-ick(a\hat{\mathbf{x}} + ib\hat{\mathbf{y}})A_0(\zeta)\exp(i\zeta)\} = ckA_0(\hat{\mathbf{x}}a\sin\zeta + \hat{\mathbf{y}}b\cos\zeta)$$

$$\mathbf{B} = \text{Re}\ \{k(b\hat{\mathbf{x}} + ia\hat{\mathbf{y}})A_0(\zeta)\exp(i\zeta)\} = kA_0(\hat{\mathbf{x}}b\cos\zeta - \hat{\mathbf{y}}a\sin\zeta).$$

(b) The linear momentum density of the field is

$$\mathbf{g}_{\text{EM}} = \epsilon_0 \mathbf{E} \times \mathbf{B} = -\epsilon_0 ck^2 A_0^2\left(a^2\sin^2\zeta + b^2\cos^2\zeta\right)\hat{\mathbf{z}}.$$

The sign of $\mathbf{g}_{\text{EM}}$ is negative because the factor $\exp(i\zeta) = \exp[ik(z+ct)]$ implies that the wave propagates in the $-z$-direction.

16.15 Fourier Uncertainty

(a) We get $\hat{h}(k) = ik\hat{f}(k)$ because

$$h(x) = \frac{df}{dx} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\hat{f}(k)ike^{ikx} = \int_{-\infty}^{\infty}\frac{dk}{2\pi}\hat{h}(k)e^{ikx}.$$

(b) Using the definitions above,

$$\begin{aligned}\frac{1}{2\pi}\int_{\infty}^{\infty} dk\hat{f}^*(k)\hat{g}(k) &= \frac{1}{2\pi}\int_{\infty}^{\infty} dk\hat{f}^*(k)\int_{-\infty}^{\infty} dxg(x)e^{-ikx} \\ &= \int_{-\infty}^{\infty} dxg(x)\int_{-\infty}^{\infty}\frac{dk}{2\pi}\hat{f}^*(k)e^{-ikx} \\ &= \int_{-\infty}^{\infty} dxg(x)f^*(x).\end{aligned}$$

(c)

$$< k >_k = \frac{\int_{-\infty}^{\infty} dk \hat{f}^*(k) k \hat{f}(k)}{\int_{-\infty}^{\infty} dk |\hat{f}(k)|^2} = \frac{-i \int_{-\infty}^{\infty} dk \hat{f}^*(k) \hat{h}(k)}{\int_{-\infty}^{\infty} dk |\hat{f}(k)|^2} = \frac{-i \int_{-\infty}^{\infty} dx f^*(x) \frac{df}{dx}}{\int_{-\infty}^{\infty} dx |\hat{f}(x)|^2} = < -i\frac{d}{dx} >_x .$$

Similarly,

$$< k^2 >_k = \frac{\int_{-\infty}^{\infty} dk \hat{f}^*(k) k^2 \hat{f}(k)}{\int_{-\infty}^{\infty} dk |\hat{f}(k)|^2} = \frac{\int_{-\infty}^{\infty} dk \hat{h}^*(k) \hat{h}(k)}{\int_{-\infty}^{\infty} dk |\hat{f}(k)|^2} = \frac{\int_{-\infty}^{\infty} dx \frac{df^*}{dx}\frac{df}{dx}}{\int_{-\infty}^{\infty} dx |\hat{f}(x)|^2}$$

$$= \frac{-\int_{-\infty}^{\infty} dx f^*(x) \frac{d^2 f}{dx^2}}{\int_{-\infty}^{\infty} dx |\hat{f}(x)|^2} = < \left(-i\frac{d}{dx}\right)^2 >_x .$$

(d) Note first that $[\tilde{A}, \tilde{B}] = [A, B]$ if $\tilde{O} = O- < O >_x$. Now define $(f, Of) = \int_{-\infty}^{\infty} dx\, f^*(x) O\, f(x)$ and let $\|f\|^2 = (f, f)$. Then, for any real number c,

$$\begin{aligned} 0 &\geq \left\| \left\{ \tilde{A} + ic\tilde{B} \right\} \right\|^2 \\ &= \left\| \tilde{A} f \right\|^2 + c^2 \left\| \tilde{B} f \right\|^2 + ic \left\{ \left(\tilde{A} f, \tilde{B} f \right) - \left(\tilde{B} f, \tilde{A} f \right) \right\} \\ &= \left\{ c^2 (\Delta B)^2 + c < i\,[A, B] >_x + (\Delta A)^2 \right\} \|f\|^2 \\ &= R(c)\, \|f\|^2 . \end{aligned}$$

Now write $R(c) = (c - c_1)(c - c_2)$ where $R(c_1) = R(c_2) = 0$. $R(c) \geq 0$ for consistency with the inequality above so the roots cannot be real and distinct. This means that the discriminant of $R(c)$ is non-positive, i.e., $< i[A, B] >_x^2 - 4(\Delta A)^2 (\Delta B)^2 \leq 0$. We conclude that

$$\Delta A\, \Delta B \geq \frac{1}{2} | < i[A, B] >_x |$$

as desired.

(e)

$$\Delta k = \sqrt{< k^2 >_k - < k >_k^2} = \sqrt{< (-id/dx)^2 >_x - < (-id/dx) >_x}$$

so

$$\Delta x\, \Delta k = \Delta x\, \Delta(-i\frac{d}{dx}) \geq \frac{1}{2} | < i[x, -i\frac{d}{dx}] >_x | = \frac{1}{2}.$$

This proves that $\Delta x\, \Delta k \geq \frac{1}{2}$ as required.

16.16 Plane Wave Packet from the Helmholtz Equation

(a) Separation of variables in the form

$$u(\mathbf{r},t) = u(\mathbf{r})T(t)$$

transforms the scalar wave equation to

$$\frac{c^2}{u}\nabla^2 u = \frac{1}{T}\frac{d^2T}{dt^2}.$$

Setting these independent ratios equal to the same constant $(-\omega^2)$ gives

$$\frac{d^2T}{dt^2} + \omega^2 T = 0 \tag{1}$$

and the Helmholtz equation

$$\nabla^2 u + \frac{\omega^2}{c^2}u = 0. \tag{2}$$

The general solution to (1) is

$$T(t|\omega) = a(\omega)e^{i\omega t} + b(\omega)e^{-i\omega t}.$$

Therefore, if $u(\mathbf{r}|\omega)$ is the general solution of (2), the general solution to the original wave equation is

$$u(\mathbf{r},t) = \int_0^\infty d\omega\, u(\mathbf{r}|\omega)T(t|\omega). \tag{3}$$

(b) In Cartesian coordinates, (2) reads

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\omega^2}{c^2}u = 0. \tag{4}$$

A separated-variable trial solution to (4) has the product form

$$u(x,y,z|\omega) = X(x)Y(y)Z(z). \tag{5}$$

Proceeding as in part (a) generates three separation constants, $-k_x^2$, $-k_y^2$, and $-k_z^2$, and three ordinary differential equations,

$$\frac{d^2X}{dx^2} + k_x^2 X = 0 \qquad \frac{d^2Y}{dy^2} + k_y^2 Y = 0 \qquad \frac{d^2Z}{dz^2} + k_z^2 Z = 0. \tag{6}$$

It is natural to define a variable k^2 using the constraint imposed by (2) on the separation constants:

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2} = k^2. \tag{7}$$

The three equations in (6) are solved by similar complex exponentials. Therefore, the wave vector $\mathbf{k} = (k_x, k_y, k_z)$ appears when we form the product (5) to get a typical solution of the form

$$\exp(ik_x x)\exp(ik_y y)\exp(ik_z z) = \exp(i\mathbf{k}\cdot\mathbf{r}). \tag{8}$$

A general solution to (2) sums terms like (8) with amplitudes $c(\mathbf{k},\omega)$ and a delta function $\delta(\omega - c|\mathbf{k}|)$ to enforce (7):

$$u(\mathbf{r}|\omega) = \int d^3k\; c(\mathbf{k},\omega)\delta(\omega - c|\mathbf{k}|)\exp(i\mathbf{k}\cdot\mathbf{r}). \tag{9}$$

Substituting (9) into (3) and using the delta function to perform the ω integral gives the final result in the form

$$u(\mathbf{r},t) = \int d^3k\; \{A(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r} + ckt)] + B(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r} - ckt)]\}\,. \tag{10}$$

(c) Equation (10) is a superposition of monochromatic plane waves propagating in the $+\mathbf{k}$ and $-\mathbf{k}$ directions. The text used only the $+\mathbf{k}$ set. However, if we change the dummy variable $\mathbf{k}$ to $-\mathbf{k}$ in the $A(\mathbf{k})$ sum, (10) becomes

$$u(\mathbf{r},t) = \int d^3k\; \{A(-\mathbf{k})\exp[-i(\mathbf{k}\cdot\mathbf{r} - ckt)] + B(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r} - ckt)]\}\,.$$

With $A = A' + iA''$ and $B = B' + iB''$, the real part of the foregoing is

$$\mathrm{Re}\,[u(\mathbf{r},t)] = \int d^3k\; \{(A' + B')\cos(\mathbf{k}\cdot\mathbf{r} - ckt) + (A'' - B'')\sin(\mathbf{k}\cdot\mathbf{r} - ckt)\}\,.$$

This is identical to

$$\mathrm{Re}\,[u(\mathbf{r},t)] = \mathrm{Re}\int d^3k\, C(\mathbf{k})\exp[i(\mathbf{k}\cdot\mathbf{r} - ckt)].$$

16.17 $\mathbf{P}_{\rm EM}$ for a Wave Packet

(a) Let $k = |\mathbf{k}|$, $k' = |\mathbf{k}'|$, $\phi = \mathbf{k}\cdot\mathbf{r} - ckt$, $\phi' = \mathbf{k}'\cdot\mathbf{r} - ck't$, and $c\mathbf{B}_\perp(\mathbf{k}) = \hat{\mathbf{k}}\times\mathbf{E}_\perp(\mathbf{k})$. Then,

$$
\begin{aligned}
\mathbf{P}_{\rm EM} &= \epsilon_0 \int d^3r\, {\rm Re}\,\mathbf{E}\times{\rm Re}\,\mathbf{B} \\
&= \frac{\epsilon_0}{4}\int d^3r\,(\mathbf{E}+\mathbf{E}^*)\times(\mathbf{B}+\mathbf{B}^*) \\
&= \frac{\epsilon_0}{4}\frac{1}{(2\pi)^3}\int d^3k\int d^3k'\int d^3r\,[\mathbf{E}_\perp(\mathbf{k})e^{i\phi}+\mathbf{E}^*_\perp(\mathbf{k})e^{-i\phi}] \\
&\qquad\qquad \times\ [\mathbf{B}_\perp(\mathbf{k}')e^{i\phi'}+\mathbf{B}^*_\perp(\mathbf{k}')e^{-i\phi'}] \\
&= \frac{\epsilon_0}{4}\frac{1}{(2\pi)^3}\int d^3k\int d^3k'\int d^3r\,\Big\{[\mathbf{E}_\perp(\mathbf{k})\times\mathbf{B}_\perp(\mathbf{k}')]e^{i(\phi+\phi')}\,. \\
&\qquad\qquad +\ [\mathbf{E}_\perp(\mathbf{k})\times\mathbf{B}^*_\perp(\mathbf{k}')]e^{i(\phi-\phi')}+c.c.\Big\}.
\end{aligned}
$$

There are four terms inside the curly brackets. Let us show that the first term (and its complex conjugate) vanish.

$$
\begin{aligned}
\mathbf{I} &= \frac{\epsilon_0}{4}\frac{1}{(2\pi)^3}\int d^3k\int d^3k'\int d^3r\,[\mathbf{E}_\perp(\mathbf{k})\times\mathbf{B}_\perp(\mathbf{k}')]\exp[i(\phi+\phi')] \\
&= \frac{\epsilon_0}{4c}\frac{1}{(2\pi)^3}\int d^3k\int d^3k'\int d^3r\,\Big\{\mathbf{E}_\perp(\mathbf{k})\times[\hat{\mathbf{k}'}\times\mathbf{E}_\perp(\mathbf{k}')]\Big\} \\
&\qquad\qquad \times\ \exp[i(\mathbf{k}+\mathbf{k}')\cdot\mathbf{r}]\exp[-ict(k+k')] \\
&= \frac{\epsilon_0}{4c}\int d^3k\int d^3k'\delta(\mathbf{k}+\mathbf{k}')\Big\{\mathbf{E}_\perp(\mathbf{k})\times[\hat{\mathbf{k}'}\times\mathbf{E}_\perp(\mathbf{k}')]\Big\}\exp[-ict(k+k')] \\
&= -\frac{\epsilon_0}{4c}\int d^3k\Big\{\mathbf{E}_\perp(\mathbf{k})\times[\hat{\mathbf{k}}\times\mathbf{E}_\perp(-\mathbf{k})]\Big\}\exp[-2ictk] \\
&= -\frac{\epsilon_0}{4c}\int d^3k\,\hat{\mathbf{k}}\,\mathbf{E}_\perp(\mathbf{k})\cdot\mathbf{E}_\perp(-\mathbf{k})\exp[-2ictk].
\end{aligned}
$$

We get $I=0$ because the last integral above is an odd function of k_x, k_y, and k_z. This leaves the third term in curly brackets in our expression for $\mathbf{P}_{\rm EM}$ and its complex conjugate. The third term is

$$
\begin{aligned}
\mathbf{T} &= \frac{\epsilon_0}{4c}\frac{1}{(2\pi)^3}\int d^3k\int d^3k'\int d^3r\,\Big\{\mathbf{E}_\perp(\mathbf{k})\times[\hat{\mathbf{k}'}\times\mathbf{E}^*_\perp(\mathbf{k}')]\Big\} \\
&\qquad\qquad \times\ \exp[i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}]\exp[-ict(k-k')] \\
&= \frac{\epsilon_0}{4c}\int d^3k\int d^3k'\delta(\mathbf{k}-\mathbf{k}')\Big\{\mathbf{E}_\perp(\mathbf{k})\times[\hat{\mathbf{k}'}\times\mathbf{E}^*_\perp(\mathbf{k}')]\Big\}\exp[-ict(k-k')] \\
&= \frac{\epsilon_0}{4c}\int d^3k\Big\{\mathbf{E}_\perp(\mathbf{k})\times[\hat{\mathbf{k}}\times\mathbf{E}^*_\perp(\mathbf{k})]\Big\} \\
&= \frac{\epsilon_0}{4c}\int d^3k\,\hat{\mathbf{k}}\,\mathbf{E}_\perp(\mathbf{k})\cdot\mathbf{E}^*_\perp(\mathbf{k}).
\end{aligned}
$$

Adding $\mathbf{T}$ to its complex conjugate gives the desired linear momentum:

$$
\mathbf{P}_{\rm EM} = \frac{\epsilon_0}{2c}\int d^3k\,\hat{\mathbf{k}}\,|\mathbf{E}_\perp(\mathbf{k})|^2.
$$

This formula is valid at any time. We have done no time-averaging.

(b) The text reports that the total electromagnetic energy is

$$U_{\rm EM} = \frac{\epsilon_0}{2}\int d^3k\,|\mathbf{E}_\perp(\mathbf{k})|^2.$$

The suggested inequality is true because the momentum adds unit vectors that generally point in different directions with the same weighting factor as the energy.

(c) $c\mathbf{P}_{\rm EM} = U_{\rm EM}$ when all the waves in the packet propagate in the same direction.

Source: E.J. Konopinski, *Electromagnetic Fields and Relativistic Particles* (McGraw-Hill, New York, 1981).

16.18 A Transverse Magnetic Beam

The Lorenz gauge satisfies the vector wave equation. With the assumed form for $\mathbf{A}$, this implies that $u(\rho, z)$ satisfies the Helmholtz equation:

$$0 = \nabla^2 u + \frac{\omega^2}{c^2}u = \frac{\partial^2 u}{\partial z^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial u}{\partial \rho}\right). \tag{1}$$

The magnetic field is

$$\mathbf{B} = \nabla\times\mathbf{A} = -\frac{\partial A_z}{\partial \rho}\hat{\boldsymbol{\phi}} = -\frac{\partial u}{\partial \rho}e^{-i\omega t}\hat{\boldsymbol{\phi}}.$$

To get the electric field, we need the scalar potential, which satisfies

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0.$$

Hence,

$$\varphi(\rho, z, t) = -i\frac{c}{\omega^2}\frac{\partial u}{\partial z}e^{-i\omega t}.$$

Therefore, using the right side of (1), we find

$$\mathbf{E} = -\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t} = \frac{ic^2}{\omega}\left[\frac{\partial^2 u}{\partial z\partial\rho}\hat{\boldsymbol{\rho}} - \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial u}{\partial\rho}\right)\hat{\mathbf{z}}\right]e^{-i\omega t}.$$

$\mathbf{E}$ and $\mathbf{B}$ at any fixed point in space do not change direction as a function of time. Therefore, both are linearly polarized. However, $\mathbf{E}$ has a z-component, which means it is not transverse.

Source: J. Lekner, *Journal of Optics A* **5**, 6 (2003).

16.19 Paraxial Fields of the Gaussian Beam

Choose a Lorenz gauge potential $\mathbf{A}_L = (u/i\omega)\hat{\mathbf{x}}$. The Lorenz gauge condition reads

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \varphi}{\partial t} = 0 \qquad \Rightarrow \qquad \nabla \cdot \mathbf{A} = i\frac{\omega}{c^2}\varphi.$$

Therefore,

$$\mathbf{E} = -\nabla\varphi = -\frac{\partial \mathbf{A}}{\partial t} = \frac{c^2}{\omega^2}\nabla\frac{\partial u}{\partial x} + u\hat{\mathbf{x}}.$$

In the paraxial approximation, $\omega = ck$ and the fields in the transverse (x and y) directions change much faster than in the longitudinal (z) direction. Therefore, the gradient may be approximated by $\partial/\partial z$. Consequently,

$$\mathbf{E} = \frac{c^2}{\omega^2}ik\frac{\partial u}{\partial x}\hat{\mathbf{z}} + u\hat{\mathbf{x}} = \frac{i}{k}\frac{\partial u}{\partial x}\hat{\mathbf{z}} + u\hat{\mathbf{x}}.$$

The magnetic field equation is Faraday's law written for time-harmonic fields.

The divergence of the electric field is

$$\nabla \cdot \mathbf{E} = \frac{i}{k}\frac{\partial^2 u}{\partial x \partial z} + \frac{\partial u}{\partial x}.$$

This is zero because $\partial u/\partial z = ik$ in the paraxial approximation. $\nabla \cdot \mathbf{B} = 0$ is an identity because the divergence of any curl is zero.

16.20 Physical Origin of the Gouy Phase

For the Gaussian beam, the transverse wave vector distribution function is

$$F(k_x, k_y) = \frac{1}{2\pi}\sqrt{\frac{2}{\pi}}\frac{1}{w}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \, \exp\left[-(x^2+y^2)/w^2\right]\exp[-i(k_x x + k_y y)].$$

By completing the square in the exponential, we easily get

$$F(k_x, k_y) = \frac{w}{\sqrt{2\pi}}\exp\left[-w^2(k_x^2+k_y^2)/4\right].$$

Then,

$$\begin{aligned}
\langle k_x^2 \rangle &= \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \, k_x^2 |F(k_x,k_y)|^2 \\
&= \frac{w^2}{2\pi}\int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \, k_x^2 \exp\left[-w^2(k_x^2+k_y^2)/2\right] = 1/w^2
\end{aligned}$$

and $\langle k_x^2 \rangle = \langle k_y^2 \rangle$ by symmetry. Therefore,

$$\int_0^z dz\,\bar{k}_z = \int_0^z dz\left\{k - \frac{2}{kw_0^2(1+z^2/z_R^2)}\right\} = kz - \tan^{-1}\frac{z}{z_R} = kz + \alpha(z).$$

No localization in the transverse direction means $\langle k_x^2 \rangle = \langle k_y^2 \rangle = 0$. Therefore, $\int_0^z dz\,\bar{k}_z = kz$ and $\alpha(z) = 0$.

Source: S. Feng and H.G. Winful, *Optics Letters* **26**, 486 (2001).

16.21 D'Alembert Solutions in Two and Three Dimensions

(a) Let $G(r,t) = F(r,t)/r$. There is no angular dependence so

$$\nabla^2 G = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial G}{\partial r}\right) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(rF' - F\right) = \frac{F''}{r}.$$

Therefore,

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = \frac{1}{r}\left[\frac{d^2 F}{dr^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}\right] = 0.$$

The quantity in square brackets is the wave operator in the one-dimensional variable r. This solves our problem because the latter is solved by $F(r \pm ct)$ where F is an arbitrary function.

(b) Let $G(\rho,t) = F(\rho,t)/\sqrt{\rho}$. There is no angular dependence so

$$\nabla^2 G = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial G}{\rho}\right) = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\sqrt{\rho}F' - \frac{1}{2}\frac{F}{\sqrt{\rho}}\right) = \frac{F''}{\sqrt{\rho}} + \frac{F}{4\rho^{5/2}}.$$

The term on the far right can be dropped when ρ is sufficiently large (see below). In that case,

$$\nabla^2 G - \frac{1}{c^2}\frac{\partial^2 G}{\partial t^2} = \frac{1}{\sqrt{\rho}}\left[\frac{d^2 F}{\rho^2} - \frac{1}{c^2}\frac{\partial^2 F}{\partial t^2}\right] = 0.$$

The quantity in square brackets is the wave operator in the one-dimensional variable ρ. This solves our problem because the latter is solved by $F(\rho \pm ct)$ where F is an arbitrary function.

Among the solutions we find, the term we dropped will be negligible when

$$\frac{F''}{\sqrt{\rho}} \gg \frac{F}{4\rho^{5/2}} \qquad \Rightarrow \qquad \rho \gg \sqrt{\frac{F}{F''}}.$$

16.22 Wave Interference

(a) The waves interfere constructively when their phases differ by a multiple of 2π. If we subsume this multiple into the definition of δ, the interference condition is

$$kx - \delta = \mathbf{k}\cdot\mathbf{r} = k\sqrt{x^2 + y^2 + z^2}.$$

Squaring both sides eliminates k^2x^2 from both sides and we get

$$x = -\frac{k}{2\delta}(y^2 + z^2) - \frac{\delta}{2k}.$$

This equation defines a family of paraboloids parameterized by the phase shift δ.

(b) Suppressing a factor of $\exp(-i\omega t)$, the electric field for the two waves can be written

$$\mathbf{E}_1 = \hat{\mathbf{y}} A \exp[ik(z\cos\theta + x\sin\theta)]$$

and

$$\mathbf{E}_2 = \hat{\mathbf{y}} A \exp[ik(z\cos\theta - x\sin\theta)].$$

The associated magnetic fields satisfy $c\mathbf{B} = \hat{\mathbf{k}} \times \mathbf{E}$. Therefore,

$$c\mathbf{B}_1 = A(\sin\theta\hat{\mathbf{z}} - \cos\theta\hat{\mathbf{x}})\exp[ik(z\cos\theta + x\sin\theta)]$$

and

$$c\mathbf{B}_2 = -A(\sin\theta\hat{\mathbf{z}} + \cos\theta\hat{\mathbf{x}})\exp[ik(z\cos\theta - x\sin\theta)].$$

The superposed waves are

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = \hat{\mathbf{y}} 2A\cos(kx\sin\theta)\exp(ikz\cos\theta)$$

and

$$c\mathbf{B} = c\mathbf{B}_1 + c\mathbf{B}_2 = 2A\left[-\hat{\mathbf{x}}\cos\theta\cos(kx\sin\theta) + i\hat{\mathbf{z}}\sin\theta\sin(kx\sin\theta)\right]\exp(ikz\cos\theta).$$

For an arbitrary angle θ, this is *not* a plane wave and it is *not* transverse.

(c) At $\theta = 0$, we have

$$\mathbf{E}(0) = \hat{\mathbf{y}} 2A\exp(ikz) \qquad \text{and} \qquad c\mathbf{B}(0) = -2A\hat{\mathbf{x}}\exp(ikz),$$

and a time-averaged Poynting vector that is uniform throughout space:

$$\langle\,\mathbf{S}(0)\,\rangle = \frac{1}{2\mu_0}\mathrm{Re}(\mathbf{E}\times\mathbf{B}^*) = \frac{2A^2}{\mu_0}\hat{\mathbf{z}}.$$

At $\theta = \pi/2$, we have a pure standing wave with

$$\mathbf{E}(\pi/2) = 2A\cos(kx)\hat{\mathbf{y}} \qquad \text{and} \qquad c\mathbf{B}(\pi/2) = i2A\sin(kx)\hat{\mathbf{z}},$$

and a time-averaged Poynting vector $\langle\,\mathbf{S}(\pi/2)\,\rangle = 0$. Finally, at $\theta = \pi/4$, the fields are

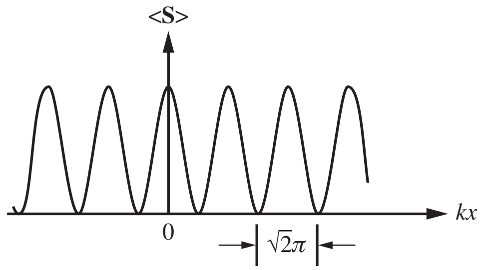
$$\mathbf{E}(\pi/4) = \hat{\mathbf{y}} 2A\cos(kx/\sqrt{2})\exp(ikz/\sqrt{2})$$

and

$$c\mathbf{B}(\pi/4) = 2A\left[-\frac{\hat{\mathbf{x}}}{\sqrt{2}}\cos(kx/\sqrt{2}) + i\frac{\hat{\mathbf{z}}}{\sqrt{2}}\sin(kx/\sqrt{2})\right]\exp(ikz/\sqrt{2}).$$

The time-averaged Poynting vector is

$$\langle\,\mathbf{S}(\pi/4)\,\rangle = \frac{1}{2\mu_0}\mathrm{Re}(\mathbf{E}\times\mathbf{B}^*) = \frac{2A^2}{\mu_0}\cos^2\left(\frac{kx}{\sqrt{2}}\right)\hat{\mathbf{z}}.$$



Source: G. Toraldo di Francia, *Electromagnetic Waves* (Interscience, New York, 1953).

16.23 Phase Velocity of Spherical Waves

The waves in question are based on the Hankel functions, $h_\ell^{(1)}(kr)$. If $\omega = ck$, solutions of the wave equation are

$$u_0(r,t) = h_0^{(1)}(kr)\exp(-i\omega t) = \frac{\exp[i(kr-\omega t)]}{ikr}$$

and

$$u_1(r,t) = h_1^{(1)}(kr)\exp(-i\omega t) = (1-ikr)\frac{\exp[i(kr-\omega t)]}{ikr}.$$

A general, complex phase function is $\phi(r,t) = \phi'(r,t) + i\phi''(r,t)$. Therefore,

$$u(r,t) = \frac{\exp[i\phi(r,t)]}{ikr} = \frac{\exp(i\phi')}{ikr}\exp(-\phi''),$$

and the phase velocity (which is in the radial direction) is

$$v_{\rm p} = -\frac{d\phi'/dt}{d\phi'/dr}. \tag{1}$$

For the $\ell = 0$ wave, $\phi'(r,t) = kr - \omega t$ and $v_{\rm p} = \omega/k = c$. For the $\ell = 1$ wave,

$$u_1(r,t) = \frac{1}{ikr}\exp[\ln(1-ikr) + i(kr-\omega t)] = \frac{\exp(i\phi)}{ikr}$$

where

$$\phi = kr - \omega t - i\ln(1-ikr) = kr - \omega t - \tan^{-1}(kr) - i\ln|1-ikr|.$$

Therefore, using (1),

$$v_{\rm p} = \frac{\omega}{k - \dfrac{k}{1+k^2r^2}} = \frac{\omega}{k}\frac{1+k^2r^2}{k^2r^2} = c\frac{1+k^2r^2}{k^2r^2}.$$

This phase velocity diverges as $r \to 0$ and goes to c as $r \to \infty$. The same general behavior may be expected for the $\ell > 1$ spherical waves.

Source: W. Gough, *European Journal of Physics* **23**, 17 (2002).

16.24 Bessel Waves

(a) Let $\psi(\rho, \phi, z, t) = f(\rho, \phi, z)\exp(-i\omega t)$. For this function to satisfy the wave equation, it is necessary that $f(\mathbf{r})$ satisfy the Helmholtz equation,

$$(\nabla^2 + k^2)f = 0,$$

where $\omega = ck$. In cylindrical coordinates, this is

$$\frac{\partial^2 f}{\partial \rho^2} + \frac{1}{\rho^2}\frac{\partial^2 f}{\partial \phi^2} + \frac{\partial^2 f}{\partial z^2} + k^2 f = 0.$$

With $f(\rho, \phi, z) = R(\rho)G(\phi)Z(z)$, separation of variables yields

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{dR}{d\rho}\right) + (\gamma^2 - m^2/\rho^2)R = 0.$$

$$\frac{d^2 Z}{dz^2} + \kappa^2 Z = 0 \qquad \text{and} \qquad \frac{d^2 G}{d\phi^2} + m^2 G = 0,$$

where $\gamma^2 = k^2 - \kappa^2$. The radial equation is Bessel's equation, which is solved by $J_m(\gamma\rho)$ and $N_m(\gamma\rho)$. The latter diverges at $\rho = 0$ so the general solution that propagates in the $+z$-direction is

$$\psi(\rho, \phi, z, t) = \sum_{m=0}^{\infty}\sum_{\kappa>0} A_m(\kappa)J_m(\gamma\rho)\left[\sin m\phi + B_m \cos m\phi\right]\exp i(\kappa z - \omega t).$$

The sum on κ is restricted to positive real numbers so we get a propagating wave in the $+z$-direction.

(b) The cylindrically symmetric solution has $m = 0$ so TE Bessel waves,

$$\mathbf{B}_{\rm TE} = \nabla \times (\nabla \times \boldsymbol{\pi}_{\rm m}) \qquad \text{and} \qquad \mathbf{E}_{\rm TE} = -\nabla \times \frac{\partial \boldsymbol{\pi}_{\rm m}}{\partial t},$$

derive from $\boldsymbol{\pi}_{\rm m} = \hat{\mathbf{z}}J_0(\gamma\rho)\exp[i(\kappa z - \omega t)]$. We get

$$\begin{aligned}
\mathbf{B}_{\rm TE} &= \nabla(\nabla \cdot \boldsymbol{\pi}_{\rm m}) - \nabla^2 \boldsymbol{\pi}_{\rm m} \\
&= \nabla(\nabla \cdot \boldsymbol{\pi}_{\rm m}) - \frac{1}{c^2}\frac{\partial^2 \boldsymbol{\pi}_{\rm m}}{\partial t^2} \\
&= [k^2 J_0(\gamma\rho)\hat{\mathbf{z}} + i\kappa\nabla J_0(\gamma\rho)]\exp[i(\kappa z - \omega t)] \\
&= [\gamma^2 J_0(\gamma\rho)\hat{\mathbf{z}} - i\kappa\gamma J_1(\gamma\rho)\hat{\boldsymbol{\rho}}]\exp[i(\kappa z - \omega t)]
\end{aligned}$$

$$\begin{aligned}\mathbf{E}_{\mathrm{TE}} &= -\nabla \times \frac{\partial}{\partial t}\hat{\mathbf{z}}J_0(\gamma\rho)\exp[i\kappa z - \omega t)] \\ &= -i\omega\frac{\partial}{\partial \rho}J_0(\gamma\rho)\exp[i\kappa z - \omega t)] \\ &= i\omega\gamma J_1(\gamma\rho)\exp[i\kappa z - \omega t)].\end{aligned}$$

The corresponding TM solutions can be obtained from the foregoing using duality:

$$\mathbf{E}_{\mathrm{TE}} \to c\mathbf{B}_{\mathrm{TM}} \qquad\qquad \mathbf{B}_{\mathrm{TE}} \to -\frac{1}{c}\mathbf{E}_{\mathrm{TM}}.$$

(c) Since $k^2 = \gamma^2 + \kappa^2$, write $\kappa = k\cos\alpha$ and $\gamma = \sin\alpha$ so

$$\psi_0(\rho, z, t) = J_0(\gamma\rho)\exp[i(\kappa z - \omega t)] = J_0(k\rho\sin\alpha)\exp[i(k\cos\alpha z - \omega t)].$$

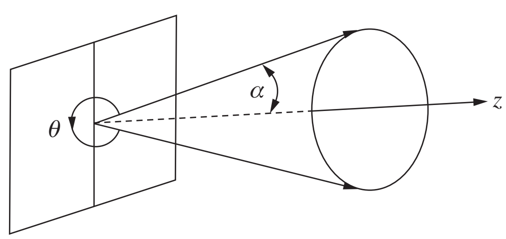
Moreover, a change of variable confirms that

$$J_0(x) = \frac{1}{2\pi}\int_0^{\pi} d\theta\exp(ix\cos\theta) = \frac{1}{2\pi}\int_0^{2\pi} d\theta\exp[ix\cos(\theta - \phi)].$$

Finally, let $x = \rho\cos\phi$ and $y = \rho\sin\phi$ so $x\cos\theta + y\sin\theta = \rho\cos(\theta - \phi)$. In that case,

$$\begin{aligned}\psi_0(\mathbf{r}, t) &= \frac{1}{2\pi}\int_0^{2\pi} d\theta\ \exp[ik\sin\alpha(x\cos\theta + y\sin\theta)]\exp[i(k\cos\alpha z - \omega t)] \\ &= \frac{1}{2\pi}\int_0^{\pi} d\theta\ \exp[i(\mathbf{q}\cdot\mathbf{r} - \omega t)],\end{aligned}$$

where $\mathbf{q} = k(\sin\alpha\cos\theta, \sin\alpha\sin\theta, \cos\alpha)$. This is indeed a set of plane wave vectors, all with the same magnitude $|\mathbf{q}| = k$, which are uniformly distributed on the surface of a cone which makes an angle α with the z-axis.



16.25 Charged Particle Motion in a Circularly Polarized Plane Wave

(a) The physical electric field is

$$\mathbf{E}(z,t) = (\hat{\mathbf{x}} + i\hat{\mathbf{y}})E_0 e^{+i(kz-\omega t)} + (\hat{\mathbf{x}} - i\hat{\mathbf{y}})E_0 e^{-i(kz-\omega t)}.$$

The corresponding magnetic field is

$$\mathbf{B}(z,t) = \frac{1}{c}\hat{\mathbf{z}} \times \mathbf{E}(z,t).$$

Therefore,

$$\begin{aligned}
\frac{d\mathbf{v}}{dt} &= \frac{q}{m}\left[\mathbf{E} + \mathbf{v} \times \frac{1}{c}(\hat{\mathbf{z}} \times \mathbf{E})\right] \\
&= \frac{q}{m}\left[(1 - \frac{v_z}{c})\mathbf{E} + \hat{\mathbf{z}}\frac{\mathbf{v}\cdot\mathbf{E}}{c}\right] \\
&= \frac{q}{m}(1 - \frac{v_z}{c})E_0\left\{(\hat{\mathbf{x}} + i\hat{\mathbf{y}})E_0 e^{+i(kz-\omega t)} + (\hat{\mathbf{x}} - i\hat{\mathbf{y}})E_0 e^{-i(kz-\omega t)}\right\} \\
&+ \hat{\mathbf{z}}\frac{qE_0}{mc}\left\{(v_x + iv_y)e^{+i(kz-\omega t)} + (v_x - iv_y)e^{-i(kz-\omega t)}\right\}.
\end{aligned}$$

From this we get, as required,

$$\begin{aligned}
\frac{dv_z}{dt} &= \frac{1}{2}\Omega\left\{v_+ e^{+i(kz-\omega t)} + v_- e^{-i(kz-\omega t)}\right\} \\
\frac{dv_\pm}{dt} &= \Omega(c - v_z)e^{\mp i(kz-\omega t)},
\end{aligned} \tag{1}$$

where $v_\pm = v_x \pm iv_y$ and $\Omega = 2qE_0/mc$.

(b) Now define $\ell_\pm = v_\pm e^{\pm i(kz-\omega t)} \pm ic\Omega/\omega$ so

$$\frac{dv_z}{dt} = \frac{1}{2}\Omega(\ell_+ + \ell_-). \tag{2}$$

On the other hand,

$$\frac{d\ell_\pm}{dt} = \frac{dv_\pm}{dt}e^{\pm i(kz-\omega t)} \mp i\omega v_\pm e^{\pm i(kz-\omega t)} = \Omega(c - v_z) \mp i\omega v_\pm e^{\pm i(kz-\omega t)}.$$

Therefore, using (1),

$$\frac{d}{dt}(\ell_- - \ell_+) = -i\omega\left[v_+ e^{i(kz-\omega t)} + v_- e^{-i(kz-\omega t)}\right] = -\frac{2i\omega}{\Omega}\frac{dv_z}{dt}.$$

We conclude that

$$\frac{d}{dt}\left\{v_z - i\frac{\Omega}{2\omega}(l_+ - \ell_-)\right\} = 0.$$

Hence, a constant of the motion is

$$K = v_z(0) - i\frac{\Omega}{2\omega}\left[l_+(0) - \ell_-(0)\right].$$

(c) Differentiating (2) gives

$$\frac{d^2 v_z}{dt^2} = \frac{\Omega}{2}\left[\frac{d\ell_+}{dt} + \frac{d\ell_-}{dt}\right] = \Omega^2(c - v_z) - \tfrac{1}{2}i\omega\Omega\left\{v_+ e^{i(kz-\omega t)} - v_- e^{-i(kz-\omega t)}\right\}.$$

But $\ell_+ - \ell_- = v_+ e^{i(kz-\omega t)} - v_- e^{-i(kz-\omega t)} + 2i\Omega c/\omega$ so

$$\frac{d^2 v_z}{dt^2} = \Omega^2(c - v_z) - \tfrac{1}{2}i\omega\Omega\left\{\ell_+ - \ell_- - 2i\Omega c/\omega\right\} = -\left(\Omega^2 + \omega^2\right)v_z + \omega^2 K.$$

Now, $v(0) = 0$ and $\ell_\pm(0) = \pm ic\Omega/\omega$, in which case, $\omega^2 K = c\Omega^2$. Hence, if we define

$$P = c\Omega^2 \qquad \text{and} \qquad \Omega_0^2 = \Omega^2 + \omega^2,$$

the equation of motion for v_z is

$$\frac{d^2 v_z}{dt^2} + \Omega_0^2 v_z = P.$$

This is solved by writing

$$\frac{d^2}{dt^2}\left(v_z - \frac{P}{\Omega_0^2}\right) + \Omega_0^2\left(v_z - \frac{P}{\Omega_0^2}\right) = 0,$$

so $v_z(t) = A\sin\Omega_0 t + B\cos\Omega_0 t + P/\Omega_0^2$. The initial conditions $v_z(0) = \dot{v}_z(0) = 0$ determine the constants and we finally get

$$v_z(t) = \frac{P}{\Omega_0^2}\left(1 - \cos\Omega_0 t\right)$$

$$a_z(t) = \frac{P}{\Omega_0^2}\sin\Omega_0 t.$$

No steady acceleration occurs; the particle cyclically accelerates and decelerates as it propagates along the z-axis.

Chapter 17: Waves in Simple Matter

17.1 Waves in Matter in the $\varphi = 0$ Gauge

(a) A gauge transformation of the potentials is

$$\varphi' = \varphi - \frac{\partial \chi}{\partial t} \qquad\qquad \mathbf{A}' = \mathbf{A} + \nabla \chi.$$

To get $\varphi' = 0$, we need

$$\varphi = \frac{\partial \chi}{\partial t} \qquad \Rightarrow \qquad \chi(\mathbf{r}, t) = \int_{-\infty}^{t} dt' \varphi(\mathbf{r}, t') + const.$$

(b) For simple matter, $\mathbf{B} = \mu \mathbf{H}$ and $\mathbf{D} = \epsilon \mathbf{E}$. Therefore,

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \qquad \Rightarrow \qquad \nabla \times \mathbf{B} = \mu \epsilon \frac{\partial \mathbf{E}}{\partial t}.$$

Substituting $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\partial \mathbf{A}/\partial t$ gives

$$\nabla \times \nabla \times \mathbf{A} = -\mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2},$$

or, using the supplied vector identity,

$$\nabla^2 \mathbf{A} - \mu \epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A}) = 0.$$

(c) The vector potential of a plane wave is $\mathbf{A} = \mathbf{A}_0 \exp(i\mathbf{k} \cdot \mathbf{r} - i\omega t)$. Substituting this into the equation just above gives

$$\left[-k^2 + \omega^2 \mu \epsilon\right] \mathbf{A} + \mathbf{k}(\mathbf{k} \cdot \mathbf{A}) = 0.$$

When $\mathbf{k} \perp \mathbf{A}$, we get waves with the vacuum dispersion relation $\omega = ck$. When $\mathbf{k} \parallel \mathbf{A}$, we get

$$\omega^2 \mu \epsilon \mathbf{A} = 0 \qquad \Rightarrow \qquad \mathbf{A} = 0.$$

17.2 Faraday Rotation During Propagation

Let $k_L = \omega n_L / c$ and $k_R = \omega n_R / c$. Left and right circularly polarized plane waves with the same amplitude and frequency propagating along the z-axis in the medium of interest are

$$\mathbf{E}_L(z,t) = E(\hat{\mathbf{x}} + i\hat{\mathbf{y}}) \exp[i(k_L z - \omega t] \qquad\qquad \mathbf{E}_R(z,t) = E(\hat{\mathbf{x}} - i\hat{\mathbf{y}}) \exp[i(k_R z - \omega t].$$

In this basis, the given electric field is

$$\mathbf{E}(z=0,t) = \frac{1}{2}[\mathbf{E}_L(z=0,t) + \mathbf{E}_R(z=0,t)].$$

Therefore, at other points in space,

$$\begin{aligned}\mathbf{E}(z,t) &= \frac{1}{2}[\mathbf{E}_L(z,t)+\mathbf{E}_R(z,t)] \\ &= \frac{1}{2}E\left[e^{ik_L z}+e^{ik_R z}\right]e^{-i\omega t}\hat{\mathbf{x}}+\frac{i}{2}E\left[e^{ik_L z}-e^{ik_R z}\right]e^{-i\omega t}\hat{\mathbf{y}}.\end{aligned}$$

The field will be linearly polarized along $\hat{\mathbf{y}}$ when

$$e^{ik_L z}=-e^{ik_R z}=e^{ik_R z}e^{\pm im\pi} \qquad m=1,3,5,\ldots$$

This occurs when z takes the values

$$z=\pm\frac{m\pi}{k_L-k_R}=\pm\frac{m\pi c/\omega}{n_L-n_R} \qquad m=1,3,5,\ldots$$

17.3 Optically Active Matter

(a) For plane wave fields, the Maxwell equations in matter are

$$\epsilon_0 i\mathbf{k}\cdot\mathbf{E}=\rho_{\rm ind}(\mathbf{k},\omega) \qquad \mathbf{k}\cdot\mathbf{B}=0 \qquad \mathbf{k}\times\mathbf{E}=\omega\mathbf{B} \qquad i\nabla\times\mathbf{B}=\mu_0\mathbf{j}_{\rm in}(\mathbf{k},\omega)-i\frac{\omega}{c^2}\mathbf{E},$$

where

$$\rho_{\rm ind}(\mathbf{r},t)=-\nabla\cdot\mathbf{P} \quad\Rightarrow\quad \rho_{\rm in}(\mathbf{k},\omega)=-i\mathbf{k}\cdot\mathbf{P}=-i(\epsilon-\epsilon_0)\mathbf{k}\cdot\mathbf{E}$$

$$\begin{aligned}\mathbf{j}_{\rm ind}(\mathbf{r},t)=\frac{\partial\mathbf{P}}{\partial t}+\nabla\times\mathbf{M} \quad\Rightarrow\quad \mathbf{j}_{\rm ind}(\mathbf{k},\omega) &= -i\omega\mathbf{P}+i\mathbf{k}\times\mathbf{M} \\ &= -i\omega(\epsilon-\epsilon_0)\mathbf{E}+i(\mu_0^{-1}-\mu^{-1})\mathbf{k}\times\mathbf{B}.\end{aligned}$$

For $\rho_{\rm in}(\mathbf{k},\omega)$, we need a scalar that is linear in the fields. Using $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{k}$, the only possibilities are $\mathbf{k}\cdot\mathbf{E}$ and $\mathbf{k}\cdot\mathbf{B}$. The latter is zero from the Maxwell equations. Hence, the form given is as general as it can be. For $\mathbf{j}_{\rm ind}(\mathbf{k},\omega)$, we need a vector that is linear in the fields. The possibilities are $\mathbf{E}$, $\mathbf{B}$, $\mathbf{k}\times\mathbf{E}$, and $\mathbf{k}\times\mathbf{B}$. The form given for $\mathbf{j}_{\rm ind}(\mathbf{k},\omega)$ includes only the first and last of these, and so is not as general as it could be.

(b) From part (a), the induced current density could include terms proportional to $\mathbf{B}$ and $\mathbf{k}\times\mathbf{E}$. But the Maxwell equations show that these are proportional. Hence, only one is needed, and the problem statement chooses $\mathbf{B}$. Including that term, we now substitute $\rho_{\rm in}(\mathbf{k},\omega)$ and $\mathbf{j}_{\rm ind}(\mathbf{k},\omega)$ into the Maxwell equations. The first gives

$$i\epsilon_0\mathbf{k}\cdot\mathbf{E}=-i(\epsilon-\epsilon_0)\mathbf{k}\cdot\mathbf{E} \quad\Rightarrow\quad \mathbf{k}\cdot\mathbf{E}=0.$$

The second gives

$$i\mathbf{k}\times\mathbf{B}=-i\omega\mu_0(\epsilon-\epsilon_0)\mathbf{E}+i\mu_0(\mu_0^{-1}+\mu^{-1})\mathbf{k}\times\mathbf{B}+\mu_0\xi\omega\mathbf{B}-i\frac{\omega}{c^2}\mathbf{E},$$

or

$$\epsilon\omega\mathbf{E}+\mu^{-1}\mathbf{k}\times\mathbf{B}+i\xi\omega\mathbf{B}=0. \tag{1}$$

(c) To study propagating waves, start with Faraday's law, $\mathbf{k}\times\mathbf{E}=\omega\mathbf{B}$, and use $\mathbf{k}\cdot\mathbf{E}=0$ from part (b) to write

$$\omega\mathbf{k}\times\mathbf{B}=\mathbf{k}\times(\mathbf{k}\times\mathbf{E})=\mathbf{k}(\mathbf{k}\cdot\mathbf{E})-k^2\mathbf{E}=-k^2\mathbf{E}.$$

Substituting this and Faraday's law into the identity proved at the end of part (b) gives

$$\left(\omega^2-\frac{k^2}{\mu\epsilon}\right)\mathbf{E}+\frac{i\xi\omega k}{\epsilon}\hat{\mathbf{k}}\times\mathbf{E}=0.$$

Operating on this equation with $\hat{\mathbf{k}}\times$ gives

$$-\frac{i\omega\xi k}{\epsilon}\mathbf{E}+\left(\omega^2-\frac{k^2}{\mu\epsilon}\right)\hat{\mathbf{k}}\times\mathbf{E}.$$

These two equations have the suggested form with

$$a=\omega^2-\frac{k^2}{\mu\epsilon}\qquad\qquad b=\frac{\omega\xi k}{\epsilon}.$$

(d) We solve the linear system by insisting that the determinant vanish:

$$\begin{vmatrix} a & -ib \\ ib & a \end{vmatrix}=0.$$

This gives $a=\pm b$, so the dispersion relations we seek are

$$\omega^2=\frac{k^2}{\mu\epsilon}\pm\frac{\xi\omega k}{\epsilon}.$$

Substituting $a=\pm b$ back into the original 2×2 system shows that the associated electric fields satisfy

$$\hat{\mathbf{k}}\times\mathbf{E}=\pm i\mathbf{E}. \tag{2}$$

These are left and right circularly polarized waves because, if $\hat{\mathbf{e}}$ is any vector perpendicular to $\mathbf{k}$, (2) is satisfied by $\mathbf{E}=\hat{\mathbf{e}}\mp i\hat{\mathbf{k}}\times\hat{\mathbf{e}}$.

(e) For a plane wave, the usual form of the Maxwell equations in matter without free charge or current is

$$\mathbf{k}\cdot\mathbf{D}=0\qquad\qquad \mathbf{k}\cdot\mathbf{B}=0\qquad\qquad \mathbf{k}\times\mathbf{E}=\omega\mathbf{B}\qquad\qquad \mathbf{k}\times\mathbf{H}=-\omega\mathbf{D}.$$

Using the suggested constitutive relations, we get

$$\mathbf{k}\cdot(\epsilon\mathbf{E}+\beta\mathbf{B})=0\qquad\Rightarrow\qquad \mathbf{k}\cdot\mathbf{E}=0$$

and

$$\frac{1}{\mu}\mathbf{k}\times(\mathbf{B}-\gamma\mathbf{E})=-\omega(\epsilon\mathbf{E}+\beta\mathbf{B})\qquad\Rightarrow\qquad \epsilon\omega\mathbf{E}+\mu^{-1}\mathbf{k}\times\mathbf{B}+\omega(\beta-\gamma/\mu)\mathbf{B}=0.$$

The last of these expressions is identical to (1) with $i\xi=\beta-\gamma/\mu$. Therefore, all the results found in parts (c) and (d) remain valid.

Source: J.F. Nieves and P.B. Pal, *American Journal of Physics* **62**, 207 (1994).

17.4 Matching Conditions

(a) The matching conditions in question derive from the Maxwell equations $\nabla \cdot \mathbf{D} = 0$ and $\nabla \cdot \mathbf{B} = 0$. However, the first is a consequence of $\nabla \times \mathbf{H} = \partial \mathbf{D}/\partial t$ and the second is a consequence of $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$. Since the divergence equations provide no new information not already contained in the curl equations, the former are not needed to derive the Fresnel equations.

(b) Let $\perp$ and $\|$ refer to components perpendicular and parallel to the interface, so

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = (\mathbf{E}_\| + \mathbf{E}_\perp)\times(\mathbf{H}_\| + \mathbf{H}_\perp) = (\mathbf{E}_\| \times \mathbf{H}_\|) + (\mathbf{E}_\| \times \mathbf{H}_\perp) + (\mathbf{E}_\perp \times \mathbf{H}_\|) + (\mathbf{E}_\perp \times \mathbf{H}_\perp).$$

The last term vanishes, so

$$\mathbf{S}_\perp = \mathbf{E}_\| \times \mathbf{H}_\| \qquad \mathbf{S}_\| = (\mathbf{E}_\| \times \mathbf{H}_\perp) + (\mathbf{E}_\perp \times \mathbf{H}_\|).$$

Since $\mathbf{E}_\|$ and $\mathbf{H}_\|$ are continuous at an interface, we conclude that $\mathbf{S}_\perp$ is continuous at an interface.

17.5 Escape from a Dielectric

The rays that escape have propagation vectors that make an angle θ with the vertical that does not exceed the critical angle for total internal reflection. Snell's law for this situation is $n \sin\theta = \sin\theta_0$. The critical angle θ_C makes $\sin\theta_0 = 1$, or

$$\theta_C = \sin^{-1}\left(\frac{1}{n}\right).$$

The waves are emitted isotropically, so the fraction of light that escape is the relative solid angle

$$f = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\sin\theta_C} d\theta \sin\theta = \frac{1}{2}\int_{\cos\theta_C}^{1} dx = \frac{1}{2}(1-\cos\theta_C) = \frac{1}{2}\left(1-\sqrt{1-1/n^2}\right).$$

17.6 Almost Total External Reflection

In the variable $\alpha = \pi/2 - \theta$, Snell's law ($\sin\theta_1 = n \sin\theta_2$) reads $\cos\alpha_1 = n\cos\alpha_2$. Since $n = 1-\delta$ and $\delta \ll 1$, we see that $\alpha_2 \ll 1$ if $\alpha_1 \ll 1$. Therefore, using $\cos\alpha \approx 1 - \frac{1}{2}\alpha^2$ on both sides of Snell's law gives

$$\alpha_1^2 - \alpha_2^2 \approx 2\delta.$$

Using the same approximations, the "s" reflection coefficient for non-magnetic matter is

$$R_{\rm s}=\left|\frac{\cos\theta_1-n\cos\theta_2}{\cos\theta_1+n\cos\theta_2}\right|^2=\left|\frac{\sin\alpha_1-n\sin\alpha_2}{\sin\alpha_1+n\sin\alpha_2}\right|^2\approx\left|\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\right|^2\approx\left|\frac{\alpha_1^2-\alpha_2^2}{4\alpha_2^2}\right|^2\approx\frac{\delta^2}{4\alpha_1^4}.$$

The "p" reflection coefficient gives the same answer,

$$R_{\rm p}=\left|\frac{n\cos\theta_1-\cos\theta_2}{n\cos\theta_1+\cos\theta_2}\right|^2=\left|\frac{n\sin\alpha_1-\sin\alpha_2}{n\sin\alpha_1+\sin\alpha_2}\right|^2\approx\left|\frac{\alpha_1-\alpha_2}{\alpha_1+\alpha_2}\right|^2\approx\left|\frac{\alpha_1^2-\alpha_2^2}{4\alpha_2^2}\right|^2\approx\frac{\delta^2}{4\alpha_1^4}.$$

Now, the vector $\mathbf{q}$ is normal to the surface with $q=(4\pi/\lambda)\sin\alpha_1\approx(4\pi/\lambda)\alpha_1$. Therefore, $R(q)\propto q^{-4}$.

Source: J. Als-Nielsen and D. McMorrow, *Elements of Modern X-ray Physics* (Wiley, New York, 2001).

17.7 Alternate Derivation of the Fresnel Equations

With the given geometry, the Maxwell matching conditions read

$$\epsilon_1[E_z^I+E_z^R]=\epsilon_2E_z^T \qquad \mu_1[H_z^I+H_z^R]=\mu_2H_z^T \tag{1}$$

and

$$\begin{aligned} E_x^I+E_x^R&=E_x^T & H_x^I+H_x^R&=H_x^T\\ E_y^I+E_y^r&=E_t^T & H_y^I+H_y^r&=H_y^T. \end{aligned} \tag{2}$$

The four equations in (2) collapse to two when we use $\mathbf{k}\cdot\mathbf{E}_1=0=\mathbf{k}\cdot\mathbf{E}_2$ and similarly for $\mathbf{H}$. Indeed, because $\mathbf{k}_I=(k_{1x},0,k_{1z})$ and $\mathbf{k}_R=(k_{1x},0,-k_{1z})$, we get

$$k_{1z}(E_z^I-E_z^R)=k_{2z}E_z^T \qquad k_{1z}(H_z^I-H_z^R)=k_{2z}H_z^T. \tag{3}$$

The left sides of (1) and (3) are two equations in two unknowns. Hence,

$$\frac{E_z^R}{E_z^I}=\frac{k_{1z}\epsilon_2-k_{2z}\epsilon_1}{k_{1z}\epsilon_2+k_{2z}\epsilon_1} \qquad \frac{E_z^T}{E_z^I}=\frac{2k_{1z}\epsilon_1}{k_{1z}\epsilon_2+k_{2z}\epsilon_1}.$$

The right sides of (1) and (3) are identical in structure except that $\mathbf{E}\to\mathbf{H}$ and $\epsilon\to\mu$. Therefore,

$$\frac{H_z^R}{H_z^I}=\frac{k_{1z}\mu_2-k_{2z}\mu_1}{k_{1z}\mu_2+k_{2z}\mu_1} \qquad \frac{H_z^T}{H_z^I}=\frac{2k_{1z}\mu_1}{k_{1z}\mu_2+k_{2z}\mu_1}.$$

Source: S.-Y. Shieh, *Physical Review* **173**, 1310 (1968).

17.8 Fresnel Transmission Amplitudes

The text gives the Fresnel transmission amplitudes as

$$t_p = \frac{2Z_2 \cos\theta_1}{Z_1 \cos\theta_1 + Z_2 \cos\theta_2} \qquad \text{and} \qquad t_s = \frac{2Z_2 \cos\theta_1}{Z_2 \cos\theta_1 + Z_1 \cos\theta_2}.$$

For non-magnetic matter, we use $Z_1/Z_2 = n_2/n_1$ to write these as

$$t_p = \frac{2n_1 \cos\theta_1}{n_2 \cos\theta_1 + n_1 \cos\theta_2} \qquad \text{and} \qquad t_s = \frac{2n_1 \cos\theta_1}{n_1 \cos\theta_1 + n_2 \cos\theta_2}.$$

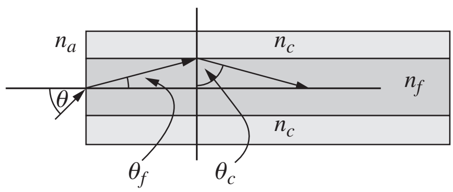
Using $n_1 \sin\theta_1 = n_2 \sin\theta_2$, we get

$$t_p = \frac{2\cos\theta_1 \sin\theta_2}{\cos\theta_1 \sin\theta_1 \cos\theta_2 \sin\theta_2} = \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1+\theta_2)\cos(\theta_1 - \cos\theta_2)} = t_{\rm TM}$$

and

$$t_s = \frac{2\cos\theta_1 \sin\theta_2}{\cos\theta_1 \sin\theta_2 \cos\theta_2 \sin\theta_1} = \frac{2\cos\theta_1 \sin\theta_2}{\sin(\theta_1+\theta_2)} = t_{\rm TE}.$$

17.9 Guidance by Total Internal Reflection

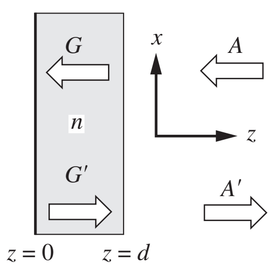


From Snell's law, $n_a \sin\theta = n_f \sin\theta_f$. We get the maximum angle θ when θ_c is the critical angle for total internal reflection. The latter is defined by $n_f \sin\theta_c = n_c$. Also, $\theta_f = 90° - \theta_c$. Combining these facts,

$$\sin\theta = \frac{n_f}{n_a}\cos\theta_c = \frac{n_f}{n_a}\sqrt{1-\sin^2\theta_c} = \frac{\sqrt{n_f^2 - n_c^2}}{n_a}.$$

17.10 Reflection from a Metal-Coated Dielectric Slab

(a) The diagram below defines the amplitudes of the four electric fields that appear in this problem.



The left-going fields in the vacuum have non-zero components

$$E_{1x} = Ae^{i(\omega t+kz)} \qquad cB_{1y} = -Ae^{i(\omega t+kz)}.$$

The right-going fields in the vacuum have non-zero components

$$E'_{1x} = A'e^{i(\omega t-kz)} \qquad cB'_{1y} = A'e^{i(\omega t-kz)}.$$

The left-going fields in the medium have non-zero components

$$E_{2x} = Ge^{i(\omega t+nkz)} \qquad cB_{2y} = -nGe^{i(\omega t+nkz)}.$$

The right-going fields in the medium have non-zero components

$$E'_{2x} = G'e^{i(\omega t-kz)} \qquad cB'_{2y} = nG'e^{i(\omega t-nkz)}.$$

The boundary condition at the metal surface is that $\mathbf{E}_{\|} = 0$. This gives

$$E_{2x}(z=0) + E'_{2x}(z=0) = 0 \qquad \Rightarrow \qquad G' = -G.$$

At the front face of the slab, we must have continuity of $\mathbf{E}_{\|}$ and continuity of $\mathbf{B}_{\|}$. The first of these gives

$$E_{1x}(z=d) + E'_{1x}(z=d) = E_{2x}(z=d) + E'_{2x}(z=d),$$

or

$$Ae^{ikd} + A'e^{-ikd} = i2G\sin(nkd). \tag{1}$$

Since $\mathbf{B} = \mu_0\mathbf{H}$ everywhere, we also get

$$B_{1y}(z=d) + B'_{1y}(z=d) = B_{2y}(z=d) + B'_{2y}(z=d),$$

or

$$-Ae^{ikd} + A'e^{-ikd} = -2nG\cos(nkd). \tag{2}$$

Let $\theta = nkd$. Adding (1) to (2) gives

$$2A'e^{-ikd} = -2G(n\cos\theta - i\sin\theta).$$

Subtracting (2) from (1) gives

$$2Ae^{ikd} = 2G(n\cos\theta + i\sin\theta).$$

Therefore,

$$\frac{A'e^{-ikd}}{Ae^{ikd}} = -\frac{n\cos\theta - i\sin\theta}{n\cos\theta + i\sin\theta}. \tag{3}$$

This shows that $|A| = |A'|$. Let δ be the amount by which the phase of the right-going wave exceeds the phase of the left-going wave in the vacuum at $z = d$. In other words,

$$A'e^{-ikd} = e^{i\delta}Ae^{ikd}.$$

Using (3), $$e^{i\delta} = -\frac{1 - i\tan(\theta)/n}{1 + i\tan(\theta)/n} = -\frac{Me^{-i\Lambda}}{Me^{i\Lambda}} = -e^{-i2\Lambda},$$

from which we conclude that $\tan\Lambda = \tan(\theta)/n$. Finally, $\exp(\pm i\pi) = -1$, so

$$\delta = \pm\pi - 2\tan^{-1}\left[\frac{\tan(\theta)}{n}\right].$$

We choose the positive sign so $\delta > 0$ in the limit when $d \to 0$.

(b) The radiation pressure $\mathcal{P}$ (time-averaged force per unit area) acting on a conductor due to the surface current density $\mathbf{K}$ induced in it by a time-harmonic magnetic field $\mathbf{B}$ is the surface value of

$$\mathcal{P} = \frac{1}{4}\mathbf{K} \times \mathbf{B}^* = \frac{1}{4\mu_0}(\hat{\mathbf{n}} \times \mathbf{B}) \times \mathbf{B}^* = -\frac{1}{4\mu_0}|\mathbf{B}|^2\,\hat{\mathbf{n}}.$$

For our problem,

$$\mathbf{B}(z=0) = [B_{2y}(z=0) + B'_{2y}(z=0)]\hat{\mathbf{y}} = \left[-\frac{nG}{c} + \frac{nG'}{c}\right]\hat{\mathbf{y}} = -\frac{2nG}{c}\hat{\mathbf{y}}.$$

Therefore, using (3), $$\mathcal{P} = -\frac{\epsilon_0 n^2 |A|^2}{n^2\cos^2\theta + \sin^2\theta}\hat{\mathbf{z}}.$$

Source: B.H. Chirgwin, C. Plumpton, and C.W. Kilmister, *Elementary Electromagnetic Theory*, Volume 3 (Pergamon, Oxford, 1973).

17.11 Fresnel's Problem for a Topological Insulator

(a) For a plane wave varying as $\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, the Maxwell equations in matter read

$$\mathbf{k}\cdot\mathbf{B} = 0 \qquad \mathbf{k}\times\mathbf{E} = \omega\mathbf{B} \qquad \mathbf{k}\cdot\mathbf{D} = 0 \qquad \mathbf{k}\times\mathbf{H} = -\omega\mathbf{D}.$$

Inserting the constitutive relations, the third of these gives

$$0 = \mathbf{k}\cdot\mathbf{D} = \epsilon\mathbf{k}\cdot\mathbf{E} - \alpha_0\mathbf{k}\cdot\mathbf{B} \quad \to \quad \mathbf{k}\cdot\mathbf{E} = 0.$$

The fourth gives

$$\mathbf{k}\times\mathbf{H} = \mathbf{k}\times\frac{\mathbf{B}}{\mu} + \alpha_0\mathbf{k}\times\mathbf{E} = -\omega(\epsilon\mathbf{E} - \alpha_0\mathbf{B}).$$

Substituting $\mathbf{k}\times\mathbf{E}$ from Faraday's law into the middle term of the preceding equation produces $\alpha_0\mu\omega\mathbf{B}$. Canceling this with an identical term in the rightmost term leaves

$$\mathbf{k}\times\mathbf{B} = -\omega\epsilon\mu\mathbf{E}.$$

Collecting results, we get conventional plane waves of $\mathbf{E}$ and $\mathbf{B}$ with wave speed $v = 1/\sqrt{\mu\epsilon}$.

(b) The matching conditions that $\mathbf{E}_\parallel$ and $\mathbf{H}_\parallel$ are continuous do not depend on the constitutive relations. Let $\mathbf{E}_I$, $\mathbf{E}_R$, and $\mathbf{E}_T$ be the amplitudes of the incident, reflected, and transmitted electric fields. At normal incidence, all of these are parallel to the interface. Therefore, the continuity of $\mathbf{E}_\parallel$ reads

$$\mathbf{E}_I + \mathbf{E}_R = \mathbf{E}_T .$$

We let $k = \omega/c$ and $k' = \omega\sqrt{\mu\epsilon}$. The reflected magnetic field points opposite to the incident magnetic field. Using the constitutive equations, the continuity of $\mathbf{H}_\parallel$ reads

$$\frac{1}{\mu_0\omega}\mathbf{k}\times(\mathbf{E}_I - \mathbf{E}_R) = \frac{1}{\mu\omega}\mathbf{k}'\times\mathbf{E}_T + \alpha_0\mathbf{E}_T .$$

We let $\hat{\mathbf{k}} = \hat{\mathbf{k}}' = \hat{\mathbf{z}}$, $\mathbf{E}_I = E_I\hat{\mathbf{x}}$, and allow for the possibility that $\mathbf{E}_T$ and $\mathbf{E}_R$ may differ in direction from $\mathbf{E}_I$. In that case, the components of the matching conditions are

$$\begin{aligned}
E_{Ix} + E_{Rx} &= E_{Tx} \\
E_{Ry} &= E_{Ty} \\
\frac{1}{\mu_0 c}(E_{Ix} - E_{Rx}) &= \frac{1}{\mu v}E_{Tx} + \alpha_0 E_{Ty} \\
\frac{1}{\mu_0 c}E_{Ry} &= -\frac{1}{\mu v}E_{Ty} + \alpha_0 E_{Tx} .
\end{aligned}$$

Substituting E_{Ry} from the second equation into the last equation gives the Faraday angle of rotation of the plane of polarization as

$$\frac{E_{Ty}}{E_{Tx}} = \frac{\alpha_0}{\sqrt{\epsilon_0/\mu_0} + \sqrt{\epsilon/\mu}} = \tan\theta_F .$$

(c) Divide the third and fourth matching conditions by E_{Rx} and use the first to eliminate E_{Ix}. This gives

$$\left(\frac{1}{\mu_0 c} - \frac{1}{\mu v}\right)\frac{E_{Tx}}{E_{Rx}} - \frac{2}{\mu_0 c} = \alpha_0\frac{E_{Ty}}{E_{Rx}}$$

$$\frac{1}{\mu_0 c}\frac{E_{Ry}}{E_{Rx}} = -\frac{1}{\mu v}\frac{E_{Ty}}{E_{Rx}} + \alpha_0\frac{E_{Tx}}{E_{Rx}} .$$

Now use the second matching condition to replace E_{Ty}/E_{Rx} by E_{Ry}/E_{Ry}:

$$\left(\frac{1}{\mu_0 c} - \frac{1}{\mu v}\right)\frac{E_{Tx}}{E_{Rx}} - \frac{2}{\mu_0 c} = \alpha_0\frac{E_{Ry}}{E_{Rx}}$$

$$\left(\frac{1}{\mu_0 c} + \frac{1}{\mu v}\right)\frac{E_{Ry}}{E_{Rx}} = \alpha_0\frac{E_{Tx}}{E_{Rx}} .$$

These are two equations in two unknowns, one of which is the tangent of the Kerr angle of rotation:

$$\frac{E_{Ry}}{E_{Rx}} = \frac{2\alpha_0/\mu_0 c}{\left(\dfrac{\epsilon_0}{\mu_0} - \dfrac{\epsilon}{\mu}\right) - \alpha_0^2} = \tan\theta_K.$$

Source: X.-L. Qi, T.L. Hughes, and S.-C. Zhang, *Physical Review* **78**, 195424 (2008).

17.12 Polarization Rotation by Reflection and Refraction

(a) Let $\hat{\mathbf{s}}$ and $\hat{\mathbf{p}}$ be unit vectors aligned with the incident electric field when the latter is purely s-polarized or p-polarized (as they are defined in the text). Then, for a general polarization, the incident electric field has the decomposition

$$\mathbf{E}_I = \sin\gamma_I\,\hat{\mathbf{s}} + \cos\gamma_I\,\hat{\mathbf{p}}.$$

By the definition of r_s and r_p, the reflected and transmitted electric fields are

$$\mathbf{E}_R = r_s\sin\gamma_I\,\hat{\mathbf{s}} + r_p\cos\gamma_I\,\hat{\mathbf{p}} \qquad \text{and} \qquad \mathbf{E}_T = t_s\sin\gamma_I\,\hat{\mathbf{s}} + t_p\cos\gamma_I\,\hat{\mathbf{p}}.$$

On the other hand, $\mathbf{E}_R = \sin\gamma_R\,\hat{\mathbf{s}} + \cos\gamma_R\,\hat{\mathbf{p}}$ and $\mathbf{E}_T = \sin\gamma_T\,\hat{\mathbf{s}} + \cos\gamma_T\,\hat{\mathbf{p}}$. Therefore,

$$\left[\frac{E_s}{E_p}\right]_R = \tan\gamma_R = \frac{r_s}{r_p}\tan\gamma_I \qquad \text{and} \qquad \left[\frac{E_s}{E_p}\right]_T = \tan\gamma_T = \frac{t_s}{t_p}\tan\gamma_I.$$

Now, the Fresnel formula use θ_1 and θ_2 for the angles of incidence and refraction, respectively, and the textbook shows that

$$r_s = -\frac{\sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)} \qquad \text{and} \qquad r_p = \frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)}.$$

Therefore, as advertised,

$$\tan\gamma_R = -\frac{\cos(\theta_1-\theta_2)}{\cos(\theta_1+\theta_2)}\tan\gamma_I.$$

Using Snell's law, $n_1\sin\theta_1 = n_2\sin\theta_2$, and a trigonometric identity, the ratio of the transmission amplitudes is

$$\begin{aligned}\frac{t_s}{t_p} &= \frac{n_2\cos\theta_1 + n_2\cos\theta_2}{n_1\cos\theta_1 + n_2\cos\theta_2} = \frac{\sin\theta_1\cos\theta_1 + \sin\theta_2\cos\theta_2}{\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1}\\ &= \frac{\sin 2\theta_1 + \sin 2\theta_2}{2\sin(\theta_1+\theta_2)} = \frac{2\sin(\theta_1+\theta_2)\cos(\theta_1-\theta_2)}{2\sin(\theta_1+\theta_2)}.\end{aligned}$$

Therefore,

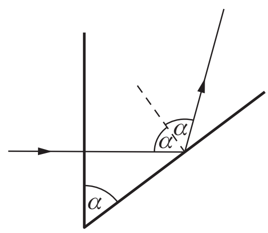
$$\tan\gamma_T = \cos(\theta_1-\theta_2)\tan\gamma_I.$$

(b) The reflected wave is more TE (s-polarized) if $\tan\gamma_R > \tan\gamma_I$. This true because $\cos(\theta_1-\theta_2) > \cos(\theta_1+\theta_2)$. Similarly, the refracted is more TM (p-polarized) when $\tan\gamma_T < \tan\gamma_I$. This is true because $\cos(\theta_1-\theta_2) < 1$.

Source: L.D. Landau and E.M. Lifshitz, *The Electrodynamics of Continuous Media* (Pergamon, Oxford, 1960).

17.13 The Fresnel Rhomb

(a) The sketch below shows that the angle of incidence at each internal interface is α:



(b) The angle of incidence is $\theta_1 = \alpha$. Then, if θ_2 denotes the (complex) angle of the transmitted wave, Snell's law reads $\sin\theta_2 = n\sin\alpha$. Since $n\sin\alpha > 1$ for total internal reflection, we get $\cos\theta_2 = i\sqrt{n^2\sin^2\alpha - 1}$. From Fresnel's equations,

$$\left.\frac{E_R}{E_I}\right]_\perp = R_\perp = -\frac{\sin(\theta_1-\theta_2)}{\sin(\theta_1+\theta_2)} = -\frac{\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2}{\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2}$$

$$= -\frac{i\sin\alpha\sqrt{n^2\sin^2\alpha-1} - n\sin\alpha\cos\alpha}{i\sin\alpha\sqrt{n^2\sin^2\alpha-1} + n\sin\alpha\cos\alpha}$$

$$= \frac{\cos\alpha - i\sqrt{\sin^2\alpha - 1/n^2}}{\cos\alpha + i\sqrt{\sin^2\alpha - 1/n^2}}.$$

This complex number has the form $(a-ib)/(a+ib)$, which has magnitude one.

(c) Similarly,

$$\left.\frac{E_R}{E_I}\right]_\parallel = R_\parallel = -\frac{\tan(\theta_1-\theta_2)}{\tan(\theta_1+\theta_2)}$$

$$= R_\perp\,\frac{\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2}{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}$$

$$= \frac{\cos\alpha - i\sqrt{\sin^2\alpha - 1/n^2}}{\cos\alpha + i\sqrt{\sin^2\alpha - 1/n^2}}\;\frac{\cos\alpha\; i\sqrt{n^2\sin^2\alpha-1} - n\sin^2\alpha}{\cos\alpha\; i\sqrt{n^2\sin^2\alpha-1} + n\sin^2\alpha}$$

$$= -\frac{\cos\alpha - in^2\sqrt{\sin^2\alpha - 1/n^2}}{\cos\alpha + in^2\sqrt{\sin^2\alpha - 1/n^2}}.$$

This is also a complex number of the form $(a - ib)/(a + ib)$ with magnitude one.

(c) Orient the coordinate axes so $\hat{\mathbf{x}}$ is parallel to the scattering plane and $\hat{\mathbf{y}}$ is perpendicular to the scattering plane. In that case, the electric field incident on the first interface where total internal reflection occurs is

$$\mathbf{E}_I = E_I(\hat{\mathbf{x}} + \hat{\mathbf{y}}).$$

With the notation given, the reflected electric field is

$$\mathbf{E}_R = E_I(R_{\|}\hat{\mathbf{x}} + R_{\perp}\hat{\mathbf{y}}) = E_I e^{-i\delta_{\perp}}\left(e^{i(\delta_{\perp}-\delta_{\|})}\hat{\mathbf{x}} + \hat{\mathbf{y}}\right).$$

This shows that the quantity $\delta_{\perp} - \delta_{\|}$ determines the polarization of the reflected wave. The same argument applies to the second internally reflected wave. Now, both $\delta_{\perp}$ and $\delta_{\|}$ have the form

$$\frac{a - ib}{a + ib} = \frac{a^2 - b^2 - 2iab}{a^2 + b^2} = \exp(-i\delta) = \cos\delta - i\sin\delta.$$

Therefore,

$$\tan\delta = \frac{2ab}{a^2 - b^2} = \frac{2b/a}{1 - b^2/a^2} = \tan 2(\delta/2) = \frac{2\tan\delta/2}{1 - \tan^2\delta/2}.$$

We infer that $\tan\frac{1}{2}\delta = b/a$, so

$$\tan\frac{1}{2}\delta_{\perp} = \frac{\sqrt{\sin^2\alpha - 1/n^2}}{\cos\alpha} \qquad \text{and} \qquad \tan\frac{1}{2}\delta_{\|} = n^2\frac{\sqrt{\sin^2\alpha - 1/n^2}}{\cos\alpha}.$$

Hence,

$$\tan\frac{1}{2}(\delta_{\perp} - \delta_{\|}) = \frac{\tan\frac{1}{2}\delta_{\perp} - \tan\frac{1}{2}\delta_{\|}}{1 + \tan\frac{1}{2}\delta_{\perp}\tan\frac{1}{2}\delta_{\|}} = -\frac{\cos\alpha\sqrt{\sin^2\alpha - 1/n^2}}{\sin^2\alpha}.$$

(d) The light is reflected twice by the rhomb. So, if $2|\delta_{\perp} - \delta_{\|}| = \pi/2$, linearly polarized light will be converted into circularly polarized light. This implies that

$$\frac{\cos\alpha\sqrt{\sin^2\alpha - 1/n^2}}{\sin^2\alpha} = \tan\frac{\pi}{8}.$$

This may be solved for $\sin\alpha$:

$$\sin^2\alpha = \frac{1 + 1/n^2 \pm \sqrt{(1 - 1/n^2) - (4/n^2)\tan\pi/8}}{2(1 + \tan\pi/8)}.$$

So, if $n = 1.5$, we get $\alpha = 53.26°$ and $\alpha = 50.23°$.

Source: M. Born and E. Wolf, *Principles of Optics* (University Press, Cambridge, 1980).

17.14 Energy Transfer to an Ohmic Medium

The time-averaged rate at which power flows through a unit area of an ohmic surface is

$$\begin{aligned}
\langle \mathbf{S} \rangle &= \langle \mathbf{E} \times \mathbf{H} \rangle|_S \\
&= \frac{1}{2}\mathrm{Re}\,\{\mathbf{E}^*(0) \times \mathbf{H}(0)\} \\
&= \frac{1}{2}\mathrm{Re}\left\{\mathbf{E}^*(0) \times \frac{\sigma\delta}{1-i}\,(\hat{\mathbf{z}} \times \mathbf{E}(0))\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{\frac{\sigma\delta}{1-i}\hat{\mathbf{z}}|\mathbf{E}(0)|^2\right\} \\
&= \frac{1}{2}\mathrm{Re}\left\{\frac{\sigma\delta(1+i)}{2}\right\}|\mathbf{E}(0)|^2\hat{\mathbf{z}} \\
&= \frac{1}{4}\sigma\delta|\mathbf{E}(0)|^2\hat{\mathbf{z}}.
\end{aligned}$$

This the total rate of Joule heating computed in the text.

17.15 Refraction into a Good Conductor

(a) We must have $\boldsymbol{\kappa} = i\kappa\hat{\mathbf{z}}$ with $\kappa > 0$ to avoid exponential runaway of the wave as $z \to \infty$ and as $x \to \pm\infty$. The phases of the incident, reflected, and transmitted waves must agree at the $z = 0$ plane where they coincide. This implies that

$$k_{I,x} = k_{R,x} = \mathrm{Re}\, k_{2,x}.$$

The left equation is the law of specular reflection. With $k_I = k_R = k_1 = \omega\sqrt{\mu\epsilon}$, the right equation is the generalization of Snell's law:

$$q \sin\theta_2 = k_1 \sin\theta_1. \tag{1}$$

(b) The dispersion relation for the conductor is $\mathbf{k}_2 \cdot \mathbf{k}_2 = \hat{\epsilon}(\omega)\mu\omega^2$. Equating real and imaginary parts of the dispersion relation gives

$$q^2 - \kappa^2 = \mu\epsilon\omega^2 = k_1^2 \tag{2}$$

and

$$2q\kappa\cos\theta_2 = \mu\sigma\omega. \tag{3}$$

Equation (2) gives $q > k_1$. Therefore, (1) guarantees that $\theta_2 \le \theta_1$.

(c) Eliminate θ_2 by adding the square of (1) to the square of (3) and use (2) to eliminate q^2. The result is

$$\kappa^4 + k_1^2 \cos^2\theta_1 \kappa^2 - \frac{1}{4}(\mu\sigma\omega)^2 = 0.$$

This is a quadratic equation in the variable κ^2 whose solution can be written using the variable $\sinh\Lambda = (\omega\epsilon/\sigma)\cos^2\theta_1$. The $\kappa^2 > 0$ solution is

$$\kappa^2 = \frac{1}{2}\mu\sigma\omega\left\{\sqrt{1+\left(\frac{\epsilon\omega}{\sigma}\cos^2\theta_1\right)^2} - \frac{\epsilon\omega}{\sigma}\cos^2\theta_1\right\} = \frac{1}{2}\mu\sigma\omega\left[\cosh\Lambda - \sinh\Lambda\right].$$

The skin depth is $\delta = \sqrt{2/\mu\sigma\omega}$, so

$$\kappa = \frac{1}{\delta(\omega)}\exp\left[-\frac{1}{2}\Lambda(\omega)\right].$$

(d) Equations (2) and (1) give the two remaining unknowns as

$$q = \sqrt{k_1^2 + \kappa^2} \qquad \text{and} \qquad \sin\theta_2 = \frac{\sin\theta_1}{\sqrt{1+(\kappa/k_1)^2}}.$$

A good conductor satisfies $\sigma/\epsilon\omega \gg 1$, so $\sinh\Lambda \approx \Lambda \ll 1$ and $q \approx \kappa \approx 1/\delta \gg k_1$. This gives $\theta_2 \ll 1$ because

$$\sin\theta_2 \approx \left(\frac{k_1}{\kappa}\right)^2 \sin\theta_1 \approx \frac{2\epsilon\omega}{\sigma}\sin\theta_1 \ll 1.$$

17.16 Phase Change for Waves Reflected from a Good Conductor

The TE reflection coefficient is

$$\frac{E_R}{E_I} = \frac{\hat{Z}_2\cos\theta_1 - \hat{Z}_1\cos\theta_2}{\hat{Z}_2\cos\theta_1 + \hat{Z}_1\cos\theta_2} = \frac{\hat{n}_1\cos\theta_1 - \hat{n}_2\cos\theta_2}{\hat{n}_1\cos\theta_1 + \hat{n}_2\cos\theta_2}.$$

The second equality above follows when $\mu_1 = \mu_2 = \mu_0$ because $\hat{Z} = \mu c/\hat{n}$. The text shows that the transmitted wave propagates normal to the interface for all angles of incidence. Therefore, $\cos\theta_2 = 1$ and, with $\hat{n}_1 = n_1$ and $\hat{n}_2 = n_2' + in_2''$,

$$\frac{E_R}{E_I} = \frac{n_1\cos\theta_1 - n_2' - in_2''}{n_1\cos\theta_1 + n_2' + in_2''} = \frac{n_1^2\cos^2\theta_1 - |\hat{n}_2|^2 - i2n_1n_2''\cos\theta_1}{n_1^2\cos^2\theta_1 + |\hat{n}_2|^2 + 2n_1n_2'\cos\theta_1}.$$

For a good conductor, $|\hat{n}_2|^2 \gg n_1^2\cos^2\theta_1$ where

$$\hat{n}_2 \approx (1+i)\frac{c}{\omega\delta(\omega)} = (1+i)n_2$$

and $n_2 \gg 1$. In that case,

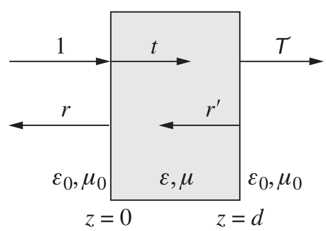
$$E_R \approx -E_I \times \frac{n_2 + in_1 \cos\theta_1}{n_2 - n_1 \cos\theta_1}.$$

The minus sign infront of E_I contributes a phase of π and the final factor contributes a factor of $\tan^{-1}(n_1 \cos\theta_1/n_2)$. Therefore, the net phase change is

$$\pi + \tan^{-1}[(\omega/c)n_1\delta(\omega)\cos\theta_1].$$

17.17 Airy's Problem Revisited

(a) The geometry in question is



The five electric field waves are

$$E = E_0 \exp(-i\omega t) \times \begin{cases} \exp(ik_0 z) + r\exp(-ik_0 z) & z \le 0, \\ t\exp(ikz) + r'\exp(-ikz) & 0 \le z \le d, \\ \mathcal{T}\exp[ik_0(z-d)] & z \ge d. \end{cases}$$

The five magnetic field waves consistent with $Z_0\mathbf{H} = \hat{\mathbf{k}}_0 \times \mathbf{E}$ in the vacuum and $Z\mathbf{H} = \hat{\mathbf{k}} \times \mathbf{E}$ in the film are

$$H = E_0 \exp(-i\omega t) \times \begin{cases} Z_0^{-1}[\exp(ik_0 z) - r\exp(-ik_0 z)] & z \le 0, \\ Z^{-1}[t\exp(ikz) - r'\exp(-ikz)] & 0 \le z \le d, \\ Z_0\mathcal{T}\exp[ik_0(z-d)] & z \ge d. \end{cases}$$

All the fields are tangential to the interfaces and therefore continuous passing through them. Continuity of $\mathbf{E}$ at $z = 0$ and $z = d$ gives

$$1 + r = t + r' \qquad te^{ikd} + r'e^{-ikd} = \mathcal{T}. \tag{1}$$

Continuity of $\mathbf{H}$ at $z = 0$ and $z = d$ gives

$$\frac{1-r}{Z_0} = \frac{t-r'}{Z} \qquad \frac{te^{ikd} - r'e^{-ikd}}{Z} = \frac{\mathcal{T}}{Z_0}. \tag{2}$$

Eliminating r' from the right sides of (1) and (2) gives

$$2te^{ikd} = Z\left(\frac{1}{Z} + \frac{1}{Z_0}\right)\mathcal{T}. \tag{3}$$

To get another relation between t and $\mathcal{T}$, we first eliminate r from the left sides of (1) and (2) to get

$$(Z - Z_0)r' = 2Z - (Z + Z_0)t. \tag{4}$$

Now use (3) to eliminate r' from the right side of (1). The result is

$$\left[(Z - Z_0)e^{ikd} - (Z + Z_0)e^{-ikd}\right]t + 2Ze^{-ikd} = (Z - Z_0)\mathcal{T}. \tag{5}$$

Finally, eliminating t between (3) and (5) gives the desired result:

$$\mathcal{T} = e^{ikd}\frac{4ZZ_0}{(Z + Z_0)^2 - (Z - Z_0)^2 e^{2ikd}}.$$

The input vacuum amplitude is unity and the output volume is also vacuum. Therefore, the fraction of power transmitted is

$$|\mathcal{T}|^2 = \left|\frac{4ZZ_0}{(Z + Z_0)^2 - (Z - Z_0)^2 e^{2ikd}}\right|^2.$$

(b) The text studies a film (medium 2) in vacuum (medium 1) with index of refraction n and shows that

$$|\mathcal{T}|^2 = \left|\frac{tt'}{1 - rr'e^{i\Delta\phi}}\right|^2,$$

where, at normal incidence, $\Delta\phi = 2n(\omega/c)d$,

$$r = \frac{Z_2 - Z_1}{Z_1 + Z_2} \quad \text{and} \quad t = \frac{2Z_2}{Z_1 + Z_2}$$

are the Fresnel reflection and transmission amplitudes for refraction into medium 2 from medium 1 and r' and t' are the same with the two media interchanged. Therefore, since $k = n\omega/c$, the textbook formula with $Z_1 \to Z_0$ and $Z_2 \to Z$ is

$$|\mathcal{T}|^2 = \left|\frac{4Z_0Z}{(Z_0 + Z)^2} \times \frac{1}{1 - \dfrac{(Z - Z_0)^2}{(Z_0 + Z)^2}e^{i2n(\omega/c)d}}\right|^2 = \left|\frac{4ZZ_0}{(Z + Z_0)^2 - (Z - Z_0)^2 e^{2ikd}}\right|^2.$$

17.18 Radiation Pressure on a Perfect Conductor

(a) The reflected electric field must be

$$\mathbf{E}_{\text{ref}}(x, z) = -\hat{\mathbf{y}}E_0 \exp[ik(z\sin\theta + x\cos\theta) - i\omega t].$$

This ensures that the total tangential electric field (the total field in this case) vanishes at $x = 0$. In other words, $\mathbf{E}_{\rm tot}(0, z) = 0$ because

$$\mathbf{E}_{\rm tot} = \mathbf{E}_{\rm inc} + \mathbf{E}_{\rm ref} = -2iE_0 \exp[i(kz\sin\theta - \omega t)] \sin(kx\cos\theta)\hat{\mathbf{y}}.$$

The curl of this field is

$$\nabla \times \mathbf{E}_{\rm tot} = -2ikE_0 \exp[i(kz\sin\theta - \omega t)] \left[\cos\theta\cos(kx\cos\theta)\hat{\mathbf{z}} - i\sin\theta\sin(kx\cos\theta)\hat{\mathbf{x}}\right].$$

From Faraday's law, $\nabla \times \mathbf{E} = i\omega\mathbf{B}$, the total magnetic field at the $x = 0$ surface is

$$c\mathbf{B}_{\rm out}(x = 0, z) = -2E_0 \exp[i(kz\sin\theta - \omega t)] \cos\theta\hat{\mathbf{z}}. \tag{1}$$

Now, since $\mu_0\mathbf{K} = \hat{\mathbf{n}} \times \mathbf{B}_{\rm out}$ is the surface current density at a conductor surface, and $\mathbf{B}_{\rm in} = 0$ is characteristic of a perfect conductor, the time-averaged force per unit area of surface is

$$\begin{aligned}
\langle \mathbf{f} \rangle &= \langle \frac{1}{2}\mathbf{K} \times (\mathbf{B}_{\rm in} + \mathbf{B}_{\rm out})\rangle\Big|_S = \frac{1}{4}\mathbf{K} \times \mathbf{B}^*_{\rm out} = \frac{1}{4}(\hat{\mathbf{n}} \times \mathbf{B}_{\rm out}) \times \mathbf{B}^*_{\rm out}\Big|_S \\
&= -\frac{1}{4\mu_0}|\mathbf{B}_{\rm out}|^2\hat{\mathbf{n}}.
\end{aligned}$$

The last equality is true because $\mathbf{B}_{\rm out}(x = 0, z)$ is perpendicular to the surface normal. This force/area is already a pressure. Therefore substitution from (1) gives the pressure exerted by the field on the surface as

$$\mathcal{P} = \epsilon_0 E_0^2 \cos^2\theta.$$

(b) From Newton's law, the we can compute the time-averaged force from the change in electromagnetic momentum suffered by the incident wave. That is, in a time Δt,

$$\langle \mathbf{P}_{\rm ref}\rangle - \langle \mathbf{P}_{\rm inc}\rangle = \langle \mathbf{F}\rangle\Delta t.$$

On the other hand, if normal incidence exposes an area A of surface,

$$\langle \mathbf{P}\rangle = \langle \mathbf{g}_{\rm EM}\rangle\Delta V = \langle \mathbf{g}_{\rm EM}\rangle c\Delta t A\cos\theta.$$

Therefore,

$$\langle \mathbf{F}\rangle = [\langle \mathbf{g}_{\rm ref}\rangle - \langle \mathbf{g}_{\rm inc}\rangle]\, cA\cos\theta.$$

For both the incident and reflected plane waves,

$$\langle \mathbf{g}\rangle = \frac{1}{2}\frac{\epsilon_0}{c}|\mathbf{E}|^2\hat{\mathbf{k}}.$$

Moreover,

$$\hat{\mathbf{k}}_{\rm inc} = -\cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}} \qquad \text{and} \qquad \hat{\mathbf{k}}_{\rm inc} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{z}}.$$

Therefore, $\hat{\mathbf{k}}_{\rm ref} - \hat{\mathbf{k}}_{\rm inc} = 2\cos\theta\hat{\mathbf{x}}$. Finally, the force exerted on the conductor by the field is the negative of the force exerted by the conductor on the field. Hence, in agreement with part (a), the pressure on the conductor is

$$\mathcal{P} = 2\cos\theta \times \frac{\epsilon_0}{2c}E_0^2\cos\theta c\cos\theta = \epsilon_0 E_0^2\cos^2\theta.$$

17.19 Phase Velocity of Evanescent Waves

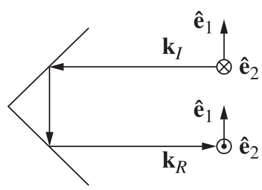
$$k^2 - q^2 = \frac{\omega^2}{c^2} n^2$$

$$v_p^2 = \frac{\omega^2}{k^2} = \frac{\omega^2}{q^2 + \kappa^2} = \frac{q^2 - \kappa^2}{q^2 + \kappa^2} \frac{c^2}{n^2}.$$

This is always less than c/n because $q^2 - \kappa^2 > 0$.

17.20 A Corner Reflector

Our discussion of polarization defines $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{k}})$ as a right-handed orthogonal triad. Moreover, back-scattering by a perfectly conducting corner reflector occurs by two reflections at $\theta_I = \pi/4$. Hence, the appropriate diagram is



LCP (upper sign) and RCP (lower sign) waves have the electric vectors

$$\mathbf{E} = E_0 \left[\hat{\mathbf{e}}_1 \cos(\omega t) \pm \hat{\mathbf{e}}_2 \sin(\omega t) \right].$$

The $\hat{\mathbf{e}}_1$ component is p-polarized and the $\hat{\mathbf{e}}_2$ component is s-polarized. In the perfect-conductor limit, the reflection amplitudes for these polarizations are $r_p = 1$ and $r_s = -1$. But there are two reflections, so the amplitudes of both components are exactly the same for the outgoing wave as for the incoming wave. Hence RCP remains RCP and LCP remains LCP.

Source: H. Mott, *Polarization in Antennas and Radar* (Wiley, New York, 1986).

17.21 Bumpy Reflection

(a) If the surface were $z = 0$, a specularly reflected wave $\mathbf{E}_R(\mathbf{r}, t) = -\hat{\mathbf{x}} E_0 \exp[i(k_y y - k_z z - \omega t)]$ added to the incident wave would give a total field

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_R = \hat{\mathbf{x}} 2i E_0 \exp[i(k_y y - \omega t)] \sin k_z z. \tag{1}$$

This field satisfies the boundary condition at the conducting surface because it is entirely tangential to, and vanishes at, the $z = 0$ surface.

(b) Now, evaluate (1) at $z = a\sin(2\pi x/d)$ and use $\omega a/c \ll 1$ to expand the result to first order in a. This gives

$$\begin{aligned} \mathbf{E}_S &= \hat{\mathbf{x}} 2iE_0 \exp[i(k_y y - \omega t)] \sin\{k_z a \sin(2\pi x/d)\} \\ &\approx \hat{\mathbf{x}} 2iE_0 \exp[i(k_y y - \omega t)] k_z a \sin(2\pi x/d). \end{aligned} \tag{2}$$

Our task is to identify a solution of the Maxwell equations in vacuum, call it $\mathbf{E}'$, with the property that $\mathbf{E}'$ reduces to $-\mathbf{E}_S$ at the corrugated surface. If we do this, $\mathbf{E} + \mathbf{E}'$ is the solution we seek. The form of $\mathbf{E}$ emerges if we rewrite (2) as

$$\mathbf{E}_S = \hat{\mathbf{x}} k_z a E_0 e^{i(k_y y - \omega t)} [e^{i2\pi x/d} - e^{-i2\pi x/d}].$$

This shows that we should choose $\mathbf{E}'$ as the sum of two plane waves,

$$\mathbf{E}' = -\hat{\mathbf{x}} k_z a E_0 \left\{ \exp[i(\mathbf{q}_+ \cdot \mathbf{r} - \omega t)] - \exp[i(\mathbf{q}_- \cdot \mathbf{r} - \omega t)] \right\},$$

where $\mathbf{q}_\pm = (\pm 2\pi/d, k_y, q)$.

(c) $\mathbf{E}'$ must satisfy the wave equation. Therefore, since part (a) tells us that the angle of incidence θ satisfies $k_y = (\omega/c)\sin\theta$,

$$\frac{\omega^2}{c^2} = \left(\frac{2\pi}{d}\right)^2 + k_y^2 + q^2 = \left(\frac{2\pi}{d}\right)^2 + \left(\frac{\omega}{c}\right)^2 \cos^2\theta + q^2.$$

Hence,

$$q = \sqrt{\frac{\omega^2}{c^2} - \left(\frac{2\pi}{d}\right)^2 - \frac{\omega^2}{c^2}\sin^2\theta} = \sqrt{\frac{\omega^2}{c^2}\cos^2\theta - \left(\frac{2\pi}{d}\right)^2}.$$

This quantity will be pure imaginary near grazing incidence ($\theta_0 \approx 90°$) and $\mathbf{E}'$ will decay exponentially from the surface into the vacuum.

17.22 Photonic Band Gap Material

(a) When ϵ is not constant, $0 = \nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = \epsilon \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \epsilon$. Also,

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{B} = -\frac{\partial}{\partial t} \mu_0 \frac{\partial \mathbf{D}}{\partial t} = -\mu_0 \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Therefore,

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} + \nabla \left(\mathbf{E} \cdot \frac{\nabla \epsilon}{\epsilon} \right) = 0.$$

The last term does not contribute if $\epsilon = \epsilon(z)$ and $\mathbf{E} = E(z,t)\hat{\mathbf{x}}$. Therefore,

$$\frac{\partial^2 E}{\partial z^2} - \mu_0 \epsilon(z) \frac{\partial^2 E}{\partial t^2} = 0.$$

(b) If $E(z,t) = E(z)\exp(-i\omega t)$, the wave equation in (a) reduces to

$$\left[\frac{d^2}{dz^2} + \mu_0 \epsilon(z)\omega^2\right] E(z) = 0.$$

Substituting $\epsilon(z) = \epsilon_0[1 + \alpha\cos(2k_0 z)]$ and $E(z) = \int\limits_{-\infty}^{\infty} dk\hat{E}(k)\cos kz$ into the wave equation gives

$$\int\limits_{-\infty}^{\infty} dk\,\hat{E}(k)\left[-k^2 + \frac{\omega^2}{c^2} + \frac{\alpha\omega^2}{c^2}\cos(2k_0 z)\right]\cos kz = 0.$$

But $\cos(k + 2k_0) + \cos(k - 2k_0) = 2\cos kz\cos(2k_0 z)$. Therefore,

$$\begin{aligned}\int\limits_{-\infty}^{\infty} dk\left(k^2 - \frac{\omega^2}{c^2}\right)\hat{E}(k)\cos kz &= \frac{\omega^2\alpha}{2c^2}\int\limits_{-\infty}^{\infty} dk\hat{E}(k)\cos[(k+2k_0)z] \\ &+ \frac{\omega^2\alpha}{2c^2}\int\limits_{-\infty}^{\infty} dk\hat{E}(k)\cos[(k-2k_0)z].\end{aligned}$$

Changing variables in the two integrals on the right and then re-naming the new variables k in both cases gives a common factor of $\cos kz$ in all three integrals. Therefore, it must be true that

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\hat{E}(k) = \frac{\omega^2\alpha}{2c^2}\left\{\hat{E}(k-2k_0) + \hat{E}(k+2k_0)\right\}.$$

(c) Let $k = q + k_0$ so the equation just above reads

$$\left[(q+k_0)^2 - \frac{\omega^2}{c^2}\right]\hat{E}(q+k_0) = \frac{\omega^2\alpha}{2c^2}\left\{\hat{E}(q-k_0) + \hat{E}(k+3k_0)\right\}. \tag{1}$$

If $\alpha = 0$, $\omega = c(q + k_0)$ where $|q| \ll k_0$. If $\alpha \ll 1$, the right side of (1) is very small. Hence, the left side must be very small. There are two ways this can happen. Either the Fourier component is very small or the quantity in the square brackets is very small. The second possibility is true for (1) as written because $\omega \approx c(q + k_0)$. Now change k_0 to $-k_0$ in (1) to get

$$\left[(q-k_0)^2 - \frac{\omega^2}{c^2}\right]\hat{E}(q-k_0) = \frac{\omega^2\alpha}{2c^2}\left\{\hat{E}(q+k_0) + \hat{E}(k-3k_0)\right\}. \tag{2}$$

The quantity in square brackets on the left side of (2) is very small for the same reason it was small in (1). Therefore, $\hat{E}(q - k_0)$ need not be small. Now change k_0 to $3k_0$ in (1). This gives

$$\left[(q+3k_0)^2 - \frac{\omega^2}{c^2}\right]\hat{E}(q+3k_0) = \frac{\omega^2\alpha}{2c^2}\left\{\hat{E}(q-3k_0) + \hat{E}(k+9k_0)\right\}. \tag{3}$$

It is still true that $\omega \approx c(q+k_0)$. This means the quantity in square brackets in (3) is *not* small. Hence, $\hat{E}(q+3k_0)$ must be small and we can drop it from (1) and (2). The 2×2 problem that remains is

$$\begin{pmatrix} (q+k_0)^2 - \dfrac{\omega^2}{c^2} & -\dfrac{\omega^2\alpha}{2c^2} \\ -\dfrac{\omega^2\alpha}{2c^2} & (q-k_0)^2 - \dfrac{\omega^2}{c^2} \end{pmatrix} \begin{pmatrix} \hat{E}(q+k_0) \\ \hat{E}(q-k_0) \end{pmatrix} = 0.$$

(d) Setting the determinant of the matrix in part (c) equal to zero gives the quadratic equation

$$\left(1-\frac{1}{4}\alpha^2\right)\left(\frac{\omega^2}{c^2}\right)^2 - \left[(q-k_0)^2+(q+k_0)^2\right]\frac{\omega^2}{c^2} + (q-k_0)^2(q+k_0)^2 = 0.$$

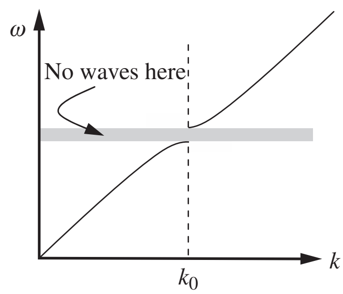
The solutions are

$$\frac{\omega_\pm^2}{c^2} = \frac{q^2+k_0^2 \pm \sqrt{(q^2+k_0^2)^2-(q^2-k_0^2)(1-\alpha^2/4)}}{1-\alpha^2/4}.$$

The key observation is apparent already from $q=0$, i.e., when $k=k_0$. In that case, the original frequency ck_0 is replaced by two frequencies,

$$\omega_\pm = ck_0\sqrt{\frac{(1\pm\alpha/2)}{(1\mp\alpha/2)}} \approx ck_0(1\pm\alpha/2).$$

Studying the dispersion relation for very small q leads to the graph of $\omega(k)$ shown below. The function $\omega(k)=ck$ except in the immediate vicinity of k_0, where there is a jump from ω_- to ω_+. Thus, there is range of frequencies (a gap) where no waves occur. A vacuum wave with a frequency in this gap, incident on a sample with this $\epsilon(z)$, would be totally reflected from its surface.



17.23 Plane Wave Amplifier

(a) The momentum of the wave field changes due to (1) reflection from the mirror and (2) the motion of the mirror. We begin with the $v=0$ contribution and note that $\mathbf{S}/c^2$ is the field linear momentum per unit volume, so $\mathbf{S}A/c^2$ is the momentum per unit length perpendicular to the mirror. The fields move at speed c so $\mathbf{S}_I A/c^2 \times c = \mathbf{S}_I A/c$ is the momentum per unit time incident on the mirror and $\mathbf{S}_R A/c$ is the momentum per unit time reflected from the mirror. Therefore,

$$\frac{d\mathbf{P}_{\rm EM}}{dt} = \frac{A}{c}(\mathbf{S}_R - \mathbf{S}_I) = \hat{\mathbf{z}}\frac{A}{c}(S_R + S_I)$$

is the rate of change of field momentum due to reflection. Now focus on the volume between the mirror and a parallel reference plane a distance L away. The total momentum stored there is $AL(\mathbf{S}_R + \mathbf{S}_I)/c^2$. The mirror moves with velocity $\boldsymbol{v} = -\hat{\mathbf{z}}dL/dt$ so the rate of change of field momentum in the volume is

$$\frac{d\mathbf{P}_{\rm EM}}{dt} = \hat{\mathbf{z}}\frac{A}{c^2}A\frac{dL}{dt}(S_R - S_I) = -\hat{\mathbf{z}}\frac{Av}{c^2}(S_R - S_I).$$

This is general because the position of the reference plane is arbitrary. By momentum conservation, the total rate of change of field momentum is the negative of the total rate of change of mirror momentum, i.e., the force on the mirror. Therefore,

$$\mathbf{F} = -\hat{\mathbf{z}}\,\frac{A}{c}\left\{(S_I + S_R) + \frac{v}{c}(S_I - S_R)\right\}.$$

(b) $A(\mathbf{S}_R + \mathbf{S}_I)\cdot\hat{\mathbf{z}} = A(S_R - S_I)$ is the rate of change of field energy in the volume between the mirror and the reference plane due to energy flow through the plane. On the other hand, the energy stored in the volume between this plane and the mirror is

$$U = AL\left\{\frac{1}{2}\epsilon_0 E^2 + \frac{1}{2}\mu_0^{-1}B^2\right\} = \frac{AL}{c}|\,\mathbf{S}\,|$$

because $E = cB$ for an plane electromagnetic wave and $\mathbf{S} = \mu_0^{-1}\mathbf{E}\times\mathbf{B}$. Therefore, the energy in the volume changes at a rate $-Av(S_R + S_I)/c$ due to the motion of the mirror. By energy conservation, the change in field energy plus the change in mechanical energy as the mirror moves against the force exerted on it must be zero. Therefore,

$$\mathbf{F}\cdot\boldsymbol{v} = A(S_I - S_R) + A\frac{v}{c}(S_I + S_R).$$

(c) From (a) and (b), we have

$$\frac{Av}{c}\left\{(S_I + S_R) + \frac{v}{c}(S_I - S_R)\right\} = (S_R - S_I) - \frac{v}{c}(S_I + S_R).$$

Collecting terms, we get $\dfrac{S_R}{S_I} = \dfrac{(1+v/c)^2}{(1-v/c)^2}$. This is the desired result because the power $P = SA$.

(d) Let the position of the mirror be $z = vt$. Since $\omega = ck$, the phase of the incident wave at the mirror surface is $k_I z + \omega_I t = \omega_I(z/c + t) = \omega_I t(v/c + 1)$. The phase of the reflected wave at the mirror surface is $k_R z - \omega_R t = \omega_R(z/c - t) = \omega_R t(v/c - 1)$. These two phases must be equal because the sum of incident and reflected waves satisfies a boundary condition on the mirror surface that is independent of time and space. Therefore, using the results of part (c),

$$\frac{\omega_R}{\omega_I} = \frac{v/c+1}{v/c-1} \Rightarrow \quad \frac{P_R}{P_I} = \left(\frac{\omega_R}{\omega_I}\right)^2 .$$

Source: J.R. Pierce, *Journal of Applied Physics* **30**, 1341 (1959).

17.24 Laser Beam Bent by a Magnetic Field

(a) A typical laboratory electromagnet produces a field of about 1 tesla (T). A typical argon-ion laser produces a continuous beam with $\sim$ 3 watts (W) of output power distributed over a beam diameter of 1 mm. Therefore, the magnitude of the beam's time-averaged Poynting vector is

$$\langle S \rangle \approx \frac{3 \text{ W}}{\pi (1 \text{ mm})^2} \approx 10^6 \text{ W/m}^2.$$

Moreover, for a monochromatic wave, $\langle u_{\text{EM}} \rangle = \frac{1}{2}\epsilon_0 |\mathcal{E}|^2 = \langle S \rangle / c$, and $c\mathcal{B} = \mathcal{E}$. Therefore,

$$\mathcal{B} = \sqrt{\frac{\mu_0 \langle S \rangle}{c}} \approx \sqrt{\frac{4\pi \times 10^{-7} \text{ N/A}^2 \times 10^6 \text{ W/m}^2}{3 \times 10^8 \text{ m/s}}} \approx 6.5 \times 10^{-5} \text{ T}.$$

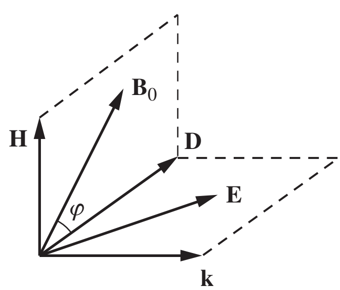
Thus, the magnetic field of the laser beam is four orders of magnitude smaller than the magnetic field produced by the electromagnet.

(b) From $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and $\mathbf{B} = \mu \mathbf{H}$, we get $\mathbf{H} = (\mathbf{k}/\mu\omega) \times \mathbf{E}$. Therefore, the Poynting vector for the electromagnetic wave is

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{k}{\mu\omega} \mathbf{E} \times (\hat{\mathbf{k}} \times \mathbf{E}) = \frac{k|\mathbf{E}|^2}{\mu\omega} \left[\hat{\mathbf{k}} - \frac{\mathbf{E}(\hat{\mathbf{k}} \cdot \mathbf{E})}{|\mathbf{E}|^2} \right].$$

To find $\hat{\mathbf{k}} \cdot \mathbf{E}$, we note that (i) $\nabla \cdot \mathbf{D} = 0$; (ii) the external magnetic field is very much larger than the optical magnetic field; and (iii) $\epsilon_0 \mathbf{E}$ differs from $\mathbf{D}$ by a factor proportional to $\mathbf{B}_0$. Therefore, since we are working to lowest order in $\mathbf{B}_0$,

$$0 = \hat{\mathbf{k}} \cdot \mathbf{D} = \epsilon \hat{\mathbf{k}} \cdot \mathbf{E} - i\gamma \hat{\mathbf{k}} \cdot (\mathbf{B} \times \mathbf{E}) \approx \epsilon \hat{\mathbf{k}} \cdot \mathbf{E} - i\gamma \hat{\mathbf{k}} \cdot (\mathbf{B}_0 \times \mathbf{E}) \approx \epsilon \hat{\mathbf{k}} \cdot \mathbf{E} - i\gamma \hat{\mathbf{k}} \cdot (\mathbf{B}_0 \times \mathbf{D}/\epsilon).$$



The diagram shows that $\mathbf{B}_0 \times \mathbf{D}$ is anti-aligned with $\mathbf{k}$. This permits us to conclude that

$$\hat{\mathbf{k}} \cdot \mathbf{E} = -i\gamma B_0 D \sin\varphi/\epsilon^2.$$

Therefore, using $\mathbf{D} \approx \epsilon\mathbf{E}$ again,

$$\mathbf{S} = \frac{k|\mathbf{E}|^2}{\mu\omega}\left[\hat{\mathbf{k}} + \frac{i\gamma B_0 D\sin\varphi}{\epsilon^2|\mathbf{E}|^2}\mathbf{E}\right] \approx \frac{k|\mathbf{E}|^2}{\mu\omega}\left[\hat{\mathbf{k}} + i\frac{\gamma}{\epsilon}\left(\frac{B_0}{D}\right)\sin\varphi\mathbf{D}\right]. \tag{1}$$

Finally, the diagram shows that $\mathbf{D}$ lies in the plane defined by $\mathbf{B}$ and $\hat{\mathbf{k}} \times \mathbf{B}_0$. Hence,

$$\mathbf{D} = \frac{(\mathbf{D}\cdot\mathbf{B}_0)}{B^2}\mathbf{B}_0 + \frac{\mathbf{D}\cdot(\hat{\mathbf{k}}\times\mathbf{B}_0)}{B^2}(\hat{\mathbf{k}}\times\mathbf{B}_0),$$

and $\mathbf{D}\cdot(\hat{\mathbf{k}}\times\mathbf{B}_0) = \hat{\mathbf{k}}\cdot(\mathbf{B}_0\times\mathbf{D}) = -B_0 D\sin\varphi$. Therefore,

$$\mathbf{D} = \frac{D}{B_0}[\cos\varphi\mathbf{B}_0 - \sin\varphi(\hat{\mathbf{k}}\times\mathbf{B}_0)]. \tag{2}$$

Substituting (2) into (1) gives

$$\mathbf{S} \approx \frac{k|\mathbf{E}|^2}{\mu\omega}\left[\hat{\mathbf{k}} + i\frac{\gamma}{\epsilon}\sin\varphi[\cos\varphi\mathbf{B}_0 - \sin\varphi(\hat{\mathbf{k}}\times\mathbf{B}_0)]\right].$$

With $\mathbf{E} = \boldsymbol{\mathcal{E}}\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, the time average of the Poynting vector is

$$\langle\mathbf{S}\rangle \approx \frac{k|\boldsymbol{\mathcal{E}}|^2}{2\mu\omega}\left[\hat{\mathbf{k}} - \frac{\mathrm{Im}\,\gamma}{\epsilon}\sin\varphi[\cos\varphi\mathbf{B}_0 - \sin\varphi(\hat{\mathbf{k}}\times\mathbf{B}_0)]\right].$$

(c) The maximum deflection of the beam from the $\hat{\mathbf{k}}$ direction occurs when the last term in the square brackets on the far right side of (1) is largest, i.e., when $\varphi = \pi/2$.

Source: D. Budker, D.F. Kimball, and D.P. DeMille, *Atomic Physics* (University Press, Oxford, 2004).

17.25 An Anisotropic Magnetic Crystal

(a) Let $\mathbf{H} = \mathbf{H}_0 \exp i(\mathbf{k}\cdot\mathbf{r} - \omega t)$. Then, because $\mathbf{H} = \boldsymbol{\mu}^{-1}\cdot\mathbf{B}$ and $\mathbf{D} = \epsilon_0\mathbf{E}$,

$$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t} \qquad \Rightarrow \quad \mathbf{k}\times(\boldsymbol{\mu}^{-1}\cdot\mathbf{B}) = -\omega\epsilon_0\mathbf{E}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t} \quad \Rightarrow \quad \mathbf{k}\times\mathbf{E} = \omega\mathbf{B}.$$

These imply that

$$\mu_0\mathbf{k}\times[\boldsymbol{\mu}^{-1}\cdot(\mathbf{k}\times\mathbf{E})] = -\omega^2\epsilon_0\mathbf{E}. \tag{1}$$

On the other hand, the data given tell us that $\mathbf{k}\times\mathbf{E} = (k_z\hat{\mathbf{y}} - k_y\hat{\mathbf{z}})E$. Therefore,

$$\mu_0\boldsymbol{\mu}^{-1}\cdot(\mathbf{k}\times\mathbf{E}) = [(k_z m - k_y m')\hat{\mathbf{y}} + (k_z m' - k_y m)\hat{\mathbf{z}}]E,$$

and, since $k^2 = k_y^2 + k_z^2$,

$$\mu_0\mathbf{k}\times\left\{\boldsymbol{\mu}^{-1}\cdot(\mathbf{k}\times\mathbf{E})\right\} = (2m'k_yk_z - mk^2)E\hat{\mathbf{x}} = -k^2(m - m'\sin 2\theta)\mathbf{E}. \qquad (2)$$

Comparing (2) with (1) gives the desired result:

$$\omega(k,\theta) = ck\sqrt{m - m'\sin 2\theta}.$$

(b) The incident wave is normal ($\theta = 0$) so $\omega = ck\sqrt{m}$ inside the medium and $\omega = ck$ in the vacuum. There is no distinction between s- and p-polarization at normal incidence. The Fresnel formula for the latter gives

$$\left.\frac{E_R}{E_I}\right|_{\parallel} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \qquad \text{and} \qquad \left.\frac{E_T}{E_i}\right|_{\parallel} = \frac{2Z_2}{Z_1 + Z_2},$$

bearing in mind that, with our conventions, the formula on the left contributes with a *negative* sign to the component of the electric field parallel to the interface. Moreover,

$$Z_1 = \mu_0 c \qquad \text{and} \qquad Z_2 = \mu_0 c/\sqrt{m}.$$

Therefore,

$$\mathbf{E}(z<0) = \hat{\mathbf{x}}E_0\left\{\exp\left[i\omega\left(\frac{z}{c} - t\right)\right] + \frac{1-\sqrt{m}}{1+\sqrt{m}}\exp\left[-i\omega\left(\frac{z}{c} + t\right)\right]\right\}$$

$$\mathbf{E}(z>0) = \hat{\mathbf{x}}\,E_0\frac{2}{1+\sqrt{m}}\exp\left[i\omega\left(\frac{z}{c\sqrt{m}} - t\right)\right].$$

Source: Prof. M.J. Cohen, University of Pennsylvania (private communication).

17.26 A Complex Dielectric Matrix

(a) We begin by looking for plane wave solutions inside the dielectric medium. With $\mathbf{E} = \mathbf{E}_0\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$, the Maxwell curl equations imply that

$$\mathbf{k}\times\mathbf{E} = \omega\mathbf{B} \qquad \text{and} \qquad \mathbf{k}\times\mathbf{B} = -\mu_0\omega\mathbf{D}.$$

Therefore,

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}) = \omega\mathbf{k}\times\mathbf{B},$$

or

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) - k^2\mathbf{E} = -\mu_0\omega^2\boldsymbol{\epsilon}\cdot\mathbf{E}. \qquad (1)$$

We are used to setting $\mathbf{k} \cdot \mathbf{E} = 0$ in isotropic media because

$$\nabla \cdot \mathbf{D} = \nabla \cdot \epsilon \mathbf{E} = 0 \rightarrow \epsilon \mathbf{k} \cdot \mathbf{E} = 0.$$

Now, however,

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\boldsymbol{\epsilon} \cdot \mathbf{E}) = 0 \rightarrow \nabla \cdot \mathbf{E} \neq 0 \rightarrow \mathbf{k} \cdot \mathbf{E} \neq 0.$$

On the other hand, we are told that $\mathbf{k} = k\hat{\mathbf{y}}$. Therefore, with $k = n\omega/c$, (1) becomes

$$\mathbf{E} - \mathbf{y}(\mathbf{E} \cdot \hat{\mathbf{y}}) = \frac{1}{n^2 \epsilon_0} \boldsymbol{\epsilon} \cdot \mathbf{E}.$$

Writing this out in detail gives

$$\begin{bmatrix} E_x \\ 0 \\ E_z \end{bmatrix} = \frac{1}{n^2} \begin{bmatrix} \alpha E_x + i\beta E_y \\ -i\beta E_x + \alpha E_y \\ \gamma E_z \end{bmatrix}.$$

This equation has two (normalized) solutions:

$$\mathbf{E}_A = E_A \hat{\mathbf{z}} \quad \text{with} \quad n_A = \sqrt{\gamma} \qquad \text{and} \qquad \mathbf{E}_B = E_B \frac{\alpha \hat{\mathbf{x}} + i\beta \mathbf{y}}{\sqrt{\alpha^2 + \beta^2}} \quad \text{with} \quad n_B = \sqrt{\frac{\alpha^2 - \beta^2}{\alpha}}.$$

There is no distinction between *s*- and *p*-polarization at normal incidence. The Fresnel formula for the latter gives

$$\left.\frac{E_R}{E_I}\right|_{||} = \frac{n_2 - n_1}{n_2 + n_1} \qquad \text{and} \qquad \left.\frac{E_T}{E_i}\right|_{||} = \frac{2n_1}{n_2 + n_1},$$

bearing in mind that, with our conventions, the formula on the left contributes with a *negative* sign to the component of the electric field parallel to the interface. Consequently, in the vacuum, the sum of the incident and reflected waves is

$$\mathbf{E}(y < 0) = E_0 \left\{ \exp\left[i\omega\left(y/c - t\right)\right] + \frac{1 - n_B}{1 + n_B} \exp\left[-i\omega\left(y/c + t\right)\right] \right\} \hat{\mathbf{x}}.$$

In the medium, we must choose E_B so $E_0 = \alpha E_B / \sqrt{\alpha^2 + \beta^2}$. In that case,

$$\mathbf{E}(y > 0) = E_0 \frac{2}{1 + n_B} \left(\hat{\mathbf{x}} + i\frac{\beta}{\alpha}\hat{\mathbf{y}} \right) \left\{ \exp\left[i\omega\left(n_B y/c - t\right)\right] \right. .$$

The "appearance" of a $\hat{\mathbf{y}}$-component in the medium does not violate any matching condition. Note also that n_B could be pure imaginary, in which case we get an evanescent wave in the medium.

(b) Superpose the $\hat{\mathbf{x}}$ solution from part (a) to a solution with a $\hat{\mathbf{z}}$-incident wave solved in exactly the same way. The result is

$$\begin{aligned}\mathbf{E}(y<0) &= \frac{E_0}{\sqrt{2}}\left\{\exp\left[i\omega\left(y/c-t\right)\right]+\frac{1-n_B}{1+n_B}\exp\left[-i\omega\left(y/c+t\right)\right]\right\}\hat{\mathbf{x}}\\ &+\frac{E_0}{\sqrt{2}}\left\{\exp\left[i\omega\left(y/c-t\right)\right]+\frac{1-n_A}{1+n_A}\exp\left[-i\omega\left(y/c+t\right)\right]\right\}\hat{\mathbf{z}}\end{aligned}$$

$$\mathbf{E}(y>0)=\frac{E_0}{\sqrt{2}}\frac{2}{1+n_B}\exp\left[i\omega\left(n_B y/c-t\right)\right]\left(\hat{\mathbf{x}}+i\frac{\beta}{\alpha}\hat{\mathbf{y}}\right)+\frac{E_0}{\sqrt{2}}\frac{2}{1+n_A}\exp\left[i\omega\left(n_A y/c-t\right)\right]\hat{\mathbf{z}}.$$

Chapter 18: Waves in Dispersive Matter

18.1 Electric Susceptibilities in Time and Frequency

The response function of interest is $\hat{\chi}(\omega) = \int\limits_{-\infty}^{\infty} dt\, \chi(t) \exp(i\omega t)$.

(a)

$$\hat{\chi}(\omega) = \int\limits_{-\infty}^{\infty} dt\, \chi_0 \delta(t) \exp(i\omega t) = \chi_0.$$

(b)

$$\begin{aligned}\hat{\chi}(\omega) &= \int\limits_{-\infty}^{\infty} dt\, \chi_0 \theta(t) \exp(i\omega t) = \lim_{\epsilon\to 0} \chi_0 \int\limits_{0}^{\infty} dt \exp[i(\omega + i\epsilon)t] \\ &= \lim_{\epsilon\to 0} i \frac{\chi_0}{\omega + i\epsilon} \frac{\omega - i\epsilon}{\omega - i\epsilon} = i\chi_0 \left[\frac{\omega}{\omega^2+\epsilon^2} - i\frac{\epsilon}{\omega^2+\epsilon^2} \right] = i\frac{\chi_0}{\omega} + \chi_0 \pi \delta(\omega).\end{aligned}$$

(c)

$$\begin{aligned}\hat{\chi}(\omega) &= \int\limits_{-\infty}^{\infty} dt\, \chi_0 \theta(t) \exp(-t/\tau) \exp(i\omega t) = \int\limits_{0}^{\infty} dt\, \chi_0 \exp(-t/\tau) \exp(i\omega t) \\ &= \frac{i\chi_0}{\omega + i/\tau} = \frac{\chi_0}{1/\tau - i\omega}.\end{aligned}$$

(d)

$$\begin{aligned}\hat{\chi}(\omega) &= \int\limits_{-\infty}^{\infty} dt\, \chi_0 \theta(t) \sin(\omega_0 t) \exp(i\omega t) \\ &= \lim_{\epsilon\to 0} \frac{\chi_0}{2i} \left[\int\limits_{0}^{\infty} dt\, \exp[i(\omega+\omega_0+i\epsilon)t] - \int\limits_{0}^{\infty} dt\, \exp[i(\omega-\omega_0+i\epsilon)t] \right] \\ &= \lim_{\epsilon\to 0} \frac{\chi_0}{2} \left[\frac{1}{\omega+\omega_0+i\epsilon} \frac{\omega+\omega_0-i\epsilon}{\omega+\omega_0-i\epsilon} - \frac{1}{\omega-\omega_0+i\epsilon} \frac{\omega-\omega_0-i\epsilon}{\omega-\omega_0-i\epsilon} \right] \\ &= \frac{\chi_0}{2} \left[\frac{1}{\omega+\omega_0} - i\pi\delta(\omega+\omega_0) - \frac{1}{\omega-\omega_0} + i\pi\delta(\omega-\omega_0) \right] \\ &= \chi_0 \left[i\frac{\pi}{2}\delta(\omega-\omega_0) - i\frac{\pi}{2}\delta(\omega+\omega_0) - \frac{\omega_0}{\omega^2-\omega_0^2} \right].\end{aligned}$$

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18.2 Magnetization and Conductivity

(a) We know that $\mathbf{M} = \chi_m \mathbf{H} = \mu^{-1}\chi_m \mathbf{B}$, $\mu = \mu_0(1+\chi_m)$, and $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$. Therefore,

$$\mathbf{M}(\mathbf{r},t) = -\frac{\chi_m}{\mu_0(1+\chi_m)} \int\limits_{-\infty}^{t} dt' \, \nabla \times \mathbf{E}(\mathbf{r},t').$$

This gives

$$\begin{aligned} \mathbf{J} &= \nabla \times \mathbf{M} = -\frac{\chi_m}{\mu_0(1+\chi_m)} \int\limits_{-\infty}^{t} dt' \, \nabla \times \nabla \times \mathbf{E}(\mathbf{r},t') \\ &= -\frac{\chi_m}{\mu_0(1+\chi_m)} \int\limits_{-\infty}^{t} dt' \, \left\{ \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} \right\} \end{aligned}$$

or

$$J_i(r,t) = \frac{\chi_m}{\mu_0(1+\chi_m)} \int\limits_{-\infty}^{t} dt' \left\{ \delta_{ij} \frac{\partial^2}{\partial x_k \partial x_k} - \frac{\partial^2}{\partial x_i \partial x_j} \right\} E_j(r,t').$$

(b) Make the Taylor expansion

$$E_i(r',t') = E_i(r,t') + \frac{\partial E_i(r,t')}{\partial x_j}(x'_j - x_j) + \frac{1}{2}\frac{\partial^2 E_i(r,t')}{\partial x_j \partial x_k}(x'_j - x_j)(x'_k - x_k) + \cdots$$

and substitute it into

$$j_i(r,t) = \int_{-\infty}^{t} dt' \int dr' \, \sigma_{ij}(r-r',t-t') E_j(r',t').$$

The result is

$$\begin{aligned} j_i(r,t) &= \int_{-\infty}^{t} dt' \left\{ \sigma_{ij}(t-t') E_j(r,t') + \Gamma_{ijk}(t-t') \frac{\partial E_j(r,t')}{\partial x_k} \right. \\ &\qquad \left. + \, \Upsilon_{ijk\ell}(t-t') \frac{\partial^2 E_j(r,t')}{\partial x_k \partial x_\ell} + \cdots \right\}, \end{aligned}$$

where

$$\sigma_{ij}(t-t') = \int dr' \, \sigma_{ij}(r-r',t-t')$$

$$\Gamma_{ijk}(t-t') = \int dr' \, \sigma_{ij}(r-r',t-t')(x'_k - x_k)$$

$$\Upsilon_{ijk\ell}(t-t') = \tfrac{1}{2} \int dr' \, \sigma_{ij}(r-r',t-t')(x'_k - x_k)(x'_\ell - x_\ell).$$

The magnetization current is a particular case of the third term where

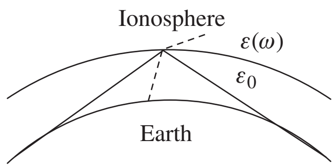
$$\Upsilon_{ijk\ell} = \frac{\chi_m}{\mu_0(1+\chi_m)} \left[\delta_{ij}\delta_{k\ell} - \delta_{ik}\delta_{j\ell} \right].$$

The first term $\sigma_{ij}(t-t')$ leads to the usual Ohm's law with a frequency-dependent conductivity. The second term $\Gamma_{ijk}(t-t')$ is present in general but vanishes if the medium has inversion symmetry.

Source: Yu.A. Il'inskii and L.V. Keldysh, *Electromagnetic Response of Material Media* (Plenum, New York, 1994).

18.3 The Radio Operator's Friend

The index of refraction for a non-magnetic medium is defined by $n^2 = \epsilon(\omega)/\epsilon_0$. Therefore, when $\omega > \Omega$, we have $n_2 < n_1$ where n_1 is the index of the lower atmosphere and n_2 is the index of the ionosphere. Therefore, from Snell's law, there will be total refection of a wave which approaches the ionosphere from the lower atmosphere if the angle of incidence exceeds $\theta = \sin^{-1}(n_2/n_1)$. Such waves return to Earth quite far from the source (solid line below). By contrast, the Fresnel equations show that the reflection coefficient is quite small, in this case, until the angle of incidence approaches the critical angle. Thus, waves which approach the ionosphere with small angles of incidence mostly continue into the atmosphere (dotted line below).



18.4 Plane Waves of Vector Potential

In the Lorenz gauge, the inhomogeneous wave equation for the vector potential is

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0 \mathbf{j}.$$

With the given constitutive relation, we get

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = k_0^2 \mathbf{A}.$$

For a plane wave

$$\mathbf{A}(\mathbf{r},t) = \mathbf{A}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)},$$

substitution into the above equation yields

$$-k^2 + \frac{\omega^2}{c^2} = k_0^2.$$

The wave vector is real (and true propagation occurs) only when $\omega > \omega_0 = ck_0$.

18.5 Plasma Sheath

(a) The field and potential are related by $\varphi(1) - \varphi(2) = \int_1^2 d\mathbf{s}\cdot\mathbf{E}$ in a quasistatic approximation. We also require the continuity of $D = \epsilon E$ across the plasma/sheath interface. These two conditions produce two equations:

$$2\ell E_S + 2LE_P = V$$

$$\epsilon_0 E_S = \epsilon E_P.$$

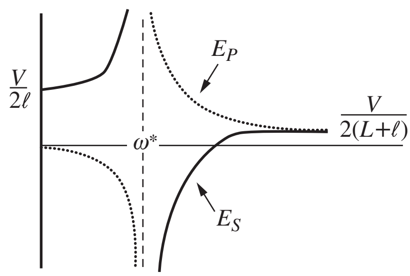
Solving for the two unknowns gives

$$E_P = \frac{V}{2\ell(\epsilon/\epsilon_0) + 2L} \qquad \text{and} \qquad E_S = \frac{(\epsilon/\epsilon_0)V}{2\ell(\epsilon/\epsilon_0) + 2L}.$$

(b) Using $\epsilon(\omega)$ as given, both fields diverge when $1 - \omega_p^2/\omega^2 + L/\ell = 0$, that is, when

$$\omega = \omega^* = \frac{\omega_p}{\sqrt{1 + L/\ell}}.$$

For the sheath, $E_S(\omega = 0) = V/2\ell$, $E_S(\omega = \omega_P) = 0$, and $E_S(\omega \to \infty) = V/2L$. As for the plasma, $E_P(\omega = 0) = 0$ and $E_P(\omega \to \infty) = V/2L$. A plot of these functions is shown below.



When $\omega \ll \omega_p$, all the voltage drops across the vacuum since the plasma acts as a perfect conductor. The voltage drops uniformly when $\omega \gg \omega_P$ because the plasma particle motion cannot follow the field. All the voltage drops across the plasma when $\omega = \omega_p$ because the plasma can support the electric field $V/2L$ needed to do this.

(c) Both fields diverge at $\omega = \omega^*$. This indicates some sort of resonant behavior. In circuit theory, amplitude divergences occur at the resonant frequency of an LC circuit (without damping), where the energy sloshes back and forth between magnetic (L) and electric (C). The same must happen here, where the electric energy is stored in the sheath and the magnetic energy is stored in the moving charged particles of the plasma.

18.6 Propagation in an Undamped Medium

The curl of Faraday's law is

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}\nabla \times \mathbf{B} = -\frac{\partial}{\partial t}\left\{\mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right\}.$$

We have $\nabla \cdot \mathbf{E} = 0$ because $\rho = 0$, and $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$. Hence,

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial \mathbf{j}}{\partial t}. \tag{1}$$

Newton's law for an electron is $\mathbf{F} = m\dot{\mathbf{v}} = -e\mathbf{E}$. Therefore, the current density associated with the entire collection of electrons is

$$\frac{\partial \mathbf{j}}{\partial t} = -ne\dot{\mathbf{v}} = \frac{ne^2}{m}\mathbf{E}. \tag{2}$$

Substituting (2) into (1) and defining $\omega_p^2 = ne^2/m\epsilon_0$ gives

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0\frac{ne^2}{m}\mathbf{E} = \frac{\omega_p^2}{c^2}\mathbf{E}. \tag{3}$$

Finally, substituting $\mathbf{E} = \mathbf{E}_0\exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)$ into (3) gives

$$-k^2 + \frac{\omega^2}{c^2} = \frac{\omega_p^2}{c^2},$$

or

$$\omega^2 = \omega_p^2 + c^2k^2.$$

Source: A.M. Portis, *Electromagnetic Fields* (Wiley, New York, 1978).

18.7 Surface Plasmon Polariton

(a) The wave equation for $z > 0$ is

$$\nabla^2\mathbf{E} + \omega^2\mu_0\hat{\epsilon}(\omega)\mathbf{E} = 0.$$

The wave equation for $z < 0$ is the same with the dielectric function replaced by ϵ_0. Therefore, substituting the trial electric field functions gives

$$\begin{aligned} \kappa_{\rm in}^2 &= q^2 - \omega^2\mu_0\hat{\epsilon}(\omega) \qquad z > 0, \\ \kappa_{\rm out}^2 &= q^2 - \omega^2/c^2 \qquad z < 0. \end{aligned} \tag{1}$$

(b) The dielectric functions do not depend on position. Therefore, in each medium $\nabla\cdot\mathbf{D} = 0$ implies that

$$\nabla\cdot\mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_z}{\partial z} = 0.$$

In other words,

$$iqE_x^{\rm in} - \kappa_{\rm in}E_z^{\rm in} = 0 \qquad \text{and} \qquad iqE_x^{\rm out} + \kappa_{\rm out}E_z^{\rm out}.$$

The electric field matching condition at $z = 0$ from $\nabla\cdot\mathbf{D} = 0$ is

$$\epsilon_0 E_z^{\rm out} = \hat{\epsilon}(\omega)E_z^{\rm in}.$$

The electric field matching condition at $z = 0$ from Faraday's law is $E_x^{\rm out} = E_x^{\rm in}$. Combining all this information gives the desired expression:

$$\hat{\epsilon}(\omega) = -\epsilon_0\frac{\kappa_{\rm in}}{\kappa_{\rm out}} < 0. \tag{2}$$

The inequality follows because, by assumption, $\kappa > 0$ in both media.

(c) Squaring (2) and using (1) to eliminate $\kappa_{\rm in}$ and $\kappa_{\rm out}$ gives

$$q^2\left[1-\frac{\hat{\epsilon}^2(\omega)}{\epsilon_0^2}\right]=\omega^2\mu_0\hat{\epsilon}(\omega)\left[1-\hat{\epsilon}(\omega)/\epsilon_0\right],$$

or

$$q^2=\frac{\omega^2}{c^2}\frac{\hat{\epsilon}(\omega)/\epsilon_0}{1+\hat{\epsilon}(\omega)/\epsilon_0}. \tag{3}$$

The inequality in (2) and $q^2>0$ imply that $1+\hat{\epsilon}(\omega)/\epsilon_0<0$. Using $\hat{\epsilon}(\omega)/\epsilon_0=1-\omega_{\rm p}^2/\omega^2$, we conclude that

$$\omega<\frac{\omega_{\rm p}}{\sqrt{2}}. \tag{4}$$

Substituting $\hat{\epsilon}(\omega)$ into (3) and rationalizing gives

$$\omega^4-\omega^2\left(\omega_{\rm p}^2+2c^2q^2\right)+\omega_{\rm p}^2c^2q^2=0.$$

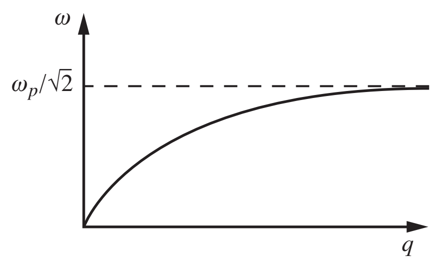
The solutions of this equation are

$$\omega_\pm^2=\frac{\omega_{\rm p}^2}{2}+c^2q^2\pm\frac{1}{2}\sqrt{\omega_{\rm p}^4+4c^4q^4}.$$

It is easy to check that only the lower solution satisfies (4). Moreover,

$$\omega_-(q\to 0)=cq \qquad \text{and} \qquad \omega_-(q\to\infty)=\frac{\omega_{\rm p}}{\sqrt{2}}.$$

The entire mode dispersion is plotted below.



18.8 Inverse Faraday Effect

(a) The time average of the current density has two terms. One term is zero because it is the time average of $-e\bar{n}\delta\mathbf{v}\exp(-i\omega t)$. The other term is the time average of the product of $-e\delta n\exp(-i\omega t)$ and $\delta\mathbf{v}\exp(-i\omega t)$, which is a familiar calculation. Hence,

$$\langle\mathbf{j}\rangle=-\frac{1}{2}\text{Re}(e\delta n\delta\mathbf{v}).$$

(b) The continuity equation is $\dfrac{\partial n}{\partial t} + \nabla \cdot \mathbf{j} = 0$. Dropping the term proportional to $\delta n \delta \mathbf{v}$, this reads

$$-i\omega \delta n + \nabla \cdot (\bar{n}\delta \mathbf{v}) = 0 \qquad \Rightarrow \qquad \delta n \approx -(i/\omega)\nabla \cdot (\bar{n}\delta \mathbf{v}).$$

On the other, the problem statement tells us that $\delta \mathbf{v} = -(\sigma/e\bar{n})\delta \mathbf{E}$. Therefore, using part (a),

$$\langle \mathbf{j} \rangle = \frac{1}{2\bar{n}e\omega}\mathrm{Re}\left[i\sigma^\star \delta \mathbf{E}^* \nabla \cdot (\sigma \delta \mathbf{E})\right] = -\frac{1}{2\bar{n}e\omega}\mathrm{Im}\left[\sigma^\star \delta \mathbf{E}^* \nabla \cdot (\sigma \delta \mathbf{E})\right].$$

It is convenient to write this in the form

$$\langle \mathbf{j} \rangle = -\frac{1}{2\bar{n}e\omega}\left[\frac{\sigma^\star \delta \mathbf{E}^* \nabla \cdot (\sigma \delta \mathbf{E}) - \sigma \delta \mathbf{E} \nabla \cdot (\sigma^* \delta \mathbf{E}^*)}{2i}\right].$$

Now use the vector identity

$$\nabla \times (\mathbf{A} \times \mathbf{A}^*) = \mathbf{A}(\nabla \cdot \mathbf{A}^*) - \mathbf{A}^*(\nabla \cdot \mathbf{A}) + (\mathbf{A}^* \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{A}^*$$

to write

$$\langle \mathbf{j} \rangle = \frac{i|\sigma|^2}{4\bar{n}e\omega}\left\{\nabla \times (\delta \mathbf{E} \times \delta \mathbf{E}^*) + (\delta \mathbf{E} \cdot \nabla)\delta \mathbf{E}^* - (\delta \mathbf{E}^* \cdot \nabla)\delta \mathbf{E}\right\}.$$

The first term has the required form of $\nabla \times \mathbf{M}$ where

$$\mathbf{M} = \frac{i|\sigma|^2}{4\bar{n}e\omega}(\delta \mathbf{E} \times \delta \mathbf{E}^*) = \frac{i\bar{n}e^3}{4m^2\omega^3}(\delta \mathbf{E} \times \delta \mathbf{E}^*) = \frac{ie\epsilon_0\omega_\mathrm{p}^2}{4m\omega^3}(\delta \mathbf{E} \times \delta \mathbf{E}^*).$$

(c) $\mathbf{M} = 0$ for linear polarization because $\delta \mathbf{E}$ is real. For a circularly polarized wave propagating along the z-axis, $\delta \mathbf{E} = A(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})$ so

$$\delta \mathbf{E} \times \delta \mathbf{E}^* = A(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}) \times A(\hat{\mathbf{x}} \mp i\hat{\mathbf{y}}) = \mp 2iA^2\hat{\mathbf{z}}.$$

Therefore, $\mathbf{M} = \pm \dfrac{e\epsilon_0\omega_\mathrm{p}^2}{2m\omega^3}A^2$.

Source: R. Hertel, *Journal of Magnetism and Magnetic Materials* **303**, L1 (2006).

18.9 The Anomalous Skin Effect

(a) If we set $\partial \mathbf{j}/\partial t = d\mathbf{j}/dt$ from above and neglect the displacement current, Ampère's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$ gives

$$\frac{\partial}{\partial t}\nabla \times \mathbf{B} = \mu_0 \frac{d\mathbf{j}}{dt} = \frac{\mu_0\sigma_0}{\tau}\mathbf{E} - \frac{\mu_0}{\tau}\mathbf{j} = \frac{\mu_0\sigma_0}{\tau}\mathbf{E} - \frac{1}{\tau}\nabla \times \mathbf{B}.$$

Taking the curl of both sides and using $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and $\nabla \cdot \mathbf{B} = 0$ gives

$$\nabla^2 \left[\mathbf{B} + \tau \frac{\partial \mathbf{B}}{\partial t} \right] = \mu_0 \sigma_0 \frac{\partial \mathbf{B}}{\partial t}. \tag{1}$$

Substituting $\mathbf{B} = \mathbf{B}_0 e^{i(kz-\omega t)}$ into (1) gives

$$k^2 = \frac{\mu_0 \sigma_0 i\omega}{1 - i\omega\tau}.$$

This is consistent with our previous treatment of ohmic matter, where we found

$$k^2 = \epsilon_0 \mu_0 \omega^2 + i\omega\sigma(\omega)\mu_0 \qquad \text{and} \qquad \sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau}$$

because, here, we neglect the displacement current term $\epsilon_0 \mu_0 \omega^2$.

(b) In this case, Ampère's law gives an extra term:

$$\frac{\partial}{\partial t} \nabla \times \mathbf{B} = \frac{\mu_0 \sigma_0}{\tau} \mathbf{E} - \frac{\mu_0}{\tau} \mathbf{j} + \mu_0 \bar{v} \frac{\partial \mathbf{j}}{\partial z}.$$

Without trouble, this leads to

$$\nabla^2 \left[\mathbf{B} - \ell \frac{\partial \mathbf{B}}{\partial z} + \tau \frac{\partial \mathbf{B}}{\partial t} \right] = \mu_0 \sigma_0 \frac{\partial \mathbf{B}}{\partial t}. \tag{2}$$

Substituting $\mathbf{B} = \mathbf{B}_0 e^{i(kz-\omega t)}$ into (2) gives the cubic equation

$$k^2 (1 - ik\ell - i\omega\tau) = i\mu_0 \sigma_0 \omega.$$

In the extreme anomalous limit, the space derivative dominates so $k\ell \gg 1$ and $k\ell \gg \omega\tau$. Therefore,

$$k(\omega) = \left(-\frac{\mu_0 \sigma_0 \omega}{\ell} \right)^{1/3} = \left(-\frac{\omega}{\Lambda^2 \bar{v}} \right)^{1/3} = \frac{2}{\delta^*(\omega)} \exp(\pm i\pi/3) = \frac{2}{\delta^*(\omega)} \left[\frac{1}{2} \pm i \frac{\sqrt{3}}{2} \right].$$

Exponential growth of the field into the medium is unphysical, so we get the anticipated behavior

$$\mathbf{B}(z,t) = \mathbf{B}_0 \exp\left[i(kz - \omega t) \right] = \mathbf{B}_0 \exp[i(z/\delta^* - \omega t)] \exp[-\sqrt{3} z/\delta^*].$$

(c) When the gradient is present, the steady-state current is found from

$$0 = \frac{d\mathbf{j}}{dt} = \frac{\sigma_0}{\tau} \mathbf{E} - \frac{\mathbf{j}}{\tau} + \bar{v} \frac{\partial \mathbf{j}}{\partial z} \quad \Rightarrow \quad \mathbf{j} = \sigma \mathbf{E} + \ell \frac{\partial \mathbf{j}}{\partial z}.$$

To a first approximation, we assume the first term is large compared to the second to get

$$\mathbf{j} \approx \sigma_0 \mathbf{E} + \ell \sigma_0 \frac{\partial \mathbf{E}}{\partial z}.$$

Finally, the order-of-magnitude estimate $\partial\mathbf{E}/\partial z \approx \mathbf{E}/\delta$ gives

$$\mathbf{j} \approx \sigma_0 \mathbf{E} + \sigma_0 \frac{\ell}{\delta}\mathbf{E}.$$

This shows that the gradient correction is negligible if $\ell \ll \delta$.

Source: P.W. Gilberd, *Journal of Physics F* **12**, 1845 (1982).

18.10 Energy Storage and Energy Loss

(a) The energy density is

$$u_{\rm EM} = \frac{1}{2}\epsilon_0 |\mathbf{E}|^2 + N\left[\frac{1}{2}m\omega_0^2 x^2 + \frac{1}{2}m\dot{x}^2\right],$$

where the complex oscillator displacement is

$$\mathbf{x}(t) = -\frac{q/m}{\omega_0^2 - \omega^2}\mathbf{E}_0 \exp(-i\omega t).$$

The time average of the energy density is

$$\langle u_{\rm EM} \rangle = \frac{1}{4}|\mathbf{E}_0|^2 \left[\epsilon(\omega) + \frac{Nq^2}{m}\frac{2\omega^2}{(\omega_0^2 - \omega^2)^2}\right],$$

where

$$\epsilon(\omega) = \epsilon_0 + \frac{Nq^2}{m}\frac{1}{\omega_0^2 - \omega^2}.$$

The time average of the energy density is

$$\langle u_{\rm EM} \rangle = \frac{1}{4}\frac{\partial}{\partial\omega}\left[\omega\epsilon'(\omega)\right]|\mathbf{E}_0|^2.$$

(b) The dielectric function is

$$\hat{\epsilon} = \epsilon_0 + i\frac{\hat{\sigma}}{\omega},$$

where

$$\hat{\sigma} = \frac{\sigma_0}{1 - i\omega\tau}.$$

The time average of the rate of work done on the particles is

$$\langle \mathcal{P} \rangle = \frac{1}{2}\text{Re}\left[\mathbf{j}(\omega)\cdot\mathbf{E}^*(\omega)\right] = \frac{1}{2}\left[\text{Re}\sigma\right]|\mathbf{E}_0|^2 = \frac{1}{2}\frac{\sigma_0}{1+\omega^2\tau^2}|\mathbf{E}_0|^2.$$

On the other hand,

$$\epsilon''(\omega) = \mathrm{Im}\left[\frac{i}{\omega}\frac{\sigma_0}{1-i\omega\tau}\right] = \frac{\sigma}{\omega(1+\omega^2\tau^2)}.$$

Therefore,

$$\langle Q(t)\rangle = \frac{1}{2}\left[\omega\epsilon''(\omega)|\mathbf{E}|^2\right] = \frac{1}{2}\frac{\sigma_0}{1+\omega^2\tau^2}|\mathbf{E}_0|^2.$$

18.11 The Lorenz-Lorentz and Drude Formulae

(a) The relevant boundary value problem is a dielectric sphere in a uniform, static electric field $\mathbf{E}_0 = E_0\hat{\mathbf{z}}$. If $\mathbf{D} = \kappa\epsilon_0\mathbf{E}$ we have $\nabla\cdot\mathbf{E} = 0$ both inside and outside the sphere. If $\mathbf{E} = -\nabla\varphi$, $\nabla^2\varphi = 0$ everywhere. Then, since $\varphi_{\mathrm{out}} \to -E_0 r\cos\theta$,

$$\varphi_{\mathrm{in}} = Ar\cos\theta$$

$$\varphi_{\mathrm{out}} = \left(\frac{(A+E_0)a^3}{r^2} - E_0 r\right)\cos\theta$$

satisfies Laplace's equation everywhere and is continuous at $r = a$. The normal (radial) component of $\mathbf{D}$ is continuous (no free surface charge) so

$$\left.\frac{\partial\varphi_{\mathrm{out}}}{\partial r}\right|_{r=a} = \kappa\left.\frac{\partial\varphi_{\mathrm{in}}}{\partial r}\right|_{r=a} \Rightarrow A = -\frac{3\epsilon_0 E_0}{\kappa+2}.$$

The polarization inside the sphere is

$$\mathbf{P} = \epsilon_0\chi_E\mathbf{E}_{\mathrm{in}} = \frac{3\epsilon_0\chi_E}{\kappa+2}\mathbf{E}_0 = 3\epsilon_0\frac{\kappa-1}{\kappa+2}\mathbf{E}_0 = 3\epsilon_0\frac{n^2-1}{n^2+2}\mathbf{E}_0.$$

In the quasistatic approximation, $E_0 \to E_0(t)$ and $n \to n(\omega)$.

(b) Since the polarization is uniform inside the sphere, the effective volume charge density $\rho = \nabla\cdot\mathbf{P} = 0$. Therefore, an electron in a tiny vacuum sphere feels only the effective surface charge densities $\sigma = \mathbf{P}\cdot\hat{\mathbf{n}}$ from two spherical surfaces. Consider the outer surface first. Since $\mathbf{P} = P\hat{\mathbf{z}}$ and $\hat{\mathbf{n}} = \hat{\mathbf{r}}$, we get $\sigma_{\mathrm{out}} = P\cos\theta$ in otherwise empty space (where Laplace's equation holds). Therefore,

$$\varphi_{\mathrm{in}} = Ar\cos\theta \qquad \text{and} \qquad \varphi_{\mathrm{out}} = A(a^3/r^2)\cos\theta.$$

The matching condition now is

$$\left[-\partial\varphi_{\mathrm{out}}/\partial r + \partial\varphi_{\mathrm{in}}/\partial r\right]_{r=a} = (P/\epsilon_0)\cos\theta.$$

This gives $A = P/2\epsilon_0$ and $\mathbf{E}_{\mathrm{inm}} = -\hat{\mathbf{z}}P/3\epsilon_0$. The charge on the surface of the inner sphere produces exactly the same electric field except that $\hat{\mathbf{n}} = -\hat{\mathbf{r}}$ in this case so the sum of the two electric fields is exactly zero. There is no force on the electron due to the sphere polarization. The only force comes from a "spring" that binds the electron to its parent atom. This gives

$$\ddot{\mathbf{r}} + \omega_0^2 r = -(e/m)\mathbf{E}_0 \cos\omega t$$

as the equation of motion, where $\mathbf{r}$ is measured from the origin of the vacuum sphere. This has the steady-state solution $(\omega_0^2 - \omega^2)r = -(e/m)\mathbf{E}_0 \cos\omega t$. Each of the N electrons leads to the same solution so the total sphere polarization (approximated by the total dipole moment per volume of the sphere) is

$$\mathbf{P}(t) = -en_0\mathbf{r}(t) = \frac{e^2 n_0}{m(\omega_0^2 - \omega^2)}\mathbf{E}(t).$$

Equating this to the polarization computed earlier gives the Lorenz-Lorentz formula,

$$\frac{n^2(\omega) - 1}{n^2(\omega) + 2} = \frac{\omega_\mathrm{p}^2}{\omega_0^2 - \omega^2}.$$

(c) The total positive charge is Ne so, from Gauss' law, the electric field inside the sphere due to the uniform positive charge of the sphere is

$$\mathbf{E} = \frac{Ne\mathbf{r}}{4\pi\epsilon_0 a^2} = ne\mathbf{r}/3\epsilon_0.$$

The equation of motion for the k^{th} electron is

$$m\ddot{\mathbf{r}}_k = -ne^2\mathbf{r}/3\epsilon_0 - \frac{\partial}{\partial \mathbf{r}_k}\frac{1}{4\pi\epsilon_0}\sum_{i\neq k}^{N}\frac{e^2}{|\mathbf{r}_k - \mathbf{r}_i|} - e\mathbf{E}_0\cos\omega t.$$

Therefore, the equation of motion for the polarization is

$$\ddot{\mathbf{P}} = -\frac{1}{3}\omega_\mathrm{p}^2\mathbf{P} + \frac{e^3}{4\pi\epsilon_0 Vm}\sum_{k=1}^{N}\sum_{i\neq k}^{N}\frac{(\mathbf{r}_k - \mathbf{r}_i)}{|\mathbf{r}_k - \mathbf{r}_i|^3} + \epsilon_0\omega_\mathrm{p}^2\mathbf{E}_0\cos\omega t,$$

where $V = \frac{4}{3}\pi a^3$. The double sum cancels out so

$$\ddot{\mathbf{P}} = -\frac{1}{3}\omega_\mathrm{p}^2\mathbf{P} + \epsilon_0\omega_\mathrm{p}^2\mathbf{E}_0\cos\omega t.$$

This equation has a steady solution,

$$\mathbf{P}(t) = \mathbf{P}\cos\omega t = \frac{\omega_\mathrm{p}^2}{\frac{1}{3}\omega_\mathrm{p}^2 - \omega^2}\epsilon_0\mathbf{E}_0(t).$$

Equating this to the polarization computed in part (a) gives the stated formula,

$$\frac{\omega_\mathrm{p}^2}{\frac{1}{3}\omega_\mathrm{p}^2 - \omega^2} = 3\frac{n^2(\omega) - 1}{n^2(\omega) + 2} \quad\Rightarrow\quad n^2(\omega) = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2}.$$

18.12 Loss and Gain Media

(a) A monochromatic plane wave propagating in the z-direction through a medium with complex index of refraction $\hat{n}(\omega) = n' + in''$ has fields of the form

$$\exp[i\omega(\hat{n}z/c - t)] = \exp[-n''\omega z/c]\exp[in'\omega z/c]\exp[-i\omega t].$$

The wave amplitude decreases if $n'' > 0$ and increases if $n'' < 0$. This can only occur if the medium absorbs energy from the wave in the first case and supplies energy to the wave in the second case. Now, because $|f| \ll 1$, the stated index of refraction is

$$\begin{aligned}\hat{n} &= n' + in'' = \sqrt{1 + \frac{f\omega_{\rm p}^2}{\omega_0^2 - \omega^2 - i\omega\Gamma}} \approx 1 + \frac{\frac{1}{2}f\omega_{\rm p}^2}{\omega_0^2 - \omega^2 - i\omega\Gamma} \\ &= 1 + \frac{\frac{1}{2}f\omega_{\rm p}^2(\omega_0^2 - \omega^2)}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2} + i\frac{f\omega_{\rm p}^2\omega\Gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\Gamma^2}.\end{aligned}$$

This shows that $f > 0$ corresponds to an absorbing medium and $f < 0$ corresponds to a gain medium.

(b) A general wave packet for one component of the electric field is

$$E(z,t) = \int_0^\infty d\omega \hat{A}(\omega)\exp\left[i\hat{n}(\omega)z\frac{\omega}{c}\right]\exp[-i\omega t].$$

The packet will emerge undistorted if the total accumulated phase $\phi = (n'_G L_G + n'_A L_A)\omega/c$ is the same as the phase $\phi_V = (L_A + L_B)\omega/c$ that would be accumulated if the packet passed through vacuum. The real parts have the form $n'_A = 1 + f_A\Lambda$ and $n'_G = 1 + f_G\Lambda$. Therefore,

$$\phi = [(1 + f_G\Lambda)L_G + (1 + f_A\Lambda)L_A]\omega/c = \phi_V + (f_G L_G + f_A L_A)\Lambda\frac{\omega}{c}.$$

Since $f_G < 0$, the condition for no distortion is $f_A L_A = |f_G|L_G$.

Source: E.L. Bolda, J.C. Garrison, and R.Y. Chiao, *Physical Review A* **49**, 2938 (1994).

18.13 A Magnetic Lorentz Model

(a) By symmetry, the magnetic field (due to its neighbors) is zero at the equilibrium position of each wire. This symmetry is broken when $u \neq 0$, but the net field will be small as long as u is small.

(b) A uniform field $B\hat{\mathbf{y}}$ exerts a force per unit length $-IB\hat{\mathbf{x}}$ on an infinite straight wire that carries a current I in the $+\hat{\mathbf{z}}$-direction. Therefore, Newton's equation of motion for the displacement u_x for a single wire is

$$m\ddot{u}_x = -ku_x - m\gamma\dot{u} - IB\exp(-i\omega t).$$

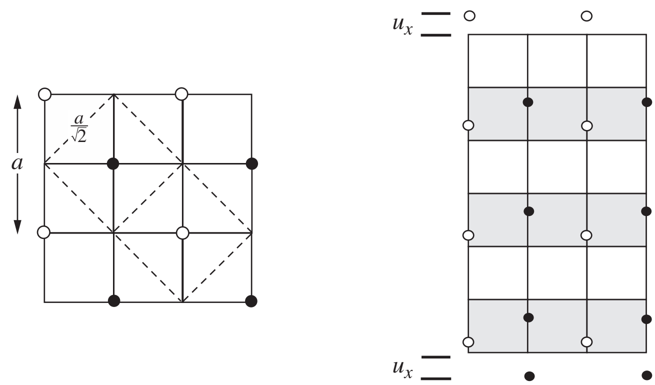
Substituting a trial solution of the form $u_x(t) = u_x \exp(-i\omega t)$ gives

$$[-m\omega^2 + k - im\gamma\omega]u_x \exp(-i\omega t) = -IB \exp(-i\omega t).$$

Therefore,

$$u_x(t) = \frac{IB}{m\omega^2 - k + im\gamma\omega} \exp(-i\omega t).$$

(c) The figure to the left below shows that there is one line of current (either positive or negative) per area $\mathcal{A} = a^2/2$ of sample cross section. The figure to the right below shows the displacements of the wires in the x-direction when $\mathbf{B}$ points in the $+y$-direction.



The internal wires pair up in the shaded regions. What remains is a sheet at the top of the sample (where $I < 0$) with current density

$$K_z = u_x \frac{I}{\mathcal{A}} = \frac{2I}{a^2} u_x,$$

and a sheet at the bottom of the sample with current density $-K_z$. From elementary magnetostatics, the magnetic field produced by these two sheets is zero outside the sample. Inside the sample, the field is

$$\mathbf{B}_{\text{ind}} = -\mu_0 K_z \hat{\mathbf{y}}.$$

The total field is the sum of the external field and the induced field just computed. This sum is the field responsible for the displacements of the wires. Therefore, if $\omega_0^2 = k/m$ and $\Omega_p^2 = 2\mu_0 I^2/ma^2$,

$$\mathbf{B} = \mathbf{B}_{\text{ext}} - \mu_0 \frac{2I}{a^2} \frac{I\mathbf{B}}{m\omega^2 - k + im\gamma\omega} = \mathbf{B}_{\text{ext}} - \frac{\Omega_p^2}{\omega^2 - \omega_0^2 + i\gamma\omega}\mathbf{B}.$$

(d) The tangential (y-component) of $\mathbf{H}$ must be continuous at the top and bottom of the sample. This tells us that

$$\frac{B_{\text{ext}}}{\mu_0} = \frac{\mathbf{B}}{\mu(\omega)}.$$

The two preceding equations above both give expressions for B/B_{ext}. Setting these equal gives the desired formula for the magnetic permeability:

$$\mu(\omega) = \frac{\mu_0}{1 + \dfrac{\Omega_p^2}{\omega^2 - \omega_0^2 + i\gamma\omega}}.$$

Source: J.B. Pendry and S. O'Brien, *Journal of Physics: Condensed Matter* **4**, 7409 (2002).

18.14 Energy Flow in the Lorentz Model

(a) We have previously derived the Poynting energy conservation statement:

$$\nabla\cdot\mathbf{S}+\frac{\partial u_{EM}}{\partial t}+\mathbf{j}\cdot\mathbf{E}=0.$$

Here, $\mathbf{j}=\partial\mathbf{P}/\partial t=-Ned\mathbf{r}/dt$ and

$$\mathbf{E}=-\frac{m}{e}\left\{\frac{d^2\mathbf{r}}{dt^2}+\frac{1}{\tau}\frac{d\mathbf{r}}{dt}+\omega_0^2\mathbf{r}\right\}.$$

Hence,

$$\mathbf{j}\cdot\mathbf{E}=Nm\dot{\mathbf{r}}\cdot\left\{\ddot{\mathbf{r}}+\tau^{-1}\dot{\mathbf{r}}+\omega_0^2\mathbf{r}\right\}=N\frac{d}{dt}\left\{\frac{1}{2}m|\dot{\mathbf{r}}|^2+\frac{1}{2}m\omega_0^2|\mathbf{r}|^2\right\}+\frac{Nm}{\tau}|\dot{\mathbf{r}}|^2.$$

The quantity in brackets is the total mechanical energy u_{mech} of a harmonic oscillator, so

$$\nabla\cdot\mathbf{S}+\frac{\partial u_{EM}}{\partial t}+\frac{\partial u_{mech}}{\partial t}+\frac{Nm}{\tau}|\dot{\mathbf{r}}|^2=0.$$

The last term represents the energy lost due to damping.

(b) We have $\mathbf{B}=\omega^{-1}\mathbf{k}\times\mathbf{E}$ and $\mathbf{k}(\omega)=\hat{\mathbf{k}}\hat{n}(\omega)\omega/c$, so

$$\langle\mathbf{S}\rangle\;=\;\frac{1}{2\mu_0}\mathrm{Re}\left\{\mathbf{E}\times\mathbf{B}^*\right\}=\frac{1}{2c\mu_0}\mathrm{Re}\left\{\mathbf{E}\times(\hat{\mathbf{k}}\times\mathbf{E}^*)\hat{n}^*\right\}=\frac{1}{2}\epsilon_0c\hat{n}_1|\mathbf{E}|^2\hat{\mathbf{k}}.$$

(c) Everything is time-harmonic so we must compute

$$\langle u\rangle\;=\;\langle u_{EM}\rangle+\langle u_{mech}\rangle\;=\frac{1}{4}\epsilon_0(\mathbf{E}\cdot\mathbf{E}^*+c^2\mathbf{B}\cdot\mathbf{B}^*)+\frac{1}{4}Nm(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}^*+\omega_0^2\mathbf{r}\cdot\mathbf{r}^*).$$

From above, $c^2\mathbf{B}\cdot\mathbf{B}^*=(c^2/\omega^2)\mathbf{k}\cdot\mathbf{k}^*|\mathbf{E}|^2=|\hat{n}|^2|\mathbf{E}|^2$, so $u_{em}=\frac{1}{4}\epsilon_0(1+|\hat{n}|^2)|\mathbf{E}|^2$. Moreover, the Lorentz model corresponds to

$$\mathbf{p}=-e\mathbf{r}=\alpha\mathbf{E}=\frac{e^2/m}{\omega_0^2-\omega^2-i\omega/\tau}\mathbf{E},$$

so $e\dot{\mathbf{r}}=i\omega\alpha\mathbf{E}$. Therefore,

$$\langle u_{mech}\rangle\;=\frac{1}{4}Nm(\dot{\mathbf{r}}\cdot\dot{\mathbf{r}}^*+\omega_0^2\mathbf{r}\cdot\mathbf{r}^*)=\frac{1}{4}N(m/e^2)(\omega^2+\omega_0^2)|\alpha|^2|\mathbf{E}|^2,$$

where

$$\frac{Nm}{4e^2}|\alpha|^2=\frac{\frac{1}{4}Ne^2/m}{(\omega_0^2-\omega^2)^2+\omega^2/\tau^2}=\frac{\frac{1}{4}\epsilon_0\omega_{\mathrm{p}}^2}{(\omega_0^2-\omega^2)^2+\omega^2/\tau^2}.$$

This gives

$$\langle u \rangle \;=\; \frac{1}{4}\epsilon_0|\mathbf{E}|^2\left\{1+|\hat{n}|^2+\frac{\omega_{\rm p}^2(\omega^2+\omega_0^2)}{(\omega_0^2-\omega^2)^2+\omega^2/\tau^2}\right\}.$$

On the other hand,

$$\hat{n}^2=\hat{n}_1^2-\hat{n}_2^2+2i\hat{n}_1\hat{n}_2=1+\frac{\omega_{\rm p}^2(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\omega^2/\tau^2}+i\frac{\omega_{\rm p}^2\omega/\tau}{(\omega_0^2-\omega^2)^2+\omega^2/\tau^2}.$$

Hence,

$$\begin{aligned}\langle u \rangle &= \frac{1}{4}\epsilon_0|\mathbf{E}|^2\left\{2\hat{n}_1^2-\frac{\omega_{\rm p}^2(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+\omega^2/\tau^2}+\frac{\omega_{\rm p}^2(\omega^2+\omega_0^2)}{(\omega_0^2-\omega^2)^2+\omega^2/\tau^2}\right\}\\ &= \frac{1}{4}\epsilon_0|\mathbf{E}|^2\left\{2\hat{n}_1^2+\frac{2\omega_{\rm p}^2\omega^2}{(\omega_0^2-\omega^2)^2+\omega^2/\tau^2}\right\}\\ &= \frac{1}{2}\epsilon_0|\mathbf{E}|^2\left\{\hat{n}_1^2+2\omega\tau\hat{n}_1\hat{n}_2\right\}\end{aligned}$$

as required.

(d) We have

$$\hat{n}^2=(\hat{n}_1+i\hat{n}_2)^2=\hat{n}_1^2-\hat{n}_2^2+2i\hat{n}_1\hat{n}_2=1+\frac{\omega_{\rm p}^2}{\omega_0^2-\omega^2-i\omega/\tau}.$$

Therefore,

$$\hat{n}_1^2-\hat{n}_2^2=1+\frac{\omega_{\rm p}^2\left(\omega_0^2-\omega^2\right)}{(\omega_0^2-\omega^2)^2+(\omega/\tau)^2}$$

$$2\hat{n}_1\hat{n}_2=\frac{\omega_{\rm p}^2\omega/\tau}{(\omega_0^2-\omega^2)^2+(\omega/\tau)^2}.$$

By definition,

$$\begin{aligned}\left(\frac{c}{v_{\rm E}}\right)^2 &= \hat{n}_1^2+4\omega\tau\hat{n}_1\hat{n}_2+4\omega^2\tau^2\hat{n}_2^2\\ &= \hat{n}_1^2-\hat{n}_2^2+4\omega\tau\hat{n}_1\hat{n}_2+(4\omega^2\tau^2+1)\hat{n}_2^2\\ &= 1+\frac{\omega_{\rm p}^2(\omega_0^2-\omega^2)}{(\omega_0^2-\omega^2)^2+(\omega/\tau)^2}+2\omega\tau\frac{\omega_{\rm p}^2\omega/\tau}{(\omega_0^2-\omega^2)^2+(\omega/\tau)^2}+(4\omega^2\tau^2+1)\hat{n}_2^2\\ &= 1+\frac{\omega_{\rm p}^2(\omega_0^2+\omega^2)}{(\omega_0^2-\omega^2)^2+(\omega/\tau)^2}+(4\omega^2\tau^2+1)\hat{n}_2^2.\end{aligned}$$

The second and third terms are positive so $v_E < c$ as required.

Source: R. Loudon, *Journal of Physics A* **3**, 233 (1970).

18.15 A Paramagnetic Microwave Amplifier

(a) When $\gamma = 0$ we have

$$\frac{dM_x}{dt} = -\frac{M_x}{\tau} \qquad \frac{dM_y}{dt} = -\frac{M_y}{\tau} \qquad \frac{dM_z}{dt} = -\frac{M_z - M}{\tau}.$$

The first two are solved by $M_x(t) = M_x(0)e^{-t/\tau}$ and $M_y(t) = M_y(0)e^{-t/\tau}$. The third is solved by $M_z(t) = M - [M - M_z(0)]e^{-t/\tau}$. Therefore, as $t \to \infty$, $M_x, M_y \to 0$ and $M_z \to M$. This is the equilibrium state.

(b) When $\gamma \neq 0$, the fact that $B_y = 0$ tells us that

$$\frac{dM_x}{dt} = \gamma M_y B_z - \frac{M_x}{\tau}$$

$$\frac{dM_y}{dt} = \gamma(M_z B_x - M_x B_z) - \frac{M_y}{\tau}$$

$$\frac{dM_z}{dt} = -\gamma M_y B_x - \frac{M_z - M}{\tau}.$$

Differentiate the $\dot{M}_x$ equation with respect to time and substitute in from the $\dot{M}_y$ equation to get

$$\frac{d^2 M_x}{dt^2} + \frac{1}{\tau}\frac{dM_x}{dt} = \gamma B_z \left[\gamma(M_z B_x - M_x B_z) - \frac{M_y}{\tau}\right].$$

Eliminate M_y in this equation using the original $\dot{M}_x$ equation to get

$$\frac{d^2 M_x}{dt^2} + \frac{2}{\tau}\frac{dM_x}{dt} + \frac{M_x}{\tau^2} + \gamma^2 B_z^2 M_x - \gamma^2 B_z M_z B_x.$$

Finally, use $\mathbf{B} = \mu_0(\mathbf{M} + \mathbf{H})$ to eliminate B_z and B_x in last two terms, respectively, and define $\omega_0^2 = \gamma^2 \mu_0 B_z H_z$ to get

$$\frac{d^2 M_x}{dt^2} + \frac{2}{\tau}\frac{dM_x}{dt} + \left(\omega_0^2 + 1/\tau^2\right) M_x = \omega_0^2 \frac{M_z}{H_z} H_x.$$

The fields in the z-direction are constant in time. Therefore, because $\mu = \mu_0(1 + \chi_M)$ and

$$M_x(t) = M_x e^{-i\omega t} = \chi(\omega) H_x(t) = \chi(\omega) H_x e^{-i\omega t},$$

we get

$$\hat{\mu}(\omega) = \mu_0 \left\{1 + \frac{\omega_0^2}{\omega_0^2 + 1/\tau^2 - \omega(\omega + 2i/\tau)} \frac{M_z}{H_z}\right\} = \hat{\mu}_1 + i\hat{\mu}_2.$$

Hence,

$$\hat{\mu}_1 = \mu_0 \left[1 + \frac{\omega_0^2(\omega_0^2 + 1/\tau^2 - \omega^2)}{(\omega_0^2 + 1/\tau^2 - \omega^2)^2 + 4\omega^2/\tau^2}\frac{M_z}{H_z}\right]$$

$$\hat{\mu}_2 = \mu_0 \left[\frac{\omega_0^2(2\omega/\tau)}{(\omega_0^2 + 1/\tau^2 - \omega^2)^2 + 4\omega^2/\tau^2}\frac{M_z}{H_z}\right].$$

(c) The Maxwell equations $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and $\nabla \times \mathbf{H} = \partial \mathbf{D}/\partial t$ in the magnet yield

$$\nabla^2 \mathbf{H} = \epsilon\hat{\mu}\partial^2 \mathbf{H}/\partial t^2 = -\hat{\mu}(\omega)\omega^2 \mathbf{H}$$

for time-harmonic solutions as assumed above. Indeed, $\mathbf{H} = \hat{\mathbf{x}}H_x \exp i(ky - \omega t)$ so $k^2 = \epsilon\hat{\mu}\omega^2$. With $k = k_1 + ik_2$, this implies that

$$k_1^2 - k_2^2 = \epsilon\hat{\mu}_1\omega^2$$

$$2k_1k_2 = \epsilon\hat{\mu}_2\omega^2.$$

For propagation in the $+y$-direction, $k_1 > 0$. Since $\epsilon > 0$, the second equation just above shows that k_2 has the same sign as $\hat{\mu}_2$. The definition of $\mu(\omega)$ given in part (b) shows that $\hat{\mu}_2$ has the same sign as M_z/H_z. Therefore, the **H**-wave decays (amplifies) exponentially as it propagates if M_z/H_z is positive (negative).

Source: A.M. Portis, *Electromagnetic Fields* (Wiley, New York, 1978).

18.16 Limits on the Photon Mass

Let the wave packet have central frequency ω_0 and width $\Delta\omega_0$. After a propagation distance $z = v_g t$, the time difference of interest is

$$\Delta t \approx \left.\frac{dt}{d\omega}\right|_0 \Delta\omega_0.$$

Now,

$$\frac{v_g}{c} = \frac{1}{c}\frac{d\omega}{dk} = \frac{1}{\sqrt{1 + (Mc/\hbar k)^2}} \approx 1 - \frac{1}{2}\left(\frac{mc^2}{\hbar\omega}\right)^2.$$

Therefore,

$$t = \frac{z}{v_g} \approx \frac{z}{c}\left[1 + \frac{1}{2}\left(\frac{mc^2}{\hbar\omega}\right)^2\right],$$

so

$$\frac{dt}{d\omega} \approx \frac{M^2c^3z}{\hbar^2\omega^3}.$$

We conclude that

$$\Delta t \approx \frac{z}{c}\frac{\Delta\omega_0}{\omega_0}\left(\frac{Mc^2}{\hbar\omega_0}\right)^2,$$

or, because it is surely true that $\Delta\omega_0/\omega_0 \approx 1$,

$$Mc^2 \approx (\hbar\omega_0)^2 \frac{\Delta t}{z/c}.$$

This shows that the smallest bound is obtained if the relevant data are collected at the lowest frequency, i.e., radio waves rather than X-rays.

18.17 Negative and Infinite Group Velocity

(a) Inside the medium, the wave propagates with phase speed c/n. When the wave emerges from the medium, it has phase speed c again, but it has accumulated a phase of $(\omega/c)(n-1)a$. Therefore,

$$E(z,t) = \begin{cases} E_0 \exp[i\omega(z/c - t)] & z < 0, \\ E_0 \exp[i\omega(nz/c - t)] & 0 < z < a, \\ E_0 \exp[i\omega na/c] \exp[-i\omega(t - (z-a)/c)] & z > a. \end{cases}$$

The slab field transforms to the post-slab field if $z \to a$ and $t \to t - (z-a)/c$.

(b) Let $n_0 = n(\omega_0)$ and insert the given group velocity approximation for $\omega n(\omega)$ into part (a) to get

$$E(z,t) = \begin{cases} E_0 \exp[i\omega z/c] \exp[-i\omega t] & z < 0, \\ E_0 \exp[i\omega_0 z(n_0/c - 1/v_{\rm g})] \exp[i\omega z/v_{\rm g}] \exp[-i\omega t] & 0 < z < a, \\ E_0 \exp[i\omega_0 a(n_0/c - 1/v_{\rm g})] \exp[i\omega(z/c - a/c - a/v_{\rm g})] \exp[-i\omega t] & z > a. \end{cases}$$

(c) We choose $\hat{E}(z,\omega) = \hat{A}(\omega)\exp(i\omega z/c)$ so

$$E(z<0,t) = f(z/c - t) = \int_{-\infty}^{\infty} d\omega\, \hat{A}(\omega) \exp(i\omega z/c) \exp(-i\omega t).$$

The values of $\hat{A}(\omega)$ are determined by $f(s)$. Therefore, for the two downstream regions, we simply replace the vacuum plane wave factor in (2) by the corresponding factors in (1). Only the ω-dependent factors remain inside the integral, so

$$E(z,t) = \begin{cases} f(z/c - t) & z < 0, \\ \exp[i\omega_0 z(n_0/c - 1/v_{\rm g})] f(z/v_{\rm g} - t) & 0 < z < a, \\ \exp[i\omega_0 a(n_0/c - 1/v_{\rm g})] f(z/c - a/c + a/v_{\rm g} - t)] & z > a. \end{cases}$$

(d) The field at $z = a$ is $\delta(a/v_g - t)$ just inside the back surface of the slab and $\delta(a/c - t - a/c + a/v_g)$ just outside the back surface of the slab. These agree that the delta function emerges from the slab at $t = a/v_g$. Since $v_g < 0$, the pulse emerges before it enters the slab at $t = 0$!

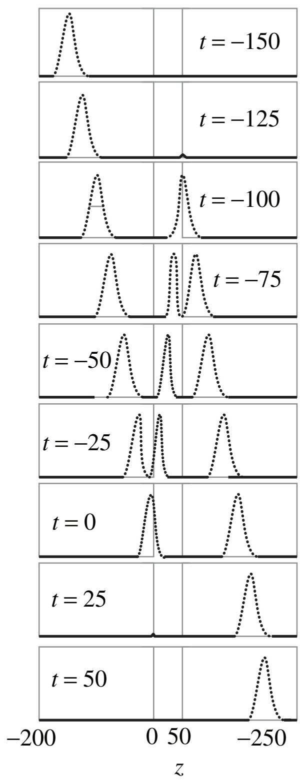
(e) Choose $f(s) = E_0 \exp(-s^2/2\tau^2)\exp(i\omega_0 s)$ and substitute into (3). There is some cancellation, and

$$E(z,t) = \begin{cases} E_0 \exp[-(z/c - t)^2/2\tau^2]\exp[i\omega_0(z/c - t)] & z < 0, \\ E_0 \exp[-(z/v_g - t)^2/2\tau^2]\exp[i\omega_0(n_0 z/c - t)] & 0 < z < a, \\ E_0 \exp[i\omega_0 a(n_0 - 1)/c]\exp[-(z/c - a/c + a/v_g - t)^2/2\tau^2]\exp[i\omega_0(z/c - t)] & \\ & z > a. \end{cases}$$

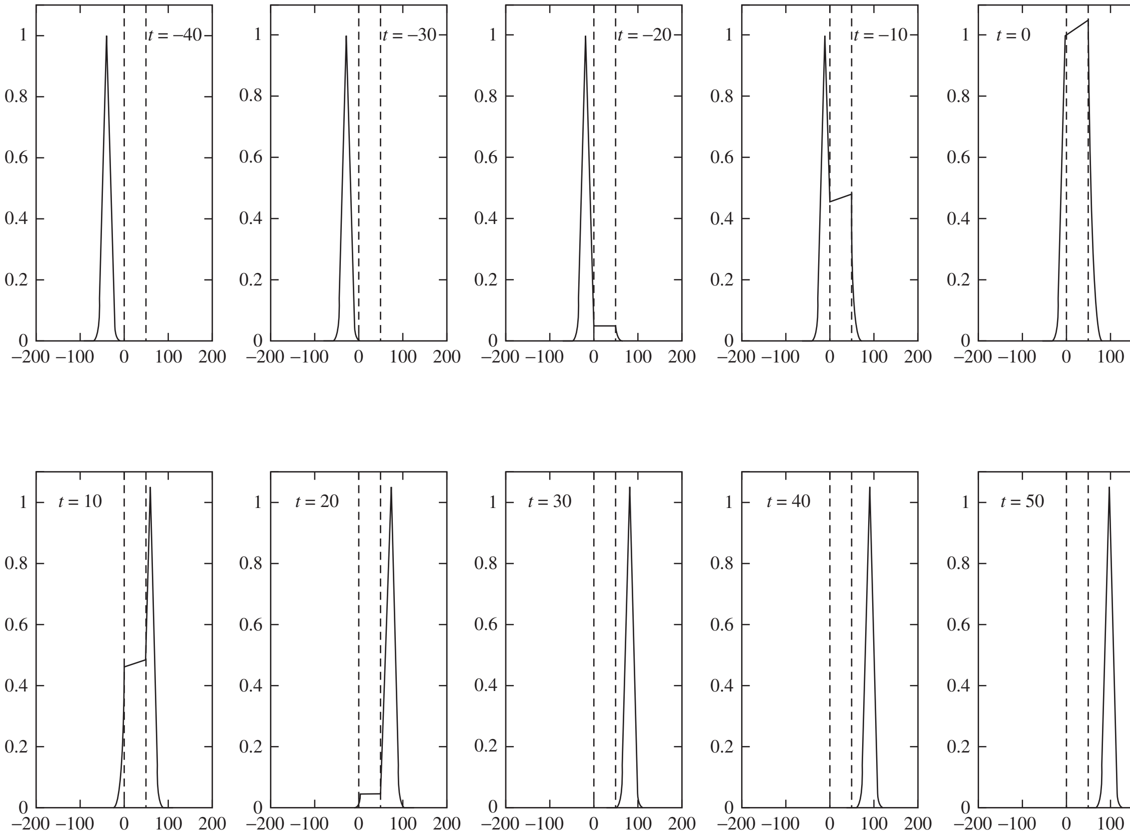
(f) Let $E_0 = 1$ and, as suggested, take out a common factor of $\exp[i\omega_0(z/c - t)]$ and write $\omega_0(n_0 - 1) = -i\epsilon$. In that case,

$$E(z,t) = \begin{cases} \exp[-(z/c - t)^2/2\tau^2] & z < 0, \\ \exp[\epsilon z/c]\exp[-(z/v_g - t)^2/2\tau^2] & 0 < z < a, \\ \exp[\epsilon a/c]\exp[-(z/c - a/c + a/v_g - t)^2/2\tau^2] & z > a. \end{cases}$$

The diagram below shows Gaussian wave packet propagation with a negative group velocity. Note that the "packet" does appear to move backward inside the medium, even as transmission proceeds.



(g) The diagram below shows Gaussian wave packet propagation when $v_g = \infty$ in the region between the vertical dashed lines. The meaning is simply that the field amplitude rises and falls uniformly throughout the slab "as if" it propagated through the slab infinitely quickly.



Source: K.T. McDonald, *American Journal of Physics* **69**, 607 (2001).

18.18 Parseval's Relation

(a) The first condition is satisfied because

$$\Delta(0) = \int_{-\infty}^{\infty} \frac{dy}{y^2} = 2\int_{0}^{\infty} \frac{dy}{y^2} \to \infty.$$

As for the second, we are told that

$$\chi'(x) = -\frac{1}{\pi}\int_{-\infty}^{\infty} dx' \frac{\chi''(x')}{x - x'} \qquad \text{and} \qquad \chi''(x) = \frac{1}{\pi}\int_{-\infty}^{\infty} dx' \frac{\chi'(x')}{x - x'}.$$

Substituting one into the other gives

$$\chi'(x) = -\frac{1}{\pi^2}\int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx'' \frac{1}{x-x'}\,\frac{\chi'(x'')}{x'-x''}.$$

Therefore,

$$\chi'(0) = \int_{-\infty}^{\infty} dx'' \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{dx'}{x'(x'-x'')}\chi'(x'').$$

This is the delta function filtering property if

$$\frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{dx'}{x'(x'-x'')} = \delta(x'').$$

(b) By direct computation,

$$\int_{-\infty}^{\infty} d\omega|\chi'(\omega)|^2 = \frac{1}{\pi^2}\int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' \frac{\chi''(\omega')}{\omega-\omega'}\int_{-\infty}^{\infty} d\omega'' \frac{\chi''(\omega'')}{\omega-\omega''}.$$

However, from part (a),

$$\delta(x-z) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{dy}{y(y-x+z)} = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \frac{ds}{(s-z)(s-x)}.$$

Therefore, as required,

$$\int_{-\infty}^{\infty} d\omega|\chi'(\omega)|^2 = \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega''\delta(\omega'-\omega'')\chi''(\omega')\chi''(\omega'') = \int_{-\infty}^{\infty} d\omega'|\chi''(\omega')|^2.$$

18.19 A Dispersive Dielectric

(a) Begin with $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = \epsilon_0\mathbf{E} + \gamma\nabla\times\mathbf{E}$. Inserting this into

$$\nabla\times\mathbf{H} = \frac{\partial\mathbf{D}}{\partial t}$$

gives

$$\nabla\times\mathbf{B} = \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} + \mu_0\gamma\nabla\times\frac{\partial\mathbf{E}}{\partial t}.$$

The time derivative of this equation, and Faraday's law, $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$, permit us to eliminate the magnetic field and get

$$-\nabla \times (\nabla \times \mathbf{E}) = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \gamma \nabla \times \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

We also know that

$$0 = \nabla \cdot \mathbf{D} = \epsilon_0 \nabla \cdot \mathbf{E} + \nabla \cdot \gamma \nabla \times \mathbf{E} = \epsilon_0 \nabla \cdot \mathbf{E}.$$

Therefore, because $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E}$, the propagation equation is

$$\nabla^2 \mathbf{E} = \frac{1}{c^2}\frac{\partial^2 \mathbf{E}}{\partial t^2} + \mu_0 \gamma \nabla \times \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

(b) Assume a plane wave $\mathbf{E} = \mathbf{E}_0 \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)]$, where $\mathbf{k} \cdot \mathbf{E}_0 = 0$. It that case, direct substitution gives

$$c^2 k^2 \mathbf{E}_0 = \omega^2 \mathbf{E}_0 + i\frac{\omega^2 \gamma}{\epsilon_0} \mathbf{k} \times \mathbf{E}_0.$$

Choose $\mathbf{k} = k\hat{\mathbf{z}}$ and write $\mathbf{E}_0 = a\hat{\mathbf{x}} + b\hat{\mathbf{y}}$. In that case, the foregoing becomes

$$(\omega^2 - c^2 k^2)a - (ik\omega^2 \gamma/\epsilon_0)b = 0$$

$$(ik\omega^2 \gamma/\epsilon_0)a + (\omega^2 - c^2 k^2)b = 0.$$

These linear equations have a solution if the determinant

$$\begin{vmatrix} \omega^2 - c^2 k^2 & -ik\omega^2 \gamma/\epsilon_0 \\ ik\omega^2 \gamma/\epsilon_0 & \omega^2 - c^2 k^2 \end{vmatrix} = 0.$$

The eigenfrequencies are

$$\omega_\pm^2 = \frac{c^2 k^2}{1 \pm k\gamma/\epsilon_0}.$$

Substituting back into the linear equation shows that the wave with frequency $\omega_\pm$ has polarization $\mathbf{E}_0 = a(\hat{\mathbf{x}} \pm i\hat{\mathbf{y}})$. These are RHC and LHC.

18.20 Lorentz-Model Sum Rule

The imaginary part of the Lorentz-model dielectric function is

$$\text{Im}\,\hat{\epsilon}(\omega) = \frac{\epsilon_0 \omega_{\rm p}^2 \omega \Gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2 \Gamma^2}.$$

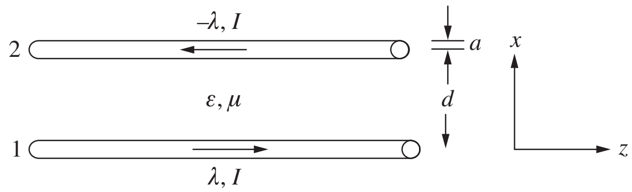
When Γ is small, the integral is dominated by contributions from $\omega \approx \omega_0$ and each of these can be substituted for the other (except when their difference is involved). Therefore,

$$
\begin{aligned}
\int_0^\infty d\omega\,\omega\,\mathrm{Im}\,\hat{\epsilon}(\omega) &= \epsilon_0\omega_\mathrm{p}^2\Gamma\int_0^\infty d\omega\,\frac{\omega^2}{(\omega_0+\omega)^2(\omega_0-\omega)^2+\omega^2\Gamma^2} \\
&\approx \epsilon_0\omega_\mathrm{p}^2\Gamma\int_0^\infty d\omega\,\frac{1}{4(\omega_0-\omega)^2+\Gamma^2} \\
&= \frac{1}{2}\epsilon_0\omega_\mathrm{p}^2\Gamma\int_0^\infty \frac{dx}{x^2+\Gamma^2} \\
&= \frac{1}{2}\epsilon_0\omega_\mathrm{p}^2\Gamma\,\tan^{-1}\,\frac{x}{\Gamma}\bigg|_0^\infty \\
&= \frac{\pi}{2}\epsilon_0\omega_\mathrm{p}^2.
\end{aligned}
$$

Chapter 19: Guided and Confined Waves

19.1 Two-Wire Transmission Line

The figure below shows the geometry of the line.



The capacitance per unit length is $\mathcal{C} = \lambda/(\varphi - \varphi_2)$ where $\varphi_1 - \varphi_2$ is the potential difference between the wires. From Gauss' law, the electric field between the wires in the plane of the wires is

$$\mathbf{E} = \frac{\lambda}{4\pi\epsilon}\left[\frac{1}{x} + \frac{1}{d-x}\right]\hat{\mathbf{x}}.$$

Therefore,

$$\varphi_1 - \varphi_2 = \int_a^{d-a} dxE = \frac{\lambda}{4\pi\epsilon}\int_a^{d-a} dx\left[\frac{dx}{x} + \frac{dx}{d-x}\right] = \frac{\lambda}{\pi\epsilon}\ln\left(\frac{d-a}{a}\right),$$

and

$$\mathcal{C} = \frac{\pi\epsilon}{\ln[(d-a)/a]}.$$

We suppose that the two wires are part of an infinite loop with wire length ℓ. The inductance per unit length is $\mathcal{L} = \Phi/I\ell$ where Φ is the total flux which passes between the wires in the plane of the wires. I is the current circulating in the loop. From Ampère's law, the magnetic field between the wires is

$$\Phi = \frac{\mu I}{2\pi}\left[\frac{1}{x} + \frac{1}{d-x}\right]\hat{\mathbf{y}}.$$

Therefore,

$$\Phi = \int d\mathbf{S}\cdot\mathbf{B} = \frac{\mu I\ell}{2\pi}\int_a^{d-a} dx\left[\frac{1}{x} + \frac{1}{d-x}\right] = \frac{\mu I\ell}{\pi}\ln[(d-a)/a],$$

and

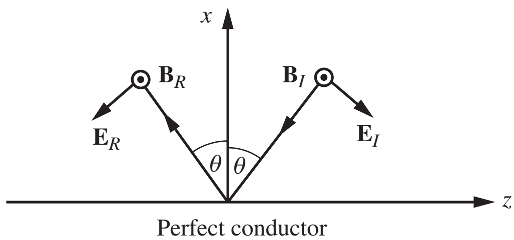
$$\mathcal{L} = \frac{\mu}{\pi}\ln[(d-a)/a].$$

These results confirm that

$$\mathcal{LC} = \mu\epsilon.$$

19.2 TM Wave Guided by a Flat Conductor

(a) The geometry defined by the statement of the problem is



We will *assume* that the magnitudes of $\mathbf{B}_I$ and $\mathbf{B}_R$ are equal and check that the boundary condition $\hat{\mathbf{n}} \times \mathbf{E}|_S = 0$ is satisfied. Hence, because the angle of incidence is equal to the angle of refraction and $\omega = ck_0$, the incident and reflected magnetic field vectors are

$$\mathbf{B}_I = B_0 \exp[-ik_0(x\cos\theta + z\sin\theta + \omega t)]\hat{\mathbf{y}}$$

and

$$\mathbf{B}_R = B_0 \exp[-ik_0(-x\cos\theta + z\sin\theta + \omega t)]\hat{\mathbf{y}}.$$

The total magnetic field is the sum of the two:

$$\mathbf{B} = 2B_0 \cos(k_0 x\cos\theta)\exp[-i(k_0 z\sin\theta + \omega t)]\hat{\mathbf{y}}.$$

To get the corresponding electric field, we use

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = -\frac{i\omega}{c^2}\mathbf{E} = \frac{\partial B_y}{\partial x}\hat{\mathbf{z}} - \frac{\partial B_y}{\partial z}\hat{\mathbf{x}}.$$

The result,

$$\mathbf{E} = 2icB_0[i\sin\theta\cos(k_0 x\cos\theta)\hat{\mathbf{x}} - \cos\theta\sin(k_0 x\cos\theta)\hat{\mathbf{z}}]\exp[-i(k_0 z\sin\theta + \omega t)],$$

satisfies the boundary condition $\hat{\mathbf{n}} \times \mathbf{E}|_S = E_z(x = 0) = 0$. Our solution is a TM wave.

(b) The time-averaged Poynting vector is

$$\langle \mathbf{S} \rangle = \frac{1}{2\mu_0}\mathrm{Re}(\mathbf{E} \times \mathbf{B}^*),$$

or

$$\begin{aligned}\langle \mathbf{S} \rangle &= \mathrm{Re}\left[\frac{2icB_0^2}{\mu_0}\cos(k_0 x\cos\theta)\left[\cos\theta\sin(k_0 x\cos\theta)\hat{\mathbf{x}} + i\sin\theta\cos(k_0 x\cos\theta)\hat{\mathbf{z}}\right]\right] \\ &= -\frac{2cB_0^2}{\mu_0}\cos^2(k_0 x\cos\theta)\sin\theta\hat{\mathbf{z}}.\end{aligned}$$

(c) The induced surface charge density is

$$\sigma = \epsilon_0 \hat{\mathbf{n}} \cdot \mathbf{E}|_S = \epsilon_0 E_x(x=0) = -2\epsilon_0 c B_0 \sin\theta \exp[-i(k_0 z \sin\theta + \omega t)].$$

The induced surface current density is

$$\mathbf{K} = \frac{1}{\mu_0}\hat{\mathbf{n}} \times \mathbf{B}|_S = \frac{1}{\mu_0} B_y(x=0) = \frac{2B_0}{\mu_0} \exp[-i(k_0 z \sin\theta + \omega t)].$$

For our geometry, the surface divergence has only one component,

$$\nabla_S \cdot \mathbf{K} = \frac{\partial K}{\partial z} = -ik_0 \sin\theta \frac{2B_0}{\mu_0} \exp[-i(k_0 z \sin\theta + \omega t)].$$

Moreover,

$$\frac{\partial \sigma}{\partial t} = ik_0 \sin\theta c^2 \epsilon_0 2B_0 \exp[-i(k_0 z \sin\theta + \omega t)].$$

Therefore, because $\mu_0 \epsilon_0 c^2 = 1$,

$$\nabla_S \cdot \mathbf{K} + \frac{\partial \sigma}{\partial t} = 0.$$

19.3 TEM Waves Guided by a Cone and a Plane

(a) For time-harmonic sources, the Maxwell curl equations are $\nabla \times \mathbf{E} = i\omega \mathbf{B}$ and $\nabla \times \mathbf{B} = -i(\omega/c^2)\mathbf{E}$. Using the information given, the components of these vector equations in a spherical coordinate system are

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_\phi)\right] = 0 \qquad \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta B_\phi)\right] = 0$$

$$-\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi) = i\omega B_\theta \qquad -\frac{1}{r}\frac{\partial}{\partial r}(rB_\phi) = -i\frac{\omega}{c^2}E_\theta$$

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\theta) = i\omega B_\phi \qquad \frac{1}{r}\frac{\partial}{\partial r}(rB_\theta) = -i\frac{\omega}{c^2}E_\phi.$$

From the bottom two lines, we see that there are two independent classes of solutions where the non-zero components of the fields are either (E_ϕ, B_θ) or (E_θ, B_ϕ).

(b) The Maxwell divergence equations read

$$\frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta E_\theta)\right] = 0 \qquad \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta B_\theta)\right] = 0.$$

These two equations have the same structure as the two equations in the first line of the curl equations in part (a). By direct integration, they show that

$$E_\theta(r,\theta) = \frac{E_\theta(r)}{\sin\theta} \qquad E_\phi(r,\theta) = \frac{E_\phi(r)}{\sin\theta} \qquad B_\theta(r,\theta) = \frac{B_\theta(r)}{\sin\theta} \qquad B_\phi(r,\theta) = \frac{B_\phi(r)}{\sin\theta}.$$

Substituting these back into the remaining curl equations gives

$$\frac{d^2}{dr^2}(rE_\theta) + \frac{\omega^2}{c^2}(rE_\theta) = 0 \qquad\qquad \frac{d^2}{dr^2}(rB_\theta) + \frac{\omega^2}{c^2}(rB_\theta) = 0.$$

Therefore,

$$E_\theta(r,\theta) = \frac{Ee^{i\omega r/c} + E'e^{-i\omega/c}}{r\sin\theta} \qquad\qquad cB_\theta(r,\theta) = \frac{Be^{i\omega r/c} + B'e^{-i\omega/c}}{r\sin\theta}.$$

The partner fields follow by integrating one curl equation:

$$cB_\phi(r,\theta) = \frac{Ee^{i\omega r/c} - E'e^{-i\omega/c}}{r\sin\theta} \qquad\qquad E_\phi(r,\theta) = -\frac{Be^{i\omega r/c} - B'e^{-i\omega/c}}{r\sin\theta}.$$

(c) To provide wave guiding, the electric field must be entirely normal to both metal surfaces. This is the case with the set (E_θ, B_ϕ) but not with the set (E_ϕ, B_θ).

Source: S.A. Schelkunoff, *Electromagnetic Waves* (Van Nostrand, New York, 1943).

19.4 The Lowest Propagating Mode of a Waveguide

(a) For a quantum particle-in-a-box, the integrated curvature of ψ is the total (kinetic) energy of the particle. A simple sketch shows that this quantity is unavoidably greater for the ground state of a particle whose wave function must go to zero at the box boundary than for the ground state of particle whose wave function need only approach the wall with zero slope. We infer from this that the lowest propagating mode of a hollow-tube waveguide will be TE.

(b) Consider a variational solution of the Helmholtz equation for the drumhead. The greater the constraints on the solution, the higher the energy will be. An elastic membrane sitting on top of a hollow cylindrical support with the same shape is less constrained than a membrane whose edges are tacked down onto the support. We infer from this that the TE mode of the waveguide will have the lowest cutoff frequency.

Source: E.T. Kornhauser and I. Stakgold, *Journal of Mathematics and Physics* **32**, 45–57 (1952).

19.5 Semi-Circular Waveguide

(a) Let us start with the modes of a waveguide with a circular cross section. The modes of the semi-circular waveguide are a subset of these modes. Our task is to solve the Helmholtz equation in plane polar coordinates,

$$\left[\nabla^2 + \gamma^2\right]\psi = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\omega^2}{c^2}\psi,$$

with the boundary condition $\psi|_S = E_z|_S = 0$ for TM and $\partial\psi/\partial n|_S = \partial B_z/\partial n|_S = 0$ for TE. Separation of variables with $\psi(r,\theta) = R(r)\Theta(\theta)$ and a separation constant m^2 produces two ordinary differential equations:

$$\frac{d^2\Theta}{d\theta^2} + m^2\Theta = 0$$

$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} + \left(\gamma^2 r^2 - m^2\right) = 0.$$

The first has linearly independent solutions $\sin(m\theta)$ and $\cos(m\theta)$. The second is Bessel's equation. The solution must be regular at $r = 0$, so the only possible solution is $J_m(\gamma r)$. We conclude that

$$\psi(r,\theta) = AJ_m(\gamma r) \times \begin{cases} \sin(m\theta) \\ \cos(m\theta). \end{cases} \tag{19.1}$$

For TM modes of the circular waveguide, we need $J_m(\gamma R) = 0$, which will be the case if we choose $\gamma_{mn} = x_{mn}/R$ where x_{mn} is the n^{th} zero of $J_m(x)$. For the semi-circular waveguide, we must ensure that $\psi = 0$ when $\theta = 0$ and when $\theta = \pi$. This will be true if we choose the sine functions in (19.1) with $m > 0$. Hence, the TM longitudinal electric fields that can exist in a semi-circular waveguide are

$$E_z(r,\theta) = E_0 J_m(\gamma_{mn} r)\sin(m\theta) \qquad m = 1, 2, \ldots \qquad n = 1, 2, \ldots$$

For a waveguide where the fields vary as $\exp[i(hz - \omega t)]$, the mode frequencies satisfy

$$\frac{\omega^2}{c^2} = \gamma^2 + h^2.$$

The cutoff frequency corresponds to $h = 0$. Therefore, the cutoff frequencies of the semi-circular waveguide are

$$\omega_{mn} = c\gamma_{mn} = c\frac{x_{mn}}{R}.$$

For TE modes of the circular waveguide, we choose $\gamma'_{mn} = x'_{mn}/R$ where x'_{mn} is the n^{th} zero of $J'_m(x)$. For the semi-circular waveguide, we note that the normal to the flat wall points in the $\hat{\boldsymbol{\theta}}$ direction. Therefore, the new boundary condition is $\partial\psi/\partial\theta = 0$ when $\theta = 0$ and when $\theta = \pi$. This will be true if we choose the $\cos(m\theta)$ functions in (19.1). Hence, the TE longitudinal magnetic fields that can exist in a semi-circular waveguide are

$$B_z(r,\theta) = B_0 J_m(\gamma'_{mn} r)\cos(m\theta) \qquad m = 0, 1, \ldots \qquad n = 1, 2, \ldots$$

The cutoff frequencies for these modes are

$$\omega'_{mn} = c\gamma'_{mn} = c\frac{x'_{mn}}{R}.$$

(b) If we look up the zeroes of $J_m(x)$ and $J'_m(x)$, we find that the smallest is $x'_{11} = 1.841$. Therefore, we need to evaluate the transverse fields of the TE_{11} mode. These are

$$\mathbf{B}_\perp = \frac{ih}{[\gamma'_{11}]^2}\nabla_\perp B_z = \frac{ihR^2}{[x'_{11}]^2}B_0\left(\hat{\boldsymbol{\rho}}\frac{x'_{11}}{R}J'_1(\gamma_{11}r/R)\cos\theta - \frac{\hat{\boldsymbol{\theta}}}{r}J_1(\gamma'_{11}r/R)\sin\theta\right)$$

$$\mathbf{E}_\perp = -\frac{\omega'_{11}}{h}\hat{\mathbf{z}}\times\mathbf{B}_\perp = -\frac{icR}{x'_{11}}B_0\left(\hat{\boldsymbol{\theta}}\frac{x'_{11}}{R}J'_1(\gamma_{11}r/R)\cos\theta + \frac{\hat{\boldsymbol{\rho}}}{r}J_1(\gamma'_{11}r/R)\sin\theta\right).$$

19.6 Whispering Gallery Modes

(a) Our general discussion of waveguide modes began with

$$\mathbf{E}(\mathbf{r},t) = [\mathbf{E}_\perp(\mathbf{r}_\perp) + \hat{\mathbf{z}}E_z(\mathbf{r}_\perp)]\exp[i(hz-\omega t)]$$

and

$$\mathbf{H}(\mathbf{r},t) = [\mathbf{H}_\perp(\mathbf{r}_\perp) + \hat{\mathbf{z}}H_z(\mathbf{r}_\perp)]\exp[i(hz-\omega t)].$$

Substituting these into the Maxwell equations produced

$$\mathbf{H}_\perp = \frac{ih}{\gamma^2}\nabla_\perp H_z + \frac{i\omega\epsilon}{\gamma^2}\hat{\mathbf{z}}\times\nabla_\perp E_z$$

and

$$\mathbf{E}_\perp = \frac{ih}{\gamma^2}\nabla_\perp E_z - \frac{i\omega\mu}{\gamma^2}\hat{\mathbf{z}}\times\nabla_\perp H_z,$$

where

$$\gamma^2 = \mu\epsilon\omega^2 - h^2$$

and

$$\left[\nabla_\perp^2 + \gamma^2\right]\left\{\begin{array}{c} E_z \\ H_z \end{array}\right\} = 0.$$

For the present problem, we set $h=0$ to eliminate the z-dependence. Therefore, TE solutions are characterized by

$$E_z = 0 \qquad \left[\nabla_\perp^2 + \gamma^2\right]H_z = 0 \qquad \mathbf{E}_\perp = -\frac{i\omega\mu}{\gamma^2}\hat{\mathbf{z}}\times\nabla_\perp H_z$$

and TM solutions are characterized by

$$H_z = 0 \qquad \left[\nabla_\perp^2 + \gamma^2\right]E_z = 0 \qquad \mathbf{H}_\perp = \frac{i\omega\epsilon}{\gamma^2}\hat{\mathbf{z}}\times\nabla_\perp E_z.$$

This obliges us to study the Helmholtz equation in two-dimensional polar coordinates:

$$[\nabla^2 + \gamma^2]\left\{\begin{array}{c} E_z \\ H_z \end{array}\right\} = \left[\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \gamma^2\right]\left\{\begin{array}{c} E_z \\ H_z \end{array}\right\} = 0.$$

The text shows that the solutions which satisfy perfect-conductor boundary conditions and which are regular at the center of the tube are

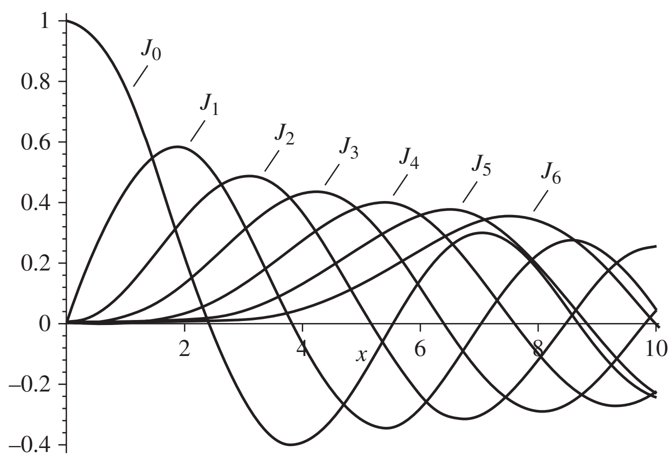
$$E_z(\rho,\phi) = E_0 J_m(\gamma_{mn}^{\rm TM}\rho)\exp[i(m\phi-\omega t)]$$

and

$$H_z(\rho,\phi) = H_0 J_m(\gamma_{mn}^{\rm TE}\rho)\exp[i(m\phi-\omega t)],$$

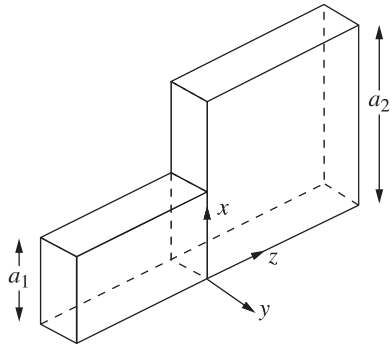
where $\gamma_{mn}^{\rm TE}R$ is the $n^{\rm th}$ zero of $J_m(x)$ and $\gamma_{mn}^{\rm TM}R$ is the $n^{\rm th}$ zero of $J'_m(x)$. These are circumferentially propagating modes because the phase is constant when ϕ increases linearly with t.

(b) The sketch below shows $J_m(x)$ for the first few values of m. The trend is clear: the weight of the function moves to larger values of x as m increases. Hence, we choose $m \gg 1$ and then γR so the first zero of the Bessel function coincides with the tube boundary. Every solution of this type has $|\mathbf{E}(\rho,\phi)| > 0$ everywhere and has almost all its weight concentrated in the immediate vicinity of $\rho = R$.



19.7 Waveguide Discontinuity

The waveguide geometry described is as follows.



For a TE_{m0} mode in waveguide 1, we have $H_z \propto \cos(m\pi x/a_1)$. The continuity of the tangential component of **E** shows that only TE_{m0} modes will propagate in waveguide 2 because the absence of y-dependence in guide 1 cannot generate y-dependence in guide 2. Our task, then, is to find the expansion coefficients H_m so

$$\sum_{m=1}^{\infty} H_m \cos\left(\frac{m\pi x}{a_2}\right) = \begin{cases} H\cos(\pi x/a_1) & 0 \le x \le a_1, \\ 0 & a_1 < x \le a_2. \end{cases}$$

This is a job for the orthogonality of the cosine functions and we find

$$\begin{aligned} H_m &= \frac{2H}{a_1}\int_0^{a_1} dx\, \cos(\pi x/a_1)\cos(m\pi x/a_2) \\ &= \frac{2Ha_2}{\pi(a_2 - ma_1)}\sin[\pi(1 - ma_1/a_2)] - \frac{2Ha_2}{\pi(a_2 + ma_1)}\sin[\pi(1 + ma_1/a_2)]. \\ &= \frac{2Hma_1a_2}{\pi(a_2^2 - m^2a_1^2)}\sin\left(m\pi\frac{a_1}{a_2}\right). \end{aligned}$$

When $a_1 = a_2$, $H_m = 0$ for $m \neq 1$. When $m = 1$, l'Hospital's rule gives the expected answer,

$$H_1 = \lim_{m\to 1}\frac{d}{dm}\frac{2H}{\pi}\frac{\sin(m\pi)}{(1-m^2)} = \lim_{m\to 1}\frac{2\pi H\cos(m\pi)}{-2m\pi} = H.$$

Source: C.G. Someda, *Electromagnetic Waves* (CRC Press, Boca Raton, FL, 2006).

19.8 A Vector-Potential Method

(a) We use $\nabla = \nabla_\perp + \hat{\mathbf{z}}\partial/\partial z$ and first compute the magnetic field:

$$\mathbf{B} = \nabla\times\mathbf{A} = -\hat{\mathbf{z}}\times\left(\nabla_\perp + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right)\left[A(\mathbf{r}_\perp)e^{i(hz-\omega t)}\right] = -\left[\hat{\mathbf{z}}\times\nabla_\perp A(\mathbf{r})\right]e^{i(hz-\omega t)}.$$

The text defines $\gamma^2 = \mu\epsilon\omega^2 - h^2$ and reports

$$\mathbf{H}_{\text{TM}} = \frac{i\omega\epsilon}{\gamma^2}\left[\hat{\mathbf{z}}\times\nabla_\perp E_z\right]e^{i(hz-\omega t)}.$$

This agrees with our calculation if

$$A = -\frac{i\omega\epsilon\mu}{k\gamma^2}E_z.$$

If so, it is necessary that $[\nabla_\perp^2 + \gamma^2]A = 0$. We confirm this by writing out the wave equation for $\mathbf{A}(\mathbf{r}, t)$ explicitly:

$$
\begin{aligned}
0 &= \left[\nabla^2 - \mu\epsilon\frac{\partial^2}{\partial t^2}\right]\mathbf{A}(\mathbf{r},t) \\
&= \left[\nabla_\perp + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right]\cdot\left[\nabla_\perp + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right]\hat{\mathbf{z}}A(\mathbf{r}_\perp)e^{i(hz-\omega t)} + \mu\epsilon\omega^2\mathbf{A} \\
&= \left[\nabla_\perp + \hat{\mathbf{z}}\frac{\partial}{\partial z}\right]\cdot\left[\nabla_\perp A + ihA\hat{\mathbf{z}}\right]e^{i(hz-\omega t)} + \mu\epsilon\omega^2\mathbf{A} \\
&= \left[\nabla_\perp^2 - h^2 + \mu\epsilon\omega^2\right]Ae^{i(hz-\omega t)}\hat{\mathbf{z}} \\
&= \left[\nabla_\perp^2 + \gamma^2\right]\mathbf{A}.
\end{aligned}
$$

We calculate the electric field from $\nabla\times\mathbf{B} = \epsilon\mu\partial\mathbf{E}/\partial t$. Specifically,

$$
\begin{aligned}
\mathbf{E} &= \frac{i}{\mu\epsilon\omega}\nabla\times\mathbf{B} \\
&= \frac{i}{\mu\epsilon\omega}\left[(\hat{\mathbf{z}}\cdot\nabla)\nabla_\perp Ae^{i(hz-\omega t)} - \hat{\mathbf{z}}\nabla\cdot\nabla_\perp Ae^{i(hz-\omega t)}\right] \\
&= \frac{i}{\mu\epsilon\omega}\left[ih\nabla_\perp Ae^{i(hz-\omega t)} - \hat{\mathbf{z}}\nabla_\perp^2 Ae^{i(hz-\omega t)}\right] \\
&= \frac{i}{\mu\epsilon\omega}\left[ih\nabla_\perp A + \hat{\mathbf{z}}\gamma^2 A\right]e^{i(hz-\omega t)} \\
&= \left[\frac{ih}{\gamma^2}\nabla_\perp E_z + \hat{\mathbf{z}}E_z\right]e^{i(hz-\omega t)}.
\end{aligned}
$$

This agrees exactly with $\mathbf{E}_{\rm TM}$ reported in the text.

(b) The fact that $\mathbf{E}_{\rm TE} = \nabla\times\tilde{\mathbf{A}}$ is not a problem because $\nabla\cdot\mathbf{E} = 0$ for the fields inside a waveguide.

19.9 Waveguide Filters

A mode transmits if its electric field is normal to every conducting surface, including the wire screen. This boundary condition is satisfied automatically by a longitudinal electric field. Otherwise, the transmitted electric field lines must be purely radial for guide (a) and purely circumferential for guide (b). The former satisfies $\nabla\times\mathbf{E}_\perp = 0$ (because it is like the field of a point charge except that the field vanishes at the origin) and the latter satisfies $\nabla\times\mathbf{E}_\perp \neq 0$ (because $\oint d\boldsymbol{\ell}\cdot\mathbf{E} \neq 0$ around a closed field line). Therefore, since $\nabla\times\mathbf{E}_\perp = i\omega B_z\hat{\mathbf{z}}$, guide (a) can only transmit a mode where $B_z = 0$, i.e., a TM mode. Conversely, guide (b) can only transmit a mode where $B_z \neq 0$, i.e., a TE mode. In fact, the modes in question are $\rm TE_{01}$ and $\rm TM_{01}$.

Source: C.G. Montgomery, R.H. Dicke, and E.M. Purcell, *Principles of Microwave Circuits* (Boston Technical Lithographers, Lexington, MA, 1963).

19.10 Waveguide Mode Orthogonality

(a) We are given

$$\nabla_\perp^2 \psi_p = \lambda_p \psi_p$$

$$\nabla_\perp^2 \psi_q = \lambda_q \psi_q .$$

Multiply the top line by ψ_q , multiply the bottom line by ψ_p , subtract, and integrate over A. The result is

$$(\lambda_q - \lambda_p) \int\limits_A d^2r\, \psi_p \psi_q = \int\limits_A d^2r\, \left\{ \psi_q \nabla_\perp^2 \psi_p - \psi_p \nabla_\perp^2 \psi_q \right\}.$$

Applying Green's second identity to the right-hand side gives

$$(\lambda_q - \lambda_p) \int\limits_A d^2r\, \psi_p \psi_q = \oint d\ell \left\{ \psi_q \hat{\mathbf{n}} \cdot \nabla \psi_p - \psi_p \hat{\mathbf{n}} \cdot \nabla \psi_q \right\} = 0.$$

The zero on the far right-hand side follows from the assumed boundary conditions for ψ_p and ψ_q.

(b) For TE modes,

$$\mathbf{B}_p = ih_p \nabla_\perp \psi_p + \hat{\mathbf{z}} \gamma_p^2 \psi_p$$

$$\mathbf{E}_p = -(\omega/h_p) \hat{\mathbf{z}} \times \mathbf{B}_p ,$$

where

$$(\nabla_\perp^2 + \gamma_p^2) \psi_p = (\nabla_\perp^2 + \omega^2/c^2 - h_p^2) \psi_p = 0 \qquad \text{and} \qquad \hat{\mathbf{n}} \cdot \nabla \psi_p |_C = 0.$$

Therefore, if C is the perimeter curve of A, Green's first identity and the boundary condition on the wall give

$$\begin{aligned}
\int\limits_A d^2r\, \mathbf{B}_p \cdot \mathbf{B}_q &= -h_p h_q \int\limits_A d^2r\, \nabla_\perp \psi_p \cdot \nabla_\perp \psi_q + \gamma_p^2 \gamma_q^2 \int\limits_A d^2r\, \psi_p\, \psi_q \\
&= h_p h_q \int\limits_A d^2r\, \psi_p \nabla^2 \psi_q - h_p h_q \oint\limits_C d\ell\, \psi_p \hat{\mathbf{n}} \cdot \nabla_\perp \psi_q + \gamma_p^2 \gamma_q^2 \int\limits_A d^2r\, \psi_p \psi_q \\
&= \gamma_q^2 (\gamma_p^2 - h_p h_q) \int\limits_A d^2r\, \psi_p \psi_q \\
&= 0.
\end{aligned}$$

The last line follows from part (a) if $\gamma_p^2 \neq \gamma_q^2$. Similarly,

$$\begin{aligned}
\int_A d^2r\,\mathbf{E}_p\cdot\mathbf{E}_q &= \frac{\omega^2}{h_p h_q}\int_A d^2r\;\{\hat{\mathbf{z}}\times\mathbf{B}_p\}\cdot\{\hat{\mathbf{z}}\times\mathbf{B}_q\} \\
&= \frac{\omega^2}{h_p h_q}\int_A d^2r\;\{\mathbf{B}_p\cdot\mathbf{B}_q-(\hat{\mathbf{z}}\cdot\mathbf{B}_p)(\hat{\mathbf{z}}\cdot\mathbf{B}_q)\} \\
&= -\omega^2\int_A d^2r\,\nabla_\perp\psi_p\cdot\nabla_\perp\psi_q \\
&= -\omega^2\gamma_p^2\int_A d^2r\,\psi_p\psi_q \\
&= 0.
\end{aligned}$$

The last line follows from part (a) if $\gamma_p^2\neq\gamma_q^2$.

(c) For TM modes, the boundary condition for the Helmholtz equation is $\psi_p|_C=0$, and

$$\mathbf{E}_p = h_p\nabla_\perp\psi_p+\hat{\mathbf{z}}\gamma_p^2\psi_p$$

$$\mathbf{B}_p = \frac{\omega}{c^3 h_p}\hat{\mathbf{z}}\times\mathbf{E}_p.$$

Exactly the same string of arguments shows that the TM electric and TM magnetic fields will be orthogonal if the conditions of part (a) are met. We need not repeat everything because duality guarantees that the algebra will be the same.

(d) For the TM-TE case, we have

$$\begin{aligned}
\int_A d^2r\,\mathbf{E}_q^{\rm TM}\cdot\mathbf{E}_p^{\rm TE} &= -i\omega h_p\int_A d^2r\,\nabla_\perp\psi_q^{\rm TM}\cdot(\hat{\mathbf{z}}\times\nabla_\perp\psi_p^{\rm TE}) \\
&= -i\omega h_p\hat{\mathbf{z}}\cdot\int_A d^2r\,\nabla_\perp\psi_p^{\rm TE}\times\nabla_\perp\psi_q^{\rm TM}. \qquad (19.2)
\end{aligned}$$

Using $\nabla\times(f\,\mathbf{g})=\nabla f\times\mathbf{g}+f\,\nabla\times\mathbf{g}$, we write the equation just above in the form

$$\int_A d^2r\,\mathbf{E}_q^{\rm TM}\cdot\mathbf{E}_p^{\rm TE}=\omega k_p\hat{\mathbf{z}}\cdot\int_A d^2r\;\left[\psi_q^{\rm TM}\nabla_\perp\times(\nabla_\perp\psi_p^{\rm TE})-\nabla_\perp\times(\psi_q^{\rm TM}\nabla_\perp\psi_p^{\rm TE})\right].$$

On the other hand,

$$\nabla_\perp\times(\nabla_\perp\psi_p^{\rm TE})=\hat{\mathbf{z}}(\partial_x\partial_y-\partial_y\partial_x)\psi_p^{\rm TE}=0.$$

Therefore,

$$\begin{aligned}
\int_A d^2r\,\mathbf{E}_q^{\rm TM}\cdot\mathbf{E}_p^{\rm TE} &= -\omega h_p\hat{\mathbf{z}}\cdot\int_A d^2r\,\nabla_\perp\times(\psi_q^{\rm TM}\nabla_\perp\psi_p^{\rm TE}) \\
&= -\omega h_p\hat{\mathbf{z}}\cdot\oint d\ell\,\hat{\mathbf{n}}\times\left[\psi_q^{\rm TM}\nabla_\perp\psi_p^{\rm TE}\right] \\
&= 0.
\end{aligned}$$

The last line follows because $\psi_p^{\rm TM}$ vanishes on the boundary. There is no requirement that γ_p and γ_q be unequal. Finally,

$$\begin{aligned}
\int_A d^2r\, \mathbf{B}_p^{\rm TE}\cdot\mathbf{B}_q^{\rm TM} &= \frac{\omega}{c^2 h_q}\int_A d^2r\, \mathbf{B}_p^{\rm TE}\cdot(\hat{\mathbf{z}}\times\mathbf{E}_q^{\rm TM}) \\
&= \frac{\omega}{c^2 h_q}\int_A d^2r\, \left\{ih_p\nabla_\perp\psi_p^{\rm TE}+\hat{\mathbf{z}}\gamma_p^2\psi_p^{\rm TE}\right\}\cdot\left\{ih_q\hat{\mathbf{z}}\times\nabla_\perp\psi_q^{\rm TM}\right\} \\
&= \frac{\omega h_p}{c^2}\hat{\mathbf{z}}\cdot\int_A d^2r\,\nabla_\perp\psi_p^{\rm TE}\times\nabla_\perp\psi_q^{\rm TM} \\
&= 0.
\end{aligned}$$

We get zero because the final integral is the same as appeared in (1).

19.11 A Waveguide with a Bend

(a) We check each free-space Maxwell equation in turn, making repeated use of the fact that $\Phi=\Phi(y,z)$:

$$\nabla\cdot\mathbf{E} = i\frac{\omega}{c}\frac{\partial\Phi}{\partial x}=0$$

$$c\nabla\cdot\mathbf{B} = -\nabla\cdot(\hat{\mathbf{x}}\times\nabla\Phi)=\hat{\mathbf{x}}\cdot\nabla\times\nabla\Phi-\nabla\Phi\cdot\nabla\times\hat{\mathbf{x}}=0$$

$$\nabla\times\mathbf{E}+\frac{\partial\mathbf{B}}{\partial t} = i\frac{\omega}{c}\nabla\times(\hat{\mathbf{x}}\Phi)+i\frac{\omega}{c}\hat{\mathbf{x}}\times\nabla\Phi=0$$

$$\nabla\times c\mathbf{B}-\frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = -\nabla\times(\hat{\mathbf{x}}\times\nabla\Phi)-\hat{\mathbf{x}}\frac{\omega^2}{c^2}\Phi=-\hat{\mathbf{x}}\left[\nabla^2+\frac{\omega^2}{c^2}\right]\Phi=0.$$

On the top and bottom walls, $\hat{\mathbf{n}}\times\mathbf{E}\propto\pm\hat{\mathbf{x}}\times\hat{\mathbf{x}}=0$ so the boundary condition is satisfied. On the side walls, $\hat{\mathbf{n}}\times\mathbf{E}\propto\pm(\hat{\mathbf{y}}\times\hat{\mathbf{x}})\Phi=0$ if $\Phi(y=0)=\Phi(y=a)=0$ is imposed as a boundary condition on the Helmholtz equation for $\Phi(y,z)$.

(b) Separating variables in the Helmholtz equation gives $\Phi(y,z)=\sin(\pi ny/a)\psi(z)$, where $n=1,2,\ldots$ and

$$\frac{d^2\psi}{dz^2}+\left[\frac{\omega^2}{c^2}-\frac{\pi^2n^2}{a^2}+\frac{\kappa^2(z)}{2}\right]\psi=0.$$

The curvature $\kappa(z)=0$ in the straight portion of the guide so, choosing $n=1$, we get a propagating (sinusoidal) solution if $\omega>\pi c/a$.

(c) In the curved portion of the guide, the Helmholtz equation is a one-dimensional Schrödinger-like equation,

$$-\frac{d^2\psi}{dz^2} + V(z)\psi = E\psi,$$

with $V(z) = -\frac{1}{2}\kappa^2(z)$ and $E = \dfrac{\omega^2}{c^2} - \dfrac{\pi^2 n^2}{a^2}$.

The potential is attractive so there is guaranteed to be at least one bound state with $E < 0$, that is, with $\omega < \pi c/a$. The corresponding spatial mode is *localized* near the bend. Like a particle-in-a-finite-well wave function, its amplitude is large in the bend and falls exponentially to zero when the bend straightens out.

Source: J. Goldstone and R.L. Jaffe, *Physical Review B* **45**, 14100 (1992).

19.12 TE and TM Modes of a Coaxial Waveguide

Let $\gamma^2 = \omega^2/c^2 - h^2$. The Helmholtz equation is

$$\left[\nabla_\perp^2 + \gamma^2\right]\psi = 0. \tag{1}$$

For TE modes, we solve (1) with the boundary condition $\partial\psi_m/\partial n|_S = 0$ and construct

$$\begin{aligned} \mathbf{B}_{\rm TE} &= \left[\frac{ih}{\gamma^2}\nabla_\perp + \hat{\mathbf{z}}\right]\psi_m e^{i(hz-\omega t)} \\ \mathbf{E}_{\rm TE} &= -v_{\rm p}\hat{\mathbf{z}} \times \mathbf{B}_{TE}. \end{aligned} \tag{2}$$

For TM modes, we solve (1) with the boundary condition $\psi_e|_S = 0$ and construct

$$\begin{aligned} \mathbf{E}_{\rm TM} &= \left[\frac{ih}{\gamma^2}\nabla_\perp + \hat{\mathbf{z}}\right]\psi_e e^{i(hz-\omega t)} \\ \mathbf{B}_{\rm TM} &= \frac{v_{\rm p}}{c^2}\hat{\mathbf{z}} \times \mathbf{E}_{\rm TM}. \end{aligned} \tag{3}$$

(a) For a coaxial guide with cylindrical symmetry, (1) becomes

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \gamma^2\right]\psi = 0. \tag{4}$$

This is Bessel's differential equation of order zero with linear independent solutions $J_0(\gamma\rho)$ and $N_0(\gamma\rho)$. The origin and infinity are excluded from $b \le \rho \le a$ so

$$\psi(\rho) = AJ_0(\gamma\rho) + BN_0(\gamma\rho).$$

<u>TM modes</u>: The boundary conditions are $\psi|_{\rho=a,b} = 0$ or

$$AJ_0(\gamma a) + BN_0(\gamma a) = 0 = AJ_0(\gamma b) + BN_0(\gamma b).$$

These linear equations will be zero if the determinant of the coefficients is zero:

$$J_0(\gamma_n a)N_0(\gamma_m b) = J_0(\gamma_m b)N_0(\gamma_m a).$$

The index m accounts for multiple solutions to the transcendental equation. The fields $\mathbf{E}_{\rm TM}$ and $\mathbf{B}_{\rm TM}$ are given by (3).

TE modes: The boundary conditions are $\partial\psi/\partial\rho|_{\rho=a,b} = 0$. By the same logic,

$$\psi(\rho) = J_0'(\gamma a)N_0'(\gamma b) = J_0'(\gamma b)N_0'(\gamma a)$$

is the transcendental equation for the TE eigenvalues $\gamma_{mn}^{\rm TE}$. The fields $\mathbf{E}_{\rm TE}$ and $\mathbf{B}_{\rm TE}$ are given by (2).

(b) When $a - b \ll \bar{\rho} = \frac{1}{2}(a+b)$ we approximate (4) with the differential equation

$$\left[\frac{1}{\bar{\rho}}\frac{\partial}{\partial\rho}\left(\bar{\rho}\frac{\partial}{\partial\rho}\right) + \gamma^2\right]\psi = 0 = \left[\frac{\partial^2}{\partial\rho^2} + \gamma^2\right]\psi.$$

This gives the solutions

$$\psi_M = \sin\frac{n\pi(\rho-b)}{a-b} \qquad \text{and} \qquad \psi_E = \cos\frac{n\pi(\rho-b)}{a-b},$$

with $\gamma_E^2 = \gamma_M^2 = \dfrac{m^2}{\bar{\rho}^2} + \left(\dfrac{n\pi}{a-b}\right)^2$.

19.13 A Baffling Waveguide

For a circular cross section, we must solve the two-dimensional Helmholtz equation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \gamma^2\psi = 0, \tag{1}$$

where

$$\gamma^2 = \frac{\omega^2}{c^2} - k^2. \tag{2}$$

Separating variables in (1) with $\psi(\rho,\phi) = F(\rho)G(\phi)$ and separation constant α^2 gives

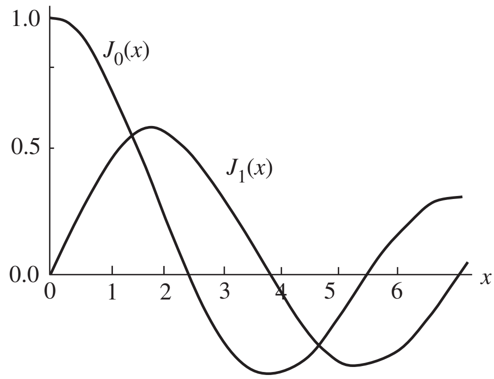
$$\frac{d^2F}{d\rho^2} + \frac{1}{\rho}\frac{dF}{d\rho} + \left(\gamma^2 - \frac{\alpha^2}{\rho^2}\right)F = 0 \tag{3}$$

$$\frac{d^2G}{d\phi^2} + \alpha^2 G = 0.$$

Equation (3) is Bessel's equation. Since $\gamma \neq 0$, the general solution that does not diverge at the origin is

$$\psi_\alpha(\rho, \phi) = AJ_\alpha(\gamma\rho)(\sin\alpha\phi + B\cos\alpha\phi).$$

The first two Bessel functions are shown below.



TM modes:

The boundary condition is

$$\psi_{\rm TM}(\gamma R) = J_\alpha(\gamma R) = 0.$$

This shows that the TM modes are indexed by an integer m such that $\gamma_m^{\rm TM} = x_m/R$ where $\{x_1, x_2, \ldots\}$ are the zeroes of the $J_\alpha(x)$ beginning with the smallest. From the diagram above, we see that the lowest TM cutoff frequency is associated with the first zero of $J_0(x)$:

$$\omega_{\rm TM} = c\gamma_1 = cx_1/R \approx 2.4c/R.$$

TE modes:

The boundary condition is

$$\left.\frac{\partial\psi_{\rm TE}}{\partial r}\right|_{r=R} = \left.\frac{\partial J_\alpha(\gamma\rho)}{\partial\rho}\right|_{\rho=R} = 0.$$

This shows that the TE modes are indexed by an integer n such that $\gamma_m^{\rm TE} = y_m/R$ where $\{y_1, y_2, \ldots\}$ are the maxima and minima of the $J_\alpha(x)$ beginning with the smallest. From the diagram, we see that the lowest TE cutoff frequency is associated with the first maximum of $J_1(x)$:

$$\omega_{\rm TE} \approx 1.8c/R.$$

(a) When the baffle is present at $\phi = 0$, we get the extra TM boundary conditions $\psi(\phi = 0) = \psi(\phi = 2\pi) = 0$. This leads to

$$\psi_\alpha^{\rm TM}(\rho, \phi) = AJ_\alpha(\gamma\rho)\sin\alpha\phi,$$

with $\alpha = m/2$ and $m = 1, 2, \ldots$ From a table of zeroes of Bessel functions of half-integer order, the smallest comes from $J_{1/2}(\pi) = 0$. Therefore the new TM cutoff frequency is $\hat{\omega}_{\rm TM} = c\pi/R > \omega_{\rm TM}$.

(b) The extra TE boundary condition is $\psi'(\phi = 0) = \psi'(\phi = 2\pi) = 0$ where the prime here denotes a derivative with respect to ϕ (the direction normal to the baffle). This leads us to choose

$$\psi_\alpha^{\rm TE} = AJ_\alpha(\gamma\rho)\cos\alpha\phi$$

and again we get $\alpha = m/2$ and $m = 1, 2, \ldots$ From a table of zeroes of the derivative of Bessel functions with half-integer order, the smallest comes from $J'_{1/2}(1.17) = 0$. Therefore, the new TE cutoff frequency is $\hat{\omega}_{\rm TE} = (1.17)c/R < \omega_{\rm TE}$.

19.14 Waveguide Charge and Current

(a) For a TM waveguide mode,

$$\begin{aligned}
\mathbf{E}_{\rm TM} &= \left[\frac{ih}{\gamma^2}\nabla_\perp E_z + E_z\hat{\mathbf{z}}\right]\exp[i(hz - \omega t)] \\
\mathbf{H}_{\rm TM} &= \frac{\omega\epsilon_0}{h}\hat{\mathbf{z}} \times \mathbf{E}.
\end{aligned}$$

The surface charge density is

$$\sigma_{\rm TM} = \epsilon_0\hat{\mathbf{n}}\cdot\mathbf{E}|_S = \frac{i\epsilon_0 h}{\gamma^2}\left.\frac{\partial E_z}{\partial n}\right|_S.$$

The corresponding surface current density is

$$\mathbf{K} = \hat{\mathbf{n}} \times \mathbf{H}|_S = \frac{\omega\epsilon_0}{h}\hat{\mathbf{n}} \times (\hat{\mathbf{z}} \times \mathbf{e})|_S = \frac{\omega\epsilon_0}{h}\hat{\mathbf{n}}\cdot\mathbf{E}\Big|_S\,\hat{\mathbf{z}} = \frac{\omega}{h}\sigma_{\rm TM}\hat{\mathbf{z}} = v_{\rm p}\sigma_{\rm TM}\hat{\mathbf{z}}.$$

(b) For a TE waveguide mode,

$$\begin{aligned}
\mathbf{E}_{\rm TE} &= -\frac{\omega\mu_0}{h}\hat{\mathbf{z}} \times \mathbf{H} \\
\mathbf{H}_{\rm TE} &= \left[\frac{ih}{\gamma^2}\nabla_\perp H_z + H_z\hat{\mathbf{z}}\right]\exp[i(hz - \omega t)].
\end{aligned}$$

The surface charge density is

$$\begin{aligned}
\sigma_{\rm TE} &= \epsilon_0\hat{\mathbf{n}}\cdot\mathbf{E}|_S = -\frac{\epsilon_0\omega\mu_0}{h}\hat{\mathbf{n}}\cdot(\hat{\mathbf{z}}\times\mathbf{H})\Big|_S = -\frac{\epsilon_0\omega\mu_0}{h}\hat{\mathbf{z}}\cdot(\hat{\mathbf{n}}\times\mathbf{H})\Big|_S \\
&= \frac{\omega}{c^2 h}\hat{\mathbf{z}}\cdot\mathbf{K}\Big|_S = \frac{v_{\rm p}}{c^2}\hat{\mathbf{z}}\cdot\mathbf{K}\Big|_S = \frac{1}{v_{\rm g}}\hat{\mathbf{z}}\cdot\mathbf{K}\Big|_S.
\end{aligned}$$

(c)

$$\hat{\boldsymbol{\tau}} \cdot \mathbf{K}_{\mathrm{TE}} = \hat{\boldsymbol{\tau}} \cdot (\hat{\mathbf{n}} \times \mathbf{H})|_S = \mathbf{H} \cdot (\hat{\boldsymbol{\tau}} \times \hat{\mathbf{n}})|_S = \hat{\mathbf{z}} \cdot \mathbf{H}|_S \neq 0.$$

(d) For TM modes, the results from part (a) tell us that

$$\frac{\partial \sigma_{\mathrm{TM}}}{\partial t} = -i\omega\sigma_{\mathrm{TM}}.$$

Moreover,

$$\nabla \cdot \mathbf{K}_{\mathrm{TM}} = \frac{\omega}{h}\frac{\partial \sigma_{\mathrm{TM}}}{\partial z} = \frac{\omega}{h} ih\sigma_{\mathrm{TM}} = i\omega\sigma_{\mathrm{TM}}.$$

This shows that

$$\nabla \cdot \mathbf{K}_{\mathrm{TM}} + \frac{\partial \sigma_{\mathrm{TM}}}{\partial t} = 0.$$

For TE modes, we have

$$\frac{\partial \sigma_{\mathrm{TE}}}{\partial t} = -i\omega\sigma_{\mathrm{TE}}.$$

Otherwise, we can rewrite one of the results from part (b) as

$$\sigma_{\mathrm{TE}} = \epsilon_0 \hat{\mathbf{n}} \cdot \mathbf{E}|_S = -\frac{\epsilon_0 \omega \mu_0}{h} \hat{\mathbf{n}} \cdot (\hat{\mathbf{z}} \times \mathbf{H})\Big|_S = -\frac{\epsilon_0 \omega \mu_0}{h} \mathbf{H} \cdot (\hat{\mathbf{n}} \times \hat{\mathbf{z}})\Big|_S = -\frac{\omega}{c^2 h} H_\tau\Big|_S. \quad (1)$$

Using part (c) and then (1),

$$\begin{aligned}
\nabla \cdot \mathbf{K}_{\mathrm{TE}} &= \frac{\partial}{\partial z}(\hat{\mathbf{z}} \cdot \mathbf{K}_{\mathrm{TE}}) + \frac{\partial}{\partial \tau}(\hat{\boldsymbol{\tau}} \cdot \mathbf{K}_{\mathrm{TE}}) \\
&= \frac{c^2 h}{\omega}\frac{\partial \sigma_{\mathrm{TE}}}{\partial z} + \frac{\partial}{\partial \tau}(\mathbf{H} \cdot \hat{\mathbf{z}}) \\
&= i\frac{c^2 h^2}{\omega}\sigma_{\mathrm{TE}} + \frac{\partial H_z}{\partial \tau} \\
&= i\frac{c^2 h^2}{\omega}\sigma_{\mathrm{TE}} + \frac{\gamma^2}{ih} H_\tau \\
&= \frac{ic^2}{\omega}\sigma_{\mathrm{TE}}\left(h^2 + \gamma^2\right) \\
&= i\omega\sigma_{\mathrm{TE}}.
\end{aligned}$$

This shows that

$$\nabla \cdot \mathbf{K}_{\mathrm{TE}} + \frac{\partial \sigma_{\mathrm{TE}}}{\partial t} = 0.$$

19.15 Cavity Modes as Harmonic Oscillators

The total electromagnetic energy for is

$$U_{\rm EM} = \frac{1}{2}\epsilon_0 \int d^3r \left[E^2 + c^2 B^2\right].$$

We work in the Coulomb gauge, $\nabla \cdot \mathbf{A} = 0$, where

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A}.$$

The vector potential satisfies the wave equation. Therefore, if

$$\mathbf{A}(\mathbf{r},t) = \sum_\lambda q_\lambda(t)\mathbf{A}_\lambda(\mathbf{r}),$$

we have

$$\nabla^2 \mathbf{A}_\lambda + \frac{\omega_\lambda^2}{c^2}\mathbf{A}_\lambda = 0. \tag{1}$$

When $\mu \neq \lambda$, the text proved that

$$\int d^3r \,(\nabla \times \mathbf{A}_\lambda)\cdot(\nabla \times \mathbf{A}_\mu) = 0 = \int d^3r \,\mathbf{A}_\lambda \cdot \mathbf{A}_\mu. \tag{2}$$

We also assume normalized mode functions, so

$$\int d^3r \,\mathbf{A}_\lambda \cdot \mathbf{A}_\lambda = 1. \tag{3}$$

Using (2), we see that the electric energy corresponds to the kinetic energy of a set of oscillators:

$$\begin{aligned} U_{\rm EM} &= \sum_{\lambda,\mu} \frac{1}{2}\epsilon_0 \int d^3r \,\left[\dot{q}_\lambda \mathbf{A}_\lambda \cdot \dot{q}_\mu \mathbf{A}_\mu + c^2(q_\lambda \nabla \times \mathbf{A}_\lambda)\cdot(q_\mu \nabla \times \mathbf{A}_\lambda)\right] \\ &= \sum_\lambda \frac{1}{2}\epsilon_0 \left[\dot{q}_\lambda^2 + c^2 \int d^3r \, q_\lambda^2 (\nabla \times \mathbf{A}_\lambda)\cdot(\nabla \times \mathbf{A}_\lambda)\right]. \end{aligned}$$

To simplify the magnetic energy, we use the identity quoted in the text,

$$\int_S d\mathbf{S} \cdot [\mathbf{a} \times (\nabla \times \mathbf{b}) + (\nabla \cdot \mathbf{b})\mathbf{a}] = \int_V d^3r \,\left[(\nabla \times \mathbf{a})\cdot(\nabla \times \mathbf{b}) + (\nabla \cdot \mathbf{a})(\nabla \cdot \mathbf{b}) + \mathbf{a}\cdot\nabla^2\mathbf{b}\right].$$

With $\mathbf{a} = \mathbf{b} = \mathbf{A}_\lambda$, the surface integral vanishes because $\nabla \cdot \mathbf{A}_\lambda = 0$ and the boundary condition on the modes is $\hat{\mathbf{n}} \times \mathbf{A}|_S = 0$. Therefore, using (1) and (2),

$$\int d^3r \,(\nabla \times \mathbf{A}_\lambda)\cdot(\nabla \times \mathbf{A}_\lambda) = \frac{\omega_\lambda^2}{c^2}.$$

Substituting this into our formula for $U_{\rm EM}$ identifies the magnetic energy as the potential energy of the oscillators and gives the desired result:

$$U_{\rm EM} = \frac{1}{2}\epsilon_0 \sum_\lambda \left[\dot{q}_\lambda^2 + \omega_\lambda^2 q_\lambda^2\right].$$

19.16 An Electromagnetic Oscillator

(a) When one sphere has charge Q, the other will have charge $-Q$. The potential of a conducting sphere with radius R and charge Q is $V = Q/4\pi\epsilon_0 R$. Therefore the self-capacitance of a single sphere is $C_0 = 4\pi\epsilon_0 R$. For the two-sphere system, we neglect the mutual capacitance so that

$$Q \approx C_0 V_1$$
$$-Q \approx C_0 V_2.$$

Therefore, $C = \dfrac{Q}{V_1 - V_2} = \dfrac{C_0}{2} = 2\pi\epsilon_0 R.$

(b) We estimate the inductance from the magnetic energy $U_B = LI^2/2$ of the rod when a current I flows through it. As long as $a < \rho \ll l$, the magnetic field is circumferential with magnitude

$$B(\rho) = \frac{\mu_0 I}{2\pi\rho}.$$

Therefore,

$$U_B = \frac{1}{2\mu_0}\int dV B^2 \approx \frac{1}{2\mu_0} 2\pi l \int_a^l \rho d\rho B^2(\rho) = \frac{\mu_0 I^2}{4\pi} l \ln(l/a)$$

and $L = \dfrac{\mu_0}{2\pi} l \ln(l/a).$

(c) Treating our system as an LC circuit, the resonant frequency is

$$\omega = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{(2\pi\epsilon_0 R)(\mu_0 l \ln(l/a)/2\pi)}} = \frac{c}{\sqrt{Rl\ln(l/a)}}.$$

19.17 A Variational Principle

(a) For any mode of the cavity, $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t = i\omega\mathbf{B}$. This gives

$$\frac{\int_V d^3r\,|\nabla\times\mathbf{E}|^2}{\int_V d^3r\,|\mathbf{E}|^2} = \omega^2 \frac{\int_V d^3r\,|\mathbf{B}|^2}{\int_V d^3r\,|\mathbf{E}|^2} = \frac{\omega^2}{c^2},$$

because the time-averaged electric energy and magnetic energy are equal for a cavity mode:

$$\int_V d^3r\,|\mathbf{E}|^2 = c^2\int_V d^3r\,|\mathbf{B}|^2.$$

Similarly, because $\nabla\times\mathbf{B} = c^{-2}\partial\mathbf{E}/\partial t = -i\omega/c^2$,

$$\frac{\int_V d^3r\,|\nabla\times\mathbf{B}|^2}{\int_V d^3r\,|\mathbf{B}|^2} = \frac{\omega^2}{c^4}\frac{\int_V d^3r\,|\mathbf{E}|^2}{\int_V d^3r\,|\mathbf{B}|^2} = \frac{\omega^2}{c^2}.$$

(b) By the quotient rule for derivatives,

$$\begin{aligned}
\delta\left\{\frac{\omega^2}{c^2}\right\} &= \delta\left\{\frac{\int_V d^3r\,|\nabla\times\mathbf{E}|^2}{\int_V d^3r\,|\mathbf{E}|^2}\right\} \\
&= \frac{\delta\left\{\int_V d^3r\,|\nabla\times\mathbf{E}|^2\right\}\int_V d^3r\,|\mathbf{E}|^2 - \delta\left\{\int_V d^3r\,|\mathbf{E}|^2\right\}\int_V d^3r\,|\nabla\times\mathbf{E}|^2}{\int_V d^3r\,|\mathbf{E}|^2\int_V d^3r\,|\mathbf{E}|^2} \\
&= \frac{\delta\int_V d^3r\,|\nabla\times\mathbf{E}|^2}{\int_V d^3r\,|\mathbf{E}|^2} - \frac{\omega^2}{c^2}\frac{\delta\int_V d^3r\,|\mathbf{E}|^2}{\int_V d^3r\,|\mathbf{E}|^2}.
\end{aligned}$$

Now,

$$\delta\int_V d^3r\,|\nabla\times\mathbf{E}|^2 \;=\; \int_V d^3r\;\left[\nabla\times(\mathbf{E}+\delta\mathbf{E})\right]\cdot\left[\nabla\times(\mathbf{E}+\delta\mathbf{E})\right] - \int_V d^3r\,|\nabla\times\mathbf{E}|^2$$

and

$$(\nabla\times\mathbf{E})\cdot(\nabla\times\delta\mathbf{E}) = \nabla\cdot(\delta\mathbf{E}\times\nabla\times\mathbf{E}) - \delta\mathbf{E}\cdot\nabla\times\nabla\times\mathbf{E} = \nabla\cdot(\delta\mathbf{E}\times\nabla\times\mathbf{E}) + \frac{\omega^2}{c^2}\delta\mathbf{E}\cdot\mathbf{E}.$$

Therefore, if $\mathbf{E} = \mathbf{E}_\parallel + \mathbf{E}_\perp$ decomposes $\mathbf{E}$ into a component parallel and perpendicular to the surface, the fact that $\delta\mathbf{E}_\parallel = 0$ at the surface of a perfect conductor gives

$$\int_V d^3r\;\nabla\cdot(\delta\mathbf{E}\times\nabla\times\mathbf{E}) = i\omega\int_S dS\cdot(\delta\mathbf{E}\times\mathbf{B}) = i\omega\int_S dS\cdot(\delta\mathbf{E}_\parallel\times\mathbf{B}_\parallel) = 0.$$

We conclude that

$$\delta\int_V d^3r\,|\nabla\times\mathbf{E}|^2 = \int_V d^3r\;\left\{|\nabla\times\delta\mathbf{E}|^2 + 2\frac{\omega^2}{c^2}\mathbf{E}\cdot\delta\mathbf{E}\right\}.$$

Much more simply,

$$\delta\int_V d^3r\,|\mathbf{E}|^2 = 2\int_V d^3r\;\mathbf{E}\cdot\delta\mathbf{E} + \int_V d^3r\;|\delta\mathbf{E}|^2.$$

This gives the final result

$$\delta\left\{\frac{\omega^2}{c^2}\right\} = \frac{\int_V d^3r\;\left\{|\nabla\times\delta\mathbf{E}|^2 - (\omega^2/c^2)|\delta\mathbf{E}|^2\right\}}{\int_V d^3r\,|\mathbf{E}|^2}$$

because the terms that are linear in $\delta\mathbf{E}$ cancel. A similar proof applies to $\delta\mathbf{B}$.

(c) If $\mathbf{B} = (\rho + a\rho^2)\hat{\boldsymbol{\phi}}$, $\nabla \times \mathbf{B} = (2 + 3a\rho)\hat{\mathbf{z}}$ in cylindrical coordinates. Therefore,

$$\Lambda = \frac{\int\limits_V d^3r\,|\nabla \times \mathbf{B}|^2}{\int\limits_V d^3r\,|\mathbf{B}|^2} \leq \frac{2\pi L \int\limits_0^R d\rho\rho(2+3a\rho)^2}{2\pi L \int\limits_0^R d\rho\rho(\rho+a\rho^2)^2} = \frac{15(8+16aR+9a^2R^2)}{R^2(15+24aR+10a^2R^2)}.$$

We want to minimize this:

$$\frac{\partial \Lambda}{\partial a} = 0 \quad \Rightarrow \quad \frac{1}{5} + \frac{11}{24}aR + \frac{7}{30}a^2R^2 = 0.$$

Therefore,

$$a = \frac{-55 \pm \sqrt{337}}{56R}.$$

The root with the plus (minus) sign gives $d^2\Lambda/da^2$ positive (negative). We want a minimum so we choose the plus sign. Therefore, $a \approx -0.654/R$ and

$$\frac{\omega}{c} = \sqrt{\Lambda} \approx \frac{2.409}{R}$$

is our estimate. The solution we guessed corresponds to a TM mode, and the lowest such mode frequency for a cylindrical cavity may be inferred from the lowest cutoff frequency for TM waves in a cylindrical waveguide. This is

$$\frac{\omega}{c} = \frac{x_{01}}{R} \approx \frac{2.405}{R},$$

where x_{01} is the first zero of the Bessel function $J_0(x)$. Our estimate is excellent!

19.18 An Asymmetric Two-Dimensional Resonant Cavity

TM modes in a cavity have the property that $\psi = 0$ on the walls of the cavity. This tells us to look for the zeroes of ψ. The waves in the sum are $\pm\mathbf{k}_0$, $\pm\mathbf{k}_1$, and $\pm\mathbf{k}_2$. Since $\omega = ck$,

$$\psi(x,y,t) = \text{Im}\left\{2i\left[\sin(\mathbf{k}_0\cdot\mathbf{r}) - \sin(\mathbf{k}_1\cdot\mathbf{r}) + \sin(\mathbf{k}_2\cdot\mathbf{r})\right]e^{-i\omega t}\right\}.$$

Moreover,

$$\mathbf{k}_0\cdot\mathbf{r} = kx$$

$$\mathbf{k}_1\cdot\mathbf{r} = \cos\left[\frac{\pi}{3}\right]kx + \sin\left[\frac{\pi}{3}\right]ky = \frac{1}{2}kx + \frac{\sqrt{3}}{2}ky$$

$$\mathbf{k}_2\cdot\mathbf{r} = \cos\left[\frac{2\pi}{3}\right]kx + \sin\left[\frac{2\pi}{3}\right]ky = -\frac{1}{2}kx + \frac{\sqrt{3}}{2}ky.$$

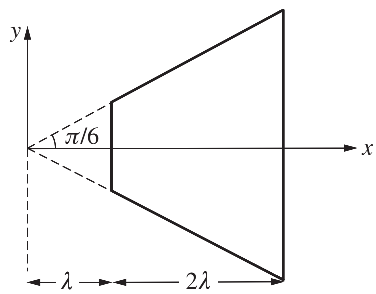
Therefore,

$$\psi(x,y,t) = 2\cos(\omega t)\left\{\sin kx - \sin\left[\frac{kx}{2} + \frac{\sqrt{3}ky}{2}\right] - \sin\left[\frac{kx}{2} - \frac{\sqrt{3}ky}{2}\right]\right\}.$$

Zeroes come from multiplicative factors. To find one, let $a = kx/2$ and $b = \sqrt{3}ky/2$ so the quantity in curly brackets is

$$\begin{aligned}\sin(a+a) - \sin(a+b) - \sin(a-b) &= [\sin a \cos a + \cos a \sin a] - [\sin a \cos b + \cos a \sin b] \\ &\qquad - [\sin a \cos b - \cos a \sin b] \\ &= 2\sin a\,[\cos a - \cos b]\,.\end{aligned}$$

This function is zero when $\sin a = 0$. Thus, $\psi = 0$ on the lines defined by $a = kx/2 = m\pi$ where m is an integer. We also get $\psi = 0$ on the lines defined by $\cos a = \cos b$. The simplest of these are $x = \pm\sqrt{3}y$. Therefore, if $\lambda = 2\pi/k$, the heavy solid lines in the figure below outline a 2D conducting cavity which will support a TM resonant mode built from $\psi(x, y, t)$.



Source: http://gregegan.customer.netspace.net.au/SCIENCE/Cavity/Simple.html

19.19 The Ark of the Covenant

The frequency ω_0 depends only on the dimensions of the Ark. Most scholars agree that the cubit used in the Hebrew bible was about 45 cm. Therefore, we take the Ark's volume as $V = abL$ where $L = 1.125$ m and $a = b = 0.675$ m. The half-width Γ depends on the skin depth and hence on the conductivity of gold. This is $\sigma = 4.5 \times 10^7\ \Omega^{-1}\text{m}^{-1}$. Omitting a factor of $\exp(-i\omega t)$, we have TE cavity modes:

$$\mathbf{E}_{\text{TE}} = -\frac{i\omega}{c}\sin\left(\frac{p\pi z}{L}\right)\hat{\mathbf{z}} \times \nabla_\perp \psi_{\text{TE}} \qquad p = 0, 1, 2, \ldots$$

$$c\mathbf{B}_{\text{TE}} = \frac{p\pi}{L}\cos\left(\frac{p\pi z}{L}\right)\nabla_\perp \psi_{\text{TE}} + \hat{\mathbf{z}}\sin\left(\frac{p\pi z}{L}\right)\gamma^2 \psi_{\text{TE}}$$

$$\psi_{\text{TE}} = \psi_0 \cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right)$$

$$\nabla_\perp \psi_{\text{TE}} = -\psi_0 \frac{m\pi}{a}\sin\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{a}\right)\hat{\mathbf{x}} - \psi_0 \frac{n\pi}{a}\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{a}\right)\hat{\mathbf{y}}$$

$$\gamma^2 = \frac{\pi^2}{a^2}(m^2 + n^2) \qquad m, n = 0, 1, 2, \ldots$$

$$\frac{\omega^2}{c^2} = \gamma^2 + \frac{p^2\pi^2}{L^2}.$$

We are interested in the lowest-frequency mode. The choice $p = 0$ gives no fields at all. Therefore, we must have $p = 1$. Then, the choices $m = 0,\ n = 1$ or $m = 1,\ n = 0$ are degenerate with frequency

$$\omega_0 = c\pi\sqrt{\frac{1}{a^2} + \frac{1}{L^2}} \approx 1.6 \text{ GHz}.$$

The lowest TM-mode frequency cannot be lower than this. We can set $\psi_0 = 1$ without loss of generality. Therefore, the fields associated with this mode are

$$\mathbf{E} = -\frac{i\omega\pi}{ac}\sin\left(\frac{\pi z}{L}\right)\sin\left(\frac{\pi y}{a}\right)\hat{\mathbf{x}}$$

$$c\mathbf{B} = -\frac{\pi^2}{aL}\cos\left(\frac{\pi z}{L}\right)\sin\left(\frac{\pi y}{a}\right)\hat{\mathbf{y}} + \frac{\pi^2}{a^2}\sin\left(\frac{\pi z}{L}\right)\cos\left(\frac{\pi y}{a}\right)\hat{\mathbf{z}}.$$

If $\langle U\rangle$ is the time-averaged energy stored in the cavity and $\langle P\rangle$ is the time-averaged power dissipated in the ohmic walls, the half-width of the resonance is $\Gamma = \omega_0/Q$, where

$$\frac{\omega_0}{Q} = \frac{\langle P\rangle}{\langle U_E\rangle + \langle U_B\rangle} = \frac{\dfrac{\delta\omega_0}{4\mu_0}\displaystyle\int_A dA\,\mathbf{B}_\parallel\cdot\mathbf{B}_\parallel^*}{\dfrac{\epsilon_0}{4}\displaystyle\int_V d^3r\,[\mathbf{E}\cdot\mathbf{E}^* + c^2\mathbf{B}\cdot\mathbf{B}^*]}.$$

Straightforward integration, and the fact that $\omega^2/c^2 = \pi^2/a^2 + \pi^2/L^2$ for this mode, gives

$$\langle U_E\rangle = \langle U_B\rangle = \frac{\pi^4\epsilon_0 L}{16a^2}\left(1 + \frac{a^2}{L^2}\right).$$

Similarly, the fields do not depend on x, $|B_y(z=0)|^2 = |B_y(z=L)|^2$, and $|B_z(y=0)|^2 = |B_z(y=a)|^2$. Therefore, suppressing the field argument,

$$\int dA\,|\mathbf{B}_\parallel|^2 = 2a\int_0^a dy\,|B_y|^2_{z=0} + 2\int_0^a dy\int_0^L dz\,\left(|B_y|^2 + |B_z|^2\right)_{x=0} + 2a\int_0^L dz\,|B_z|^2_{y=0}.$$

Therefore,

$$\langle P\rangle = \frac{\delta\omega_0}{4\mu_0}\frac{\pi^4}{c^2a^2}\left[\frac{a^2}{L^2} + \frac{a}{2L} + \frac{3}{2}\frac{L}{a}\right]$$

and

$$\Gamma = \frac{\omega_0}{Q} = \frac{\dfrac{\delta\omega_0}{4\mu_0}\dfrac{\pi^4}{c^2a^2}\left[\dfrac{a^2}{L^2} + \dfrac{a}{2L} + \dfrac{3}{2}\dfrac{L}{a}\right]}{\dfrac{\pi^4\epsilon_0 L}{8a^2}\left(1 + \dfrac{a^2}{L^2}\right)} = 2\frac{\delta\omega_0}{L}\frac{\dfrac{a^2}{L^2} + \dfrac{a}{2L} + \dfrac{3}{2}\dfrac{L}{a}}{1 + \dfrac{a^2}{L^2}}.$$

At the resonance frequency, the skin depth is

$$\delta = \sqrt{\frac{2}{\mu_0\omega_0\sigma}} \approx 16\times 10^{-6}\text{ m}.$$

For the Ark of the Covenant, $a/L \approx 0.6$. These numbers give the estimate

$$\Gamma \approx 6.6 \times 10^{-5}\omega_0 \approx 0.1 \text{ MHz}.$$

Source: J. Franklin, *Classical Electromagnetism* (Pearson, San Francisco, 2005).

19.20 Perturbation of a Cavity Resonator

The theorem states that $\langle U_{\rm EM}\rangle T =$ constant. Therefore, if $T = 2\pi/\omega$,

$$\frac{\delta\langle U_{\rm EM}\rangle}{\langle U_{\rm EM}\rangle} = \frac{\delta\omega}{\omega}.$$

In our chapter on dielectrics, we learned that the energy change when a field $\mathbf{E}_0$ polarizes a dielectric is

$$\delta U_E = -\frac{1}{2}\int d^3r\, \mathbf{P}\cdot\mathbf{E}_0.$$

Similarly, the energy change when the field $\mathbf{B}_0$ magnetizes an object is

$$\delta U_B = -\frac{1}{2}\int d^3r\, \mathbf{M}\cdot\mathbf{E}_0.$$

The adiabatic theorem makes sense when the object is small and the (barely perturbed) field is almost constant over the volume of the object. Therefore, if an asterix denotes complex conjugation, we take the field out of the integrals and use the time-averaging theorem to write

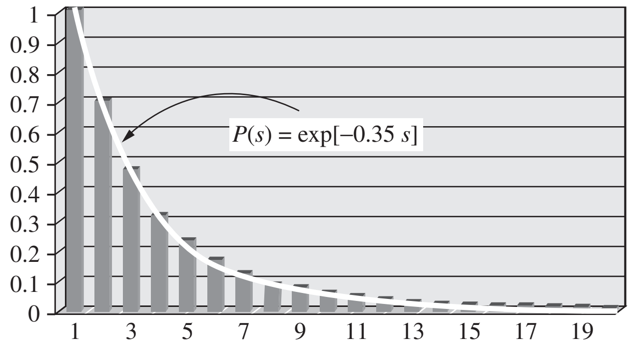
$$\frac{\delta\omega}{\omega} = -\frac{\text{Re}\left[\mathbf{p}^*\cdot\mathbf{E}_0 + \mathbf{m}^*\cdot\mathbf{B}_0\right]}{4\langle U_{\rm EM}\rangle}.$$

This is called the Müller-Slater formula.

Source: C.H. Papas, *Journal of Applied Physics* **25**, 1552 (1954).

19.21 Resonant-Frequency Differences for a Cavity

For a box with volume abc, the (un-normalized) resonant frequencies are $\omega^2_{m,n,k} = n^2/a^2 + m^2/b^2 + k^2/c^2$. This equation defines an *ellipsoid* in (n, m, k) space rather than a cube. Therefore, to capture *all* the frequencies less than a fixed value $\omega_{\rm max}$ it is necessary to use different maximum values for n, m, and k. The following (normalized) results were obtained from $85,532$ mode frequencies for a cavity with dimensions $a = e$, $b = \pi$, and $c = \sqrt{17}$. The white line is a fit to a Poisson distribution.



19.22 The Panofsky-Wenzel Theorem

(a) The fields inside the cavity can be described by $\mathbf{E} = -\partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla\times\mathbf{A}$. Moreover, the particle velocity is not a function of position. Hence,

$$\mathbf{v}\times\mathbf{B} = \mathbf{v}\times(\nabla\times\mathbf{A}) = \nabla(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A}.$$

Now, using the hint for a velocity $\mathbf{v} = v\hat{\mathbf{z}}$,

$$\Delta\mathbf{p}_\perp = -\frac{q}{v}\int_0^L dz\left[\frac{\partial\mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A} - \nabla(\mathbf{v}\cdot\mathbf{A})\right]_\perp = -\frac{q}{v}\int_0^L dz\left[\frac{d\mathbf{A}_\perp}{dt} - v\nabla_\perp A_z\right].$$

Because $dz = vdt$ and the cavity electric field is $\mathbf{E} = i\omega\mathbf{A}$,

$$\Delta\mathbf{p}_\perp = -q\int_{\mathbf{A}_\perp(0)}^{\mathbf{A}_\perp(L)} d\mathbf{A}_\perp + q\int_0^L dz\nabla_\perp A_z = \frac{iq}{\omega}\int_{\mathbf{E}_\perp(0)}^{\mathbf{E}_\perp(L)} d\mathbf{E}_\perp - \frac{iq}{\omega}\int_0^L dz\nabla_\perp E_z.$$

In this problem, the transverse direction is parallel to the $z = 0$ and $z = L$ walls. If the holes in the cavity are negligibly small, the component of the electric field parallel to these walls vanishes and we get the advertised result.

(b) $E_z = 0$ for a TE mode so $\Delta\mathbf{p}_\perp = 0$. This can be true only if the transverse force from the TE mode electric field exactly cancels the transverse Lorentz force from the TE mode magnetic field.

Source: T.P. Wangler, *Principles of RF Linear Accelerators* (Wiley, New York, 1998).

19.23 Forces on Resonant-Cavity Walls

Every component of $\mathbf{E}$ satisfies the wave equation. Therefore

$$\nabla^2 E_z + \frac{1}{c^2}\frac{\partial E_z}{\partial t^2} = 0 \quad \Rightarrow \quad 2\frac{\pi^2}{a^2} = \frac{\omega^2}{c^2}.$$

We calculate the remaining non-zero components of $\mathbf{B}$ from $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = i\omega\mathbf{B}$. The result is

$$B_x = -i\frac{\pi}{a\omega}E_0 \sin\frac{\pi x}{a}\cos\frac{\pi y}{a}\exp(-i\omega t) = -i\frac{E_0}{\sqrt{2}c}\sin\frac{\pi x}{a}\cos\frac{\pi y}{a}\exp(-i\omega t)$$

$$B_y = -i\frac{\pi}{ac}E_0 \cos\frac{\pi x}{a}\sin\frac{\pi y}{a}\exp(-i\omega t) = -i\frac{E_0}{\sqrt{2}c}\cos\frac{\pi x}{a}\sin\frac{\pi y}{a}\exp(-i\omega t).$$

The Poynting vector contribution to the force density vanishes upon time-averaging because

$$\left\langle \frac{\partial \mathbf{S}}{\partial t} \right\rangle \propto \int_0^{2\pi/\omega} dt \frac{\partial}{\partial t}\left[\mathrm{Re}\,\mathbf{E}(\mathbf{r},t) \times \mathrm{Re}\,\mathbf{B}(\mathbf{r},t)\right] = \mathbf{E}(\mathbf{r})\cos(\omega t) \times \mathbf{B}(\mathbf{r})\sin(\omega t)\Big|_0^{2\pi/\omega} = 0.$$

Otherwise, we treat each interior wall of the cavity separately and choose the volume V in the force integral as the space between two parallel walls. One wall lies inside the "meat" of the conducting wall, where the fields are zero. The other wall (call it S) lies in the vacuum interior of the cavity adjacent to the wall of interest. Therefore, the divergence theorem gives the force on the wall parallel to S in term of the normal $\hat{\mathbf{n}}$ to S (which points into the cavity):

$$\mathbf{F} = \int_S dS\, \hat{\mathbf{n}} \cdot \mathbf{T} = \epsilon_0 \int_S dS \left[(\hat{\mathbf{n}} \cdot \mathbf{E})\mathbf{E} + c^2(\hat{\mathbf{n}} \cdot \mathbf{B})\mathbf{B} - \frac{1}{2}\hat{\mathbf{n}}\left(E^2 + c^2 B^2\right)\right].$$

The cavity cannot exert a net force on itself. Therefore, the force on the $x = 0$ wall is equal and opposite to the force on the $x = a$ wall. The same argument applies to the $y = 0$ and $y = a$ walls and to the $z = 0$ and $z = h$ walls.

The time-averaged force on the $x = 0$ wall is

$$\langle \mathbf{F} \rangle_{x=0} = \hat{\mathbf{x}}\frac{\epsilon_0}{4}\int_{x=0} dS \left(c^2|B_x|^2 - |E_z|^2 - c^2|B_y|^2\right) = -\hat{\mathbf{x}}\frac{\epsilon_0}{4}\int_0^h dz \int_0^a dy\, c^2|B_y(x=0)|^2.$$

This gives an outward force

$$\langle \mathbf{F} \rangle_{x=0} = -\hat{\mathbf{x}}\frac{\epsilon_0 h}{4}\frac{E_0^2}{2}\int_0^a dy\, \sin^2(\pi y/a) = -\hat{\mathbf{x}}\frac{1}{16}\epsilon_0 a h E_0^2 = -\langle \mathbf{F} \rangle_{x=a}.$$

An exactly similar calculation gives

$$\langle \mathbf{F} \rangle_{y=0} = \hat{\mathbf{y}}\frac{\epsilon_0}{4}\int_{y=0} dS \left(c^2|B_y|^2 - |E_z|^2 - c^2|B_x|^2\right) = -\hat{\mathbf{y}}\frac{\epsilon_0}{4}\int_0^h dz \int_0^a dx\, c^2|B_x(y=0)|^2.$$

Hence,

$$\langle \mathbf{F} \rangle_{y=0} = -\hat{\mathbf{y}} \frac{\epsilon_0 h}{4} \frac{E_0^2}{2} \int_0^a dx \, \sin^2(\pi x/a) = -\hat{\mathbf{y}} \frac{1}{16} \epsilon_0 a h E_0^2 = -\langle \mathbf{F} \rangle_{y=a}.$$

Finally,

$$\begin{aligned}
\langle \mathbf{F} \rangle_{z=0} &= \hat{\mathbf{z}} \frac{\epsilon_0}{4} \int_{z=0} dS \, \left(|E_z|^2 - c^2 |B_x|^2 - c^2 |B_y|^2 \right) \\
&= \hat{\mathbf{y}} \frac{\epsilon_0 E_0^2}{4} \int_0^a dx \int_0^a dy \left\{ \sin^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{a} - \frac{1}{2} \sin^2 \frac{\pi x}{a} \cos^2 \frac{\pi y}{a} - \frac{1}{2} \cos^2 \frac{\pi x}{a} \sin^2 \frac{\pi y}{a} \right\} \\
&= 0.
\end{aligned}$$

By the previous argument, $\langle \mathbf{F} \rangle_{z=h} = 0$ as well.

Source: Prof. K.T. McDonald, Princeton University, http://cosmology.princeton.edu/∼mcdonald/examples/

19.24 Graded Index Fiber

(a) There is no free charge, so

$$0 = \nabla \cdot \mathbf{D} = \nabla \left[\epsilon(\mathbf{r}) \mathbf{E} \right] = \epsilon \nabla \cdot \mathbf{E} + \mathbf{E} \cdot \nabla \epsilon.$$

Also,

$$\nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = \nabla \times \nabla \times \mathbf{E} = -\frac{\partial}{\partial t} \nabla \times \mathbf{B} = -\mu \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2}.$$

Therefore, if the medium is non-magnetic so $\mu = \mu_0$ and $n^2(\mathbf{r}) = \epsilon(\mathbf{r})/\epsilon_0$,

$$\nabla^2 \mathbf{E} - \frac{n^2(\mathbf{r})}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} + \nabla \left[\mathbf{E} \cdot \frac{\nabla \epsilon}{\epsilon} \right] = 0.$$

This reduces to the stated equation if the last term can be neglected.

(b) We need to suppose that $L \nabla \epsilon \ll 1$ where L is a typical scale for the variation of $\mathbf{E}$. This, in turn, requires α to be "small", so a good approximation is

$$n^2(\mathbf{r}) = n_0^2 (1 - \alpha^2 \rho^2)^2 \approx n_0^2 (1 - 2\alpha^2 \rho^2).$$

Using this, we write out the wave equation in cylindrical coordinates to get

$$\frac{\partial^2 \mathbf{E}}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial \mathbf{E}}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 \mathbf{E}}{\partial \phi^2} + \frac{\partial^2 \mathbf{E}}{\partial z^2} + \frac{n_0^2 (1 - 2\alpha^2 \rho^2)}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0.$$

The proposed solution satisfies $\nabla \cdot \mathbf{E} = 0$. With $k_0 = n_0 \omega / c$, substituting it into the wave equation above gives

$$\frac{d^2E}{d\rho^2} + \frac{1}{\rho}\frac{dE}{d\rho} - h^2 + k_0^2(1 - 2\alpha^2\rho^2)E = 0.$$

Now we try an exponential, $E(\rho) = \exp(-\beta\rho)$. This gives

$$\left[\beta^2 - \frac{\beta}{\rho} - h^2 + k_0^2(1 - 2\alpha^2\rho^2)\right] E \neq 0.$$

Now try a Gaussian, $E(\rho) = \exp(-\beta^2\rho^2)$, where

$$\frac{dE}{d\rho} = -2\rho\beta^2 E \qquad \text{and} \qquad \frac{d^2E}{d\rho^2} = -2\beta^2 E + 4\beta^4\rho^2 E.$$

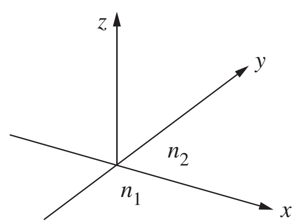
This gives

$$\left[-2\beta^2 + 4\beta^4\rho^2 - 2\beta^2 - h^2 + k_0^2(1 - 2\alpha^2\rho^2)\right] E = 0,$$

which is a solution if $\beta^2 = \alpha k_0/\sqrt{2}$ and

$$k_0^2 = n_0^2\frac{\omega^2}{c^2} = 4\beta^2 + h^2 = \frac{4}{\sqrt{2}}\alpha k_0 + h^2.$$

19.25 Interfacial Guided Waves



(a) The field is time-harmonic so Faraday's law and the fact that the fields have no z-dependence gives

$$\mathbf{H} = \frac{1}{i\omega\mu}\nabla\times\mathbf{E} = \frac{1}{i\omega\mu}\left[\hat{\mathbf{x}}\partial_y E_z - \hat{\mathbf{y}}\partial_x E_z + \hat{\mathbf{z}}(\partial_x E_y - \partial_y E_x)\right].$$

Accordingly,

$$\mathbf{H}_1 = \frac{1}{i\omega\mu_1}\left[E_{1z}(\alpha\hat{\mathbf{x}} - ik\hat{\mathbf{y}}) + iE_{1y}(k - \alpha^2)\right] e^{\alpha y}\exp[i(kx - \omega t)].$$

On the other hand, the field satisfies the wave equation

$$\nabla^2\mathbf{E} - \frac{n^2}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2} = 0,$$

where $n^2 = c^2\epsilon\mu$. Therefore,

$$-k^2 + \alpha^2 + \frac{n_1^2\omega^2}{c^2} = 0 \qquad \text{and} \qquad -k^2 + \beta^2 + \frac{n_2^2\omega^2}{c^2} = 0.$$

Hence,

$$\mathbf{H}_1 = \left[-\frac{E_{1z}}{\omega\mu_1}(i\alpha\hat{\mathbf{x}} + k\hat{\mathbf{y}}) + \hat{\mathbf{z}}E_{1y}\frac{\epsilon_1\omega}{k}\right] e^{\alpha y}\exp[i(kx-\omega t)]$$

and

$$\mathbf{H}_2 = \left[-\frac{E_{2z}}{\omega\mu_2}(-i\beta\hat{\mathbf{x}} + k\hat{\mathbf{y}}) + \hat{\mathbf{z}}E_{2y}\frac{\epsilon_2\omega}{k}\right] e^{-\beta y}\exp[i(kx-\omega t)].$$

(b) We have not yet enforced the Maxwell equation $\nabla \cdot \mathbf{D} = 0$. A brief calculation using the given forms of the electric field yields

$$E_{1x} = i\frac{\alpha}{k}E_{1y} \qquad \text{and} \qquad E_{2x} = -i\frac{\beta}{k}E_{2y}. \tag{1}$$

Using these, the $y = 0$ electric field matching conditions $E_{1x} = E_{2x}$ and $E_{1z} = E_{2z}$ imply that

$$\alpha E_{1y} = -\beta E_{2y} \qquad \text{and} \qquad E_{1z} = E_{2z}. \tag{2}$$

Similarly, the $y = 0$ magnetic field matching conditions $H_{1z} = H_{2z}$ and $H_{1x} = H_{2x}$ imply that

$$\epsilon_1 E_{1y} = \epsilon_2 E_{2y} \qquad \text{and} \qquad \frac{\alpha}{\mu_1}E_{1z} = -\frac{\beta}{\mu_2}E_{2z}. \tag{3}$$

Since α and β are positive, (2) and (3) cannot be satisfied simultaneously except in two special cases:

$$\frac{\alpha}{\beta} = -\frac{\epsilon_1}{\epsilon_2} \qquad \text{and} \qquad E_{1z} = E_{2z} = 0$$

or

$$\frac{\alpha}{\beta} = -\frac{\mu_1}{\mu_2} \qquad \text{and} \qquad E_{1x} = E_{2x} = E_{1y} = E_{2y} = 0.$$

In the case when $E_z = 0$, (1) shows that the fields are elliptically polarized.

(c) The results for $\mathbf{E}_1$ and $\mathbf{E}_2$ are identical; we will display the details for $\mathbf{E}_1$ only. The first case has $E_{1z} = 0$ and $\mathbf{H}_1 = H_{1z}\hat{\mathbf{z}}$. The time-averaged Poynting vector is

$$\begin{aligned}
\langle \mathbf{S}_1 \rangle &= \frac{1}{2}\text{Re}(\mathbf{E}_1 \times \mathbf{H}_1^*) \\
&= \frac{1}{2}\text{Re}\left\{(E_{1x}\hat{\mathbf{x}} + E_{1y}\hat{\mathbf{y}}) \times \hat{\mathbf{z}}E_{1y}^*\frac{\epsilon_1\omega}{k}\right\} e^{2\alpha y} \\
&= \frac{1}{2}\frac{\epsilon_1\omega}{k}\text{Re}\left\{-\hat{\mathbf{y}}E_{1x}E_{1y}^* + \hat{\mathbf{x}}|E_{1y}|^2\right\} e^{2\alpha y}.
\end{aligned}$$

A glance at (1) shows that the first term in curly brackets above is pure imaginary and thus does not contribute when the real part is taken. Hence, $\langle \mathbf{S}_1 \rangle$ points along $\hat{\mathbf{x}}$. The second case has $\mathbf{E}_1 = E_{1z}\hat{\mathbf{z}}$ and $H_{1z} = 0$. We find $\langle \mathbf{S}_1 \rangle \propto \hat{\mathbf{x}}$ here too because

$$\begin{aligned}
\langle \mathbf{S}_1 \rangle &= \frac{1}{2}\mathrm{Re}(\mathbf{E}_1 \times \mathbf{H}_1^*) \\
&= \frac{1}{2}\mathrm{Re}\left\{ E_{1z}\hat{\mathbf{z}} \times \frac{E_{iz}^*}{\omega\mu_1}(i\alpha\hat{\mathbf{x}} - k\hat{\mathbf{y}}) \right\} e^{2\alpha y} \\
&= \frac{1}{2}\frac{1}{\omega\mu_1}\mathrm{Re}\left\{ \hat{\mathbf{y}}|E_{1z}|^2 i\alpha + +\hat{\mathbf{x}}k|E_{1z}|^2 \right\} e^{2\alpha y}.
\end{aligned}$$

Source: A.M. Portis, *Electromagnetic Fields* (Wiley, New York, 1978).

Chapter 20: Retardation and Radiation

20.1 Poynting Flux above a Thunderstorm

The frequency band given corresponds to wavelengths between 10^5 and 10^7 meters. A thunderhead rises to about 10^4 meters. Therefore, we are in the long-wavelength limit and can regard the lightning antenna as a vertically oriented dipole oscillator. The measurements are performed in the near zone of the dipole. The observation point lies above the dipole, which means that the electric field is approximately

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2p(t)}{z^3}\hat{\mathbf{z}},$$

where z is measured from the position of the dipole. The magnetic near-field comes from the displacement current on the right side of

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

Using the integral form of this equation, we calculate the line integral of $\mathbf{B}$ around a circle of radius ρ with z as its symmetry axis. This gives

$$2\pi\rho B_\phi = \frac{1}{c^2}\dot{E}_z\pi\rho^2 = -i\frac{\omega}{c^2}E_z\pi\rho^2,$$

or

$$\mathbf{B} = -i\frac{\omega\rho}{c^2}\frac{1}{4\pi\epsilon_0}\frac{p(t)}{z^3}\hat{\boldsymbol{\phi}}.$$

The Poynting vector is $\mathbf{S} = \epsilon_0\mathbf{E}\times\mathbf{B}$. From the foregoing, this will be radial and increase linearly with frequency.

Source: W.M. Farrell *et al.*, *Radio Science* **41**, RS3008 (2006).

20.2 Fields from an Alternating Current in an Ohmic Wire

(a) We have $\mathbf{E} = E\hat{\mathbf{z}}$ so $\nabla\cdot\mathbf{E} = 0 \;\Rightarrow\; E(\rho,t) = \mathrm{Re}\left\{E(\rho)e^{-i\omega t}\right\}$. With $\mathbf{j} = \sigma\mathbf{E}$ for each Fourier component, the Maxwell equations yield the Helmholtz equation $\nabla^2 E + k^2 E = 0$ with

$$k^2 = \omega^2/c^2 + i\sigma_0\omega\mu_0. \tag{1}$$

In cylindrical coordinates, the general solution that is regular at the origin is

$$\mathbf{E}_{\mathrm{in}}(\rho) = A\,J_0(k\rho)\hat{\mathbf{z}}, \tag{2}$$

where $J_0(x)$ is the zero-order Bessel function. The value of the coefficient A is fixed by the fact that the total current in the wire is I_0. Therefore,

$$I_0 = 2\pi\int_0^a d\rho\rho j(\rho) = 2\pi\sigma A\int_0^a d\rho\rho J_0(k\rho) \;\Rightarrow\; A = kI_0/2\pi a\sigma_0 J_1(ka)$$

because $\int_0^1 dx\,x\,J_0(\gamma x) = \gamma^{-1}J_1(\gamma)$.

(b) Outside the wire, we have $\nabla^2 E + k_0^2 E = 0$ where $\omega = ck_0$. This means that $E(\rho)$ is a linear combination of $J_0(k_0\rho)$ and $N_0(k_0\rho)$ or, more conveniently, a linear combination of the Hankel functions $H_\nu^{(1)} = J_\nu + iN_\nu$ and $H_\nu^{(2)} = J_\nu - iN_\nu$:

$$\mathbf{E}_{\rm out}(\rho) = [CH_0^{(1)}(k_0\rho) + DH_0^{(2)}(k_0\rho)]\hat{\mathbf{z}}. \tag{3}$$

The coefficients C and D follow from the matching conditions at the surface of the wire, $\rho = a$. First, the tangential component of $\mathbf{E}$ (E itself in this case) is continuous. Second, the tangential component of $\mathbf{B}$ (only a perfect conductor supports a surface current) is continuous. The latter reduces to the continuity of $\partial E/\partial\rho$ because E is a function of ρ only and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \;\Rightarrow\; \mathbf{B} = \frac{i}{\omega}\frac{\partial E}{\partial \rho}\hat{\boldsymbol{\phi}}. \tag{4}$$

In any event, because $J_0'(x) = -J_1(x)$ and $N_0'(x) = -N_1(x)$, the two matching conditions yield

$$C = \frac{I_0 k_0 k}{8\sigma}\left[\frac{J_0(ka)}{J_1(ka)}H_1^{(2)}(k_0 a) - \frac{k}{k_0}H_0^{(2)}(k_0 a)\right] \tag{5}$$

and

$$D = \frac{I_0 k_0 k}{8\sigma}\left[-\frac{J_0(ka)}{J_1(ka)}H_1^{(1)}(k_0 a) + \frac{k}{k_0}H_0^{(1)}(k_0 a)\right].$$

Explicit formulae for $\mathbf{E}$ and $\mathbf{B}$ follow from (3) and (4).

(c) Poynting's theorem for any surface A concentric with the wire is

$$-\int_A d\mathbf{A}\cdot\mathbf{S} = \frac{dU_{\rm EM}}{dt} + \int_V d^3r\,\mathbf{j}\cdot\mathbf{E}.$$

The total energy is time-harmonic if the fields are time-harmonic. Therefore, the time-derivative term integrates to zero when we time-average. Otherwise, since $\mathbf{j} = \sigma_0\mathbf{E}$, we conclude from

$$-\int_A d\mathbf{A}\cdot\langle\,\mathbf{S}\,\rangle \;=\frac{1}{2}\sigma_0\int_V d^3r\,|\mathbf{E}|^2 > 0$$

that the normal component of the time-averaged Poynting vector $\langle\,\mathbf{S}\,\rangle$ always points *toward* the z-axis.

(d) To get the rate of energy loss due to resistive heating, $P_{\rm res}$, we use a cylindrical surface whose radius is infinitesimally less than a. Per unit length of wire, the rate of energy loss to ohmic processes is

$$P_{\rm res} = 2\pi a\langle\,S\,\rangle = \frac{\pi a}{\mu_0}{\rm Re}\,\{E_{\rm in}(a)B_{\rm in}^*(a)\}\,,$$

where $E_{\rm in}$ is given by (2) and $B_{\rm in}$ is computed from (4) as

$$B_{\rm in}(a) = \frac{i}{\omega}Ak\frac{\partial J_0(k\rho)}{\partial\rho}\bigg|_{\rho=a} = -\frac{i}{\omega}AkJ_1(ka).$$

Remembering from (1) that the wave vector k is complex, the final result is

$$P_{\rm res} = \frac{\pi a|k|^2}{\mu_0\omega}\left(\frac{I_0}{2\pi a\sigma}\right)^2 {\rm Im}\left\{\frac{k^* J_0(ka)}{J_1(ka)}\right\}.$$

(e) To get the rate of energy loss per unit length due to radiation, $P_{\rm rad}$, we begin with a cylindrical surface whose radius ρ will go to infinity and compute

$$\langle\, S\,\rangle = \frac{1}{2\mu_0}{\rm Re}\,\{E_{\rm out}(\rho)B^*_{\rm out}(\rho)\}\,.$$

The calculation requires (3), (4), and the following asymptotic forms for the Hankel functions:

$$\lim_{\rho\to\infty} H_0^{(1)}(k_0\rho) = \sqrt{\frac{2}{\pi k_0\rho}}e^{-i\pi/4}e^{ik_0\rho} \qquad \lim_{\rho\to\infty} H_0^{(2)}(k_0\rho) = \sqrt{\frac{2}{\pi k_0\rho}}e^{+i\pi/4}e^{-ik_0\rho}.$$

Only the outgoing wave can contribute to energy loss by radiation. Comparison with (3) then tells us that

$$P_{\rm rad} = \lim_{\rho\to\infty} 2\pi\rho\langle\, S\,\rangle = \frac{2|C|^2}{\mu_0\omega},$$

where C is given by (5). This is consistent with part (c) because the incoming wave supplies the net flow of energy into the wire.

20.3 Free-Space Green Function in Two Dimensions

In plane polar coordinates, the Green function for the Helmholtz equation is

$$\left[\nabla^2 + k^2\right] G_0(\boldsymbol{\rho},\boldsymbol{\rho}') = -\delta(\boldsymbol{\rho}-\boldsymbol{\rho}').$$

We put $\boldsymbol{\rho}' = 0$ and note that $G_0(\rho,\phi) = G_0(\rho)$ for the infinite plane. Therefore, the defining equation for the free-space Green function is

$$\nabla^2 G_0(\rho) + k^2 G_0(\rho) = -\frac{\delta(\rho)}{2\pi\rho}. \qquad (1)$$

The homogeneous equation is a form of Bessel's equation of order zero, for which the text establishes that $H_0^{(1)}(k\rho)$ is an outgoing-wave solutions. Otherwise, integrate (1) over the volume V of a cylinder with radius $\rho = \epsilon$ and unit length in the z-direction. The integral over the delta function gives -1. Because $G_0(\rho)$ does not depend on z, the integral over the Laplacian term gives

$$\int_V d^3r\,\nabla^2 G_0 = \int_S d\mathbf{S}\cdot\nabla G_0 = \oint d\boldsymbol{\ell}\frac{\partial G_0}{\partial\rho} = 2\pi\rho\frac{dG_0}{d\rho}\bigg|_{\rho=\epsilon}.$$

The small-argument behavior of the Hankel function is

$$\lim_{x\to 0} H_0^{(1)}(x) = \frac{2i}{\pi}\log x.$$

Therefore, using the proposed Green function,

$$2\pi\rho \frac{dG_0}{d\rho}\bigg|_{\rho=\epsilon} = 2\pi\epsilon \times \left[\frac{i}{4}\frac{2i}{\pi}\frac{1}{\epsilon}\right] = -1.$$

This proves the assertion because the integral of the remaining term in (1) over a disk of radius ϵ is

$$k \cdot \pi\epsilon^2 \cdot \frac{i}{4} H_0^{(1)}(k\epsilon) = -k\pi\epsilon^2 \frac{1}{2\pi}\log k\epsilon,$$

which goes to zero as $\epsilon \to 0$ (by l'Hospital's rule).

20.4 The Method of Descent

(a) Begin with the three-dimensional equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G(x,y,z,t) = -\delta(x)\delta(y)\delta(z)\delta(t).$$

Using this, we see that

$$\begin{aligned}
&\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] \int dz\, G(x,y,z,t) \\
&\qquad = -\int dz \frac{\partial^2}{\partial z^2} G(x,y,z,t) - \int dz\, \delta(x)\delta(y)\delta(z)\delta(t) \\
&\qquad = -\frac{\partial G}{\partial z}\bigg|_{z=-\infty}^{z=\infty} - \delta(x)\delta(y)\delta(t) \\
&\qquad = -\delta(x)\delta(y)\delta(t).
\end{aligned}$$

The last line follows because the z-derivative of $G(r,t) = \delta(t - r/c)/4\pi r$ vanishes at $z = \pm\infty$ for any finite time. This proves the assertion because, by definition,

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right] G_2(x,y,t) = -\delta(x)\delta(y)\delta(t).$$

(b) $G(r,t)$ is an even function of z. Therefore, using the delta function rule,

$$\int_a^b dx g(x)\delta[f(x)] = \sum_k \frac{g(x_k)}{|f'(x_k)|} \qquad \text{where} \qquad f(x_k) = 0 \qquad \text{and} \qquad a < x_k < b,$$

we find

$$
\begin{aligned}
G(x,y,t) &= \int_{-\infty}^{\infty} dz\, G(x,y,z,t) \\
&= \frac{1}{2\pi}\int_0^{\infty} dz\, \frac{\delta(t-\sqrt{z^2+\rho^2}\big/c)}{\sqrt{z^2+\rho^2}} \\
&= \frac{1}{2\pi}\Theta(t-\rho/c)\, \frac{c}{z}\bigg|_{z=\sqrt{c^2t^2-\rho^2}} \\
&= \Theta(t-\rho/c)\frac{1}{2\pi}\frac{1}{\sqrt{t^2-\rho^2/c^2}}.
\end{aligned}
$$

The theta function is required to keep the Green function real.

Source: G. Barton, *Elements of Green's Functions and Propagation* (Clarendon, Oxford, 1989).

20.5 Retarded Fields from Non-Retarded Potentials

(a) The key observation is that $\mathbf{j}_\perp(\mathbf{r},t) \neq 0$ at *every* point in space, even if $\mathbf{j}(\mathbf{r},t)$ is localized and vanishes outside a finite volume of space. On the other hand, $\mathbf{j}_\perp(\mathbf{r},t)$ gets contributions from every part of the physical current density, and all of these occur at the instantaneous time t. Focus now on the integral for $\mathbf{A}_C(\mathbf{r},t)$ at a point $\mathbf{r}$ which lies far outside the physical source current. One contribution to this integral comes from $\mathbf{r}' = \mathbf{r}$ where $t_{\rm ret} = t$ and $\mathbf{j}_\perp(\mathbf{s},t) \neq 0$. However, the latter depends on values of the physical current which are far from the observation point but which occur at the instantaneous time. These contributions are not retarded, so $\mathbf{A}_C$ is not retarded.

(b) Inserting the retardation with a delta function, the magnetic field is

$$
\begin{aligned}
\mathbf{B}(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int d^3r' \int_{-\infty}^{\infty} dt'\, \nabla\times\left[\frac{\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}\right]\mathbf{j}_\perp(\mathbf{r}',t') \\
&= -\frac{\mu_0}{4\pi}\int d^3r' \int_{-\infty}^{\infty} dt'\, \nabla'\times\left[\frac{\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}\right]\mathbf{j}_\perp(\mathbf{r}',t') \\
&= \frac{\mu_0}{4\pi}\int d^3r' \int_{-\infty}^{\infty} dt'\, \left[\frac{\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|}\right]\nabla'\times\mathbf{j}_\perp(\mathbf{r}',t').
\end{aligned}
$$

The last line above follows by partial integration because $\mathbf{j}_\perp/|\mathbf{r}-\mathbf{r}'|$ goes to zero at infinity. However, $\nabla\times\mathbf{j} = \nabla\times\mathbf{j}_\perp + \nabla\times\mathbf{j}_\| = \nabla\times\mathbf{j}_\perp$. Therefore,

$$\mathbf{B}(\mathbf{r},t) = \frac{\mu_0}{4\pi}\int d^3r' \int_{-\infty}^{\infty} dt'\,\delta(t' - t + |\mathbf{r}-\mathbf{r}'|/c)\frac{\nabla'\times\mathbf{j}_\perp(\mathbf{r}',t')}{|\mathbf{r}-\mathbf{r}'|},$$

which is manifestly causal.

(c) The Amprère-Maxwell equation is

$$\nabla\times\mathbf{B} = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}.$$

This can be integrated to

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}(\mathbf{r},t_0) + c^2\int_{t_0}^{t} dt'\,[\nabla\times\mathbf{B}(\mathbf{r}) - \mu_0\mathbf{j}(\mathbf{r})],$$

where $t_0 < t$. This expression involves functions evaluated only at the observation point $\mathbf{r}$ (for which $t_{\rm ret} = t$) and sums only over times which are earlier than t. Hence, the electric field is properly retarded.

Source: C.W. Gardiner and P.D. Drummond, *Physical Review A* **38**, 4897 (1988).

20.6 Radiation from a Magnetized Electron Gas

The Larmor power emitted by a single electron in a uniform magnetic field is $P_L = e^2v_\perp^2B_0^2/6\pi\epsilon_0m^2c^4$, where $v_\perp$ is the component of the electron velocity perpendicular to $\mathbf{B}_0$. Under the conditions stated, we can treat the electrons independently and simply sum the Larmor power from each electron, weighted by the Maxwell distribution

$$n(\mathbf{v}) = n_0\left(\frac{m}{2\pi k_BT}\right)^{3/2}\exp(-mv^2/2k_BT).$$

Since $\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\|$, we can use cylindrical coordinates $(v_\perp,\phi,v_\|)$ to perform the sum:

$$\begin{aligned}\frac{dP}{dV} &= n_0\left(\frac{m}{2\pi k_BT}\right)^{3/2}\frac{e^2B_0^2}{6\pi\epsilon_0m^2c^4}\int_0^{2\pi}d\phi\int_{-\infty}^{\infty}dv_\|\int_0^{\infty}dv_\perp v_\perp^3\exp[-m(v_\perp^2+v_\|^2)/2k_BT]\\ &= n_0\left(\frac{m}{2\pi k_BT}\right)^{3/2}\frac{e^2B_0^2}{6\pi\epsilon_0m^2c^4}\times 2\pi\times\sqrt{\frac{2\pi k_BT}{m}}\times\frac{2k_B^2T^2}{m^2}\\ &= \frac{n_0e^2B_0^2}{3\pi\epsilon_0m^2c^4}\frac{k_BT}{m}.\end{aligned}$$

Source: Prof. M. Gedalin, Ben-Gurion University (public communication).

20.7 Energy Flow from a Point Electric Dipole

The magnetic and electric fields of a point electric dipole are

$$\mathbf{B} = -\frac{\mu_0}{4\pi}\hat{\mathbf{r}} \times \left\{ \frac{\dot{\mathbf{p}}_{\text{ret}}}{r^2} + \frac{\ddot{\mathbf{p}}_{\text{ret}}}{cr} \right\}$$

and

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left\{ \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p}_{\text{ret}}) - \mathbf{p}_{\text{ret}}}{r^3} + \frac{3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\dot{\mathbf{p}}_{\text{ret}}) - \dot{\mathbf{p}}_{\text{ret}}}{cr^2} + \frac{\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}_{\text{ret}}) - \ddot{\mathbf{p}}_{\text{ret}}}{c^2 r} \right\}.$$

We are interested in the radial component of the Poynting vector cross product because

$$\frac{dU}{dt} = \int d\mathbf{A}\cdot\mathbf{S} = \frac{1}{\mu_0}R^2 \int d\Omega\, (\mathbf{E}\times\mathbf{B})\cdot\hat{\mathbf{r}}\big|_{r=R}\,.$$

Therefore, it is convenient to write

$$3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p}) - \mathbf{p} = 3\hat{\mathbf{r}}\times(\mathbf{r}\times\mathbf{p}) + 2\mathbf{p} \qquad \text{and} \qquad \hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}) - \ddot{\mathbf{p}} = \hat{\mathbf{r}}\times(\mathbf{r}\times\ddot{\mathbf{p}})$$

and use the “BAC-CAB” rule to write

$$(\hat{\mathbf{r}}\times\dot{\mathbf{p}})\times[3\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\mathbf{p})-\mathbf{p}] = (\hat{\mathbf{r}}\times\dot{\mathbf{p}})\times[3\hat{\mathbf{r}}\times(\mathbf{r}\times\mathbf{p})+2\mathbf{p}] = 3(\hat{\mathbf{r}}\times\dot{\mathbf{p}})\cdot(\hat{\mathbf{r}}\times\mathbf{p})\hat{\mathbf{r}}+2\dot{\mathbf{p}}(\mathbf{p}\cdot\hat{\mathbf{r}})-2(\mathbf{p}\cdot\dot{\mathbf{p}})\hat{\mathbf{r}} \quad (1)$$

and

$$(\hat{\mathbf{r}}\times\dot{\mathbf{p}})\times[\hat{\mathbf{r}}(\hat{\mathbf{r}}\cdot\ddot{\mathbf{p}}) - \ddot{\mathbf{p}}] = (\hat{\mathbf{r}}\times\dot{\mathbf{p}})\times[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\ddot{\mathbf{p}})] = (\hat{\mathbf{r}}\times\dot{\mathbf{p}})\cdot(\hat{\mathbf{r}}\times\ddot{\mathbf{p}})\cdot\hat{\mathbf{r}}. \qquad (2)$$

Four of the terms that appear in the Poynting vector have the structure of (1). The remaining two terms have the structure of (2). Therefore, with $\hat{\mathbf{p}}\times\hat{\mathbf{r}} = \sin\theta$ and $\hat{\mathbf{p}}\cdot\hat{\mathbf{r}} = \cos\theta$,

$$\begin{aligned}\frac{dU}{dt} &= \frac{1}{4\pi\epsilon_0}\frac{1}{4\pi}2\pi\int_0^\pi d\theta\sin^3\theta\left[\frac{p\dot{p}}{R^3} + \frac{\dot{p}^2}{cR^2} + \frac{\dot{p}\ddot{p}}{c^2R} + \frac{p\ddot{p}}{cR^2} + \frac{\dot{p}\ddot{p}}{c^2R} + \frac{\ddot{p}^2}{c^3R}\right]_{\text{ret}} \\ &= \frac{2}{3}\frac{1}{4\pi\epsilon_0}\left[\frac{d}{dt}\left\{\frac{p^2}{2R^3} + \frac{p\dot{p}}{cR^2} + \frac{\dot{p}^2}{c^2R}\right\}_{\text{ret}} + \frac{\ddot{p}^2_{\text{ret}}}{c^3}\right].\end{aligned}$$

Source: L. Mandel, *Journal of the Optical Society of America* **62**, 1011 (1972).

20.8 A Point Charge Blinks On

(a) By symmetry, the electric field must be spherically symmetric and radial. Therefore, from the integral form of Gauss' law,

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{4\pi\epsilon_0}\frac{q(t)}{r^2}\hat{\mathbf{r}}.$$

The magnetic field satisfies

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

The last term is radial and, by symmetry, the current density which makes $q(t)$ change must be radial also. However, if the curl of $\mathbf{B}$ is to have a radial part, $\mathbf{B}$ itself must be a function of the angular variables. This cannot be true by symmetry. Therefore,

$$\mathbf{B}(\mathbf{r}, t) = 0.$$

(b) With $\rho(\mathbf{r}, t) = q(t)\delta(\mathbf{r})$, the Coulomb gauge scalar potential is

$$\varphi_C(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho(\mathbf{r}, t)}{|\mathbf{r} - \mathbf{r}'|} = \frac{q(t)}{4\pi\epsilon_0 r}.$$

The Coulomb gauge vector potential is

$$\mathbf{A}_C(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int d^3r' \, \frac{\mathbf{j}_\perp(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|},$$

where the transverse current density is defined by

$$\mathbf{j}_\perp(\mathbf{r}, t) = \nabla \times \frac{1}{4\pi}\int d^3r' \, \frac{\nabla' \times \mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}.$$

The current density for this problem must satisfy the continuity equation,

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0.$$

By inspection, we must have

$$\mathbf{j}(\mathbf{r}, t) = -\frac{\dot{q}}{4\pi}\frac{\hat{\mathbf{r}}}{r^2}.$$

The curl of this current density is zero, so $\mathbf{A}_C = 0$ and we reproduce the magnetic field from part (a), $\mathbf{B} = \nabla \times \mathbf{A}_C = 0$. Because the vector potential is zero, we also reproduce the electric field from part (a):

$$\mathbf{E}(\mathbf{r}, t) = -\nabla\varphi_C(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\frac{q(t)}{r^2}\hat{\mathbf{r}}.$$

(c) With $\rho(\mathbf{r}, t) = q(t)\delta(\mathbf{r})$, the Lorenz gauge scalar potential is

$$\varphi_L(\mathbf{r}, t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \, \frac{\rho(\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/c)}{|\mathbf{r} - \mathbf{r}'|} = \frac{q(t - r/c)}{4\pi\epsilon_0 r}. \tag{1}$$

With the current density computed in part (b), we use the hint and write the Lorenz gauge vector potential:

$$\begin{aligned}
\mathbf{A}_L(\mathbf{r},t) &= \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}',t-|\mathbf{r}-\mathbf{r}'|/c)}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{\mu_0}{4\pi}\int d^3r'\int dt'\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)\frac{\dot{q}(t')}{4\pi}\frac{\mathbf{r}'}{r'^3}\frac{1}{|\mathbf{r}-\mathbf{r}'|} \\
&= -\frac{\mu_0}{(4\pi)^2}\int dt'\frac{\dot{q}(t')}{c|t-t'|}\left[\int d^3r'\frac{\mathbf{r}'}{r'^3}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c)\right]. \qquad (2)
\end{aligned}$$

Focus on the space integral in brackets, integrate by parts, and let $\mathbf{s} = \mathbf{r} - \mathbf{r}'$. This gives

$$\begin{aligned}
\int d^3r'\frac{\mathbf{r}'}{r'^3}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c) &= -\int d^3r'\nabla'\left(\frac{1}{r'}\right)\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c) \\
&= \int d^3r'\frac{1}{r'}\nabla'\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c) \\
&= -\nabla\int d^3r'\frac{1}{r'}\delta(t'-t+|\mathbf{r}-\mathbf{r}'|/c) \\
&= -\nabla\int d^3s\frac{\delta(t-t'-s/c)}{|\mathbf{r}-\mathbf{s}|}. \qquad (3)
\end{aligned}$$

We do the space integral in (3) using

$$\frac{1}{|\mathbf{r}-\mathbf{s}|} = \sum_{\ell=0}^{\infty}\frac{r_<^\ell}{r_>^{\ell+1}}P_\ell(\cos\theta),$$

where $r_<\,(r_>)$ is the lesser (greater) of r and s. The integral does not depend on the angle θ between $\mathbf{r}$ and $\mathbf{s}$, so (by orthogonality of the Legendre polynomials) only the $\ell = 0$ term in the sum survives the integration. Therefore, with $\tau = t - t'$,

$$\begin{aligned}
\int d^3s\frac{\delta(\tau-s/c)}{|\mathbf{r}-\mathbf{s}|} &= 4\pi c\int_0^\infty ds\frac{s^2}{r_>}\delta(s-c\tau) \\
&= 4\pi c\int_0^r ds\frac{s^2}{r}\delta(s-c\tau) + 4\pi c\int_r^\infty ds s\delta(s-c\tau) \\
&= 4\pi[\frac{c^3\tau^2}{r}\Theta(r-c\tau)\Theta(\tau) + c^2\tau\Theta(c\tau-r)]. \qquad (4)
\end{aligned}$$

Inserting (4) into (3) and (3) into (2) gives

$$\begin{aligned}\frac{\partial \mathbf{A}_L(\mathbf{r},t)}{\partial t} &= \frac{1}{4\pi\epsilon_0}\nabla\int_{-\infty}^{\infty} dt'\dot{q}(t')\frac{\partial}{\partial t}[\frac{c\tau}{r}\Theta(r-c\tau)\Theta(\tau)+\Theta(c\tau-r)] \\ &= \frac{1}{4\pi\epsilon_0}\nabla\int_{-\infty}^{\infty} dt'\dot{q}(t')\left\{\frac{1}{r}\Theta(r-c\tau)\Theta(\tau)+\frac{\tau}{r}\Theta(r-c\tau)\delta(\tau)\right. \\ &\qquad \left. -\frac{c\tau}{r}\delta(r-c\tau)+\delta(c\tau-r)\right\}.\end{aligned}$$

The last two terms in the brackets cancel and the second term is zero. Otherwise, the product of the theta functions requires that $t - r/c \le t' \le t$. Therefore,

$$\frac{\partial \mathbf{A}_L(\mathbf{r},t)}{\partial t} = \frac{1}{4\pi\epsilon_0}\nabla\int_{t-r/c}^{t} dt'\frac{dq(t')/dt'}{r} = \frac{1}{4\pi\epsilon_0}\nabla\left[\frac{q(t)}{r}-\frac{q(t-r/c)}{r}\right].$$

Combining this with the scalar potential in (1) shows that

$$\mathbf{E} = -\nabla\varphi_L - \frac{\partial \mathbf{A}_L}{\partial t} = -\nabla\frac{q(t-r/c)}{4\pi\epsilon_0 r} - \frac{1}{4\pi\epsilon_0}\nabla\left[\frac{q(t)}{r}-\frac{q(t-r/c)}{r}\right] = -\frac{1}{4\pi\epsilon_0}\nabla\frac{q(t)}{r}.$$

Source: P.R. Berman, *American Journal of Physics* **76**, 48 (2008).

20.9 The Birth of Radiation

(a) The electric field lines pinch off at points where $\mathbf{E} = 0$ and two electric fields cross. The necessary conditions are

$$\frac{dR}{dz} = 0 \qquad\qquad \frac{dR}{d\rho} = 0.$$

We get $dR/dz = 0$ when $z = 0$ so it is sufficient to study

$$R(\rho,t) = R(\rho, z=0, t) = \frac{p(t-\rho/c)}{\rho} + \frac{\dot{p}(t-\rho/c)}{c}.$$

In particular, we demand

$$0 = \frac{d}{d\rho}R(\rho,t) = \frac{d}{d\rho}\left\{\frac{p(t-\rho/c)}{\rho}+\frac{\dot{p}(t-\rho/c)}{c}\right\} = -\frac{R}{\rho} - \frac{d}{dt}\frac{\dot{p}(t-\rho/c)}{c^2}.$$

This gives the required formula,

$$\frac{R(\rho,t)}{\rho} + \frac{\ddot{p}(t-\rho/c)}{c^2} = 0.$$

(b) With $p(t) = p_0\cos\omega t$ the expressions above for $R(\rho,t)$ and $dR/d\rho$ are

$$R_0 = \frac{p_0}{\rho}\cos\Omega - \frac{p_0\omega}{c}\sin\Omega$$

and

$$\frac{R_0}{\rho} - \frac{p_0 \omega^2}{c^2} \cos \Omega = 0,$$

where $\Omega = \omega t - \rho\omega/c$. These two must be solved simultaneously for the unknown detachment radius ρ. Eliminating the sinusoids gives

$$(\bar{R}_0^2 - 1)\bar{\rho}^4 - \bar{R}_0^2 \bar{\rho}^2 + \bar{R}_0^2 = 0,$$

in terms of the dimensionless lengths $\bar{R}_0 = R_0 c/p_0\omega$ and $\bar{\rho} = \omega c/\rho$. This is a quadratic equation in the variable $\bar{\rho}^2$. We get real solutions only if the discriminant is non-negative, i.e., $\bar{R}_0^4 - 4\bar{R}_o^2(\bar{R}_0^2 - 1) \geq 0$ or

$$|R_0| \; \leq \frac{2}{\sqrt{3}} \frac{p_0 \omega}{c}.$$

When this is true, one of the two roots

$$\bar{\rho}^2 = \frac{1}{1/\bar{R}_0^2 - 1} \frac{-1 \pm \sqrt{4/\bar{R}_0^2 - 3}}{2}$$

is always positive.

Source: G. Scharf, *From Electrostatics to Optics* (Springer, Berlin, 1994).

20.10 An Electrically Short Antenna

(a) The angular distribution of interest is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 c I_0^2}{8\pi^2} \left[\frac{\cos(kd\cos\theta) - \cos kd}{\sin\theta} \right]^2.$$

When $kd \ll 1$, we use $\cos x \approx 1 - \frac{1}{2}x^2$ to get

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 c I_0^2}{32\pi^2} (kd)^4 \sin^2\theta.$$

(b) A time-harmonic polarization current obeys

$$\mathbf{j} = \frac{\partial \mathbf{P}}{\partial t} = -i\omega \mathbf{P}.$$

The associated electric dipole moment is

$$\mathbf{p} = \int d^3r \, \mathbf{P} = \frac{i}{\omega} \int d^3r \, \mathbf{j}.$$

For our dipole antenna aligned with the z-axis with current $I(z) = I_0 \sin(kd - k|z|)$, we get $\mathbf{p} = p\hat{\mathbf{z}}$ and the foregoing simplifies to

$$p = \frac{i}{\omega} \int_{-d}^{d} dz I(z) = \frac{iI_0}{ck} \left[\int_{-d}^{0} dz \sin[k(d+z)] + \int_{0}^{d} dz \sin[k(d-z)] \right] = \frac{2iI_0}{ck^2}(1 - \cos kd).$$

In the long-wavelength limit, $p \approx iI_0 d^2/c$. Inserting this into the time-averaged angular distribution of power radiated by a point electric dipole gives

$$\left\langle \frac{dP}{d\Omega} \right\rangle_{\text{dipole}} = \frac{ck^4}{32\pi^2 \epsilon_0} |p|^2 \sin^2\theta = \frac{\mu_0 c I_0^2}{32\pi^2} (kd)^4 \sin^2\theta. \tag{1}$$

This the same answer as part (a).

(c) We have $I(z) = I_0 \sin(kd - k|z|)$, so $I(0) = I_0 \sin kd \approx I_0 kd$ when $kd \ll 1$. In that case,

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0 c I^2(0)}{32\pi^2} (kd)^2 \sin^2\theta. \tag{2}$$

By assumption, $I(z) = I(0) = const.$ for a point dipole modeled using two point charges. In that case, the formula derived in part (b) gives the dipole moment as

$$p = \frac{i}{\omega} \int_{-d}^{d} dz I(z) = i\frac{2I(0)d}{\omega}.$$

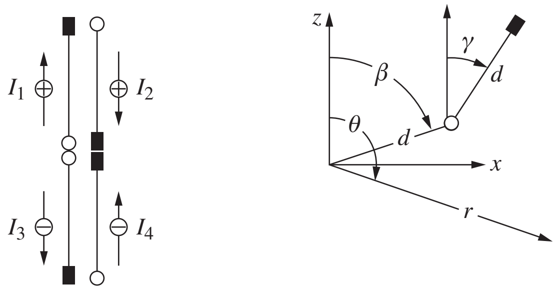
Substituting this into the point dipole expression for $\langle dP/d\Omega \rangle$ in (1) gives

$$\left\langle \frac{dP}{d\Omega} \right\rangle_{\text{dipole}} = \frac{ck^4}{32\pi^2 \epsilon_0} \frac{4I^2(0)d^2}{\omega^2} \sin^2\theta = \frac{\mu_0 c I_0^2}{8\pi^2} (kd)^2 \sin^2\theta.$$

This agrees with (2) except for a factor of 4.

20.11 The Time-Domain Electric Field of a Dipole Antenna

The left panel of the figure below redraws from the text the four linear end-fed antennas used to model a dipole antenna. The right panel indicates the position of the feed point (open circle) and the orientation of each antenna with respect to antenna 1. Specifically, the origins of antennas 2 and 4 are shifted up the z-axis by a distance d and the angle $\beta = \pi$ applies to antenna 4. The angle $\gamma = \pi$ applies to antennas 2 and 3. Plus charge is carried by antennas 1 and 2. Negative charge is carried by antennas 3 and 4. Finally, the launch time is delayed by d/c for antennas 2 and 4.



Taking account of the above factors when evaluating the vector potential integral for each antenna gives the total electric field as the sum of four terms:

$$\begin{aligned}
\mathbf{E} &= \frac{\mu_0 c}{4\pi}\frac{\sin\theta}{1-\cos\theta}\left[\frac{I_S(t-r/c)}{r} - \frac{I_S(t-d/c-(r-d\cos\theta)/c)}{r}\right]\hat{\boldsymbol{\theta}} \\
&+ \frac{\mu_0 c}{4\pi}\frac{\sin(\theta-\pi)}{1-\cos(\theta-\pi)}\left[\frac{I_S(t-d/c-(r-d\cos\theta)/c)}{r}\right. \\
&\qquad \left. - \frac{I_S(t-d/c-(r-d\cos\theta)/c-d[1-\cos(\theta-\pi)]/c)}{r}\right]\hat{\boldsymbol{\theta}} \\
&- \frac{\mu_0 c}{4\pi}\frac{\sin(\theta-\pi)}{1-\cos(\theta-\pi)}\left[\frac{I_S(t-r/c)}{r} - \frac{I_S(t-r/c-d[1-\cos(\theta-\pi)/c]}{r}\right]\hat{\boldsymbol{\theta}} \\
&- \frac{\mu_0 c}{4\pi}\frac{\sin\theta}{1-\cos\theta}\left[\frac{I_S(t-d/c-[r-d\cos(\theta-\pi)]/c)}{r}\right. \\
&\qquad \left. - \frac{I_S(t-d/c-[r-d\cos(\theta-\pi)]/c-d[1-\cos\theta]/c)}{r}\right]\hat{\boldsymbol{\theta}}.
\end{aligned}$$

Now, $\cos(\theta-\pi) = -\cos\theta$, so

$$\begin{aligned}
\mathbf{E} &= \frac{\mu_0 c}{4\pi r}\left\{\frac{\sin\theta}{1-\cos\theta}\left[I_S(t-r/c) - I_S(t-r/c-d[1-\cos\theta]/c)\right]\right. \\
&- \frac{\sin\theta}{1+\cos\theta}\left[I_S(t-r/c-d[1-\cos\theta]/c) - I_S(t-r/c-2d/c)\right] \\
&+ \frac{\sin\theta}{1+\cos\theta}\left[I_S(t-r/c) - I_S(t-r/c-d[1+\cos\theta]/c)\right] \\
&\left. - \frac{\sin\theta}{1-\cos\theta}\left[I_S(t-r/c-d[1+\cos\theta]/c) - I_S(t-r/c-2d/c)\right]\right\}\hat{\boldsymbol{\theta}}.
\end{aligned}$$

Combining like terms,

$$\begin{aligned}
\mathbf{E} &= \frac{\mu_0 c}{4\pi r}\sin\theta\left[I_S(t-r/c)\left(\frac{1}{1-\cos\theta}+\frac{1}{1+\cos\theta}\right)\right. \\
&+ I_S(t-r/c-2d/c)\left(\frac{1}{1+\cos\theta}+\frac{1}{1-\cos\theta}\right) \\
&+ I_S(t-r/c-d[1-\cos\theta]/c)\left(\frac{-1}{1-\cos\theta}-\frac{1}{1+\cos\theta}\right) \\
&\left. + I_S(t-r/c-d[1+\cos\theta]/c)\left(\frac{-1}{1+\cos\theta}-\frac{1}{1-\cos\theta}\right)\right]\hat{\boldsymbol{\theta}}.
\end{aligned}$$

Finally, we recover the expression quoted in the text,

$$\begin{aligned}\mathbf{E} &= \frac{\mu_0 c}{2\pi \sin\theta}\left\{\frac{I_S(t-r/c)}{r} + \frac{I_S(t-r/c-2d/c)}{r}\right. \\ &\quad - \frac{I_S[t-d/c-(r-d\cos\theta)/c]}{r} \\ &\quad \left. - \frac{I_S[t-d/c-(r+d\cos\theta)/c]}{r}\right\}\hat{\boldsymbol{\theta}}.\end{aligned}$$

Source: G.S. Smith, *An Introduction to Classical Electromagnetic Radiation* (University Press, Cambridge, 1997).

20.12 Radiation Recoil

(a) A quantity independent of distance from the source is r^2 multiplied by the density of electromagnetic linear momentum in the radiation zone:

$$r^2\mathbf{g}_{\rm rad} = r^2\mathbf{r}\times(\epsilon_0\mathbf{E}_{\rm rad}\times\mathbf{B}_{\rm rad}).$$

But $c\mathbf{B}_{\rm rad} = \hat{\mathbf{r}}\times\mathbf{E}_{\rm rad}$ and $\hat{\mathbf{r}}\cdot\mathbf{E}_{\rm rad}$. Therefore,

$$r^2\mathbf{g}_{\rm rad} = \frac{\epsilon_0}{c}\hat{\mathbf{r}}|\mathbf{E}_{\rm rad}|^2 \propto \frac{dP}{d\Omega}\hat{\mathbf{r}}.$$

In other words, the angular dependence of $r^2\mathbf{g}_{\rm rad}$ is determined by the angular distribution of radiated power. Hence, any source which radiates the same amount of power in the $\hat{\mathbf{r}}$ direction as in the $-\hat{\mathbf{r}}$ direction cannot experience recoil. This is the case for dipole radiation.

(b) It is sufficient to break the symmetry indicated in part (a). For example, place a perfect mirror in the vicinity of a dipole radiator.

20.13 Non-Radiating Sources

The Fourier transforms of $\mathbf{f}(\mathbf{r})$ and $\mathbf{j}(\mathbf{r}|\omega)$ are

$$\mathbf{f}(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d^3k\,\hat{\mathbf{f}}(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}) \qquad \text{and} \qquad \mathbf{j}(\mathbf{r}|\omega) = \frac{1}{(2\pi)^3}\int d^3k\,\hat{\mathbf{j}}(\mathbf{k}|\omega)\exp(i\mathbf{k}\cdot\mathbf{r}).$$

Substituting these into the given equation shows that

$$\frac{1}{i\omega}\left[i\mathbf{k}\times(i\mathbf{k}\times\hat{\mathbf{f}}) - \frac{\omega^2}{c^2}\hat{\mathbf{f}}\right] = \hat{\mathbf{j}}(\mathbf{k}|\omega).$$

Using the BAC-CAB rule, this equation reads

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\hat{\mathbf{f}} - \mathbf{k}(\hat{\mathbf{f}}\cdot\mathbf{k}) = i\omega\hat{\mathbf{j}}(\mathbf{k}|\omega).$$

Finally, take the cross product of this equation with $\mathbf{k}$. This gives

$$\left(k^2 - \frac{\omega^2}{c^2}\right)\mathbf{k}\times\hat{\mathbf{f}} = i\omega\mathbf{k}\times\hat{\mathbf{j}}(\mathbf{k}|\omega) = i\omega\hat{\mathbf{j}}_\perp(\mathbf{k}|\omega).$$

Therefore, if $\omega = ck$, the transverse component of the Fourier transform of the current density vanishes. The text showed that this condition guarantees that no radiation is produced. Combining the two Maxwell equations,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = i\omega \mathbf{B} \qquad \text{and} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{j} = -\frac{i\omega}{c^2}\mathbf{E},$$

for time-harmonic fields gives

$$\mu_0 \mathbf{j}(\mathbf{r}|\omega) = \frac{1}{i\omega}\left\{\nabla \times [\nabla \times \mathbf{E}(\mathbf{r})] - \frac{\omega^2}{c^2}\mathbf{E}(\mathbf{r})\right\}.$$

Comparing this with the given equation shows that $\mathbf{f}(\mathbf{r})\exp(-i\omega t)$ is the electric field produced by this non-radiating current density.

Source: A.J. Devaney and E. Wolf, *Physical Review D* **8**, 1044 (1973).

20.14 Lorentz Reciprocity

(a) The Maxwell curl equations for a time-harmonic sources are

$$\nabla \times \mathbf{E}_1 = i\omega \mathbf{B}_1 \qquad \text{and} \qquad \nabla \times \mathbf{B}_1 = \mu_0 \mathbf{j}_1 - i\frac{\omega}{c^2}\mathbf{E}_1.$$

Take the dot product of the rightmost equation with $\mathbf{E}_2$, subtract from this the same expression with 1 and 2 exchanged, and integrate over a volume V. The result is

$$\int_V d^3r\,(\mathbf{E}_2 \cdot \nabla \times \mathbf{B}_1 - \mathbf{E}_1 \cdot \nabla \times \mathbf{B}_2) = \mu_0 \int_V d^3r\,(\mathbf{j}_1 \cdot \mathbf{E}_2 - \mathbf{j}_2 \cdot \mathbf{E}_1).$$

Now, form the dot product of Faraday's law above with $\mathbf{B}_2$, subtract from this the same expression with 1 and 2 exchanged, and integrate over the volume V. The result is

$$\int_V d^3r\,(\mathbf{B}_2 \cdot \nabla \times \mathbf{E}_1 - \mathbf{B}_1 \cdot \nabla \times \mathbf{E}_2) = 0.$$

Adding the preceding two equations gives

$$\int_V d^3r\,\nabla \cdot (\mathbf{E}_1 \times \mathbf{B}_2 - \mathbf{E}_2 \times \mathbf{B}_1) = \mu_0 \int_V d^3r\,(\mathbf{j}_1 \cdot \mathbf{E}_2 - \mathbf{j}_2 \cdot \mathbf{E}_1).$$

An application of the divergence equation produces the desired expression:

$$\mu_0 \int_V d^3r\,(\mathbf{E}_2 \cdot \mathbf{j}_1 - \mathbf{E}_1 \cdot \mathbf{j}_2) = \int_S d\mathbf{S} \cdot (\mathbf{E}_1 \times \mathbf{B}_2 - \mathbf{E}_2 \times \mathbf{B}_1).$$

(b) Under the stated conditions, $d\mathbf{S} = \hat{\mathbf{r}}dS$ and all radiation fields satisfy $\hat{\mathbf{r}} \times \mathbf{E} = c\mathbf{B}$. This relation and $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b})$ imply that the surface integral in part (a) vanishes. Therefore,

$$\int_V d^3r\,\mathbf{E}_2 \cdot \mathbf{j}_1 = \int_V d^3r\,\mathbf{E}_1 \cdot \mathbf{j}_2.$$

(c) The polarization of a point dipole at $\mathbf{r}_k$ is $\mathbf{P}_k(t) = \mathbf{p}_k(t)\delta(\mathbf{r}-\mathbf{r}_k)$. The associated current density is

$$\mathbf{j}_k = \frac{\partial \mathbf{P}_k}{\partial t} = -i\omega\mathbf{P}_k = -i\omega\mathbf{p}_k\delta(\mathbf{r}-\mathbf{r}_k).$$

Substituting this into the formula in part (b) gives

$$\mathbf{p}_1 \cdot \mathbf{E}_2(\mathbf{r}_1) = \mathbf{p}_2 \cdot \mathbf{E}_1(\mathbf{r}_2).$$

Source: L.D. Landau and E.M. Lifshitz, *The Electrodynamics of Continuous Media* (Pergamon, Oxford, 1960).

20.15 Radiation from a Phased Array

(a) For a single time-harmonic source, the radiation vector is

$$\boldsymbol{\alpha}_0 = \frac{d}{dt}\int d^3r'\,\mathbf{j}_0(\mathbf{r}', t-r/c+\hat{\mathbf{r}}\cdot\mathbf{r}'/c) = -i\omega\exp[i(kr-\omega t)]\int d^3r'\,\mathbf{j}_0(\mathbf{r}')\exp(-i\mathbf{k}\cdot\mathbf{r}').$$

For the case at hand, the total current density is $\mathbf{j}(\mathbf{r}) = \sum_{k=1}^{N}\mathbf{j}_0(\mathbf{r}-\mathbf{R}_k)\exp(-i\delta_k)$. Therefore,

$$\begin{aligned}
\boldsymbol{\alpha} &= -i\omega\exp[i(kr-\omega t)]\sum_{k=1}^{N}\exp(-i\delta_k)\mathbf{j}(\mathbf{r}'-\mathbf{R}_k)\exp(-i\mathbf{k}\cdot\mathbf{r}') \\
&= -i\omega\exp[i(kr-\omega t)]\sum_{k=1}^{N}\exp[i(\mathbf{k}\cdot\mathbf{R}_k+\delta_k)]\int d^3r'\,\mathbf{j}_0(\mathbf{r}'-\mathbf{R}_k)\exp[-i\mathbf{k}\cdot(\mathbf{r}'-\mathbf{R}_k)] \\
&= \left[\sum_{k=1}^{N}\exp[i(\mathbf{k}\cdot\mathbf{R}_k+\delta_k)]\right]\boldsymbol{\alpha}_0.
\end{aligned}$$

This result implies that the angular distribution of power has the form

$$\frac{dP}{d\Omega} = \left|\sum_{k=1}^{N}\exp[i(\mathbf{k}\cdot\mathbf{R}_k+\delta_k)]\right|^2 \left.\frac{dP}{d\Omega}\right|_0.$$

(b) For observation points in the x-z plane, $\mathbf{k} = k(\hat{\mathbf{x}}\sin\theta+\hat{\mathbf{z}}\cos\theta)$ where $\omega = ck$. Let the current loop lie in the $z=0$ plane. In that case,

$$\begin{aligned}
\int d^3r'\,\mathbf{j}_0(\mathbf{r}')\exp(i\mathbf{k}\cdot\mathbf{r}') &= I\oint d\mathbf{s}\exp(i\mathbf{k}\cdot\mathbf{s}) \\
&= Ia\hat{\mathbf{y}}\left[\exp(-\tfrac{1}{2}ika\sin\theta)-\exp(\tfrac{1}{2}ika\sin\theta)\right] \\
&= -2iIa\sin(\tfrac{1}{2}ka\sin\theta)\hat{\mathbf{y}},
\end{aligned}$$

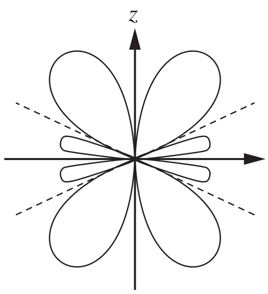
because $\mathbf{k} \cdot \mathbf{s} = 0$ for the two legs parallel to the x-axis and their contributions cancel. The two loops are really at $z = \pm a/2$ and $\delta_1 = \delta_1 = 0$ so the coherence factor is

$$|\exp(\frac{1}{2}ika\cos\theta) + \exp(\frac{1}{2}ika\cos\theta)|^2 = 4\cos^2(\frac{1}{2}ka\cos\theta).$$

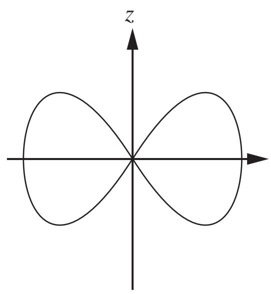
Hence, the time-averaged angular distribution of power is

$$\begin{aligned}\left\langle \frac{dP}{d\Omega} \right\rangle &= 4\cos^2(\frac{1}{2}ka\cos\theta) \times \frac{I^2a^2\omega^2}{2\pi c}\frac{\mu_0}{4\pi} \times \sin^2(\frac{1}{2}ka\sin\theta) \\ &= \frac{\mu_0\omega^2a^2I^2}{2\pi^2c}\sin^2(\frac{1}{2}ka\sin\theta)\cos^2(\frac{1}{2}ka\cos\theta).\end{aligned}$$

When $ka \ll 1$, $dP/d\Omega \propto \sin^2\theta$. The long-wavelength radiation is magnetic dipole.



$ka = 2\pi$



$ka \ll 1$

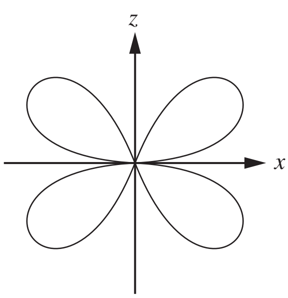
(c) The only change here is that $\delta_1 = 0$ and $\delta_2 = \pi$ so the coherence factor changes to

$$|\exp(\frac{1}{2}ika\cos\theta) - \exp(\frac{1}{2}ika\cos\theta)|^2 = 4\sin^2(\frac{1}{2}ka\cos\theta).$$

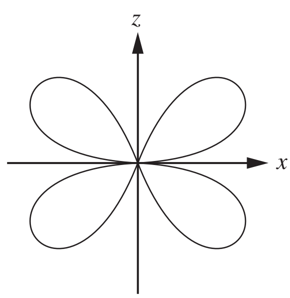
This gives

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0\omega^2a^2I^2}{2\pi^2c}\sin^2(\frac{1}{2}ka\sin\theta)\sin^2(\frac{1}{2}ka\cos\theta).$$

When $ka \ll 1$, $dP/d\Omega \propto \sin^2 2\theta$. The long-wavelength radiation is magnetic quadrupole, i.e., two magnetic dipoles that (almost) cancel.



$ka = 2\pi$



$ka \ll 1$

20.16 Radiation from a Square Loop

(a) The factor $\mathbf{r}' \cdot \hat{\mathbf{r}} = x' \sin\theta$ is the same for the two current legs parallel to the x-axis. But the current runs in opposite directions so their contributions to the integral cancel. The contribution from the remaining two legs is

$$\boldsymbol{\alpha}(\mathbf{r}, t) = \hat{\mathbf{y}} \frac{d}{dt} \int_{-a}^{a} dy I[t - r/c + (a/c)\sin\theta] + \hat{\mathbf{y}} \frac{d}{dt} \int_{a}^{-a} dy I[t - r/c - (a/c)\sin\theta,$$

or

$$\boldsymbol{\alpha} = \hat{\mathbf{y}} 2a \left\{ \dot{I}[t - r/c + (a/c)\sin\theta] - \dot{I}[t - r/c - (a/c)\sin\theta] \right\}.$$

(b) The current is

$$I(t) = (I_0 t/\tau)[\Theta(t) - \Theta(t - \tau)] + I_0 \Theta(t - \tau).$$

Therefore,

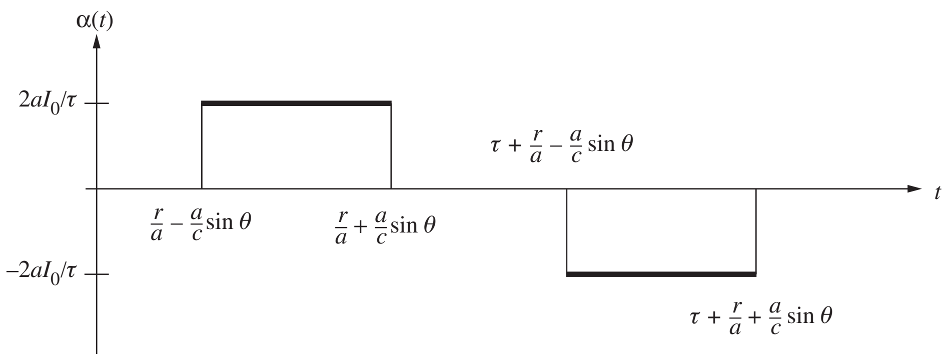
$$\dot{I}(t) = \frac{I_0}{\tau}[\Theta(t) - \Theta(t - \tau)] + \frac{I_0 t}{\tau}\delta(t) - \frac{I_0 t}{\tau}\delta(t - \tau) + I_0 \delta(t - \tau) = \frac{I_0}{\tau}[\Theta(t) - \Theta(t - \tau)].$$

We thus get two terms for the scalar $\alpha(r, \theta, t)$:

(1) $2aI_0/\tau$ in the interval $r/c - (a/c)\sin\theta < t < \tau + r/c - (a/c)\sin\theta$

(2) $-2aI_0/\tau$ in the interval $r/c + (a/c)\sin\theta < t < \tau + r/c + (a/c)\sin\theta$.

But $\tau > 2a/c$, so $r/c + (a/c)\sin\theta < r/c - (a/c)\sin\theta + \tau$. Hence, the graph is as follows.



Source: Prof. M.J. Cohen, University of Pennsylvania (private communication).

20.17 Linear Antenna Radiation

(a) Because $\hat{\mathbf{z}} = \hat{\mathbf{r}}\cos\theta - \hat{\boldsymbol{\theta}}\sin\theta$ and $\boldsymbol{\alpha} \parallel \hat{\mathbf{z}}$,

$$\mathbf{E}_{\rm rad}(r,t) = \frac{\mu_0}{4\pi r}\hat{\mathbf{r}}\times\hat{\mathbf{r}}\times\boldsymbol{\alpha}(\mathbf{r},t) = \hat{\boldsymbol{\theta}}\frac{\mu_0\sin\theta}{4\pi r}\alpha(r,t),$$

where

$$\alpha(r,t) = \frac{d}{dt}\int d^3r'\, I(\mathbf{r}', t - r/c + \hat{\mathbf{r}}\cdot\mathbf{r}'/c) = \int_{-h}^{h} dz'\,\frac{d}{dt}I(z',\, t - r/c + z'\cos\theta/c).$$

When $I(z,t) = A\delta(t)$, this gives

$$\begin{aligned}\mathbf{E}_{\rm rad}(\mathbf{r},t) &= \hat{\boldsymbol{\theta}}\frac{A\mu_0 c\sin\theta}{4\pi r}\int_{-h}^{h} dz'\frac{d}{dt}\delta(t - r/c + z'\cos\theta/c) \\ &= \hat{\boldsymbol{\theta}}\frac{A\mu_0 c\tan\theta}{4\pi r}\int_{-h}^{h} dz'\frac{d}{dz'}\delta(t - r/c + z'\cos\theta/c) \\ &= \hat{\boldsymbol{\theta}}\frac{A\mu_0 c\tan\theta}{4\pi r}\left[\delta(t - r/c + h\cos\theta/c) - \delta(t - r/c - h\cos\theta/c)\right].\end{aligned}$$

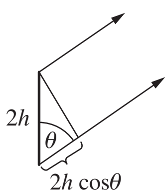
The field is non-zero when $r_\pm = \pm h\cos\theta + ct$. From the derivation above, we suspect that these arise from point sources located at each end of the antenna. In the $y = 0$ plane, this would correspond to circular wave fronts defined by $x^2 + (z \mp h)^2 = c^2t^2$. To check this, expand the square and use $r^2 = x^2 + z^2$ and $z = r\cos\theta$ to get

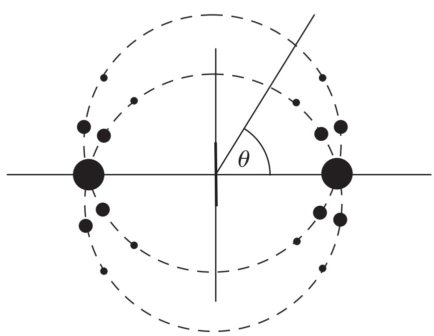
$$r^2 \mp 2h\cos\theta r + h^2 - c^2t^2 = 0.$$

This is solved by $r = \pm h\cos\theta + \sqrt{c^2t^2 - h^2\sin^2\theta}$. This gives the desired formula in the limit $ct \gg h$ which is appropriate for radiation fields. The time delay between the signals is

$$\Delta t = (t - r/c + h\cos\theta/c) - (t - r/c - h\cos\theta/c) = 2h\cos\theta/c,$$

which is sensible on geometrical grounds (see figure below at left). The figure below at right shows the antenna, the two circular wave fronts, the signal time delay, and the effect of the $\tan\theta$ pre-factor on the amplitude of the electric field.

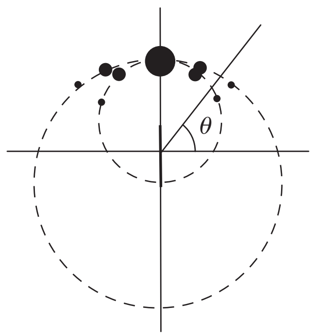




(b) When $I(z,t) = \bar{A}\delta(t - z/c)$, we get

$$\begin{aligned}
\mathbf{E}_{\rm rad}(\mathbf{r},t) &= \hat{\boldsymbol{\theta}}\frac{\bar{A}\mu_0 \sin\theta}{4\pi r}\int_{-h}^{h} dz'\,\frac{d}{dt}\delta(t - z'/c - r/c + z'\cos\theta/c) \\
&= \hat{\boldsymbol{\theta}}\frac{\bar{A}\mu_0 c\sin\theta}{4\pi r(\cos\theta - 1)}\int_{-h}^{h} dz'\,\frac{d}{dz'}\delta(t - z'/c - r/c + z'\cos\theta/c) \\
&= \hat{\boldsymbol{\theta}}\frac{\bar{A}\mu_0 c\sin\theta}{4\pi r(1-\cos\theta)}\left\{\delta[t - r/c + h(1-\cos\theta)/c]\right. \\
&\qquad \left. -\ \delta[t - r/c - h(1-\cos\theta)/c]\right\}.
\end{aligned}$$

Now the field is non-zero when $r_\pm = \pm h(1-\cos\theta) + ct$. Again, in the radiation zone, a circular wave front is centered at each end of the antenna. But now, $r_+ = r_- = ct$ at $\theta = 0$ so the radii of the circles differ by $2h$. The amplitude factor is $\sin\theta/(1-\cos\theta)$ so the electric field pulses appear as shown below. The time delay is $\Delta t = 2h(1-\cos\theta)/c$ which is $2h/c$ larger than the previous case. This is reasonable because the travelling wave of current propagates in the $+z$-direction (up the antenna) so the signal from the top end is delayed (see figure below).



(c) When $I(z,t) = A\exp(-i\omega t)$, define $k = \omega/c$ so

$$\begin{aligned}
\hat{\mathbf{r}}\times\boldsymbol{\alpha} &= \hat{\boldsymbol{\phi}}Ac\tan\theta\{\exp[-i\omega(t - r/c + h\cos\theta/c)] - \exp[-i\omega(t - r/c - h\cos\theta/c)]\} \\
&= \hat{\boldsymbol{\phi}}2Ac\sin(kh\cos\theta)\tan\theta\exp[i(kr-\omega t)].
\end{aligned}$$

$$\frac{dP}{d\Omega} \sim |\hat{\mathbf{r}}\times\boldsymbol{\alpha}|^2 \sim \tan^2\theta\sin^2(kh\cos\theta),$$

so, with $z = kh\cos\theta$, the radiated power is

$$P \sim \int_0^\pi d\theta\sin\theta\tan^2\theta\sin^2(kh\cos\theta) \sim kh\int_{-kh}^{kh} dz\frac{\sin^2 z}{z^2} - \frac{1}{kh}\int_{-kh}^{kh} dz\sin^2 z.$$

The second term dominates when $kh \ll 1$ and we get

$$P \sim \frac{\sin 2kh}{2kh} - 1 \sim k^2h^2 \sim \omega^2\tau^2.$$

The first term dominates when $kh \gg 1$ and we get

$$P \sim kh \int_{-\infty}^{\infty} dx \frac{\sin^2 x}{x^2} \sim kh \sim \omega\tau.$$

(d) When $I(z,t) = \bar{A}\exp[i(kz - \omega t)]$,

$$\begin{aligned} \hat{\mathbf{r}} \times \boldsymbol{\alpha} &= \hat{\boldsymbol{\phi}}\bar{A}\frac{c\sin\theta}{1-\cos\theta}\{\exp[-i\omega(t - r/c + h(\cos\theta - 1)/c)] \\ &\quad - \exp[-i\omega(t - r/c - h(\cos\theta - 1)/c)]\} \\ &= \hat{\boldsymbol{\phi}}2i\bar{A}\frac{c\sin\theta}{1-\cos\theta}\sin[kh(1-\cos\theta)]\exp[i(kr - \omega t)]. \end{aligned}$$

$$\frac{d\bar{P}}{d\Omega} \sim |\hat{\mathbf{r}} \times \boldsymbol{\alpha}|^2 \sim \frac{\sin^2\theta}{(1-\cos\theta)^2}\sin^2[kh(1-\cos\theta)],$$

so with $z = kh(1-\cos\theta)$, the radiated power is

$$\bar{P} \sim \int_0^{\pi} d\theta \frac{\sin^3\theta}{(1-\cos\theta)^2}\sin^2[kh(1-\cos\theta) \sim 2\int_0^{2kh} dz \frac{\sin^2 z}{z} - \frac{1}{kh}\int_0^{2kh} dz \sin^2 z.$$

The second term dominates when $kh \ll 1$ and we get

$$\bar{P} \sim \frac{\sin 4kh}{4kh} - 1 \sim k^2h^2 \sim \omega^2\tau^2.$$

The first term dominates when $kh \gg 1$ and we get

$$\bar{P} \sim \int_0^{2kh} dx \frac{\sin^2 x}{x} \sim [\ln x - \mathrm{Ci}(2x)]_0^{2kh} \sim \ln kh \sim \ln \omega\tau,$$

because $\mathrm{Ci}(x \to 0) \to \ln x$ and $\mathrm{Ci}(x \to \infty) \to 0$.

Source: G.S. Smith, *An Introduction to Classical Electromagnetic Radiation* (University Press, Cambridge, 1997).

20.18 Radiation from a Filamentary Current

(a) The wire is neutral so there is no scalar potential. The vector potential in the filamentary limit is

$$\mathbf{A}(\rho,t) = \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\int_{-\infty}^{\infty} dz\, \frac{I(t - R/c)}{R}.$$

Since $R^2 = z^2 + \rho^2$ and $I(t) = 0$ when $t < 0$, the only portions of the wire that contribute to the field at $(\rho, 0)$ at time t satisfy $z^2 < c^2t^2 - \rho^2$. Therefore, $\mathbf{A}(\rho > ct) = 0$ and

$$\mathbf{A}(\rho, t) = \hat{\mathbf{z}}\frac{\mu_0}{2\pi} \int_0^{\sqrt{c^2t^2-\rho^2}} \frac{dz}{\sqrt{z^2+\rho^2}} \qquad \rho < ct.$$

Performing the integral, we conclude that

$$\mathbf{A}(\rho, t) = \hat{\mathbf{z}}\frac{\mu_0 I_0}{2\pi} \ln\left[\frac{\sqrt{c^2t^2 - \rho^2} + ct}{\rho}\right] \Theta(ct - \rho).$$

Apart from a $\delta(ct - \rho)$ "burst" contribution that comes from taking the derivative of the theta function in $\mathbf{A}(\rho, t)$, the electromagnetic fields are zero for $\rho > ct$, with

$$\mathbf{E}(\rho < ct) = -\frac{\partial A}{\partial t}\hat{\mathbf{z}} = -\frac{\mu_0 I_0}{2\pi}\frac{c}{\sqrt{c^2t^2 - \rho^2}}\hat{\mathbf{z}}$$

and

$$\mathbf{B}(\rho < ct) = -\frac{\partial A}{\partial \rho}\hat{\boldsymbol{\phi}} = \frac{\mu_0 I_0}{2\pi}\frac{ct}{\rho}\frac{1}{\sqrt{c^2t^2 - \rho^2}}\hat{\boldsymbol{\phi}}.$$

(b) The $t \to \infty$ limit of these formulae gives zero electric field and the usual magnetostatic formula for the magnetic field:

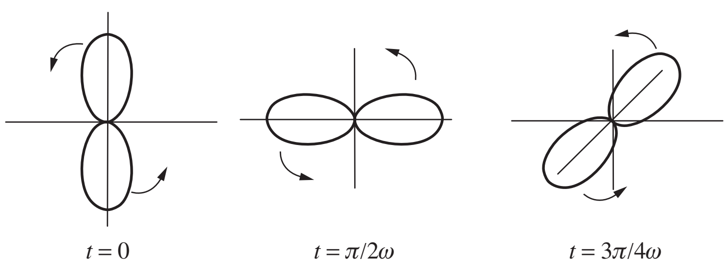
$$\mathbf{B}(t \to \infty) = \frac{\mu_0 I_0}{2\pi\rho}\hat{\boldsymbol{\phi}}.$$

20.19 Crossed and Oscillating Electric Dipoles

(a) Choose $\mathbf{p}(t) = p(\hat{\mathbf{x}} \cos\omega t + \hat{\mathbf{y}} \sin\omega t)$ so $\boldsymbol{\alpha}(t) = \ddot{\mathbf{p}}(t_R) = -\omega^2 p(\hat{\mathbf{x}} \cos\omega t_R + \hat{\mathbf{y}} \sin\omega t_R)$ where $t_R = t - r/c$. The angular distribution of power in the x-y plane is

$$\frac{dP}{d\Omega} \sim |\hat{\mathbf{r}} \times \boldsymbol{\alpha}|^2 \sim |(\hat{\mathbf{x}}\cos\varphi + \hat{\mathbf{y}}\sin\varphi) \times (\hat{\mathbf{x}}\cos\omega t_R + \hat{\mathbf{y}}\sin\omega t_R)|^2 \sim \sin^2(\omega t_R - \varphi).$$

This is an emission pattern that rotates in the plane as shown below.

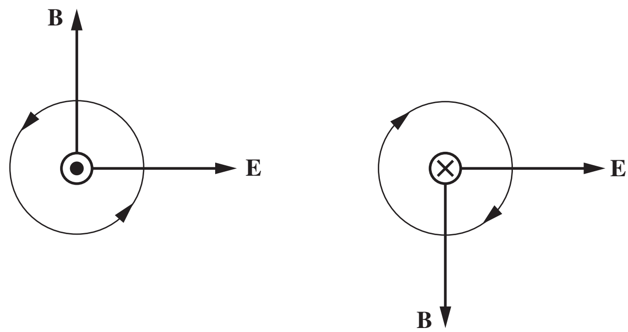


(b) For an observation point along the $\pm z$-axis,

$$\mathbf{B}_{\text{rad}} \propto \mp\hat{\mathbf{z}} \times \boldsymbol{\alpha} \propto \pm(\hat{\mathbf{y}}\cos\omega t_R - \hat{\mathbf{x}}\sin\omega t_R)$$

$$\mathbf{E}_{\text{rad}} \propto \mp\hat{\mathbf{z}} \times \mathbf{B}_{\text{rad}} \propto \hat{\mathbf{x}}\cos\omega t_R + \hat{\mathbf{y}}\sin\omega t_R.$$

This is left circular polarization for emission along $+z$ and right circular polarization for emission along $-z$.



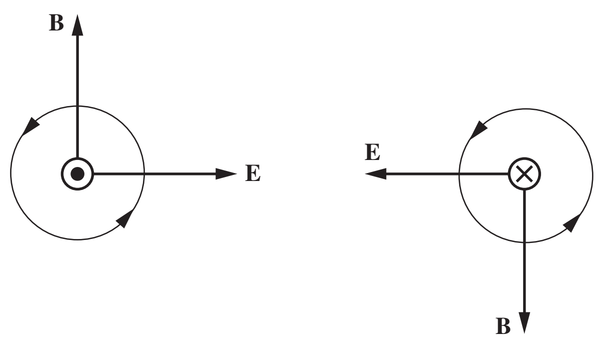
(c) Now,

$$\boldsymbol{\alpha} = -\omega^2 p[\hat{\mathbf{x}}\cos(\omega t - \omega|z|/c) + \hat{\mathbf{y}}\cos(\omega t - \omega|z + \lambda/4|/c)],$$

so, along the $\pm z$-axis in the radiation zone,

$$\begin{aligned}\mathbf{B}_{\text{rad}} &\propto \mp\hat{\mathbf{z}} \times \boldsymbol{\alpha} \\ &\propto \pm\{\hat{\mathbf{y}}\cos[\omega(t - |z|/c)] - \hat{\mathbf{x}}\cos[\omega(t - |z|/c) \pm \pi/2]\} \\ &\propto \pm\{\hat{\mathbf{y}}\cos[\omega(t - |z|/c)] \mp \hat{\mathbf{x}}\sin[\omega(t - |z|/c)]\}.\end{aligned}$$

$\mathbf{E}_{\text{rad}} \propto \mp\hat{\mathbf{z}} \times \mathbf{B}_{\text{rad}}$. This is left circular polarization for emission along both $+z$ and $-z$.



20.20 An Uncharged Rotor

(a) The time-dependent dipole moment of the rotating rod is

$$\mathrm{Re}\{\mathbf{p}(t)\} = \mathrm{Re}\{p_0(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\exp(-i\omega t)\} = p_0\cos\omega t\,\hat{\mathbf{x}} + p_0\sin\omega t\,\hat{\mathbf{y}}.$$

Therefore, the (complex) electric field in the radiation zone is

$$\mathbf{E}_{\mathrm{rad}} = \hat{\mathbf{r}} \times \frac{\mu_0}{4\pi r}\left[\hat{\mathbf{r}} \times \frac{d^2}{dt^2}\mathbf{p}(t-r)\right] = -\frac{k^2 p_0}{4\pi\epsilon_0}\frac{e^{i(kr-\omega t)}}{r}[\hat{\mathbf{r}} \times \{\hat{\mathbf{r}} \times (\hat{\mathbf{x}} + i\hat{\mathbf{y}})\}].$$

Using

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\hat{\mathbf{r}} + \cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}} \qquad \text{and} \qquad \hat{\mathbf{y}} = \sin\theta\sin\phi\hat{\mathbf{r}} + \cos\theta\sin\phi\hat{\boldsymbol{\theta}} + \cos\phi\hat{\boldsymbol{\phi}}$$

shows that

$$\hat{\mathbf{r}} \times \{\hat{\mathbf{r}} \times (\hat{\mathbf{x}} + i\hat{\mathbf{y}})\} = -e^{i\phi}(\cos\theta\hat{\boldsymbol{\theta}} + i\hat{\boldsymbol{\phi}}).$$

Therefore,

$$\mathbf{E}_{\mathrm{rad}}(r,\theta,\phi,t) = \frac{k^2 p_0}{4\pi\epsilon_0}\left(\cos\theta\hat{\boldsymbol{\theta}} + i\hat{\boldsymbol{\phi}}\right)\frac{e^{i(kr-\omega t+\phi)}}{r}.$$

The observer's azimuthal coordinate ϕ occurs in the phase because the dipole is rotating in the ϕ direction. Hence, observers at different ϕ see the dipole at different points in its oscillation cycle.

(b) On the x-axis, $\theta = \pi/2$, $\phi = 0$, and $y = z = 0$, so $\hat{\mathbf{r}} = \hat{\mathbf{x}}$, $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{z}}$, and $\hat{\boldsymbol{\phi}} = \hat{\mathbf{y}}$. Therefore, the real electric field is

$$\mathbf{E}(\mathbf{r},t) = \frac{k^2 p_0}{4\pi\epsilon_0}\mathrm{Re}\left\{i\hat{\mathbf{y}}\frac{e^{i(kx-\omega t)}}{x}\right\} = \frac{k^2 p_0}{4\pi\epsilon_0}\frac{\sin(\omega t - kx)}{x}\hat{\mathbf{y}}.$$

This is linear polarization along $\hat{\mathbf{y}}$, which makes sense because an observer on the x-axis sees only a projection of the rotating dipole along the y-axis. On the y-axis, $\theta = \pi/2$, $\phi = \pi/2$, and $x = z = 0$, so $\hat{\mathbf{r}} = \hat{\mathbf{y}}$, $\hat{\boldsymbol{\theta}} = -\hat{\mathbf{z}}$, and $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}}$. The real electric field is

$$\mathbf{E}(\mathbf{r},t) = \frac{k^2 p_0}{4\pi\epsilon_0}\mathrm{Re}\left\{-i\hat{\mathbf{x}}\frac{e^{i(ky-\omega t)}}{y}\right\} = \frac{k^2 p_0}{4\pi\epsilon_0}\frac{\cos(\omega t - ky)}{y}\hat{\mathbf{x}}.$$

As in the previous case, this is linear polarization because only the projection of the rotation along the x-axis is visible to an observer on the y-axis. On the z-axis, $\theta = 0$, $\phi = 0$, and $x = y = 0$, so $\hat{\mathbf{r}} = \hat{\mathbf{z}}$, $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}}$, and $\hat{\boldsymbol{\phi}} = \hat{\mathbf{y}}$. The real electric field is

$$\mathbf{E}(\mathbf{r},t) = \frac{k^2 p_0}{4\pi\epsilon_0}\mathrm{Re}\left\{(\hat{\mathbf{x}} + i\hat{\mathbf{y}})\frac{e^{i(kz-\omega t)}}{z}\right\} = \frac{k^2 p_0}{4\pi\epsilon_0}\frac{\cos(\omega t - kz)\hat{\mathbf{x}} + \sin(\omega t - kz)\hat{\mathbf{y}}}{z}.$$

This is right circular polarization, as might be expected because the observer sees the rotator in full.

(c) The time-averaged angular distribution of power is

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{r^2}{2c\mu_0}\mathbf{E}\cdot\mathbf{E}^* = \frac{k^4p_0^2}{2c\mu_0}\frac{1}{(4\pi\epsilon_0)^2}\left(1+\cos^2\theta\right) = \frac{\mu_0}{4\pi}\frac{\omega^4p_0^2}{8\pi c}\left(1+\cos^2\theta\right).$$

Then, since $\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta(1+\cos^2\theta) = 16\pi/3$, the time-averaged total power radiated is

$$\langle P\rangle = \int d\Omega \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0}{4\pi}\frac{\omega^4p_0^2}{8\pi c}\frac{16\pi}{3} = \frac{\mu_0}{4\pi}\frac{2\omega^4p_0^2}{3c}.$$

Source. Prof. C. Caves, University of New Mexico (public communication).

20.21 Pulsar Radiation

The time-averaged power radiated by a magnetic dipole pulsar is

$$< P > = \frac{dU}{dt} = \frac{\mu_0}{4\pi}\frac{m^2}{3c^3}\omega^4$$

where $\omega = 2\pi/T$. If this energy is derived from a decrease in rotational kinetic energy,

$$\frac{dU}{dt} = -\frac{d}{dt}\left(\frac{1}{2}I\omega^2\right) = \frac{2}{5}MR^2\omega|\dot{\omega}|$$

if we take $I = \frac{2}{5}MR^2$. We conclude that

$$m^2 = \frac{6c^3}{5\pi\mu_0}MR^2T|\dot{T}|$$

because $|\dot{\omega}| = 2\pi|\dot{T}|\big/T^2$. The "near field" is

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3},$$

so at the pulsar surface ($r = R$) we get a maximum field of $B = \mu_0 m\big/2\pi R^3$. Hence,

$$\begin{aligned} B^2 &= \frac{6}{5\pi^2}\frac{\mu_0}{4\pi}\frac{c^3}{R^4}MT|\dot{T}| = \frac{6}{5\pi^2}\cdot 10^{-7}\cdot\frac{(3\cdot 10^8)^3}{(10^4)^4}(2.8\cdot 10^{30})(7.5)(8\cdot 10^{-11}) \\ &= 5.5\cdot 10^{22}\ T^2 \end{aligned}$$

or $B \approx 2.3\cdot 10^{15}\ G$.

20.22 Neutron Radiation

The magnetic moment of a neutron is $\mathbf{m} = \gamma\mathbf{S}$ where γ is its gyromagnetic ratio and $\mathbf{S}$ is its spin angular momentum. The precession of the angular momentum is driven by the torque $\mathbf{N} = \mathbf{m}\times\mathbf{B}$. Therefore,

$$\frac{d\mathbf{S}}{dt} = \mathbf{N} = \mathbf{m} \times \mathbf{B} = \gamma \mathbf{S} \times \mathbf{B}.$$

We have solved this equation in the past. If $\omega = \gamma B$ and Θ is the angle of precession measured from the polar **B**-axis,

$$\begin{aligned} m_z &= m \cos\Theta \\ m_y &= m \sin\Theta \cos(\omega t) \\ m_x &= m \sin\Theta \sin(\omega t). \end{aligned}$$

The initial conditions for our problem are satisfied by the choice $\Theta(t = 0) = \pi/2$. On the other hand, the precessing dipole radiates and the (time-averaged) rate at which power is lost is

$$\langle P \rangle = \int d\Omega \, \langle \frac{dP}{d\Omega} \rangle = \frac{\mu_0}{4\pi} \frac{1}{4\pi c^3} \int d\Omega \langle |\hat{\mathbf{r}} \times \boldsymbol{\alpha}|^2 \rangle$$

where

$$|\hat{\mathbf{r}} \times \boldsymbol{\alpha}|^2 = \frac{\mathbf{1}}{c^2} |\hat{\mathbf{r}} \times (\ddot{\mathbf{m}} \times \hat{\mathbf{r}})|^2 = \frac{\mathbf{1}}{c^2} |\ddot{\mathbf{m}} - \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \ddot{\mathbf{m}})|^2 = \frac{|\ddot{\mathbf{m}}|^2 - (\hat{\mathbf{r}} \cdot \ddot{\mathbf{m}})^2}{c^2}.$$

If we neglect $d\Theta/dt$ compared to ω, $\ddot{\mathbf{m}} = -m\omega^2 \sin\Theta(\hat{x} \sin\omega t + \hat{y} \cos\omega t)$. Also, $\hat{\mathbf{r}} = \hat{\mathbf{x}} \sin\vartheta \cos\varphi + \hat{\mathbf{y}} \sin\vartheta \sin\varphi + \hat{\mathbf{z}} \cos\vartheta$. Then, because $\langle \sin^2 \omega t \rangle = \langle \cos^2 \omega t \rangle = 1/2$ and $\langle \sin\omega t \cos\omega t \rangle = 0$, we find

$$\langle |\hat{\mathbf{r}} \times \boldsymbol{\alpha}|^2 \rangle = \frac{m^2 \omega^4}{c^2} \sin^2\Theta \left[1 - \frac{1}{2} \sin^2\vartheta \right].$$

Consequently, integrating over the emission angles ϑ and φ,

$$\langle P \rangle = \frac{m^2 \omega^4 \sin^2\Theta}{6\pi\epsilon_0 c^5}.$$

Finally, by conservation of energy, the magnetic potential energy $U = -\mathbf{m} \cdot \mathbf{B}$ is connected to the radiated power by $dU/dt = -\langle P \rangle$. Therefore, since $\omega = \gamma B$ and $m = \gamma S = \gamma \hbar/2$, we get

$$\frac{d\Theta}{dt} = -\frac{\sin\Theta}{\tau} \qquad \text{with} \qquad \frac{1}{\tau} = \frac{m\omega^4}{6\pi\epsilon_0 c^5 B} = \frac{8m^5 B^3}{3\pi\epsilon_0 c^5 \hbar^4}.$$

It remains only to integrate the Θ equation. This is most easily done by writing

$$-\frac{dt}{\tau} = \frac{d\Theta}{\sin\Theta} = -\frac{d(\cos\Theta)}{1 - \cos^2\Theta},$$

so $t/\tau = \tanh^{-1}[\cos\Theta] + \mathit{const}$. With the initial condition $\Theta(0) = \pi/2$, we get

$$\Theta(t) = \cos^{-1}[\tanh(t/\tau)]$$

as suggested. Our solution says that $d\Theta/dt \sim 1/\tau$ so the assumption made above will be valid if $\omega\tau \gg 1$. That is, the precession is very fast compared to the decay of the tilt angle.

Source: D.R. Stump and G.L. Pollack, *European Journal of Physics* **19**, 591 (1998).

20.23 Radiation Interference

(a) Since the two sources emit in phase,

$$\frac{dP}{d\Omega} \propto |\hat{\mathbf{r}} \times \mathbf{p} + \hat{\mathbf{r}} \times (\mathbf{m} \times \hat{\mathbf{r}})|^2.$$

The cross (interference) term is

$$(\hat{\mathbf{r}} \times \mathbf{p}) \cdot [\hat{\mathbf{r}} \times (\mathbf{m} \times \hat{\mathbf{r}})] = (\hat{\mathbf{r}} \times \mathbf{p}) \cdot [\mathbf{m} - \hat{\mathbf{r}}(\hat{\mathbf{r}} \cdot \mathbf{m})] = (\hat{\mathbf{r}} \times \mathbf{p}) \cdot \mathbf{m} = \hat{\mathbf{r}} \cdot (\mathbf{p} \times \mathbf{m}).$$

This is non-zero unless $\mathbf{p}$ and $\mathbf{m}$ are collinear.

(b)

$$P = \int d\Omega \frac{dP}{d\Omega} \propto \int_0^{2\pi} d\phi \int_0^{\pi} d\theta \sin\theta \, \hat{\mathbf{r}} \cdot (\mathbf{p} \times \mathbf{m}) \propto (\mathbf{p} \times \mathbf{m}) \int_{-1}^{1} d(\cos\theta) \cos\theta = 0.$$

20.24 Wire Radiation

(a) Choose a cylindrical surface concentric with the wire. For radiation, the flux of $\mathbf{S} = (1/\mu_0)\mathbf{E}\times\mathbf{B}$ through the surface area element $dS = \rho d\theta dz$ must be constant. Moreover, $|\mathbf{E}| = c|\mathbf{B}|$. Therefore, both $\mathbf{E}$ and $\mathbf{B}$ must vary as $1/\sqrt{\rho}$.

(b) $\rho(\mathbf{r}, t) = -\mathbf{p}(t) \cdot \nabla\delta(\mathbf{r})$ is the dipole charge density. From the continuity equation, $\mathbf{j}(\mathbf{r}, t) = \dot{\mathbf{p}}(t)\delta(\mathbf{r})$ is its current density. Therefore, using superposition, the current density of the entire wire is equivalent to the current density obtained when a a point electric dipole $\mathbf{p}(t) = \hat{\mathbf{z}}(I/A)\exp(-i\omega t)$ sits at every point on the z-axis.

(c) Let the observation point be $(\rho, 0, 0)$. For a point dipole at the point z on the z-axis, the variable $r = \rho\hat{\boldsymbol{\rho}} + z\hat{\mathbf{z}}$ so

$$\mathbf{B}_{\text{rad}} \propto \int_{-\infty}^{\infty} dz \frac{\rho\hat{\boldsymbol{\phi}}}{\rho^2 + z^2} \exp\left\{i(\omega/c)\sqrt{\rho^2 + z^2}\right\}.$$

Now assume that $z \ll \rho$ (even though the limit on the integration goes to infinity) so we can ignore z^2 compared to ρ^2 in the denominator and put $\sqrt{\rho^2 + z^2} \approx \rho + z^2/2\rho$ in the exponential. This gives

$$\mathbf{B}_{\text{rad}} \propto \frac{\hat{\boldsymbol{\phi}}}{\rho} e^{i\rho\omega/c} \int_{\infty}^{\infty} dz \, e^{-z^2\omega/2ic\rho} \sim \frac{\hat{\boldsymbol{\phi}}}{\rho} e^{i\rho\omega/c} \sqrt{\frac{2ic\rho}{\omega}}.$$

Therefore, $\mathbf{B}_{\text{rad}} \sim 1/\sqrt{\rho}$ as obtained in part (a). The Gaussian integral is dominated by values of z where $z \ll \rho$. This justifies the approximation used.

20.25 A Charged Rotor

(a) The electric dipole moment is

$$\mathbf{p}(t) = \int d^3r\, \mathbf{r}\rho(\mathbf{r},t) = 0$$

because the charge density is an even function of r. There is no electric dipole radiation.

(b) The current density is $j(\mathbf{r},t) = \mathbf{v}\rho(\mathbf{r},t)$ where $\mathbf{v}$ points (locally) in the direction of the particle motion. This gives a magnetic dipole moment

$$\mathbf{m}(t) = \frac{1}{2}\int d^3r\, \mathbf{r}\times\mathbf{j}(\mathbf{r},t) = \frac{1}{2}\int d^3r\, \mathbf{r}\times\mathbf{v}\rho(\mathbf{r},t) = \frac{1}{2}\ell^2\frac{\omega}{2}\hat{\mathbf{z}}\int d^3r\, \rho(\mathbf{r},t) = \frac{1}{2}\omega q\ell^2\hat{\mathbf{z}}.$$

This quantity is time-independent so there is no magnetic dipole radiation.

(c) The electric quadrupole moment is

$$\mathbf{Q} = \frac{1}{2}\int d^3r\, \rho(\mathbf{r})\ \cdot \begin{bmatrix} x^2 & xy & xz \\ yx & y^2 & yz \\ zx & zy & z^2 \end{bmatrix},$$

where the charge density (in cylindrical coordinates) is

$$\rho(\mathbf{r},t) = q\delta(z)\frac{\delta(r-\ell)}{r}\left[\delta(\phi - \frac{1}{2}\omega t) + \delta(\phi - \pi - \frac{1}{2}\omega t)\right].$$

We have $z=0$ so, because $x = r\cos\phi$ and $y = r\sin\phi$, the integration above gives

$$\mathbf{Q}(t) = q\ell^2\begin{bmatrix} \cos^2\frac{1}{2}\omega t & \cos\frac{1}{2}\omega t\,\sin\frac{1}{2}\omega t & 0 \\ \cos\frac{1}{2}\omega t\,\sin\frac{1}{2}\omega t & \sin^2\frac{1}{2}\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{2}q\ell^2\begin{bmatrix} 1+\cos\omega t & \sin\omega t & 0 \\ \sin\omega t & 1-\cos\omega t & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

(d) For quadrupole radiation, we need the components of $\mathbf{Q}\cdot\hat{\mathbf{r}}$. It is simplest to begin with

$$\begin{aligned} \mathbf{Q}\cdot\mathbf{r} &= \frac{1}{2}\hat{\mathbf{x}}\left[Q_{xx}x + Q_{xy}y\right] + \frac{1}{2}\hat{\mathbf{y}}\left[Q_{yx}x + Q_{yy}y\right] \\ &= \frac{1}{2}q\ell^2\hat{\mathbf{x}}\left\{x(1+\cos\omega t) + y\sin\omega t\right\} + \frac{1}{2}q\ell^2\hat{\mathbf{y}}\left\{x\sin\omega t + y(1-\cos\omega t)\right\}. \end{aligned}$$

Therefore,

$$r\boldsymbol{\alpha}(\mathbf{r},t) = \frac{1}{c}\frac{d^3}{dt^3}\mathbf{Q}(t)\cdot\mathbf{r} = \frac{q\omega^3\ell^2}{2c}\left[(x\sin\omega t - y\cos\omega t)\hat{\mathbf{x}} - (x\cos\omega t + y\sin\omega t)\hat{\mathbf{y}}\right].$$

From this we compute

$$\begin{aligned} \mathbf{r}\times r\boldsymbol{\alpha} &= (xz\cos\omega t + yz\sin\omega t)\hat{x} + (xz\sin\omega t - yz\cos\omega t)\hat{y} \\ &\quad + \left[(y^2 - x^2)\cos\omega t - 2xy\sin\omega t\right]\hat{z}, \end{aligned}$$

so

$$\langle |\mathbf{r}\times r\boldsymbol{\alpha}|^2\rangle = (x^2+y^2)z^2\langle\cos^2\omega t+\sin^2\omega t\rangle + (y^2-x^2)^2\langle\cos^2\omega t\rangle + 4x^2y^2\langle\sin^2\omega t\rangle$$

$$- 2xy(y^2-x^2)\langle\sin\omega t\cos\omega t\rangle.$$

The time average of the last term is zero so

$$\langle|\hat{\mathbf{r}}\times\boldsymbol{\alpha}|^2\rangle = \frac{(x^2+y^2)z^2}{r^4}+\frac{1}{2}\frac{(x^2+y^2)^2}{r^4} = \sin^2\cos^2\theta+\frac{1}{2}\sin^4\theta = \frac{1}{2}(1-\cos^4\theta).$$

We conclude that

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{\mu_0}{4\pi}\frac{1}{4\pi c}\langle|\hat{\mathbf{r}}\times\boldsymbol{\alpha}|^2\rangle = \frac{\mu_0}{4\pi}\frac{q^2\omega^6\ell^4}{32\pi c^3}\left(1-\cos^4\theta\right).$$

20.26 Rotating-Triangle Radiation

Let the distance from the axis of rotation to the charges be R. The electric dipole moment is

$$\mathbf{p} = \int d^3r\,\mathbf{r}\rho(\mathbf{r},t).$$

The positions of the charges are $x_i = R\cos(\omega t+\phi_i)$ and $y_i = R\sin(\omega t+\phi_i)$ where $\phi_1 = 0$, $\phi_2 = 2\pi/3$, and $\phi_2 = 4\pi/3$. We therefore have

$$p_x = q\sum_{i=1}^{3} x_i = Rq\left(\cos(\omega t)+\cos(\omega t+2\pi/3)+\cos(\omega t-2\pi/3)\right) = 0$$

and, similarly, $p_y = 0$. Therefore, there is no electric dipole radiation.

The current density is $\mathbf{j}(\mathbf{r},t) = \mathbf{v}\rho(\mathbf{r},t)$ where the velocity $\mathbf{v}$ points (locally) in the direction of the particle motion with magnitude $v = R\omega$. The magnetic dipole moment is

$$\mathbf{m} = \frac{1}{2c}\int d^3r\,\mathbf{r}\times\mathbf{j} = \frac{1}{2c}\int d^3r\,\mathbf{r}\times\mathbf{v}\rho(\mathbf{r},t) = \frac{1}{2c}R^2\omega\hat{\mathbf{z}}\int d^3r\,\rho(\mathbf{r},t) = \frac{3\omega qR^2}{2c}\hat{\mathbf{z}}.$$

The magnetic dipole moment is time-independent. Therefore, there is no magnetic dipole radiation.

The components of the electric quadrupole tensor are

$$Q_{ij} = \frac{1}{2}\int d^3r\rho(\mathbf{r},t)r_ir_j.$$

The symmetry of ρ with respect to the x-axis dictates that $Q_{xy} = 0$. Since the charges all lie in the plane $z = 0$, $Q_{xz} = Q_{yz} = Q_{zz} = 0$. Furthermore,

$$\begin{aligned} Q_{xx} &= \frac{q}{2}\sum_{i=1}^{3}x_i^2 = \frac{R^2q}{2}\left(\cos^2(\omega t)+\cos^2(\omega t+2\pi/3)+\cos^2(\omega t-2\pi/3)\right) \\ &= \frac{R^2q}{4}\left(\cos(2\omega t)+1+\cos(2\omega t+4\pi/3)+1+\cos(2\omega t-4\pi/3)+1\right) \\ &= \frac{3}{4}R^2q+\frac{R^2q}{4}\left(\cos(2\omega t)+\cos(2\omega t-2\pi/3)+\cos(2\omega t+2\pi/3)\right) = \frac{3}{4}R^2q \qquad (20.1) \end{aligned}$$

and, similarly, $Q_{yy} = Q_{xx} = \frac{3}{4}R^2 q$. Since Q is time-independent, there is no electric quadrupole radiation either.

20.27 Collision Radiation

The dipole moment for this situation is given by

$$\mathbf{p} = q_1 \mathbf{r}_1 + q_3 \mathbf{r}_2 .$$

There will be no dipole radiation if $\ddot{\mathbf{p}} = 0$, i.e., if

$$q_1 \ddot{\mathbf{r}}_1 + q_2 \ddot{\mathbf{r}}_2 = 0.$$

On the other hand, conservation of momentum tells us that

$$m_1 \dot{\mathbf{r}}_1 + m_2 \dot{\mathbf{r}}_2 = \text{constant} \Rightarrow m_1 \ddot{\mathbf{r}}_1 + m_2 \ddot{\mathbf{r}}_2 = 0.$$

Consequently, there will be no dipole radiation if

$$\frac{q_1}{m_1} = \frac{q_2}{m_2}.$$

20.28 Radiation of Linear Momentum

(a) If $\mathbf{S} = \mu_0^{-1} \mathbf{E} \times \mathbf{B}$, the law for for conservation of energy for a spherical volume with radius r is

$$\frac{dU_{\text{tot}}}{dt} = -\int d\mathbf{A} \cdot \mathbf{S} = -\int d\Omega r^2 \hat{\mathbf{r}} \cdot \mathbf{S}.$$

The angular distribution of the rate of radiated energy is defined as

$$\frac{dP}{d\Omega} = \lim_{r \to \infty} r^2 \hat{\mathbf{r}} \cdot \mathbf{S}.$$

If $\mathbf{T} = \epsilon_0 [\mathbf{EE} + c^2 \mathbf{BB} - \frac{1}{2}\mathbf{I}(E^2 + c^2 B^2)]$, the corresponding law for conservation of linear momentum is

$$\frac{d\mathbf{P}_{\text{tot}}}{dt} = \int d\mathbf{A} \cdot \mathbf{T} = \int d\Omega r^2 \hat{\mathbf{r}} \cdot \mathbf{T}.$$

Therefore, it makes sense to define the angular distribution of the rate of radiated linear momentum as

$$\frac{d\mathbf{P}_{\text{EM}}}{dt\, d\Omega} = -\lim_{r \to \infty} r^2 \hat{\mathbf{r}} \cdot \mathbf{T}.$$

(b) Because $c\mathbf{B}_{\text{rad}} = \hat{\mathbf{r}} \times \mathbf{E}_{\text{rad}}$ and $\hat{\mathbf{r}} \cdot \mathbf{E}_{\text{rad}} = \hat{\mathbf{r}} \cdot \mathbf{B}_{\text{rad}} = 0$,

$$\frac{dP}{d\Omega} = \lim_{r \to \infty} r^2 \hat{\mathbf{r}} \cdot (\mathbf{E}_{\text{rad}} \times \mathbf{B}_{\text{rad}}) = \lim_{r \to \infty} \frac{r^2}{c\mu_0} \mathbf{E}_{\text{rad}} \cdot \mathbf{E}_{\text{rad}}.$$

Similarly,

$$\begin{aligned}\frac{d\mathbf{P}_{\rm EM}}{dt\,d\Omega} &= -\lim_{r\to\infty} r^2\hat{\mathbf{r}}\cdot\epsilon_0[\mathbf{E}_{\rm rad}\mathbf{E}_{\rm rad}+c^2\mathbf{B}_{\rm rad}\mathbf{B}_{\rm rad}-\frac{1}{2}\mathbf{I}(E^2_{\rm rad}+c^2B^2_{\rm rad})] \\ &= \lim_{r\to\infty} r^2\frac{\epsilon_0}{2}[E^2_{\rm rad}+c^2B^2_{\rm rad}]\hat{\mathbf{r}}.\end{aligned}$$

The last equality follows because $I_{ij}=\delta_{ij}$ are the components of $\mathbf{I}$ so $\hat{r}_i\delta_{ij}=\hat{r}_j$ means that $\hat{\mathbf{r}}\cdot\mathbf{I}=\hat{\mathbf{r}}$. Moreover, $\mathbf{E}_{\rm rad}\cdot\mathbf{E}_{\rm rad}=c^2\mathbf{B}_{\rm rad}\cdot\mathbf{B}_{\rm rad}$, so

$$\frac{d\mathbf{P}_{\rm EM}}{dt\,d\Omega}=\lim_{r\to\infty} r^2\epsilon_0[\mathbf{E}_{\rm rad}\cdot\mathbf{E}_{\rm rad}]\hat{\mathbf{r}}.$$

(c) Since $P=dU_{\rm EM}/dt$, we see from part (b) that

$$\frac{d\mathbf{P}_{\rm EM}}{dt\,d\Omega}=\frac{\hat{\mathbf{r}}}{c}\frac{dU_{\rm EM}}{dt\,d\Omega}.$$

This is consistent with the plane wave result that the linear momentum density $\mathbf{g}_{\rm EM}=\dfrac{u_{\rm EM}}{c}\hat{\mathbf{k}}$.

20.29 Angular Momentum of Electric Dipole Radiation

(a) By direct computation,

$$\frac{d\mathbf{L}}{dt}=\epsilon_0\int_V d^3r\left\{\mathbf{r}\times\left(\frac{\partial\mathbf{E}}{\partial t}\times\mathbf{B}\right)+\mathbf{r}\times\left(\mathbf{E}\times\frac{\partial\mathbf{B}}{\partial t}\right)\right\}+\sum_{i=1}^N\mathbf{v}_i\times\mathbf{p}_i+\sum_{i=1}^N\mathbf{r}_i\times\mathbf{F}_i,$$

where $\mathbf{v}_i=d\mathbf{r}_i/dt$ and $\mathbf{F}_i=d\mathbf{p}_i/dt$. The next-to-last term is zero because $\mathbf{v}_i\parallel\mathbf{p}_i$ and, in what follows, we will write the last term in the form

$$\sum_{i=1}^N\mathbf{r}_i\times\mathbf{F}_i=\int_V d^3r'\,\mathbf{r}\times\{\rho\mathbf{E}+\mathbf{j}\times\mathbf{B}\}.$$

Substituting the Maxwell equations

$$\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}=\nabla\times\mathbf{B}-\mu_0\mathbf{j}\qquad\text{and}\qquad\frac{\partial\mathbf{B}}{\partial t}=-\nabla\times\mathbf{E}$$

gives

$$\frac{d\mathbf{L}}{dt}=\epsilon_0\int_V d^3r\,\left\{\mathbf{r}\times[(\nabla\times c\mathbf{B})\times c\mathbf{B}]-\mathbf{r}\times(c^2\mu_0\mathbf{j}\times\mathbf{B})\right\}.$$

The current terms cancel so, using a vector identity,

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \epsilon_0 \int_V d^3r\, \mathbf{r} \times \left\{ (c\mathbf{B}\cdot\nabla)c\mathbf{B} - \frac{1}{2}\nabla(cB)^2 \right\} \\ &\quad + \epsilon_0 \int_V d^3r\, \mathbf{r} \times \left\{ (\mathbf{E}\cdot\nabla)\mathbf{E} - \frac{1}{2}\nabla E^2 \right\} + \int_V d^3r\, \mathbf{r}\times\rho\mathbf{E}.\end{aligned}$$

Now use the vector identity $\mathbf{r}\times(\mathbf{a}\cdot\nabla)\mathbf{a} = \mathbf{a}\cdot\nabla(\mathbf{r}\times\mathbf{a})$ to write

$$\begin{aligned}\epsilon_0 \int_V d^3r\, \mathbf{r} &\times (\mathbf{E}\cdot\nabla)\mathbf{E} + \int_V d^3r\, (\mathbf{r}\times\mathbf{E})\rho \\ &= \epsilon_0 \int_V d^3r\, \{\mathbf{E}\cdot\nabla(\mathbf{r}\times\mathbf{E}) + (\mathbf{r}\times\mathbf{E})\nabla\cdot\mathbf{E}\} \\ &= \epsilon_0 \int_V d^3r\, \nabla\cdot\{\mathbf{E}\,(\mathbf{r}\times\mathbf{E})\} \\ &= \epsilon_0 \int_S d\mathbf{S}\cdot\mathbf{E}\,(\mathbf{r}\times\mathbf{E}).\end{aligned} \tag{1}$$

Because $\nabla\cdot\mathbf{B} = 0$, the magnetic term can be written similarly:

$$\epsilon_0 \int_V d^3r\, \mathbf{r}\times(c\mathbf{B}\cdot\nabla)c\mathbf{B} = \epsilon_0 \int_S d\mathbf{S}\cdot c\mathbf{B}\,(\mathbf{r}\times c\mathbf{B}). \tag{2}$$

Finally, because $\nabla\times\mathbf{r} = 0$,

$$\frac{1}{2}\int_V d^3r\, \nabla(E^2 + c^2B^2)\times\mathbf{r} = \frac{1}{2}\int_V d^3r\, \nabla\times\left\{\mathbf{r}(E^2+c^2B^2)\right\} = \frac{1}{2}\int_S d\mathbf{S}\times\mathbf{r}(E^2+c^2B^2). \tag{3}$$

Combining (1), (2), and (3) gives the desired result,

$$\frac{d\mathbf{L}}{dt} = \epsilon_0 \int_S d\mathbf{S}\cdot\{c\mathbf{B}(\mathbf{r}\times c\mathbf{B}) + \mathbf{E}(\mathbf{r}\times\mathbf{E})\} + \frac{1}{2}\epsilon_0 \int_S d\mathbf{S}\times\mathbf{r}\left\{E^2 + c^2B^2\right\}. \tag{4}$$

This is a continuity equation which relates the time rate of change of angular momentum in a volume to an "angular momentum current" through the surface of the volume.

(b) For a spherical volume, $d\mathbf{S} = \hat{\mathbf{r}}R^2d\Omega$ and the last term in (4) always vanishes. But in the radiation zone, $\mathbf{E}_{\rm rad}(\mathbf{r},t)$ and $\mathbf{B}_{\rm rad}(\mathbf{r},t)$ for a time-dependent dipole are both *transverse* to $\hat{\mathbf{r}}$. This means that the first term in (4) vanishes also. The hard-to-believe conclusion is that the radiation fields carry no angular momentum out of the volume V.

(c) The foregoing suggests that we must keep terms other than just the radiation fields in the first integral in (4). The structure of the integrand is

$$r\,(field)\,(field)\,r^2d\Omega,$$

so only terms where $(field)\,(field) \sim 1/r^3$ will survive in the limit where the sphere radius goes to infinity. We have seen that the exact $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$ for an oscillating electric dipole contain terms that vary as $1/r$ (radiation zone), $1/r^2$ (intermediate zone), and $1/r^3$ (near zone). Hence, the surviving terms will be

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \epsilon_0 \int_S d\mathbf{S}\cdot\{c\mathbf{B}_{\text{rad}}(\mathbf{r}\times c\mathbf{B}_{\text{int}}) + c\mathbf{B}_{\text{int}}(\mathbf{r}\times c\mathbf{B}_{\text{rad}}) \\ &\quad + \mathbf{E}_{\text{rad}}(\mathbf{r}\times\mathbf{E}_{\text{int}}) + \mathbf{E}_{\text{int}}(\mathbf{r}\times\mathbf{E}_{\text{rad}})\}.\end{aligned}$$

But $d\mathbf{S}\cdot\mathbf{E}_{\text{rad}} = d\mathbf{S}\cdot\mathbf{B}_{\text{rad}} = 0$ as explained above so we need only

$$\frac{d\mathbf{L}}{dt} = \epsilon_0 \int_S d\mathbf{S}\cdot\{c\mathbf{B}_{\text{int}}(\mathbf{r}\times c\mathbf{B}_{\text{rad}}) + \mathbf{E}_{\text{int}}(\mathbf{r}\times\mathbf{E}_{\text{rad}})\}.$$

On the other hand,

$$\mathbf{B}_{\text{int}} = \frac{\mu_0}{4\pi r^2}\dot{\mathbf{p}}(t-r/c)\times\hat{\mathbf{r}},$$

so $d\mathbf{S}\cdot\mathbf{B}_{\text{int}} = 0$ as well. This leaves us with

$$\frac{d\mathbf{L}}{dt} = \epsilon_0\int_S d\mathbf{S}\cdot\mathbf{E}_{\text{int}}(\mathbf{r}\times\mathbf{E}_{\text{rad}}) = \epsilon_0\int_S (d\mathbf{S}\cdot\mathbf{E}_{\text{int}})cr\mathbf{B}_{\text{rad}} = \epsilon_0\int_S d\Omega\,\hat{\mathbf{r}}\cdot\mathbf{E}_{\text{int}}cr^3\mathbf{B}_{\text{rad}}.$$

Inserting the appropriate fields for a point electric dipole gives

$$\begin{aligned}\frac{d\mathbf{L}}{dt} &= \frac{\epsilon_0}{c}\frac{1}{4\pi\epsilon_0}\frac{\mu_0}{4\pi}\int_S d\Omega\,\{3(\hat{\mathbf{r}}\cdot[\dot{\mathbf{p}}]_{\text{ret}}) - \hat{\mathbf{r}}\cdot[\dot{\mathbf{p}}]_{\text{ret}}\}\,[\ddot{\mathbf{p}}]_{\text{ret}}\times\hat{\mathbf{r}} \\ &= \frac{\mu_0}{4\pi}\frac{1}{2\pi c}\int_S d\Omega\,([\ddot{\mathbf{p}}]_{\text{ret}}\times\hat{\mathbf{r}})\,(\hat{\mathbf{r}}\cdot[\dot{\mathbf{p}}]_{\text{ret}}).\end{aligned}$$

(d) If we write out $\hat{\mathbf{r}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$, we find that the only contributions to $(\ddot{\mathbf{p}}\times\hat{\mathbf{r}})\,(\hat{\mathbf{r}}\cdot\dot{\mathbf{p}})$ that survive the angular integration are

$$\begin{aligned}&\hat{\mathbf{x}}(\ddot{p}_y\dot{p}_z\cos^2\theta - \ddot{p}_z\dot{p}_y\sin^2\theta\sin^2\phi) \\ &+\hat{\mathbf{y}}(\ddot{p}_z\dot{p}_x\sin^2\theta\cos^2\phi - \ddot{p}_x\dot{p}_z\cos^2\theta) \\ &+\hat{\mathbf{z}}(\ddot{p}_x\dot{p}_y\sin^2\theta\sin^2\phi - \ddot{p}_y\dot{p}_x\sin^2\theta\cos^2\phi).\end{aligned}$$

Every one of the non-zero angular integrals gives a factor of $4\pi/3$. Consequently,

$$\frac{d\mathbf{L}}{dt} = \frac{\mu_0}{6\pi c}\,[\ddot{\mathbf{p}}]_{\text{ret}}\times[\dot{\mathbf{p}}]_{\text{ret}}.$$

20.30 Dipole Moment of the Slotted Sphere

The text gives the transverse electric field of the slotted sphere as

$$E_\theta = c\sum_{\ell=1}^{\infty}A_\ell\frac{1}{r}\frac{d}{dr}\left[rh_\ell^{(1)}(kr)\right]\frac{d}{d\theta}P_\ell(\cos\theta),$$

where

$$A_\ell = -\frac{V}{c}\frac{2\ell+1}{2\ell(\ell+1)}\frac{\sin\theta_0 P_\ell^1(\cos\theta_0)}{\frac{d}{dr}\left[rh_\ell^{(1)}(kr)\right]_{r=R}}.$$

In the numerator, we get the radiation field by using the $kr \gg 1$ limit,

$$\lim_{kr\gg 1} h_\ell^{(1)}(kr) = \frac{1}{kr}\exp\{i[kr - \tfrac{1}{2}(\ell+1)\pi]\}.$$

In the denominator, we use the $kR \ll 1$ long-wavelength limit,

$$\lim_{kr\ll 1} h_\ell^{(1)}(kr) = -i\frac{(2\ell-1)!!}{(kr)^{\ell+1}}.$$

The result is

$$E_\theta \approx V\sum_{\ell=1}^{\infty}\frac{2\ell+1}{2\ell(\ell+1)}\frac{(kR)^{\ell+1}}{\ell(2\ell-1)!!}\sin\theta_0 P_\ell^1(\cos\theta_0)\frac{d}{d\theta}P_\ell(\cos\theta)\frac{\exp(ikr)}{r}.$$

Since $kR \ll 1$, only the $\ell = 1$ term survives. Moreover, $P_1^1(\cos\theta) = \sin\theta$. Therefore,

$$E_\theta \approx -\frac{3}{4}(kR)^2 V\sin^2\theta_0\sin\theta\frac{\exp(ikr)}{r}.$$

This may be compared to the general time-harmonic electric dipole field with $\mathbf{p} = p\hat{\mathbf{z}}$:

$$\mathbf{E} = -\frac{k^2}{4\pi\epsilon_0}[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{p})]\frac{\exp(ikr)}{r} = -\frac{k^2 p}{4\pi\epsilon_0}\sin\theta\frac{\exp(ikr)}{r}\hat{\boldsymbol{\theta}}.$$

Therefore,

$$\mathbf{p} = 3\pi\epsilon_0 R^2 V\sin^2\theta_0\hat{\mathbf{z}}.$$

Source: J. Van Bladel, *Electromagnetic Fields* (McGraw-Hill, New York, 1964).

Chapter 21: Scattering and Diffraction

21.1 Scattering from a Bound Electron

The total scattering cross section is

$$\sigma_{\rm scatt} = \frac{\langle P \rangle}{\frac{1}{2}\epsilon_0 c E_0^2},$$

where $\langle P \rangle$ is the total power radiated by the electron and the electric field of the incident circularly polarized wave propagating in the z-direction is

$$\mathbf{E}(\mathbf{r}, t) = E_0(\cos\omega\hat{\mathbf{x}} + \sin\omega t\hat{\mathbf{y}})\exp(ik_0 z).$$

The latter sets the electron into small-amplitude motion around $z = 0$ according to

$$m\ddot{\mathbf{r}} + k\mathbf{r} = -e\mathbf{E}(z = 0, t) = -e\mathbf{E}(t).$$

The steady-state solution of this differential equation is

$$\mathbf{r}(t) = \frac{e}{m\omega^2 - k}\mathbf{E}(t).$$

For the radiated power, we use the cycle-average of Larmor's formula:

$$\langle P \rangle = \frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}\langle |\mathbf{a}|^2 \rangle = \frac{1}{4\pi\epsilon_0}\frac{e^2}{3c^3}|\mathbf{a}|^2.$$

If $\omega_0 = k/m$, the acceleration we need is

$$\mathbf{a}(t) = \ddot{\mathbf{r}}(t) = -\frac{\omega^2 e}{m\omega^2 - k}\mathbf{E}(t) = -\frac{e}{m\left(1 - \omega_0^2/\omega^2\right)}\mathbf{E}(t).$$

Therefore,

$$\sigma_{\rm scatt} = \frac{1}{4\pi\epsilon_0}\frac{e^2}{3c^3}\frac{e^2 E_0^2}{m^2\left(1 - \omega_0^2/\omega^2\right)^2}\frac{1}{\frac{1}{2}\epsilon_0 c E_0^2} = \frac{8\pi}{3}r_e^2\frac{1}{\left(1 - \omega_0^2/\omega^2\right)^2}.$$

Here, $r_e = e^2/4\pi\epsilon_0 mc^2$ is the classical radius of the electron which appears in the Thomson scattering cross section.

21.2 Scattering from a Hydrogen Atom

According to Example 1.2, the cross section for scattering from an ensemble of electrons with number density $n(\mathbf{r})$ is

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm ensemble} = \frac{d\sigma_{\rm Thom}}{d\Omega} \times |n(\mathbf{q})|^2,$$

where

$$n(\mathbf{q}) = \int d^3r\, n(\mathbf{r}) \exp(-i\mathbf{q}\cdot\mathbf{r}).$$

The electron number density associated with the 1s orbital of hydrogen is

$$n(\mathbf{r}) = \left| \frac{1}{\sqrt{\pi a_{\rm B}^3}} \exp(-r/a_{\rm B}) \right|^2 = \frac{1}{\pi a_{\rm B}^3} \exp(-2r/a_{\rm B}).$$

Therefore,

$$\begin{aligned} n(\mathbf{q}) &= \frac{2\pi}{\pi a_{\rm B}^3} \int_0^\infty dr r^2 \exp(-2r/a_{\rm B}) \int_0^\pi d\theta \sin\theta \exp(iqr\cos\theta) \\ &= \frac{2}{a_{\rm B}^3} \int_0^\infty dr r^2 \exp(-2r/a_{\rm B}) \frac{2\sin qr}{qr} \\ &= \frac{4}{a_{\rm B}^3}\frac{1}{q} \mathrm{Im} \left\{ \int_0^\infty dr\, r \exp[-r(2/a_{\rm B} - iq)] \right\}. \end{aligned}$$

Integration by parts gives

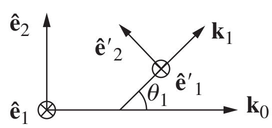
$$n(\mathbf{q}) = \frac{4}{a_{\rm B}^3}\frac{1}{q}\mathrm{Im}\left\{\frac{1}{(2/a_{\rm B} - iq)^2}\right\} = \frac{1}{[1+(qa_{\rm B}/2)^2]^2}.$$

The cross section is proportional to the absolute square of this quantity, which is the desired result.

Source: J. Als-Nielsen and D. McMorrow, *Elements of Modern X-ray Physics* (Wiley, New York, 2001).

21.3 Double Scattering

This is elastic scattering, so $k_0 = k_1 = k_2 = k$. Three orthogonal triads of unit vectors are $(\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{k}}_0)$, $(\hat{\mathbf{e}}'_1, \hat{\mathbf{e}}'_2, \hat{\mathbf{k}}_1)$, and $(\hat{\mathbf{e}}''_1, \hat{\mathbf{e}}''_2, \hat{\mathbf{k}}_2)$. The out-of-plane vectors satisfy $\hat{\mathbf{e}}'_1 = \hat{\mathbf{e}}_1 = \hat{\mathbf{e}}''_1$. Consider the first scattering event shown below.



If the incident wave is $\hat{\mathbf{e}}_0 E_0 \exp[i(\mathbf{k}_0\cdot\mathbf{r} - \omega t)]$, the radiated electric field is

$$\mathbf{E} = \frac{k^2\alpha E_0}{4\pi r_1} \exp[i(kr_1 - \omega t)] \left[(\hat{\mathbf{k}}_1 \times \hat{\mathbf{e}}_0) \times \hat{\mathbf{k}}_1 \right].$$

For left circular polarization, our convention is $\hat{\mathbf{e}}_0 = (\hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2)/\sqrt{2}$. Therefore,

$$\frac{[\hat{\mathbf{k}}_1 \times (\hat{\mathbf{e}}_1 + i\hat{\mathbf{e}}_2)] \times \hat{\mathbf{k}}_1}{\sqrt{2}} = \frac{\hat{\mathbf{e}}'_2 \times \hat{\mathbf{k}}_1}{\sqrt{2}} + i\frac{-\hat{\mathbf{e}}_1 \cos\theta_1 \times \hat{\mathbf{k}}_1}{\sqrt{2}} = \frac{\hat{\mathbf{e}}'_1 + i\hat{\mathbf{e}}'_2 \cos\theta_1}{\sqrt{2}},$$

and the electric field after the first scattering event is

$$\mathbf{E} = \frac{k^2 \alpha E_0}{4\pi r_1} \exp[i(kr_1 - \omega t)] \frac{\hat{\mathbf{e}}'_1 + i\hat{\mathbf{e}}'_2 \cos\theta_1}{\sqrt{2}}.$$

The second scattering event is exactly like the first. Therefore, iteration of the preceding calculation with a distance r_2 and a scattering angle θ_2 gives the observed electric field as

$$\mathbf{E} = \frac{k^4 \alpha^2 E_0}{16\pi^2 r_1 r_2} \exp[i(kr_1 + kr_2 - \omega t)] \frac{\hat{\mathbf{e}}''_1 + i\hat{\mathbf{e}}''_2 \cos\theta_1 \cos\theta_2}{\sqrt{2}}.$$

Source: Prof. C. Baird, University of Massachusetts, Lowell (public communication).

21.4 Rayleigh Scattering à la Rayleigh

Let $\mathcal{A} = |\mathbf{E}_{\text{scatt}}|/|\mathbf{E}_{\text{inc}}|$ be the amplitude ratio. The speed of light has dimensions of length/time and none of the other listed quantities has time as a dimension to cancel it out. Therefore, $\mathcal{A}$ cannot depend on c. Otherwise, we must have $\mathcal{A} \propto 1/r$ so the flux of energy is the same for all distant observers. It also stands to reason that the scattering increases as V increases. The simplest guess is $\mathcal{A} \propto V$. Therefore, if N is an integer, the combination

$$\mathcal{A} \propto \frac{V}{r}\lambda^N$$

must be dimensionless. This implies that $N = -2$. Therefore, the scattered intensity

$$I = |\mathcal{A}|^2 \propto \lambda^{-4}.$$

This is Rayleigh's law.

Source: Lord Rayleigh, *Philosophical Magazine* **41**, 107 (1871).

21.5 Rayleigh Scattering from a Conducting Sphere

(a) Let $\mathbf{E}_{\text{ind}}$ be the field produced by the dipole $\mathbf{p}$ at the center of the sphere. The boundary condition is that the tangential electric field vanishes. In other words,

$$0 = \hat{\mathbf{r}} \times \{\mathbf{E}_0 + \mathbf{E}_{\text{ind}}\}_S = \hat{\mathbf{r}} \times \left\{\mathbf{E}_0 + \frac{1}{4\pi\epsilon_0}\left[\frac{3\hat{\mathbf{r}}(\mathbf{r}\cdot\mathbf{p}) - \mathbf{p}}{r^3}\right]_S\right\} = \hat{\mathbf{r}} \times \left\{\mathbf{E}_0 - \frac{\mathbf{p}}{4\pi\epsilon_0 a^3}\right\}.$$

Therefore,

$$\mathbf{p} = 4\pi\epsilon_0 a^2 \mathbf{E}_0.$$

(b) Similarly, let $\mathbf{B}_{\rm ind}$ be the field produced by the dipole $\mathbf{m}$ at the center of the sphere. The boundary condition is that the normal component of the magnetic field vanishes at the conductor's surface. In other words,

$$0 = \hat{\mathbf{r}} \cdot \{\mathbf{B}_0 + \mathbf{B}_{\rm ind}\}_S = \hat{\mathbf{r}} \cdot \left\{\mathbf{B}_0 + \frac{\mu_0}{4\pi}\left[\frac{3\hat{\mathbf{r}}(\mathbf{r}\cdot\mathbf{m}) - \mathbf{m}}{r^3}\right]_S\right\} = \hat{\mathbf{r}} \cdot \left\{\mathbf{B}_0 + \frac{2\mu_0 \mathbf{m}}{4\pi a^3}\right\}.$$

Therefore,

$$\mathbf{m} = -\frac{2\pi a^3}{\mu_0}\mathbf{B}_0.$$

(c) The scattering cross section is

$$\left\langle \frac{d\sigma_{\rm scatt}}{d\Omega} \right\rangle = \left(\frac{k_0^2}{4\pi\epsilon_0 E_0 c}\right)^2 \left|\hat{\mathbf{k}} \times \mathbf{m} + \hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times c\mathbf{p})\right|^2, \tag{1}$$

where $\mathbf{p} = 4\pi\epsilon_0 a^3 E_0 \hat{\mathbf{e}}_0$ and

$$\mathbf{m} = -\frac{2\pi a^2}{\mu_0}\mathbf{B}_0 = -\frac{2\pi a^3 E_0}{c\mu_0}(\mathbf{k}_0 \times \hat{\mathbf{e}}_0).$$

Substituting these into (1) gives

$$\left\langle \frac{d\sigma_{\rm scatt}}{d\Omega} \right\rangle = a^2 (k_0 a)^4 \left|\hat{\mathbf{k}} \times (\hat{\mathbf{k}}_0 \times \hat{\mathbf{e}}_0) - \frac{1}{2}\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \hat{\mathbf{e}}_0)\right|^2. \tag{2}$$

There is no loss of generality if we choose $\hat{\mathbf{k}}_0 = \hat{\mathbf{z}}$ and write

$$\mathbf{k} = \cos\theta\hat{\mathbf{z}} + \sin\theta\hat{\mathbf{x}}$$

for the scattered wave vector and

$$\hat{\mathbf{e}}_0 = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$$

for the incident wave polarization vector. We find straightforwardly that

$$\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \hat{\mathbf{e}}_0) = -\cos^2\theta\cos\phi\hat{\mathbf{x}} - \sin\phi\hat{\mathbf{y}} + \sin\theta\cos\theta\cos\phi\hat{\mathbf{z}}$$

and

$$\hat{\mathbf{k}} \times (\hat{\mathbf{k}}_0 \times \hat{\mathbf{e}}_0) = -\cos\theta\cos\phi\hat{\mathbf{x}} - \cos\theta\sin\phi\hat{\mathbf{y}} + \sin\theta\cos\phi\hat{\mathbf{z}}.$$

Using these to evaluate (2) gives

$$\begin{aligned}\left\langle \frac{d\sigma_{\rm scatt}}{d\Omega} \right\rangle &= a^2 (k_0 a)^4 \left|\cos\theta\cos\phi(2\cos\theta - 1)\hat{\mathbf{x}} + \sin\phi(2 - \cos\theta)\hat{\mathbf{y}}\right. \\ &\quad \left. + \sin\theta\cos\phi(1 - 2\cos\theta)\hat{\mathbf{z}}\right|^2. \end{aligned} \tag{3}$$

Ultimately, we are interested in the average over the direction of polarization:

$$\left\langle \frac{d\sigma_{\rm scatt}}{d\Omega} \right\rangle_{\rm unpol} = \frac{1}{2\pi}\int_0^{2\pi} d\phi \left\langle \frac{d\sigma_{\rm scatt}}{d\Omega} \right\rangle.$$

Therefore, after performing the absolute square in (3), we may replace factors of $\sin^2\phi$ and $\cos^2\phi$ by 1/2. The final result is as advertised:

$$\left\langle \frac{d\sigma_{\rm scatt}}{d\Omega} \right\rangle_{\rm unpol} = a^2(k_0a)^4\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta\right].$$

(d) We return to (2) and the use the circular polarization unit vectors

$$\hat{\mathbf{e}}_0 = \frac{\hat{\mathbf{x}} \pm i\hat{\mathbf{y}}}{\sqrt{2}}.$$

Now,

$$\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \hat{\mathbf{e}}_0) = \frac{-\cos^2\theta\hat{\mathbf{x}} \mp i\hat{\mathbf{y}} + \sin\theta\cos\theta\hat{\mathbf{z}}}{\sqrt{2}}$$

and

$$\hat{\mathbf{k}} \times (\hat{\mathbf{k}}_0 \times \hat{\mathbf{e}}_0) = \frac{-\cos\theta\hat{\mathbf{x}} \mp i\cos\theta\hat{\mathbf{y}} + \sin\theta\hat{\mathbf{z}}}{\sqrt{2}}.$$

Using these to evaluate (2) gives

$$\left\langle \frac{d\sigma_{\rm scatt}}{d\Omega} \right\rangle_{\pm} = \frac{1}{2}a^2(k_0a)^4\left|\cos\theta\left(\frac{1}{2}\cos\theta - 1\right)\hat{\mathbf{x}} \pm i\left(\frac{1}{2} - \cos\theta\right)\hat{\mathbf{y}} + \sin\theta\left(1 - \frac{1}{2}\cos\theta\right)\hat{\mathbf{z}}\right|^2.$$

Being careful to use the complex conjugate when evaluating the absolute square gives

$$\left\langle \frac{d\sigma_{\rm scatt}}{d\Omega} \right\rangle_{\pm} = a^2(k_0a)^4\left[\frac{5}{8}(1+\cos^2\theta) - \cos\theta\right] = \left\langle \frac{d\sigma_{\rm scatt}}{d\Omega} \right\rangle_{\rm unpol}.$$

Source: D.S. Jones, *The Theory of Electromagnetism* (Macmillan, New York, 1964).

21.6 Scattering from a Molecular Rotor

Let the incident field be $\mathbf{E} = \mathbf{E}_0 \exp[i(\mathbf{k}\cdot\mathbf{r} - \omega t)]$. The dipole torque $\mathbf{N} = \mathbf{p} \times \mathbf{E}$ sets the molecule into rotation and we assume that $\mathbf{r} = 0$ is the position of the center of mass. The angular distribution of electric dipole radiation is

$$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2 c}|\hat{\mathbf{r}} \times \ddot{\mathbf{p}}_{\rm ret}|^2. \tag{1}$$

To calculate $\ddot{\mathbf{p}}$, we note that the torque equation of motion for the moment is

$$I\frac{d\mathbf{\Omega}}{dt} = \mathbf{p} \times \mathbf{E}. \tag{2}$$

In addition, we are told that

$$\frac{d\mathbf{p}}{dt} = \mathbf{\Omega} \times \mathbf{p}. \tag{3}$$

Using (2) and (3),

$$\ddot{\mathbf{p}} = \dot{\mathbf{\Omega}} \times \mathbf{p} + \mathbf{\Omega} \times \dot{\mathbf{p}} = \left(\frac{\mathbf{p} \times \mathbf{E}}{I}\right) \times \mathbf{p} + \mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{p}).$$

We are advised to drop the term quadratic in Ω. Therefore,

$$\ddot{\mathbf{p}} \approx \frac{p^2\mathbf{E} - (\mathbf{E} \cdot \mathbf{p})\mathbf{p}}{I}. \tag{4}$$

Retardation plays no significant role here. Therefore, if θ is the angle between $\hat{\mathbf{r}}$ and $\mathbf{p}$, the differential cross section is

$$\frac{d\sigma_{\text{scatt}}}{d\Omega} = \frac{1}{|\langle \mathbf{S}_{\text{inc}} \rangle|}\langle \frac{dP}{d\Omega} \rangle = \frac{1}{\frac{1}{2}\epsilon_0 c E_0^2}\frac{\mu_0}{16\pi^2 c}\langle |\ddot{\mathbf{p}}|^2 \rangle \sin^2\theta = \frac{\mu_0^2 \langle |\ddot{\mathbf{p}}|^2 \rangle}{8\pi^2 E_0^2}\sin^2\theta. \tag{5}$$

Integrating (5) over all observation directions gives a factor of $8\pi/3$ and it remains only to average over all orientations of $\mathbf{p}$ to get the total scattering cross section:

$$\sigma_{\text{scatt}} = \frac{\mu_0^2}{3\pi E_0^2} \times \frac{1}{4\pi}\int d\Omega_{\mathbf{p}} \langle |\ddot{\mathbf{p}}|^2 \rangle. \tag{6}$$

If Θ is the angle between $\mathbf{E}$ and $\mathbf{p}$, (4) gives

$$|\ddot{\mathbf{p}}|^2 = \frac{1}{I^2}\left[|\mathbf{E}|^2 p^4 + |(\mathbf{E} \cdot \mathbf{p})|^2 p^2 - 2|(\mathbf{E} \cdot \mathbf{p})|^2 p^2\right] = \frac{E^2 p^4}{I^2}(1 - \cos^2\Theta).$$

The average of $\cos^2\Theta$ over a sphere is 1/3. Therefore, taking account of a factor of 1/2 from the time-averaged $\langle |\ddot{\mathbf{p}}| \rangle$, (6) becomes

$$\sigma_{\text{scatt}} = \frac{\mu_0^2}{3\pi E_0^2}\frac{E_0^2 p^4}{I^2}\frac{1}{3} = \frac{\mu_0^2 p^4}{9\pi I^2}.$$

Source: L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1962).

21.7 Preservation of Polarization I

From the data given in Problem 21.5, the electric and magnetic moments induced in the sphere are $\mathbf{p} = a\mathbf{E}_0$ and $\mathbf{m} = b\mathbf{B}_0$ where $b = -a/2$. Therefore, if $\mathbf{k}$ is the scattered wave vector, the sum of the electric and magnetic dipole radiation fields produced by the sphere has the form

$$\mathbf{E}_{\text{rad}} \propto \hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{p}) + \hat{\mathbf{k}} \times \mathbf{m}/c = \hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times a\mathbf{E}_0) + \hat{\mathbf{k}} \times \gamma \mathbf{B}_0/c.$$

On the other hand, $\hat{\mathbf{k}}_0 \times \mathbf{E}_0 = c\mathbf{B}_0$ defines the incident wave. Therefore,

$$\mathbf{E}_{\text{rad}} \propto \hat{\mathbf{k}} \times \left[(\alpha\hat{\mathbf{k}} \times \gamma\hat{\mathbf{k}}_0) \times \mathbf{E}_0\right] = \mathbf{E}_{\text{rad}} \propto \hat{\mathbf{k}} \times \left[(\hat{\mathbf{k}} - \frac{1}{2}\hat{\mathbf{k}}_0) \times \mathbf{E}_0\right],$$

or, because $\hat{\mathbf{k}} \cdot \hat{\mathbf{k}}_0 = \cos\theta$,

$$\mathbf{E}_{\text{rad}} \propto (\hat{\mathbf{k}} - \frac{1}{2}\hat{\mathbf{k}}_0)\hat{\mathbf{k}} \cdot \mathbf{E}_0 - \mathbf{E}_0 \left[\hat{\mathbf{k}} \cdot (\hat{\mathbf{k}} - \frac{1}{2}\hat{\mathbf{k}}_0)\right] = (\hat{\mathbf{k}} - \frac{1}{2}\hat{\mathbf{k}}_0)\hat{\mathbf{k}} \cdot \mathbf{E}_0 - \mathbf{E}_0(1 - \frac{1}{2}\cos\theta).$$

This shows that the scattered electric field vector is parallel to $\mathbf{E}_0$ if

$$\hat{\mathbf{k}} = \frac{1}{2}\hat{\mathbf{k}}_0 + \frac{\sqrt{3}}{2}\hat{\mathbf{E}}_0.$$

Hence, the deflection angle from the incident direction is

$$\theta = \cos^{-1}(\hat{\mathbf{k}}_0 \cdot \hat{\mathbf{k}}) = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}.$$

Source: D.S. Jones, *The Theory of Electromagnetism* (Macmillan, New York, 1964).

21.8 Scattering and Absorption by an Ohmic Sphere

(a) From Section 17.6, the dielectric constant of a simply conducting (ohmic) sphere is

$$\epsilon(\omega) = \epsilon_0 + i\frac{\sigma}{\omega}.$$

The skin depth is large, so the (static) dipole moment calculation we need is very much like the one done in Example 6.3 of Section 6.5.5. Assume a static field $\mathbf{E} = E_0\hat{\mathbf{z}}$. The potentials inside and outside the sphere are

$$\varphi_{\text{in}} = Ar\cos\theta \qquad \text{and} \qquad \varphi_{\text{out}} = [-E_0 r + (B/r^2)]\cos\theta.$$

The matching conditions are

$$\varphi_{\text{in}}(a) = \varphi_{\text{out}}(a) \qquad \text{and} \qquad \epsilon_0 \frac{\partial \varphi_{\text{out}}}{\partial r}\bigg|_{r=a} = \epsilon(\omega)\frac{\partial \varphi_{\text{in}}}{\partial r}\bigg|_{r=a}.$$

These conditions are satisfied if

$$A = -\frac{3\epsilon_0}{\epsilon(\omega) + 2\epsilon_0}E_0 \qquad \text{and} \qquad B = \frac{\epsilon(\omega) - \epsilon_0}{\epsilon(\omega) + 2\epsilon_0}a^3 E_0.$$

Now, the potential outside an electric dipole aligned with the polar axis is

$$\varphi_{\text{dipole}} = \frac{1}{4\pi\epsilon_0}\frac{p\cos\theta}{r^2}.$$

Therefore, the electric moment of the ohmic sphere is

$$\mathbf{p} = 4\pi\epsilon_0 \frac{\epsilon(\omega)-\epsilon_0}{\epsilon(\omega)+2\epsilon_0} a^3 \mathbf{E}_0. \tag{1}$$

(b) From the text, the absorption cross section for an ohmic ($\mathbf{j} = \sigma\mathbf{E}$) volume V is

$$\sigma_{\text{abs}} = \frac{1}{|\langle \mathbf{S}_{\text{inc}}\rangle|}\int_V d^3r\,\langle \mathbf{j}\cdot\mathbf{E}\rangle = \frac{1}{\frac{1}{2}\epsilon_0 c E_0^2}\frac{\sigma}{2}\int_V d^3r\,|\mathbf{E}|^2.$$

From part (a), the electric field inside the sphere is

$$\mathbf{E} = -A\hat{\mathbf{z}} = \frac{3\epsilon_0}{\epsilon(\omega)+2\epsilon_0}E_0.$$

Substituting this into the foregoing, we get

$$\sigma_{\text{abs}} = 12\pi a^3 \frac{\sigma/\epsilon_0 c}{9+(\sigma/\epsilon_0\omega)^2}.$$

(c) The radiated electric field due to dipole scattering is

$$\mathbf{E}_{\text{rad}} = -\frac{k^2}{4\pi\epsilon_0}\frac{\exp[i(kr-\omega t)]}{r}[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{p})] = E_0\frac{\exp[i(kr-\omega t)]}{r}\mathbf{f}(\hat{\mathbf{r}}).$$

Therefore, using (1), the scattering amplitude is

$$\mathbf{f}(\hat{\mathbf{r}}) = -\frac{k^2}{4\pi\epsilon_0 E_0}\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{p}) = -k^2\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}a^3\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\hat{\mathbf{e}}_0).$$

The forward direction is $\hat{\mathbf{r}} = \hat{\mathbf{k}}_0$ where $\hat{\mathbf{k}}_0\cdot\hat{\mathbf{e}}_0 = 0$. Therefore,

$$\mathbf{f}(\hat{\mathbf{k}}_0) = k^2\frac{\epsilon-\epsilon_0}{\epsilon+2\epsilon_0}a^3\hat{\mathbf{e}}_0.$$

Moreover, since $\epsilon = \epsilon_0 + i\epsilon''$ for our problem,

$$\operatorname{Im}\frac{\epsilon(\omega)-\epsilon_0}{\epsilon(\omega)+2\epsilon_0} = \frac{3\epsilon_0\epsilon''}{9\epsilon_0^2+\epsilon''^2}.$$

Collecting results, the optical theorem asks us to compute

$$\frac{4\pi}{k}\operatorname{Im}\left[\mathbf{f}(\hat{\mathbf{k}}_0)\cdot\hat{\mathbf{e}}_0^*\right] = \frac{4\pi}{k}k^2a^3\frac{3\epsilon_0\sigma/\omega}{9\epsilon_0^2+\sigma^2/\omega^2} = 12\pi a^3\frac{\sigma/\epsilon_0 c}{9+(\sigma/\epsilon_0\omega)^2}.$$

This is indeed σ_{abs} computed in part (b) above.

(d) Our dipole scattering approximation is most correct when $ka \to 0$. In that case, $\sigma_{\rm abs} \propto (ka)^2$ completely dominates the Rayleigh scattering result that $\sigma_{\rm scatt} \propto (ka)^4$. The latter plays a role in the optical theorem at higher frequency when it is necessary to go beyond the dipole approximation.

Source: J.D. Jackson, *Classical Electrodynamics*, 3rd edition (Wiley, New York, 1999).

21.9 Scattering from a Dielectric Cylinder

By symmetry, the scattered electric field has a z-component only. Moreover, none of the fields can depend on z. This is the same situation as was studied in the text for the conducting cylinder. Therefore, E_z is a linear combination of Bessel functions in the radial variable and a complex exponential in the angular variable. There are no sources, so the Ampère-Maxwell law gives the associated magnetic field components as

$$B_\rho = -\frac{i}{\omega\rho}\frac{\partial E_z}{\partial \phi} \qquad \text{and} \qquad B_\phi = \frac{i}{\omega}\frac{\partial E_z}{\partial \rho}.$$

Inside the cylinder, we let $k = \omega^2 \epsilon\mu_0$ and write

$$E_{\rm z,in} = \sum_{m=-\infty}^{\infty} c_m J_m(k\rho)\exp(im\phi) \qquad \rho < a. \tag{1}$$

Outside the cylinder, we let $\omega = ck_0$ and write the electric field as the sum of the incident wave with amplitude

$$E_{z,\rm inc} = E_0 \exp(i\mathbf{k}_0 \cdot \boldsymbol{\rho}) = E_0 \sum_{m=-\infty}^{\infty} i^m J_m(k_0\rho)\exp[im(\phi-\phi_0)]$$

and an outgoing scattered wave with amplitude

$$E_{\rm z,out} = \sum_{m=-\infty}^{\infty} b_m H_m^{(1)}(k_0\rho)\exp(im\phi) \qquad \rho > a. \tag{2}$$

The matching conditions are continuity of the tangential field components E_z and B_ϕ at $\rho = a$. Using all the foregoing, these conditions read

$$\begin{aligned}\sum_{m=-\infty}^{\infty} c_m J_m(ka)\exp(im\phi) &= E_0 \sum_{m=-\infty}^{\infty} i^m J_m(k_0 a)\exp[im(\phi-\phi_0)] \\ &+ \sum_{m=-\infty}^{\infty} b_m H_m^{(1)}(k_0 a)\exp(im\phi)\end{aligned}$$

and

$$\begin{aligned}k\sum_{m=-\infty}^{\infty} c_m J'_m(ka)\exp(im\phi) &= E_0 k_0 \sum_{m=-\infty}^{\infty} i^m J'_m(k_0 a)\exp[im(\phi-\phi_0)] \\ &+ k_0 \sum_{m=-\infty}^{\infty} b_m H_m'^{(1)}(k_0 a)\exp(im\phi).\end{aligned}$$

Each of the Fourier components (in the variable ϕ) are linearly independent. Therefore, the two conditions above reduce to

$$J_m(ka)c_m - H_m^{(1)}(k_0a)b_m = i^m J_m(k_0a)\exp(-im\phi_0)E_0$$

and

$$\frac{k}{k_0}J_m'(ka)c_m - H_m'^{(1)}(k_0a)b_m = i^m J_m'(k_0a)\exp(-im\phi_0)E_0.$$

These are two linear equations in two unknowns. Hence, we find without difficulty that

$$b_m = \frac{J_m(ka)J_m'(k_0a) - (k/k_0)J_m'(ka)J_m(k_0a)}{(k/k_0)H_m^{(1)}(k_0a)J_m'(ka) - H_m'^{(1)}(k_0a)J_m(ka)} i^m \exp(-im\phi_0)E_0$$

and

$$c_m = \frac{H_m^{(1)}(k_0a)J_m'(k_0a) - H_m'^{(1)}(k_0a)J_m(k_0a)}{(k/k_0)H_m^{(1)}(k_0a)J_m'(ka) - H_m'^{(1)}(k_0a)J_m(ka)} i^m \exp(-im\phi_0)E_0.$$

The numerator in the c_m expression is a Wronskian relation with the value $-2/\pi k_0 a$. Otherwise, substituting these formulas for b_m and c_m into (2) and (1) completes the solution.

21.10 Preservation of Polarization II

Physical optics assumes that the current density on the illuminated surface of a perfectly conducting object is $\mathbf{K} = 2\hat{\mathbf{n}} \times \mathbf{B}_0$, where $\mathbf{B}_0$ is the incident magnetic field. The current density is assumed to be zero on the non-illuminated portion of the surface. Now, the electric field radiated by a time-harmonic plane wave is

$$\mathbf{E}_{\rm rad} = -\frac{ik}{4\pi\epsilon_0 c}\hat{\mathbf{k}} \times \left[\hat{\mathbf{k}} \times \int d^3r'\, \mathbf{j}(\mathbf{r}'|\omega)\exp(-i\mathbf{k}\cdot\mathbf{r}')\right]\frac{\exp[i(kr-\omega t)]}{r}.$$

Therefore, in the physical optics approximation, the integrand contains the factor

$$\hat{\mathbf{k}} \times [\hat{\mathbf{k}} \times (\hat{\mathbf{n}} \times \mathbf{B}_0)] = \hat{\mathbf{k}} \times [\hat{\mathbf{n}}(\hat{\mathbf{k}}\cdot\mathbf{B}_0) - \mathbf{B}_0(\hat{\mathbf{n}}\cdot\hat{\mathbf{k}})].$$

In the backward direction, $\hat{\mathbf{k}} = -\mathbf{k}_0$. Also, all radiation fields are transverse. Therefore,

$$\hat{\mathbf{k}}\cdot\mathbf{B}_0 = -\hat{\mathbf{k}}_0\cdot\mathbf{B}_0 = 0,$$

because the incident plane wave is also transverse. Using these two facts again, we see that the integrand contains the factor

$$-(\hat{\mathbf{k}} \times \mathbf{B}_0)(\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}) = (\hat{\mathbf{k}}_0 \times \mathbf{B}_0)(\hat{\mathbf{n}}\cdot\hat{\mathbf{k}}) = -\mathbf{E}_0(\hat{\mathbf{n}}\cdot\mathbf{k}).$$

This proves the assertion because all contributions to the integral are proportional to $\mathbf{E}_0$.

Source: E.F. Knott and T.B.A. Senior, *Electronics Letters* **7**, 184 (1971).

21.11 Scattering from a Conducting Strip

(a) The diagram shows that

$$\mathbf{k}_0 = k_0\cos(\pi-\phi_0)\hat{\mathbf{x}} - k_0\sin(\pi-\phi_0)\hat{\mathbf{y}} = -k_0\cos\phi_0\hat{\mathbf{x}} - k_0\sin\phi_0\hat{\mathbf{y}}.$$

Therefore the incident electric field is

$$\mathbf{E}_0 = \hat{\mathbf{z}}E_0\exp(i\mathbf{k}_0\cdot\mathbf{r}) = \hat{\mathbf{z}}E_0\exp[-ik_0(x\cos\phi_0 + y\sin\phi_0)]$$

and the incident magnetic field is

$$c\mathbf{B}_0 = \hat{\mathbf{k}}_0\times\mathbf{E}_0 = (-\sin\phi_0\hat{\mathbf{x}} + \cos\phi_0\hat{\mathbf{y}})E_0\exp[-ik_0(x\cos\phi_0 + y\sin\phi_0)].$$

With $Z_0 = \sqrt{\mu_0/\epsilon_0}$, the physical optics current density on the top surface of the strip is

$$\mathbf{K}_{\rm PO} = 2\hat{\mathbf{y}}\times\mathbf{B}_0/\mu_0|_{y=0} = \hat{\mathbf{z}}\frac{2E_0}{Z_0}\sin\phi_0\exp(-ik_0x\cos\phi_0).$$

(b) The exact vector potential produced by a time-harmonic surface current density is

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int dS'\frac{\mathbf{K}(\mathbf{r}')\exp(ik|\mathbf{r}-\mathbf{r}'|)}{|\mathbf{r}-\mathbf{r}'|}.$$

If we write $\mathbf{r} = \boldsymbol{\rho} + z\hat{\mathbf{z}}$ and similarly for $\mathbf{r}'$, substituting the physical optics current density from part (a) gives

$$\mathbf{A}(\mathbf{r}) = \hat{\mathbf{z}}\frac{\mu_0}{4\pi}\int_0^w dx'\frac{2E_0}{Z_0}\sin\phi_0\exp(-ikx'\cos\phi_0)\int_{-\infty}^{\infty} dz'\frac{\exp\left(ik\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z-z')^2}\right)}{\sqrt{|\boldsymbol{\rho}-\boldsymbol{\rho}'|^2+(z-z')^2}}.$$

From the given integral, a change of variable shows that

$$i\pi H_0^{(1)}(\alpha x) = \int_{-\infty}^{\infty} dy\frac{\exp(i\alpha\sqrt{x^2+y^2})}{\sqrt{x^2+y^2}}.$$

Therefore, with $x = |\boldsymbol{\rho}-\boldsymbol{\rho}'|$ and $y = z - z'$, we get

$$\mathbf{A}(\mathbf{r}) = \hat{\mathbf{z}}\frac{i\mu_0E_0}{2Z_0}\sin\phi_0\int_0^w dx' H_0^{(1)}(k|\boldsymbol{\rho}-\boldsymbol{\rho}'|)\exp(-ikx'\cos\phi_0).$$

(c) The asymptotic behavior of the Hankel function is

$$\lim_{x\to\infty} H_0^{(1)}(x) = \sqrt{\frac{2}{\pi x}}\exp[i(x-\pi/4)].$$

When $\rho \gg \rho'$,

$$|\boldsymbol{\rho} - \boldsymbol{\rho}'|^2 = \rho^2 + \rho'^2 - 2\rho\rho' \cos\phi \approx \rho^2 \left(1 - 2\frac{x'}{\rho}\cos\phi\right).$$

Therefore, $k|\boldsymbol{\rho} - \boldsymbol{\rho}'| \approx \rho - x'\cos\phi$. The x' dependence can be dropped in the square-root pre-factor of the Hankel function, so

$$\lim_{\rho \gg \rho'} H_0^{(1)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|) = \sqrt{\frac{2}{i\pi k\rho}} \exp[ik(\rho - x'\cos\phi)].$$

Substituting this into the vector potential expression of part (b) gives

$$\mathbf{A}_{\rm rad}(\rho, \phi) = \sqrt{\frac{i}{2\pi}} \frac{\mu_0 E_0}{Z_0} \sin\phi_0 \frac{\exp(ik\rho)}{\sqrt{k\rho}} \int_0^w dx' \exp[-ikx'(\cos\phi + \cos\phi_0)].$$

With $\sigma = \cos\phi + \cos\phi_0$, the integral is

$$w \exp\left(\frac{i}{2} kw\sigma\right) \frac{\sin\left(\frac{1}{2}kw\sigma\right)}{\frac{1}{2}kw\sigma}.$$

Finally, $\mathbf{E}_{\rm rad} = -i\omega\mathbf{A}_{\rm rad}$ and the two-dimensional cross section is

$$\frac{d\sigma}{d\phi} = \rho\frac{|\mathbf{E}_{\rm rad}|^2}{E_0^2} = \frac{2}{\pi k} \sin^2\phi_0 \frac{\sin^2\left[\frac{1}{2}kw(\cos\phi + \cos\phi_0)\right]}{(\cos\phi + \cos\phi_0)^2}.$$

Source: C.A. Balanis, *Advanced Engineering Electromagnetics* (Wiley, New York, 1989).

21.12 Physical Optics Backscattering

(a) We begin with the expression derived in the text for the differential cross section, specialized to the case of a conductor where all the current flows on the surface:

$$\frac{d\sigma_{\rm scatt}}{d\Omega} = \left(\frac{k_0}{4\pi\epsilon_0 E_0 c}\right)^2 \left|\hat{\mathbf{k}} \times \int dS'\, \mathbf{K}(\mathbf{r}'|\omega) \exp(-i\mathbf{k}\cdot\mathbf{r}')\right|^2.$$

Substituting into this formula the physical optics approximation for the surface current density,

$$\mu_0\mathbf{K} = \begin{cases} 2\hat{\mathbf{n}} \times \mathbf{B}_{\rm inc} & \text{at illuminated surface points} \\ 0 & \text{at shadowed surface points,} \end{cases}$$

calls for $\hat{\mathbf{k}} \times (\hat{\mathbf{n}} \times \mathbf{B}_{\rm inc}) = \hat{\mathbf{n}}(\hat{\mathbf{k}} \cdot \mathbf{B}_{\rm inc}) - \mathbf{B}_{\rm inc}(\hat{\mathbf{n}} \cdot \hat{\mathbf{k}})$. However, the backscattering wave vector $\mathbf{k}$ is simply the negative of the incident wave vector. Therefore, $\hat{\mathbf{k}} \cdot \mathbf{B}_{\rm inc} = 0$, where $\mathbf{B}_{\rm inc} = \mathbf{B}_0 \exp(-i\mathbf{k}\cdot\mathbf{r})$ and $c|\mathbf{B}_0| = E_0$. Hence,

$$\sigma_{\rm R} = \frac{k_0^2}{4\pi^2} \left| \int_S dS' \, \hat{\mathbf{k}} \cdot \hat{\mathbf{n}}' \exp(-2i\mathbf{k} \cdot \mathbf{r}') \right|^2 .$$

(b) The backscattered wave propagates along the direction $\hat{\mathbf{k}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$. Because $\hat{\mathbf{n}'} = \hat{\mathbf{z}}$,

$$\begin{aligned} \sigma_{\rm R} &= \frac{k_0^2}{4\pi^2}\cos^2\theta \left| \int_{-a/2}^{a/2} dx \int_{-b/2}^{b/2} dy \, \exp[-i(2k_0 x \sin\theta\cos\phi + 2k_0 y \sin\theta\sin\phi)] \right|^2 \\ &= \frac{k_0^2}{4\pi^2}\cos^2\theta \left| \frac{\sin(k_0 a \sin\theta\cos\phi)}{k_0 \sin\theta\cos\phi} \times \frac{\sin(k_0 b \sin\theta\sin\phi)}{k_0 \sin\theta\sin\phi} \right|^2 . \end{aligned}$$

The plate area is $A = ab$ and $k_0 = 2\pi/\lambda$. Therefore

$$\sigma_{\rm R} = \frac{A^2}{\lambda^2}\cos^2\theta \left| \frac{\sin(k_0 a \sin\theta\cos\phi)}{k_0 a \sin\theta\cos\phi} \times \frac{\sin(k_0 b \sin\theta\sin\phi)}{k_0 b \sin\theta\sin\phi} \right|^2 .$$

Source: A. Ishimaru, *Electromagnetic Wave Propagation, Radiation, and Scattering* (Prentice-Hall, Upper Saddle River, NJ, 1991).

21.13 Born Scattering from a Dielectric Cube

(a) If $\mathbf{q} = \mathbf{k} - \mathbf{k}_0$ is the difference between the incoming and outgoing scattering wave vectors ($\omega = ck = ck_0$), we learned in Example 21.3 that the Born approximation to the differential cross section is

$$\frac{d\sigma_{\rm Born}}{d\Omega} = \left(\frac{k_0^2 \chi_e}{4\pi}\right)^2 |\hat{\mathbf{k}} \times \hat{\mathbf{E}}_0|^2 \left| \int_V d^3r' \, \exp(i\mathbf{q}\cdot\mathbf{r}') \right|^2 .$$

The integral of interest is

$$\int_V d^3r' \, \exp(i\mathbf{q}\cdot\mathbf{r}') = \int_0^a dx' \, \exp(iq_x x') \times \int_0^a dy' \, \exp(iq_y y') \times \int_0^a dz' \, \exp(iq_z z').$$

Moreover,

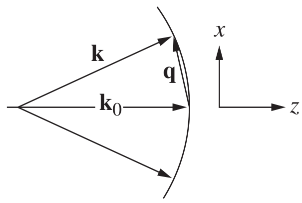
$$\int_0^a dx' \, \exp(iq_x x') = a\exp(-iq_x a/2)\frac{\sin(q_x a/2)}{q_x a/2}.$$

Therefore,

$$\frac{d\sigma_{\rm Born}}{d\Omega} = \left(\frac{k_0^2 V \chi_e}{4\pi}\right)^2 |\hat{\mathbf{k}} \times \hat{\mathbf{E}}_0|^2 \left[\frac{\sin(q_x a/2)}{q_x a/2}\frac{\sin(q_y a/2)}{q_y a/2}\frac{\sin(q_z a/2)}{q_z a/2}\right]^2 .$$

(b) Let $\hat{\mathbf{k}}_0 = \hat{\mathbf{z}}$ and $\mathbf{E}_0 = \hat{\mathbf{x}}$. When $ka \gg 1$, near-forward scattering dominates and $|\hat{\mathbf{k}} \times \hat{\mathbf{E}}_0| \approx |\hat{\mathbf{k}}_0 \times \hat{\mathbf{E}}_0| = 1$. The diagram below shows that, in the same limit, the area element $k^2 d\Omega_{\rm k}$ is essentially the area $dq_x dq_y$ of a circular disk perpendicular to $\mathbf{k}_0$:

$$dS_{\mathbf{k}} = k^2 d\Omega_{\mathbf{k}} \approx dq_x dq_y.$$



Therefore, in the $ka \gg 1$ limit when $q_z \to 0$, the fact that $k_0 = k$ means that

$$\lim_{ka\gg 1} \sigma_{\rm Born} = \lim_{ka\gg 1} \int d\Omega_{\rm k} \frac{d\sigma_{\rm Born}}{d\Omega_{\rm k}} \approx \lim_{ka\gg 1} \left(\frac{kV\chi_e}{4\pi}\right)^2 \int dq_x \frac{\sin^2(q_x a/2)}{(q_x a/2)^2} \int dq_y \frac{\sin^2(q_y a/2)}{(q_y a/2)^2}.$$

The integrals are dominated by contributions when $q_x, q_y \sim 1/a$ so the limits can be extended to $\pm\infty$ with little loss of accuracy. Therefore,

$$\lim_{ka\gg 1} \sigma_{\rm Born} = \frac{k^2 a^4 \chi_e^2}{4\pi^2} \left[\int_{-\infty}^{\infty} \frac{\sin^2 u}{u^2}\right]^2 = \frac{k^2 a^4 \chi_e^2}{4}.$$

(c) From the definition of the cross section and the result of part (a),

$$\frac{E_{\rm rad}}{E_0} \approx \frac{1}{r}\sqrt{\frac{d\sigma}{d\Omega}} \approx \frac{k^2 a^3 \chi_e}{4\pi r} \left| \frac{\sin(q_x a/2)}{q_x a/2} \frac{\sin(q_y a/2)}{q_y a/2} \frac{\sin(q_z a/2)}{q_z a/2} \right|.$$

The absolute-value term gets no larger than one. Therefore, with $r = a$, the weak scattering criterion is indeed

$$1 \gg k^2 a^2 \chi_e = \sigma_{\rm Born}/a^2 \chi_e.$$

Source: Prof. K. Likharev, SUNY Stony Brook (public communication).

21.14 Scattering from a Short Conducting Wire

(a) We need the time-harmonic scattered field, $\mathbf{E}_{\rm scatt} = -\nabla\varphi + i\omega\mathbf{A}$, in the immediate vicinity of the wire. In that case, $k|\mathbf{r} - \mathbf{r}'| = kR \ll 1$, so

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int dS' \, \sigma(\mathbf{r}') \frac{\exp(ikR)}{R} \approx \frac{1}{4\pi\epsilon_0} \int d^3 r' \, \frac{\rho(\mathbf{r}')}{R} \left[1 + ikR - \frac{1}{2}(kR)^2 - \frac{i}{6}(kR)^3\right]$$

and

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\int dS'\,\mathbf{K}(\mathbf{r}')\,\frac{\exp(ikR)}{R} \approx \frac{\mu_0}{4\pi}\int d^3r'\,\frac{\mathbf{j}(\mathbf{r}')}{R}\left[1 + ikR - \frac{1}{2}(kR)^2 - \frac{i}{6}(kR)^3\right].$$

Therefore,

$$\begin{aligned}\hat{\mathbf{z}}\cdot\mathbf{E}_{\rm scatt} &\approx \frac{1}{4\pi\epsilon_0}\int dS'\sigma(\mathbf{r}')\left[\frac{1}{R^2} + \frac{1}{2}k^2 + \frac{i}{3}k^3R + \cdots\right]\hat{\mathbf{R}}\cdot\hat{\mathbf{z}} \\ &\quad + \frac{i\omega\mu_0}{4\pi}\int dS'\,K(\mathbf{r}')\left[\frac{1}{R} + ik - \frac{1}{2}k^2R - \frac{i}{6}k^3R^2 + \cdots\right].\end{aligned}$$

Moreover, from the continuity equation,

$$\frac{\partial\sigma}{\partial t} = -i\omega\sigma = -\nabla\cdot\mathbf{K} = -\frac{d}{dz}\frac{I_0}{2\pi a}\left[1 - (z/h)^2\right],$$

we deduce that

$$\sigma(z) = \frac{iI_0 z}{\pi a h^2\omega}.$$

The perfect-conductor condition requires $\mathbf{E}_{\rm scatt}$ at the $\mathbf{r}=0$ center of the wire. From that particular point, $R = \sqrt{a^2 + (z')^2}$, $\hat{\mathbf{R}}\cdot\hat{\mathbf{z}} = -z'/R$, and the integrand is an even function of z'. Therefore, with $s = z'/h$ and $R_0 = \sqrt{s^2 + (a/h)^2}$,

$$\begin{aligned}\hat{\mathbf{z}}\cdot\mathbf{E}_{\rm scatt}(0) &\approx -\frac{iI_0}{\pi\epsilon_0\omega h^2}\left\{\int_0^1 ds s^2\left[\frac{1}{R_0^3} + \frac{(kh)^2}{2R_0} + \frac{i}{3}(kh)^3\right]\right. \\ &\qquad \left. - \frac{1}{2}(kh)^2\int_0^1 ds\left[\frac{1}{R_0} + ikh\right](1 - s^2)\right\}.\end{aligned}$$

Performing the various integrals, the perfect conductor condition sets $\hat{\mathbf{z}}\cdot\mathbf{E}_0$ equal to

$$\begin{aligned}&\frac{iI_0}{\pi\epsilon_0\omega h^2}\left(\left\{\frac{1}{2}\ln\left[\frac{\sqrt{1+(a/h)^2}+1}{\sqrt{1+(a/h)^2}-1}\right] - \frac{1}{\sqrt{1+(a/h)^2}}\right\}\right. \\ &\qquad\left. \times\left\{1 - \frac{1}{2}(kh)^2[1 + (a/h)^2]\right\} - i\frac{2}{9}(kh)^3\right).\end{aligned}$$

Using $a \ll h$, we finally find that

$$I_0 = \frac{-i\pi\epsilon_0\omega h^2}{\{[\ln(2h/a) - 1][1 - \frac{1}{2}(kh)^2] - i\frac{2}{9}(kh)^3\}}\hat{\mathbf{z}}\cdot\mathbf{E}_0.$$

(b) Dropping the indicated terms, the surface charge density on the wire is

$$\sigma(z) = \frac{iI_0 z}{\pi a h^2\omega} = \epsilon_0\frac{z}{a}\frac{1}{[\ln(2h/a) - 1]}\hat{\mathbf{z}}\cdot\mathbf{E}_0.$$

The corresponding electric dipole moment is

$$\mathbf{p} = \hat{\mathbf{z}} \int dS\sigma(z) = 2\pi a \int_{-h}^{h} dz \, \frac{z^2}{a} \frac{1}{[\ln(2h/a) - 1]} (\hat{\mathbf{z}} \cdot \mathbf{E}_0)\hat{\mathbf{z}} = \frac{4\pi\epsilon_0 h^3}{3[\ln(2h/a) - 1]} (\hat{\mathbf{z}} \cdot \mathbf{E}_0)\hat{\mathbf{z}}.$$

For a unit-amplitude electric field, the total cross section for electric dipole scattering is

$$\sigma_{\rm scatt} = \frac{8\pi}{3} \left(\frac{\mu_0}{4\pi} \omega^2 \right)^2 |\mathbf{p}|^2,$$

where the factor of $8\pi/3$ comes from integrating over all directions of the outgoing wave vector. Inserting $\mathbf{p}$ from above gives

$$\sigma_{\rm scatt} = \frac{8\pi (hk)^4 h^2 \sin^2\theta_0}{27[\ln(2h/a) - 1]^2}.$$

(c) A perfectly conducting rod does not absorb energy. Therefore, the optical theorem says that

$$\sigma_{\rm scatt} = \frac{4\pi}{k} {\rm Im}[\hat{\mathbf{E}}_0^* \cdot \mathbf{f}(\mathbf{k}_0)],$$

where

$$\mathbf{E}_{\rm rad} = E_0 \mathbf{f}(\theta, \phi) \frac{\exp[i(kr - \omega t)]}{r}.$$

For our problem,

$$\mathbf{f} = -\frac{\mu_0}{4\pi} \omega^2 \hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{p}),$$

where $\mathbf{p} = -p \sin\theta_0 \hat{\mathbf{z}}$. The forward direction corresponds to a scattering angle of $\theta = \pi - \theta_0$. Therefore,

$$\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{p}) = -\hat{\mathbf{E}}_0 p \sin\theta_0 \sin(\pi - \theta_0) = -\hat{\mathbf{E}}_0 p \sin^2\theta_0.$$

Using the complete value of I_0 computed in part (a), the optical theorem reads

$$\sigma_{\rm scatt} = \frac{4\pi}{k} \frac{\mu_0 \omega^2}{4\pi} \frac{4\pi}{3} \epsilon_0 h^3 \sin^2\theta_0 {\rm Im} \left\{ \frac{1}{\{[\ln(2h/a) - 1][1 - \frac{1}{2}(kh)^2] - i\frac{2}{9}(kh)^3\}} \right\},$$

or

$$\sigma_{\rm scatt} = \frac{4\pi}{3} (kh) h^2 \sin^2\theta_0 \frac{(2/9)(kh)^3}{[\ln(2h/a) - 1]^2 [1 - \frac{1}{2}(kh)^2]^2 + [(2/9)(kh)^3]^2}.$$

Since $kh \ll 1$, this is the same as the result of part (b):

$$\sigma_{\rm scatt} = \frac{8\pi (hk)^4 h^2 \sin^2\theta_0}{27[\ln(2h/a) - 1]^2}.$$

Source: G.S. Smith, *An Introduction to Classical Electromagnetic Radiation* (University Press, Cambridge, 1997).

21.15 Absorption Cross Section for a Microscopic Object

From the text, the absorption cross section can be written in terms of the time-averaged work done by the field on the current density:

$$\sigma_{\rm abs} = \frac{1}{|\langle \mathbf{S}_{\rm inc}\rangle|}\int d^3r\,\langle \mathbf{j}\cdot\mathbf{E}\rangle = \frac{1}{|\langle \mathbf{S}_{\rm inc}\rangle|}\frac{1}{2}{\rm Re}\int d^3r\,\mathbf{j}\cdot\mathbf{E}^*.$$

The time-averaged incident flux in the denominator is $\frac{1}{2}\epsilon_0 cE_0^2$ and we can replace the macroscopic field $\mathbf{E}$ by $\mathbf{E}_0$ when the "target" is a microscopic object. The current density of interest is the polarization current density,

$$\mathbf{j}_{\rm P} = \frac{\partial \mathbf{P}}{\partial t} = -i\omega\mathbf{P}.$$

Therefore,

$$\sigma_{\rm abs} = \frac{1}{\epsilon_0 cE_0^2}{\rm Re}\left[-i\omega\int d^3r\,\mathbf{P}\right]\cdot\mathbf{E}_0^*.$$

The external macroscopic field amplitude $\mathbf{E}_0$ is constant over the volume of a microscopic object and the volume integral of the polarization $\mathbf{P}$ is the electric dipole moment $\mathbf{p}$ of the object. Therefore,

$$\sigma_{\rm abs} = -\frac{\omega}{\epsilon_0 cE_0^2}{\rm Re}\left[i\mathbf{p}\cdot\mathbf{E}_0^*\right].$$

In terms of the polarizability α of the object, the dipole moment is

$$\mathbf{p} = \alpha\epsilon_0\mathbf{E}_0.$$

Substituting into the foregoing gives the desired result,

$$\sigma_{\rm abs} = -\frac{\omega}{cE_0^2}{\rm Re}\left[i\alpha|\mathbf{E}_0|^2\right] = \frac{\omega}{c}{\rm Im}\,\alpha.$$

21.16 Absorption Sum Rule for a Lorentz Oscillator

Because $|\langle \mathbf{S}_{\rm inc}\rangle| = \frac{1}{2}\epsilon_0 cE_0^2$, the absorption cross section is

$$\sigma_{\rm abs} = \frac{1}{|\langle \mathbf{S}_{\rm inc}\rangle|}\int_V d^3r\,\langle \mathbf{j}\cdot\mathbf{E}\rangle = \frac{1}{\epsilon_0 cE_0^2}{\rm Re}\int_V d^3r\,\mathbf{j}\cdot\mathbf{E}^*. \tag{1}$$

The current density is

$$\mathbf{j} = -e\mathbf{v} = -e\dot{\mathbf{r}} = ie\omega\mathbf{r}, \tag{2}$$

where $\mathbf{r}$ is the displacement of the electron. This is small, so we let $\mathbf{r}=0$ in $\mathbf{E} = \mathbf{E}_0\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$ and write the equation of motion of the electron in the presence of the field as

$$\ddot{\mathbf{r}} + \gamma\dot{\mathbf{r}} + \omega_0^2\mathbf{r} = -\frac{e}{m}\mathbf{E}_0\exp(-i\omega t). \tag{3}$$

Substituting the guess $\mathbf{r}(t) = \hat{\mathbf{r}}(\omega)\exp(-i\omega t)$ into (3) gives

$$\hat{\mathbf{r}}(\omega) = \frac{e\mathbf{E}_0}{m}\frac{1}{\omega^2 - \omega_0^2 + i\omega\gamma}. \tag{4}$$

Substituting (4) into (2) and the latter into (1) gives the cross section as

$$\sigma_{\rm abs}(\omega) = \frac{e^2}{\epsilon_0 mc}\mathrm{Re}\frac{i\omega}{\omega^2 - \omega_0^2 + i\omega\gamma} = \frac{e^2}{\epsilon_0 mc}\frac{\gamma\omega^2}{(\omega^2 - \omega_0^2)^2 + \omega^2\gamma^2}.$$

The damping is small, so the resonance is narrow and we can set $\omega^2 = \omega_0^2$ in the numerator, and $\omega^2 - \omega_0^2 = 2\omega_0(\omega - \omega_0)$ and $\gamma^2\omega^2 = \gamma^2\omega_0^2$ in the denominator. Therefore, with $y = \omega - \omega_0$,

$$\int_0^\infty d\omega\,\sigma_{\rm abs}(\omega) = \frac{e^2\gamma\omega_0^2}{\epsilon_0 mc}\int_{-\omega_0}^\infty \frac{dy}{(2\omega_0 y)^2 + \omega_0^2\gamma^2} \approx \frac{e^2\gamma\omega_0^2}{\epsilon_0 mc}\int_{-\infty}^\infty \frac{dy}{(2\omega_0 y)^2 + \omega_0^2\gamma^2}.$$

The integral is $\pi/2\omega_0^2\gamma$ so

$$\int_0^\infty d\omega\,\sigma_{\rm abs}(\omega) = \frac{e^2\pi}{2\epsilon_0 mc} = 2\pi^2 r_e c,$$

as advertised.

Source: W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd edition (Addison-Wesley, Reading, MA, 1962).

21.17 The Optical Theorem in Two Dimensions

(a) Let us write H_m for $H_m^{(1)}$ in this problem. The differential cross section for $\mathbf{E}$ parallel to the cylinder axis is

$$\frac{d\sigma_\parallel}{d\phi} = \frac{2}{\pi k}\left|\sum_{m=-\infty}^{\infty}\frac{J_m(ka)}{H_m(ka)}\exp(im\phi)\right|^2,$$

which we write out in the form

$$\frac{d\sigma_\parallel}{d\phi} = \frac{2}{\pi k}\sum_{m=-\infty}^{\infty}\sum_{m'=-\infty}^{\infty}\frac{J_m(ka)J_{m'}(ka)}{|H_m(ka)|^2}\exp[i(m-m')\phi)].$$

The ϕ integration generates a factor of $2\pi\delta_{m,m'}$. Therefore, because $H_m = J_m + iN_m$,

$$\sigma_\parallel = \int_0^{2\pi} d\phi\,\frac{d\sigma_\parallel}{d\phi} = \frac{4}{k}\sum_{m=-\infty}^{\infty}\frac{J_m^2(ka)}{J_m^2(ka) + N_m^2(ka)}.$$

The differential cross section for **E** perpendicular to the cylinder axis is

$$\frac{d\sigma_\perp}{d\phi} = \frac{2}{\pi k}\left|\sum_{m=-\infty}^{\infty}\frac{J_m'(ka)}{H_m'(ka)}\exp(im\phi)\right|^2.$$

The calculation proceeds exactly as the previous case. Therefore,

$$\sigma_\perp = \frac{4}{k}\sum_{m=-\infty}^{\infty}\frac{J_m'^2(ka)}{J_m'^2(ka)+N_m'^2(ka)}.$$

(b) From the text, the radiation electric field for this polarization is

$$\mathbf{E}_{\text{rad}} = -E_0\hat{\mathbf{e}}_0\sum_{m=-\infty}^{\infty} i^m\frac{J_m(ka)}{H_m(ka)}\exp(im\phi)H_m(k\rho).$$

Into this we substitute the asymptotic form of the Hankel function,

$$H_m(k\rho)\to\sqrt{\frac{2}{\pi k\rho}}\exp(ik\rho)\exp(-im\pi/2)\exp(-i\pi/4).$$

Then, because $i^m\exp(-im\pi/2)=1$ and $\exp(-i\pi/4)=\sqrt{1/i}$,

$$\mathbf{E}_{\text{rad}}\to -E_0\hat{\mathbf{e}}_0\sqrt{\frac{2}{\pi ik\rho}}\exp(ik\rho)\sum_{m=-\infty}^{\infty}\frac{J_m(ka)}{H_m(ka)}\exp(im\phi).$$

Comparing this to the formula in the statement of the problem shows that

$$\mathbf{f}(k,0) = i\sqrt{\frac{2}{\pi}}\sum_{m=-\infty}^{\infty}\frac{J_m(ka)}{J_m(ka)+iN_m(ka)}\hat{\mathbf{e}}_0.$$

Therefore,

$$\operatorname{Im}\mathbf{f}(k,0) = \sqrt{\frac{2}{\pi}}\sum_{m=-\infty}^{\infty}\frac{J_m^2(ka)}{J_m^2(ka)+N_m^2(ka)}\hat{\mathbf{e}}_0.$$

Comparing this to the scattering cross section computed in part (a) (there is no absorption) confirms the proposed statement of the optical theorem in two dimensions.

Source: S.K. Adhikari, *American Journal of Physics* **54**, 362 (1986).

21.18 The Optical Theorem for Pedestrians

When $\theta \ll 1$, we may assume that $x, y \ll z$ and approximate $r=\sqrt{x^2+y^2+z^2}$ by

$$r = z\sqrt{1+\frac{x^2+y^2}{z^2}} \approx z+\frac{\rho^2}{2z},$$

where $\rho^2 = x^2 + y^2$. Therefore,

$$\psi(\mathbf{r}) \approx \exp(ikz) + \frac{1}{z}\exp(ikz)\exp\left[i\frac{k\rho^2}{2z}\right]f(0) + O\left(\frac{1}{z^2}\right).$$

We can drop the last term in the asymptotic limit and conclude that

$$|\psi|^2 \approx 1 + \frac{2}{z}\mathrm{Re}\left\{f(0)\exp\left[i\frac{k\rho^2}{2z}\right]\right\}.$$

Integrating this quantity over a flat disk of radius R in a $z \gg R$ plane gives

$$\int_{\text{screen}} dS = \pi R^2 + \frac{4\pi}{z}\mathrm{Re}\left\{f(0)\int_0^R d\rho\rho\exp\left[i\frac{k\rho^2}{2z}\right]\right\}.$$

Changing variables to $u = \rho^2$ and using a convergence factor to perform the exponential integral when $R^2 \gg z/k$ gives the advertised result,

$$\int_{\text{screen}} dS = \pi R^2 - \frac{4\pi}{k}\mathrm{Im}f(0).$$

The area πR^2 is the cross section of the beam in the absence of the scatterer. Therefore, any energy removed from the beam is the result of either scattering or absorption. In other words,

$$\sigma_{\text{tot}} = \sigma_{\text{scatt}} + \sigma_{\text{abs}} = \frac{4\pi}{k}\mathrm{Im}f(0).$$

Source: H.C. van de Hulst, *Light Scattering by Small Particles* (Wiley, New York, 1957).

21.19 Total Cross Section Sum Rule

The real part on the left-hand side suggests that we exploit the Kramers-Krönig relation,

$$\chi'(\omega) = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} d\omega' \frac{\chi''(\omega')}{\omega - \omega'}.$$

For our problem, we use $\omega = ck = ck_0$ and write this in the form

$$\frac{\mathrm{Re}\,[\mathbf{f}(k, \hat{\mathbf{k}}_0) \cdot \hat{\mathbf{e}}_0^*]}{k^2} = -\frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{dk'}{k - k'} \frac{\mathrm{Im}[\mathbf{f}(k', \hat{\mathbf{k}}_0) \cdot \hat{\mathbf{e}}_0^*]}{k'^2}.$$

The limit $k \to 0$ is

$$\lim_{k\to 0} \frac{\mathrm{Re}\,[\mathbf{f}(k, \hat{\mathbf{k}}_0) \cdot \hat{\mathbf{e}}_0^*]}{k^2} = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{dk'}{k'} \frac{\mathrm{Im}[\mathbf{f}(k', \hat{\mathbf{k}}_0) \cdot \hat{\mathbf{e}}_0^*]}{k'^2}.$$

The imaginary part of any causal response function is an odd function of its argument. Therefore, $\text{Im}[\mathbf{f}\cdot\hat{\mathbf{e}}_0^*]/k'^2 \to 0$ as $k' \to 0$ and the integrand above (which is simultaneously established to be an even function of k') is not singular at $k'=0$. Therefore, we can remove the principal value and write

$$\lim_{k\to 0}\frac{\text{Re}\,[\mathbf{f}(k,\hat{\mathbf{k}}_0)\cdot\hat{\mathbf{e}}_0^*]}{k^2} = \frac{2}{\pi}\int_0^\infty \frac{dk'}{k'}\frac{\text{Im}[\mathbf{f}(k',\hat{\mathbf{k}}_0)\cdot\hat{\mathbf{e}}_0^*]}{k'^2}.$$

Therefore, using the optical theorem, $\sigma_{\text{tot}} = (4\pi/k)\text{Im}\,[\mathbf{f}(\mathbf{k}_0)\cdot\hat{\mathbf{e}}_0^*]$,

$$\lim_{k\to 0}\frac{\text{Re}\,[\mathbf{f}(k,\hat{\mathbf{k}}_0)\cdot\hat{\mathbf{e}}_0^*]}{k^2} = \frac{1}{2\pi^2}\int_0^\infty \frac{dk'}{k'^2}\,\sigma_{\text{tot}}(k').$$

Changing to wavelength variable $\lambda = 2\pi/k$, we have $dk/k^2 = -d\lambda/2\pi$. The advertised result follows:

$$\lim_{\lambda\to\infty}\text{Re}\,[\mathbf{f}(\lambda,\mathbf{k}_0)\cdot\hat{\mathbf{e}}_0^*] = \frac{1}{\pi\lambda^2}\int_0^\infty d\lambda'\,\sigma_{\text{tot}}(\lambda').$$

Source: H.M. Nussenzveig, *Causality and Dispersion Relations* (Academic, New York, 1972).

21.20 The Index of Refraction

(a) The geometry and $R^2 = \rho^2 + z^2$ tell us that $\rho d\rho = RdR$ and $\cos\theta = z/R$. Therefore,

$$\mathbf{E}_{\text{rad}} = \delta N E_0 \delta t \int_0^{2\pi} d\phi \int_z^\infty dR \exp(ikR)\,\mathbf{f}\left(\cos^{-1}\frac{1}{\eta},\phi\right).$$

Changing variables to $\eta = R/z$ transforms the integral to

$$\mathbf{E}_{\text{rad}} = \delta N E_0 \delta t \int_0^{2\pi} d\phi \mathbf{J}(z,\phi),$$

where

$$\mathbf{J}(z,\phi) = z\int_1^\infty d\eta\,\exp(ikz\eta)\,\mathbf{f}\left(\cos^{-1}\frac{1}{\eta},\phi\right). \tag{1}$$

Integrating by parts gives

$$\mathbf{J}(z,\phi) = \frac{1}{ik}\exp(ikz\eta)\,\mathbf{f}\left(\cos^{-1}\frac{1}{\eta}\right)\Bigg|_1^\infty - \frac{1}{ik}\int_1^\infty d\eta\exp(ikz\eta)\frac{d}{d\eta}\,\mathbf{f}\left(\cos^{-1}\frac{1}{\eta},\phi\right). \tag{2}$$

The ratio of the integral in (2) to the original integral (1) goes like $1/kz$, which vanishes as $kz \to \infty$. Therefore, (1) is equal to the fully integrated term in (2). For the latter, we use the convergence factor $\lim_{\beta\to 0}\exp(i\beta R)$ to eliminate the contribution at ∞. The result is

$$\mathbf{J} = \exp(ikz)\frac{i}{k}\,\mathbf{f}\big(\cos^{-1}1,\phi\big) = \exp(ikz)\frac{i}{k}dz\,\mathbf{f}(0,\phi).$$

The scattering amplitude does not depend on ϕ when $\theta = 0$. Therefore,

$$\mathbf{E}_{\rm rad}(z) = \frac{2\pi i}{k}\delta N E_0 \delta t\exp(ikz)\,\mathbf{f}(0) \qquad\qquad kz \gg 1.$$

(b) Using the results of part (a), the total field in the far zone is

$$\mathbf{E}(z) = E_0\exp(ikz)\left[\hat{\mathbf{e}}_0 + i\frac{2\pi\delta N\delta t}{k}\mathbf{f}(0)\right].$$

This is just a plane wave in the inhomogeneous medium, with a z-independent change in its phase, amplitude, and polarization due to the density inhomogeneity. Therefore, a perfectly acceptable solution of the Maxwell equations at $z = \delta t$ is

$$\hat{\mathbf{e}}_0^*\cdot\mathbf{E}(\delta t) = E_0\exp(ik\delta t)\left[1 + i\frac{2\pi\delta N\delta t}{k}\hat{\mathbf{e}}_0^*\cdot\mathbf{f}(0)\right] \approx E_0\exp[i(k+\delta k)\delta t],$$

where

$$\delta k = \frac{2\pi\delta N}{k}\hat{\mathbf{e}}_0^*\cdot\mathbf{f}(0).$$

Multiply both sides by k and integrate the right side from 0 to N and the left side from k_0 to k. The result is

$$\frac{1}{2}(k^2 - k_0^2) = 2\pi N\hat{\mathbf{e}}_0^*\cdot\mathbf{f}(0).$$

Because $k = nk_0$, we get the desired result,

$$n^2 = 1 + \frac{4\pi N}{k_0^2}\hat{\mathbf{e}}_0^*\cdot\mathbf{f}(0).$$

Source: R. Serber, *Serber Says: About Nuclear Physics* (Singapore, World Scientific, 1987).

21.21 Radiation Pressure from Scattering

The general expression for the electromagnetic force on the contents of a spherical volume V is

$$\mathbf{F} = \int_V d^3r\left[-\frac{1}{c^2}\frac{\partial\mathbf{S}}{\partial t} + \nabla\cdot\mathbf{T}\right].$$

The Poynting vector term disappears after averaging over one period. Therefore, if S is the surface of a sphere of radius r,

$$\langle \mathbf{F} \rangle = r^2 \int d\Omega\, \hat{\mathbf{r}} \cdot \langle \mathbf{T} \rangle = \frac{\epsilon_0 r^2}{2} \text{Re} \int d\Omega\, \hat{\mathbf{r}} \cdot \left[\mathbf{E}\mathbf{E}^* + c^2 \mathbf{B}\mathbf{B}^* - \frac{1}{2} \mathbf{I} \left(|\mathbf{E}|^2 + c^2 |\mathbf{B}|^2 \right) \right]. \quad (1)$$

In the radiation zone, $\mathbf{E} = \mathbf{E}_{\text{inc}} + \mathbf{E}_{\text{rad}}$ and $\mathbf{B} = \mathbf{B}_{\text{inc}} + \mathbf{B}_{\text{rad}}$, where (dropping the time dependence)

$$\mathbf{E}_{\text{rad}} = E_0 \frac{\exp(ikr)}{r} \mathbf{f}(\mathbf{k}), \quad (2)$$

and we know from (21.59) that

$$\lim_{r \to \infty} \mathbf{E}_0 \exp(i\mathbf{k}_0 \cdot \mathbf{r}) = 2\pi i \left[\frac{\exp(-ik_0 r)}{k_0 r} \delta(\hat{\mathbf{r}} + \hat{\mathbf{k}}_0) - \frac{\exp(ik_0 r)}{k_0 r} \delta(\hat{\mathbf{r}} - \hat{\mathbf{k}}_0) \right] E_0 \hat{\mathbf{e}}_0. \quad (3)$$

Of course, $\hat{\mathbf{k}} \cdot \mathbf{f} = 0$ and $\mathbf{k}_0 \cdot \hat{\mathbf{e}}_0 = 0$. Also, $c\mathbf{B}_{\text{rad}} = \hat{\mathbf{k}} \times \mathbf{E}_{\text{rad}}$ and $c\mathbf{B}_{\text{inc}} = \hat{\mathbf{k}}_0 \times \mathbf{E}_{\text{inc}}$. Substituting (2) and (3) into (1) and using the fact that the delta functions make $\hat{\mathbf{r}} = \pm\hat{\mathbf{k}}_0$, we see that the transverse nature of the fields implies that only the term with the unit dyadic $\hat{\mathbf{I}}$ is not zero. Indeed, because $\hat{\mathbf{r}} \cdot \mathbf{I} = \hat{\mathbf{r}}$,

$$\begin{aligned} \langle \mathbf{F} \rangle &= -\frac{1}{4} \epsilon_0 r^2 \text{Re} \int d\Omega\, \hat{\mathbf{r}} \left(|\mathbf{E}_{\text{inc}}|^2 + |\mathbf{E}_{\text{rad}}|^2 + \mathbf{E}_{\text{inc}} \cdot \mathbf{E}^*_{\text{rad}} + \mathbf{E}^*_{\text{inc}} \cdot \mathbf{E}_{\text{rad}} \right) \\ &= -\frac{1}{4} \epsilon_0 r^2 \text{Re} \int d\Omega\, \hat{\mathbf{r}} \left(|\mathbf{B}_{\text{inc}}|^2 + |\mathbf{B}_{\text{rad}}|^2 + \mathbf{B}_{\text{inc}} \cdot \mathbf{B}^*_{\text{rad}} + \mathbf{B}^*_{\text{inc}} \cdot \mathbf{B}_{\text{rad}} \right). \end{aligned}$$

The integrals with the factors $|\mathbf{E}_{\text{inc}}|^2$ and $|\mathbf{B}_{\text{inc}}|^2$ are zero because these field magnitudes are constants. Otherwise substituting the fields into the preceding equation and using $|\mathbf{E}| = c|\mathbf{B}|$ gives

$$\langle \mathbf{F} \rangle = \frac{2\pi \epsilon_0 E_0^2}{k} \hat{\mathbf{k}}_0 \text{Im} \left[\mathbf{f}(\mathbf{k}_0) \cdot \hat{\mathbf{e}}_0^* \right] - \frac{\epsilon_0 E_0^2}{2} \int d\Omega\, \hat{\mathbf{r}}\, |\mathbf{f}|^2.$$

The intensity of the incident beam is $I_{\text{inc}} = \frac{1}{2} \epsilon_0 c E_0^2$. Therefore, using the optical theorem and the definition of the differential scattering cross section, we get the advertised formula,

$$\langle \mathbf{F} \rangle = \frac{I_{\text{inc}}}{c} \left[\sigma_{\text{tot}} \hat{\mathbf{k}}_0 - \int d\Omega\, \hat{\mathbf{r}} \frac{d\sigma_{\text{scatt}}}{d\Omega} \right].$$

Source: M.I. Mishchenko, L.D. Travis, and A.A. Lacis, *Scattering, Absorption, and Emission of Light by Small Particles* (University Press, Cambridge, 2002).

21.22 A Backscatter Theorem

(a) For monochromatic fields, the Maxwell equations in matter are

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = i\omega\mu \mathbf{H}$$

and

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} = -i\omega\epsilon\mathbf{E}.$$

Hence,

$$\nabla \times (\nabla \times \mathbf{E}) = i\omega\nabla \times (\mu\mathbf{H}) = i\omega\mathbf{H} \times \nabla\mu - i\omega\mu\nabla \times \mathbf{H} = (\nabla \times \mathbf{E}) \times \frac{\nabla\mu}{\mu} + \omega\mu\epsilon\mathbf{E}.$$

Similarly,

$$\nabla \times (\nabla \times \mathbf{H}) = -i\omega\nabla \times (\epsilon\mathbf{E}) = -i\omega\mathbf{E} \times \nabla\epsilon + i\omega\epsilon\nabla \times \mathbf{E} = (\nabla \times \mathbf{H}) \times \frac{\nabla\epsilon}{\epsilon} + \omega\mu\epsilon\mathbf{H}.$$

These equations are the same if $\epsilon(\mathbf{r}) = \mu(\mathbf{r})$. In fact, only the weaker condition $(\nabla\mu)/\mu = (\nabla\epsilon)/\epsilon$ is necessary.

(b) The body produces the scattered wave

$$\mathbf{E}^{\rm scatt}(x,y,z) = E_x^{\rm scatt}(x,y,z)\hat{\mathbf{x}} + E_y^{\rm scatt}(x,y,z)\hat{\mathbf{y}} + E_z^{\rm scatt}(x,y,z)\hat{\mathbf{z}}.$$

The scattering body and the vacuum are both invariant to space rotations around the z-axis by 90°. Moreover, $\mathbf{E}^{\rm scatt}$ and $\mathbf{H}^{\rm scatt}$ satisfy the same generalized wave equation. Therefore, the latter can be obtained from the former by a 90° rotation around the z-axis, which takes $x \to y$ and $y \to -x$. The unit vectors rotate in the same way. Hence, as suggested,

$$\mathbf{H}^{\rm scatt}(x,y,z) = -E_y^{\rm scatt}(y,-x,z)\hat{\mathbf{x}} + E_x^{\rm scatt}(y,-x,z)\hat{\mathbf{y}} + E_z^{\rm scatt}(y,-x,z)\hat{\mathbf{z}}.$$

(c) The backward direction is $x = y = 0$. For this special case, the foregoing can be written

$$\mathbf{H}^{\rm scatt}(0,0,z) = \hat{\mathbf{z}} \times \mathbf{E}^{\rm scatt}(0,0,z) + \hat{\mathbf{z}}E_z^{\rm scatt}(0,0,z).$$

The last term is limited to the near field because it is not transverse. Therefore, the time-averaged, far-field Poynting vector is

$$\langle \mathbf{S} \rangle = \frac{1}{2}{\rm Re}(\mathbf{E} \times \mathbf{H}^*) = \frac{1}{2}|\mathbf{E}^{\rm scatt}|^2\hat{\mathbf{z}}.$$

But this carries energy toward $+z$ rather than toward $-z$, which is required for backscattering. Hence, we must have $\mathbf{E}^{\rm scatt}(0,0,z) = 0$ in the backward direction. This proves the theorem.

Source: V.H. Weston, *IEEE Transactions on Antennas and Propagation* **11**, 578 (1963).

21.23 The Angular Spectrum of Plane Waves in Two Dimensions

(a) The proposed form for $\mathbf{E}_{\rm TE}(x,z,\omega)$ is general for a field polarized in the $\hat{\mathbf{y}}$-direction because it is a superposition of plane wave solutions for all possible values of the wave vector $\mathbf{k} = k_x\hat{\mathbf{x}} + k_z\hat{\mathbf{z}}$. The sum is on k_x only because k_z is fixed by $\omega = ck_0 = c\sqrt{k_x^2 + k_z^2}$:

$$k_z = \begin{cases} \sqrt{k_0^2 - k_x^2} & k_x \le k_0, \\ i\sqrt{k_x^2 - k_0^2} & k_x > k_0. \end{cases}$$

The positive imaginary sign is chosen so the real exponential wave remains bounded as it propagates in the half-space $z > 0$. Notice that these exponentially decaying waves are required to obtain a complete solution.

Using the initial data,

$$\bar{E}_y(x) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, \Lambda_{TE}(k_x) \exp(ik_x x),$$

so the Fourier inversion theorem gives the scalar function $\Lambda_{TE}(k_x)$ as

$$\Lambda_{TE}(k_x) = \int_{-\infty}^{\infty} dx\, \bar{E}_y(x) \exp(-ik_x x).$$

Faraday's law is $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$ so

$$\mathbf{B}_{\rm TE} = -\frac{i}{\omega}\nabla \times \mathbf{E}_{\rm TE} = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi c} \left[\frac{k_x}{k_0}\hat{\mathbf{z}} - \frac{k_z}{k_0}\hat{\mathbf{x}}\right] \Lambda_{TE}(k_x) \exp(i\mathbf{k}\cdot\mathbf{r}).$$

From this we read off

$$\mathbf{\Gamma}_{TE}(k_x) = \frac{1}{c}\left[\frac{k_x}{k_0}\hat{\mathbf{z}} - \frac{k_z}{k_0}\hat{\mathbf{x}}\right] \Lambda_{TE}(k_x).$$

(b) The proposed form for $\mathbf{B}_{\rm TM}$ is appropriate as in part (a). The Ampère-Maxwell law is $\nabla \times \mathbf{B} = -(i\omega/c^2)\mathbf{E}$. Therefore,

$$\mathbf{E}_{TM}(x, z, \omega) = \frac{ic^2}{\omega}\nabla \times \mathbf{B}_{TM} = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \left[\hat{\mathbf{x}} - \frac{k_x}{k_z}\hat{\mathbf{z}}\right] \Lambda_{TM}(k_x) \exp(i\mathbf{k}\cdot\mathbf{r}).$$

From this we read off

$$\mathbf{\Gamma}_{TM}(k_x) = \left[\hat{\mathbf{x}} - \frac{k_x}{k_z}\hat{\mathbf{z}}\right] \Lambda_{TM}(k_x).$$

Now use the initial data:

$$\hat{\mathbf{x}} \cdot \mathbf{E}_{TM}(x, z = 0, t = 0) = \bar{E}_x(x) = \int_{-\infty}^{\infty} \frac{dk_x}{2\pi}\, \Lambda_{TM}(k_x) \exp(ik_x x).$$

Hence,

$$\Lambda_{TM}(k_x) = \int_{-\infty}^{\infty} dx\, \bar{E}_x(x) \exp(-ik_x x).$$

(c) The time-averaged Poynting vector is

$$< \mathbf{S} > = \frac{1}{2\mu_0} \mathrm{Re}\left\{ (\mathbf{E}_{TE} + \mathbf{E}_{TM}) \times (\mathbf{B}_{TE} + \mathbf{B}_{TM})^* \right\}.$$

But the cross terms do not point in the $\hat{\mathbf{z}}$-direction. Therefore,

$$\begin{aligned} P_z &= \int_{-\infty}^{\infty} dx \, < \hat{\mathbf{z}} \cdot \mathbf{S} > \\ &= \frac{1}{2\mu_0 c} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} \frac{dk_x}{2\pi} \int_{-\infty}^{\infty} \frac{dk'_x}{2\pi} \, \mathrm{Re}\left\{ \left(\frac{k_z^*}{k_0} \Lambda_{\mathrm{TE}}(k_x) \Lambda_{\mathrm{TE}}^*(k_x) + \frac{k_0}{k_z^*} \Lambda_{\mathrm{TM}}(k_x) \Lambda_{\mathrm{TM}}^*(k_x) \right) \right\} \\ &\qquad \times \exp[i(k_x - k'_x)x] \exp[i(k_z - k_z^{'*})z] \end{aligned}$$

Then x-integration gives $2\pi\delta(k_x - k'_x)$. Therefore,

$$P_z = \frac{1}{2\mu_0 c} Re \int_{-\infty}^{\infty} dk_x \left\{ \frac{k_z^*}{k_0} |\Lambda_{\mathrm{TE}}|^2 + \frac{k_0}{k_z^*} |\Lambda_{\mathrm{TE}}|^2 \right\} \exp i(k_z - k_z^*) z.$$

Split the integral into two parts. The portion where $k_x > k_0$ is pure imaginary and so does not contribute. This is physically correct because the exponentially damped evanescent waves do not carry energy down the z-axis. Otherwise, k_z is real so

$$P_z = \frac{1}{2\pi\mu_0 c} \int_{-k_0}^{k_0} \frac{dk_x}{2\pi} \left\{ \frac{k_z}{k_0} |\Lambda_{\mathrm{TE}}|^2 + \frac{k_0}{k_z} |\Lambda_{\mathrm{TE}}|^2 \right\},$$

as required.

Source: G.S. Smith, *An Introduction to Classical Electromagnetic Radiation* (University Press, Cambridge, 1997).

21.24 Weyl's Identity

(a) Let

$$G_0(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3k \, \hat{G}_0(\mathbf{k}) \exp(i\mathbf{k} \cdot \mathbf{r}).$$

Substituting this into $(\nabla^2 + k_0^2) G_0(\mathbf{r}) = -\delta(\mathbf{r})$ and using

$$\delta(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3k \, \exp(i\mathbf{k} \cdot \mathbf{r})$$

shows that $\hat{G}_0(\mathbf{k}) = (k^2 - k_0^2)^{-1}$. Substituting this back into the original Fourier integral gives

$$G_0(\mathbf{r}) = \frac{1}{(2\pi)^3} \int d^3k \, \frac{\exp(i\mathbf{k} \cdot \mathbf{r})}{k^2 - k_0^2}.$$

(b) Let $\mathbf{k}_\perp = k_x\hat{\mathbf{x}} + k_y\hat{\mathbf{y}}$ and $\mathbf{r}_\perp = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$. Then,

$$G_0(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d^2k_\perp \exp(i\mathbf{k}_\perp\cdot\mathbf{r}_\perp)\int_{-\infty}^{\infty} dk_z \frac{\exp(ik_z z)}{k_z^2 - (k_0^2 - k_\perp^2)}.$$

The poles of the integrand occur at $k_z = \pm K = \pm\sqrt{k_0^2 - k_\perp^2}$. If k_0 has a small positive imaginary part, $+K$ is in the first quadrant and $-K$ is in the third quadrant. The factor $\exp(ik_z z)$ converges for $z > 0$ if we close the contour in the upper half-plane. It converges for $z < 0$ if we close the contour in the lower half-plane. Therefore, the residue theorem picks up the first (third)-quadrant pole only when $z > 0$ ($z < 0$). The result in both cases is

$$G_0(\mathbf{r}) = \frac{1}{(2\pi)^3}\int d^2k_\perp \exp(i\mathbf{k}_\perp\cdot\mathbf{r}_\perp)\times\frac{2\pi i}{2K}\exp(iK|z|).$$

With k_z defined as in the statement of the problem, we recover the Weyl identity because $K = k_z$ guarantees convergence of the integral.

21.25 Radiation from an Open Waveguide

With respect to an origin at the center of the aperture, the electric field of a TE_{10} mode in the plane of the aperture is

$$\mathbf{E}(x,y,z=0) = \begin{cases} E_0\cos(\pi x/a)\hat{\mathbf{y}} & |x| < a/2, \quad |y| < a/2, \\ 0 & |x| \geq a/2, \quad |y| \geq a/2. \end{cases}$$

The Fraunhofer limit of the Smythe integral is

$$\mathbf{E}_{\mathrm{rad}}(\mathbf{r}_\perp, z\geq 0) = ik_0\frac{\exp(ik_0 r)}{2\pi r}\hat{\mathbf{r}}\times\int_{z'=0} d^2r'_\perp\ [\hat{\mathbf{z}}\times\mathbf{E}(\mathbf{r}'_\perp)]\exp(-ik_0\hat{\mathbf{r}}\cdot\mathbf{r}'_\perp).$$

For our problem, this reduces to

$$\mathbf{E}(\mathbf{r}) = ik_0\frac{\exp(ik_0 r)}{2\pi r}[\hat{\mathbf{r}}\times(\hat{\mathbf{z}}\times\hat{\mathbf{y}})]\int_{-a/2}^{a/2} dx' \int_{-b/2}^{b/2} dy'\, E_0\cos(\pi x'/a)\exp(-ik_0\hat{\mathbf{r}}\cdot\mathbf{r}').$$

Using

$$\hat{\mathbf{x}} = \sin\theta\cos\phi\hat{\mathbf{r}} + \cos\theta\cos\phi\hat{\boldsymbol{\theta}} - \sin\phi\hat{\boldsymbol{\phi}}$$

and

$$\hat{\mathbf{y}} = \sin\theta\sin\phi\hat{\mathbf{r}} + \cos\theta\sin\phi\hat{\boldsymbol{\theta}} + \cos\phi\hat{\boldsymbol{\phi}},$$

we find without difficulty that

$$\hat{\mathbf{r}}\times(\hat{\mathbf{z}}\times\mathbf{y}) = -(\sin\phi\hat{\boldsymbol{\theta}} + \cos\theta\cos\phi\hat{\boldsymbol{\phi}})$$

and

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}' = x'\hat{\mathbf{x}} + y'\hat{\mathbf{y}} = x' \sin\theta \cos\phi + y' \sin\theta \sin\phi.$$

The double integral we must do is

$$I = I_x \times I_y = \int_{-a/2}^{a/2} dx' \, \cos(\pi x'/a) \exp(-ik_0 x' \sin\theta \cos\phi) dx' \int_{-b/2}^{b/2} dy' \, \exp(-ik_0 y' \sin\theta \sin\phi).$$

These are straightforward (if tedious) exponential integrals and we find

$$I_y = \frac{2}{k_0 \sin\theta \sin\phi} \sin[k_0(b/2)\sin\theta \sin\phi]$$

and

$$I_x = \frac{\cos[k_0(a/2)\sin\theta\cos\phi]}{k_0 \sin\theta\cos\phi + \pi/a} - \frac{\cos[k_0(a/2)\sin\theta\cos\phi]}{k_0\sin\theta\cos\phi - \pi/a} = -\frac{2\pi}{a}\frac{\cos[k_0(a/2)\sin\theta\cos\phi]}{(k_0\sin\theta\cos\phi)^2 - (\pi/a)^2}.$$

Therefore, using the notation sinc$x = \sin x/x$, the radiated electric field is

$$\mathbf{E}(\mathbf{r}) = i\frac{\exp(ik_0 r)}{r}(\sin\phi\hat{\boldsymbol{\theta}} + \cos\theta\cos\phi\hat{\boldsymbol{\phi}})\frac{E_0 k_0 ab \cos[k_0(a/2)\cos\theta\cos\phi]\text{sinc}[k_0(b/2)\sin\theta\sin\phi]}{(k_0 a \sin\theta\cos\phi)^2 - \pi^2}.$$

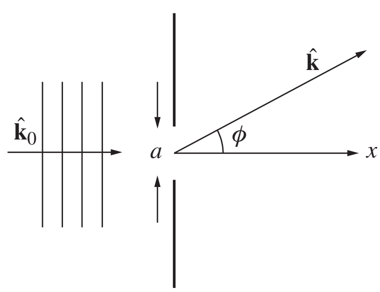
Source: Prof. G.S. Smith, Georgia Institute of Technology (private communication).

21.26 Diffraction from a Slit

The Huygens representation of the exact field diffracted by an aperture in a plane is

$$\mathbf{E}(\mathbf{r}_\perp, z \geq 0) = -2\int d^2 r'_\perp \, [\hat{\mathbf{n}} \times \mathbf{E}] \times \nabla G_0(\mathbf{r}, \mathbf{r}_\perp).$$

Our two-dimensional geometry is



Therefore, using the Kirchoff approximation and the two-dimensional Green function from part (a),

$$\mathbf{E}(\rho,\phi) = -\frac{i}{2}(\hat{\mathbf{x}}\times\mathbf{E}_0)\times\int\limits_{-a/2}^{a/2} dy'\,\nabla H_0^{(1)}(k\sqrt{x^2+(y-y')^2}).$$

Let $s=\sqrt{x^2+(y-y')^2}$. The asymptotic behavior of the Hankel function is

$$\lim_{ks\to\infty} H_0^{(1)}(ks) = \sqrt{\frac{2}{\pi ks}}\exp[i(ks-\pi/4)].$$

In the same limit,

$$\nabla H_0^{(1)} = \frac{\partial H_0^{(1)}}{\partial s}\nabla s \approx ikH_0^{(1)}(ks)\hat{\boldsymbol{\rho}} \approx i\mathbf{k}H_0^{(1)}(ks) \approx i\mathbf{k}\sqrt{\frac{2}{\pi k\rho}}\exp[i(ks-\pi/4)].$$

Moreover,

$$s = (x^2+y^2-2yy'+y'^2)^{1/2} \approx \rho(1-2yy'/\rho^2)^{1/2} \approx \rho - yy'/\rho \approx \rho - y'\sin\phi.$$

Therefore,

$$\mathbf{E}(\rho,\phi) = -\frac{1}{2}\mathbf{k}\times(\hat{\mathbf{x}}\times\mathbf{E}_0)\sqrt{\frac{2}{\pi k\rho}}\exp[i(k\rho-\pi/4)]\int\limits_{-a/2}^{a/2} dy'\,\exp[-iky'\sin\phi].$$

or

$$\mathbf{E}(\rho,\phi) = -\frac{a}{2}\mathbf{k}\times(\hat{\mathbf{x}}\times\mathbf{E}_0)\sqrt{\frac{2}{\pi k\rho}}\exp[i(k\rho-\pi/4)]\frac{\sin(\frac{1}{2}ka\sin\phi)}{\frac{1}{2}ka\sin\phi}.$$

21.27 Diffraction of a Beam by a Large Aperture

(a) The calculation proceeds exactly as in the text except that $E_0 \to E_0\exp(-\rho^2/w^2)$ in the integrand of (21.101). Using (21.102) and (21.103) to evaluate the ϕ' integral in (21.101) gives

$$\mathbf{E} = -ik_0E_0\frac{\exp(ik_0r)}{r}(\sin\phi\hat{\boldsymbol{\theta}}+\cos\phi\cos\theta\hat{\boldsymbol{\phi}})\int\limits_0^a d\rho'\rho'\exp(-\rho'^2/w^2)J_0(k_0\rho'\sin\theta).$$

Because $w \ll a$, we can safely extend the upper limit of the integral to infinity. This gives a tabulated integral [see 6.631 and 9.215 of I.S. Gradshteyn and I.M. Ryzhik, *Tables of Integrals, Series, and Products* (1980)], namely,

$$\int\limits_0^\infty d\rho'\rho'\exp(-\rho'^2/w^2)J_0(k_0\rho'\sin\theta) = \frac{1}{2}w^2\exp(-k_0^2w^2\sin^2\theta/4).$$

Because $\sin\theta = \rho/r$, the diffracted electric field still has a Gaussian profile:

$$\mathbf{E} = -ik_0w^2E_0\frac{\exp(ik_0r)}{2r}\exp(-k_0^2w^2\rho^2/4r^2)(\sin\phi\hat{\boldsymbol{\theta}}+\cos\phi\cos\theta\hat{\boldsymbol{\phi}}).$$

(b) Start this time with (21.98):

$$\mathbf{E}_{\rm rad}(\mathbf{r}_\perp, z \geq 0) = ik_0 \frac{\exp(ik_0 r)}{2\pi r}\hat{\mathbf{r}} \times \int\limits_{z'=0} d^2 r'_\perp \; [\hat{\mathbf{z}} \times \mathbf{E}(\mathbf{r}'_\perp)] \exp(-ik_0 \hat{\mathbf{r}} \cdot \mathbf{r}'_\perp).$$

The vector part of $\hat{\mathbf{r}} \times [\hat{\mathbf{z}} \times \mathbf{E}_{\rm inc}]_{z=0} = -E_0 \exp(-\rho'^2/w^2) \sin\phi \hat{\boldsymbol{\theta}} + \cos\phi \cos\theta \hat{\boldsymbol{\phi}})$ is the same as in (21.100). But for this geometry we use Cartesian variables and write

$$\hat{\mathbf{r}} \cdot \mathbf{r}'_\perp = \frac{xx' + yy'}{r}.$$

Therefore, because $\rho^2 = x^2 + y^2$,

$$\begin{aligned} \mathbf{E} &= -ik_0 E_0 \frac{\exp(ik_0 r)}{2\pi r}(\sin\phi \hat{\boldsymbol{\theta}} + \cos\phi \cos\theta \hat{\boldsymbol{\phi}}) \\ & \times \int\limits_{-a/2}^{a/2} dx' \; \exp(-ik_0 xx'/r) \exp(-x'^2/w^2) \\ & \times \int\limits_{-a/2}^{a/2} dy' \; \exp(-ik_0 yy'/r) \exp(-y'^2/w^2). \end{aligned}$$

The two integrals have the same structure and, when $w \ll a$, we may extend the limits of integration to $\pm\infty$ without concern. In that case, completing the square in the argument gives a Gaussian integral:

$$\int\limits_{-\infty}^{\infty} dx \exp(-ax^2 + bx) = \exp(b^2/4a) \int\limits_{-\infty}^{\infty} du \exp(-au^2) = \sqrt{\frac{\pi}{a}} \exp(b^2/4a).$$

Using $a = 1/w^2$ and first $b = -ik_0 x/r$ and then $b = -ik_0 y/r$, the result is exactly the same as found in part (a),

$$\mathbf{E} = -ik_0 w^2 E_0 \frac{\exp(ik_0 r)}{2r} \exp(-k_0^2 w^2 \rho^2/4r^2)(\sin\phi \hat{\boldsymbol{\theta}} + \cos\phi \cos\theta \hat{\boldsymbol{\phi}}).$$

Source: S. Ramo, J.R. Whinnery, and T. Van Duzer, *Fields and Waves in Communication Electronics* (Wiley, New York, 1994).

21.28 Effective Aperture Dipoles I

For reference, we record the radiated electric fields for dipole sources with $\exp(-i\omega t)$ time dependence:

$$\mathbf{E}_{\rm M1} = -\frac{\mu_0}{4\pi}\frac{\omega^2}{c}\frac{\exp(ikr)}{r}(\hat{\mathbf{r}} \times \mathbf{m}) \qquad \text{and} \qquad \mathbf{E}_{\rm E1} = -\frac{\mu_0}{4\pi}\omega^2 \frac{\exp(ikr)}{r}[\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{p})]. \quad (1)$$

The far-field limit of Smythe's formula derived in the text is

$$\mathbf{E}_{\rm rad}(\mathbf{r}) = ik_0\frac{\exp(ikr)}{2\pi r}\hat{\mathbf{r}}\times\int\limits_{z'=0} d^2r'_\perp\,\hat{\mathbf{z}}\times\mathbf{E}(\mathbf{r}'_\perp)\exp[-ik_0\hat{\mathbf{r}}\cdot\mathbf{r}'{}_\perp].$$

The integrand involves only the tangential component of the electric field, $\mathbf{E}_\parallel$, so there is no loss of generality if we replace the foregoing with

$$\mathbf{E}_{\rm rad}(\mathbf{r}) = ik_0\frac{\exp(ikr)}{2\pi r}\hat{\mathbf{r}}\times\int\limits_{z'=0} d^2r'_\perp\,\hat{\mathbf{z}}\times\mathbf{E}_\parallel(\mathbf{r}'_\perp)\exp[-ik_0\hat{\mathbf{r}}\cdot\mathbf{r}'{}_\perp].$$

Expanding the exponential to two terms gives

$$\mathbf{E}_{\rm rad} \approx ik_0\frac{\exp(ikr)}{2\pi r}\hat{\mathbf{r}}\times\int\limits_{z'=0} d^2r'_\perp\,\hat{\mathbf{z}}\times\mathbf{E}_\parallel(\mathbf{r}'_\perp) + k_0^2\frac{\exp(ikr)}{2\pi r}\hat{\mathbf{r}}\times\int\limits_{z'=0} d^2r'_\perp\,\hat{\mathbf{z}}\times\mathbf{E}_\parallel(\mathbf{r}'_\perp)\hat{\mathbf{r}}\cdot\mathbf{r}'{}_\perp. \quad (2)$$

The first term in (2) has the form of $\mathbf{E}_{\rm M1}$ in (1) if $\mathbf{m}$ is

$$\mathbf{m} = \frac{2}{i\omega\mu_0}\int\limits_{\rm aperture} d^2r_\perp\,\hat{\mathbf{z}}\times\mathbf{E}_\parallel.$$

The direction of $\mathbf{m}$ is consistent with Figure 21.23.

To make further progress, we recall a vector identity we have used in previous multipole expansions to separate the magnetic dipole from the electric quadrupole:

$$\mathbf{j}(\mathbf{r}'\cdot\hat{\mathbf{r}}) = -\hat{\mathbf{r}}\times\frac{1}{2}(\mathbf{r}'\times\mathbf{j}) + \frac{1}{2}\left[(\hat{\mathbf{r}'}\cdot\hat{\mathbf{r}})\mathbf{j} + (\hat{\mathbf{r}}\cdot\mathbf{j})\hat{\mathbf{r}}'\right].$$

Here, replace $\mathbf{j}$ with $\hat{\mathbf{z}}\times\mathbf{E}_\parallel$ and use

$$\mathbf{r}'_\perp\times(\hat{\mathbf{z}}\times\mathbf{E}_\parallel) = \hat{\mathbf{z}}(\mathbf{r}'_\perp\cdot\mathbf{E}_\parallel) - \mathbf{E}_\parallel(\mathbf{r}'_\perp\cdot\hat{\mathbf{z}}) = \hat{\mathbf{z}}(\mathbf{r}'_\perp\cdot\mathbf{E}_\parallel)$$

to get

$$(\hat{\mathbf{z}}\times\mathbf{E}_\parallel)(\hat{\mathbf{r}}\cdot\mathbf{r}'_\perp) = -\hat{\mathbf{r}}\times\frac{1}{2}[\hat{\mathbf{z}}(\mathbf{r}'_\perp\cdot\mathbf{E}_\parallel)] + \frac{1}{2}\left\{\mathbf{r}'_\perp[\hat{\mathbf{r}}\cdot(\hat{\mathbf{z}}\times\mathbf{E}_\parallel)] + \hat{\mathbf{r}}[\mathbf{r}'_\perp\cdot(\hat{\mathbf{z}}\times\mathbf{E}_\parallel)]\right\}. \quad (3)$$

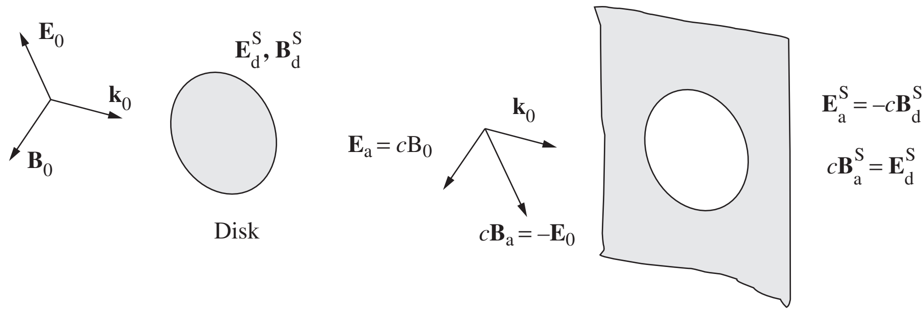
The left side of (3) is the integrand of the second term in (2). The second term on the right-hand side of (3) contributes to an effective quadrupole moment. Substituting the first term on the right-hand side of (3) into the second term of (2) has the form of $\mathbf{E}_{\rm E1}$ in (1) if

$$\mathbf{p} = \epsilon_0\hat{\mathbf{z}}\int\limits_{\rm aperture} d^2r_\perp\,\mathbf{r}\cdot\mathbf{E}_\parallel.$$

The direction of $\mathbf{p}$ is consistent with Figure 21.21.

Source: H. Bethe, *Physical Review* **66**, 163 (1944).

21.29 Effective Aperture Dipoles II



According to Babinet's principle, the fields diffracted by the aperture, $\mathbf{E}_{\rm a}^S$ and $\mathbf{B}_{\rm a}^S$, are related to the fields scattered by the disk, $\mathbf{E}_{\rm d}^S$ and $\mathbf{B}_{\rm d}^S$, by

$$\mathbf{E}_{\rm a}^S = -c\mathbf{B}_{\rm d}^S \qquad \text{and} \qquad \mathbf{B}_{\rm a}^S = \frac{1}{c}\mathbf{E}_{\rm d}^S.$$

The disk-scattered fields are dipolar. Therefore, from the electric moment for the disk,

$$\mathbf{E}_{\rm a}^S = -c\frac{\mu_0}{4\pi}\frac{\omega^2}{c}(\hat{\mathbf{k}} \times \mathbf{p}_{\rm d})\frac{\exp i(kr - \omega t)}{r}$$

and

$$\mathbf{B}_{\rm a}^s = -\frac{1}{c}\frac{\mu_0}{4\pi}[\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{p}_{\rm d})]\frac{\exp i(kr - \omega t)}{r}.$$

Comparing these to the fields produced by a magnetic dipole shows that the effective aperture magnetic dipole moment is

$$\mathrm{m}_{\rm a} = c\mathbf{p}_{\rm d} = -\frac{16}{3}a^3 c\epsilon_0 \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{E}_0) = \frac{16}{3\mu_0}a^3 \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{B}_{\rm a}).$$

Similarly, from the magnetic moment of the disk,

$$\mathbf{E}_{\rm a}^S = -c\frac{\mu_0}{4\pi}[\hat{\mathbf{r}} \times (\mathbf{m}_{\rm d} \times \hat{\mathbf{k}})]\frac{\exp i(kr - \omega t)}{r}$$

and

$$\mathbf{B}_{\rm a}^S = -\frac{1}{c}\frac{\mu_0}{4\pi}\frac{\omega^2}{c}(\hat{\mathbf{k}} \times \mathbf{m}_{\rm d})\frac{\exp i(kr - \omega t)}{r}.$$

Comparing these to the fields produced by an electric dipole shows that the effective electric dipole moment is

$$\mathbf{p}_{\rm a} = -\frac{1}{c}\mathbf{m}_{\rm d} = \frac{8}{3c\mu_0}a^3(\hat{\mathbf{n}} \cdot \mathbf{B}_0)\hat{\mathbf{n}} = \frac{8}{3}\epsilon_0 a^3(\hat{\mathbf{n}} \cdot \mathbf{E}_{\rm a})\hat{\mathbf{n}}.$$

Source: G.S. Smith, *An Introduction to Classical Electromagnetic Radiation* (University Press, Cambridge, 1997).

21.30 Kirchhoff's Approximation for Complementary Scatterers

(a) We start with the form (21.94) of Smythe's formula,

$$\mathbf{E}(\mathbf{r}_\perp, z \geq 0) = -2\int_{z'=0} d^2r'_\perp \, [\hat{\mathbf{z}} \times \mathbf{E}(\mathbf{r}'_\perp)] \times \nabla G_0(\mathbf{r}, \mathbf{r}'_\perp).$$

With $\mathbf{R} = \mathbf{r} - \mathbf{r}'$, (21.96) gives the exact gradient of the free-space Green function as

$$\nabla G_0 = (ik_0 R - 1)\frac{G_0(R)}{R}\hat{\mathbf{R}}.$$

For our problem , $\mathbf{R} = \mathbf{r} = z\hat{\mathbf{z}} - \rho'\hat{\boldsymbol{\rho}}$ is a vector with magnitude $r = \sqrt{\rho'^2 + z^2}$ which points from the screen to the observation point on the z-axis. Therefore,

$$\mathbf{E}(0,0,z) = 2ik\int_0^{2\pi} d\phi' \int_0^\infty d\rho'\rho'\hat{\mathbf{r}} \times [\hat{\mathbf{z}} \times \mathbf{E}(\rho', \phi', 0)] \left(1 + \frac{i}{kr}\right)\frac{\exp(ikr)}{4\pi r}.$$

The incident electric field is $\mathbf{E}_{\rm inc} = E_0 \exp(ikz)\hat{\mathbf{y}}$. For a conducting disk of radius a, the Kirchoff approximation for the tangential component of $\mathbf{E}(\rho, \phi, 0)$ is (i) zero for $\rho < a$ and (ii) $\mathbf{E}_0$ for $\rho > a$. Therefore,

$$\mathbf{E}(0,0,z) = -\frac{ikE_0}{2\pi}\int_0^{2\pi} d\phi' \int_a^\infty d\rho'\rho' \left(\rho' \sin\phi'\hat{\mathbf{z}} + z\hat{\mathbf{y}}\right)\left(1 + \frac{i}{kr}\right)\frac{\exp(ikr)}{r^2}$$

or

$$\mathbf{E}(0,0,z) = -ikzE_0\int_a^\infty d\rho'\rho' \left(1 + \frac{i}{kr}\right)\frac{\exp(ikr)}{r^2}\hat{\mathbf{y}}.$$

Now change integration variables from ρ' to r so $\rho' d\rho' = r dr$, let $r_0 = \sqrt{a^2 + z^2}$, and integrate the second integral by parts. Using a convergence factor to make the upper limit of the integrated term vanish, we find

$$\mathbf{E}(0,0,z) = -ikzE_0\left(\int_{r_0}^\infty dr\frac{\exp(ikr)}{r} + \left[\frac{-i\exp(ikr)}{kr}\right]_{r_0}^\infty - \int_{r_0}^\infty dr\frac{\exp(ikr)}{r}\right)\hat{\mathbf{y}} \qquad (1)$$

or

$$\mathbf{E}_{\rm disk}(0,0,z) = E_0\left(\frac{z}{r_0}\right)\exp(ikr_0)\hat{\mathbf{y}}.$$

(b) The calculation for the circular aperture is exactly the same except that the Kirchoff approximation puts the tangential component of $\mathbf{E}(\rho, \phi, 0)$ equal to (i) zero for $\rho > a$ and (ii) $\mathbf{E}_0$ for $\rho < a$. This changes the lower limit of integration in (1) to z and the upper limit of integration to r_0. Therefore,

$$\mathbf{E}_{\text{aperture}}(0,0,z) = E_0 \left[\exp(ikz) - \frac{z}{r_0} \exp(ikr_0) \right] \hat{\mathbf{y}}.$$

(c) The formula immediately above is indeed

$$\mathbf{E}_{\text{aperture}}(0,0,z) = \mathbf{E}_{\text{disk}}(0,0,z) - \mathbf{E}_{\text{inc}}.$$

However, Babinet's principle is not at work here, because that principle relates the electric field of one problem to the *magnetic* field of the complementary problem.

Source: G.S. Smith, *European Journal of Physics* **27**, L21 (2006).

Chapter 22: Special Relativity

22.1 Low-Velocity Limit

For a boost along z, the intervals Δz and Δt transform according to

$$\Delta z' = \gamma(\Delta z - v\Delta t) \qquad\qquad \Delta t' = \gamma\left(\Delta t - \frac{v}{c^2}\Delta z\right).$$

When $v \ll c$, the factor $\gamma = (1 - v^2/c^2)^{-1/2} \approx 1$ and

$$\Delta z' \approx \Delta z - v\Delta t \qquad\qquad \Delta t' = \left(\Delta t - \frac{v}{c^2}\Delta z\right).$$

The left equation is the Galilean limit. The right equation is not, because, no matter how small we make v/c, it is always possible to choose Δz large enough so the second term is not small compared to the first. Indeed, we must have $\Delta z \gg c\Delta t$, which means that the events are very space-like.

Source: R. Baierlein, *American Journal of Physics* **74**, 193 (2006).

22.2 Linearity of the Lorentz Transformation

Let $x_4 = ct$ and $v_4 = c$. Rectilinear motion in frame S means that $x_i = x_{i0} + v_i(t - t_0)$ where $i = 1, 2, 3, 4$. Therefore, under the proposed transformation,

$$x_i' = \frac{A_{ij}[x_{j0} + v_j(t - t_0)] + b_i}{c_j[x_{j0} + v_j(t - t_0)] + d} \qquad i = 1, 2, 3, \tag{1}$$

and

$$ct' = \frac{A_{4j}[x_{j0} + v_j(t - t_0)] + b_4}{c_j[x_{j0} + v_j(t - t_0)] + d}. \tag{2}$$

Our task is to check whether these formulae imply that $x_i' = x_{i0}' + v_i'(t' - t_0')$, which is the equation for rectilinear motion in S'. This is straightforward because inverting (2) shows that

$$t - t_0 = \frac{A_{4j}x_{j0} + b_4 - ct'(c_j x_{j0} + d)}{ct' c_j v_j - A_{4j} v_j}. \tag{3}$$

Substituting (3) into (1) gives

$$x_i' = \frac{A_{ij}\left[x_{j0} + v_j \dfrac{A_{4j}x_{j0} + b_4 - ct'(c_j x_{j0} + d)}{ct' c_j v_j - A_{4j} v_j}\right] + b_i}{c_j\left[x_{j0} + v_j \dfrac{A_{4j}x_{j0} + b_4 - ct'(c_j x_{j0} + d)}{ct' c_j v_j - A_{4j} v_j}\right] + d} \qquad i = 1, 2, 3. \tag{4}$$

Rationalizing the numerator of (4) produces a quotient of two linear functions of t'. Rationalizing the denominator of (4) produces a similar quotient. The two quotients have the same denominator (by construction) so the latter cancels out. The numerator of the numerator quotient is a linear function of t'. We will have accomplished our task if the numerator of the denominator quotient is independent of t'. Carrying out the rationalization of the denominator of (4) produces a quotient whose numerator is a linear function of t':

$$c_j \{x_{j0}(ct'c_jv_j - A_{4j}v_j) + v_j[A_{4j}x_{j0} + b_4 - ct'(c_jx_{j0} + d)]\} + d(ct'c_jv_j - A_{4j}v_j).$$

However, the coefficients of t' sum to zero in this expression, as required. Finally, the original transformation maps (x, y, z, t) to infinity if $c_jx_j + d = 0$. Physically, any transformation between two inertial frames must map finite points to finite points and infinity to infinity. Therefore, we must have $c_j = 0$.

Source: V. Fock, *The Theory of Spacetime and Gravitation* (Pergamon, London, 1959), Section 8 and Appendix A.

22.3 Velocity Addition

(a) The text writes the Lorentz transformation for the four-vector $(\mathbf{r}, ict)$ where $\mathbf{r} = \mathbf{r}_{\|} + \mathbf{r}_{\perp}$. By linearity, these apply equally well to the differential elements dt and $d\mathbf{r}$. Here, we need only

$$d\mathbf{r}_{\|} = \gamma(d\mathbf{r}'_{\|} + \boldsymbol{\beta}cdt')$$

$$d\mathbf{r}_{\perp} = d\mathbf{r}'_{\perp}$$

$$cdt = \gamma(cdt' + \boldsymbol{\beta} \cdot d\mathbf{r}'_{\|}).$$

Consequently,

$$\frac{\mathbf{u}_{\|}}{c} = \frac{1}{c}\frac{d\mathbf{r}_{\|}}{dt} = \frac{\gamma(d\mathbf{r}'_{\|} + \boldsymbol{\beta}cdt')}{\gamma(cdt' + \boldsymbol{\beta} \cdot d\mathbf{r}'_{\|})} \qquad \text{and} \qquad \frac{\mathbf{u}_{\perp}}{c} = \frac{1}{c}\frac{d\mathbf{r}_{\perp}}{dt} = \frac{d\mathbf{r}'_{\perp}}{\gamma(cdt' + \boldsymbol{\beta} \cdot d\mathbf{r}'_{\|})}.$$

This gives the advertised relativistic law of velocity addition:

$$\mathbf{u}_{\|} = \frac{\mathbf{u}'_{\|} + \mathbf{v}}{1 + \dfrac{\mathbf{v} \cdot \mathbf{u}'_{\|}}{c^2}} \qquad \text{and} \qquad \mathbf{u}_{\perp} = \frac{\mathbf{u}'_{\perp}}{\gamma(v)\left[1 + \dfrac{\mathbf{v} \cdot \mathbf{u}'_{\|}}{c^2}\right]}.$$

(b) If the direction of $\mathbf{v}$ defines a polar axis, $u_{\|} = u\cos\theta$ and $u_{\perp} = u\sin\theta$ and similarly for the primed variables. Therefore,

$$u\cos\theta = \frac{u'\cos\theta' + v}{1 + u'v\cos\theta'/c^2} \qquad \text{and} \qquad u\sin\theta = \frac{u'\sin\theta'}{\gamma(1 + u'v\cos\theta'/c^2)}.$$

Using $\gamma = 1/\sqrt{1 - v^2/c^2}$, we divide one of these equations by the other, and also square and add the equations to get

$$\tan\theta = \frac{u'\sin\theta'}{\gamma(u'\cos\theta' + v)} \qquad \text{and} \qquad u = \frac{\left[u'^2 + v^2 + 2u'v\cos\theta' - (u'v\sin\theta'/c)^2\right]^{1/2}}{1 + u'v\cos\theta'/c^2}.$$

(c) Evaluating the previous formula in the limit when $v \to c$ gives

$$u = \frac{\sqrt{u'^2 + c^2 + 2u'c\cos\theta' - u'^2\sin^2\theta'}}{1 + (u'/c)\cos\theta'} = c\frac{\sqrt{[1 + (u'/c)\cos\theta']^2}}{1 + (u'/c)\cos\theta'} = c.$$

This confirms that c is the limiting speed for any particle.

22.4 Invariance of the Scalar Product

The transformation law for a four-vector is

$$\begin{aligned}
\mathbf{A}'_\perp &= \mathbf{A}_\perp \\
\mathbf{A}'_\parallel &= \gamma(\mathbf{A}_\parallel + i\boldsymbol{\beta}A_4) \\
A'_4 &= \gamma(A_4 - i\boldsymbol{\beta}\cdot\mathbf{A}_\parallel).
\end{aligned}$$

By direct substitution,

$$\begin{aligned}
\mathbf{a}'\cdot\mathbf{b}' + a'_4b'_4 &= \mathbf{a}'_\perp\cdot\mathbf{b}'_\perp + \mathbf{a}'_\parallel\cdot\mathbf{b}'_\parallel - a'_4b'_4 \\
&= \mathbf{a}_\perp\cdot\mathbf{b}_\perp + \gamma(\mathbf{a}_\parallel + i\boldsymbol{\beta}a_4)\cdot\gamma(\mathbf{b}_\parallel + i\boldsymbol{\beta}b_4) + \gamma(a_4 - i\boldsymbol{\beta}\cdot\mathbf{a}_\parallel)\gamma(b_4 - i\boldsymbol{\beta}\cdot\mathbf{b}_\parallel) \\
&= \mathbf{a}_\perp\cdot\mathbf{b}_\perp + \gamma^2\left[\mathbf{a}_\parallel\cdot\mathbf{b}_\parallel - \beta^2a_4b_4 + ib_4\mathbf{a}_\parallel\cdot\boldsymbol{\beta} + ia_4\mathbf{b}_\parallel\cdot\boldsymbol{\beta}\right] \\
&\quad + \gamma^2\left[a_4b_4 - \beta^2\mathbf{a}_\parallel\cdot\mathbf{b}_\parallel - ia_4\mathbf{b}_\parallel\cdot\boldsymbol{\beta} - ib_4\mathbf{a}_\parallel\cdot\boldsymbol{\beta}\right] \\
&= \mathbf{a}_\perp\cdot\mathbf{b}_\perp + \gamma^2(1-\beta^2)a_4b_4 + \gamma^2(1-\beta^2)\mathbf{a}_\parallel\cdot\mathbf{b}_\parallel \\
&= \mathbf{a}\cdot\mathbf{b} + a_4b_4.
\end{aligned}$$

22.5 The Quotient Theorem for a Four-Vector

Assume that $A_\mu B_\mu = A'_\mu B'_\mu$. Substituting A'_μ (from the Lorentz transformation rule for four-vectors) into the right member of this equation gives

$$\begin{aligned}
A_\mu B_\mu &= A'_1B'_1 + A'_2B'_2 + A'_3B'_3 + A'_4B'_4 \\
&= A_1B'_1 + A_2B'_2 + \gamma(A_3 + i\beta A_4)B'_3 + \gamma(A_4 - i\beta A_3)B'_4.
\end{aligned}$$

Rearranging terms on the right side of this equation produces

$$A_\mu B_\mu = A_1B'_1 + A_2B'_2 + A_3\gamma(B'_3 - i\beta B'_4) + a_4\gamma(B'_4 + i\beta B'_3).$$

The coefficients of A_μ must be the same on both sides if the preceding equation is to be true for an arbitrary choice of $\vec{A}$. In other words,

$$B_1 = B_1' \qquad B_2 = B_2' \qquad B_3 = \gamma(B_3' - i\beta B_4') \qquad B_4 = \gamma(B_4' + i\beta B_3').$$

This is the inverse of the Lorentz transformation rule for four-vectors, which proves that the ordered set (B_1, B_2, B_3, B_4) is indeed a four-vector $\vec{B}$.

22.6 Transformation Law for $\mathbf{E}_\perp$

By direct computation,

$$\begin{aligned}
\mathbf{E}'_\perp &= -\nabla'_\perp - \frac{\partial \mathbf{A}'_\perp}{\partial t'} \\
&= -\nabla_\perp \gamma(\varphi - \mathbf{v}\cdot\mathbf{A}) - \gamma\left(\frac{\partial}{\partial t} + \mathbf{v}\cdot\nabla\right)\mathbf{A}_\perp \\
&= \gamma\left[\left(-\nabla_\perp\varphi - \frac{\partial \mathbf{A}_\perp}{\partial t}\right) + \nabla_\perp(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A}_\perp\right] \\
&= \gamma\left[\mathbf{E}_\perp + \nabla_\perp(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\nabla)\mathbf{A}_\perp\right]. \qquad (1)
\end{aligned}$$

Because $\mathbf{v}$ is constant, we observe that

$$\nabla_\perp(\mathbf{v}\cdot\mathbf{A}) = \nabla_\perp(\mathbf{v}\cdot\mathbf{A}_\parallel) = \mathbf{v}\times(\nabla_\perp\times\mathbf{A}_\parallel) + (\mathbf{v}\cdot\nabla_\perp)\mathbf{A}_\parallel = \mathbf{v}\times(\nabla_\perp\times\mathbf{A}_\parallel).$$

On the other hand,

$$\nabla_\parallel(\mathbf{v}\cdot\mathbf{A}_\perp) = \mathbf{v}\times(\nabla_\parallel\times\mathbf{A}_\perp) + (\mathbf{v}\cdot\nabla_\parallel)\mathbf{A}_\perp.$$

But the left-hand side of the preceding equation is zero because $\mathbf{v}\cdot\mathbf{A}_\perp = 0$. Therefore,

$$-(\mathbf{v}\cdot\nabla_\parallel)\mathbf{A}_\perp = \mathbf{v}\times(\nabla_\parallel\times\mathbf{A}_\perp).$$

Combining these side calculations, we conclude that

$$\nabla_\perp(\mathbf{v}\cdot\mathbf{A}) - (\mathbf{v}\cdot\mathbf{A}_\parallel) = \mathbf{v}\times[\nabla_\perp\times\mathbf{A}_\parallel + \nabla_\parallel\times\mathbf{A}_\perp] = \mathbf{v}\times(\nabla\times\mathbf{A})_\perp = \mathbf{v}\times\mathbf{B}_\perp = (\mathbf{v}\times\mathbf{B})_\perp.$$

Inserting this result into (1) completes the proof.

22.7 Transformation of Force

(a) From Gauss' law, the electric field inside the electron column is

$$\mathbf{E} = \frac{\rho_0 r}{2\epsilon_0}\hat{\mathbf{r}} \qquad r < a.$$

Therefore the force on a electron at $r < a$ is

$$\mathbf{F} = -eq\mathbf{E}(r) = -\frac{e\rho_0 r}{2\epsilon_0}\hat{\mathbf{r}}.$$

(b) In the laboratory frame of the observer,

$$\mathbf{E}_\perp = \gamma(\mathbf{E}' - \mathbf{v} \times \mathbf{B}')_\perp \qquad \mathbf{E}'_\| = \mathbf{E}_\|,$$

$$\mathbf{B}_\perp = \gamma(\mathbf{B}' + (\mathbf{v}/c^2) \times \mathbf{E}')_\perp \qquad \mathbf{B}'_\| = \mathbf{B}_\|.$$

There is no magnetic field in the rest frame of the electrons and the rest-frame electric field [computed in part (a)] is entirely transverse. Therefore, the force of interest is

$$\mathbf{F} = -e\mathbf{E} = -e[\mathbf{E} + \mathbf{v} \times \mathbf{B}] = -e\gamma\mathbf{E}'_\perp - e\mathbf{v} \times \gamma[(\mathbf{v}/c^2) \times \mathbf{E}'_\perp] = -\frac{e\mathbf{E}'_\perp}{\gamma}.$$

22.8 A Charged, Current-Carrying Wire

(a) The transformation laws for charge density and current density are

$$\mathbf{j}_\| = \gamma(\mathbf{j}'_\| + \rho'\mathbf{v}) \qquad \mathbf{j}_\perp = \mathbf{j}'_\perp \qquad \rho = \gamma(\rho' + \mathbf{v} \cdot \mathbf{j}'/c^2).$$

For this geometry, I and λ are related to $\mathbf{j}$ and ρ by the same invariant transverse area. Therefore, we can immediately write

$$\mathbf{I} = \gamma(\mathbf{I}' + \lambda'\mathbf{v}) \qquad \lambda = \gamma\left(\lambda' + \frac{\mathbf{v} \cdot \mathbf{I}'}{c^2}\right).$$

(b) To eliminate the magnetic field, we need $\mathbf{I} = 0$. This happens if $\mathbf{v} = -\mathbf{I}'/\lambda'$. In that case,

$$\lambda = \gamma\left(\lambda' - \frac{I'^2}{\lambda' c^2}\right) = \frac{\lambda'}{\sqrt{1 - v^2/c^2}}\left(1 - \frac{v^2}{c^2}\right) = \lambda'\sqrt{1 - v^2/c^2} = \sqrt{\lambda'^2 - \frac{I'^2}{c^2}}.$$

To eliminate the electric field, we need $\lambda = 0$. This happens if $\mathbf{v} = -(\lambda' c^2/I')\hat{\mathbf{I}}$. In that case,

$$I = \gamma\left(I' - \frac{\lambda'^2 c^2}{I'}\right) = \frac{I'}{\sqrt{1 - v^2/c^2}}\left(1 - \frac{v^2}{c^2}\right) = I'\sqrt{1 - v^2/c^2} = \sqrt{I'^2 - \lambda'^2 c^2}.$$

Only one of these is possible because either $I' < c\lambda'$ or $I' > c\lambda'$. This means that only one of the two velocities computed above is less than c. Equivalently, if I is real, λ is imaginary and vice versa.

Source: E.M. Purcell, *Electricity and Magnetism* (McGraw-Hill, New York, 1985).

22.9 Poynting in the Wrong Direction?

(a) Let K' be the rest frame of the charge distribution so $\mathbf{E}' \neq 0$ and $\mathbf{B}' = 0$. Then,

$$\mathbf{B}_{\|} = \mathbf{B}'_{\|} = 0 \qquad c\mathbf{B}_{\perp} = \gamma\left(c\mathbf{B}' + \boldsymbol{\beta} \times \mathbf{E}'\right)_{\perp} = \gamma\beta\hat{\mathbf{x}} \times \mathbf{E}'_{\perp}$$

$$\mathbf{E}_{\|} = \mathbf{E}'_{\|} \qquad \mathbf{E}_{\perp} = \gamma(\mathbf{E}' - \boldsymbol{\beta} \times c\mathbf{B}')_{\perp} = \gamma\mathbf{E}'_{\perp}.$$

In the lab frame,

$$\begin{aligned} u_{\rm EM} &= \frac{1}{2}\epsilon_0\left\{|\mathbf{E}|^2 + c^2|\mathbf{B}|^2\right\} = \frac{1}{2}\epsilon_0\left\{|\mathbf{E}'_{\|}|^2 + \gamma^2|\mathbf{E}'_{\perp}|^2 + \gamma^2\beta^2|\hat{\mathbf{x}} \times \mathbf{E}'_{\perp}|^2\right\} \\ &= \frac{1}{2}\epsilon_0\left\{|\mathbf{E}'_{\|}|^2 + \frac{1+\beta^2}{1-\beta^2}|\mathbf{E}'_{\perp}|^2\right\}. \end{aligned}$$

This quantity does not depend on t', so $\partial u_{\rm EM}/\partial t' = 0$. Therefore, using the transformation law for the time derivative and the fact that $\mathbf{S}_0 = u_{\rm EM}\mathbf{v}$,

$$0 = \frac{\partial u_{\rm EM}}{\partial t'} = \gamma\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)u_{\rm EM} = \gamma\left[\frac{\partial u_{\rm EM}}{\partial t} + \frac{\partial(vu_{\rm EM})}{\partial x}\right] = \gamma\left[\frac{\partial u_{\rm EM}}{\partial t} + \nabla\cdot\mathbf{S}_0\right].$$

(b) Let $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$. Poynting's theorem says that, at all points outside the moving charge distribution (where $\mathbf{j} \neq 0$),

$$\nabla\cdot\mathbf{S} + \frac{\partial u_{\rm EM}}{\partial t} = 0.$$

Given the final result in part (b), it must be the case that $\nabla\cdot\mathbf{S} = \nabla\cdot\mathbf{S}_0$. Now, $\mathbf{B}_{\|} = 0$, so $\mathbf{S} = (\mathbf{E}_{\|} \times \mathbf{B}_{\perp} + \mathbf{E}_{\perp} \times \mathbf{B}_{\perp})/\mu_0$ cannot be entirely in the $\hat{\mathbf{x}}$-direction. On the other hand, $\mathbf{S}_0 \parallel \hat{\mathbf{x}}$, so we must conclude that $\mathbf{S} \neq \mathbf{S}_0$.

Source: W. Gough, *European Journal of Physics* **3**, 83 (1982).

22.10 Boost the Electromagnetic Field

(a) Observer A evaluates the two electromagnetic field invariants and finds the values

$$\mathbf{E}\cdot\mathbf{B} = \alpha^2/c \qquad \text{and} \qquad E^2 - c^2B^2 = \alpha^2 - c^2(\alpha^2/c^2 + 4\alpha^2/c^2) = -4\alpha^2.$$

Observer B evaluates the same invariants and finds

$$\mathbf{E}'\cdot\mathbf{B}' = E'_x\alpha/c + B'_y\alpha \qquad \text{and} \qquad \begin{aligned} E'^2 - c^2B'^2 &= E'^2_x + \alpha^2 - c^2(2\alpha^2/c^2 + B'^2_y) \\ &= E'^2_x - c^2B'^2_y - \alpha^2. \end{aligned}$$

Setting these invariants equal in the two frames gives

$$\begin{aligned} E'_x + cB'_y &= \alpha \\ E'^2_x - c^2B'^2_y &= -3\alpha^2. \end{aligned}$$

Solving these equations, we find $E'_x = -\alpha$ and $B'_y = 2\alpha/c$. Therefore,

$$\mathbf{E}' = (-\alpha, \alpha, 0) \qquad \text{and} \qquad \mathbf{B}' = (\alpha/c, 2\alpha/c, \alpha/c).$$

(b) The fields transform according to

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \qquad \mathbf{E}'_{\perp} = \gamma(\mathbf{E} + \boldsymbol{\beta} \times c\mathbf{B})_{\perp}$$

and

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \qquad c\mathbf{B}'_{\perp} = \gamma(c\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E})_{\perp}.$$

Therefore,

$$\begin{aligned}
\mathbf{E}'' &= -\alpha\hat{\mathbf{x}} + \gamma[\mathbf{E}'_{\perp} + v\hat{\mathbf{x}} \times (B'_y\hat{\mathbf{y}} + B'_z\hat{\mathbf{z}})] \\
&= -\alpha\hat{\mathbf{x}} + \gamma[\mathbf{E}'_{\perp} + vB'_y\hat{\mathbf{z}} - vB'_z\hat{\mathbf{y}}] \\
&= -\alpha\hat{\mathbf{x}} + \gamma(\alpha - v\alpha/c)\hat{\mathbf{y}} + 2\gamma v\alpha/c\hat{\mathbf{z}} \\
&= -\alpha\hat{\mathbf{x}} + \gamma\alpha(1-\beta)\hat{\mathbf{y}} + 2\gamma\alpha\beta\hat{\mathbf{z}}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{B}'' &= \alpha/c\hat{\mathbf{x}} + \gamma[\mathbf{B}'_{\perp} - (v/c^2)\hat{\mathbf{x}} \times (E'_y\hat{\mathbf{y}} + E'_z\hat{\mathbf{z}})] \\
&= \alpha/c\hat{\mathbf{x}} + \gamma[\mathbf{B}'_{\perp} - (v/c^2)E'_y\hat{\mathbf{z}} + (v/c^2)E'_z\hat{\mathbf{y}}] \\
&= \alpha/c\hat{\mathbf{x}} + 2\gamma\alpha/c\hat{\mathbf{y}} + \gamma(\alpha/c - v\alpha/c^2)\hat{\mathbf{z}} \\
&= \alpha/c\hat{\mathbf{x}} + 2\gamma\alpha/c\hat{\mathbf{y}} + \gamma\alpha(1-\beta)/c\hat{\mathbf{z}}.
\end{aligned}$$

Source: Prof. J. Mickelsson, KTH, Stockholm (public communication).

22.11 Covariance of the Maxwell Equations

The transformation law for the derivatives and for the charge and current density are

$$\frac{\partial}{\partial \mathbf{r}'_{\parallel}} = \gamma\left(\frac{\partial}{\partial \mathbf{r}_{\parallel}} + \frac{\boldsymbol{\beta}}{c}\frac{\partial}{\partial t}\right) \qquad \frac{\partial}{\partial \mathbf{r}'_{\perp}} = \frac{\partial}{\partial \mathbf{r}_{\perp}} \qquad \frac{\partial}{\partial t'} = \gamma\left(c\boldsymbol{\beta}\cdot\frac{\partial}{\partial \mathbf{r}_{\parallel}} + \frac{\partial}{\partial t}\right)$$

and

$$\mathbf{j}'_{\parallel} = \gamma(\mathbf{j}_{\parallel} - \rho\mathbf{v}) \qquad \mathbf{j}'_{\perp} = \mathbf{j}_{\perp} \qquad \rho' = \gamma(\rho - \mathbf{v}\cdot\mathbf{j}/c^2).$$

The transformation laws for the fields are

$$\mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \qquad c\mathbf{B}'_{\perp} = \gamma\left(c\mathbf{B} - \boldsymbol{\beta} \times \mathbf{E}\right)_{\perp}$$

$$\mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \qquad \mathbf{E}'_{\perp} = \gamma(\mathbf{E} + \boldsymbol{\beta} \times c\mathbf{B})_{\perp}.$$

No-Monopole Law:

Because $\nabla \cdot \mathbf{B} = 0$ and $\partial \mathbf{B}/\partial t + \nabla \times \mathbf{E} = 0$,

$$\begin{aligned}
\nabla' \cdot \mathbf{B}' &= \nabla'_{\parallel} \cdot \mathbf{B}'_{\parallel} + \nabla'_{\perp} \cdot \mathbf{B}'_{\perp} \\
&= \gamma(\nabla_{\parallel} + c^{-2}\mathbf{v}\partial_t) \cdot \mathbf{B}_{\parallel} + \nabla_{\perp} \cdot \gamma(\mathbf{B}_{\perp} - c^{-2}\mathbf{v} \times \mathbf{E}_{\perp}) \\
&= \gamma \nabla \cdot \mathbf{B} + \gamma c^{-2}\mathbf{v} \cdot \left[\frac{\partial \mathbf{B}_{\perp}}{\partial t} + \nabla_{\perp} \times \mathbf{E}_{\perp}\right] \\
&= \gamma \nabla \cdot \mathbf{B} + \gamma c^{-2}\mathbf{v} \cdot \left[\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}\right]_{\perp} \\
&= 0.
\end{aligned}$$

Faraday's Law: Parallel component first:

$$\begin{aligned}
\left[\frac{\partial \mathbf{B}'}{\partial t'} + \nabla' \times \mathbf{E}'\right]_{\parallel} &= \frac{\partial \mathbf{B}'_{\parallel}}{\partial t'} + \nabla'_{\perp} \times \mathbf{E}'_{\perp} \\
&= \gamma(\partial_t + \mathbf{v} \cdot \nabla)\mathbf{B}_{\parallel} + \nabla_{\perp} \times \gamma[\mathbf{E}_{\perp} + (\mathbf{v} \times \mathbf{B})_{\perp}] \\
&= \gamma \left[\frac{\partial \mathbf{B}_{\parallel}}{\partial t} + \nabla_{\perp} \times \mathbf{E}_{\perp}\right] + \gamma\left[(\mathbf{v} \cdot \nabla)\mathbf{B}_{\parallel} + \nabla_{\perp} \times (\mathbf{v} \times \mathbf{B})_{\perp}\right] \\
&= \gamma \left[\frac{\partial \mathbf{B}}{\partial t} + \nabla \times \mathbf{E}\right]_{\parallel} + \gamma\left[(\mathbf{v} \cdot \nabla)\mathbf{B}_{\parallel} + \{\nabla \times (\mathbf{v} \times \mathbf{B})\}_{\parallel}\right] \\
&= \gamma\left[(\mathbf{v} \cdot \nabla)\mathbf{B}_{\parallel} + \mathbf{v}(\nabla \cdot \mathbf{B}) - (\mathbf{v} \cdot \nabla)\mathbf{B}_{\parallel}\right] \\
&= 0,
\end{aligned}$$

because $\nabla \cdot \mathbf{B} = 0$ and $\partial \mathbf{B}/\partial t + \nabla \times \mathbf{E} = 0$. Now the transverse component:

$$\begin{aligned}
\left[\frac{\partial \mathbf{B}'}{\partial t'} + \nabla' \times \mathbf{E}'\right]_{\perp} &= \gamma(\partial_t + \mathbf{v} \cdot \nabla)\gamma(\mathbf{B}_{\perp} - c^{-2}\mathbf{v} \times \mathbf{E}_{\perp}) + \nabla'_{\parallel} \times \mathbf{E}'_{\perp} + \nabla'_{\perp} \times E'_{\parallel} \\
&= \gamma^2 \left[\frac{\partial \mathbf{B}_{\perp}}{\partial t} - \frac{\mathbf{v}}{c^2} \times \frac{\partial \mathbf{E}_{\perp}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{B}_{\perp} - (\mathbf{v} \cdot \nabla)\frac{\mathbf{v}}{c^2} \times \mathbf{E}_{\perp}\right] \\
&\quad + \gamma(\nabla_{\parallel} + c^{-2}\mathbf{v}\partial_t) \times \gamma[\mathbf{E}_{\perp} + (\mathbf{v} \times \mathbf{B})_{\perp}] + \nabla_{\perp} \times \mathbf{E}_{\parallel} \\
&= \gamma^2 \left[\frac{\partial \mathbf{B}_{\perp}}{\partial t} + \frac{\mathbf{v}}{c^2} \times \left(\mathbf{v} \times \frac{\partial \mathbf{B}_{\perp}}{\partial t}\right) + (\mathbf{v} \cdot \nabla_{\parallel})\mathbf{B}_{\perp} + \nabla_{\parallel} \times (\mathbf{v} \times \mathbf{B})_{\perp}\right. \\
&\quad \left. + \nabla_{\parallel} \times \mathbf{E}_{\perp} - (\mathbf{v} \cdot \nabla_{\parallel})\frac{\mathbf{v}}{c^2} \times \mathbf{E}_{\perp}\right] + \nabla_{\perp} \times \mathbf{E}_{\parallel}. \qquad (1)
\end{aligned}$$

However, $(\mathbf{v}\times\mathbf{B})_\parallel = 0$ and $\nabla\cdot\mathbf{B}=0$. Therefore, we add zero to get

$$\nabla_\parallel \times (\mathbf{v}\times\mathbf{B})_\perp = \nabla_\parallel \times (\mathbf{v}\times\mathbf{B})_\perp + \nabla_\perp \times (\mathbf{v}\times\mathbf{B})_\parallel = [\nabla\times(\mathbf{v}\times\mathbf{B})]_\perp = -(\mathbf{v}\cdot\nabla)\mathbf{B}_\perp.$$

This shows that the third and fourth terms cancel in the brackets in (1). Also,

$$\mathbf{v}\times\left(\mathbf{v}\times\frac{\partial\mathbf{B}_\perp}{\partial t}\right) = -v^2\frac{\partial\mathbf{B}_\perp}{\partial t} \qquad \text{and} \qquad v^2\nabla_\parallel\times\mathbf{E}_\perp = (\mathbf{v}\cdot\nabla)\mathbf{v}\times\mathbf{E}_\perp.$$

Therefore,

$$\left[\frac{\partial\mathbf{B}'}{\partial t'} + \nabla'\times\mathbf{E}'\right]_\perp = \gamma^2(1-\beta^2)\left[\frac{\partial\mathbf{B}_\perp}{\partial t} + \nabla_\parallel\times\mathbf{E}_\perp\right] + \nabla_\perp\times\mathbf{E}_\parallel = \left[\frac{\partial\mathbf{B}}{\partial t} + \nabla\times\mathbf{E}\right]_\perp = 0.$$

Gauss' Law: We note first that

$$\nabla_\perp\cdot(\mathbf{v}\times\mathbf{B})_\perp = \nabla_\perp\cdot(\mathbf{v}\times\mathbf{B})_\perp + \nabla_\parallel\cdot(\mathbf{v}\times\mathbf{B})_\parallel = \nabla\cdot(\mathbf{v}\times\mathbf{B}) = -\mathbf{v}\cdot(\nabla\times\mathbf{B}).$$

Therefore, because $\nabla\cdot\mathbf{E} = \rho/\epsilon_0$ and $\nabla\times\mathbf{B} - c^{-2}\partial\mathbf{E}/\partial t = \mu_0\mathbf{j}$,

$$\begin{aligned}
\nabla'\cdot\mathbf{E}' - \rho'/\epsilon_0 &= \nabla'_\perp\cdot\mathbf{E}'_\perp + \nabla'_\parallel\cdot\mathbf{E}'_\parallel - \rho'/\epsilon_0 \\
&= \nabla_\perp\cdot\gamma(\mathbf{E}+\mathbf{v}\times\mathbf{B})_\perp + \gamma(\nabla_\parallel + c^{-2}\mathbf{v}\partial_t)\cdot\mathbf{E}_\parallel - \gamma(\rho - c^{-2}\mathbf{v}\cdot\mathbf{j})/\epsilon_0 \\
&= \gamma(\nabla\cdot\mathbf{E} - \rho/\epsilon_0) - \gamma\mathbf{v}\cdot\left[\nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t} - \mu_0\mathbf{j}\right] \\
&= 0.
\end{aligned}$$

Ampère-Maxwell Law: We begin with the parallel component and add zero as above to get the identity

$$\nabla_\perp\times(\mathbf{v}\times\mathbf{E}_\perp) = \nabla_\perp\times(\mathbf{v}\times\mathbf{E})_\perp = \nabla_\perp\times(\mathbf{v}\times\mathbf{E})_\perp = [\nabla\times(\mathbf{v}\times\mathbf{E})]_\parallel = \mathbf{v}(\nabla\cdot\mathbf{E}) - (\mathbf{v}\cdot\nabla)\mathbf{E}_\parallel.$$

Therefore, because $\nabla\times\mathbf{B} - \mu_0\mathbf{j} - (1/c^2)\partial\mathbf{E}/\partial t = 0$ and $\nabla\cdot\mathbf{E} = \rho/\epsilon_0$,

$$\begin{aligned}
\left[\nabla'\times\mathbf{B}' - \mu_0\mathbf{j}' - \frac{1}{c^2}\frac{\partial\mathbf{E}'}{\partial t'}\right]_\parallel &= \nabla'_\perp\times\mathbf{B}'_\perp - \mu_0\mathbf{j}'_\parallel - \frac{1}{c^2}\frac{\partial\mathbf{E}'_\parallel}{\partial t'} \\
&= \nabla_\perp\times\gamma(\mathbf{B}_\perp - c^{-2}\mathbf{v}\times\mathbf{E}_\perp) - \mu_0\gamma(\mathbf{j}_\parallel - \mathbf{v}\rho) \\
&\quad - \frac{\gamma}{c^2}(\partial_t + \mathbf{v}\cdot\nabla)\mathbf{E}_\parallel \\
&= \gamma\left[\nabla\times\mathbf{B} - \mu_0\mathbf{j} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right]_\parallel - \gamma\left[\nabla_\perp\times c^{-2}(\mathbf{v}\times\mathbf{E}_\perp)\right. \\
&\quad \left. -\ \mu_0\mathbf{v}\rho + c^{-2}(\mathbf{v}\cdot\nabla)\mathbf{E}_\parallel\right] \\
&= \gamma\mathbf{v}c^{-2}(\nabla\cdot\mathbf{E} - \rho/\epsilon_0) \\
&= 0.
\end{aligned}$$

Finally, using the previous identities, the transverse component is

$$\begin{aligned}
\left[\nabla'\times\mathbf{B}'-\mu_0\mathbf{j}'-\frac{1}{c^2}\frac{\partial\mathbf{E}'}{\partial t'}\right]_\perp &= \nabla'_\perp\times\mathbf{B}'_\parallel+\nabla'_\parallel\times\mathbf{B}'_\perp-\mu_0\mathbf{j}_\perp \\
&\quad -(\gamma^2/c^2)(\partial_t+\mathbf{v}\cdot\nabla)(\mathbf{E}+\mathbf{v}\times\mathbf{B})_\perp \\
&= \nabla_\perp\times\mathbf{B}_\parallel+\gamma^2(\nabla_\parallel+c^{-2}\mathbf{v}\,\partial_t)\times(\mathbf{B}-c^{-2}\mathbf{v}\times\mathbf{E})_\perp \\
&\quad -\mu_0\mathbf{j}_\perp-\frac{\gamma^2}{c^2}(\partial_t+\mathbf{v}\cdot\nabla)(\mathbf{E}+\mathbf{v}\times\mathbf{B})_\perp \\
&= \nabla_\perp\times\mathbf{B}_\parallel+\gamma^2(1-\beta^2)\left[\nabla_\parallel\times\mathbf{B}_\perp-\frac{1}{c^2}\frac{\partial\mathbf{E}_\perp}{\partial t}\right]-\mu_0\mathbf{j}_\perp \\
&= \left[\nabla\times\mathbf{B}-\mu_0\mathbf{j}-\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right]_\perp \\
&= 0.
\end{aligned}$$

22.12 Transformations of E and B

(a)

$$\begin{aligned}
\mathbf{E}'\cdot\mathbf{B}' &= \mathbf{E}'_\parallel\cdot\mathbf{B}'_\parallel+\mathbf{E}'_\perp\cdot\mathbf{B}'_\perp=\mathbf{E}_\parallel\cdot\mathbf{B}_\parallel+\gamma(\mathbf{E}_\perp+\mathbf{v}\times\mathbf{B}_\perp)\cdot\gamma(\mathbf{B}_\perp-c^{-2}\mathbf{v}\times\mathbf{E}_\perp) \\
&= \mathbf{E}_\parallel\cdot\mathbf{B}_\parallel+\gamma^2\left\{\mathbf{E}_\perp\cdot\mathbf{B}_\perp-c^{-2}(\mathbf{v}\times\mathbf{B}_\perp)\cdot(\mathbf{v}\times\mathbf{E}_\perp)\right\} \\
&= \mathbf{E}_\parallel\cdot\mathbf{B}_\parallel+\gamma^2\mathbf{E}_\perp\cdot\mathbf{B}_\perp(1-v^2/c^2)=\mathbf{E}_\parallel\cdot\mathbf{B}_\parallel+\mathbf{E}_\perp\cdot\mathbf{B}_\perp=\mathbf{E}\cdot\mathbf{B}.
\end{aligned}$$

(b) Let $\mathbf{E}=E\hat{\mathbf{y}}$, $\mathbf{B}=B\hat{\mathbf{z}}$, and $\mathbf{v}=v\hat{\mathbf{x}}$ so

$$\mathbf{E}'_\perp=\gamma(\mathbf{E}_\perp+\mathbf{v}\times\mathbf{B}_\perp)=\gamma(E-vB)\hat{\mathbf{y}}$$

$$\mathbf{B}'_\perp=\gamma(\mathbf{B}_\perp-c^{-2}\mathbf{v}\times\mathbf{E}_\perp)=\gamma(B-c^{-2}vE)\hat{\mathbf{z}}.$$

To eliminate **E**, we let $v=v_E=E/B$. To eliminate **B**, we let $v=v_B=c^2B/E$. Since $E\neq cB$, one of these must be less than c because $v_Ev_B=c^2$ and $v_E+v_B>2c$.

(c) We must have $\mathbf{v}=v\hat{\mathbf{x}}$ because the parallel components of the field are invariant. Then,

$$\mathbf{E}'_\perp=\gamma\left[E_0\hat{\mathbf{y}}+\mathbf{v}\times\mathbf{B}_\perp\right]=\gamma E_0\left[(1-v\sin\theta/c)\hat{\mathbf{y}}+(v/c)\cos\theta\hat{\mathbf{z}}\right]$$

$$\mathbf{B}'_\perp=\gamma\left[\mathbf{B}_\perp-c^{-2}\mathbf{v}\times\mathbf{E}_\perp\right]=\gamma(E_0/c)[\cos\theta\hat{\mathbf{y}}+(\sin\theta-v/c)\cos\theta\hat{\mathbf{z}}]$$

so

$$\frac{\mathbf{E}'_\perp}{E'_\perp}=\frac{1-(v/c)\sin\theta}{\sqrt{(1-v\sin\theta/c)^2+(v/c)^2\cos^2\theta}}\hat{\mathbf{y}}+\frac{(v/c)\cos\theta}{\sqrt{(1-v\sin\theta/c)^2+(v/c)^2\cos^2\theta}}\hat{\mathbf{z}}$$

$$\frac{B'_\perp}{B'_\perp} = \frac{\cos\theta}{\sqrt{\cos^2\theta + (\sin\theta - v_0/c)^2}}\hat{\mathbf{y}} + \frac{\sin\theta - v_0/c}{\sqrt{\cos^2\theta + (\sin\theta - v_0/c)^2}}\hat{\mathbf{z}}.$$

Now, the denominators above are the same,

$$\begin{aligned}\sqrt{(1 - v\sin\theta/c)^2 + (v/c)^2\cos^2\theta} &= \sqrt{1 + (v/c)^2 - 2(v/c\sin\theta)} \\ &= \sqrt{\cos^2\theta + (\sin\theta - v/c)^2}.\end{aligned}$$

Therefore, to make $\mathbf{E}'_\perp \parallel \mathbf{B}'_\perp$ we need either

(i) $1 - (v/c)\sin\theta = \cos\theta$ and $(v/c)\cos\theta = \sin\theta - v/c$

or

(ii) $1 - (v/c\sin\theta = -\cos\theta$ and $(v/c)\cos\theta = (v/c)\cos\theta - \sin\theta.$

From (i), we get the parallel case:

$$\frac{v}{c} = \frac{\sin\theta}{1 + \cos\theta} = \frac{1 - \cos\theta}{\sin\theta} = \tan\frac{\theta}{2} \qquad \mathrm{d}\ \left(-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\right).$$

From (ii), we get the anti-parallel case:

$$\frac{v}{c} = \frac{\sin\theta}{1 - \cos\theta} = \frac{1 + \cos\theta}{\sin\theta} = \cot\frac{\theta}{2} \qquad \left(\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\right).$$

For the <u>parallel</u> case:

$$\mathbf{E}'_\perp = \gamma E_0(1 - (v/c)\sin\theta)\hat{\mathbf{y}} + \gamma E_0(v/c)\cos\theta\hat{\mathbf{z}} = \gamma E_0\cos\theta\hat{\mathbf{y}} + \gamma E_0\cos\theta\tan(\theta/2)\hat{\mathbf{z}}$$

$$c\mathbf{B}'_\perp = \mathbf{E'}_\perp,$$

with

$$E'_\perp = \gamma E_0\cos\theta\sqrt{1 + \tan^2(\theta/2)} = \gamma E_0\cos\theta\sec\frac{\theta}{2} = E_0\frac{\cos\theta\sec(\theta/2)}{\sqrt{1 - \tan^2(\theta/2)}} = E_0\sqrt{\cos\theta}.$$

For the <u>anti-parallel</u> case:

$$\mathbf{E}'_\perp = -\gamma E_0\cos\theta\hat{\mathbf{y}} + \gamma E_0\cos\theta\cot(\theta/2)\hat{\mathbf{z}}$$

$$c\mathbf{B}'_\perp = \gamma E_0\cos\theta\hat{\mathbf{y}} - \gamma E_0\cos\theta\cot(\theta/2)\hat{\mathbf{z}},$$

with

$$E'_\perp = \gamma E_0\cos\theta\sqrt{1 + \cot^2(\theta/2)} = E_0\frac{\cos\theta\csc(\theta/2)}{\sqrt{1 - \cot^2(\theta/2)}} = E_0\sqrt{-\cos\theta}.$$

22.13 Covariant Charge and Current Density

The delta function relation we are asked to prove follows from the transformation

$$\mathbf{r}_{\parallel} = \gamma(\mathbf{r}'_{\parallel} + \mathbf{v}_0 t) \qquad \mathbf{r}_{\perp} = \mathbf{r}'_{\perp} \qquad t = \gamma(t' + \mathbf{v}_0 \cdot \mathbf{r}'/c^2),$$

which permits us to regard $\mathbf{R} = \mathbf{r} - \mathbf{r}_k(t)$ as a function of $\mathbf{r}'$ and t'. We will need two facts. First, $\mathbf{r} = \mathbf{r}_k(t)$ is the same event in space-time as $\mathbf{r}' = \mathbf{r}'_k(t')$. Second, the volume element d^3R is related to the volume element d^3r' by

$$d^3r' = \frac{d^3R}{|\mathbf{J}(\mathbf{R}, \mathbf{r}')|},$$

where the Jacobian determinant is

$$|\mathbf{J}(\mathbf{R}, \mathbf{r}')| = \begin{vmatrix} \partial R_x/\partial x' & \partial R_x/\partial y' & \partial R_x/\partial z' \\ \partial R_y/\partial x' & \partial R_y/\partial y' & \partial R_y/\partial z' \\ \partial R_z/\partial x' & \partial R_z/\partial y' & \partial R_z/\partial z' \end{vmatrix}.$$

Using this information, we choose an arbitrary function $F(\mathbf{r}')$ and evaluate the delta function integral

$$\begin{aligned} I &= \int d^3r'\, F(\mathbf{r}')\delta[\mathbf{r} - \mathbf{r}_k(t)] = \int d^3r'\, F(\mathbf{r}')\delta(\mathbf{R}) = \int \frac{d^3R}{|\mathbf{J}(\mathbf{R}, \mathbf{r}')|}\, F(\mathbf{r}')\delta(\mathbf{R}) \\ &= \left[F(\mathbf{r}')\frac{1}{|\mathbf{J}(\mathbf{R}, \mathbf{r}')|}\right]_{\mathbf{r}=\mathbf{r}_k(t)} = \left[F(\mathbf{r}')\frac{1}{|\mathbf{J}(\mathbf{R}, \mathbf{r}')|}\right]_{\mathbf{r}'=\mathbf{r}'_k(t')} \\ &= \int \frac{d^3r'}{|\mathbf{J}(\mathbf{R}, \mathbf{r}')|}\, F(\mathbf{r}')\delta[\mathbf{r}' - \mathbf{r}'_k(t')]. \end{aligned}$$

Comparing the first integral to the last shows that

$$\delta[\mathbf{r} - \mathbf{r}_k(t)] = \frac{\delta[\mathbf{r}' - \mathbf{r}'_k(t')]}{|\mathbf{J}(\mathbf{R}, \mathbf{r}')|}.$$

To evaluate the Jacobian, we use the transformation law above to write

$$\mathbf{R} = \mathbf{r} - \mathbf{r}_k(t) = \mathbf{r}_{\parallel} + \mathbf{r}_{\perp} - \mathbf{r}_k(t) = \gamma(\mathbf{r}'_{\parallel} + \mathbf{v}_0 t') + \mathbf{r}'_{\perp} - \mathbf{r}_k(t = \gamma t' + \gamma \mathbf{v}_0 \cdot \mathbf{r}'_{\parallel}/c^2).$$

Carrying out the derivatives (choosing one axis as the boost direction) shows that

$$|\mathbf{J}| = \gamma(1 - \mathbf{v}_k \cdot \mathbf{v}_0/c^2).$$

Therefore, we get the advertised result,

$$\delta[\mathbf{r} - \mathbf{r}_k(t)] = \frac{\delta[\mathbf{r}' - \mathbf{r}'_k(t')]}{\gamma(1 - \mathbf{v}_k \cdot \mathbf{v}_0/c^2)}.$$

The transformation laws for the charge/current density are

$$\mathbf{j}'_{\parallel} = \gamma(\mathbf{j}_{\parallel} - \rho\mathbf{v}) \qquad \mathbf{j}'_{\perp} = \mathbf{j}_{\perp} \qquad \rho' = \gamma(\rho - \mathbf{v}\cdot\mathbf{j}/c^2).$$

To prove the covariance of the charge density formula, we substitute ρ and $\mathbf{j}$ from the statement of the problem, and use the invariance of charge ($q_k = q'_k$) and the just-proved delta function identity. The result is

$$\begin{aligned}
\rho(\mathbf{r}', t') &= \gamma(\rho(\mathbf{r}, t) - \mathbf{v}_0 \cdot \mathbf{j}(\mathbf{r}, t)/c^2) \\
&= \gamma \sum_k q_k \left(1 - \frac{\mathbf{v}_0 \cdot \mathbf{v}_k(t)}{c^2}\right) \delta[(\mathbf{r} - \mathbf{r}_k(t)] \\
&= \sum_k q'_k \delta[\mathbf{r}' - \mathbf{r_k}'(t')].
\end{aligned}$$

Turning to the current density, we make essential use of the transformation law for velocities. First, for the component of $\mathbf{j}$ parallel to the boost,

$$\begin{aligned}
\mathbf{j}'_{\parallel}(\mathbf{r}', t') &= \gamma[\mathbf{j}_{\parallel}(\mathbf{r}, t) - \mathbf{v}_0 \rho(\mathbf{r}, t)] \\
&= \gamma \sum_k q_k (\mathbf{v}_{k,\parallel} - \mathbf{v}_0) \delta[\mathbf{r} - \mathbf{r}_k(t)] \\
&= \gamma \sum_k q_k \frac{\mathbf{v}_{k,\parallel} - \mathbf{v}_0}{\gamma(1 - \mathbf{v}_k \cdot \mathbf{v}_0/c^2)} \delta[\mathbf{r}' - \mathbf{r_k}'(t')] \\
&= \sum_k q'_k \mathbf{v}'_{k,\parallel} \delta[\mathbf{r}' - \mathbf{r_k}'(t')].
\end{aligned}$$

Then, for the component of $\mathbf{j}$ perpendicular to the boost,

$$\begin{aligned}
\mathbf{j}'_{\perp}(\mathbf{r}', t') &= \mathbf{j}_{\perp}(\mathbf{r}, t) \\
&= \sum_k q_k \mathbf{v}_{k,\perp} \delta[\mathbf{r} - \mathbf{r}_k(t)] \\
&= \sum_k q_k \frac{\mathbf{v}_{k,\perp}}{\gamma(1 - \mathbf{v}_k \cdot \mathbf{v}_0/c^2)} \delta[\mathbf{r}' - \mathbf{r_k}'(t')] \\
&= \sum_k q'_k \mathbf{v}'_{k,\perp} \delta[\mathbf{r}' - \mathbf{r_k}'(t')].
\end{aligned}$$

Source: B. Podolsky and K.S. Kunz, *Fundamentals of Electrodynamics* (Marcel Dekker, New York, 1969).

22.14 A Relativistic Particle in a Constant Electric Field

The equation of motion is

$$\frac{d\mathbf{p}}{dt} = q\mathbf{E}.$$

Therefore, if $p_0 = \gamma m u_0$ is the initial momentum,

$$\mathbf{p}(t) = p_0\,\hat{\mathbf{y}} + qEt\hat{\mathbf{z}}.$$

The initial (total) energy of the particle is $\mathcal{E}_0 = \sqrt{c^2p_0^2 + m^2c^4}$. Therefore, the instantaneous velocity of the particle is

$$\mathbf{u} = \frac{c^2\mathbf{p}}{\mathcal{E}} = \frac{c^2\mathbf{p}}{\sqrt{c^2p^2 + m^2c^4}} = \frac{p_0\,\hat{\mathbf{y}} + qEt\hat{\mathbf{z}}}{\sqrt{\mathcal{E}_0^2 + c^2q^2E^2t^2}}c^2.$$

The particle speed $u \to c$ as $t \to \infty$ and one time integration of $\mathbf{u}(t)$ gives

$$\mathbf{r}(t) = \hat{\mathbf{y}}\frac{cp_0}{qE}\sinh^{-1}\left(\frac{cqEt}{\mathcal{E}_0}\right) + \hat{\mathbf{z}}\frac{1}{qE}\sqrt{\mathcal{E}_0^2 + c^2q^2E^2t^2}.$$

We have chosen the origin of coordinates so the integration constants are zero. Eliminating t and using the properties of $\sinh x$ and $\cosh y$ shows that the particle trajectory is

$$z = \frac{\mathcal{E}_0}{qE}\cosh\left(\frac{qEy}{cp_0}\right).$$

The non-relativistic limit is $u \ll c$ or $cqEt \ll \mathcal{E}_0$. We recover the expected parabolic trajectory in this limit because $\cosh x \approx 1 + \frac{1}{2}x^2$ when $x \ll 1$.

Source: R.D. Sard, *Relativistic Mechanics* (W.A. Benjamin, New York, 1970).

22.15 A Charged Particle in Uniform Motion Revisited

(a) In the rest frame, the electromagnetic potentials are $\varphi' = \dfrac{q}{4\pi\epsilon_0 r'}$ and $\mathbf{A}' = 0$. Therefore, since the Lorentz transformation is

$$\mathbf{r}'_\perp = \mathbf{r}_\perp \qquad \text{and} \qquad \mathbf{r}'_\parallel = \gamma(\mathbf{r}_\parallel - \mathbf{v}t),$$

the potentials in the laboratory frame are

$$\varphi = \gamma(\varphi' + \mathbf{v}\cdot\mathbf{A}') = \gamma\,\varphi' = \frac{\gamma q}{4\pi\epsilon_0 r'} = \frac{\gamma q}{4\pi\epsilon_0}\frac{1}{[r_\perp^2 + \gamma^2(r_\parallel - vt)^2]^{1/2}}$$

$$\mathbf{A}_\parallel = \gamma(\mathbf{A}'_\parallel + \frac{\mathbf{v}}{c^2}\varphi') = \frac{\gamma\mathbf{v}}{c^2}\frac{q}{4\pi\epsilon_0 r'} = \frac{\gamma q}{4\pi\epsilon_0}\frac{\mathbf{v}/c^2}{[r_\perp^2 + \gamma^2(r_\parallel - vt)^2]^{1/2}}$$

$$\mathbf{A}_\perp = \mathbf{A}'_\perp = 0.$$

(b)

$$
\begin{aligned}
\mathbf{E} &= -\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t} \\
&= -\nabla_\perp \varphi - \nabla_\parallel \varphi - \frac{\partial \mathbf{A}_\parallel}{\partial t} \\
&= \frac{\gamma q}{4\pi\epsilon_0} \frac{\mathbf{r}_\perp + \gamma^2(\mathbf{r}_\parallel - \mathbf{v}t) - \beta^2\gamma^2(\mathbf{r}_\parallel - \mathbf{v}t)}{[r_\perp^2 + \gamma^2(r_\parallel - vt)^2]^{3/2}} \\
&= \frac{\gamma q}{4\pi\epsilon_0} \frac{\mathbf{r}_\perp + \mathbf{r}_\parallel - \mathbf{v}t}{[r_\perp^2 + \gamma^2(r_\parallel - vt)^2]^{3/2}} = \frac{\gamma q}{4\pi\epsilon_0} \frac{\mathbf{r} - \mathbf{v}t}{[r_\perp^2 + \gamma^2(r_\parallel - vt)^2]^{3/2}}
\end{aligned}
$$

$$
\mathbf{B} = \nabla \times \mathbf{A} = (\nabla_\perp + \nabla_\parallel) \times (\mathbf{A}_\perp + \mathbf{A}_\parallel) = \nabla_\perp \times \mathbf{A}_\parallel = \frac{\mu_0}{4\pi} \frac{\gamma q \mathbf{v} \times \mathbf{r}_\perp}{[r_\perp^2 + \gamma^2(r_\parallel - vt)^2]^{3/2}}.
$$

(c) $\mathbf{E}_\parallel \to 0$ trivially when $\mathbf{r}_\parallel = \mathbf{v}t$ for any $\mathbf{v}$. When $\mathbf{r}_\parallel \neq \mathbf{v}t$, $\mathbf{E}_\parallel \to 0$ anyway when $v \to c$, i.e., when $\gamma^2 \to \infty$. Similarly, $\mathbf{E}_\perp \to 0$ when $v \to c$ if $\mathbf{r}_\parallel \neq \mathbf{v}t$. However, $\mathbf{E}_\perp \to \infty$ when $v \to c$ if $\mathbf{r}_\parallel = \mathbf{v}t$. On the other hand,

$$
\int d(r_\parallel - ct)\mathbf{E}_\perp = \frac{\gamma q \mathbf{r}_\perp}{2\pi\epsilon_0} \int_0^\infty \frac{dy}{[r_\perp^2 + \gamma^2 y^2]^{3/2}} = \frac{\gamma q \mathbf{r}_\perp}{2\pi\epsilon_0} \left. \frac{y}{r_\perp^2 \sqrt{r_\perp^2 + \gamma^2 y^2}} \right|_0^\infty = \frac{q \mathbf{r}_\perp}{2\pi\epsilon_0 r_\perp^2}.
$$

Therefore,

$$
\lim_{v \to c} \mathbf{E} = \frac{1}{2\pi\epsilon_0} \frac{\mathbf{r}_\perp}{r_\perp^2} q\delta(r_\parallel - ct).
$$

Since $\mathbf{B} = \dfrac{\mathbf{v}}{c^2} \times \mathbf{E}_\perp$ for any $\mathbf{v}$, it must be the case that

$$
\lim_{v \to c} \mathbf{B} = \frac{\mu_0}{2\pi} \frac{\mathbf{v} \times \mathbf{r}_\perp}{r_\perp^2} q\delta(r_\parallel - ct).
$$

(d) The fields computed in part (c) satisfy $|\mathbf{E}| = c|\mathbf{B}|$. In addition, **E, B,** and **v** form a right-handed, orthogonal triad. Radiation fields have the same characteristics (with $\hat{\mathbf{r}}$ playing the role of $\hat{\mathbf{v}}$). The only difference is that radiation fields decay as $1/r$ whereas the relativistic particle fields decay as $1/r_\perp$.

Source: J.D. Jackson, *Classical Electrodynamics*, 3rd edition (Wiley, New York, 1999).

22.16 The Four-Potential

(a) If K' is the rest frame, we know that

$$\mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v}_0 \times \mathbf{B}) \qquad \mathbf{E}'_\parallel = \mathbf{E}_\parallel$$

$$\mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - c^{-2}\mathbf{v}_0 \times \mathbf{B}) \qquad \mathbf{B}'_\parallel = \mathbf{B}_\parallel.$$

Given this, and the fact that $\vec{\nabla}$ is a four-vector, we see that

$$\mathbf{B}'_\parallel = \nabla'_\perp \times \mathbf{A}'_\perp = \nabla_\perp \times \mathbf{A}'_\perp = \mathbf{B}_\parallel = \nabla_\perp \times \mathbf{A}_\perp.$$

This allows us to conclude that $\mathbf{A}'_\perp = \mathbf{A}_\perp$. On the other hand,

$$\mathbf{E}'_\parallel \;=\; -\nabla'_\parallel \varphi' - \frac{\partial \mathbf{A}'_\parallel}{\partial t'} = -\gamma\left(\nabla_\parallel + \frac{1}{c^2}\mathbf{v}_0 \frac{\partial}{\partial t}\right)\varphi' - \gamma\left(\frac{\partial}{\partial t} + \mathbf{v}_0 \cdot \nabla_\parallel\right)\mathbf{A}'_\parallel.$$

Therefore, it must be the case that $\mathbf{A}_\parallel = \gamma(\mathbf{A}'_\parallel + c^{-2}\mathbf{v}_0\varphi')$ and $\varphi = \gamma(\varphi' + \mathbf{v}_0 \cdot \mathbf{A}')$. This proves that $(\mathbf{A}, i\varphi/c)$ is a four-vector. The same conclusion can be reached by using the perpendicular components $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$.

(b) In the particle's rest frame, we have

$$\varphi' = \frac{q}{4\pi\varepsilon_0}\frac{1}{(x'^2 + y'^2 + z'^2)^{1/2}} \qquad \mathbf{A}' = 0.$$

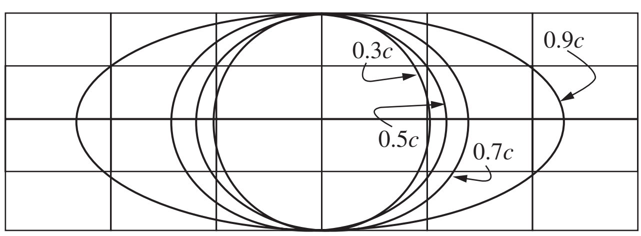
Lorentz transforming to the lab frame gives

$$\varphi = \gamma(\varphi' + \mathbf{v}_0 \cdot \mathbf{A}') = \gamma\varphi' = \frac{\gamma q}{4\pi\epsilon_0}\frac{1}{(\gamma^2(x - v_0 t)^2 + y^2 + z^2)^{1/2}}.$$

Now put $\varphi = \varphi_0$ to get

$$\frac{(x - v_0 t)^2}{\left(\dfrac{q}{4\pi\epsilon_0\varphi_0}\right)^2} + \frac{y^2}{\left(\dfrac{\gamma q}{4\pi\epsilon_0\varphi_0}\right)^2} + \frac{z^2}{\left(\dfrac{\gamma q}{4\pi\epsilon_0\varphi_0}\right)^2} = 1.$$

This is an ellipsoid centered at $(v_0 t, 0, 0)$ with major axes $\gamma q/4\pi\epsilon_0\varphi_0$ and minor axis $\gamma/4\pi\epsilon_0\varphi_0$. For the indicated values, γ varies from 1.05 to 2.3. The equipotentials are plotted below.



22.17 A Moving Current Loop

(a) In the rest frame, $\varphi' = 0$ and

$$\mathbf{A}'(\mathbf{r}') = \frac{\mu_0}{4\pi}\frac{\mathbf{m}' \times \mathbf{r}'}{r'^3}.$$

Therefore,

$$\mathbf{A}'_\perp = \frac{\mu_0}{4\pi}\frac{\mathbf{m}'_\perp \times \mathbf{r}'_\parallel + \mathbf{m}'_\parallel \times \mathbf{r}'_\perp}{r'^3} \qquad \text{and} \qquad \mathbf{A}'_\parallel = \frac{\mu_0}{4\pi}\frac{\mathbf{m}'_\perp \times \mathbf{r}'_\perp}{r'^3}.$$

On the other hand,

$$\mathbf{A}_\perp = \mathbf{A}'_\perp$$

$$\mathbf{A}_\parallel = \gamma(\mathbf{A}'_\parallel + \mathbf{v}_0\varphi'/c^2) = \gamma\mathbf{A}'_\parallel$$

$$\varphi = \gamma(\varphi' + \mathbf{v}_0 \cdot \mathbf{A}') = \gamma\mathbf{v}_0 \cdot \mathbf{A}'.$$

Hence,

$$\mathbf{A}_\perp = \frac{\mu_0}{4\pi}\frac{\mathbf{m}_\perp \times \gamma(\mathbf{r}_\parallel - \mathbf{v}_0 t) + \gamma\mathbf{m}_\parallel \times \mathbf{r}_\perp}{\left\{\gamma^2(\mathbf{r}_\parallel - \mathbf{v}_0 t)^2 + \mathbf{r}_\perp^2\right\}^{3/2}} = \frac{\gamma\mu_0}{4\pi}\frac{(\mathbf{m}\times\mathbf{R})_\perp}{\left\{\gamma^2\mathbf{R}_\parallel^2 + \mathbf{R}_\perp^2\right\}^{3/2}}$$

$$\mathbf{A}_\parallel = \frac{\gamma\mu_0}{4\pi}\frac{\mathbf{m}'_\perp \times \mathbf{r}'_\perp}{\left\{\gamma^2(\mathbf{r}_\parallel - \mathbf{v}_0 t)^2 + \mathbf{r}_\perp^2\right\}^{3/2}} = \frac{\gamma\mu_0}{4\pi}\frac{(\mathbf{m}\times\mathbf{R})_\parallel}{\left\{\gamma^2\mathbf{R}_\parallel^2 + \mathbf{R}_\perp^2\right\}^{3/2}},$$

so

$$\mathbf{A} = \frac{\gamma\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{R}}{\left\{\gamma^2\mathbf{R}_\parallel^2 + \mathbf{R}_\perp^2\right\}^{3/2}}.$$

Similarly,

$$\varphi = \mathbf{v}_0 \cdot \mathbf{A}_\parallel = \frac{\gamma\mu_0}{4\pi}\frac{v_0 \cdot (\mathbf{m}\times\mathbf{R})}{\left\{\gamma^2\mathbf{R}_\parallel^2 + \mathbf{R}_\perp^2\right\}^{3/2}}.$$

(b) In the non-relativistic limit, $\gamma \to 1$, so

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{R}}{R^3}$$

$$\varphi = \frac{\mu_0}{4\pi}\frac{\mathbf{v}_0 \cdot (\mathbf{m}\times\mathbf{R})}{R^3} = \frac{1}{4\pi\epsilon_0}\frac{(\mathbf{v}_0 \times \mathbf{m})\cdot\mathbf{R}/c^2}{R^3}.$$

These are the vector and scalar potentials for a system moving at a velocity v_0 with a magnetic dipole moment $\mathbf{m}$ and an electric dipole moment $\mathbf{p} = (\mathbf{v}_0 \times \mathbf{m})/c^2$.

Source: J.D. Jackson, *Classical Electrodynamics*, 3rd edition (Wiley, New York, 1999).

22.18 Transformation of Dipole Moments

Let inertial frame K' move with velocity $\mathbf{v}$ with respect to the laboratory frame K. The text gives the transformation laws for the polarization and magnetization as

$$\mathbf{P}_{\parallel} = \mathbf{P'}_{\parallel} \qquad \mathbf{P}_{\perp} = \gamma\left(\mathbf{P'} + \frac{\mathbf{v}}{c^2} \times \mathbf{M'}\right)_{\perp}$$

$$\mathbf{M}_{\parallel} = \mathbf{M'}_{\parallel} \qquad \mathbf{M}_{\perp} = \gamma\left(\mathbf{M'} - \mathbf{v} \times \mathbf{P'}\right)_{\perp}.$$

Let K' be the rest frame of the body. Then, because the volume element suffers length contraction,

$$\begin{aligned}
\mathbf{m} &= \int d^3r\,(\mathbf{M}_{\perp} + \mathbf{M}_{\parallel}) \\
&= \int \frac{d^3r'}{\gamma}\,(\gamma\mathbf{M'}_{\perp} + \mathbf{M'}_{\parallel} - \gamma\mathbf{v} \times \mathbf{P'}) \\
&= \int d^3r' \left(\mathbf{M'}_{\perp} + \frac{\mathbf{M'}_{\parallel}}{\gamma} - \mathbf{v} \times \mathbf{P'}\right) \\
&= \mathbf{m'}_{\perp} + \frac{\mathbf{m'}_{\parallel}}{\gamma} - \mathbf{v} \times \mathbf{p'}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbf{p} &= \int d^3r\,(\mathbf{P}_{\perp} + \mathbf{P}_{\parallel}) \\
&= \int \frac{d^3r'}{\gamma}\,(\gamma\mathbf{P'}_{\perp} + \mathbf{P'}_{\parallel} + \gamma\frac{\mathbf{v}}{c^2} \times \mathbf{M'}) \\
&= \int d^3r' \left(\mathbf{P'}_{\perp} + \frac{\mathbf{P'}_{\parallel}}{\gamma} + \frac{\mathbf{v}}{c^2} \times \mathbf{M'}\right) \\
&= \mathbf{p'}_{\perp} + \frac{\mathbf{p'}_{\parallel}}{\gamma} + \frac{\mathbf{v}}{c^2} \times \mathbf{m'}.
\end{aligned}$$

Source: V.V. Batygin and I.N. Toptygin, *Problems in Electrodynamics* (Academic, London, 1978).

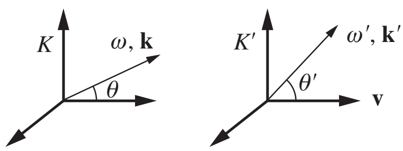
22.19 TE and TM Modes of a Waveguide

A TE mode has zero longitudinal electric field, a TM mode has zero longitudinal magnetic field, and a Lorentz transformation leaves $\mathbf{E}_{\parallel}$ and $\mathbf{B}_{\parallel}$ invariant. Therefore, a Lorentz boost of a TE mode does not generate a longitudinal electric field and a Lorentz boost of a TM mode does not generate a longitudinal magnetic field. The TE/TM classification is a Lorentz invariant concept.

Source: M. Aalund and G. Johannsen, *Journal of Applied Physics* **42**, 2669 (1971).

22.20 Stellar Aberration

The geometry is



The transformation law for the four-vector $\vec{k}$ is

$$\mathbf{k}_\perp = \mathbf{k}'_\perp$$

$$\mathbf{k}_\parallel = \gamma(\mathbf{k}'_\parallel + \boldsymbol{\beta} k'_0)$$

$$k_4 = \gamma(k'_4 + \boldsymbol{\beta} \cdot \mathbf{k}'_\parallel).$$

Therefore,

$$k_\parallel = \gamma(k'_\parallel + v\omega'/c^2) \qquad \text{and} \qquad \mathbf{k}_\perp = \mathbf{k}'_\perp.$$

It is most convenient to compute the inverse. Using $\omega' = ck'$,

$$\cot\theta = \frac{k_\parallel}{k_\perp} = \frac{\gamma k'_\parallel + \gamma v c k'/c^2}{k'_\perp} = \gamma\cot\theta' + \frac{\gamma v k'}{ck'\sin\theta'} = \gamma\left(\frac{\cos\theta'}{\sin\theta'} + \frac{\beta}{\sin\theta'}\right).$$

Therefore,

$$\tan\theta = \frac{\sin\theta'}{\gamma(\cos\theta' + \beta)}.$$

22.21 Reflection from a Rotating Mirror

The frequency ω and wave vector $\mathbf{k}$ of a monochromatic plane wave form a four-vector. Therefore, if the inertial frame S' moves with velocity $\mathbf{v}$ with respect to the (lab) frame S,

$$\omega' = \gamma(\omega - \mathbf{v} \cdot \mathbf{k})$$

$$\mathbf{k}'_\parallel = \gamma(\mathbf{k}_\parallel - \mathbf{v}\omega/c^2)$$

$$\mathbf{k}'_\perp = \mathbf{k}_\perp.$$

Our strategy is to (i) transform to the mirror frame; (ii) apply Snell's law of reflection; (iii) transform back to the lab frame. However, $\mathbf{v}$ lies in the plane of the mirror so $\mathbf{k}_\perp$ is the normal component of the incident wave vector. Thus, the law of reflection changes the sign of $\mathbf{k}_\perp$. But the latter is otherwise *unaltered* by the above Lorentz transformation so step (3) merely reverses the effect of step (1). We conclude that $\mathbf{v} \neq 0$ has absolutely no effect on either the frequency or the wave vector of the reflected ray.

Source: P. Hickson, R Bhatia, and A. Iovino, *Astronomy and Astrophysics* **303**, L37 (1995).

22.22 Reflection from a Moving Mirror Revisited

Let the reflected wave be $\mathbf{E} = \hat{\mathbf{x}}E\exp[-i(kz+\omega t)]$. The electric field is entirely tangential to the mirror surface. The key observation is that the usual matching condition that the total electric field amplitude vanish at the mirror surface must be replaced by the moving interface matching condition from (2.50),

$$\hat{\mathbf{n}}_2 \times [\mathbf{E}_1 - \mathbf{E}_2] = (\hat{\mathbf{n}}_2 \cdot \mathbf{v})[\mathbf{B}_1 - \mathbf{B}_2].$$

Medium 1 is the vacuum space $z < 0$ where $c\mathbf{B}_0 = \hat{\mathbf{z}} \times \mathbf{E}_0$ and $c\mathbf{B} = -\hat{\mathbf{z}} \times \mathbf{E}$. Medium 2 is the mirror, so $\hat{\mathbf{n}}_2 = -\hat{\mathbf{z}}$ and $\mathbf{E}_2 = \mathbf{B}_2 = 0$. Substituting this information into the matching condition with $z = vt$ gives all terms proportional to the factor $\hat{\mathbf{z}} \times \hat{\mathbf{x}}$. What remains is

$$\begin{aligned} &-\{E_0\exp[i(k_0v-\omega_0)t] + E\exp[-i(kv+\omega)t]\} \\ &\qquad = -\frac{v}{c}\{E_0\exp[i(k_0v-\omega_0)t] - E\exp[-i(kv+\omega)t]\}\,. \end{aligned} \tag{1}$$

This expression must be true at all times. Therefore, because $\omega_0 = ck_0$ and $\omega = ck$,

$$\frac{\omega_0}{c}v - \omega_0 = -\frac{\omega}{c}v - \omega$$

or

$$\omega = \frac{1-\beta}{1+\beta}\omega_0.$$

The amplitudes must be equal on both sides of (1) also. Therefore,

$$-E_0 - E = -\beta E_0 + \beta E$$

or

$$E = \frac{\beta - 1}{\beta + 1}E_0.$$

These are the results derived in the text using special relativity.

22.23 Transformation of Phase and Group Velocity

(a) The notation is simplified if we treat vectors transverse to the boost velocity $\mathbf{v}$ as scalars. Thus, we write

$$\delta\omega = \mathbf{u}\cdot\delta\mathbf{k} = u_\parallel\delta k_\parallel + u_\perp\delta k_\perp \tag{1}$$

instead of

$$\delta\omega = \mathbf{u}\cdot\delta\mathbf{k} = u_{\|}\delta k_{\|} + \mathbf{u}_{\perp}\cdot\delta\mathbf{k}_{\perp}.$$

Our task, then, is to compute

$$u'_{\|} = \left.\frac{\delta\omega'}{\delta k'_{\|}}\right|_{k'_{\perp}} \qquad \text{and} \qquad u'_{\perp} = \left.\frac{\delta\omega'}{\delta k'_{\perp}}\right|_{k'_{\|}}.$$

The transformation law for the $(\omega, c\mathbf{k})$ four-vector is

$$\omega' = \gamma(\omega - vk_{\|})$$

$$k'_{\|} = \gamma(k_{\|} - v\omega/c^2)$$

$$k'_{\perp} = k_{\perp}.$$

This shows that

$$\delta\omega' = \gamma(\delta\omega - v\delta k_{\|}) = \gamma\delta k_{\|}(u_{\|} - v)$$

$$\delta k_{\|} = \gamma(\delta k_{\|} - v\delta\omega/c^2) = \gamma\delta k_{\|}(1 - vu_{\|}/c^2).$$

Dividing one of these by the other gives

$$u'_{\|} = \left.\frac{\delta\omega'}{\delta k'_{\|}}\right|_{k'_{\perp}} = \frac{u_{\|} - v}{1 - vu_{\|}/c^2}. \tag{2}$$

Turning to the perpendicular components, we use (1) to write

$$u'_{\perp} = \left.\frac{\delta\omega'}{\delta k'_{\perp}}\right|_{k'_{\|}} = \frac{\gamma(\delta\omega - v\delta k_{\|})}{\delta k_{\perp}} = \frac{\gamma[(u_{\|} - v)\delta k_{\|} + u_{\perp}\delta k_{\perp}]}{\delta k_{\perp}}. \tag{3}$$

This shows that we need to express $\delta k_{\|}$ in term of $\delta k_{\perp}$. We do this using the fact that $k'_{\|}$ is held constant during the differentiation in (3). Therefore,

$$0 = \delta k'_{\|} = \gamma(\delta k_{\|} - v\delta\omega/c^2),$$

so

$$\delta k_{\|} = \frac{v}{c^2}\delta\omega = \frac{v}{c^2}\left[u_{\|}\delta k_{\|} + u_{\perp}\delta k_{\perp}\right].$$

Hence,

$$\delta k_{\|} = \frac{(v/c^2)u_{\perp}}{1 - vu_{\|}/c^2}\delta k_{\perp}. \tag{4}$$

Substituting (4) into (3) and rationalizing the result gives

$$u'_{\perp} = \frac{u_{\perp}}{\gamma(1 - vu_{\|}/c^2)}. \tag{5}$$

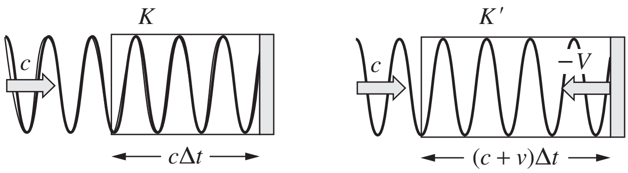
The transformation laws (2) and (5) are the same as the transformation laws for a particle velocity.

(b) When $\omega = ck$, the group velocity and phase velocity are the same. Hence the latter transforms exactly like the former.

Source: G. Barton, *Introduction to the Relativity Principle* (Wiley, New York, 1999).

22.24 The Invariance of $U_{\rm EM}/\omega$ Revisited

The left panel of the figure below shows the physical situation in the lab frame K. In a time Δt, the volume of wave field which enters the detector is $c\Delta tA$ because the velocity difference $v_{\rm wave} - v_{\rm detector} = c$. Therefore, the total energy absorbed is $U_{\rm EM} = u_{\rm EM}c\Delta tA$, where the energy density is $u_{\rm EM} = \epsilon_0 E^2$.



The right panel of the figure shows the physical situation in a frame K' which moves in the direction of the wave with speed v. Using the electric field transformation, $\mathbf{E}'_\perp = \gamma(\mathbf{E} + \boldsymbol{\beta} \times c\mathbf{B})_\perp$, the electric field and energy density in this frame are

$$E' = \gamma(E - \beta E) = \gamma(1 - \beta)E$$

and

$$u'_{\rm EM} = \epsilon_0 E'^2 = \epsilon_0 \gamma^2 (1-\beta)^2 E^2 = \gamma^2(1-\beta)^2 u_{\rm EM}.$$

The volume of wave field which enters the detector in K is $(c + v)\Delta tA$ because $v_{\rm wave} - v_{\rm detector} = c - (-v)$. Compared to the rest frame, the transverse dimensions are invariant ($A = A'$). The detector is moving in K', so the time interval is dilated ($\Delta t' = \gamma\Delta t$). Therefore, the energy absorbed by the detector in K' is

$$U'_{\rm EM} = u'_{\rm EM}(c+v)\Delta t' A' = \gamma^2(1-\beta)^2 u_{\rm EM} c(1+\beta)\gamma\Delta tA = \gamma(1-\beta)U_{\rm EM}.$$

Combining this with the frequency transformation formula,

$$\omega' = \gamma(\omega - \mathbf{v}\cdot\mathbf{k}) = \gamma(1-\beta)\omega,$$

produces the advertised result:

$$\frac{U_{\rm EM}}{\omega} = \frac{U'_{\rm EM}}{\omega'}.$$

Source: G. Margaritondo, *European Journal of Physics* **16**, 169 (1995).

22.25 Conservation of Energy-Momentum

The covariant equation of motion in question is is

$$\frac{dp_\mu}{d\tau} = qU_\nu F_{\mu\nu}.$$

Multiply the equation of motion by p_μ and sum over the repeated index to get

$$p_\mu \frac{dp_\mu}{d\tau} = qp_\mu U_\nu F_{\mu\nu}.$$

But $p_\nu = mU_\nu$, so

$$p_\mu \frac{dp_\mu}{d\tau} = (q/m)p_\mu p_\nu F_{\mu\nu}.$$

The right side of this equation vanishes because $F_{\mu\nu} = -F_{\mu\nu}$ is anti-symmetric. This proves the result because $\vec{p}\cdot\vec{p} = p_\mu p_\mu$ and

$$0 = p_\mu \frac{dp_\mu}{d\tau} = \frac{1}{2}\frac{d}{d\tau} p_\mu p_\mu.$$

This shows that $p_\mu p_\mu$ is a constant, independent of the proper time τ.

22.26 Gauge Freedom and Lorentz Invariance

(a) If Λ is a gauge function, $A'_\mu = A_\mu + \partial_\mu \Lambda$ produces the same electric and magnetic field as A_μ. To prove this, we recall that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field tensor and check that

$$F'_{\mu\nu} = \partial_\mu(A_\nu + \partial_\nu \Lambda) - \partial_\nu(A_\mu + \partial_\mu \Lambda) = F_{\mu\nu} + \partial_\mu\partial_\nu\Lambda - \partial_\nu\partial_\mu\Lambda = F_{\mu\nu}.$$

(b) The Lorenz gauge constraint is $\partial_\mu A_\mu = 0$. Imposing this constraint on A'_μ also means that

$$\partial_\mu A'_\mu = \partial_\mu(A_\mu + \partial_\mu\Lambda) = \partial_\mu A_\mu + \partial_\mu\partial_\mu\Lambda = \partial_\mu\partial_\mu\Lambda = 0.$$

In other words, Λ must satisfy the wave equation. Now, the space part of the change-of-gauge expression is

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda = \mathbf{A} + i\mathbf{k}\Lambda,$$

and its dot product with the wave vector is

$$\mathbf{k}\cdot\mathbf{A}' = \mathbf{k}\cdot\mathbf{A} + ik^2\Lambda.$$

Therefore, if a Lorentz boost causes $\mathbf{k}\cdot\mathbf{e} \neq 0$ (so $\mathbf{k}\cdot\mathbf{A} \neq 0$), we can restore $\mathbf{k}\cdot\mathbf{A}' = 0$ simply by choosing the gauge function as

$$\Lambda = \frac{i\mathbf{k}\cdot\mathbf{A}}{k^2}.$$

This choice of gauge function satisfies the Lorenz gauge constraint because, being proportional to $\mathbf{A}$, it is a plane wave.

Source: R.P. Feynman, *Quantum Electrodynamics* (W.A. Benjamin, New York, 1962).

22.27 Covariant Properties of a Plane Wave

The four-vector $k_\mu = (\mathbf{k}, i\omega/c)$ and

$$F_{\mu\nu} = \begin{bmatrix} 0 & B_z & -B_y & -iE_x/c \\ -B_z & 0 & B_x & -iE_y/c \\ B_y & -B_x & 0 & -iE_z/c \\ iE_x/c & iE_y/c & iE_z/c & 0 \end{bmatrix}$$

$$G_{\mu\nu} = \begin{bmatrix} 0 & -E_z/c & E_y/c & -iB_x \\ E_z/c & 0 & -E_x/c & -iB_y \\ -E_y/c & E_x/c & 0 & -iB_z \\ iB_x & iB_y & iB_z & 0 \end{bmatrix}.$$

By direct calculation, $G_{\mu\nu}F_{\mu\nu} = -4\mathbf{E}\cdot\mathbf{B}/c$. This is zero in one frame, and therefore zero in every frame. Similarly,

$$k_\mu F_{\mu i} = (\mathbf{B}\times\mathbf{k})_i - (\omega/c^2)E_i \qquad \text{and} \qquad k_\mu F_{\mu 4} = -(i/c)\mathbf{k}\cdot\mathbf{E}.$$

Because $\omega = c|\mathbf{k}|$, all of these are zero also. Therefore, $G_{\mu\nu}F_{\mu\nu} = 0$ and $k_\mu F_{\mu\nu} = 0$ in every inertial frame.

It remains only to prove that $\mathbf{k}\cdot\mathbf{B} = 0$ in every frame. To do this, square the leftmost equation above to get

$$\mathbf{E}\cdot\mathbf{E} = (c\mathbf{B}\times\hat{\mathbf{k}})\cdot(c\mathbf{B}\times\hat{\mathbf{k}}) = c^2\mathbf{B}^2 - c^2(\mathbf{B}\cdot\hat{\mathbf{k}})^2.$$

Therefore, the assertion will be proved if $\mathbf{E}^2 = c^2\mathbf{B}^2$ is a Lorentz invariant statement. This is true because the text shows that $F_{\mu\nu}F_{\mu\nu} = 2(\mathbf{B}^2 - \mathbf{E}^2/c^2)$ is a Lorentz scalar and this combination vanishes in one frame.

Source: Y.-K. Lim, *Problems and Solutions on Electromagnetism* (World Scientific, River Edge, NJ, 1993).

22.28 A Stress-Energy Invariant

The stress-energy tensor for the electromagnetic field is

$$\mathbf{\Theta} = \begin{bmatrix} -T_{xx} & -T_{xy} & -T_{xz} & icg_x \\ -T_{xy} & -T_{yy} & -T_{yz} & icg_y \\ -T_{zx} & -T_{yz} & -T_{zz} & icg_z \\ icg_x & icg_y & icg_z & -u_{\rm EM} \end{bmatrix},$$

where

$$\Theta_{ij} = -T_{ij} = -\epsilon_0 \left[E_i E_j + c^2 B_i B_j - \tfrac{1}{2}\delta_{ij}(E^2 + c^2 B^2)\right] = -\epsilon_0(E_i E_j + c^2 B_i B_j) + \delta_{ij} u_{\rm EM}.$$

Therefore,

$$\begin{aligned}
\theta_{\mu\nu}\theta_{\mu\nu} &= T_{ij}T_{ij} - 2c^2 g^2 + u_{\rm EM}^2 \\
&= \left[\epsilon_0\left(E_i E_j + c^2 B_i B_j - \delta_{ij} u_{\rm EM}\right)\right]\left[\epsilon_0\left(E_i E_j + c^2 B_i B_j - \delta_{ij} u_{\rm EM}\right)\right] \\
&\quad - 2c^2\epsilon_0^2 |\mathbf{E}\times\mathbf{B}|^2 + u_{\rm EM}^2 \\
&= \epsilon_0 E^2 E^2 + \epsilon_0^2 c^2 (\mathbf{E}\cdot\mathbf{B})^2 - \epsilon_0 E^2 u_{\rm EM} + \epsilon^2 c^2 (\mathbf{E}\cdot\mathbf{B})^2 + \epsilon_0^2 c^4 B^2 B^2 \\
&\quad - \epsilon_0 c^2 B^2 u_{\rm EM} - \epsilon_0 E^2 u_{\rm EM} - \epsilon_0 c^2 B^2 u_{\rm EM} + \delta_{ij}\delta_{ij} u_{\rm EM}^2 - 2\epsilon_0^2 c^2 E^2 B^2 \\
&\quad - 2\epsilon_0^2 c^2 (\mathbf{E}\cdot\mathbf{B})^2 + u_{\rm EM}^2 \\
&= \epsilon_0^2 (E^4 + c^4 B^4) - 2\epsilon_0 u_{\rm EM}(E^2 + c^2 B^2) + u_{\rm EM}^2 + \delta_{ii} u_{\rm EM}^2 - 2\epsilon_0^2 c^2 E^2 B^2 \\
&= \epsilon_0^2 (E^4 + c^4 B^4) - 4u_{\rm EM}^2 + u_{\rm EM}^2 + 3u_{\rm EM}^2 - 2\epsilon_0^2 c^2 E^2 B^2 \\
&= \epsilon_0^2 (E^4 + c^4 B^4) - 2\epsilon_0^2 c^2 E^2 B^2 \\
&= \epsilon_0^2 (E^2 - c^2 B^2)^2.
\end{aligned}$$

This invariant is zero for, say, a transverse electromagnetic wave where $|\mathbf{E}| = c|\mathbf{B}|$.

22.29 Diagonalize the Stress-Energy Tensor

(a) To simplify writing, we temporarily let $\epsilon_0 = \mu_0 = c = 1$ and restore these variables at the end. Under the stated conditions, the elements of the symmetric matrix $\Theta_{\mu\nu}$ are

$$\begin{aligned}
\Theta_{11} &= -\Theta_{22} = -\frac{1}{2}(E_1^2 - E_2^2 + B_1^2 - B_2^2) \\
\Theta_{33} &= -\Theta_{44} = \frac{1}{2}(E^2 + B^2) = \frac{1}{2}(E_1^2 + E_2^2 + B_1^2 + B_2^2) \\
\Theta_{12} &= -(E_1 E_2 + B_1 B_2) \\
\Theta_{13} &= \Theta_{23} = \Theta_{14} = \Theta_{24} = 0 \\
\Theta_{34} &= -i(E_2 B_1 - E_1 B_2).
\end{aligned}$$

To find the eigenvalues λ of $\Theta_{\mu\nu}$, we solve the determinant equation

$$\begin{vmatrix} \Theta_{11}-\lambda & \Theta_{12} & 0 & 0 \\ \Theta_{12} & \Theta_{22}-\lambda & 0 & 0 \\ 0 & 0 & \Theta_{33}-\lambda & \Theta_{34} \\ 0 & 0 & \Theta_{4}3 & \Theta_{44}-\lambda \end{vmatrix} = 0.$$

The determinant is block diagonal, so it is sufficient to require

$$(\Theta_{11}-\lambda_1)(\Theta_{22}-\lambda_1)-\Theta_{12}^2 = 0 = (\Theta_{11}-\lambda_1)(\Theta_{11}+\lambda_1)+\Theta_{12}^2$$

and

$$(\Theta_{33}-\lambda_2)(\Theta_{44}-\lambda_2)-\Theta_{34}^2 = 0 = (\Theta_{33}-\lambda_2)(\Theta_{33}+\lambda_2)+\Theta_{34}^2.$$

Therefore,

$$\lambda_1^2 = \Theta_{11}^2+\Theta_{12}^2 = \frac{1}{4}\left[(E_1^2-E_2^2+B_1^2-B_2^2)^2+4(E_1E_1+B_2B_2)^2\right]$$

and

$$\lambda_2^2 = \Theta_{33}^2+\Theta_{34}^2 = \frac{1}{4}\left[(E^2+B^2)^2-4(E_1B_2-E_2B_1)^2\right].$$

However, direct calculation confirms that $\lambda_1^2=\lambda_2^2=\lambda^2$, where

$$\lambda^2 = \frac{1}{4}\left[(E^2-B^2)^2+4(E_1B_1+E_2B_2)^2\right].$$

We now recall the two Lorentz invariants associated with the stress tensor and its dual:

$$F_{\mu\nu}F_{\mu\nu} = 2(B^2-E^2/c^2) \qquad \text{and} \qquad F_{\mu\nu}G_{\mu\nu} = -4\mathbf{E}\cdot\mathbf{B}/c = -4(E_1B_1+E_2B_2)/c.$$

Hence, putting back the dimensional factors so λ has the correct dimensions,

$$\lambda = \pm\frac{1}{4\mu_0}\sqrt{(F_{\mu\nu}F_{\mu\nu})^2+(F_{\mu\nu}G_{\mu\nu})^2}.$$

On the other hand,

$$\begin{aligned}(FF)^2+(FG)^2 &= 4(B^4+E^4-2E^2B^2)+16E^2B^2\cos^2\theta \\ &= 4(B^2+E^4)-8E^2B^2+8E^2B^2(1+\cos 2\theta) \\ &= 4[B^4+E^4+2E^2B^2\cos(2\theta)].\end{aligned}$$

Therefore, putting in the dimensional factors again,

$$\lambda = \pm\frac{1}{2}\epsilon_0\sqrt{E^4+c^4B^4+2E^2B^2c^2\cos(2\theta)}.$$

(b) Because $\cos(2\theta) \le 1$, an immediate inequality is

$$|\lambda| \le \frac{1}{2}\epsilon_0(E^2 + c^2B^2) = u_{\rm EM}.$$

Therefore, either $u_{\rm EM} = 0$ (if the fields vanish) or $u_{\rm EM}$ is bounded from below by λ. This is true in every inertial frame because λ is a Lorentz scalar.

Source: J.L. Synge, *Relativity: The Special Theory* (North-Holland, Amsterdam, 1956).

22.30 Stress-Energy Tensor for Matter

(a) Using the relativistic identity $c^2\vec{p}_k = \mathcal{E}_k(d\vec{r}_k/dt)$, the given stress-energy tensor takes the manifestly symmetric form

$$\Theta^{\rm mat}_{\alpha\beta}(\mathbf{s},t) = c^2\sum_k \frac{p_{k,\alpha}p_{k,\beta}}{\mathcal{E}_k}\delta[\mathbf{s}-\mathbf{r}_k(t)].$$

(b) Following the hint, we compute the space divergence:

$$\begin{aligned}
\frac{\partial\Theta^{\rm mat}_{\alpha i}(\mathbf{s},t)}{\partial s_i} &= \sum_k p_{k,\alpha}\frac{dr_{k,i}}{dt}\frac{\partial}{\partial s_i}\delta[\mathbf{s}-\mathbf{r}_k(t)] \\
&= -\sum_k p_{k,\alpha}\frac{dr_{k,i}}{dt}\frac{\partial}{\partial r_{k,i}}\delta[\mathbf{s}-\mathbf{r}_k(t)] \\
&= -\sum_k p_{k,\alpha}\frac{\partial}{\partial t}\delta[\mathbf{s}-\mathbf{r}_k(t)] \\
&= -\left[\sum_k \frac{\partial}{\partial t}\{p_{k,\alpha}\delta[\mathbf{s}-\mathbf{r}_k(t)]\} - \sum_k \delta[\mathbf{s}-\mathbf{r}_k(t)]\frac{dp_{k,\alpha}}{dt}\right] \\
&= -\frac{\partial\Theta^{\rm mat}_{\alpha 4}}{\partial(ict)} + \sum_k \frac{dp_{k,\alpha}}{dt}\,\delta[\mathbf{s}-\mathbf{r}_k(t)].
\end{aligned}$$

Therefore, moving the derivative with respect to $s_4 = ict$ to the left-hand side,

$$\frac{\partial\Theta^{\rm mat}_{\alpha\beta}(\mathbf{s},t)}{\partial s_\beta} = \sum_k \frac{dp_{k,\alpha}}{dt}\,\delta[\mathbf{s}-\mathbf{r}_k(t)].$$

Now, the equation of motion of a particle with charge q_k in an electromagnetic field is

$$\frac{dp_{k,\alpha}}{d\tau} = q_k U_{k,\nu}F_{\alpha\nu} = q_q\frac{dr_{k,\nu}}{d\tau}F_{\alpha\nu}.$$

Therefore,

$$\frac{dp_{k,\alpha}}{dt} = q_k \frac{dr_{k,\nu}}{dt} F_{\alpha\nu}.$$

Substituting this into the four-divergence just above gives

$$\partial_\beta \Theta^{\rm mat}_{\alpha\beta}(\mathbf{s}, t) = \sum_k q_k \frac{dr_{k,\nu}}{dt} \delta[\mathbf{s} - \mathbf{r}_k(t)] F_{\alpha,\nu} = j_\nu F_{\alpha\nu}.$$

The last equality is true because the four-current density is (see Example 22.2)

$$\vec{j}(\mathbf{s}, t) = \sum_k q_k \frac{d\vec{r}_k}{dt} \delta[\mathbf{s} - \mathbf{r}_k(t)].$$

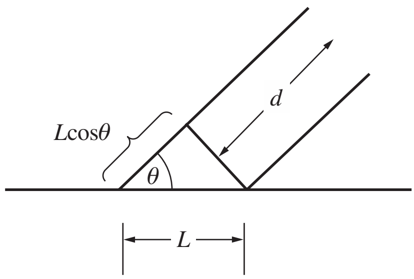
Source: S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972).

Chapter 23: Fields from Moving Charges

23.1 Smith-Purcell and Undulator Radiation

(a) The electric field of the passing electron induces electrons at the surface of each metal strip to accelerate and hence radiate. The bursts of radiation from successive strips are identical except they are separated by the time interval L/v. Therefore, to get constructive interference, we need the path difference between waves received from successive periods to be an integer multiple of the wavelength:

$$\Delta = (d + cL/v) - (d + L\cos\theta) = L(c/v - \cos\theta) = n\lambda.$$



(b) Each magnet induces a force which causes the passing electron to accelerate (in the transverse direction) and hence radiate. The bursts of radiation produced as the charge passes by successive magnets are identical except they are separated by the time interval L/v. The argument for the wavelengths emitted is then the same as part (a).

Source: S.J. Smith and E.M. Purcell, Physical Review **92**, 1069 (1953).

23.2 Gauss' Law for a Moving Charge

Let the charge move up the z-axis with speed $v = \beta c$. Then, if θ is the usual polar angle,

$$\mathbf{E}(\mathbf{r}) = \hat{\mathbf{r}}\frac{q}{4\pi\epsilon_0 r^2}\frac{(1-\beta^2)}{(1-\beta^2\sin^2\theta)^{3/2}}.$$

Choosing as a Gaussian surface S a sphere centered at q, the azimuthal part of the electric flux integral can be done immediately:

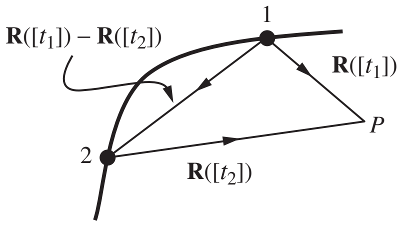
$$\Phi_E = \int_S d\mathbf{S}\cdot\mathbf{E} = \frac{q(1-\beta^2)}{2\epsilon_0}\int_0^\pi d\theta\frac{\sin\theta}{(1-\beta^2\sin^2\theta)^{3/2}} = \frac{q(1-\beta^2)}{2\epsilon_0}\int_0^\pi d\theta\frac{\sin\theta}{(1-\beta^2+\beta^2\cos^2\theta)^{3/2}}.$$

Changing variables to $x = \beta\cos\theta$ gives the desired result,

$$\Phi_E = \frac{q(1-\beta^2)}{2\epsilon_0}\frac{1}{\beta}\int_{-\beta}^{\beta}\frac{dx}{(1-\beta^2+x^2)^{3/2}} = \frac{q(1-\beta^2)}{2\epsilon_0}\frac{1}{\beta}\left.\frac{x}{(1-\beta^2)\sqrt{x+1-\beta^2}}\right|_{-\beta}^{\beta} = \frac{q}{\epsilon_0}.$$

23.3 The Retarded Time

Let the observation point be P in the diagram below and suppose there are two solutions, $[t_1]$ and $[t_2]$. The diagram shows the corresponding vectors $\mathbf{R}([t_1]) = \mathbf{r} - \mathbf{r}_0([t_1])$ and $\mathbf{R}([t_2]) = \mathbf{r} = \mathbf{r}_0([t_2])$.



By assumption, $[t_1] = t - R([t_1])/c$ and $[t_2] = t - R([t_2])/c$. Therefore,

$$c|[t_2] - [t_1]| = |R([t_1]) - R([t_2])|.$$

On the other hand, it is a consequence of the triangle inequality that

$$|R([t_2]) - R([t_1])| \leq |\mathbf{R}([t_2]) - \mathbf{R}([t_1])|.$$

Therefore,

$$c|[t_2] - [t_1]| \leq |\mathbf{R}([t_2]) - \mathbf{R}([t_1])| \leq S, \tag{1}$$

where S is the distance traveled by the particle between the two points labeled 1 and 2 in the diagram. But (1) is impossible to satisfy if the particle speed is strictly less that c. Therefore, the original assumption that $[t] = t - R([t])$ has two solutions cannot be correct.

23.4 The Direction of the Velocity Field

The velocity field is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{n}} - \boldsymbol{\beta}}{\gamma^2 g^3 R^2}\right]_{\text{ret}}.$$

The direction of this field is the same as the direction of the vector

$$[\mathbf{R} - \boldsymbol{\beta}R]_{\text{ret}} = \mathbf{r} - \mathbf{r}_0(t_{\text{ret}}) - \boldsymbol{\beta}|\mathbf{r} - \mathbf{r}_0(t_{\text{ret}})|.$$

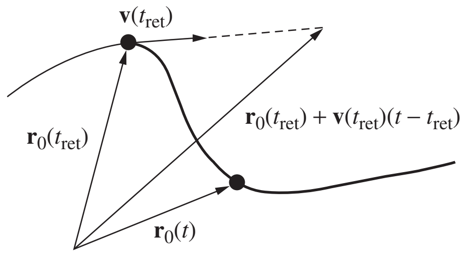
Let the observer sit at the origin ($\mathbf{r} = 0$). Then, because the retarded time is defined by the relation

$$t_{\text{ret}} + r_0(t_{\text{ret}})/c - t = 0,$$

the velocity electric field,

$$\begin{aligned}\mathbf{E} \propto -\mathbf{r}_0(t_{\text{ret}}) - \frac{\mathbf{v}(t_{\text{ret}})}{c} r_0(t_{\text{ret}}) &= -\mathbf{r}_0(t_{\text{ret}}) - \mathbf{v}(t_{\text{ret}})(t - t_{\text{ret}}) \\ &= -[\mathbf{r}_0(t_{\text{ret}}) + \mathbf{v}(t_{\text{ret}})(t - t_{\text{ret}})] = -\mathbf{r}_A.\end{aligned}$$

The diagram below shows that this proves the assertion.



Source: W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd edition (Addison-Wesley, Reading, MA, 1962).

23.5 Inverting the Retarded Field

(a) We drop the subscript "ret" for convenience. The equation given implies that

$$c\hat{\mathbf{n}} \times \mathbf{B} = \hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \mathbf{E}) = \hat{\mathbf{n}}(\hat{\mathbf{n}} \cdot \mathbf{E}) - \mathbf{E}.$$

Therefore,

$$c\mathbf{E} \cdot (\hat{\mathbf{n}} \times \mathbf{B}) = (\hat{\mathbf{n}} \cdot \mathbf{E})^2 - E^2.$$

On the other hand, $c\mathbf{B} \cdot (\mathbf{E} \times \hat{\mathbf{n}}) = -cB^2$. Therefore, $(\hat{\mathbf{n}} \cdot \mathbf{E})^2 = E^2 - c^2B^2$. This gives $\hat{\mathbf{n}} \cdot \mathbf{E} = \pm\sqrt{E^2 - c^2B^2}$. To get the sign, we recall the Liénard-Wiechert electric field,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{(\hat{\mathbf{n}} - \boldsymbol{\beta})(1-\beta^2)}{R^2(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^3} + \frac{\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \mathbf{a}]}{c^2R(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^3}\right]_{\text{ret}}.$$

By direction computation,

$$\hat{\mathbf{n}} \cdot \mathbf{E} = \frac{q}{4\pi\epsilon_0 R^2}\frac{1-\beta^2}{(1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta})^2}.$$

This shows that the dot product has same sign as the charge itself. Hence,

$$\hat{\mathbf{n}}_{\text{ret}} \cdot \mathbf{E} = \frac{q}{|q|}\sqrt{E^2 - c^2B^2}.$$

(b) Using the formula given in part (a),

$$c\mathbf{E} \times \mathbf{B} = \mathbf{E} \times (\hat{\mathbf{n}} \times \mathbf{E}) = \hat{\mathbf{n}}E^2 - \mathbf{E}(\mathbf{E} \cdot \hat{\mathbf{n}}).$$

Therefore,

$$\hat{\mathbf{n}} = \frac{\mathbf{E} \times \mathbf{B} + \mathbf{E}(q/|q|)\sqrt{E^2 - c^2B^2}}{E^2}. \tag{1}$$

(c) The Heaviside-Feynman electric field is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\rm ret} + \frac{R_{\rm ret}}{c}\frac{d}{dt}\left[\frac{\hat{\mathbf{n}}}{R^2}\right]_{\rm ret} + \frac{1}{c^2}\frac{d^2\hat{\mathbf{n}}_{\rm ret}}{dt^2}\right\}$$

or

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{n}}}{R^2} + \frac{R}{c}\frac{d}{dt}\frac{1}{R^2} + \frac{1}{cR}\frac{d\hat{\mathbf{n}}}{dt} + \frac{1}{c^2}\frac{d^2\hat{\mathbf{n}}}{dt^2}\right].$$

Therefore,

$$c\mathbf{B} = \hat{\mathbf{n}} \times \mathbf{E} = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{n}}}{cR} \times \frac{d\hat{\mathbf{n}}}{dt} + \frac{1}{c^2}\frac{d^2\hat{\mathbf{n}}}{dt^2}\right].$$

The dot product of this with $\mathbf{B}$ is

$$cB^2 = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{cR}\mathbf{B}\cdot(\hat{\mathbf{n}} \times \dot{\mathbf{n}}) + \frac{1}{c^2}\mathbf{B}\cdot(\hat{\mathbf{n}} \times \ddot{\mathbf{n}})\right],$$

so we can solve for R to get

$$R = \frac{q\,\mathbf{B}\cdot(\hat{\mathbf{n}} \times \dot{\mathbf{n}})}{4\pi\epsilon_0 c^2 B^2 - (q/c)\mathbf{B}\cdot(\hat{\mathbf{n}} \cdot \ddot{\mathbf{n}})}. \tag{2}$$

Restoring the "ret" subscript to R and $\hat{\mathbf{n}}$, we conclude from (1) and (2) that

$$\mathbf{R}_{\rm ret} = R_{\rm ret}\hat{\mathbf{n}}_{\rm ret} = \frac{q\,\hat{\mathbf{n}}_{\rm ret}\mathbf{B}\cdot(\hat{\mathbf{n}} \times \dot{\mathbf{n}})_{\rm ret}}{4\pi\epsilon_0 c^2 B^2 - (q/c)\mathbf{B}\cdot(\hat{\mathbf{n}} \cdot \ddot{\mathbf{n}})_{\rm ret}}.$$

(d) The retarded time and present time are connected by

$$t = t_{\rm ret} + R_{\rm ret}/c. \tag{3}$$

But (2) gives $R_{\rm ret}$ in terms of present-time quantities. Therefore, (3) is an explicit formula for $t_{\rm ret}$ in terms of present-time quantities.

Source: V.Ya. Epp and T.G. Mitrofanova, *Physics Letters A* **330**, 7 (2004).

23.6 The Covariant Liénard-Wiechert Field

(a) Begin with the hint and recall from (23.17) that

$$R_\nu = (x - x_0, y - y_0, z - z_0, ic(t - t_{\rm ret})) \qquad \text{and} \qquad R_\sigma R_\sigma = 0, \tag{1}$$

where $\mathbf{r}_0 = (x_0, y_0, z_0)$ is the position of the charge q at the retarded time $t_{\rm ret}$. By direct computation,

$$\begin{aligned}\frac{\partial}{\partial r_\mu}(R_\sigma R_\sigma) &= 2R_\sigma\frac{\partial R_\sigma}{\partial r_m u} = 2R_\sigma\frac{\partial}{\partial r_\mu}(r_\sigma - r_{0,\sigma}) = 2R_\sigma\left(\delta_{\sigma\mu} - \frac{\partial r_{0,\sigma}}{\partial \tau}\frac{\partial \tau}{\partial r_\mu}\right)\\ &= 2R_\sigma\left(\delta_{\sigma\mu} - U_\sigma\frac{\partial \tau}{\partial r_\mu}\right).\end{aligned}$$

The foregoing is zero from the right side of (1). Therefore, as advertised,

$$R_\mu = R_\sigma U_\sigma \frac{\partial \tau}{\partial r_\mu} \quad \Rightarrow \quad \frac{\partial \tau}{\partial r_\mu} = \frac{R_\mu}{R_\sigma U_\sigma}. \tag{2}$$

The Liénard-Wiechert potential is

$$A_\nu = -\frac{q}{4\pi\epsilon_0} \frac{U_\nu}{cR_\sigma U_\sigma}.$$

This formula, and the hint result (2), establish that

$$\begin{aligned} F_{\mu\nu} &= \frac{\partial A_\nu}{\partial r_\mu} - \frac{\partial A_\mu}{\partial r_\nu} = \frac{\partial A_\nu}{\partial \tau}\frac{\partial \tau}{\partial r_\mu} - \frac{\partial A_\mu}{\partial \tau}\frac{\partial \tau}{\partial r_\nu} \\ &= -\frac{q}{4\pi\epsilon_0 c}\left[\frac{\partial}{\partial \tau}\left(\frac{U_\nu}{R_\sigma U_\sigma}\right)\frac{R_\mu}{R_\lambda U_\lambda} - \frac{\partial}{\partial \tau}\left(\frac{U_\mu}{R_\sigma U_\sigma}\right)\frac{R_\nu}{R_\lambda U_\lambda}\right]. \end{aligned} \tag{3}$$

Consider the first term in the square brackets in (3). Because $\partial_\tau R_\lambda = -U_\lambda$ and $U_\sigma U_\sigma = -c^2$, this is

$$\begin{aligned} \frac{\partial}{\partial \tau}\left(\frac{U_\nu}{R_\sigma U_\sigma}\right)\frac{R_\mu}{R_\lambda U_\lambda} &= \frac{1}{(R_\sigma U_\sigma)^2}\left[\frac{\partial U_\nu}{\partial \tau}R_\lambda U_\lambda - U_\nu \frac{\partial}{\partial \tau}(R_\lambda U_\lambda)\right]\frac{R_\mu}{R_\alpha U_\alpha} \\ &= \frac{1}{(R_\sigma U_\sigma)^2}\left[\frac{\partial U_\nu}{\partial \tau}R_\lambda U_\lambda + U_\nu(U_\lambda U_\lambda) - U_\nu R_\lambda \frac{\partial U_\lambda}{\partial \tau}\right]\frac{R_\mu}{R_\alpha U_\alpha} \\ &= \frac{R_\mu}{(R_\sigma U_\sigma)^2}\frac{\partial U_\nu}{\partial \tau} - \frac{c^2 U_\nu R_\mu}{(R_\sigma U_\sigma)^3} - \frac{R_\mu U_\nu}{(R_\sigma U_\sigma)^3}R_\lambda \frac{\partial U_\lambda}{\partial \tau}. \end{aligned}$$

The second term in square brackets in (3) is the same as the foregoing with μ and ν exchanged. Therefore, we conclude that

$$\begin{aligned} F_{\mu\nu} = -\frac{q}{4\pi\epsilon_0 c}&\left[\frac{1}{(R_\sigma U_\sigma)^2}\left(R_\mu \frac{\partial U_\nu}{\partial \tau} - R_\nu \frac{\partial U_\mu}{\partial \tau}\right)\right. \\ &\left. - \frac{1}{(R_\sigma U_\sigma)^3}(R_\mu U_\nu - R_\nu U_\mu)\left(c^2 + R_\lambda \frac{\partial U_\lambda}{\partial \tau}\right)\right]. \end{aligned} \tag{4}$$

(b) Each term in (4) is a second-rank tensor. Therefore, the sum of the terms which do *not* depend on $\partial U_\nu/\partial\tau$ are the covariant velocity field. The sum of the terms which *do* depend on this four-acceleration are the covariant acceleration field.

(c) The component of the electric field are $E_k = -icF_{4k}$. If $(k\ell m)$ is a cyclic permutation, the components of the magnetic field are $B_m = F_{k\ell}$. Now, from (22.57) the four-acceleration $\vec{\mathcal{A}} = d\vec{U}/d\tau$ has space and time components

$$\boldsymbol{\mathcal{A}} = \frac{\mathbf{a}}{1-u^2/c^2} + \frac{\mathbf{u}(\mathbf{u}\cdot\mathbf{a})/c^2}{(1-u^2/c^2)^2} \qquad \text{and} \qquad \mathcal{A}_4 = \frac{i(\mathbf{u}\cdot\mathbf{a})/c}{(1-u^2/c^2)^2}.$$

In the rest frame of q, this reduces to $\vec{\mathcal{A}} = (\mathbf{a}, 0)$. Similarly, $\vec{U} = (0, ic)$ and $\vec{R} = (\mathbf{R}, iR)$ so $\vec{R}\cdot\vec{U} = -cR$ and $\vec{R}\cdot\vec{\mathcal{A}} = \mathbf{a}\cdot\mathbf{R}$. Therefore, because $\mathcal{A}_4 = U_k = 0$, we find

$$E_k = -icF_{4k} = -\frac{q}{4\pi\epsilon_0}\frac{a_k}{c^2 R} + \frac{q}{4\pi\epsilon_0}\left(\frac{R_k}{c^2 R^3}\right)(c^2 + \mathbf{a}\cdot\mathbf{R}).$$

Therefore, as expected when $\boldsymbol{\beta} = 0$,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{R}}{r^3} + \frac{q}{4\pi\epsilon_0 R}[\hat{\mathbf{R}} \times (\hat{\mathbf{R}} \times \mathbf{a})].$$

Similarly,

$$B_m = F_{k\ell} = -\frac{q}{4\pi\epsilon_0 c}\frac{R_k a_\ell - R_\ell a_k}{c^2 R^2}.$$

Therefore,

$$\mathbf{B} = \frac{\mu_0 q}{4\pi}\frac{\mathbf{a}\times\hat{\mathbf{R}}}{cR}.$$

Source: W. Pauli, *Theory of Relativity* (Dover, New York, 1958).

23.7 N Charges Moving in a Circle I

For a single charged particle, the Liénard-Wiechert electric field is

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{(\hat{\mathbf{n}} - \boldsymbol{\beta})}{\gamma^2(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3 R^2} + \frac{\hat{\mathbf{n}} \times \left\{(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right\}}{c(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3 R}\right]_{\rm ret},$$

where $\mathbf{R}(t) = R(t)\hat{\mathbf{n}} = \mathbf{r} - \mathbf{r}_0(t)$, $\boldsymbol{\beta} = \dot{\mathbf{r}}_0(t)/c$, and $\gamma^2 = 1/(1-\beta^2)$. On the symmetry axis, the position vector of the $k^{\rm th}$ particle is

$$\mathbf{R}_k(t) = -a\cos(\omega t + \phi_k)\hat{\mathbf{x}} - a\sin(\omega t + \phi_k)\hat{\mathbf{y}} + z\,\hat{\mathbf{z}},$$

where $v = a\omega$ and $\phi_k = 2\pi k/N$. From the definitions above, $R_k = \sqrt{a^2+z^2} = R$ is the *same* for all the particles and

$$\hat{\mathbf{n}}_k = -\frac{a}{R}[\cos(\omega t + \phi_k)\hat{\mathbf{x}} + \sin(\omega t + \phi_k)\hat{\mathbf{y}}] + \frac{z}{R}\hat{\mathbf{z}}$$

$$\boldsymbol{\beta}_k = -\dot{\mathbf{R}}_k/c = \beta[-\sin(\omega t + \phi_k)\hat{\mathbf{x}} + \cos(\omega t + \phi_k)\hat{\mathbf{y}}]$$

$$\dot{\boldsymbol{\beta}}_k = -\omega\beta[\cos(\omega t + \phi_k)\hat{\mathbf{x}} + \sin(\omega t + \phi_k)\hat{\mathbf{y}}].$$

We have $\hat{\mathbf{n}}_k \cdot \boldsymbol{\beta}_k = 0$ so $\hat{\mathbf{n}}_k \times \left\{(\hat{\mathbf{n}}_k - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}\right\} = (\hat{\mathbf{n}}_k - \boldsymbol{\beta})(\hat{\mathbf{n}}_k \cdot \dot{\boldsymbol{\beta}}) - \dot{\boldsymbol{\beta}}$ and $\hat{\mathbf{n}}_k \cdot \dot{\boldsymbol{\beta}}_k = c\beta^2/R$. Since $\gamma^{-2} + \beta^2 = 1$, this reduces the electric field to

$$\mathbf{E}(z,t) = \frac{q}{4\pi\epsilon_0}\sum_{k=1}^{N}\left[\frac{(\hat{\mathbf{n}}_k - \boldsymbol{\beta}_k)}{R^2}\left\{\frac{1}{\gamma^2} + \beta^2\right\} - \frac{\dot{\boldsymbol{\beta}}_k}{cR}\right]_{\rm ret} = \frac{q}{4\pi\epsilon_0}\sum_{k=1}^{N}\left[\frac{(\hat{\mathbf{n}}_k - \boldsymbol{\beta}_k)}{R^2} - \frac{\dot{\boldsymbol{\beta}}_k}{cR}\right]_{\rm ret}. \tag{1}$$

The x- and y-components of this electric field vanish because, when $N > 1$,

$$\sum_{k=1}^{N} \cos(\omega t + \phi_k) = \sum_{k=1}^{N} \sin(\omega t + \phi_k) = 0. \tag{2}$$

The proof that both sums in (2) vanish is straightforward:

$$\sum_{k=1}^{N} e^{i(\omega t+\phi_k)} = e^{i(\omega t+2\pi/N)} \sum_{k=1}^{N} e^{i(k-1)2\pi/N} = e^{i(\omega t+2\pi/N)} \frac{1-e^{i2\pi}}{1-e^{i2\pi/N}} = 0.$$

The only term in (1) which survives is the z-component of $\hat{\mathbf{n}}_k$. Hence, the electric field on the symmetry axis is

$$\mathbf{E}(z,t) = \hat{z}\frac{q}{4\pi\epsilon_0}\sum_{k=1}^{N}\left[\frac{z}{R^3}\right]_{\rm ret} = \frac{qNz}{4\pi\epsilon_0 R^3}\hat{\mathbf{z}} \qquad (N > 1).$$

23.8 Energy Loss from Gyro-Radiation

The relativistic Larmor formula is

$$P = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c}\gamma^6\left[\dot{\beta}^2 - (\boldsymbol{\beta}\times\dot{\boldsymbol{\beta}})^2\right]_{\rm ret}.$$

Using the Lorentz force, the relativistic equation of motion for the charge is

$$\frac{d}{dt}(\gamma\boldsymbol{\beta}) = \frac{q}{m}\boldsymbol{\beta}\times\mathbf{B} = \dot{\gamma}\boldsymbol{\beta} + \gamma\dot{\boldsymbol{\beta}}. \tag{1}$$

Taking the dot product of $\boldsymbol{\beta}$ with the rightmost equation in (1) gives

$$\dot{\gamma}\beta^2 + \gamma\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}} = \frac{q}{m}\boldsymbol{\beta}\cdot(\boldsymbol{\beta}\times\mathbf{B}) = 0. \tag{2}$$

On the other hand, because $\gamma = (1-\boldsymbol{\beta}\cdot\boldsymbol{\beta})^{-1/2}$, the time derivative is

$$\dot{\gamma} = \gamma^3\boldsymbol{\beta}\cdot\dot{\boldsymbol{\beta}}. \tag{3}$$

Comparing (3) to (2) shows that $\dot{\gamma} = 0$ and we conclude from (1) that

$$\dot{\boldsymbol{\beta}} = \frac{q}{m\gamma}\boldsymbol{\beta}\times\mathbf{B}.$$

This is the information we need to evaluate the Larmor formula because

$$\dot{\boldsymbol{\beta}}^2 = \left(\frac{q}{m\gamma}\right)^2(\boldsymbol{\beta}\times\mathbf{B})\cdot(\boldsymbol{\beta}\times\mathbf{B}) = \left(\frac{q}{m\gamma}\right)^2\left[\beta^2B^2 - (\boldsymbol{\beta}\cdot\mathbf{B})^2\right] = \left(\frac{q}{m\gamma}\right)^2\beta^2B^2$$

and

$$\boldsymbol{\beta} \times \dot{\boldsymbol{\beta}} = \frac{q}{m\gamma}\boldsymbol{\beta} \times (\boldsymbol{\beta} \times \mathbf{B}) = \frac{q}{m\gamma}\left[\boldsymbol{\beta}(\boldsymbol{\beta} \cdot \mathbf{B}) - \beta^2 \mathbf{B}\right] = -\frac{q}{m\gamma}\beta^2 \mathbf{B}.$$

Using these, the power radiated is

$$P = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c}\gamma^6 \left(\frac{q}{m\gamma}\beta B\right)^2 (1 - \beta^2) = \frac{1}{4\pi\epsilon_0}\frac{2q^4\beta^2\gamma^2 B^2}{3m^2 c}.$$

23.9 The Path of Minimum Radiation

Larmor's formula for non-relativistic power radiated is

$$\frac{dU}{dt} = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0}\frac{a(t)^2}{c^3}.$$

(a) Since the particle begins at rest, its displacement under constant acceleration a is $s(t) = \frac{1}{2}at^2$. Therefore, because the particle must return to rest by uniform deceleration $-a$,

$$\frac{1}{2}d = \frac{1}{2}a\left(\frac{T}{2}\right)^2.$$

Therefore, $a = 4d/T^2$, and the total amount of energy radiated is

$$U = \int_0^T dt\, \frac{dU}{dt} = \int_0^T dt\, \frac{2}{3}\frac{q^2}{4\pi\epsilon_0 c^3}\frac{16d^2}{T^4} = \frac{8}{3}\frac{q^2}{\pi\epsilon_0 c^3}\frac{d^2}{T^3}.$$

(b) The problem is to minimize the functional

$$I[x(t)] = \int_0^T dt\, \ddot{x}^2(t),$$

subject to the constraints that $\dot{x}(0) = \dot{x}(T) = 0$ and $x(0) = 0$ with $x(T) = d$. In classical mechanics, we get Lagrange's equation when we minimize a functional like (1) when the integrand is $L(x, \dot{x})$. When the integrand is $G(x, \dot{x}, \ddot{x})$, a straightforward generalization gives

$$\frac{\partial G}{\partial x} - \frac{d}{dt}\frac{\partial G}{\partial \dot{x}} - \frac{d^2}{dt^2}\frac{\partial G}{\partial \ddot{x}} = 0.$$

For our problem, $G = \ddot{x}^2$, so (2) gives

$$\frac{d^4 x}{dt^4} = 0.$$

Consequently, the equation of motion can contain terms no higher than cubic in time:

$$x(t) = A + Bt + Ct^2 + Dt^3.$$

We get $A = B = 0$ from the boundary conditions $x(0) = \dot{x}(0) = 0$. Imposing $\dot{x}(T) = 0$ and $x(T) = d$ leads to

$$x(t) = \frac{3d}{T^2}t^2 - \frac{2d}{T^3}t^3.$$

The corresponding acceleration function is

$$a(t) = \frac{6d}{T^2} - \frac{12d}{T^3}t = \frac{6d}{T^2}\left(1 - \frac{2t}{T}\right).$$

Source: Prof. K.T. McDonald, http://cosmology.princeton.edu/∼mcdonald/examples/

23.10 Radiation Energy Loss from Coulomb Repulsion

The problem is one-dimensional (in the radial coordinate) and the non-relativistic equation of motion is

$$m\dot{v} = -\frac{dV}{dr} = \frac{Ze^2}{r^2}.$$

From Larmor's formula, the power radiated is

$$P = \frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}\dot{v}^2 = \frac{1}{4\pi\epsilon_0}\frac{2e^6Z^2}{3c^3m^2r^4}.$$

Hence, the total energy radiated is

$$\Delta E = \int_{-\infty}^{\infty} dt\, P = \frac{1}{4\pi\epsilon_0}\frac{2e^6Z^2}{3c^3m^2}\int_{-\infty}^{\infty}\frac{dt}{r^4}.$$

In classical mechanics, we do integrals of this kind using $v = dr/dt$ and knowledge of the velocity function $v(r)$. The latter we get from conservation of energy:

$$\frac{1}{2}mv_0^2 = \frac{1}{2}mv(r)^2 + \frac{Ze^2}{r}.$$

Therefore,

$$v(r) = v_0\sqrt{1 - \frac{s}{r}} \qquad \text{where} \qquad s = \frac{2Ze^2}{mv_0^2}$$

and the particle travels from $r = +\infty$ to $r = s$ and then back to $r = +\infty$. Accordingly,

$$\begin{aligned}\Delta E &= \frac{1}{4\pi\epsilon_0}\frac{4e^6Z^2}{3c^3m^2}\int_s^{\infty}\frac{dr}{vr^4} = \frac{1}{4\pi\epsilon_0}\frac{4e^6Z^2}{3c^3m^2v_0}\int_s^{\infty}\frac{dr}{r^4\sqrt{1 - s/r}}\\ &= \frac{1}{4\pi\epsilon_0}\frac{4e^6Z^2}{3c^3m^2v_0s^3}\int_1^{\infty}\frac{dx}{x^4\sqrt{1 - 1/x}}.\end{aligned}$$

The integral is

$$\begin{aligned}\int_1^\infty \frac{dx}{x^4\sqrt{1-1/x}} &= \int_0^\infty \frac{dy}{\sqrt{y}(y+1)^{7/2}} \qquad (x=y+1)\\ &= 2\int_0^\infty \frac{dz}{(1+z^2)^{7/2}} \qquad (y=z^2)\\ &= \int_{-\infty}^\infty \frac{d\theta}{\cosh^6\theta} \qquad (z=\sinh\theta)\\ &= \left[\frac{1}{5}\tanh^5\theta - \frac{2}{3}\tanh^3\theta + \tanh\theta\right]_{-\infty}^{\infty} = \frac{16}{15}.\end{aligned}$$

Therefore, because $s^{-3} = m^3 v_0^6/8Z^3e^6$,

$$\Delta E = \frac{2mv_0^5}{45Z\pi\epsilon_0 c^3}.$$

Source: J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).

23.11 Frequency of Dipole Radiation

Since $\rho(\mathbf{r},t) = q\delta[\mathbf{r}-\mathbf{r}_0(t)]$, the dipole moment of the moving charge is

$$\mathbf{p}(t) = \int d^3r\,\mathbf{r}\rho(\mathbf{r},t) = qR[\hat{\mathbf{x}}\cos(\omega_1 t)\cos(\omega_2 t)]\hat{\mathbf{x}} + qR\sin(\omega_2 t)\hat{\mathbf{y}}.$$

But $\cos(\omega_1 t)\cos(\omega_2 t) = \frac{1}{2}\cos[(\omega_1+\omega_2)t] + \frac{1}{2}\cos[(\omega_1-\omega_2)t]$. Therefore, the dipole moment is the sum of three time-harmonic dipoles, each of which emits at its natural frequency. Hence, the moving particle emits dipole radiation at frequencies $\omega_1+\omega_2$, $|\omega_1-\omega_2|$, and ω_2.

23.12 Larmor's Formula with Fields Displayed

The covariant form of the Larmor's formula derived in the text is

$$P = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3m^2c^3}\frac{dp_\mu}{d\tau}\frac{dp_\mu}{d\tau}.$$

To express this in terms of the fields, we use the covariant form of Newton's law for a particle with charge q moving in an arbitrary electromagnetic field. This was given in Example 22.5 as

$$\frac{dp_\mu}{d\tau} = qU_\nu F_{\mu\nu}.$$

Therefore, the desired expression is

$$P = \frac{1}{4\pi\epsilon_0}\frac{2q^4}{3m^2c^3}U_\nu F_{\mu\nu}U_\nu F_{\mu\sigma}U_\sigma. \tag{1}$$

The evaluation of (1) in an arbitrary inertial frame reduces to matrix multiplication. Thus,

$$F_{\mu\sigma}U_\sigma = \begin{bmatrix} 0 & B_z & -B_y & -iE_x/c \\ -B_z & 0 & B_x & -iE_y/c \\ B_y & -B_x & 0 & -iE_z/c \\ iE_x/c & iE_y/c & iE_z/c & 0 \end{bmatrix}\begin{bmatrix} \gamma v_x \\ \gamma v_y \\ \gamma v_z \\ i\gamma c \end{bmatrix} = \gamma\begin{bmatrix} (\mathbf{v}\times\mathbf{B}+\mathbf{E})_x \\ (\mathbf{v}\times\mathbf{B}+\mathbf{E})_y \\ (\mathbf{v}\times\mathbf{B}+\mathbf{E})_z \\ i\boldsymbol{\beta}\cdot\mathbf{E} \end{bmatrix}.$$

Therefore,

$$U_\nu F_{\mu\nu}U_\nu F_{\mu\sigma}U_\sigma = \gamma^2\{|\mathbf{E}+\mathbf{v}\times\mathbf{B}|^2 - |\boldsymbol{\beta}\cdot\mathbf{E}|^2\},$$

and

$$P = \frac{1}{4\pi\epsilon_0}\frac{2q^2\gamma^2}{3m^2c^3}\{|\mathbf{E}+\mathbf{v}\times\mathbf{B}|^2 - |\boldsymbol{\beta}\cdot\mathbf{E}|^2\}.$$

Source: L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1962).

23.13 Emission Rates by Lorentz Transformation

In the (momentary) rest frame of the electron, the particle acceleration is $a' = eE'/m$ and Larmor's formula gives the exact rate at which the particle radiates energy:

$$P' = \frac{dU'}{dt'} = \frac{1}{4\pi\epsilon_0}\frac{2e^2a'^2}{3c^3} = \frac{1}{4\pi\epsilon_0}\frac{2e^2}{3c^3}\frac{e^2E'^2}{m^2}.$$

Therefore, the total energy lost to radiation is

$$\Delta U' = P't' = \frac{1}{4\pi\epsilon_0}\frac{2e^4E'^2t'}{3m^2c^3}.$$

There is no preferred direction in the rest frame, so $\Delta P' = 0$. Transforming to the laboratory frame,

$$\Delta U = \gamma(\Delta U' + v\Delta P') = \gamma\Delta U' = \gamma\frac{1}{4\pi\epsilon_0}\frac{2e^4E'^2t'}{3m^2c^3}.$$

The electric field is parallel to the boost, so $E = E'$. The electron transit time through the capacitor is $t = d/v$ and $t = \gamma t'$ by time dilation. Therefore,

$$\Delta U = \gamma\frac{1}{4\pi\epsilon_0}\frac{2e^4E'^2t'}{3m^2c^3} = \frac{1}{4\pi\epsilon_0}\frac{2e^4E^2d}{3m^2c^3v}.$$

The associated total momentum radiated is

$$\Delta P = \gamma(\Delta P' + v\Delta U'/c^2) = \frac{\gamma v \Delta U'}{c^2} = \frac{v\Delta U}{c^2} = \frac{1}{4\pi\epsilon_0}\frac{2e^4E^2d}{3m^2c^5}.$$

Source: Prof. L. Levitov, MIT (public communication).

23.14 Emission Rates by Explicit Integration

(a) Since the Poynting vector is $\mathbf{S} = (1/\mu_0 c)\hat{\mathbf{r}}\,|\mathbf{E}_{\rm rad}|^2$, and the Liénard-Wiechert radiation field is

$$\mathbf{E}_{\rm rad}(\mathbf{r},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{n}}\times\left\{(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right\}}{c(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3 R}\right]_{\rm ret},$$

the rate of energy emission as measured by the particle itself is

$$\frac{dU_{\rm EM}}{dt} = \frac{q^2}{16\pi^2c^3\epsilon_0}\int d\Omega\,\frac{\{\hat{\mathbf{n}}\times[(\hat{\mathbf{n}}-\boldsymbol{\beta})\times\mathbf{a}]\}^2}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^5}. \tag{1}$$

Writing out the numerator gives

$$\begin{aligned}\frac{dU_{\rm EM}}{dt} &= \frac{q^2}{16\pi^2c^3\epsilon_0}\int\frac{d\Omega}{g^5}\left\{g^2a^2+2g(\boldsymbol{\beta}\cdot\mathbf{a})(\hat{\mathbf{n}}\cdot\mathbf{a})-(1-\beta^2)(\hat{\mathbf{n}}\cdot\mathbf{a})^2\right\}\\ &= \frac{q^2}{16\pi^2c^3\epsilon_0}\left\{a^2I+2(\boldsymbol{\beta}\cdot\mathbf{a})a_iJ_i-(1-\beta^2)a_ia_jK_{ij}\right\},\end{aligned}$$

with I, J_i, K_{ij} defined as above. Now we evaluate

$$I = \int\frac{d\Omega}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3} = 2\pi\int_{-1}^{1}\frac{dx}{(1-\beta x)^3} = \frac{\pi}{\beta}\left\{\frac{1}{(1-\beta)^2}-\frac{1}{(1+\beta)^2}\right\} = \frac{4\pi}{(1-\beta^2)^2}$$

and

$$\begin{aligned}J_i &= \int d\Omega\frac{n_i}{(1-n_k\beta_k)^4} = \frac{1}{3}\frac{\partial}{\partial\beta_i}\int\frac{d\Omega}{(1-n_k\beta_k)^3} = \frac{1}{3}\frac{\partial I}{\partial\beta_i}\\ &= \frac{1}{3}\frac{\partial}{\partial\beta_i}\frac{4\pi}{(1-\beta_k\beta_k)^2} = \frac{16\pi\beta_i}{3(1-\beta^2)^3}.\end{aligned}$$

Similarly,

$$\begin{aligned}K_{ij} &= \int\frac{n_in_j\,d\Omega}{(1-n_k\beta_k)^5} = \frac{1}{4}\frac{\partial}{\partial\beta_j}\int\frac{n_i\,d\Omega}{(1-n_k\beta_k)^4} = \frac{1}{4}\frac{\partial J_i}{\partial\beta_j}\\ &= \frac{1}{4}\frac{\partial}{\partial\beta_j}\frac{16\pi\beta_i}{3(1-\beta_k\beta_k)^3} = \frac{4\pi}{3}\frac{\delta_{ij}}{(1-\beta^2)^3}+\frac{8\pi\beta_i\beta_j}{(1-\beta^2)^4}.\end{aligned}$$

Substituting all of this into (1) gives the desired result:

$$\frac{dU_{\rm EM}}{dt} = \frac{2}{3}\frac{\gamma^4}{c^3}\frac{q^2}{4\pi\epsilon_0}\left[a^2 + \frac{(\mathbf{a}\cdot\boldsymbol{\beta})^2}{1-\beta^2}\right] = \frac{2}{3}\frac{\gamma^6}{c^3}\frac{q^2}{4\pi\epsilon_0}\left[a^2 - |\mathbf{a}\times\boldsymbol{\beta}|^2\right].$$

(b) We need to compute

$$\begin{aligned}\frac{dP_{\mathrm{EM},j}}{dt} &= -\int_A d[\mathbf{A}\cdot g\mathbf{T}]_j = -\int_A dA\, g\, \hat{n}_i T_{ij} \\ &= -\epsilon_0 \int_A dA\, g\, \hat{n}_i \left\{E_iE_j + c^2 B_i B_j - \frac{1}{2}\delta_{ij}(E^2 + c^2B^2)\right\}.\end{aligned}$$

But $\hat{\mathbf{n}}\cdot\mathbf{E}_{\rm rad} = 0 = \hat{\mathbf{n}}\cdot\mathbf{B}_{\rm rad}$ and $\mathbf{E}_{\rm rad}\cdot\mathbf{E}_{\rm rad} = c^2\mathbf{B}_{\rm rad}\cdot\mathbf{B}_{\rm rad}$. Therefore,

$$\begin{aligned}\frac{dP_i}{dt} &= \epsilon_0\int dA\, g\, n_i |\mathbf{E}_{\rm rad}|^2 \\ &= \frac{q^2}{16\pi^2c^4\epsilon_0}\int \frac{d\Omega}{g^5}\hat{n}_i\left[g^2a^2 + 2g(\boldsymbol{\beta}\cdot\mathbf{a})(\hat{\mathbf{n}}\cdot\mathbf{a}) - (1-\beta^2)(\hat{\mathbf{n}}\cdot\mathbf{a})^2\right] \\ &= \frac{q^2}{16\pi^2c^4\epsilon_0}\left[a^2A_i + 2(\boldsymbol{\beta}\cdot\mathbf{a})a_jB_{ij} - (1-\beta^2)a_ja_kL_{ijk}\right],\end{aligned} \tag{2}$$

where

$$\begin{aligned}A_i &= \int d\Omega\frac{n_i}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^3} = \frac{1}{2}\frac{\partial M}{\partial\beta_i} = \frac{4\pi\beta_i}{(1-\beta^2)^2} \\ B_{ij} &= \int d\Omega\frac{n_in_j}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^4} = \frac{1}{3}\frac{\partial A_i}{\partial\beta_j} = \frac{4\pi}{3}\frac{\delta_{ij}}{(1-\beta^2)^2} + \frac{16\pi}{3}\frac{\beta_i\beta_j}{(1-\beta^2)^3} \\ L_{ijk} &= \int d\Omega\frac{n_in_jn_k}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^5} = \frac{1}{4}\frac{\partial B_{ij}}{\partial\beta_k} = \frac{4\pi}{3}\frac{\delta_{ij}\beta_k + \delta_{ik}\beta_j + \delta_{jk}\beta_i}{(1-\beta^2)^3} + \frac{8\pi\beta_i\beta_j\beta_k}{(1-\beta^2)^4}\end{aligned}$$

and

$$M = \int\frac{d\Omega}{(1-\hat{\mathbf{n}}\cdot\boldsymbol{\beta})^2} = \frac{4\pi}{1-\beta^2}.$$

Substituting all of this into the momentum-loss formula (2) gives

$$\frac{dP_{\rm EM}}{dt} = \frac{2}{3}\frac{\gamma^4}{c^4}\frac{q^2}{4\pi\epsilon_0}\left[a^2 + \frac{(\mathbf{a}\cdot\boldsymbol{\beta})^2}{1-\beta^2}\right]\beta = \frac{\boldsymbol{\beta}}{c}\frac{dU_{\rm EM}}{dt}.$$

Source: R. Napolitano and S. Ragusa, *American Journal of Physics* **67**, 997 (1999).

23.15 The Radiated Power Spectrum of a Linear Oscillator

The power spectrum formula (23.91) from the text is

$$\frac{dP_m}{d\Omega} = \frac{\mu_0 q^2 m^2\omega_0^4}{32\pi^4 c}\left|\hat{\mathbf{r}}\times\int_0^{2\pi/\omega_0} d\tau\,\mathbf{v}(\tau)\exp\{-im\omega_0[\hat{\mathbf{r}}\cdot\mathbf{r}_0(\tau)/c - \tau]\}\right|^2.$$

By symmetry, we can take the observation point in the x-z plane so $\hat{\mathbf{r}} = (\sin\theta, 0, \cos\theta)$. The velocity function is $\mathbf{v} = -a\omega_0 \sin\omega_0 t\hat{\mathbf{z}}$. Therefore, with $\beta_0 = a\omega_0/c$, we find without difficulty that

$$\frac{dP_m}{d\Omega} = \frac{\mu_0 q^2 m^2 \omega_0^4}{32\pi^4 c} a^2 \omega_0^2 \sin^2\theta \left| \int_0^{2\pi/\omega_0} d\tau \, \sin\omega_0\tau \exp\{im\omega_0(\tau - \beta_0 \cos\theta \cos\omega_0\tau)\} \right|^2 .$$

Changing variables to $\phi = \omega_0\tau$ and $x = m\beta_0\cos\theta$,

$$\frac{dP_m}{d\Omega} = \frac{\mu_0 q^2 m^2 \omega_0^4}{32\pi^4 c} a^2 \sin^2\theta \left| \int_0^{2\pi} d\phi \, \sin\phi \exp\{im(\phi - x\cos\phi)\} \right|^2 . \tag{1}$$

Now, using the Bessel function information in the statement of the problem,

$$\begin{aligned}
&\int_0^{2\pi} d\phi \, \sin\phi \exp\{im(\phi - x\cos\phi)\} \\
&= \frac{1}{2i} \int_0^{2\pi} d\phi \, [\exp(i\phi) - \exp(-i\phi)] \exp\{im(\phi - x\cos\phi)\} \\
&= \frac{1}{2i} \left[\int_0^{2\pi} d\phi \, \exp\{i[(m+1)\phi - x\cos\phi]\} - \int_0^{2\pi} d\phi \, \exp\{i[(m-1)\phi - x\cos\phi]\} \right] \\
&= -\frac{\pi}{i^m} [J_{m+1}(x) + J_{m-1}(x)] \\
&= -\frac{\pi}{i^m} \frac{2m}{x} J_m(x).
\end{aligned}$$

Inserting this result into (1) gives the power spectrum as

$$\frac{dP_m}{d\Omega} = \frac{\mu_0 c q^2 m^2 \omega_0^2}{8\pi^2} \tan^2\theta J_m^2(m\beta_0\cos\theta).$$

The non-relativistic limit is $\beta_0 \ll 1$ or $x \ll 1$ so $J_m(x) \approx x^m/2^n n!$. This shows that emission at ω_0 $(m = 1)$ dominates.

Source: G.A. Schott, *Electromagnetic Radiation* (University Press, Cambridge, 1912).

23.16 The Radiation Spectrum of Beta Decay

(a) One of our expressions for the angular spectrum of radiated energy is

$$\frac{dI(\omega)}{d\Omega} = \frac{\mu_0 q^2 \omega^2}{16\pi^3 c} \left| \hat{\mathbf{r}} \times \int_{-\infty}^{\infty} dt \, \mathbf{v}(t) \exp[-i(\mathbf{k}\cdot\mathbf{r}_0(t) - \omega t)] \right|^2 .$$

For the present problem, the integral is non-zero between $t = 0$ and $t = \infty$ using the velocity function $\mathbf{v}(t) = \mathbf{v}$ and the trajectory function $\mathbf{r}_0(t) = \mathbf{v}t$. Therefore, because $\mathbf{k} = (\omega/c)\hat{\mathbf{r}}$,

$$\frac{dI(\omega)}{d\Omega} = \frac{\mu_0 q^2 \omega^2}{16\pi^3 c}|\hat{\mathbf{r}} \times \mathbf{v}|^2 \left| \int_0^\infty dt\, \exp[-i\omega(\hat{\mathbf{r}} \cdot \boldsymbol{\beta} - 1)t] \right|^2 .$$

The integral does not converge at its upper limit. To make sense of the integral, we insert a convergence factor $\exp(-\epsilon t)$ and let $\epsilon \to 0$ at the end. Writing θ for the angle between $\hat{\mathbf{r}}$ and $\boldsymbol{\beta}$, we find that the distribution does not depend on frequency:

$$\frac{dI(\omega)}{d\Omega} = \frac{\mu_0 q^2 c}{16\pi^3} \frac{\beta^2 \sin^2\theta}{(1 - \beta\cos\theta)^2}.$$

(b) The integral we need to do is

$$I = \int d\Omega \frac{\sin^2\theta}{(1 - \beta\cos\theta)^2} = 2\pi \left[\int_{-1}^{1} \frac{dx}{(1 - \beta x)^2} - \int_{-1}^{1} dx \frac{x^2}{(1 - \beta x)^2} \right] = \frac{4\pi}{\beta^2}\left[\frac{1}{\beta}\ln\frac{1+\beta}{1-\beta} - 2\right].$$

Therefore,

$$I(\omega) = \frac{\mu_0 q^2 c}{4\pi^2}\left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 2\right].$$

(c) Our spectrum does not depend on frequency because we assumed that the electron was created with velocity $\mathbf{v}$. In reality, the electron created by beta decay is accelerated up to this velocity in some very short time interval Δt. In that case, general Fourier considerations tell us that the spectrum function will be negligibly small when $\omega \gg 1/\Delta t$.

Source: J.D. Jackson, *Classical Electrodynamics* (Wiley, New York, 1962).

23.17 Energy Loss and Electric Field Spectrum

For non-relativistic and small-amplitude excursions of the bound electron it is sufficient to write the one-dimensional equation of motion,

$$\ddot{\mathbf{r}} + \Gamma\dot{\mathbf{r}} + \omega_0^2\mathbf{r} = -\frac{e}{m}\mathbf{E}(t),$$

where $\mathbf{E}(t) = \mathbf{E}(\mathbf{r} = 0, t)$. The frequency-domain solution of this equation is

$$\hat{\mathbf{r}}(\omega) = \frac{e/m}{\omega^2 - \omega_0^2 + i\omega\Gamma}\hat{\mathbf{E}}(\omega),$$

and the corresponding transform of the velocity is

$$\hat{\mathbf{v}}(\omega) = -i\omega\hat{\mathbf{r}}(\omega).$$

Using Parseval's theorem, the energy transferred to the bound electron is

$$\Delta E = -e\int_{-\infty}^{\infty} dt\,\mathbf{E}(t)\cdot\mathbf{v}(t) = -\frac{e}{2\pi}\int_{-\infty}^{\infty} d\omega\,\hat{\mathbf{E}}^*(\omega)\cdot\hat{\mathbf{v}}(\omega).$$

Substituting from above,

$$\begin{aligned}\Delta E &= \frac{e^2}{2\pi m}\int_{-\infty}^{\infty} d\omega\,|\hat{\mathbf{E}}(\omega)|^2\frac{i\omega}{\omega^2-\omega_0^2+i\omega\Gamma}\\ &= \frac{e^2}{2\pi m}\int_{-\infty}^{\infty} d\omega\,|\hat{\mathbf{E}}(\omega)|^2\frac{i\omega(\omega^2-\omega_0^2-i\omega\Gamma)}{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}\\ &= \frac{e^2}{m\pi}\int_{0}^{\infty} d\omega\,|\hat{\mathbf{E}}(\omega)|^2\frac{\omega^2\Gamma}{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2}.\end{aligned}$$

To study the $\Gamma \to 0$ limit, we note that

$$\lim_{\Gamma\to 0}\frac{\omega^2\Gamma}{(\omega^2-\omega_0^2)^2+\omega^2\Gamma^2} = \frac{\pi}{2}\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right].$$

Therefore,

$$\Delta E = \frac{e^2}{2m}|\hat{\mathbf{E}}(\omega_0)|^2.$$

Source: W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd edition (Addison-Wesley, Reading, MA, 1962).

23.18 Angular Distribution of Radiated Frequency Harmonics

We follow the discussion in the text which Fourier analyzes the time-periodic electric field as

$$\mathbf{E}(t) = \sum_{m=-\infty}^{\infty}\hat{\mathbf{E}}_m(\mathbf{r})\exp(-im\omega_0 t) \qquad \hat{\mathbf{E}}_m(\mathbf{r}) = \frac{\omega_0}{2\pi}\int_0^{2\pi/\omega_0} dt\,\mathbf{E}(t)\exp(im\omega_0 t), \tag{1}$$

and defines $dP_m/d\Omega$ from

$$\left\langle\frac{dP}{d\Omega}\right\rangle = \frac{1}{T}\int_0^T dt\,\frac{dP}{d\Omega} = \sum_{m=1}^{\infty}\frac{dP_m}{d\Omega}, \tag{2}$$

where

$$\frac{dP}{d\Omega} = \epsilon_0 cr^2 |\mathbf{E}_{\rm rad}|^2. \tag{3}$$

Substituting (1) into (3) and performing the integral in (2) gives

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \epsilon_0 cr^2 \frac{\omega_0}{2\pi} \int_0^{2\pi/\omega} dt \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \hat{\mathbf{E}}_m(\mathbf{r})\hat{\mathbf{E}}_n^*(\mathbf{r}) \exp[-i(m-n)\omega_0 t].$$

The time integral is $(2\pi/\omega)\delta_{mn}$ and $\mathbf{E}(\mathbf{r},t) = \mathbf{E}^*(\mathbf{r},t)$ implies that $\mathbf{E}_m(\mathbf{r}) = -\mathbf{E}_m^*(\mathbf{r})$. Therefore,

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \epsilon_0 cr^2 \sum_{m=-\infty}^{\infty} |\hat{\mathbf{E}}_m(\mathbf{r})|^2 = 2\epsilon_0 cr^2 \sum_{m=1}^{\infty} |\hat{\mathbf{E}}_m(\mathbf{r})|^2. \tag{4}$$

We omit $m = 0$ because $\omega = 0$ is not germane to radiation. To evaluate $\hat{\mathbf{E}}_m(\mathbf{r}$ in (1), we use the Liénard-Wiechert radiation field,

$$\mathbf{E}_{\rm rad} = \frac{q}{4\pi\epsilon_0} \left[\frac{\hat{\mathbf{n}} \times \left\{ (\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right\}}{cg^3 R} \right]_{\rm ret},$$

where $g_{\rm ret} = [1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta}]_{\rm ret}$. Following the text, we change the integration variable from t to $t_{\rm ret} = \tau$, use $t = \tau + R(\tau)/c$, and use the long-distance approximation $R \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}_0(\tau)$. The result is

$$|\hat{\mathbf{E}}_m(\mathbf{r})|^2 = \left| \frac{\omega_0}{2\pi} \frac{q}{4\pi\epsilon_0} \frac{1}{cr} \int_0^{2\pi/\omega_0} d\tau \, \frac{\hat{\mathbf{n}} \times [(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}}]}{g^2} \exp\{im\omega_0[\tau - \hat{\mathbf{r}} \cdot \mathbf{r}_0(\tau)/c]\} \right|^2.$$

Using

$$\frac{\hat{\mathbf{n}} \times \left[(\hat{\mathbf{n}} - \boldsymbol{\beta}) \times \dot{\boldsymbol{\beta}} \right]}{(1 - \boldsymbol{\beta} \cdot \hat{\mathbf{n}})^2} = \frac{d}{d\tau} \left[\frac{\hat{\mathbf{n}} \times (\hat{\mathbf{n}} \times \boldsymbol{\beta})}{1 - \hat{\mathbf{n}} \cdot \boldsymbol{\beta}} \right]$$

to integrate by parts (the integrated part vanishes exactly) gives

$$|\hat{\mathbf{E}}_m(\mathbf{r})|^2 = \left| \frac{\omega_0}{2\pi} \frac{q}{4\pi\epsilon_0} \frac{-im\omega_0}{cr} \int_0^{2\pi/\omega_0} d\tau \, \hat{\mathbf{n}} \times (\hat{\mathbf{n}} - \boldsymbol{\beta}) \exp\{im\omega_0[\tau - \hat{\mathbf{r}} \cdot \mathbf{r}_0(\tau)/c]\} \right|^2. \tag{5}$$

In our approximation $\hat{\mathbf{n}} \approx \hat{\mathbf{r}}$. Therefore, substituting (5) into (4) shows that

$$\frac{dP_m}{d\Omega} = \frac{\mu_0 q^2 m^2 \omega_0^4}{32\pi^4 c} \left| \hat{\mathbf{r}} \times \hat{\mathbf{r}} \times \int_0^{2\pi/\omega_0} d\tau \, \mathbf{v}(\tau) \exp\{-im\omega_0[\hat{\mathbf{r}} \cdot \mathbf{r}_0(\tau)/c - \tau]\} \right|^2.$$

But $|\hat{\mathbf{r}} \times (\hat{\mathbf{r}} \times \mathbf{s})|^2 = |\hat{\mathbf{r}} \times \mathbf{s}|^2$ for any vector $\mathbf{s}$. Therefore, as required,

$$\frac{dP_m}{d\Omega} = \frac{\mu_0 q^2 m^2 \omega_0^4}{32\pi^4 c} \left| \hat{\mathbf{r}} \times \int_0^{2\pi/\omega_0} d\tau\, \mathbf{v}(\tau) \exp\{-im\omega_0[\hat{\mathbf{r}} \cdot \mathbf{r}_0(\tau)/c - \tau]\} \right|^2 .$$

Source: G.A. Schott, *Electromagnetic Radiation* (University Press, Cambridge, 1912).

23.19 N Charges Moving in a Circle II

(a) The generalization of the text expression (23.91) for the power spectrum due to N identical charges with trajectories $\mathbf{r}_j(t)$ and velocities $\mathbf{v}_j(t)$ is

$$\left.\frac{dP_m}{d\Omega}\right|_N = \frac{\mu_0 q^2 m^2 \omega_0^4}{32\pi^4 c} \left| \sum_{j=1}^N \int_0^{2\pi/\omega_0} d\tau\, \hat{\mathbf{r}} \times \mathbf{v}_j(\tau) \exp\{-im\omega_0[\hat{\mathbf{r}} \cdot \mathbf{r}_j(\tau)/c - \tau]\} \right|^2 . \quad (1)$$

Because all the charges move at the same speed, the key observation is that

$$\mathbf{r}_j(t) = \mathbf{r}_1\left(t + \frac{\theta_j - \theta_1}{\omega_0}\right) \qquad \text{and} \qquad \mathbf{v}_j(t) = \mathbf{v}_1\left(t + \frac{\theta_j - \theta_1}{\omega_0}\right).$$

Substituting these into (1) and changing the integration variable to $\tau' = \tau + (\theta_j - \theta_1)/\omega_0$ gives

$$\left.\frac{dP_m}{d\Omega}\right|_N = \frac{\mu_0 q^2 m^2 \omega_0^4}{32\pi^4 c} \left| \sum_{j=1}^N \int_{(\theta_j-\theta_1)/\omega_0}^{2\pi/\omega_0 + (\theta_j-\theta_1)/\omega_0} d\tau'\, \hat{\mathbf{r}} \times \mathbf{v}_1(\tau') \right.$$
$$\left. \times \exp\{im\omega_0[\tau' + (\theta_j - \theta_1)/\omega_0 + \hat{\mathbf{r}} \cdot \mathbf{r}_1(\tau')/c]\} \right|^2 .$$

The integrand is periodic, so the limits of integration can be shifted back to the interval $[0, 2\pi/\omega_0]$. Hence,

$$\left.\frac{dP_m}{d\Omega}\right|_N = \frac{\mu_0 q^2 m^2 \omega_0^4}{32\pi^4 c} \left| \sum_{j=1}^N \exp[im(\theta_1 - \theta_j)] \int_0^{2\pi/\omega_0} d\tau'\, \hat{\mathbf{r}} \times \mathbf{v}_1(\tau') \right.$$
$$\left. \times \exp\{im\omega_0[\tau' + \hat{\mathbf{r}} \cdot \mathbf{r}_1(\tau')/c]\} \right|^2 .$$

Because $|\exp(im\theta_1)|^2 = 1$, we get the advertised formula,

$$\left.\frac{dP_m}{d\Omega}\right|_{\mathrm{N}} = \left| \sum_{j=1}^N \exp(-im\theta_j) \right|^2 \left.\frac{dP_m}{d\Omega}\right|_1 .$$

(b) The case of equally spaced charges means that $\theta_j = 2\pi j/N$. In that case, we get a geometric series:

$$\begin{aligned}
\left|\sum_{j=1}^{N} \exp(-im\theta_j)\right|^2 &= \left|\sum_{j=1}^{N} \exp(-im2\pi j/N)\right|^2 \\
&= \left|\sum_{j=1}^{N} [\exp(-i2\pi m/N)]^j\right|^2 \\
&= \left|\exp(-i2\pi m/N) \sum_{j=0}^{N-1} [\exp(-i2\pi m/N)]^j\right|^2 \\
&= \left|\exp(-i2\pi m/N) \frac{1-\exp(-i2\pi m}{1-\exp(-i2\pi m/N)}\right|^2 \\
&= \left|\exp(-i2\pi m/N)\exp(-i\pi m)\frac{\sin \pi m}{\sin(\pi m/N)}\right|^2 \\
&= \frac{\sin^2 \pi m}{\sin^2(\pi m/N)}.
\end{aligned}$$

This "array" factor is zero because m is an integer and the numerator is zero *unless* the denominator is zero also. The latter happens when m is an integer multiple of N. This proves the assertion.

(c) When the θ_j are random,

$$\left|\sum_{j=1}^{N} \exp(-im\theta_j)\right|^2 = \sum_{j=1}^{N}\sum_{k=1}^{N} \exp[-im(\theta_j - \theta_k)] = N + \sum_{j\neq k} \exp[-im(\theta_j - \theta_k)].$$

Hence, the radiation intensity is always of order N.

Source: G.A. Schott, *Electromagnetic Radiation* (University Press, Cambridge, 1912).

23.20 Covariant Radiation of Energy-Momentum

The four-acceleration is $\mathcal{A}_\nu = dU_\nu/d\tau$ and a formula derived in the text is

$$\frac{dU_{\rm rad}}{dt} = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c^3}\mathcal{A}_\nu\mathcal{A}_\nu. \tag{1}$$

Another result from the text is

$$\frac{d\mathbf{P}_{\rm rad}}{dt} = \frac{\mathbf{v}}{c^2}\frac{dU_{\rm rad}}{dt}. \tag{2}$$

With the definition of $\vec{P}$ given in the problem statement, dimensional analysis suggests that a fully covariant statement of the rate of change of energy-momentum is

$$\frac{dP_\mu}{d\tau} = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c^5}\mathcal{A}_\nu\mathcal{A}_\nu U_\mu. \tag{3}$$

To check this, we use $d\tau = dt/\gamma$ and $U_4 = i\gamma c$ to write out the time component as (3) as

$$\gamma\frac{dP_4}{dt} = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c^3}\mathcal{A}_\nu\mathcal{A}_\nu\frac{1}{c^2}(i\gamma c).$$

But $P_4 = (i/c)U_{\rm rad}$ and, using (1), the foregoing becomes an identity:

$$\frac{i}{c}\frac{dU_{\rm rad}}{dt} = \frac{i}{c}\frac{dU_{\rm rad}}{dt}.$$

Similarly, the space components of P_μ are $\mathbf{P}_{\rm rad}$ and $\vec{U} = \gamma(\mathbf{v}, ic)$. Therefore, the space components of (3) are

$$\gamma\frac{d\mathbf{P}_{\rm rad}}{dt} = \frac{1}{4\pi\epsilon_0}\frac{2q^2}{3c^3}\mathcal{A}_\nu\mathcal{A}_\nu\frac{1}{c^2}(\gamma\mathbf{v}).$$

This gives

$$\frac{d\mathbf{P}_{\rm rad}}{dt} = \frac{\mathbf{v}}{c^2}\frac{dU_{\rm rad}}{dt},$$

which is exactly (2).

23.21 Lorentz Transformation of $dP/d\Omega$

Consider the energy $dU'_{\rm rad}$ emitted into a solid angle $d\Omega' = d(\cos\theta')d\phi'$ in the instantaneous rest frame K' of the particle. Our interest is the transformation law for

$$\frac{dP'}{d\Omega'} = \frac{d^2U'_{\rm rad}}{dt'd\Omega'}.$$

If the charge moves with velocity $\mathbf{v} = v\hat{\mathbf{z}}$, the radiated energy transforms like

$$dU_{\rm rad} = \gamma(dU'_{\rm rad} + \mathbf{v}\cdot\mathbf{P}_{\rm rad}) = \gamma(dU'_{\rm rad} + vdP'_{\rm rad}\cos\theta').$$

However, $dU_{\rm rad} = cP_{\rm rad}$ for plane wave radiation. Therefore,

$$dU_{\rm rad} = \gamma(1+\beta\cos\theta')dU'_{\rm rad}. \tag{1}$$

The transformation law for the solid angle follows from $\omega = ck$ and the fact $(\mathbf{k}, i\omega/c)$ is a four-vector. Therefore,

$$k_\parallel = k\cos\theta = \gamma(k'\cos\theta' + v\omega/c) = \gamma(\cos\theta' + \beta)k'$$

and

$$\omega = \gamma(\omega' + vk'\cos\theta') = \gamma(1+\beta\cos\theta')\omega'.$$

Combining these gives one form of the law of aberration:

$$\cos\theta = \frac{\cos\theta' + \beta}{1+\beta\cos\theta'}. \tag{2}$$

Then, because $1-\beta^2 = 1/\gamma^2$,

$$d(\cos\theta) = \frac{d(\cos\theta')}{1+\beta\cos\theta'} - \frac{\beta(\cos\theta'+\beta)d(\cos\theta')}{(1+\beta\cos\theta')^2} = \frac{d(\cos\theta')}{\gamma^2(1+\beta\cos\theta')^2}. \tag{3}$$

Now, $d\phi = d\phi'$ because the azimuthal direction involves directions transverse to the boost. Therefore, using (1) and (3),

$$\frac{dU_{\rm rad}}{d\Omega} = \gamma^3(1+\beta\cos\theta')^3\frac{dU'_{\rm rad}}{d\Omega'}.$$

Finally, time dilation says that $dt = \gamma dt'$. This gives the advertised result:

$$\frac{dP}{d\Omega} = \gamma^2(1+\beta\cos\theta')^3\frac{dP'}{d\Omega'}.$$

To derive the alternative formula give in the problem statement, we need the reverse of the transformation (2), namely,

$$\cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta}. \tag{2}$$

Using this, we find

$$1+\beta\cos\theta' = \frac{1-\beta^2}{1-\beta\cos\theta} = \frac{1}{\gamma^2(1-\beta\cos\theta)}.$$

This produces the desired formula because

$$\gamma^2(1+\beta\cos\theta')^3 = \frac{1}{\gamma^4(1+\beta\cos\theta)^3}.$$

Source: L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1962).

23.22 Cyclotron Motion with Radiation Reaction

The equation of motion including radiation reaction is

$$m\dot{\mathbf{v}} = q\mathbf{v}\times\mathbf{B} + m\tau_0\ddot{\mathbf{v}}. \tag{1}$$

The motion is circular (not helical) so $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$ and (1) becomes

$$\dot{v}_x - \tau_0\ddot{v}_x = qv_yB/m$$

$$\dot{v}_y - \tau_0 \ddot{v}_y = -q v_x B/m.$$

We may assume $\mathbf{v} = \mathbf{v}_0 \exp(-i\omega t)$ if the motion remains nearly circular. The cyclotron frequency is $\omega_c = qB/m$, so the equations of motion take the form

$$-i\omega(1 + i\omega\tau_0)v_x = \omega_c v_y$$

$$-i\omega(1 + i\omega\tau_0)v_y = -\omega_c v_x.$$

Eliminating v_y produces an equation with v_x on both sides. Canceling this gives

$$\omega^2(1 + i\omega\tau_0)^2 = \omega_c^2 \qquad \Rightarrow \qquad i\tau_0\omega^2 + \omega - \omega_c = 0.$$

Because we must get $\omega = \omega_c$ as $\tau_0 \to 0$, the solution of this quadratic equation is

$$\omega = \frac{1}{2i\tau_0}\left[-1 + \sqrt{1 + 4i\omega_c\tau_0}\,\right].$$

Weak damping means $\omega_c\tau_0 \ll 1$. This justifies the expansion

$$\sqrt{1 + 4i\omega_c\tau_0} = 1 + \tfrac{1}{2}(4i\omega_c\tau_0) - \tfrac{1}{8}(4i\omega_c\tau_0)^2 + \cdots.$$

Hence, the velocity damps according to

$$\mathbf{v}(t) = \mathbf{v}_0 \exp(-i\omega_c t)\exp(-\omega_c^2\tau_0 t),$$

and the damping time constant is $1/\omega_c^2\tau_0$.

23.23 Radiation Pressure Due to Radiation Reaction

Let the electron site at $\mathbf{r} = 0$. The equation of motion,

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E}_0 \exp(-i\omega t),$$

implies that the electron acquires a velocity

$$\mathbf{v} = -\frac{ie}{m\omega}\mathbf{E}_0 \exp(-i\omega t). \tag{1}$$

If we include the effect of radiation reaction, the equation of motion is

$$m\frac{d\mathbf{v}}{dt} = -e\mathbf{E}_0 \exp(-i\omega t) + m\tau_0\ddot{\mathbf{v}}. \tag{2}$$

We guess a solution for (2) of the form

$$\mathbf{v} = -\frac{ie}{m\omega}\mathbf{E}_0 \exp(-i\omega t) + \delta\mathbf{v}, \tag{3}$$

and substitute this back into (2). Because radiation reaction is small, we neglect the contribution from $\delta\ddot{\mathbf{v}}$ to conclude that

$$\delta\dot{\mathbf{v}} = \frac{ie\omega\tau_0}{m}\mathbf{E}_0\exp(-i\omega t).$$

Hence,

$$\delta\mathbf{v} = -\frac{e\tau_0}{m}\mathbf{E}_0\exp(-i\omega t),$$

and the cycle-averaged Lorentz force on the electron is

$$\langle\mathbf{F}\rangle = -e\tfrac{1}{2}\mathrm{Re}\left[(\mathbf{v}^*\times\mathbf{B}_0\right] = -e\tfrac{1}{2}\mathrm{Re}\left[(\delta\mathbf{v}^*\times\mathbf{B}_0\right] = \frac{e^2\tau_0}{2mc}E_0^2\hat{\mathbf{k}}.$$

Note that there is no self Lorentz force on the electron without radiation reaction because the pre-factor of $\exp(-i\omega t)$ in (1) is pure imaginary. The Thomson cross section is $\sigma_{\mathrm{T}} = (8\pi/3)r_e^2 = 6\pi(c\tau_0)^2$. Therefore,

$$\langle\mathbf{F}\rangle = \tfrac{1}{2}\epsilon_0\sigma_{\mathrm{T}}E_0^2\hat{\mathbf{k}}.$$

This force is sensibly interpreted as a pressure because it pushes the electron in the direction of wave propagation.

Source: G. Stupakov, *Lecture Notes on Classical Mechanics and Electromagnetism in Accelerator Physics*, US Particle Accelerator School, Albuquerque, NM, June 2009.

23.24 Angular Momentum Decay by Radiation Reaction

If $a = v^2/r$ is the centripetal acceleration of the orbiting particle, and $\tau_0 = \mu_0 q^2/6\pi mc$, the Larmor rate at which the atom loses energy by radiation is

$$\frac{dE}{dt} = -\frac{1}{4\pi\epsilon_0}\frac{2q^2a^2}{3c^3} = -\frac{\mu_0 q^2}{6\pi c}\left(\frac{v^2}{r}\right)^2 = -\frac{m\tau_0 v^4}{r^2}. \tag{1}$$

The rate of change of the angular momentum is

$$\frac{d\mathbf{L}}{dt} = \frac{d}{dt}(\mathbf{r}\times m\mathbf{v}) = \mathbf{v}\times m\mathbf{v} + \mathbf{r}\times m\dot{\mathbf{v}} = \mathbf{r}\times m\dot{\mathbf{v}}. \tag{2}$$

To compute this quantity, we appeal to the equation of motion of the orbiting charge including the effect of radiation reaction:

$$m\dot{\mathbf{v}} - m\tau_0\ddot{\mathbf{v}} = \mathbf{F}_{\mathrm{Coul}} = -\frac{Zq^2}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}. \tag{3}$$

Taking the cross product of this expression with $\mathbf{r}$ gives

$$m\mathbf{r}\times\dot{\mathbf{v}} - m\tau_0\mathbf{r}\times\ddot{\mathbf{v}} = 0.$$

Comparing this with (1) shows that

$$\frac{d\mathbf{L}}{dt} = m\tau_0 \mathbf{r} \times \ddot{\mathbf{v}} = m\tau \left[\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{v}}) - \mathbf{v} \times \dot{\mathbf{v}} \right]. \tag{4}$$

Now, use the hint and employ (3) to eliminate $m\dot{\mathbf{v}}$ on the far right side of (4) in favor of $\mathbf{F}_{\text{Coul}}$. This gives

$$\frac{d\mathbf{L}}{dt} = \tau \frac{d}{dt}(\mathbf{r} \times \mathbf{F}_{\text{Coul}}) - \tau_0 \mathbf{v} \times \mathbf{F}_{\text{Coul}}.$$

In other words,

$$\frac{dL_z}{dt} = -\frac{Zq^2 v\tau}{4\pi\epsilon_0 r^2}. \tag{5}$$

Combining (5) with (1) gives

$$\frac{dL_z}{dE} = \frac{dL_z/dt}{dE/dt} = \frac{Zq^2 v\tau_0}{4\pi\epsilon_0 r^2} \frac{r^2}{m\tau_0 v^4} = \frac{Zq^2}{4\pi\epsilon_0 m v^3} = \frac{r}{v}, \tag{6}$$

where the last equality follows from the centripetal force equation,

$$\frac{mv^2}{r} = \frac{Zq^2}{4\pi\epsilon_0 r^2}. \tag{7}$$

Now, the virial theorem tells us that the total energy is

$$E = -\frac{Zq^2}{8\pi\epsilon_0 r}, \tag{8}$$

and, using (7), the angular momentum is

$$L_z = mvr = \frac{Zq^2}{4\pi\epsilon_0 v}. \tag{9}$$

Combining (8) and (9) gives

$$\frac{L_z}{|E|} = \frac{2r}{v}.$$

Comparing this to (6) produces the advertised result,

$$\frac{dL_z}{dE} = \frac{1}{2}\frac{L_z}{|E|}.$$

Source: E.J. Konopinski, *Electromagnetic Fields and Relativistic Particles* (McGraw-Hill, New York, 1981).

23.25 Covariant Landau-Lifshitz Equation

The equation of motion for a point charge in an electromagnetic field without radiation damping is

$$m\dot{U}_\mu = eU_\alpha F_{\mu\alpha}. \tag{1}$$

The corresponding Lorentz-Abraham-Dirac equation is

$$m\dot{U}_\mu = eU_\alpha F_{\mu\alpha} + m\tau_0 \ddot{U}_\mu + \frac{m\tau_0}{c^2} U_\mu U_\nu \ddot{U}_\nu. \tag{2}$$

The first iterate of this equation uses the original equation of motion to compute

$$\ddot{U}_\mu = \frac{e}{m}\dot{U}_\beta F_{\mu\beta} + \frac{e}{m} U_\beta \dot{F}_{\mu\beta} \tag{3}$$

and substitutes this back into (2). A moment's reflection shows that the covariant version of the convective derivative is

$$\frac{d}{d\tau} = U_\alpha \partial_\alpha.$$

Substituting this and (1) into (3) gives

$$\ddot{U}_\mu = \frac{e^2}{m^2} U_\alpha F_{\beta\alpha} F_{\mu\beta} + \frac{e}{m} U_\beta U_\alpha \partial_\alpha F_{\mu\beta}. \tag{4}$$

Substituting (4) into (2) gives

$$\begin{aligned} m\dot{U}_\alpha &= eU_\alpha F_{\mu\alpha} + m\tau_0 \left[\frac{e^2}{m^2} U_\alpha F_{\beta\alpha} F_{\mu\beta} + \frac{e}{m} U_\beta U_\alpha \partial_\alpha F_{\mu\beta}\right] \\ &\quad + \frac{m\tau_0}{c^2} U_\mu U_\nu \left[\frac{e^2}{m} U_\alpha F_{\beta\alpha} F_{\nu\beta} + \frac{e}{m} U_\beta U_\alpha \partial_\alpha F_{\nu\beta}\right]. \end{aligned} \tag{5}$$

The last term in (5) is zero because $F_{\nu\beta} = -F_{\beta\nu}$ implies that $U_\nu U_\beta F_{\nu\beta} = 0$. Otherwise, we use the asymmetry of $F_{\mu\nu}$ twice to conclude that

$$m\dot{U}_\mu = eU_\alpha F_{\mu\alpha} - \frac{e^2\tau_0}{m} U_\alpha F_{\alpha\beta} F_{\mu\beta} + e\tau_0 U_\beta U_\alpha \partial_\alpha F_{\mu\beta} - \frac{e^2\tau_0}{c^2}(F_{\beta\alpha}U_\alpha)^2 U_\mu.$$

Source: L. Landau and E.M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1962).

Chapter 24: Lagrangian and Hamiltonian Methods

24.1 Working Backward

(a) Use the first Helmholtz relation three times to get

$$\frac{\partial}{\partial \dot{r}_k}\frac{\partial F_i}{\partial \dot{r}_j} = -\frac{\partial}{\partial \dot{r}_k}\frac{\partial F_j}{\partial \dot{r}_i} = -\frac{\partial}{\partial \dot{r}_i}\frac{\partial F_j}{\partial \dot{r}_k} = \frac{\partial}{\partial \dot{r}_i}\frac{\partial F_k}{\partial \dot{r}_j} = \frac{\partial}{\partial \dot{r}_j}\frac{\partial F_k}{\partial \dot{r}_i} = -\frac{\partial}{\partial \dot{r}_j}\frac{\partial F_i}{\partial \dot{r}_k}.$$

This implies that $\partial^2 F_i/\partial \dot{r}_j \partial \dot{r}_k = 0$. Integrating the latter equation once gives

$$\frac{\partial F_i}{\partial \dot{r}_j} = \epsilon_{ijk} Q_k(\mathbf{r}, t),$$

where $Q_k(\mathbf{r}, t)$ is an arbitrary function of the particle coordinates and time. The Levi-Cività symbol is necessary to guarantee anti-symmetry with respect $i \leftrightarrow j$ interchange, which is a property of the first Helmholtz relation. Integrating again gives

$$F_i = P_i(\mathbf{r}, t) + \epsilon_{ijk} \dot{r}_j Q_k(\mathbf{r}, t),$$

where $P_i(\mathbf{r}, t)$ is another arbitrary function of the particle coordinates and time.

(b) Write the second Helmholtz relation out for one case:

$$\frac{\partial F_x}{\partial y} - \frac{\partial F_y}{\partial x} = \frac{1}{2}\frac{d}{dt}\left(\frac{\partial F_x}{\partial \dot{y}} - \frac{\partial F_y}{\partial \dot{x}}\right).$$

This is

$$\frac{\partial P_x}{\partial y} + \frac{\partial}{\partial y}(\dot{\mathbf{r}} \times \mathbf{Q})_x - \frac{\partial P_y}{\partial x} - \frac{\partial}{\partial x}(\dot{\mathbf{r}} \times \mathbf{Q})_y = \frac{1}{2}\frac{d}{dt}\left[\frac{\partial}{\partial \dot{y}}(\dot{y}Q_z - \dot{z}Q_y) - \frac{\partial}{\partial \dot{x}}(\dot{z}Q_x - \dot{x}Q_z)\right]$$

or

$$-(\nabla \times \mathbf{P})_z + \dot{y}\frac{\partial Q_z}{\partial y} - \dot{z}\frac{\partial Q_y}{\partial y} - \dot{z}\frac{\partial Q_x}{\partial x} + \dot{x}\frac{\partial Q_z}{\partial x} = \frac{dQ_z}{dt}.$$

Using the convective derivative, $dQ_z/dt = \partial Q_z/\partial t + (\mathbf{v} \cdot \nabla)Q_z$,

$$-(\nabla \times \mathbf{P})_z + \dot{y}\frac{\partial Q_z}{\partial y} + \dot{x}\frac{\partial Q_z}{\partial x} + \dot{z}\frac{\partial Q_z}{\partial z} - \dot{z}(\nabla \cdot \mathbf{Q}) = \left[\frac{\partial}{\partial t} + (\dot{\mathbf{r}} \cdot \nabla)\right] Q_z.$$

Rearranging terms, we finally get

$$\left(\nabla \times \mathbf{P} + \frac{\partial \mathbf{Q}}{\partial t}\right)_z + \dot{z}(\nabla \cdot \mathbf{Q}) = 0,$$

and similarly for the other two components. This must be true for arbitrary values of $\dot{\mathbf{r}}$, so we get the two homogeneous equations

$$\nabla \times \mathbf{P} + \frac{\partial \mathbf{Q}}{\partial t} = 0 \qquad \text{and} \qquad \nabla \cdot \mathbf{Q} = 0.$$

(c) To check the Helmholtz relations, write **F** in the form

$$\begin{aligned} F_i &= \epsilon_{ijk} r_j \dot{\omega}_k + 2\epsilon_{ijk} \dot{r}_j \dot{\omega}_k + \epsilon_{ijk} \omega_j \epsilon_{k\ell m} r_l \omega_m \\ &= \epsilon_{ijk} r_j \dot{\omega}_k + 2\epsilon_{ijk} \dot{r}_j \dot{\omega}_k + (\delta_{il}\delta_{jm} - \delta_{im}\delta_{\ell j})\, \omega_j r_l \omega_m \\ &= \epsilon_{ijk} r_j \dot{\omega}_k + 2\epsilon_{ijk} \dot{r}_j \dot{\omega}_k + \omega_j r_i \omega_j - \omega_j r_j \omega_i . \end{aligned}$$

The foregoing implies that

$$\partial F_i / \partial \dot{r}_j = 2\epsilon_{ijk} \omega_k .$$

This expression satisfies the Helmholtz relation. Otherwise, $\partial F_i / \partial r_j = \epsilon_{ijk} \dot{\omega}_k - \omega_j \omega_i$. Therefore,

$$\frac{\partial F_i}{\partial r_j} - \frac{\partial F_j}{\partial r_i} = \epsilon_{ijk} \dot{\omega}_k - \epsilon_{jik} \dot{\omega}_k = 2\epsilon_{ijk} \dot{\omega}_k .$$

Similarly,

$$\frac{\partial F_i}{\partial \dot{r}_j} - \frac{\partial F_j}{\partial \dot{r}_i} = 4\epsilon_{ijk} \omega_k .$$

This shows that the second Helmholtz relation is satisfied. We can read off the "fields" from the equation of motion, namely,

$$\mathbf{P} = \mathbf{r} \times \dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{r} \times \boldsymbol{\omega}) \qquad \text{and} \qquad \mathbf{Q} = 2\boldsymbol{\omega}.$$

The frequency $\boldsymbol{\omega}$ does not depend on position. Therefore, $\nabla \cdot \mathbf{Q} = 0$. Using the expression above,

$$P_i = \varepsilon_{ijk} r_j \dot{\omega}_k + \omega_j r_i \omega_j - \omega_j r_j \omega_i .$$

Therefore,

$$\begin{aligned} (\nabla \times \mathbf{P})_p &= \epsilon_{pqi} \partial_q P_i \\ &= \epsilon_{pqi} \epsilon_{jki} \dot{\omega}_k \partial_q r_j + w^2 \epsilon_{pqi} \partial_q r_i - \omega_j \omega_i \varepsilon_{pqi} \partial_q r_j \\ &= (\delta_{pj}\delta_{qk} - \delta_{pk}\delta_{qj})\, \dot{\omega}_k \delta_{qj} + w^2 \epsilon_{pqi} \delta_{iq} - \omega_j \omega_i \epsilon_{pqi} \delta_{qj} \\ &= \dot{\omega}_p - 3\dot{\omega}_p = -2\dot{\omega}_p = -\partial Q_p / \partial t. \end{aligned}$$

This proves the claim because the last line is the p^{th} component of $-\partial \mathbf{Q}/\partial t$.

24.2 An Effective Nuclear Force

(a) With the given Hamiltonian, one of Hamilton's equations gives the velocity as

$$v = \frac{\partial H}{\partial p} = \frac{p}{m} + 2pf(r).$$

Solving this for the canonical momentum gives

$$p = \frac{mv}{1+2mf(r)}. \tag{1}$$

By definition, the Lagrangian is

$$L = pv - H = \frac{mv^2}{1+2mf(r)} - \frac{p^2}{2m} - g(r) - p^2 f(r). \tag{2}$$

The Lagrangian $L = L(r, v)$ is not a function of p, so we use (1) to eliminate p from (2) in favor of $v = \dot{r}$. The result is

$$L(r, \dot{r}) = \frac{\frac{1}{2}m\dot{r}^2}{1+2mf(r)} - g(r).$$

(b) The Lagrange equation is

$$\frac{dp}{dt} = \frac{\partial L}{\partial r}.$$

For our problem, this reads

$$\frac{d}{dt}\left(\frac{m\dot{r}}{1+2mf}\right) = \frac{\partial}{\partial r}\left(\frac{\frac{1}{2}m\dot{r}^2}{1+2mf}\right) - g'(r). \tag{3}$$

Using $\dot{A}(r) = vA'(r)$ for any function $A(r)$ of the radial distance,

$$\frac{d}{dt}\left(\frac{m\dot{r}}{1+2mf}\right) = \frac{m\ddot{r}}{1+2mf} + m\dot{r}^2\frac{d}{dr}\left(\frac{1}{1+2mf}\right).$$

In addition,

$$\frac{m\ddot{r}}{1+2mf} = m\ddot{r}\left(1 - \frac{2mf}{1+2mf}\right).$$

Therefore, (3) can be rearranged into the Newton's-law form

$$m\ddot{r} = \frac{2m^2 f}{1+2mf}\ddot{r} - \frac{m}{2}\frac{d}{dr}\left(\frac{1}{1+2mf}\right)\dot{r}^2 - g'(r) = F(r, \dot{r}, \ddot{r}).$$

Source: E. Boridy and J.M. Pearson, *Physical Review Letters* **27**, 203 (1971).

24.3 Relativistic Lagrangian

The Lagrangian is

$$L = -mc^2\sqrt{1 - \dot{\mathbf{r}}\cdot\dot{\mathbf{r}}/c^2} + e\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r}, t) - e\varphi(\mathbf{r}, t).$$

Therefore, the conjugate momentum is

$$\mathbf{p} = \frac{\partial L}{\partial \dot{\mathbf{r}}} = \frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} + e\mathbf{A}.$$

The Lagrange equation is

$$\frac{d\mathbf{p}}{dt} = \frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}} + e\mathbf{A}\right) = \frac{\partial L}{\partial \mathbf{r}} = e\mathbf{v}\cdot\nabla\mathbf{A} - e\nabla\varphi.$$

Using the convective derivative,

$$\frac{d\mathbf{A}}{dt} = \frac{\partial \mathbf{A}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{A},$$

the foregoing becomes

$$\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}\right) = e\left(-\nabla\varphi - \frac{\partial \mathbf{A}}{\partial t}\right) + e\left(\mathbf{v}\cdot\nabla\mathbf{A} - (\mathbf{v}\cdot\nabla)\,\mathbf{A}\right).$$

Since $\mathbf{B} = \nabla\times\mathbf{A}$, an elementary vector identity gives

$$\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1 - v^2/c^2}}\right) = e\left(\mathbf{E} + \mathbf{v}\times(\nabla\times\mathbf{A})\right) = e\left(\mathbf{E} + \mathbf{v}\times\mathbf{B}\right).$$

This is the correct relativistic equation of motion.

24.4 A Relativistic Particle Coupled to a Scalar Field

The first term must be the action of a point particle in isolation. The Lagrangian of the latter is $L_p = -mc^2/\gamma$. Therefore, in terms of the proper time differential $d\tau = dt/\gamma$,

$$S_p = -mc^2\int\frac{dt}{\gamma} = -mc^2\int d\tau = -mc\int d(c\tau).$$

We conclude that $ds = cd\tau$ and we can write the total Lagrangian as a function of time:

$$L = -\frac{mc^2}{\gamma} - \frac{gc}{\gamma}\varphi(\mathbf{r}(t)).$$

Lagrange's equation is

$$\frac{d}{dt}\frac{\partial L}{\partial \mathbf{v}} = \frac{\partial L}{\partial \mathbf{r}},$$

and

$$\frac{\partial}{\partial \mathbf{v}}\frac{1}{\gamma} = \frac{\partial}{\partial \mathbf{v}}\sqrt{1 - \mathbf{v}\cdot\mathbf{v}/c^2} = -\frac{\mathbf{v}/c^2}{\sqrt{1 - v^2/c^2}} = -\gamma\mathbf{v}/c^2.$$

Therefore,

$$\frac{\partial L}{\partial \mathbf{v}} = \gamma m \mathbf{v} + \gamma g \frac{\mathbf{v}}{c} \varphi,$$

and the equation of motion is

$$\frac{d}{dt}(\gamma m \mathbf{v}) = -\frac{d}{dt}\left(g\gamma \frac{\mathbf{v}}{c}\varphi\right) - \frac{gc}{\gamma}\nabla\varphi.$$

The last term on the right is electric field-like. The first term on the right has an entirely different character.

24.5 The Clausius and Darwin Lagrangians

(a) The Lagrangian which describes the non-relativistic motion of the α^{th} particle is

$$L_\alpha = \tfrac{1}{2} m_\alpha v_\alpha^2 + q_\alpha \mathbf{v}_\alpha \cdot \mathbf{A}(\mathbf{r}_\alpha) - q_\alpha \varphi(\mathbf{r}_\alpha). \tag{1}$$

We substitute into (1) the static forms for the scalar and vector potentials created by the remaining particles,

$$\begin{aligned} \varphi(\mathbf{r}, t) &= \frac{1}{4\pi\epsilon_0} \sum_{\beta\neq\alpha}^{N} \frac{q_\beta}{|\mathbf{r} - \mathbf{r}_\beta(t)|} \\ \mathbf{A}(\mathbf{r}, t) &= \frac{\mu_0}{4\pi} \sum_{\beta\neq\alpha}^{N} \frac{q_\beta \mathbf{v}_\beta(t)}{|\mathbf{r} - \mathbf{r}_\beta(t)|}, \end{aligned}$$

to get

$$L_\alpha = \tfrac{1}{2} m_\alpha v_\alpha^2 + \frac{\mu_0}{4\pi} \sum_{\beta\neq\alpha}^{N} q_\alpha q_\beta \frac{\mathbf{v}_\alpha \cdot \mathbf{v}_\beta}{|\mathbf{r}_\alpha - \mathbf{r}_\beta|} - \frac{1}{4\pi\epsilon_0} \sum_{\beta\neq\alpha}^{N} \frac{q_\alpha q_\beta}{|\mathbf{r}_\alpha - \mathbf{r}_\beta|}.$$

The last two terms are symmetrical in α and β and thus account for the interaction between particle α and particle β in total. For this reason, these terms must be multiplied by 1/2 to avoid double-counting when we sum over α to form the total Clausius Lagrangian,

$$L_C = \frac{1}{2} \sum_{\alpha=1}^{N} m_\alpha v_\alpha^2 - \frac{1}{8\pi\epsilon_0} \sum_{\alpha=1}^{N} \sum_{\beta\neq\alpha}^{N} \frac{q_\alpha q_\beta}{r_{\alpha\beta}} \left(1 - \frac{\mathbf{v}_\alpha \cdot \mathbf{v}_\beta}{c^2}\right).$$

We can see by inspection that the static scalar and vector potentials above satisfy the Lorenz gauge condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2} \frac{\partial \varphi}{\partial t} = 0.$$

(b) From Section 15.3.2, the transverse current density is

$$\mathbf{j}_\perp(\mathbf{r}, t) = \mathbf{j}(\mathbf{r}, t) + \nabla \frac{1}{4\pi} \int d^3 r' \, \frac{\nabla' \cdot \mathbf{j}(\mathbf{r}', t)}{|\mathbf{r} - \mathbf{r}'|}.$$

The hint suggests we should pull the gradient outside the integral. To do this, we use the identity

$$\frac{\nabla' \cdot \mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} = \nabla' \cdot \left[\frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right] + \mathbf{j}(\mathbf{r}') \cdot \nabla \frac{1}{|\mathbf{r}-\mathbf{r}'|}$$

and note that the integral of the total divergence vanishes for a localized current distribution. Therefore,

$$\mathbf{j}_\perp(\mathbf{r},t) = \mathbf{j}(\mathbf{r},t) + \nabla\nabla_m \frac{1}{4\pi}\int d^3r' \, \frac{j_m(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}.$$

Inserting into this the current density,

$$\mathbf{j}(\mathbf{r},t) = \sum_\beta q_\beta \mathbf{v}_\beta \delta(\mathbf{r}-\mathbf{r}_\beta),$$

gives

$$\mathbf{j}_\perp(\mathbf{r},t) = \sum_\alpha q_\beta \mathbf{v}_\beta \delta(\mathbf{r}-\mathbf{r}_\beta) + \nabla\nabla_m \frac{1}{4\pi}\sum_\beta \frac{q_\beta v_{\beta,m}}{|\mathbf{r}-\mathbf{r}_\beta|}. \tag{2}$$

The Clausius Lagrangian shows that the vector potential contribution is already of $O(v^2/c^2)$ compared to the scalar potential term. If we adopt the Coulomb gauge, this means that retardation of the vector potential can be safely neglected. Hence, the vector potential we need to substitute into (1) is

$$\mathbf{A}(\mathbf{r}_\alpha,t) = \frac{\mu_0}{4\pi}\int d^3r \, \frac{\mathbf{j}_\perp(\mathbf{r},t)}{|\mathbf{r}_\alpha-\mathbf{r}|}. \tag{3}$$

Specifically, substituting (2) into (3) and using the hint shows that

$$\begin{aligned} q_\alpha \mathbf{v}_\alpha \cdot \mathbf{A}(\mathbf{r}_\alpha,t) &= \frac{\mu_0}{4\pi}\sum_{\beta\neq\alpha}^{N} q_\alpha q_\beta \frac{\mathbf{v}_\alpha\cdot\mathbf{v}_\beta}{|\mathbf{r}_\alpha-\mathbf{r}_\beta|} \\ &\quad + \frac{1}{2}\frac{\mu_0}{4\pi}\sum_{\beta\neq\alpha}^{N} q_\alpha q_\beta \left[\frac{\mathbf{v}_\alpha\cdot(\mathbf{r}_\alpha-\mathbf{r}_\beta)\mathbf{v}_\beta\cdot(\mathbf{r}_\alpha-\mathbf{r}_\beta)}{|\mathbf{r}_\alpha-\mathbf{r}_\beta|^3} - \frac{\mathbf{v}_\alpha\cdot\mathbf{v}_\beta}{|\mathbf{r}_\alpha-\mathbf{r}_\beta|}\right]. \end{aligned}$$

The relativistic particle Lagrangian is

$$L_p = -mc^2\sqrt{1-v_\alpha^2/c^2} = -mc^2\left(1 - v_\alpha^2/2c^2 - v_\alpha^4/8c^4 + \cdots\right)$$

and the scalar potential contribution to the Lagrangian is the same as in the Clausius expression. Combining all the above and dropping the inessential constant $-mc^2$ gives

$$L_\alpha = \frac{1}{2}m_\alpha v_\alpha^2\left(1+\frac{v_\alpha^2}{4c^2}\right) - \frac{1}{4\pi\epsilon_0}\sum_{\beta\neq\alpha}^{N}\frac{q_\alpha q_\beta}{r_{\alpha\beta}}\left(1 - \frac{\mathbf{v}_\alpha\cdot\mathbf{v}_\beta + (\mathbf{v}_\alpha\cdot\hat{\mathbf{r}}_{\alpha\beta})(\mathbf{v}_\beta\cdot\hat{\mathbf{r}}_{\alpha\beta})}{2c^2}\right).$$

Adding the Lagrangian for each particle and dividing the interaction term by two to avoid double-counting finally gives the desired Darwin Lagrangian:

$$\begin{aligned} L_D &= \frac{1}{2}\sum_{\alpha=1}^{N} m_\alpha v_\alpha^2\left(1+\frac{v_\alpha^2}{4c^2}\right) \\ &\quad - \frac{1}{8\pi\epsilon_0}\sum_{\alpha=1}^{N}\sum_{\beta\neq\alpha}^{N}\frac{q_\alpha q_\beta}{r_{\alpha\beta}}\left(1 - \frac{\mathbf{v}_\alpha\cdot\mathbf{v}_\beta + (\mathbf{v}_\alpha\cdot\hat{\mathbf{r}}_{\alpha\beta})(\mathbf{v}_\beta\cdot\hat{\mathbf{r}}_{\alpha\beta})}{2c^2}\right). \end{aligned}$$

24.6 Equivalent Lagrangians

(a) The Lagrange equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial L}{\partial q_k}.$$

If $L \to L + d\Lambda/dt$ and $\Lambda = \Lambda(\dot{q}_k, q_k, t)$, the change in the left side of this equation is

$$\begin{aligned}
\frac{d}{dt}\frac{\partial \dot{\Lambda}}{\partial \dot{q}_k} &= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_k}\left(\frac{\partial \Lambda}{\partial q_m}\dot{q}_m + \frac{\partial \Lambda}{\partial \dot{q}_m}\ddot{q}_m + \frac{\partial \Lambda}{\partial t}\right) \\
&= \frac{d}{dt}\left[\frac{\partial^2 \Lambda}{\partial \dot{q}_k \partial q_m}\dot{q}_m + \frac{\partial \Lambda}{\partial q_k} + \frac{\partial^2 \Lambda}{\partial \dot{q}_m \partial \dot{q}_k}\ddot{q}_m + \frac{\partial^2 \Lambda}{\partial \dot{q}_k \partial t}\right] \\
&= \frac{\partial \dot{\Lambda}}{\partial q_k} + \frac{d}{dt}\left[\frac{\partial^2 \Lambda}{\partial \dot{q}_k \partial q_m}\dot{q}_m + \frac{\partial^2 \Lambda}{\partial \dot{q}_m \partial \dot{q}_k}\ddot{q}_m + \frac{\partial^2 \Lambda}{\partial \dot{q}_k \partial t}\right].
\end{aligned}$$

The terms in the square brackets in the last line above must be absent to make this equal to the change in the right side of the original Lagrange equations. Therefore, the Lagrange equations are not preserved when Λ is velocity-dependent.

(b) The Lagrange equation for $L = \dot{x}\dot{y} - xy$ is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = \ddot{q} + q = 0$$

for $q = x$ and $q = y$. This is the harmonic oscillator equation of motion. Therefore, the T–V Lagrangian

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}\left(x^2 + y^2\right)$$

will produce the same equations of motion. This Lagrangian and the original Lagrangian do not differ by a total time derivative.

(c) The change in the Lagrangian is

$$\Delta L = \int d^3r\, \Delta\mathcal{L} = \int d^3r\, \nabla \cdot \boldsymbol{\zeta}[\psi(\mathbf{r}, t)] = \int \boldsymbol{\zeta} \cdot d\mathbf{S}.$$

The Lagrangian, and hence the equations of motion, will be unchanged if $\boldsymbol{\zeta}$ is such that the surface integral vanishes.

24.7 Practice with Lagrangian Densities

(a) The Lagrange equation is

$$\partial_t \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} + \partial_k \frac{\partial \mathcal{L}}{\partial(\partial_k \phi)} = 0.$$

Using

$$\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} = \dot{\phi} \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial(\partial_k \phi)} = -\partial_k \phi,$$

we find the equation of motion is the wave equation,

$$\nabla^2 \phi - \ddot{\phi} = 0.$$

(b) The Lagrange equation is

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

Using

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \partial_\mu \phi \qquad \text{and} \qquad \frac{\partial \mathcal{L}}{\partial \phi} = -\sigma\phi,$$

we find the equation of motion is

$$\partial_\mu \partial_\mu + \sigma\phi = 0.$$

Source: Prof. N. Buttimore, Trinity College, Dublin (private communication).

24.8 One-Dimensional Massive Scalar Field

The Lagrangian density is

$$\mathcal{L} = \frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial \varphi}{\partial t}\right)^2 - \left(\frac{\partial \varphi}{\partial x}\right)^2 - m^2\varphi^2\right].$$

The Lagrange equation of motion for $\varphi(x,t)$ is

$$\begin{aligned} 0 &= \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}} - \frac{\partial \mathcal{L}}{\partial \varphi} + \frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial(\partial\varphi/\partial x)} \\ &= \frac{1}{c^2}\frac{d}{dt}\dot{\varphi} + m^2\varphi - \frac{\partial}{\partial x}\frac{\partial \varphi}{\partial x} \\ &= \frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} - \frac{\partial^2 \varphi}{\partial x^2} + m^2\varphi. \end{aligned} \tag{1}$$

The generalized momentum is $\pi = \partial\mathcal{L}/\partial\dot{\varphi} = \dot{\varphi}/c^2$. Therefore, the Hamiltonian density is

$$\begin{aligned} H &= \pi\dot{\varphi} - \mathcal{L} \\ &= \pi\frac{\partial \varphi}{\partial t} - \frac{1}{2}\left[\frac{1}{c^2}\left(\frac{\partial \varphi}{\partial t}\right)^2 - \left(\frac{\partial \varphi}{\partial x}\right)^2 - m^2\varphi^2\right] \\ &= \frac{1}{2}\left[c^2\pi^2 + (\partial\varphi/\partial x)^2 + m^2\varphi^2\right]. \end{aligned}$$

Hamilton's equations are

$$\begin{aligned}\dot{\pi} &= -\frac{\partial\mathcal{H}}{\partial\varphi}+\frac{\partial}{\partial x}\frac{\partial\mathcal{H}}{\partial(\partial\varphi/\partial x)}=-m^2\varphi+\frac{\partial^2\varphi}{\partial x^2}\\ \dot{\varphi} &= +\frac{\partial\mathcal{H}}{\partial\pi}-\frac{\partial}{\partial x}\frac{\partial\mathcal{H}}{\partial(\partial\pi/\partial x)}=c^2\pi.\end{aligned}$$

Combining the two Hamilton equations gives

$$\frac{1}{c^2}\ddot{\varphi}=-m^2\varphi+\frac{\partial^2\varphi}{\partial x^2}.$$

This is the same as (1).

24.9 Proca Electrodynamics

(a) The Lagrangian density

$$\mathcal{L}=\mathbf{j}\cdot\mathbf{A}-\rho\varphi+\frac{1}{2}\epsilon_0\left[\left(\nabla\varphi+\frac{\partial\mathbf{A}}{\partial t}\right)^2-c^2\left(\nabla\times\mathbf{A}\right)^2\right]-\frac{1}{2\mu_0\ell^2}\left[\mathbf{A}^2-(\varphi/c)^2\right]$$

does not introduce any additional dependence on the generalized velocities $\dot{\mathbf{A}}$ and $\dot{\varphi}$. Therefore, the canonical momenta are the same as in Maxwell theory, namely,

$$\boldsymbol{\pi}=\frac{\partial\mathcal{L}}{\partial\dot{\mathbf{A}}}=-\epsilon_0\mathbf{E} \qquad \text{and} \qquad \pi_0=\frac{\partial\mathcal{L}}{\partial\dot{\varphi}}=0.$$

The generalized Lagrange equation is

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_k}=\frac{\partial\mathcal{L}}{\partial q_k}-\partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i q_k)}. \tag{1}$$

The text evaluated (1) for $\boldsymbol{\pi}$ and found

$$\frac{d\pi_k}{dt}=j_k-\frac{1}{\mu_0}\epsilon_{kij}\partial_i B_j. \tag{2}$$

The Proca term makes a contribution of $-(1/\mu_0\ell^2)A_k$ to the right side of (2). Therefore, the Ampère-Maxwell law changes to

$$\nabla\times\mathbf{B}+\frac{\mathbf{A}}{\ell^2}=\mu_0\mathbf{j}+\frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}. \tag{3}$$

The text also evaluated (1) For π_0 and found

$$\frac{d\pi_0}{dt}=0=-\rho+\epsilon\nabla\cdot\mathbf{E}. \tag{4}$$

The Proca term makes a contribution of $\epsilon_0\varphi/\ell^2$ to the right side of (4). Therefore, Gauss' law changes to

$$\nabla\cdot\mathbf{E}+\frac{\varphi}{\ell^2}=\frac{\rho}{\epsilon_0}. \tag{5}$$

The homogeneous Maxwell equations do not change because we represent the fields in the terms of the potentials:

$$\mathbf{E}=-\nabla\varphi-\frac{\partial\mathbf{A}}{\partial t} \qquad \text{and} \qquad \mathbf{B}=\nabla\times\mathbf{A}.$$

(b) Conservation of charge will be respected if we require that the continuity equation be satisfied. Using (3) and (5), this condition reads

$$0=\nabla\cdot\mathbf{j}+\frac{\partial\rho}{\partial t}=\frac{1}{\mu_0\ell^2}\nabla\cdot\mathbf{A}-\frac{1}{\mu_0c^2}\nabla\cdot\frac{\partial\mathbf{E}}{\partial t}+\epsilon_0\frac{\partial}{\partial t}(\nabla\cdot\mathbf{E})+\frac{\epsilon_0}{\ell^2}\frac{\partial\varphi}{\partial t}=\frac{1}{\mu_0\ell^2}\nabla\cdot\mathbf{A}+\frac{\epsilon_0}{\ell^2}\frac{\partial\varphi}{\partial t}.$$

In other words, charge is conserved only if the potentials satisfy the Lorenz gauge constraint,

$$\nabla\cdot\mathbf{A}+\frac{1}{c^2}\frac{\partial\varphi}{\partial t}=0.$$

(c) For a static point charge, $\rho(\mathbf{r})=q\delta(\mathbf{r})$ and $\mathbf{E}=-\nabla\varphi$. Therefore, (5) takes the form

$$\nabla^2\varphi-\frac{\varphi}{\ell^2}=-q\delta(\mathbf{r}).$$

This is exactly (5.88) from Section 5.7, which has the solution (5.91), namely,

$$\varphi(\mathbf{r})=\frac{q}{4\pi\epsilon_0 r}\exp(-r/\ell).$$

24.10 Podolsky Electrodynamics

(a) The text derived the Lagrange equations,

$$\frac{\partial\mathcal{L}}{\partial A_\alpha}-\partial_\mu\left\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\alpha)}\right\}=0, \tag{1}$$

with no dependence of $\mathcal{L}$ on $\partial_\mu\partial_\nu A_\alpha$. If this dependence exists, there is an additional contribution to the variation of the action equal to

$$\delta S=\int dt\int d^3r\,\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu A_\alpha)}\delta(\partial_\mu\partial_\nu A_\alpha)$$

or

$$\delta S=\int dt\int d^3r\left\{\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu A_\alpha)}\delta(\partial_\nu A_\alpha)\right]-\left[\partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_\nu A_\alpha)}\right]\delta(\partial_\nu A_\alpha)\right\}.$$

We assume that the total derivative term vanishes by virtue of the vanishing of the variation $\delta(\partial_\nu A_\alpha)$ at infinity. The term which remains is the same as

$$\delta S = \int dt \int d^3r \left\{ -\partial_\nu \left[\left(\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu A_\alpha)} \right) \delta A_\alpha \right] + \left[\partial_\mu \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu A_\alpha)} \right] \delta A_\alpha . \right\}.$$

The total derivative term vanishes again and we are left with the final term alone. Therefore, the Lagrange equation (1) generalizes to

$$\frac{\partial \mathcal{L}}{\partial A_\alpha} - \partial_\mu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\alpha)} \right\} + \partial_\mu \partial_\nu \left\{ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu A_\alpha)} \right\} = 0.$$

(b) The Lagrangian is

$$\mathcal{L}_P = j_\mu A_\mu - \frac{1}{4\mu_0} F_{\sigma\beta} F_{\sigma\beta} - \frac{a^2}{2\mu_0} (\partial_\lambda F_{\beta\lambda})(\partial_\rho F_{\beta\rho}).$$

We already know the Lagrange equation when $a = 0$:

$$0 = j_\alpha + \frac{1}{\mu_0} \partial_\mu F_{\mu\alpha}. \tag{2}$$

Therefore, we focus on the Podolsky term and compute

$$\begin{aligned}
\frac{\partial}{\partial(\partial_\mu \partial_\nu A_\alpha)} \left[-\partial_\lambda F_{\beta\lambda} \partial_\rho F_{\beta\rho} \right] &= -2\partial_\lambda F_{\beta\lambda} \frac{\partial}{\partial(\partial_\mu \partial_\nu A_\alpha)} \partial_\rho F_{\beta\rho} \\
&= -2\partial_\lambda F_{\beta\lambda} \frac{\partial}{\partial(\partial_\mu \partial_\nu A_\alpha)} (\partial_\rho \partial_\beta A_\rho - \partial_\rho \partial_\rho A_\beta) \\
&= -2\partial_\lambda F_{\nu\lambda} \delta_{\mu\rho} \delta_{\alpha\rho} + 2\partial_\lambda F_{\alpha\lambda} \delta_{\mu\rho} \delta_{\nu\rho} \\
&= -2\partial_\lambda F_{\nu\lambda} \delta_{\alpha\mu} + 2\partial_\lambda F_{\alpha\lambda} \delta_{\mu\nu}.
\end{aligned}$$

In the Lagrange equation, we apply the operation $\partial_\mu \partial_\nu$ to both terms. This causes the first term to vanish because $F_{\nu\lambda} = -F_{\lambda\nu}$ causes $\partial_\nu \partial_\lambda F_{\nu\lambda}$ to vanish. Taking account of the second term generalizes (2) to

$$0 = j_\alpha + \frac{1}{\mu_0} \partial_\mu F_{\mu\alpha} + \frac{a^2}{\mu_0} \partial_\mu \partial_\mu \partial_\lambda F_{\alpha\lambda}$$

or

$$\left[(1 - a^2 \partial_\mu \partial_\mu) \right] \partial_\lambda F_{\lambda\alpha} = -\mu_0 j_\alpha. \tag{3}$$

(c) We know that the $a = 0$ version of (3) produces the inhomogeneous Maxwell equations. Therefore, we can write down the generalized Gauss and Ampère-Maxwell laws by inspection:

$$\left[1 - a^2 \left(\nabla^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) \right] \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

$$\left[1 - a^2\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\right]\left[\nabla\times\mathbf{B} - \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}\right] = \mu_0\mathbf{j}.$$

Source: B. Podolsky, *Physical Review* **62**, 68 (1942).

24.11 Chern-Simons Electrodynamics

(a) A change of gauge where $\varphi \to \varphi + \dot{\Lambda}$ and $\mathbf{A} \to \mathbf{A} - \nabla\Lambda$ leaves the Maxwell fields unchanged, but changes the Lagrangian density by

$$\Delta\mathcal{L} = \rho\dot{\Lambda} + \mathbf{j}\cdot\nabla\Lambda + \frac{1}{2}\left[\mathbf{d}\cdot(\mathbf{E}/c\times\nabla\Lambda) - d_0\nabla\Lambda\cdot\mathbf{B} - \dot{\Lambda}\left(\mathbf{d}\cdot\mathbf{B}/c\right)\right]. \tag{1}$$

The text (Section 24.4.1) writes the first two terms as

$$\nabla(\mathbf{j}\Lambda) + \frac{\partial}{\partial t}(\rho\Lambda) - \Lambda\left[\nabla\cdot\mathbf{j} + \frac{\partial\rho}{\partial t}\right].$$

The first two terms vanish in the action integral from end-point restrictions on a variation. The last two terms vanish from charge conservation. In (1), we proceed similarly and write the remaining terms using two divergences and one time derivative. We assume these vanish also from end-point restrictions. The terms which remain are

$$\begin{aligned}\Delta\mathcal{L} &= \frac{\Lambda}{2}\left[\nabla\cdot(d_0\mathbf{B}) + \partial_t\left(\mathbf{d}\cdot\dot{\mathbf{B}}\right) - \epsilon_{ijk}\partial_k\left(d_iE_j\right)\right]\\ &= \frac{\Lambda}{2}\left[d_0\left(\nabla\cdot\mathbf{B}\right) + \left(\mathbf{B}\cdot\nabla\right)d_0 + \mathbf{B}\cdot\dot{\mathbf{d}} - \mathbf{E}\cdot(\nabla\times\mathbf{d}) + \mathbf{d}\cdot\left(\dot{\mathbf{B}} + \nabla\times\mathbf{E}\right)\right].\end{aligned}$$

The first and last terms in the square brackets are zero by the definition of the fields in terms of the potentials. Otherwise, if the change in the action is to be zero for arbitrary Λ, we must have

$$\nabla d_0 = 0 \qquad \nabla\times\mathbf{d} = 0 \qquad \dot{\mathbf{d}} = 0.$$

(b) The conditions above are surely satisfied if we choose $(\mathbf{d}, id_0)$ as a *constant* four-vector. The canonical momenta are

$$\begin{aligned}\pi_0 &= \frac{\partial\mathcal{L}}{\partial\dot{\varphi}} = 0\\ \boldsymbol{\pi} &= \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{A}}} = -\epsilon_0\mathbf{E} - \tfrac{1}{2}\mathbf{d}\times\mathbf{A}.\end{aligned}$$

The Lagrange equation,

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\dot{q}_k} = \frac{\partial\mathcal{L}}{\partial q_k} - \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i q_k)},$$

for the scalar potential is

$$0 = -\rho + \epsilon_0 \nabla \cdot \mathbf{E} - \tfrac{1}{2}\mathbf{d} \cdot \mathbf{B} + \tfrac{1}{2}\nabla \cdot (\mathbf{d} \times \mathbf{A})\,;$$

or, because $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{d}$ is a constant vector, the Chern-Simons Gauss' law is

$$\epsilon_0 \nabla \cdot \mathbf{E} = \rho + \mathbf{p} \cdot \mathbf{B}. \tag{2}$$

The Lagrange equation for the vector potential is

$$\begin{aligned} -\epsilon_0 \dot{\mathbf{E}} - \tfrac{1}{2}\mathbf{d} \times \dot{\mathbf{A}} &= \tfrac{1}{2}\mathbf{E} \times \mathbf{d} + \tfrac{1}{2}d_0 \mathbf{B} + \mathbf{j} - \epsilon_0 c^2 \nabla \times \mathbf{B} \\ &\quad -\partial_i \frac{\partial}{\partial\left(\partial_i \mathbf{A}\right)} \left\{\tfrac{1}{2}\left(d_0 \mathbf{A} - \varphi\, \mathbf{d}\right) \cdot (\nabla \times \mathbf{A})\right\} \\ &= \tfrac{1}{2}\mathbf{E} \times \mathbf{d} + \tfrac{1}{2}d_0 \mathbf{B} + \mathbf{j} - \epsilon_0 c^2 \nabla \times \mathbf{B} \\ &\quad +\tfrac{1}{2}d_0 \mathbf{B} + \tfrac{1}{2}\mathbf{d} \times \nabla\varphi. \end{aligned}$$

Combining terms gives the Chern-Simons Ampère-Maxwell law,

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \mu_0\left(\mathbf{j} + d_0 \mathbf{B} - \mathbf{d} \times \mathbf{E}\right).$$

These modified Maxwell equations are gauge invariant because only the Maxwell fields $\mathbf{E}$ and $\mathbf{B}$ appear in them. They cannot be Lorentz covariant when p_0 and $\mathbf{p}$ are constants. This is most clear from (2) because a Lorentz transformation mixes $\mathbf{E}$ and $\mathbf{B}$.

Source: S.M. Carroll, G.B. Field, and R. Jackiw, *Physical Review D* **41**, 1231 (1990).

24.12 First-Order Lagrangian

(a) For the given Lagrangian density, evaluate the Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{\partial \mathcal{L}}{\partial q_k} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i q_k)}.$$

For **E:**

$$0 = -\mathbf{E} - \nabla\varphi - \dot{\mathbf{A}},$$

for **B:**

$$0 = \mathbf{B} - \nabla \times \mathbf{A},$$

for φ:

$$0 = -\rho + \epsilon_0 \nabla \cdot \mathbf{E},$$

for **A:**

$$-\frac{d}{dt}E_j = j_i \partial_i \frac{\partial \mathcal{L}}{\partial\left(\partial_i A_j\right)} c^2 B_k \varepsilon_{k\ell m} \partial_\ell A_m = j_i + c^2 \partial_\ell B_k \varepsilon_{k\ell j} = -c^2 \varepsilon_{j\ell k} \partial_\ell B_k.$$

The first two equations above are equivalent to the two homogeneous Maxwell equations

$$\nabla \cdot \mathbf{B} = 0 \qquad \text{and} \qquad \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}.$$

The second two are the inhomogeneous Maxwell equations

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \qquad \text{and} \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}.$$

(b) The ten canonical momenta each produce a primary constraint because no time derivative appears anywhere:

$$\begin{aligned}
\pi_\varphi &= \partial\mathcal{L}/\partial\dot{\varphi} = 0 \\
\boldsymbol{\pi}_{\mathbf{A}} &= \partial\mathcal{L}/\partial\dot{\mathbf{A}} = -\epsilon_0 \mathbf{E} \\
\boldsymbol{\pi}_{\mathbf{E}} &= \partial\mathcal{L}/\partial\dot{\mathbf{E}} = 0 \\
\boldsymbol{\pi}_{\mathbf{B}} &= \partial\mathcal{L}/\partial\dot{\mathbf{B}} = 0.
\end{aligned}$$

24.13 Primary Hamiltonian

The Lagrange equation is

$$\frac{dp}{dt} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

The Hamiltonian is $H = p\dot{q} - L$ so

$$\begin{aligned}
dH &= p d\dot{q} + \dot{q} dp - \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial \dot{q}} d\dot{q} \\
&= \dot{q} dp - \frac{\partial L}{\partial q} dq = \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial p} dp,
\end{aligned}$$

using the definition of p. Using the Lagrange equation gives

$$\left(\frac{\partial H}{\partial q} + \dot{p}\right) dq + \left(\frac{\partial H}{\partial p} - \dot{q}\right) dp = 0.$$

But $\Psi(p,q) = 0$ implies

$$\frac{\partial \Psi}{\partial q} dq + \frac{\partial \Psi}{\partial p} dp = 0,$$

which can be multiplied by an arbitrary function $-u(p,q)$ to get

$$-u\frac{\partial \Psi}{\partial q} dq - u\frac{\partial \Psi}{\partial p} dp = 0.$$

Equating the coefficients of dq and dp in the equations above gives

$$\begin{aligned} \dot{q} &= \frac{\partial H}{\partial p} + u\frac{\partial \Psi}{\partial p} = \frac{\partial H_P}{\partial p} \\ \dot{p} &= -\frac{\partial H}{\partial q} - u\frac{\partial \Psi}{\partial q} = -\frac{\partial H_P}{\partial q}, \end{aligned}$$

where $H_P = H + u\Psi$.

24.14 Gauge Fixing à la Fermi

(a) The canonical momenta are

$$\boldsymbol{\pi} = \frac{\partial \mathcal{L}}{\partial \dot{\mathbf{A}}} = \epsilon_0 \left(\nabla\varphi + \frac{\partial \mathbf{A}}{\partial t}\right) = -\epsilon_0 \mathbf{E}$$

and

$$\pi_0 = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = -\frac{\lambda}{\mu_0}\Omega\frac{\partial \Omega}{\partial \dot{\varphi}} = -\epsilon_0 \lambda \Omega.$$

The Lagrange equations are

$$\begin{aligned} \frac{d\pi_i}{dt} &= \frac{\partial \mathcal{L}}{\partial A_i} - \partial_j \frac{\partial \mathcal{L}}{\partial(\partial_j A_i)} \\ \frac{d\pi_0}{dt} &= \frac{\partial \mathcal{L}}{\partial \varphi} - \partial_i \frac{\partial \mathcal{L}}{\partial(\partial_i \varphi)}. \end{aligned}$$

Writing this out for π_0 gives

$$-\epsilon_0 \lambda \dot{\Omega} = -\rho - \epsilon_0 \partial_i \left(\nabla_i \varphi + \frac{\partial A_i}{\partial t}\right). \tag{1}$$

Adding and subtracting $(1/c^2)\partial^2\varphi/\partial t^2$ puts this in the form of a modified inhomogeneous wave equation:

$$\nabla^2\varphi - \frac{1}{c^2}\frac{\partial^2\varphi}{\partial t^2} + (1-\lambda)\frac{\partial\Omega}{\partial t} = -\frac{\rho}{\epsilon_0}. \qquad (2)$$

Writing out the $\dot{\boldsymbol{\pi}}$ equation of motion similarly gives

$$\epsilon_0\frac{\partial}{\partial t}\left(\nabla_i\varphi + \frac{\partial A_i}{\partial t}\right) = j_i + \partial_j\left[\frac{1}{\mu_0}(\nabla\times\mathbf{A})_k\epsilon_{k\ell m}\frac{\partial}{\partial(\partial_j A_i)}(\partial_\ell A_m)\right] + \frac{\lambda}{\mu_0}\partial_j\left[\Omega\frac{\partial\Omega}{\partial(\partial_j A_i)}\right], \qquad (3)$$

which can also be manipulated into an inhomogeneous wave equation:

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} + (\lambda-1)\nabla\Omega = -\mu_0\mathbf{j}. \qquad (4)$$

(b) Using $\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t$ and $\mathbf{B} = \nabla\times\mathbf{A}$ in (1) and (3), we find without difficulty that

$$\nabla\cdot\mathbf{E} + \lambda\dot{\Omega} = \rho/\epsilon_0 \qquad \text{and} \qquad \nabla\times\mathbf{B} - \lambda\nabla\Omega = \mu_0\mathbf{j} + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t}.$$

(c) Apply the operator $(1/c^2)\partial^2/\partial t^2$ to (2) and add this to the divergence of (4). The result is

$$-\lambda\left[\nabla^2\Omega - \frac{1}{c^2}\frac{\partial^2\Omega}{\partial t^2}\right] = \mu_0\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j}\right).$$

The right side of this equation is zero by conservation of charge. Therefore, Ω satisfies a homogeneous wave equation. The latter is second order in time. Therefore, if we impose the initial conditions $\Omega = \dot{\Omega} = 0$, we may conclude that the Lorenz gauge condition

$$\Omega(\mathbf{r},t) = \nabla\cdot\mathbf{A} + \frac{1}{c^2}\frac{\partial\varphi}{\partial t} = 0$$

at all times. This means that $\dot{\Omega} = \nabla\Omega = 0$ also and the field equations in (b) reduce to the usual Maxwell equations.