APPENDIX I

Complex Numbers

"Old Macdonald had a farm,
Minus E-squared O."

A mathematically simplified children's song

Many of the problems to which mathematics is applied involve the solution of equations. Over the centuries the number system had to be expanded many times to provide solutions for more and more kinds of equations. The natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

are inadequate for the solutions of equations of the form

$$x + n = m, \qquad (m, n \in \mathbb{N}).$$

Zero and negative numbers can be added to create the integers

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$$

in which that equation has the solution $x = m - n$ even if $m < n$. (Historically, this extension of the number system came much later than some of those mentioned below.) Some equations of the form

$$nx = m, \qquad (m, n \in \mathbb{Z}, \quad n \neq 0),$$

cannot be solved in the integers. Another extension is made to include numbers of the form m/n, thus producing the set of rational numbers

$$\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z}, \quad n \neq 0\right\}.$$

Every linear equation

$$ax = b, \qquad (a, b \in \mathbb{Q}, \quad a \neq 0),$$

has a solution $x = b/a$ in $\mathbb{Q}$, but the quadratic equation

$$x^2 = 2$$

has no solution in $\mathbb{Q}$, as was shown in Section P.1. Another extension enriches the rational numbers to the real numbers $\mathbb{R}$ in which some equations like $x^2 = 2$ have solutions. However, other quadratic equations, for instance,

$$x^2 = -1$$

do not have solutions, even in the real numbers, so the extension process is not complete. In order to be able to solve any quadratic equation, we need to extend the real number system to a larger set, which we call the **complex number system**. In this appendix we will define complex numbers and develop some of their basic properties.

Definition of Complex Numbers

We begin by defining the symbol i, called **the imaginary unit,**[1] to have the property

$$i^2 = -1.$$

Thus, we could also call i the **square root of** -1 and denote it $\sqrt{-1}$. Of course, i is not a real number; no real number has a negative square.

DEFINITION 1

A **complex number** is an expression of the form

$$a + bi \qquad \text{or} \qquad a + ib,$$

where a and b are *real numbers*, and i is the imaginary unit.

For example, $3 + 2i$, $\frac{7}{2} - \frac{2}{3}i$, $i\pi = 0 + i\pi$, and $-3 = -3 + 0i$ are all complex numbers. The last of these examples shows that every real number can be regarded as a complex number. (We will normally use $a + bi$ unless b is a complicated expression, in which case we will write $a + ib$ instead. Either form is acceptable.)

It is often convenient to represent a complex number by a single letter; w and z are frequently used for this purpose. If a, b, x, and y are real numbers, and

$$w = a + bi \qquad \text{and} \qquad z = x + yi,$$

then we can refer to the complex numbers w and z. Note that $w = z$ if and only if $a = x$ and $b = y$. Of special importance are the complex numbers

$$0 = 0 + 0i, \qquad 1 = 1 + 0i, \qquad \text{and} \qquad i = 0 + 1i.$$

DEFINITION 2

If $z = x + yi$ is a complex number (where x and y are real), we call x the **real part** of z and denote it $\mathrm{Re}(z)$. We call y the **imaginary part** of z and denote it $\mathrm{Im}(z)$:

$$\mathrm{Re}(z) = \mathrm{Re}(x + yi) = x, \qquad \mathrm{Im}(z) = \mathrm{Im}(x + yi) = y.$$

Note that both the real and imaginary parts of a complex number are real numbers:

$$\begin{aligned} &\mathrm{Re}(3 - 5i) = 3 && \mathrm{Im}(3 - 5i) = -5 \\ &\mathrm{Re}(2i) = \mathrm{Re}(0 + 2i) = 0 && \mathrm{Im}(2i) = \mathrm{Im}(0 + 2i) = 2 \\ &\mathrm{Re}(-7) = \mathrm{Re}(-7 + 0i) = -7 && \mathrm{Im}(-7) = \mathrm{Im}(-7 + 0i) = 0. \end{aligned}$$

Graphical Representation of Complex Numbers

Since complex numbers are constructed from pairs of real numbers (their real and imaginary parts), it is natural to represent complex numbers graphically as points in a Cartesian plane. We use the point with coordinates (a, b) to represent the complex number $w = a + ib$. In particular, the origin $(0, 0)$ represents the complex number 0, the point $(1, 0)$ represents the complex number $1 = 1 + 0i$, and the point $(0, 1)$ represents the point $i = 0 + 1i$. (See Figure I.1.)

[1] In some fields, for example, electrical engineering, the imaginary unit is denoted j instead of i. Like "negative," "surd," and "irrational," the term "imaginary" suggests the distrust that greeted the new kinds of numbers when they were first introduced.

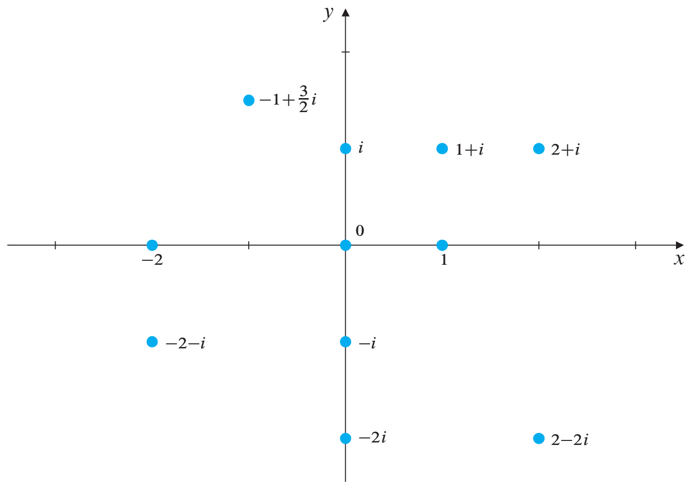

Figure I.1 An Argand diagram representing the complex plane

Such a representation of complex numbers as points in a plane is called an **Argand diagram**. Since each complex number is represented by a unique point in the plane, the set of all complex numbers is often referred to as the **complex plane**. The symbol $\mathbb{C}$ is used to represent the set of all complex numbers and, equivalently, the complex plane:

$$\mathbb{C} = \{x + yi \; : \; x, \; y, \in \; \mathbb{R}\}.$$

The points on the x-axis of the complex plane correspond to real numbers ($x = x + 0i$), so the x-axis is called the **real axis**. The points on the y-axis correspond to **pure imaginary** numbers ($yi = 0 + yi$), so the y-axis is called the **imaginary axis**.

It can be helpful to use the *polar coordinates* of a point in the complex plane.

DEFINITION 3

The distance from the origin to the point (a, b) corresponding to the complex number $w = a + bi$ is called the **modulus** of w and is denoted by $|w|$ or $|a + bi|$:

$$|w| = |a + bi| = \sqrt{a^2 + b^2}.$$

DEFINITION 4

If the line from the origin to (a, b) makes angle θ with the positive direction of the real axis (with positive angles measured counterclockwise), then we call θ an **argument** of the complex number $w = a + bi$ and denote it by $\arg(w)$ or $\arg(a + bi)$. (See Figure I.2.)

The modulus of a complex number is always real and nonnegative. It is positive unless the complex number is 0. Modulus plays a similar role for complex numbers that absolute value does for real numbers. Indeed, sometimes modulus is called absolute value.

Arguments of complex numbers are not unique. If $w = a + bi \neq 0$, then any two possible values for $\arg(w)$ differ by an integer multiple of 2π. The symbol $\arg(w)$ actually represents not a single number, but a set of numbers. When we write $\arg(w) = \theta$, we are saying that the set $\arg(w)$ contains all numbers of the form $\theta + 2k\pi$, where k is an integer. Similarly, the statement $\arg(z) = \arg(w)$ says that two sets are identical.

If $w = a + bi$, where $a = \mathrm{Re}\,(w) \neq 0$, then

$$\tan \arg(w) = \tan \arg(a + bi) = \frac{b}{a}.$$

This means that $\tan\theta = b/a$ for every θ in the set $\arg(w)$.

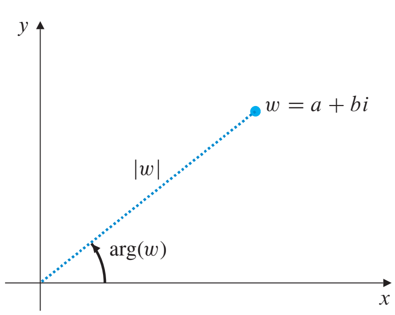

Figure I.2 The modulus and argument of a complex number

It is sometimes convenient to restrict $\theta = \arg(w)$ to an interval of length 2π, say, the interval $0 \le \theta < 2\pi$, or $-\pi < \theta \le \pi$, so that nonzero complex numbers will have unique arguments. We will call the value of $\arg(w)$ in the interval $-\pi < \theta \le \pi$ the **principal argument** of w and denote it $\operatorname{Arg}(w)$. Every complex number w except 0 has a unique principal argument $\operatorname{Arg}(w)$.

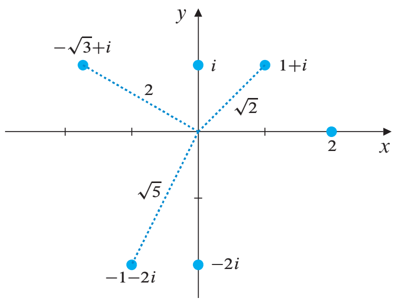

Figure I.3 Some complex numbers with their moduli

EXAMPLE 1 **(Some moduli and principal arguments)** See Figure I.3.

$$\begin{aligned}
&|2| = 2 && \operatorname{Arg}(2) = 0\\
&|1+i| = \sqrt{2} && \operatorname{Arg}(1+i) = \pi/4\\
&|i| = 1 && \operatorname{Arg}(i) = \pi/2\\
&|-2i| = 2 && \operatorname{Arg}(-2i) = -\pi/2\\
&|-\sqrt{3}+i| = 2 && \operatorname{Arg}(-\sqrt{3}+i) = 5\pi/6\\
&|-1-2i| = \sqrt{5} && \operatorname{Arg}(-1-2i) = -\pi + \tan^{-1}(2).
\end{aligned}$$

BEWARE! Review the cautionary remark at the end of the discussion of the arctangent function in Section 3.5; different programs implement the two-variable arctangent using different notations and/or order of variables.

Remark If $z = x + yi$ and $\operatorname{Re}(z) = x > 0$, then $\operatorname{Arg}(z) = \tan^{-1}(y/x)$. Many computer spreadsheets and mathematical software packages implement a two-variable arctan function denoted $\operatorname{atan2}(x, y)$, which gives the polar angle of (x, y) in the interval $(-\pi, \pi]$. Thus,

$$\operatorname{Arg}(x + yi) = \operatorname{atan2}(x, y).$$

Given the modulus $r = |w|$ and any value of the argument $\theta = \arg(w)$ of a complex number $w = a + bi$, we have $a = r\cos\theta$ and $b = r\sin\theta$, so w can be expressed in terms of its modulus and argument as

$$w = r\cos\theta + i\,r\sin\theta.$$

The expression on the right side is called the **polar representation** of w.

DEFINITION 5

The **conjugate** or **complex conjugate** of a complex number $w = a + bi$ is another complex number, denoted $\overline{w}$, given by

$$\overline{w} = a - bi.$$

EXAMPLE 2 $\overline{2-3i} = 2 + 3i, \qquad \overline{3} = 3, \qquad \overline{2i} = -2i.$

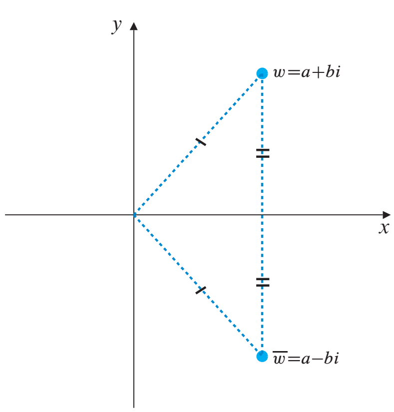

Figure I.4 A complex number and its conjugate are mirror images of each other in the real axis

Observe that

$$\begin{aligned}
&\operatorname{Re}(\overline{w}) = \operatorname{Re}(w) && |\overline{w}| = |w|\\
&\operatorname{Im}(\overline{w}) = -\operatorname{Im}(w) && \arg(\overline{w}) = -\arg(w).
\end{aligned}$$

In an Argand diagram the point $\overline{w}$ is the reflection of the point w in the real axis. (See Figure I.4.)

Note that w is real ($\operatorname{Im}(w) = 0$) if and only if $\overline{w} = w$. Also, w is pure imaginary ($\operatorname{Re}(w) = 0$) if and only if $\overline{w} = -w$. (Here, $-w = -a - bi$ if $w = a + bi$.)

Complex Arithmetic

Like real numbers, complex numbers can be added, subtracted, multiplied, and divided. Two complex numbers are added or subtracted as though they are two-dimensional vectors whose components are their real and imaginary parts.

The sum and difference of complex numbers

If $w = a + bi$ and $z = x + yi$, where a, b, x, and y are real numbers, then

$$w + z = (a + x) + (b + y)i$$
$$w - z = (a - x) + (b - y)i.$$

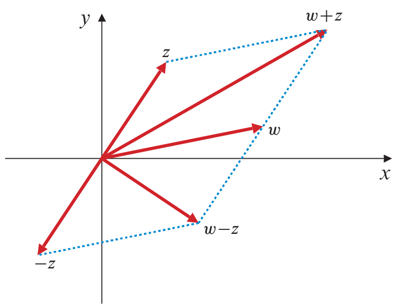

Figure I.5 Complex numbers are added and subtracted vectorially. Observe the parallelograms

In an Argand diagram the points $w+z$ and $w-z$ are the points whose position vectors are, respectively, the sum and difference of the position vectors of the points w and z. (See Figure I.5.) In particular, the complex number $a + bi$ is the sum of the real number $a = a + 0i$ and the pure imaginary number $bi = 0 + bi$.

Complex addition obeys the same rules as real addition: if w_1, w_2, and w_3 are three complex numbers, the following are easily verified:

$w_1 + w_2 = w_2 + w_1$	Addition is commutative.
$(w_1 + w_2) + w_3 = w_1 + (w_2 + w_3)$	Addition is associative.
$\|w_1 \pm w_2\| \le \|w_1\| + \|w_2\|$	the triangle inequality

Note that $|w_1 - w_2|$ is the distance between the two points w_1 and w_2 in the complex plane. Thus, the triangle inequality says that in the triangle with vertices w_1, $\mp w_2$ and 0, the length of one side is less than the sum of the other two.

It is also easily verified that the conjugate of a sum (or difference) is the sum (or difference) of the conjugates:

$$\overline{w + z} = \overline{w} + \overline{z}.$$

EXAMPLE 3 (a) If $w = 2 + 3i$ and $z = 4 - 5i$, then

$$w + z = (2 + 4) + (3 - 5)i = 6 - 2i$$
$$w - z = (2 - 4) + (3 - (-5))i = -2 + 8i.$$

(b) $3i + (1 - 2i) - (2 + 3i) + 5 = 4 - 2i$.

Multiplication of the complex numbers $w = a + bi$ and $z = x + yi$ is carried out by formally multiplying the binomial expressions and replacing i^2 by -1:

$$\begin{aligned} wz &= (a + bi)(x + yi) = ax + ayi + bxi + byi^2 \\ &= (ax - by) + (ay + bx)i. \end{aligned}$$

The product of complex numbers

If $w = a + bi$ and $z = x + yi$, where a, b, x, and y are real numbers, then

$$wz = (ax - by) + (ay + bx)i.$$

EXAMPLE 4 (a) $(2 + 3i)(1 - 2i) = 2 - 4i + 3i - 6i^2 = 8 - i$.

(b) $i(5 - 4i) = 5i - 4i^2 = 4 + 5i$.

(c) $(a + bi)(a - bi) = a^2 - abi + abi - b^2i^2 = a^2 + b^2$.

Part (c) of the example above shows that the square of the modulus of a complex number is the product of that number with its complex conjugate:

$$w\,\overline{w} = |w|^2.$$

Complex multiplication shares many properties with real multiplication. In particular, if w_1, w_2, and w_3 are complex numbers, then

$w_1 w_2 = w_2 w_1$	Multiplication is commutative.
$(w_1 w_2) w_3 = w_1 (w_2 w_3)$	Multiplication is associative.
$w_1 (w_2 + w_3) = w_1 w_2 + w_1 w_3$	Multiplication distributes over addition.

The conjugate of a product is the product of the conjugates:

$$\overline{wz} = \overline{w}\,\overline{z}.$$

To see this, let $w = a + bi$ and $z = x + yi$. Then

$$\begin{aligned}\overline{wz} &= \overline{(ax - by) + (ay + bx)i}\\ &= (ax - by) - (ay + bx)i\\ &= (a - bi)(x - yi) = \overline{w}\,\overline{z}.\end{aligned}$$

It is particularly easy to determine the product of complex numbers expressed in polar form. If

$$w = r(\cos\theta + i\sin\theta) \qquad \text{and} \qquad z = s(\cos\phi + i\sin\phi),$$

where $r = |w|$, $\theta = \arg(w)$, $s = |z|$, and $\phi = \arg(z)$, then

$$\begin{aligned}wz &= rs(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\ &= rs\big((\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \cos\theta\sin\phi)\big)\\ &= rs\big(\cos(\theta + \phi) + i\sin(\theta + \phi)\big).\end{aligned}$$

(See Figure I.6.) Since arguments are only determined up to integer multiples of 2π, we have proved that

The modulus and argument of a product

$$|wz| = |w||z| \qquad \text{and} \qquad \arg(wz) = \arg(w) + \arg(z).$$

The second of these equations says that the set $\arg(wz)$ consists of all numbers $\theta + \phi$, where θ belongs to the set $\arg(w)$ and ϕ to the set $\arg(z)$.

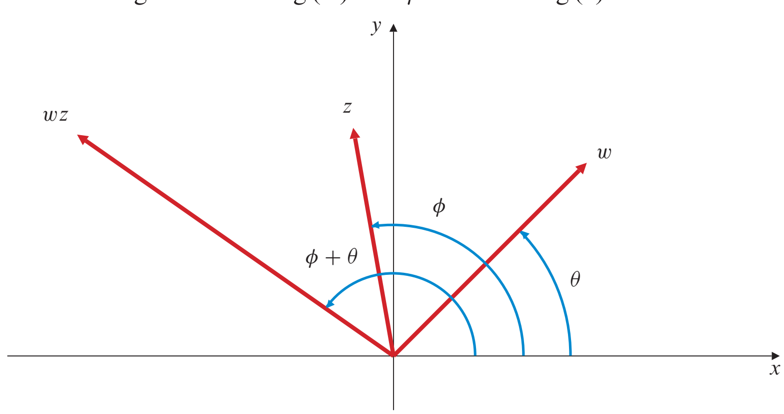

Figure I.6 The argument of a product is the sum of the arguments of the factors

More generally, if $w_1, w_2, \ldots, w_n$ are complex numbers, then

$$\begin{aligned}|w_1 w_2 \cdots w_n| &= |w_1||w_2|\cdots|w_n|\\ \arg(w_1 w_2 \cdots w_n) &= \arg(w_1) + \arg(w_2) + \cdots + \arg(w_n).\end{aligned}$$

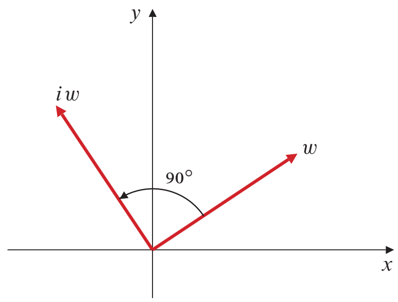

Figure I.7 Multiplication by i corresponds to counterclockwise rotation by 90°

Multiplication of a complex number by i has a particularly simple geometric interpretation in an Argand diagram. Since $|i| = 1$ and $\arg(i) = \pi/2$, multiplication of $w = a + bi$ by i leaves the modulus of w unchanged but increases its argument by $\pi/2$. (See Figure I.7.) Thus, multiplication by i rotates the position vector of w counterclockwise by 90° about the origin.

Let $z = \cos\theta + i\sin\theta$. Then $|z| = 1$ and $\arg(z) = \theta$. Since the modulus of a product is the product of the moduli of the factors and the argument of a product is the sum of the arguments of the factors, we have $|z^n| = |z|^n = 1$ and $\arg(z^n) = n\arg(z) = n\theta$. Thus,

$$z^n = \cos n\theta + i\sin n\theta,$$

and we have proved de Moivre's Theorem.

THEOREM 1

de Moivre's Theorem

$$\left(\cos\theta + i\sin\theta\right)^n = \cos n\theta + i\sin n\theta.$$

Remark Much of the study of complex-valued functions of a complex variable is beyond the scope of this book. However, in Appendix II we will introduce a complex version of the exponential function having the following property: if $z = x+iy$ (where x and y are real), then

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i\sin y).$$

Thus, the modulus of e^z is $e^{\mathrm{Re}(z)}$, and $\mathrm{Im}(z)$ is a value of $\arg(e^z)$. In this context, de Moivre's Theorem just says

$$(e^{i\theta})^n = e^{in\theta}.$$

EXAMPLE 5 Express $(1+i)^5$ in the form $a + bi$.

Solution Since $|(1+i)^5| = |1+i|^5 = \left(\sqrt{2}\right)^5 = 4\sqrt{2}$, and $\arg\left((1+i)^5\right) = 5\arg(1+i) = \dfrac{5\pi}{4}$, we have

$$(1+i)^5 = 4\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = 4\sqrt{2}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right) = -4 - 4i.$$

de Moivre's Theorem can be used to generate trigonometric identities for multiples of an angle. For example, for $n = 2$ we have

$$\cos 2\theta + i\sin 2\theta = \left(\cos\theta + i\sin\theta\right)^2 = \cos^2\theta - \sin^2\theta + 2i\cos\theta\sin\theta.$$

Thus, $\cos 2\theta = \cos^2\theta - \sin^2\theta$, and $\sin 2\theta = 2\sin\theta\cos\theta$.

The **reciprocal** of the nonzero complex number $w = a + bi$ can be calculated by multiplying the numerator and denominator of the reciprocal expression by the conjugate of w:

$$w^{-1} = \frac{1}{w} = \frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2} = \frac{\overline{w}}{|w|^2}.$$

Since $|\overline{w}| = |w|$, and $\arg(\overline{w}) = -\arg(w)$, we have

$$\left|\frac{1}{w}\right| = \frac{|\overline{w}|}{|w|^2} = \frac{1}{|w|} \qquad \text{and} \qquad \arg\left(\frac{1}{w}\right) = -\arg(w).$$

The **quotient** z/w of two complex numbers $z = x + yi$ and $w = a + bi$ is the product of z and $1/w$, so

$$\frac{z}{w} = \frac{z\overline{w}}{|w|^2} = \frac{(x+yi)(a-bi)}{a^2+b^2} = \frac{xa+yb+i(ya-xb)}{a^2+b^2}.$$

We have

The modulus and argument of a quotient

$$\left|\frac{z}{w}\right| = \frac{|z|}{|w|} \qquad \text{and} \qquad \arg\left(\frac{z}{w}\right) = \arg(z) - \arg(w).$$

The set $\arg(z/w)$ consists of all numbers $\theta - \phi$ where θ belongs to the set $\arg(z)$ and ϕ to the set $\arg(w)$.

EXAMPLE 6 Simplify (a) $\dfrac{2+3i}{4-i}$ and (b) $\dfrac{i}{1+i\sqrt{3}}$.

Solution

(a) $$\frac{2+3i}{4-i} = \frac{(2+3i)(4+i)}{(4-i)(4+i)} = \frac{8-3+(2+12)i}{4^2+1^2} = \frac{5}{17} + \frac{14}{17}i.$$

(b) $$\frac{i}{1+i\sqrt{3}} = \frac{i(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{\sqrt{3}+i}{1^2+3} = \frac{\sqrt{3}}{4} + \frac{1}{4}i.$$

Alternatively, since $|1+i\sqrt{3}| = 2$ and $\arg(1+i\sqrt{3}) = \tan^{-1}\sqrt{3} = \dfrac{\pi}{3}$, the quotient in (b) has modulus $\dfrac{1}{2}$ and argument $\dfrac{\pi}{2} - \dfrac{\pi}{3} = \dfrac{\pi}{6}$. Thus,

$$\frac{i}{1+i\sqrt{3}} = \frac{1}{2}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{\sqrt{3}}{4} + \frac{1}{4}i.$$

Roots of Complex Numbers

If a is a positive real number, there are two distinct real numbers whose square is a. These are usually denoted

$\sqrt{a}$ (the positive square root of a) and

$-\sqrt{a}$ (the negative square root of a).

Every nonzero complex number $z = x + yi$ (where $x^2 + y^2 > 0$) also has two square roots; if w_1 is a complex number such that $w_1^2 = z$, then $w_2 = -w_1$ also satisfies $w_2^2 = z$. Again, we would like to single out one of these roots and call it $\sqrt{z}$.

Let $r = |z|$, so that $r > 0$. Let $\theta = \operatorname{Arg}(z)$. Thus, $-\pi < \theta \le \pi$. Since

$$z = r\big(\cos\theta + i\sin\theta\big),$$

the complex number

$$w = \sqrt{r}\left(\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right)$$

clearly satisfies $w^2 = z$. We call this w the **principal square root** of z and denote it $\sqrt{z}$. The two solutions of the equation $w^2 = z$ are, thus, $w = \sqrt{z}$ and $w = -\sqrt{z}$.

Observe that the real part of $\sqrt{z}$ is always nonnegative, since $\cos(\theta/2) \ge 0$ for $-\pi/2 < \theta \le \pi/2$. In this interval $\sin(\theta/2) = 0$ only if $\theta = 0$, in which case $\sqrt{z}$ is real and positive.

EXAMPLE 7 (a) $\sqrt{4} = \sqrt{4(\cos 0 + i \sin 0)} = 2.$

(b) $\sqrt{i} = \sqrt{1\left(\cos\frac{\pi}{2} + i \sin\frac{\pi}{2}\right)} = \cos\frac{\pi}{4} + i \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i.$

(c) $$\sqrt{-4i} = \sqrt{4\left[\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right]} = 2\left[\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right)\right]$$
$$= \sqrt{2} - i\sqrt{2}.$$

(d) $$\sqrt{-\frac{1}{2} + i\frac{\sqrt{3}}{2}} = \sqrt{\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}} = \cos\frac{\pi}{3} + i \sin\frac{\pi}{3} = \frac{1}{2} + \frac{\sqrt{3}}{2} i.$$

Given a nonzero complex number z, we can find n distinct complex numbers w that satisfy $w^n = z$. These n numbers are called nth roots of z. For example, if $z = 1 = \cos 0 + i \sin 0$, then each of the numbers

$$\begin{aligned}
w_1 &= 1 \\
w_2 &= \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n} \\
w_3 &= \cos\frac{4\pi}{n} + i \sin\frac{4\pi}{n} \\
w_4 &= \cos\frac{6\pi}{n} + i \sin\frac{6\pi}{n} \\
&\vdots \\
w_n &= \cos\frac{2(n-1)\pi}{n} + i \sin\frac{2(n-1)\pi}{n}
\end{aligned}$$

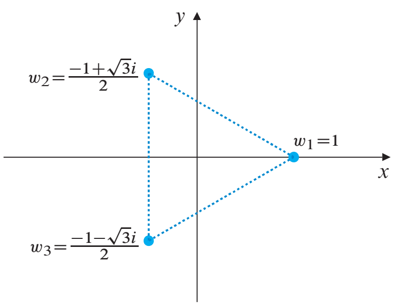

Figure I.8 The cube roots of unity

satisfies $w^n = 1$ so it is an nth root of 1. (These numbers are usually called the nth roots of unity.) Figure I.8 shows the three cube roots of 1. Observe that they are at the three vertices of an equilateral triangle with centre at the origin and one vertex at 1. In general, the n nth roots of unity lie on a circle of radius 1 centred at the origin, and at the vertices of a regular n-sided polygon with one vertex at 1.

If z is any nonzero complex number, and θ is the principal argument of z ($-\pi < \theta \le \pi$), then the number

$$w_1 = |z|^{1/n}\left(\cos\frac{\theta}{n} + i \sin\frac{\theta}{n}\right)$$

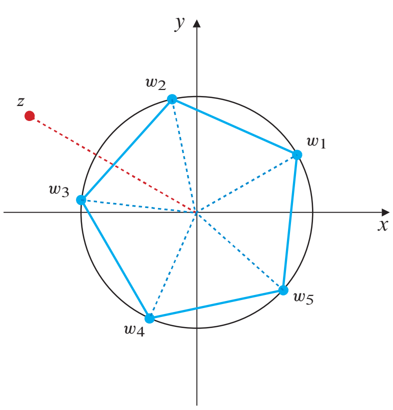

Figure I.9 The five 5th roots of z

is called the **principal** nth root of z. All the nth roots of z are on the circle of radius $|z|^{1/n}$ centred at the origin and are at the vertices of a regular n-sided polygon with one vertex at w_1. (See Figure I.9.) The other nth roots are

$$\begin{aligned}
w_2 &= |z|^{1/n}\left(\cos\frac{\theta + 2\pi}{n} + i \sin\frac{\theta + 2\pi}{n}\right) \\
w_3 &= |z|^{1/n}\left(\cos\frac{\theta + 4\pi}{n} + i \sin\frac{\theta + 4\pi}{n}\right) \\
&\vdots \\
w_n &= |z|^{1/n}\left(\cos\frac{\theta + 2(n-1)\pi}{n} + i \sin\frac{\theta + 2(n-1)\pi}{n}\right).
\end{aligned}$$

We can obtain all n of the nth roots of z by multiplying the principal nth root by the nth roots of unity.

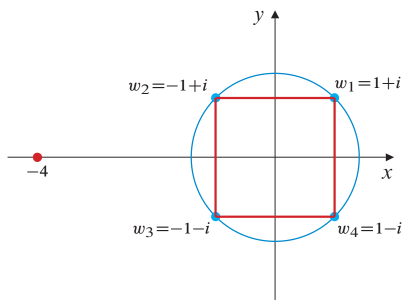

Figure I.10 The four 4th roots of -4

EXAMPLE 8 Find the 4th roots of -4. Sketch them in an Argand diagram.

Solution Since $|-4|^{1/4} = \sqrt{2}$ and $\arg(-4) = \pi$, the principal 4th root of -4 is

$$w_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = 1 + i.$$

The other three 4th roots are at the vertices of a square with centre at the origin and one vertex at $1+i$. (See Figure I.10.) Thus, the other roots are

$$w_2 = -1 + i, \qquad w_3 = -1 - i, \qquad w_4 = 1 - i.$$

EXERCISES: APPENDIX I

In Exercises 1–4, find the real and imaginary parts ($\mathrm{Re}(z)$ and $\mathrm{Im}(z)$) of the given complex numbers z, and sketch the position of each number in the complex plane (i.e., in an Argand diagram).

1. $z = -5 + 2i$ **2.** $z = 4 - i$

3. $z = -\pi i$ **4.** $z = -6$

In Exercises 5–15, find the modulus $r = |z|$ and the principal argument $\theta = \mathrm{Arg}(z)$ of each given complex number z, and express z in terms of r and θ.

5. $z = -1 + i$ **6.** $z = -2$

7. $z = 3i$ **8.** $z = -5i$

9. $z = 1 + 2i$ **10.** $z = -2 + i$

11. $z = -3 - 4i$ **12.** $z = 3 - 4i$

13. $z = \sqrt{3} - i$ **14.** $z = -\sqrt{3} - 3i$

15. $z = 3\cos\dfrac{4\pi}{5} + 3i\sin\dfrac{4\pi}{5}$

16. If $\mathrm{Arg}(z) = 3\pi/4$ and $\mathrm{Arg}(w) = \pi/2$, find $\mathrm{Arg}(zw)$.

17. If $\mathrm{Arg}(z) = -5\pi/6$ and $\mathrm{Arg}(w) = \pi/4$, find $\mathrm{Arg}(z/w)$.

In Exercises 18–23, express in the form $z = x + yi$ the complex number z whose modulus and argument are given.

18. $|z| = 2, \quad \arg(z) = \pi$ **19.** $|z| = 5, \quad \arg(z) = \tan^{-1}\dfrac{3}{4}$

20. $|z| = 1, \quad \arg(z) = \dfrac{3\pi}{4}$ **21.** $|z| = \pi, \quad \arg(z) = \dfrac{\pi}{6}$

22. $|z| = 0, \quad \arg(z) = 1$ **23.** $|z| = \dfrac{1}{2}, \quad \arg(z) = -\dfrac{\pi}{3}$

In Exercises 24–27, find the complex conjugates of the given complex numbers.

24. $5 + 3i$ **25.** $-3 - 5i$

26. $4i$ **27.** $2 - i$

Describe geometrically (or make a sketch of) the set of points z in the complex plane satisfying the given equations or inequalities in Exercises 28–33.

28. $|z| = 2$ **29.** $|z| \le 2$

30. $|z - 2i| \le 3$ **31.** $|z - 3 + 4i| \le 5$

32. $\arg z = \dfrac{\pi}{3}$ **33.** $\pi \le \arg(z) \le \dfrac{7\pi}{4}$

Simplify the expressions in Exercises 34–43.

34. $(2 + 5i) + (3 - i)$ **35.** $i - (3 - 2i) + (7 - 3i)$

36. $(4 + i)(4 - i)$ **37.** $(1 + i)(2 - 3i)$

38. $(a + bi)\overline{(2a - bi)}$ **39.** $(2 + i)^3$

40. $\dfrac{2 - i}{2 + i}$ **41.** $\dfrac{1 + 3i}{2 - i}$

42. $\dfrac{1 + i}{i(2 + 3i)}$ **43.** $\dfrac{(1 + 2i)(2 - 3i)}{(2 - i)(3 + 2i)}$

44. Prove that $\overline{z + w} = \overline{z} + \overline{w}$.

45. Prove that $\overline{\left(\dfrac{z}{w}\right)} = \dfrac{\overline{z}}{\overline{w}}$.

46. Express each of the complex numbers $z = 3 + i\sqrt{3}$ and $w = -1 + i\sqrt{3}$ in polar form (i.e., in terms of its modulus and argument). Use these expressions to calculate zw and z/w.

47. Repeat Exercise 46 for $z = -1 + i$ and $w = 3i$.

48. Use de Moivre's Theorem to find a trigonometric identity for $\cos 3\theta$ in terms of $\cos\theta$ and one for $\sin 3\theta$ in terms of $\sin\theta$.

49. Describe the solutions, if any, of the equations (a) $\overline{z} = 2/z$ and (b) $\overline{z} = -2/z$.

50. For positive real numbers a and b it is always true that $\sqrt{ab} = \sqrt{a}\sqrt{b}$. Does a similar identity hold for $\sqrt{zw}$, where z and w are complex numbers? *Hint:* Consider $z = w = -1$.

51. Find the three cube roots of -1.

52. Find the three cube roots of $-8i$.

53. Find the three cube roots of $-1 + i$.

54. Find all the fourth roots of 4.

55. Find all complex solutions of the equation $z^4 + 1 - i\sqrt{3} = 0$.

56. Find all solutions of $z^5 + a^5 = 0$, where a is a positive real number.

! **57.** Show that the sum of the n nth roots of unity is zero. *Hint:* Show that these roots are all powers of the principal root.