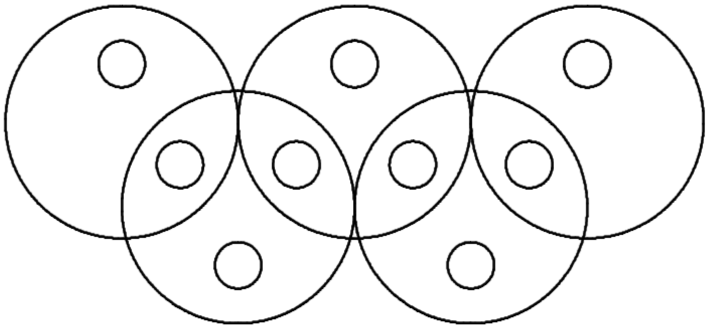
# 2007 Changchun Invitational World Youth Mathematics Intercity Competition

# Team Contest

2007/7/23 Changchun, China

**Team**: ________________________ **Score**: ________________

1. Use each of the numbers 1, 2, 3, 4, 5, 6, 7, 8 and 9 exactly once to fill in the nine small circles in the Olympic symbol below, so that the numbers inside each large circle is 14.



**Solution:**

The sum of the nine numbers is 1+2+3+4+5+6+7+8+9=45. The total sum of the numbers in the five large circles is 14×5=70. The difference 70 - 45=25 is the sum of the four numbers in the middle row, because each appears in two large circles. The two numbers at one end must be 9 and 5 while the two numbers at the other end must be 8 and 6. Consider the two numbers at the end of the middle row. Clearly they cannot be 5 and 6. If they are 5 and 8, the other two numbers must sum to 12. With 5 and 8 gone, the only possibility is 9 and 3, but 9 cannot be in the inner part of the middle row. If they are 9 and 8, the other two numbers must sum to 8. Since neither 5 nor 6 can appear in the inner part of the middle row, the only possibility is 7 and 1. However, 7 cannot be in the same large circle with either 9 or 8. It follows that the two numbers at the end of the middle row are 9 and 6, and the other two numbers sum to 10. The only possibility is 7 and 3, and 7 must be in the same large circle with 6. The remaining numbers can now be filled in easily, and the 9-digit number formed is 861743295.

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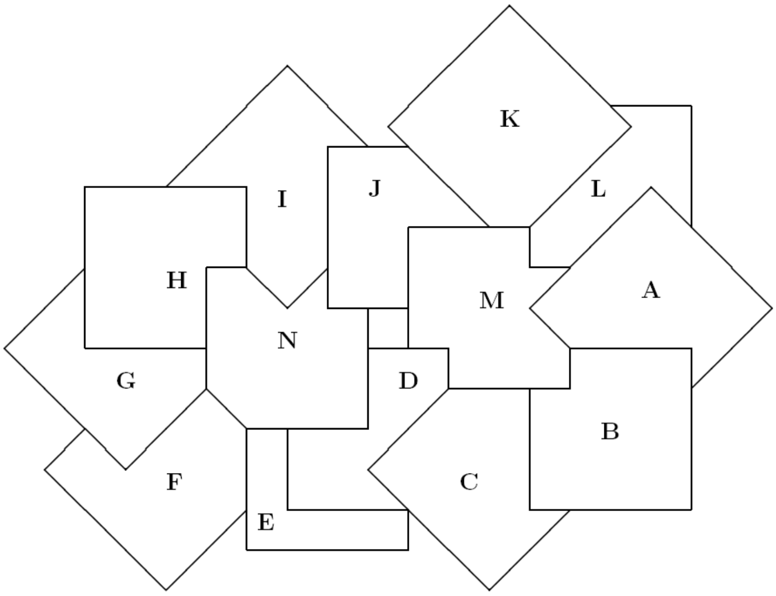
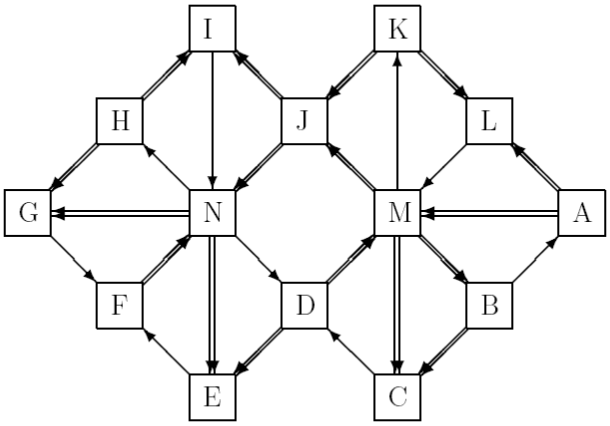
2. The diagram below shows fourteen pieces of paper stacked on top of one another. Beginning on the pieces marked B, move from piece to adjacent piece in order to finish at the piece marked F. The path must alternately climb up to a piece of paper stacked higher and come down to a piece of paper stacked lower. The same piece may be visited more than once, and it is not necessary to visit every piece. List the pieces of paper in the order visited.




**Solution:**

We construct below a diagram which is easier to use. An arrow from one piece of paper to another represents coming down from the first to the second. Note that each of $M$ and $N$ is connected to 7 other pieces, each of $D$ and $J$ is connected to 4 other pieces, while each of the others is connected to 3 other pieces. The path we seek consists of alternately going along with the arrow and going against it. Of the three arrows at $A$, the one from $B$ cannot be used as otherwise we would be stuck at $A$. Equally useless are the arrows from $L$ to $M$, from $M$ to $K$, from $I$ to $N$, from $N$ to $H$, from $G$ to $F$, from $E$ to $F$, and from $C$ to $D$. In the diagram below, they are drawn as single arrows while the useable ones are drawn as double arrows.

From *B*, if we climb up to *C*, then we will go down to *M*, but we could have gone down to *M* directly. Once in *M*, we have a choice of *C* or *J*, but *C* leads back to *B*, and we will get stuck there. From *J*, we have to move onto *K*, *L*, *A*, *M*, *D*, *E*, *N*, *G*, *H*, *I*, *J*, *N* and *F*. Thus the path is *BMJKLAMDENGHIJNF*.



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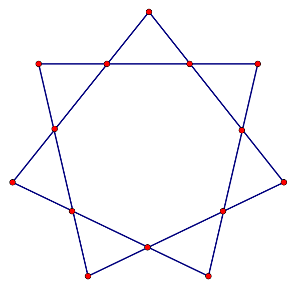
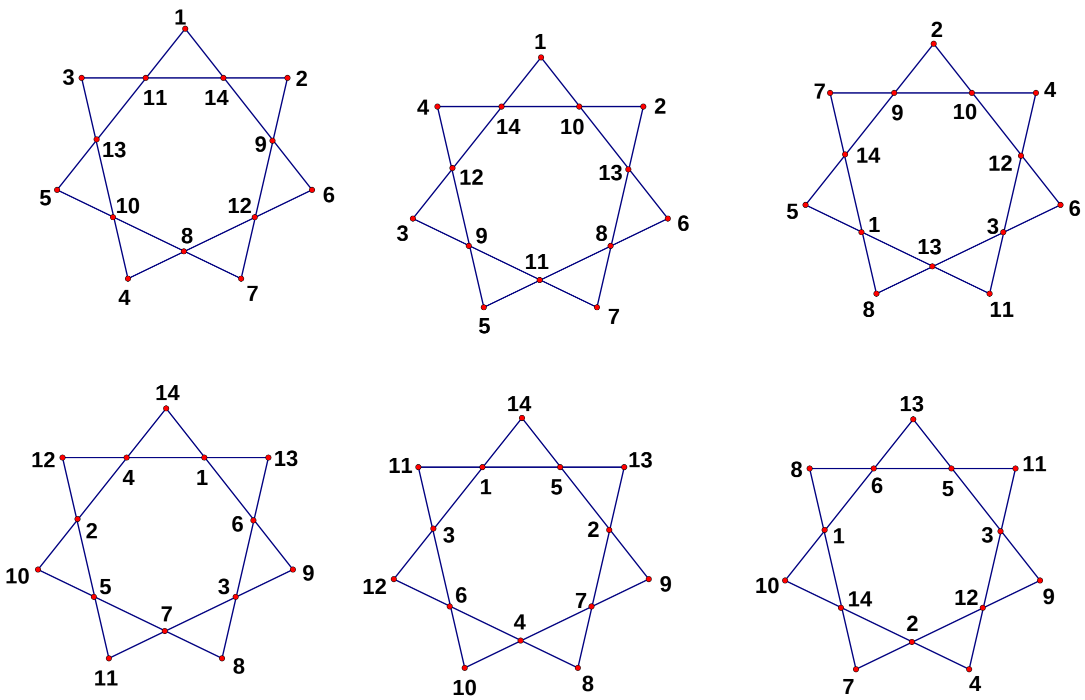
3. There are 14 points of intersection in the seven-pointed star in the diagram on the right. Label these points with the numbers 1, 2, 3, …, 14 such that the sum of the labels of the four points on each line is the same. Give one of Solution.



**Solution:**

Each of the points of intersection lies on exactly two lines. Hence the common sum is given by 2(1+2+3+…+14)/7=30. We claim that the smallest label on the same line with 14 is 1 or 2. Otherwise, the sum of the labels of the two lines is at least 14+14+3+4+5+6+7+8=61>30+30, which is a contradiction. Similarly, the smallest label on the same line with 13 is 1, 2 or 3. In view of these observations, we put 14 on a line with 1 and on another line with 2, and 13 on a line with 1 and on another line with 2 or 3. This leads to the three labeling on the top row. The labeling on the bottom row are the complements of the corresponding ones in the top row, that is, each label k is replaced by 15 - k.



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4. Mary found a 3-digit number that, when multiplied by itself, produced a number which ended in her 3-digit number. What is the sum of the numbers which have this property?

**Solution:**

Since $1\times1=1$, $5\times5=25$ and $6\times6=36$, the last digit of the 3-digit number must be either 1, 5 or 6.

No 2-digit number with units digit 1 whose square ends with that number.

There is only one 2-digit number with units digit 5 whose square ends with that number and that is 25.

There is only one 2-digit number with units digit 6 whose square ends with that number and that is 76.

There is only one 3-digit number which ends in 25 whose square ends with that number and that is 625.

There is only one 3-digit number which ends in 76 whose square ends with that number and that is 376

The sum of these two numbers is 625+376=1001.

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5. Determine all positive integers $m$ and $n$ such that $m^2+1$ is a prime number and $10(m^2+1)=n^2+1$.

**Solution:**

From the given condition, $9(m^2+1)=(n+m)(n-m)$. Note that $m^2+1$ is a prime number not equal to 3. Hence there are four cases.

(1) $n-m=1$, $n+m=9(m^2+1)$.

Subtraction yields $9m^2+8=2m$, which is impossible.

(2) $n-m=3$, $n+m=3(m^2+1)$.

Subtraction yields $3m^2=2m$, which is impossible.

(3) $n-m=9$, $n+m=m^2+1$.

Subtraction yields $2m=m^2-8$, so that $m=4$ and $n=13$. Note that $m^2+1=17$ is indeed a prime number.

(4) $n-m=m^2+1$, $n+m=9$.

Subtraction yields $-2m=m^2-8$, so that $m=2$ and $n=7$. Note that $m^2+1=5$ is indeed a prime number.

In summary, there are two solutions, $(m,n)=(2,7)$ or $(4,13)$.

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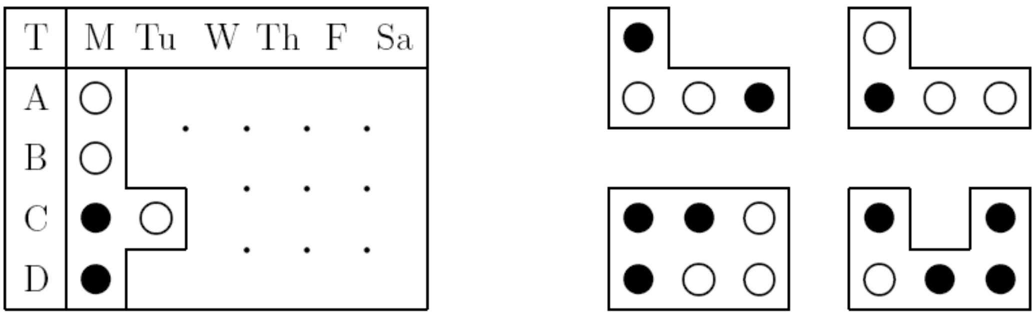
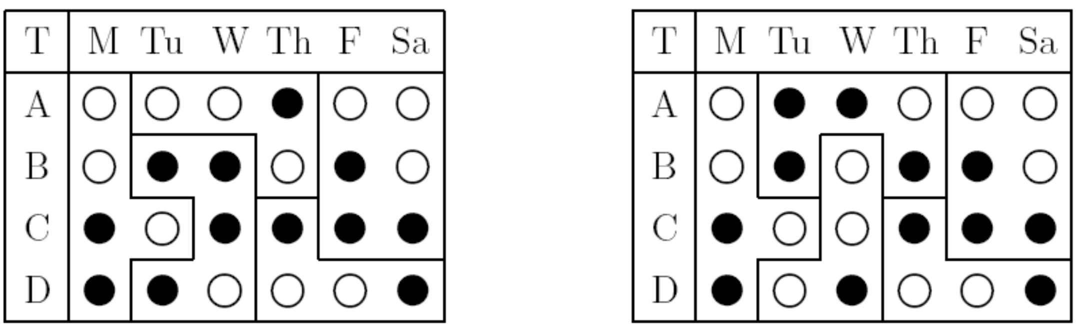
6. Four teams take part in a week-long tournament in which every team plays every other team twice, and each team plays one game per day. The diagram below on the left shows the final scoreboard, part of which has broken off into four pieces, as shown on the diagram below on the right. These pieces are printed only on one side. A black circle indicates a victory and a white circle indicates a defeat. Which team wins the tournament?




**Solution:**

When reconstructing the broken scoreboard, there are two positions where the U-shaped piece can be placed. So as to leave room for the 3×2 rectangle. Once it is in place, the positions for the remaining pieces are determined. They are placed so that there are two black circles and two white circles in each column. There are two possibilities, as shown in the diagrams below, but in either case, the winner of the tournament is Team C.



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7. Let $A$ be a 3 by 3 array consisting of the numbers 1, 2, 3, …, 9. Compute the sum of the three numbers on the $i$-th row of $A$ and the sum of the three numbers on the $j$-th column of $A$. The number at the intersection of the $i$-th row and the $j$-th column of a 3 by 3 array $B$ is equal to the absolute difference of these two sums. For example, $b_{12}=\left|(a_{11}+a_{12}+a_{13})-(a_{12}+a_{22}+a_{32})\right|$.
Is it possible to arrange the numbers in $A$ so that the numbers in $B$ are also 1, 2, 3, …, 9?

| | | |
|---|---|---|
| $a_{11}$ | $a_{12}$ | $a_{13}$ |
| $a_{21}$ | $a_{22}$ | $a_{23}$ |
| $a_{31}$ | $a_{32}$ | $a_{33}$ |

$A$

| | | |
|---|---|---|
| $b_{11}$ | $b_{12}$ | $b_{13}$ |
| $b_{21}$ | $b_{22}$ | $b_{23}$ |
| $b_{31}$ | $b_{32}$ | $b_{33}$ |

$B$

**Solution:**

Let C be defined just like B, except that we use the actual difference instead of the absolute difference. Compute the sum of the nine numbers in C. Each number in A appears twice in this sum, once with a positive sign and once with a negative sign. Hence this sum is 0. It follows that among the nine numbers in C, the number of those which are odd is even. The same is true of the nine numbers in B, since taking the absolute value does not affect parity. Thus it is not possible for the nine numbers in B be 1, 2, 3, …, 9.

| | | |
|---|---|---|
| $c_{11}$ | $c_{12}$ | $c_{13}$ |
| $c_{21}$ | $c_{22}$ | $c_{23}$ |
| $c_{31}$ | $c_{32}$ | $c_{33}$ |

$C$

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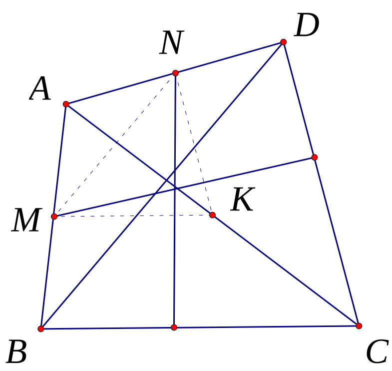
8. The diagonals $AC$ and $BD$ of a convex quadrilateral are perpendicular to each other. Draw a line that passes through point $M$, the midpoint of $AB$ and perpendicular to $CD$, draw another line through point $N$, the midpoint of $AD$ and perpendicular to $CB$. Prove that the point of intersection of these two lines lies on the line $AC$.

**Solution:**

Let $M$, $K$ and $N$ be the respective midpoints of $AB$, $AC$ and $AD$. Then $MN$ is parallel to $BD$, $MK$ is parallel to $BC$ and $NK$ is parallel to $CD$. Hence $AC$ and the two lines in question are the altitudes of triangle $MNK$, and are therefore concurrent.



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9. The positive integers from 1 to $n$ (where $n$>1 are arranged in a line such that the sum of any two adjacent numbers is a square. What is the minimum value of $n$?

**Solution:**

The minimum value of $n$ is 15. Since $n$>1, we must include 2, so that $n$≧7 because 2+7=9. For $n$=7, we have three separate lines (1,3,6), (2,7) and (4,5). Adding 8 only lengthen the first to (8,1,3,6). Adding 9 now only lengths the second to (2,7,9). Hence $n$≧10. Now 8, 9 and 10 all have to be at the end if we have a single line, because we can only have 8+1=9, 9+7=16 and 10+6=16. The next options are 8+17=25, 9+16=25 and 10+15=25. Hence $n$≧15. For $n$=15, we have the arrangement 8, 1, 15, 10, 6, 3, 13, 12, 4, 5, 11, 14, 2, 7 and 9.

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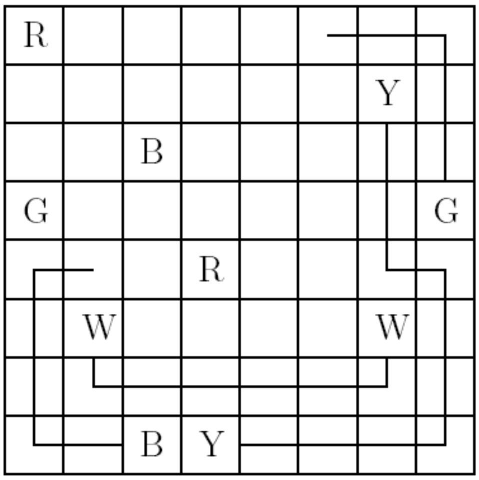
10. Use one of the five colours (R represent red, Y represent yellow, B represent blue, G represent green and W represent white) to paint each square of an 8×8 chessboard, as shown in the diagram below. Then paint the rest of the squares so that all the squares of the same colour are connected to one another edge to edge. What is the largest number of squares of the same colour as compare to the other colours?

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| R | | | | | | | |
| | | | | | | Y | |
| | | B | | | | | |
| G | | | | | | | G |
| | | | R | | | | |
| | W | | | | | W | |
| | | | | | | | |
| | | B | Y | | | | |

**Solution:**

While it may tempting to colour the entire fourth row green, this will divide the red squares, the yellow squares and the blue squares into two disconnected parts. Obviously, the northeast corner is to be used to allow the green path to get around the yellow path. Similarly, the southwest corner is to be used to allow the blue path to get around the white path, and the southeast corner is to be used to allow the yellow path to get around the white path. In fact, we can complete the entire yellow path. Also, the white path may as well make full use of the seventh row. This brings us to the configuration as shown in the diagram below on the left. It is now not hard to complete the entire configuration, which is shown in the diagram below on the right. The longest path is the green one, and the number of green squares is 24.




| | | | | | | | |
|---|---|---|---|---|---|---|---|
| R | R | R | R | R | G | G | G |
| G | G | G | G | R | G | Y | G |
| G | B | B | G | R | G | Y | G |
| G | B | G | G | R | G | Y | G |
| B | B | G | R | R | G | Y | Y |
| B | W | G | G | G | G | W | Y |
| B | W | W | W | W | W | W | Y |
| B | B | B | Y | Y | Y | Y | Y |