Using the "Freshman's Dream" to Prove Combinatorial Congruences

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Abstract. Recently, William Y.C. Chen, Qing-Hu Hou, and Doron Zeilberger developed an algorithm for finding and proving congruence identities (modulo primes) of indefinite sums of many combinatorial sequences, namely those (like the Catalan and Motzkin sequences) that are expressible in terms of constant terms of powers of Laurent polynomials. We first give a leisurely exposition of their approach and then extend it in two directions. The Laurent polynomials may be of several variables, and instead of single sums we have multiple sums. In fact, we even combine these two generalizations. We conclude with some super-challenges.

1. INTRODUCTION. In the article [**4**], the following type of quantities were considered:

$$\left(\sum_{k=0}^{rp-1} a(k)\right) \bmod p\,,$$

where,

1. $a(k)$ is a combinatorial sequence, expressible as the constant term of a power of a Laurent polynomial of a single variable (for example, the central binomial coefficient $\binom{2k}{k}$ is the coefficient of x^0 in $(x+\frac{1}{x})^{2k}$).
2. r is a specific positive integer.
3. p is an arbitrary prime.

Let $x \equiv_p y$ mean $x \equiv y \pmod{p}$, in other words, that $x - y$ is divisible by p. The method in [**4**], while ingenious, is very elementary. The main "trick" is:

The freshman's dream identity ([**10**]): $(a+b)^p \equiv_p a^p + b^p$.

Recall that the easy proof follows from the binomial theorem and noting that $\binom{p}{k}$ is divisible by p except when $k = 0$ and $k = p$. This also leads to one of the many proofs of the grandmother of all congruences, Fermat's little theorem, $a^p \equiv_p a$, by starting with $0^p \equiv_p 0$ and applying induction to $(a+1)^p \equiv_p a^p + 1^p$.

The second ingredient in the method in [**4**] is even more elementary. It is:

Sum of a Geometric Series: $\displaystyle\sum_{i=0}^{n-1} z^i = \frac{z^n - 1}{z - 1}.$

The focus in the Chen–Hou–Zeilberger article [**4**] was both computer-algebra implementation and proving a general theorem about a wide class of sums. Their paper is rather technical, hence the first purpose of the present article is to give a leisurely

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introduction to their method, and illustrate it with numerous simple examples. The second, main, purpose, however, is to extend the method in two directions. The summand $a(k)$ may be the constant term of a Laurent polynomial of several variables, and instead of a single summation sign, we can have *multisums*. In fact we can combine these two.

2. NOTATION. The constant term of a Laurent polynomial $P(x_1, x_2, \ldots, x_n)$, alias the coefficient of $x_1^0 x_2^0 \cdots x_n^0$, is denoted by $CT[P(x_1, x_2, \ldots, x_n)]$. The general coefficient of $x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}$ in $P(x_1, x_2, \ldots, x_n)$ is denoted by

$$[x_1^{m_1} x_2^{m_2} \cdots x_n^{m_n}]\, P(x_1, x_2, \ldots, x_n)\,.$$

Example 1.

$$CT\left[\frac{1}{xy} + 3 + 5xy - x^3 + 6y^2\right] = 3\,, \quad [xy]\left[\frac{1}{xy} + 3 + 5xy + x^3 + 6y^2\right] = 5.$$

We use the symmetric representation of integers in $(-\frac{p}{2}, \frac{p}{2}]$ when reducing modulo a prime p.

Example 2. $6 \pmod 5 = 1$ and $4 \pmod 5 = -1$.

3. REVIEW OF THE CHEN–HOU–ZEILBERGER SINGLE VARIABLE CASE. In order to motivate our generalization, we will first review, in more detail than given in [**4**], some of the results of [**4**]. Let's start with the central binomial coefficients, sequence A000984 in the great OEIS ([**7**], https://oeis.org/A000984).

Proposition 1. *For any prime $p \geq 5$, we have*

$$\sum_{n=0}^{p-1} \binom{2n}{n} \equiv_p \begin{cases} 1, & \text{if } p \equiv 1 \mod 3 \\ -1, & \text{if } p \equiv 2 \mod 3. \end{cases}$$

Proof. Using the fact that

$$\binom{2n}{n} = CT\left[\frac{(1+x)^{2n}}{x^n}\right],$$

and the freshman's dream identity, $(a+b)^p \equiv_p a^p + b^p$, we have

$$\sum_{n=0}^{p-1} \binom{2n}{n} = \sum_{n=0}^{p-1} CT\left[\left(\frac{(1+x)^{2n}}{x^n}\right)\right] = \sum_{n=0}^{p-1} CT\left[\left(2 + x + \frac{1}{x}\right)^n\right]$$

$$= CT\left[\frac{\left(2 + x + \frac{1}{x}\right)^p - 1}{2 + x + \frac{1}{x} - 1}\right] \equiv_p CT\left[\frac{2^p + x^p + \frac{1}{x^p} - 1}{1 + x + \frac{1}{x}}\right]$$

(by freshman's dream)

$$\equiv_p CT\left[\frac{2 + x^p + \frac{1}{x^p} - 1}{1 + x + \frac{1}{x}}\right] \qquad \text{(by Fermat's little theorem)}$$

$$= CT\left[\frac{1+x^p+\frac{1}{x^p}}{1+x+\frac{1}{x}}\right] = CT\left[\frac{1+x^p+x^{2p}}{(1+x+x^2)x^{p-1}}\right]$$
$$= [x^{p-1}]\left[\frac{1}{1+x+x^2}\right] = [x^{p-1}]\left[\frac{1-x}{1-x^3}\right]$$
$$= [x^p]\left(\sum_{i=0}^{\infty} x^{3i+1}\right) + [x^p]\left(\sum_{i=0}^{\infty}(-1)\cdot x^{3i+2}\right).$$

The result follows from extracting the coefficient of x^p in the above geometric series. ■

Proposition 1′.

$$\sum_{n=0}^{2p-1}\binom{2n}{n} \equiv_p \begin{cases} 3, & \text{if } p \equiv 1 \mod 3 \\ -3, & \text{if } p \equiv 2 \mod 3. \end{cases}$$

Proof.

$$\sum_{n=0}^{2p-1}\binom{2n}{n} = \sum_{n=0}^{2p-1} CT\left[\left(2+x+\frac{1}{x}\right)^n\right] = CT\left[\frac{\left(2+x+\frac{1}{x}\right)^{2p}-1}{2+x+\frac{1}{x}-1}\right]$$
$$= CT\left[\frac{\left(6+4x+\frac{4}{x}+x^2+\frac{1}{x^2}\right)^p-1}{2+x+\frac{1}{x}-1}\right]$$
$$\equiv_p CT\left[\frac{\left(6+4x^p+\frac{4}{x^p}+x^{2p}+\frac{1}{x^{2p}}\right)-1}{2+x+\frac{1}{x}-1}\right].$$

Obviously, only the terms $\frac{4}{x^p}$ and $\frac{1}{x^{2p}}$ contribute to the constant term. Discarding all the other ones and simplifying, we get that this equals

$$[x^{2p-1}]\left[\frac{1+4x^p}{1+x+x^2}\right] = [x^{2p-1}]\left[\frac{1}{1+x+x^2}\right] + 4\cdot[x^{p-1}]\left[\frac{1}{1+x+x^2}\right]$$
$$= [x^{2p-1}]\left[\frac{1-x}{1-x^3}\right] + 4\cdot[x^{p-1}]\left[\frac{1-x}{1-x^3}\right]$$
$$= [x^{2p-1}]\left[\frac{1}{1-x^3}\right] + [x^{2p-1}]\left[\frac{-x}{1-x^3}\right]$$
$$+ 4\cdot[x^{p-1}]\left[\frac{1}{1-x^3}\right] + 4\cdot[x^{p-1}]\left[\frac{-x}{1-x^3}\right]$$
$$= [x^{2p}]\left[\sum_{i=0}^{\infty} x^{3i+1}\right] + [x^{2p}]\left[\sum_{i=0}^{\infty}(-1)\cdot x^{3i+2}\right]$$
$$+ 4\cdot[x^p]\left[\sum_{i=0}^{\infty} x^{3i+1}\right] + 4\cdot[x^p]\left[\sum_{i=0}^{\infty}(-1)\cdot x^{3i+2}\right].$$

The result follows from extracting the coefficients of x^{2p} in the first two geometric series above and the coefficient of x^p in the last two. ■

The same method (of [**4**]) can be used to find the "mod p" of $\sum_{n=0}^{rp-1}\binom{2n}{n}$ for any positive integer r. This leads to the following proposition, whose somewhat tedious proof we omit.

Proposition 1″. *For any prime $p \geq 5$ and any positive integer r,*

$$\sum_{n=0}^{rp-1}\binom{2n}{n} \equiv_p \begin{cases} \alpha_r, & \text{if } p \equiv 1 \mod 3 \\ -\alpha_r, & \text{if } p \equiv 2 \mod 3, \end{cases}$$

where

$$\alpha_r = \sum_{n=0}^{r-1}\binom{2n}{n}.$$

For the record, here are the first ten terms of the integer sequence α_r:

$$1, 3, 9, 29, 99, 351, 1275, 4707, 17577, 66187.$$

The sequence α_r is Sequence A6134 ([**7**], https://oeis.org/A006134). Note that α_r is the number of ways of tossing a coin less than $2r$ times and getting as many Heads as Tails.

The most *ubiquitous* sequence in combinatorics is sequence A000108 in the OEIS ([**7**], https://oeis.org/A000108, that, according to Neil Sloane, is the longest entry), the super-famous *Catalan numbers*, $C_n := \dfrac{(2n)!}{n!(n+1)!}$, that count *zillions* of combinatorial families (see [**8**] for some of the more interesting ones).

Proposition 2. *Let C_n denote the nth Catalan number. Then, for every prime $p \geq 5$,*

$$\sum_{n=0}^{p-1} C_n \equiv_p \begin{cases} 1, & \text{if } p \equiv 1 \mod 3 \\ -2, & \text{if } p \equiv 2 \mod 3. \end{cases}$$

Proof. Since $C_n = \binom{2n}{n} - \binom{2n}{n-1}$, it is readily seen that

$$C_n = CT\left[(1-x)\left(2+x+\frac{1}{x}\right)^n\right].$$

We have

$$\sum_{n=0}^{p-1} C_n = \sum_{n=0}^{p-1} CT\left[(1-x)\left(2+x+\frac{1}{x}\right)^n\right] = CT\left[\frac{(1-x)\left(\left(2+x+\frac{1}{x}\right)^p - 1\right)}{2+x+\frac{1}{x}-1}\right]$$

$$\equiv_p CT\left[\frac{(1-x)\left(\left(2+x^p+\frac{1}{x^p}\right)-1\right)}{2+x+\frac{1}{x}-1}\right] \quad \text{(by freshman's dream).}$$

Since only the term $\frac{1}{x^p}$ in the numerator contributes to the constant term, this equals

$$\begin{aligned}[x^{p-1}]\left[\frac{1-x}{1+x+x^2}\right] &= [x^{p-1}]\left[\frac{(1-x)^2}{1-x^3}\right] \\ &= [x^p]\left[\frac{x}{1-x^3}\right] + [x^p]\left[\frac{-2x^2}{1-x^3}\right] + [x^p]\left[\frac{x^3}{1-x^3}\right] \\ &= [x^p]\left[\sum_{i=0}^{\infty} 1\cdot x^{3i+1}\right] + [x^p]\left[\sum_{i=0}^{\infty}(-2)\cdot x^{3i+2}\right] + [x^p]\left[\sum_{i=0}^{\infty} 1\cdot x^{3i+3}\right],\end{aligned}$$

and the result follows from extracting the coefficient of x^p from the first or second geometric series above. (Note that we would never have to use the third geometric series since $p > 3$.) ■

The same method (of [**4**]) can be used to find the mod p of $\sum_{n=0}^{rp-1} C_n$ for any *specific* positive integer r. In fact, one can keep r general, but then the proof is rather tedious, and we will spare the readers (and ourselves, from typing it).

Proposition 2′. *Let C_n denote the nth Catalan number. Then, for any positive integer r, we have*

$$\sum_{n=0}^{rp-1} C_n \equiv_p \begin{cases} \beta_r, & \text{if } p \equiv 1 \mod 3 \\ -\gamma_r, & \text{if } p \equiv 2 \mod 3, \end{cases}$$

where

$$\beta_r = \sum_{n=0}^{r-1} C_n, \quad \gamma_r = \sum_{n=0}^{r-1}(3n+2)C_n.$$

For the record, the first ten terms of the sequence of integer pairs $[\beta_r, -\gamma_r]$ are $[1,-2]$, $[2,-7]$, $[4,-23]$, $[9,-78]$, $[23,-274]$, $[65,-988]$, $[197,-3628]$, $[626,-13495]$, $[2076,-50675]$, $[6918,-191673]$. We note that the sequence β_r is sequence A014137 in the OEIS ([**7**], https://oeis.org/A014137) but at this time of writing (June 9, 2016), the sequence γ_r is not there (yet).

Not as famous as the Catalan numbers, but not exactly obscure, are the *Motzkin numbers*, M_n, sequence A001006 in the OEIS ([**7**], https://oeis.org/A001006), that may be defined by the constant term formula

$$M_n = CT\left[(1-x^2)\left(1+x+\frac{1}{x}\right)^n\right].$$

Proposition 3. *Let M_n denote the nth Motzkin number. Then, for any prime $p \geq 3$, we have*

$$\sum_{n=0}^{p-1} M_n \equiv_p \begin{cases} 2, & \text{if } p \equiv 1 \mod 4 \\ -2, & \text{if } p \equiv 3 \mod 4. \end{cases}$$

Proof.

$$\sum_{n=0}^{p-1} M_n = \sum_{n=0}^{p-1} CT\left[(1-x^2)\left(1+x+\frac{1}{x}\right)^n\right] = CT\left[\frac{(1-x^2)\left(\left(1+x+\frac{1}{x}\right)^p-1\right)}{1+x+\frac{1}{x}-1}\right]$$

$$\equiv_p CT\left[\frac{(1-x^2)\left(1+x^p+\frac{1}{x^p}-1\right)}{1+x+\frac{1}{x}-1}\right] = CT\left[\frac{(1-x^2)\left(x^p+\frac{1}{x^p}\right)}{x+\frac{1}{x}}\right]$$

$$= CT\left[\frac{x(1-x^2)\left(x^p+\frac{1}{x^p}\right)}{1+x^2}\right]$$

$$= [x^{p-1}]\left[\frac{1-x^2}{1+x^2}\right] = [x^p]\left[\frac{x}{1+x^2}\right] - [x^p]\left[\frac{x^3}{1+x^2}\right]$$

$$= [x^p]\left[\sum_{i=0}^{\infty}(-1)^i x^{2i+1}\right] + [x^p]\left[\sum_{i=0}^{\infty}(-1)^{i+1}x^{2i+3}\right],$$

and the result follows from extracting the coefficient of x^p from the first and second geometric series above by noting that when $p \equiv 1 \pmod 4$, i is even in the first series and odd in the second one, and vice versa when $p \equiv 3 \pmod 4$. ∎

The same method, applied to a general r, yields the following.

Proposition 3′. *Let M_n denote the nth Motzkin number. Then, for any prime $p \geq 3$:*

$$\sum_{n=0}^{rp-1} M_n \equiv_p \begin{cases} 2\delta_r, & \text{if } p \equiv 1 \mod 4 \\ -2\delta_r, & \text{if } p \equiv 3 \mod 4, \end{cases}$$

where δ_r is the sequence of partial sums of the central trinomial coefficients, sequence A097893 in the OEIS([7], https://oeis.org/A097893) whose generating function is

$$\sum_{r=0}^{\infty} \delta_r x^r = \frac{1}{(1-x)\sqrt{(1+x)(1-3x)}}.$$

4. MULTISUMS AND MULTIVARIABLES. We now extend the Chen–Hou–Zeilberger method for discovery and proof of congruence theorems to multisums and multivariables.

Proposition 4. *Let $p \geq 5$ be prime; then*

$$\sum_{n=0}^{p-1}\sum_{m=0}^{p-1}\binom{n+m}{m}^2 \equiv_p \begin{cases} 1, & \text{if } p \equiv 1 \mod 3 \\ -1, & \text{if } p \equiv 2 \mod 3. \end{cases}$$

Proof. Let $P(x,y) = (1+y)\left(1+\frac{1}{x}\right)$ and $Q(x,y) = (1+x)\left(1+\frac{1}{y}\right)$. Then

$$\binom{n+m}{m}^2 = \binom{n+m}{m}\binom{n+m}{n} = CT\left[P(x,y)^n Q(x,y)^m\right].$$

We have

$$\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\binom{m+n}{m}^2 = \sum_{m=0}^{p-1}\sum_{n=0}^{p-1} CT\left[P(x,y)^n Q(x,y)^m\right]$$
$$= CT\left[\sum_{m=0}^{p-1}\left[\frac{(P(x,y)^p-1)Q(x,y)^m}{P(x,y)-1}\right]\right]$$
$$= CT\left[\left(\frac{P(x,y)^p-1}{P(x,y)-1}\right)\left(\frac{Q(x,y)^p-1}{Q(x,y)-1}\right)\right].$$

Using the freshman's dream, $(a+b)^p \equiv_p a^p + b^p$, we can pass to mod p as above and get

$$\sum_{m=0}^{p-1}\sum_{n=0}^{p-1}\binom{m+n}{m}^2 \equiv_p CT\left[\left(\frac{P(x^p,y^p)-1}{P(x,y)-1}\right)\left(\frac{Q(x^p,y^p)-1}{Q(x,y)-1}\right)\right]$$
$$= CT\left[\frac{(1+y^p+x^py^p)(1+x^p+x^py^p)}{(1+y+xy)(1+x+xy)x^{p-1}y^{p-1}}\right]$$
$$= [x^{p-1}y^{p-1}]\left[\frac{(1+y^p+x^py^p)(1+x^p+x^py^p)}{(1+y+xy)(1+x+xy)}\right]$$
$$= [x^{p-1}y^{p-1}]\left[\frac{1}{(1+y+xy)(1+x+xy)}\right].$$

It is possible to show that the coefficient of $x^n y^n$ in the Maclaurin expansion of the rational function $\frac{1}{(1+y+xy)(1+x+xy)}$ is 1 when $n \equiv 0 \pmod 3$, -1 when $n \equiv 1 \pmod 3$, and 0 when $n \equiv 2 \pmod 3$. One way is to do a partial fraction decomposition and extract the coefficient of x^n, getting a certain expression in y and n, and then extract the coefficient of y^n. Another way is by using the Apagodu–Zeilberger algorithm ([**2**]), which outputs that the sequence of diagonal coefficients, let's call them $a(n)$, satisfy the recurrence equation $a(n+2)+a(n+1)+a(n)=0$, with initial conditions $a(0)=1, a(1)=-1$. ■

A bit of more work, which we omit, leads to the following.

Proposition 4′. *For any prime $p \geq 5$ and any pair of positive integers r, s, we have*

$$\sum_{n=0}^{rp-1}\sum_{m=0}^{sp-1}\binom{n+m}{m}^2 \equiv_p \begin{cases} \epsilon_{rs}, & \text{if } p \equiv 1 \mod 3 \\ -\epsilon_{rs}, & \text{if } p \equiv 2 \mod 3, \end{cases}$$

where

$$\epsilon_{rs} = \sum_{m=0}^{r-1}\sum_{n=0}^{s-1}\binom{n+m}{m}^2.$$

We finally consider partial sums of *trinomial coefficients*.

Proposition 5. *Let $p > 2$ be prime; then we have*

$$\sum_{m_1=0}^{p-1}\sum_{m_2=0}^{p-1}\sum_{m_3=0}^{p-1}\binom{m_1+m_2+m_3}{m_1,m_2,m_3} \equiv_p 1.$$

Proof. First observe that $\binom{m_1+m_2+m_3}{m_1,m_2,m_3} = CT\left[\frac{(x+y+z)^{m_1+m_2+m_3}}{x^{m_1}y^{m_2}z^{m_3}}\right]$. Hence,

$$\sum_{0\le m_1,m_2,m_3\le p-1}\binom{m_1+m_2+m_3}{m_1,m_2,m_3} = \sum_{0\le m_1,m_2,m_3\le p-1} CT\left[\frac{(x+y+z)^{m_1+m_2+m_3}}{x^{m_1}y^{m_2}z^{m_3}}\right]$$

$$= CT\left[\sum_{0\le m_1,m_2,m_3\le p-1}\frac{(x+y+z)^{m_1+m_2+m_3}}{x^{m_1}y^{m_2}z^{m_3}}\right]$$

$$= CT\left[\left(\sum_{m_1=0}^{p-1}\left(\frac{x+y+z}{x}\right)^{m_1}\right)\left(\sum_{m_2=0}^{p-1}\left(\frac{x+y+z}{x}\right)^{m_2}\right)\left(\sum_{m_3=0}^{p-1}\left(\frac{x+y+z}{x}\right)^{m_3}\right)\right]$$

$$= CT\left[\frac{(\frac{x+y+z}{x})^p-1}{\frac{x+y+z}{x}-1}\cdot\frac{(\frac{x+y+z}{y})^p-1}{\frac{x+y+z}{y}-1}\cdot\frac{(\frac{x+y+z}{z})^p-1}{\frac{x+y+z}{z}-1}\right]$$

$$= [x^{p-1}y^{p-1}z^{p-1}]\left[\frac{(x+y+z)^p-x^p}{y+z}\cdot\frac{(x+y+z)^p-y^p}{x+z}\cdot\frac{(x+y+z)^p-z^p}{x+y}\right].$$

So far this is true for all p, not only p prime. Now take it mod p and get, using the freshman's dream in the form $(x+y+z)^p \equiv_p x^p+y^p+z^p$, that

$$\sum_{m_1=0}^{p-1}\sum_{m_2=0}^{p-1}\sum_{m_3=0}^{p-1}\binom{m_1+m_2+m_3}{m_1,m_2,m_3} \equiv_p [x^{p-1}y^{p-1}z^{p-1}]\left(\frac{y^p+z^p}{y+z}\cdot\frac{x^p+z^p}{x+z}\cdot\frac{x^p+y^p}{x+y}\right)$$

$$= [x^{p-1}y^{p-1}z^{p-1}]\left(\sum_{i=0}^{p-1}(-1)^i y^i z^{p-1-i}\right)\left(\sum_{j=0}^{p-1}(-1)^j z^j x^{p-1-j}\right)\left(\sum_{k=0}^{p-1}(-1)^k x^k y^{p-1-k}\right)$$

$$= [x^{p-1}y^{p-1}z^{p-1}]\left[\sum_{0\le i,j,k<p}(-1)^{i+j+k}x^{p-1-j+k}y^{i+p-1-k}z^{p-1-i+j}\right].$$

The only contributions to the coefficient of $x^{p-1}y^{p-1}z^{p-1}$ in the above triple sum come when $i=j=k$, so the desired coefficient of $x^{p-1}y^{p-1}z^{p-1}$ is

$$\sum_{i=0}^{p-1}(-1)^{3i} = \sum_{i=0}^{p-1}(-1)^i = (1-1+1-1+\cdots+1-1)+1 = 1. \qquad \blacksquare$$

With more effort, one can get the following generalization.

Proposition 5′. *Let $p \geq 3$ be prime, and let r, s, t be any positive integers. Then,*

$$\sum_{m_1=0}^{rp-1} \sum_{m_2=0}^{sp-1} \sum_{m_3=0}^{tp-1} \binom{m_1+m_2+m_3}{m_1, m_2, m_3} \equiv_p \kappa_{rst},$$

where

$$\kappa_{rst} = \sum_{m_1=0}^{r-1} \sum_{m_2=0}^{s-1} \sum_{m_3=0}^{t-1} \binom{m_1+m_2+m_3}{m_1, m_2, m_3}.$$

The same method of proof used in Proposition 5 yields (with a little more effort) a *multinomial* generalization.

Proposition 6. *Let $p \geq 3$ be prime, then*

$$\sum_{m_1=0}^{p-1} \cdots \sum_{m_n=0}^{p-1} \binom{m_1+\cdots+m_n}{m_1, \ldots, m_n} \equiv_p 1.$$

In fact, the following also holds.

Proposition 6′. *Let $p \geq 3$ be prime, and let $r_1, \ldots, r_n$ be positive integers. Then we have*

$$\sum_{m_1=0}^{r_1 p-1} \cdots \sum_{m_n=0}^{r_n p-1} \binom{m_1+\cdots+m_n}{m_1, \ldots, m_n} \equiv_p \kappa_{r_1 \ldots r_n},$$

where

$$\kappa_{r_1 \ldots r_n} = \sum_{m_1=0}^{r_1-1} \cdots \sum_{m_n=0}^{r_n-1} \binom{m_1+\cdots+m_n}{m_1, \ldots, m_n}.$$

5. SUPER-CONGRUENCES. If a congruence identity that is valid modulo a prime p is also valid modulo p^2 (or better still, modulo p^3 and beyond), then we have a *super-congruence*. The grandmother of all super-congruences is Wolstenholme's theorem ([**11**]; see also [**9**]) that asserts that

$$\binom{2p-1}{p-1} \equiv_{p^3} 1,$$

and that improves on the weaker version $\binom{2p-1}{p-1} \equiv_{p^2} 1$, first proved by Charles Babbage ([**3**]), better known for more impressive innovations.

To our surprise, most (but not all!) of the above congruences have super-congruence extensions. The method of [**4**], as it stands now, is not applicable since the "freshman's dream" is *only* valid modulo p, hence we have no clue how to prove the extensions. We leave them as challenges to our readers.

Super-conjecture 1. *For any prime $p \geq 5$,*

$$\sum_{n=0}^{p-1} \binom{2n}{n} \equiv_{p^2} \begin{cases} 1, & \text{if } p \equiv 1 \mod 3 \\ -1, & \text{if } p \equiv 2 \mod 3. \end{cases}$$

More generally,

Super-conjecture 1″. *For any prime $p \geq 5$ and any positive integer r,*

$$\sum_{n=0}^{rp-1} \binom{2n}{n} \equiv_{p^2} \begin{cases} \alpha_r, & \text{if } p \equiv 1 \mod 3 \\ -\alpha_r, & \text{if } p \equiv 2 \mod 3, \end{cases}$$

where

$$\alpha_r = \sum_{n=0}^{r-1} \binom{2n}{n}.$$

Super-conjecture 2. *Let C_n denote the nth Catalan number. Then, for every prime $p \geq 5$,*

$$\sum_{n=0}^{p-1} C_n \equiv_{p^2} \begin{cases} 1, & \text{if } p \equiv 1 \mod 3 \\ -2, & \text{if } p \equiv 2 \mod 3. \end{cases}$$

More generally,

Super-conjecture 2′. *Let C_n denote the nth Catalan number. Then, for any positive integer r,*

$$\sum_{n=0}^{rp-1} C_n \equiv_{p^2} \begin{cases} \beta_r, & \text{if } p \equiv 1 \mod 3 \\ -\gamma_r, & \text{if } p \equiv 2 \mod 3, \end{cases}$$

where

$$\beta_r = \sum_{n=0}^{r-1} C_n, \quad \gamma_r = \sum_{n=0}^{r-1} (3n+2) C_n.$$

[Added in revision: Conjectures 1″ and 2′ have now been proved; see [**6**].]

We note that poor Motzkin does not seem to have a super-extension, but Proposition 4 sure does.

Super-conjecture 4. *Let $p \geq 5$ be prime; then*

$$\sum_{n=0}^{p-1} \sum_{m=0}^{p-1} \binom{n+m}{m}^2 \equiv_{p^2} \begin{cases} 1, & \text{if } p \equiv 1 \mod 3 \\ -1, & \text{if } p \equiv 2 \mod 3. \end{cases}$$

More generally,

Super-conjecture 4′. *For any prime $p \geq 5$, and any pair of positive integers r, s, we have*

$$\sum_{n=0}^{rp-1} \sum_{m=0}^{sp-1} \binom{n+m}{m}^2 \equiv_{p^2} \begin{cases} \epsilon_{rs}, & \text{if } p \equiv 1 \mod 3 \\ -\epsilon_{rs}, & \text{if } p \equiv 2 \mod 3, \end{cases}$$

where

$$\epsilon_{rs} = \sum_{m=0}^{r-1} \sum_{n=0}^{s-1} \binom{n+m}{m}^2.$$

The most pleasant surprise is that Propositions 5 and 5′ can be "upgraded" to a cubic super-congruence, i.e., it is still true modulo p^3.

Super-conjecture 5. *Let $p > 2$ be prime. Then, we have*

$$\sum_{m_1=0}^{p-1} \sum_{m_2=0}^{p-1} \sum_{m_3=0}^{p-1} \binom{m_1+m_2+m_3}{m_1, m_2, m_3} \equiv_{p^3} 1.$$

More generally,

Super-conjecture 5′. *Let $p \geq 3$ be prime, and let r, s, t be any positive integers. Then*

$$\sum_{m_1=0}^{rp-1} \sum_{m_2=0}^{sp-1} \sum_{m_3=0}^{tp-1} \binom{m_1+m_2+m_3}{m_1, m_2, m_3} \equiv_{p^3} \kappa_{rst},$$

where

$$\kappa_{rst} = \sum_{m_1=0}^{r-1} \sum_{m_2=0}^{s-1} \sum_{m_3=0}^{t-1} \binom{m_1+m_2+m_3}{m_1, m_2, m_3}.$$

[Added in revision: Conjectures 4′ and 5′ have now been proved; see [**1**].]

To our bitter disappointment, Propositions 6 and 6′, for $n \geq 4$ summation signs, do not have super-upgrades.

6. LOTS AND LOTS OF COMBINATORIAL CHALLENGES. Perhaps the nicest proof of Fermat's little theorem, $a^p \equiv_p a$, is Golomb's ([**5**]) combinatorial proof that notes that a^p is the number of (straight) necklaces with p beads, using beads of a different colors, and hence $a^p - a$ is the number of such (straight) necklaces that are not all of the same color. For any such necklace, all its p circular rotations are distinct (since p is prime), hence the set of such necklaces can be divided into families, each of them with p members, and hence there are $\dfrac{a^p - a}{p}$ "circular" necklaces (without clasp), and this must be an integer.

Each and every quantity in the propositions and conjectures above counts a natural combinatorial family. For example, $\displaystyle\sum_{n=0}^{p-1} \binom{2n}{n}$ counts the number of binary sequences

with the same number of 0's and 1's whose length is less than $2p$. Can you find a member of this set that when you remove it, and $p \equiv 1 \pmod 3$, you can partition that set into families each of them with exactly p (or better still, for the super-congruence, p^2) members? And when $p \equiv 2 \pmod 3$, can you find two such members?

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