CHAPTER 7

CHAPTER 7

• Applications of Integration price and **price to the said.**

This like this,' he said. 'When you go after honey with a balloon, the

great thing is not to let the bees know you're coming. Now if you have

a green balloon, they might think you were onl

If 't's like this,' he said. 'When you go after honey with a balloon, the great thing is not to let the bees know you're coming. Now if you have a green balloon, they might think you were only part of the tree and not noti plications of
the exact to let the bees know you after honey with a balloon, the
great thing is not to let the bees know you're coming. Now if you have
a green balloon, they might think you were only part of the tree and
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if thing is not to let the bees know you're coming. Now if you have
a green balloon, they might think you were only part of the tree **Prications of**
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We said. When you go after honey with a balloon, the
great thing is not to let the bees know you're coming. Now if you have
a green balloon, they might think you were only part of the tree and
not notice you, an The Pooh of the said. When you go after honey with a balloon, the great thing is not to let the bees know you're coming. Now if you have a green balloon, they might think you were only part of the tree and not notice you, The State of the University of the tree and
A. Mow if you have
A. Milne 1982–1956
The A. A. Milne 1882–1956
The Milne 1882–1956
The Milne the Pooh a green balloon, they might think you were only part of the tree and
not notice you, and if you have a blue balloon, they might think you
were only part of the sky and not notice you, and the question' [said
Winnie the Poo

"

from Winnie the Pooh The entire world believes [in the Normal distribution], Mr. Lippmann told me one day, because the experimentalists believe that it is a theorem of mathematics, and mathematicians believe it is an experimental not notice you, and if you have a blue balloon, they might think you
were only part of the sky and not notice you, and the question' [said
Winnie the Pooh] 'is: Which is most likely?'
A. A. Milne 1882–1956
from *Winnie t* were only part of the sky and not notice you, and the question' [said Winnie the Pooh] 'is: Which is most likely?'
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 Form Winnie the Pooh

The entire world believes [in the Normal distribution], M fact. ne 1882–1956
Winnie the Pooh
Lippmann
t is a theo-
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Henri Poincaré
tés, 1896, p.149

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Example 18 Second Terms
 Example 18 Second Probabilités, 1896, p.149

in mathematics, physics, economics,

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measuring plane areas, the problem

al, we can use these integrals to extold me one day, because the experimentalists believe that it is a theorem of mathematics, and mathematicians believe it is an experimental

fact.
 Exacut des Probabilités, 1896, p.149
 Exacut des Probabilités, 1896, p and mathematicians believe it is an experimental
Henri Poincaré
Calcul des Probabilités, 1896, p.149
Numerous quantities in mathematics, physics, economics,
biology, and indeed any quantitative science can be con-
gr **vention intervention and the probabilities, 1896, p.149**
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 exacut des Probabil Example 19 Follows
 Example 19 Follows Calcul des Probabilités, 1896, p. 149
 Example 10 Follows biology, and indeed any quantitative science can be conveniently represented by integrals. In addition to measuring **Example 19 Follows Calcul des Probabilités, 1896, p. 149**
 Probabilités, 1896, p. 149
 Probabilités, biology, and indeed any quantitative science can be conveniently represented by integrals. In addition to measuring pressure, probabilities, dollar values of a stream of payments, and a variety of other **Calcul des Probabilités, 1896, p.149**
 Calcul des Probabilités, 1896, p.149
 Concernity represented by integrals. In addition to measuring plane areas, the problem

that motivated the definition of the definite integ **TOOUCTION** Numerous quantities in mathematics, physics, economics,
 TOOUCTION biology, and indeed any quantitative science can be con-

ently represented by integrals. In addition to measuring plane areas, the problem
 comomist are expressed in the solution of the later sections in the independent of one another and the statement of the definition of the definite integral, we can use these integrals to express volumes of solids, length **Problems.** In this chapter that the solution is a key tool integration is a key toolonical proper serves of solids, lengths. In addition to measuring plane areas, the problem that motivated the definition of the definite **LTO COCUTT** biology, and indeed any quantitative science can be con-
ently represented by integrals. In addition to measuring plane areas, the problem
motivated the definition of the definite integral, we can use these i veniently represented by integrals. In addition to measuring plane areas, the problem
that motivated the definition of the definite integral, we can use these integrals to ex-
press volumes of solids, lengths of curves, ar

that motivated the definition of the definite integral, we can use these integrals to express volumes of solids, lengths of curves, areas of surfaces, forces, work, energy, pressure, probabilities, dollar values of a strea press volumes of solids, lengths of curves, areas of surfaces, forces, work, energy, pressure, probabilities, dollar values of a stream of payments, and a variety of other quantities that are in one sense or another equiva calculus. 11 Moltumes by Slicing—Solids of Revolution
11 In this section word are expressed in terms of differential eq
12 problems. Indefinite integration is a key tool in the solution of
11 this chapter we examine some of these ap In this chapter we examine some of these applications. For the most part they are independent of one another, and for that reason some of the later sections in this chapter can be regarded as optional material. The materia integrals are considered as optical and for that reason some of the later sections in this chapter
can be regarded as optional material. The material of Sections 7.1–7.3, however, should
be regarded as core because these i the regarded as optional material. The material of Sections 7.1–7.3, however, should
be regarded as optional material. The material of Sections 7.1–7.3, however, should
be regarded as core because these ideas will arise ag

4 CHAPTER 7 Applications of Integration

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For example, if the base

therefore area $A = lw$),

If l , w , and h are measured

what is a cylinder? The word

"cylinder" has two different but

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For example, if the base of

therefore area $A = lw$), an

If l , w , and h are measure
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 What is a cylinder? objects to provide e

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for example, if the

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For example, if the base

therefore area $A = lw$

If l , w , and h are mean

what is a cylinder? The word

"cylinder" has two different but

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"cylinder" has two different but

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 What is a cylinder? The word

"cylinder" has two different but

related meanings in Mathe-

matics. As used in this Section,

ti is a *solid object* lying between

c **Example 19 and the sum of the same of th Example 19 and the set of the segments in Matheumatics. As used in this Section, it is a** *solid obj* **Example 18 CENT CONFIDENT CONFIDENT CONFIDENT CONFIDENT CONFIDENT CONFIDENT CONFIDENT CONFIDENT CONFIDENCIAL CONFIDENCIAL CONFIDENCIAL CONFIDENCIAL CONFIDENCIAL CONFIDENCIAL CONFIDENCIAL CONFIDENCIAL CONFIDENCIAL CONFIDE What is a cylinder?** The word

"cylinder" has two different but

related meanings in Mathematics. As used in this Section,

it is a *solid object* lying between

congruent bases in two parallel

planes and inside a surfa "cylinder" has two different but

related meanings in Mathematics. As used in this Section,

it is a *solid object* lying between

congruent bases in two parallel

planes and inside a surface (the

cylindrical wall) consi Frelated meanings in Mathe-

matics. As used in this Section,

it is a *solid object* lying between

congruent bases in two parallel

planes and inside a surface (the

explindical wall) consisting of

the two bases. A cyl matics. As used in this Section,

it is a *solid object* lying between

congruent bases in two parallel

planes and inside a surface (the

cylindrical wall) consisting of

parallel line segments joining

corresponding poi it is a *solid object* lying between

congruent bases in two parallel

planes and inside a surface (the

cylindrical wall) consisting of

the two bases. A cylinder

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planes and inside a surface (the

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cylindrical wall) consisting of

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second meaning for "cy Explindirical wall) consisting of

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orresponding points on the

boundaries of those bases. The

and the area of a basecond meaning for "cylinder"
 $V = Ah$ cubic parallel line segments joining

parallel line segments joining

boundaries of those bases. The

second meaning for "cylinder"

that we will encounter in Chapter

10 and later, extends the concept

of the cylindrical wall corresponding points on the

boundaries of those bases. The

second meaning for "cylinder"

that we will encounter in Chapter

that we will encounter in Chapter

that we will encounter in Chapter

of the cylinder. We use boundaries of those bases. The
second meaning for "cylinder"
that we will encounter in Chapter
that we will encounter in Chapter
of the cylindrical wall of a solid
of the cylindrical wall of a solid
cylinder. It is a *sur* second meaning for "cylinder"

that we will encounter in Chapter

10 and later, extends the concept

of the cylindrical wall of a solid

cylinder. It is a *surface*

consisting of a family of parallel

straight lines in t

objects to provide enough insight for us to specify the volumes of certain simple solids.
For example, if the base of a rectangular box is a rectangle of length *l* and width *w* (and therefore area $A = lw$), and if the box objects to provide enough insight for us to specify the volumes of certain simple solids.
For example, if the base of a rectangular box is a rectangle of length l and width w (and
therefore area $A = lw$), and if the box has objects to provide enough insight for us to specify the volumes of certain simple solids.
For example, if the base of a rectangular box is a rectangle of length l and width w (and
therefore area $A = lw$), and if the box has objects to provide enough insight for us to specify the volumes of certain sincular For example, if the base of a rectangular box is a rectangle of length l and v therefore area $A = lw$), and if the box has height h, then i *cubic units* (cubic centimetres, or cm³). cts to provide enough insight for us to specify the volumes of certain simple solids.

Example, if the base of a rectangular box is a rectangle of length *l* and width *w* (and

fore area $A = lw$), and if the box has height

objects to provide enough insight for us to specify the volumes of certain simple solids.
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For example, if the base of a rectangular box is a rectangle of length l and width w (and therefore area $A = lw$), and if the box has For example, if the base of a rectangular box is a rectangle of length *l* and width *w* (and therefore area $A = lw$), and if the box has height *h*, then its volume is $V = Ah = lwh$.
If *l*, *w*, and *h* are measured in *units* therefore area $A = lw$), and if the box has height h , then its volume is $V = Ah = lwh$.
If l , w , and h are measured in *units* (e.g., centimetres), then the volume is expressed in
cubic units (cubic centimetres, or cm If *l*, *w*, and *h* are measured in *units* (e.g., centimetres), then the volume is expressed in *cubic units* (cubic centimetres, or cm³).
A rectangular box is a special case of a solid called a **cylinder**. (See Figur cubic units (cubic centimetres, or cm³).

A rectangular box is a special case of a solid called a **cylinder**. (See Figure 7.1.)

Such a solid has a flat base occupying a region R in a plane, and consists of all points
 A rectangular box is a special case of a solid called a **cylinder**. (See Figure 7.1.)
Such a solid has a flat base occupying a region R in a plane, and consists of all points
on parallel straight line segments having one Such a solid has a flat base occupying a region R in a p
Such a solid has a flat base occupying a region R in a p
on parallel straight line segments having one end in R a
ily congruent) region in a second plane parallel t arallel straight line segments having one end in R and the other end in a (necessar-
ongruent) region in a second plane parallel to the plane of the base. Either of these
ons can be called the **base** of the cylinder. Th ily congruent) region in a second plane parallel to the plane of the base. Either of these
regions can be called the **base** of the cylinder. The **cylindrical wall** is the surface con-
sisting of the parallel line segments regions can be called the **base** of the cylinder. The **cylindrical wall** is the surface consisting of the parallel line segments joining corresponding points on the boundaries of the two bases. A cylinder having a polygon

sisting of the parallel line segments joining corresponding points on the boundaries of
the two bases. A cylinder having a polygonal base (i.e., one bounded by straight lines)
is usually called a **prism**. The height of an the two bases. A cylinder having a polygonal ba
is usually called a **prism**. The height of any c
distance between the parallel planes containing
and the area of a base is A square units, then t
 $V = Ah$ cubic units.
We use t inder having a polygonal base (i.e., one bounded by straight lines)
 rism. The height of any cylinder or prism is the perpendicular

parallel planes containing the two bases. If this height is h units

se is A square un is usually called a **prism**. The height of any cylinder or prism is the perpendicular distance between the parallel planes containing the two bases. If this height is *h* units and the area of a base is *A* square units, base.

base): $V = Ah$

 $\begin{array}{c|c|c|c|c|c} a & b & b & c \end{array}$ Volumes by Slicing

Explores the volume of a cylinder enables us to determ

general solids. We can divide solids into thin "slices"

Volumes by Slicing

Knowing the volume of a cylinder enables us to determine the volumes of some more

general solids. We can divide solids into thin "slices" by parallel planes. (Think of a

loaf of sliced bread.) Each s **Volumes by Slicing**
 Example 3 Example 1 and the solution of a cylinder enables us to determine the volumes of some more general solids. We can divide solids into thin "slices" by parallel planes. (Think of a loaf of s **Volumes by Slicing**
 Knowing the volume of a cylinder enables us to determine the volumes of some more general solids. We can divide solids into thin "slices" by parallel planes. (Think of a loaf of sliced bread.) Each Volumes by Slicing
 Knowing the volume of a cylinder enables us to determine the volumes of some more

general solids. We can divide solids into thin "slices" by parallel planes. (Think of a

loaf of sliced bread.) Ea

SECTION 7.1: Volumes by Slicing—Solids of Revolution **395**
horizontally in the x direction. If we know the cross-sectional area of each slice, we
can determine its volume and sum these volumes to find the volume of the so

SECTION 7.1: Volumes by Slicing—Solids of Revolution **395**
horizontally in the x direction. If we know the cross-sectional area of each slice, we
can determine its volume and sum these volumes to find the volume of the so SECTION 7.1: Volumes by Slicing—Solids of Revolution **395**

Exontally in the x direction. If we know the cross-sectional area of each slice, we

determine its volume and sum these volumes to find the volume of the solid.
 sectrion 7.1: Volumes by Slicing—Solids of Revolution **395**

horizontally in the *x* direction. If we know the cross-sectional area of each slice, we

can determine its volume and sum these volumes to find the volume of t SECTION 7.1: Volumes by Slicing—Solids of Revolution **395**

horizontally in the x direction. If we know the cross-sectional area of each slice, we

can determine its volume and sum these volumes to find the volume of the SECTION 7.1: Volumes by Slicing—Solids of Revolution **395**

horizontally in the *x* direction. If we know the cross-sectional area of each slice, we

can determine its volume and sum these volumes to find the volume of th l area of each slice, we
lume of the solid.
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 $n \le x \le b$. We assume
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nto n subintervals, and
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 $\le x \le b$. We assume
 $\cdot < x_{n-1} < x_n = b$,
to n subintervals, and
the solid into n slices
 ΔV_i of that slice lies
lues of x in [x_{i-1}, x_i SECTION 7.1: Volumes by Slicing—Solids of Revolution **395**

horizontally in the *x* direction. If we know the cross-sectional area of each slice, we

can determine its volume and sum these volumes to find the volume of th SECTION 7.1: Volumes by Slicing—Solids of Revolution **395**

horizontally in the x direction. If we know the cross-sectional area of each slice, we

can determine its volume and sum these volumes to find the volume of the horizontally in the *x* direction. If we know the cross-sectional area of each slice, we
can determine its volume and sum these volumes to find the volume of the solid.
To be specific, suppose that the solid *S* lies betw horizontally in the x direction. If we know the cross-sectional area of each slice, we
can determine its volume and sum these volumes to find the volume of the solid.
To be specific, suppose that the solid S lies between for values of Revolution
ional area of each slice, we
evolume of the solid.
planes perpendicular to the
tional area of S in the plane
for $a \le x \le b$. We assume
 $\lt \cdots \lt x_{n-1} \lt x_n = b$,
 b] into *n* subintervals, and
divide l_l can determine its volume and sum these volumes to find the volume of the solid.

To be specific, suppose that the solid *S* lies between planes perpendicular to the *x*-axis at positions $x = a$ and $x = b$ and that the cross rea of each slice, we
me of the solid.
perpendicular to the
area of S in the plane
 $\le x \le b$. We assume
 $\lt x_{n-1} \lt x_n = b$,
 $\ge n$ subintervals, and
the solid into *n* slices
 ΔV_i of that slice lies
ues of x in [x_{i-1}, x_i] between the maximum and minimum values of $A(x) \Delta x_i$ for values of x in $[x_{i-1}, x_i]$
(Figure 7.3), so, by the Intermediate-Value Theorem, for some c_i in $[x_{i-1}, x_i]$, x-axis at positions $x = a$ and $x = b$ and that the cross-sectional area of S in the plane
perpendicular to the x-axis at x is a known function $A(x)$, for $a \le x \le b$. We assume
that $A(x)$ is continuous on [a, b]. If $a = x_0 < x_$ the planes perpendicular to the *x*-axis at *x*₁, *x*₂, ..., *x_n*-1 divide the solid into *n* slices
of which the *i*th has thickness $\Delta x_i = x_i - x_{i-1}$. The volume ΔV_i of that slice lies
between the maximum and mi that $A(x)$ is continuous on $[a, b]$. If $a = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = b$,
then $P = \{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\}$ is a partition of $[a, b]$ into *n* subintervals, and
the planes perpendicular to the x -axis at $x_1, x_2, \ldots, x_{$

 $\Delta V_i = A(c_i) \, \Delta x_i.$

$$
V = \sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} A(c_i) \Delta x_i.
$$

In the volume of the solid is therefore the value Theorem, for some c_i in $[x_{i-1}, x_i]$,
 $\Delta V_i = A(c_i) \Delta x_i$.

wolume of the solid is therefore given by the Riemann sum
 $V = \sum_{i=1}^n \Delta V_i = \sum_{i=1}^n A(c_i) \Delta x_i$.

ng *n* approach $\Delta V_i = A(c_i) \Delta x_i.$
volume of the solid is therefore given by the Riemann
 $V = \sum_{i=1}^{n} \Delta V_i = \sum_{i=1}^{n} A(c_i) \Delta x_i.$
ng *n* approach infinity in such a way that max Δx_i
inte integral of $A(x)$ over [*a*, *b*] as the limit of t ΔV_i = $\sum_{i=1}^{n} A(c_i) \Delta x_i$.

Solid is therefore given by the Riemann sum
 $\Delta V_i = \sum_{i=1}^{n} A(c_i) \Delta x_i$.

Solid is therefore given by the Riemann sum
 ΔV_i approach infinity in such a way that max Δx_i approach and of

$$
V = \int_{a}^{b} A(x) \, dx.
$$

Letting *n* approach infinity in such a way that max Δx_i approaches 0, we obtain the definite integral of $A(x)$ over [a, b] as the limit of this Riemann sum. Therefore:

The volume V of a solid between $x = a$ and $x = b$ definite integral of $A(x)$ over [a, b] as the limit of this Riemann sum. Therefore:

The volume V of a solid between $x = a$ and $x = b$ having cross-sectional

area $A(x)$ at position x is
 $V = \int_a^b A(x) dx$.

There is another wa The volume V of a solid between $x = a$ and $x = b$ having cross-sectional
area $A(x)$ at position x is
 $V = \int_a^b A(x) dx$.
There is another way to obtain this formula and others of a similar nature. Consider
a slice of the solid b The volume V of a solid between $x = a$ and $x = b$ having cross-sectional
area $A(x)$ at position x is
 $V = \int_a^b A(x) dx$.
There is another way to obtain this formula and others of a similar nature. Consider
a slice of the solid b area $A(x)$ at position x is
 $V = \int_{a}^{b} A(x) dx$.

There is another way to obtain this formula and others of a si

a slice of the solid between the planes perpendicular to the x-
 $x + \Delta x$. Since $A(x)$ is continuous, it doesn $V = \int_a^b A(x) dx$.

There is another way to obtain this formula and others of a similar nature. Consider

a slice of the solid between the planes perpendicular to the x-axis at positions x and
 $x + \Delta x$. Since $A(x)$ is continu

 $\Delta V \approx A(x) \Delta x$.

There is another way to obtain this formula and others of a similar nature. Consider
a slice of the solid between the planes perpendicular to the x-axis at positions x and
 $x + \Delta x$. Since $A(x)$ is continuous, it doesn't ch $V = \int_a^b A(x) dx$.

There is another way to obtain this formula and others of a similar nature. Consider

a slice of the solid between the planes perpendicular to the x-axis at positions x and
 $x + \Delta x$. Since $A(x)$ is continu There is another way to obtain this formula and others of a similar nature. Consider
a slice of the solid between the planes perpendicular to the x-axis at positions x and
 $x + \Delta x$. Since $A(x)$ is continuous, it doesn't ch a slice of the solid between the planes perpendicular to th $x + \Delta x$. Since $A(x)$ is continuous, it doesn't change much
is small, then the slice has volume ΔV approximately equal
of base area $A(x)$ and height Δx :
 Δ way to obtain this formula and others of a similar nature. Consider
id between the planes perpendicular to the x-axis at positions x and
(x) is continuous, it doesn't change much in a short interval, so if Δx
slice has $\Delta V \approx A(x) \Delta x$.

The error in this approximation is small compared to the size of ΔV . This suggests,

correctly, that the **volume element**, that is, the volume of an infinitely thin slice of

thickness dx is $dV = A(x) dx$, The error in this approximation is small compared to the size of ΔV . This suggests,
correctly, that the **volume element**, that is, the volume of an infinitely thin slice of
thickness dx is $dV = A(x) dx$, and that the volum correctly, that the **volume element**, that is, the volume of an infinitely thin slice of thickness dx is $dV = A(x) dx$, and that the volume of the solid is the "sum" (i.e., the integral) of these volume elements between the t

$$
V = \int_{x=a}^{x=b} dV, \qquad \text{where} \qquad dV = A(x) \, dx.
$$

thickness dx is $dV = A(x) dx$, and that the volume of the solid is the "sum" (i.e.,
the integral) of these volume elements between the two ends of the solid, $x = a$ and
 $x = b$ (see Figure 7.4):
 $V = \int_{x=a}^{x=b} dV$, where $dV = A(x) dx$ the integral) of these volume elements between the
 $x = b$ (see Figure 7.4):
 $V = \int_{x=a}^{x=b} dV$, where $dV = A(x) d$

We will use this *differential element* approach to m

in integrals rather than setting up explicit Riemann

 $V = \int_{x=a}^{x=b} dV$, where $dV = A(x) dx$.
We will use this differential element approach to model other applications that result
in integrals rather than setting up explicit Riemann sums each time. Even though this
argument does $V = \int_{x=a} dV$, where $dV = A(x) dx$.
We will use this *differential element* approach to model other applications that result
in integrals rather than setting up explicit Riemann sums each time. Even though this
argument does $x=a$
We will use this *differential element* approach to model other applications that result
in integrals rather than setting up explicit Riemann sums each time. Even though this
argument does *not* constitute a proof of We will use this *differential element* approach to model other applications that result
in integrals rather than setting up explicit Riemann sums each time. Even though this
argument does *not* constitute a proof of the Alf in the disk (Figure 7.5(a)). Similarly, a solid representation and the set of the same experiment does *not* constitute a proof of the formula, you are strongly encouraged to think of the formula this way; the volume arisement does *not* constitute a proof of the formula, you are strongly encouraged to think of the formula this way; the volume is the integral of the volume elements.
 Solids of Revolution

Many common solids have cir will use this *differential element* approach to model other applications that result
tegrals rather than setting up explicit Riemann sums each time. Even though this
ment does *not* constitute a proof of the formula, you **Solids of Revolution**
Many common solids have circular cross-sections in planes perpendicular to some
axis. Such solids are called **solids of revolution** because they can be generated by
rotating a plane region about an argument does *not* constitute a proof of the formula, you are strongly encouraged to think of the formula this way; the volume is the integral of the volume elements.
Solids of Revolution
Many common solids have circul e formula this way; the volume is the integ
 Revolution

In solids have circular cross-sections in

h solids are called **solids of revolution** be

plane region about an axis in that plane so

a solid ball is generated b lids have circular cross-sections in planes perpendicular to some
are called **solids of revolution** because they can be generated by
gion about an axis in that plane so that it sweeps out the solid. For
all is generated b

 $A(x) = \pi (f(x))^{2}$, so the volume of the

$$
V = \pi \int_{a}^{b} (f(x))^{2} dx.
$$

 $V = \pi \int_{a}^{b} (f(x))^{2} dx$.
 EXAMPLE 1 (The volume of a ball) Find the volume of a solid ball having radius *a*.
 Solution The ball can be generated by rotating the half-disk, $0 \le y \le \sqrt{a^{2} - x^{2}}$, $-a \le x \le a$ about the *x* **Solution** The ball can be generated by rotating the half-disk, $0 \le y \le \sqrt{a^2 - x^2}$, $-a \le x \le a$ about the *x*-axis. See the cutaway view in Figure 7.5(a). Therefore, its $V = \pi \int_{a}^{b} (f(x))^{2} dx$.
 EXAMPLE 1 (The volume of a ball) Find the volume of a solid ball having radius *a*.
 Solution The ball can be generated by rotating the half-disk, $0 \le y \le \sqrt{a^{2} - x^{2}}$, $-a \le x \le a$ about the *x* $V = \pi \int_{a}^{b} (f(x))^{2} dx.$
 EXAMPLE 1 (The volume of a ball) Fi

radius *a*.
 Solution The ball can be generated by rotatin
 $-a \le x \le a$ about the *x*-axis. See the cutaway

volume is
 $V = \pi \int_{-a}^{a} (\sqrt{a^{2} - x^{2}})^{2} dx = 2\pi \$ the volume of a solid ball having
 \therefore half-disk, $0 \le y \le \sqrt{a^2 - x^2}$,

w in Figure 7.5(a). Therefore, its
 \therefore $\frac{dy}{dx}$

EXAMPLE 1 (The volume of a ball) Find the volume of a solid ball having radius *a*.
\n**ition** The ball can be generated by rotating the half-disk,
$$
0 \le y \le \sqrt{a^2 - x^2}
$$
, $\le x \le a$ about the *x*-axis. See the cutaway view in Figure 7.5(a). Therefore, its line is
\n
$$
V = \pi \int_{-a}^{a} (\sqrt{a^2 - x^2})^2 dx = 2\pi \int_{0}^{a} (a^2 - x^2) dx
$$
\n
$$
= 2\pi \left(a^2 x - \frac{x^3}{3} \right) \Big|_{0}^{a} = 2\pi \left(a^3 - \frac{1}{3} a^3 \right) = \frac{4}{3} \pi a^3
$$
 cubic units.

(a) (b)
 EXAMPLE 2 (**The volume of a right-circular cone**) Find the volume of the

by rotating the triangle with vertices (0, 0), (*h*, 0), and (*h*, *r*) about the *x*-axis.
 Solution The line from (0, 0) to (*h*, *r*

by rotating the triangle with vertices (0, 0), (*h*, 0), and (*h*, *r*) about the *x*-axis.
\n**Solution** The line from (0, 0) to (*h*, *r*) has equation
$$
y = rx/h
$$
. Thus, the volume
\nthe cone (see the cutaway view in Figure 7.5(b)) is
\n
$$
V = \pi \int_0^h \left(\frac{rx}{h}\right)^2 dx = \pi \left(\frac{r}{h}\right)^2 \frac{x^3}{3} \Big|_0^h = \frac{1}{3} \pi r^2 h
$$
 cubic units.
\nImproper integrals can represent volumes of unbounded solids. If the improper integ
\nconverges, the unbounded solid has a finite volume.
\n**EXAMPLE 3** Find the volume of the infinitely long horn that is generated
\nrotating the region bounded by $y = 1/x$ and $y = 0$ and lying

olution The line from (0, 0) to (h, r) has equation $y = rx/h$. Thus, the volume of e cone (see the cutaway view in Figure 7.5(b)) is
 $V = \pi \int_0^h \left(\frac{rx}{h}\right)^2 dx = \pi \left(\frac{r}{h}\right)^2 \frac{x^3}{3} \Big|_0^h = \frac{1}{3} \pi r^2 h$ cubic units.
 o taway view in Figure 7.5(b)) is
 $\left(\frac{r}{h}\right)^2 \frac{x^3}{3} \Big|_0^h = \frac{1}{3} \pi r^2 h$ cubic units.

an represent volumes of unbounded solids. If the improper integral

unded solid has a finite volume.

Find the volume of the infin $V = \pi \int_0^h \left(\frac{rx}{h}\right)^2 dx = \pi \left(\frac{r}{h}\right)^2 \frac{x^3}{3} \Big|_0^h = \frac{1}{3} \pi r^2 h$ cubic units.

Improper integrals can represent volumes of unbounded solids. If the improper integral

converges, the unbounded solid has a finite volum For $V = \pi \int_0^{\pi} \left(\frac{h}{h}\right)^2 dx = \pi \left(\frac{h}{h}\right)^2 \frac{1}{3} \Big|_0^{\pi} = \frac{\pi}{3} \pi r^n$ denote the sum expression in the sum and solid has a finite volume.
 EXAMPLE 3 Find the volume of the infinitely long horn that is gereated to r e infinitely long horn that is generated by

unded by $y = 1/x$ and $y = 0$ and lying to

unit is illustrated in Figure 7.6.
 $\frac{1}{x^2}$
 $\frac{1}{x^2}$
 $\frac{1}{x^2}$
 π cubic units.

$$
V = \pi \int_1^{\infty} \left(\frac{1}{x}\right)^2 dx = \pi \lim_{R \to \infty} \int_1^R \frac{1}{x^2} dx
$$

= $-\pi \lim_{R \to \infty} \frac{1}{x} \Big|_1^R = -\pi \lim_{R \to \infty} \left(\frac{1}{R} - 1\right) = \pi$ cubic units.

- Figure 7.5

(a) The ball is generated by rotating the

red half-disk $0 \le y \le \sqrt{a^2 x^2}$ (a) The ball is generated by rotating the
red half-disk $0 \le y \le \sqrt{a^2 - x^2}$
- triangle $0 \le x \le h$, $0 \le y \le rx/h$
about the x-axis e 7.5

The ball is generated by rotating the

red half-disk $0 \le y \le \sqrt{a^2 - x^2}$

about the x-axis

The cone of base radius r and height

h is generated by rotating the red

triangle $0 \le x \le h, 0 \le y \le rx/h$

about the x-axis

SECTION 7.1: Volumes by Slicing—Solids of Revolution **397**
It is interesting to note that this finite volume arises from rotating a region that itself
has infinite area: $\int_1^{\infty} dx/x = \infty$. We have a paradox: it takes an i SECTION 7.1: Volumes by Slicing—S.
It is interesting to note that this finite volume arises from rotati
has infinite area: $\int_1^{\infty} dx/x = \infty$. We have a paradox: it takes
paint to paint the region but only a finite amount has infinite area: $\int_1^{\infty} dx/x = \infty$. We have a paradox: it takes an infinite amount of paint to paint the region but only a finite amount to fill the horn obtained by rotating SECTION 7.1: Volumes by Slicing—Solids of Revolution **397**
It is interesting to note that this finite volume arises from rotating a region that itself
has infinite area: $\int_1^\infty dx/x = \infty$. We have a paradox: it takes an inf SECTION 7.1: Volumes by Slicing—Solids of Revolution
It is interesting to note that this finite volume arises from rotating a region the
has infinite area: $\int_{1}^{\infty} dx/x = \infty$. We have a paradox: it takes an infinite an
pa

Figure 7.6 Cutaway view of an infinitely
long horn
is

element. EXAMPLE 4 A ring-shaped solid is generated by rotating the finite plane region $y = 2$. Find its volume.

The following example shows how to deal with a problem where the axis of rotation is not the *x*-axis. Just rotate a ple shows how to deal with a problem where the axis of rotate a suitable area element about the axis to form a ve
A ring-shaped solid is generated by rotating the finite plane r
R bounded by the curve $y = x^2$ and the line bolem where the axis of rotation
about the axis to form a volume
by rotating the finite plane region
and the line $y = 1$ about the line
 $= x²$ and $y = 1$ to obtain the The following example shows how to deal with a probl
is not the *x*-axis. Just rotate a suitable area element ab
element.
EXAMPLE 4 A ring-shaped solid is generated by
 $y = 2$. Find its volume.
Solution First, we solve

radius 1. Thus, The following example shows how to deal with a problem where the axis of rotation
is not the x-axis. Just rotate a suitable area element about the axis to form a volume
element.
EXAMPLE 4 A ring-shaped solid is generate where the axis of rotation
the axis to form a volume
ting the finite plane region
 $e \text{ line } y = 1$ about the line
and $y = 1$ to obtain the
these two values of x. The
 dx extending upward from
this area element sweeps The following example shows how to deal with a problem where the axis of rotation
is not the x-axis. Just rotate a suitable area element about the axis to form a volume
element.

EXAMPLE 4 A ring-shaped solid is generat **EXAMPLE 4**

and the x-axis. Just rotate a suitable area element about the axis to form a volume

element.
 EXAMPLE 4

A ring-shaped solid is generated by rotating the finite plane region
 $y = 2$. Find its volume.
 Sol $y = x^2$ to $y = 1$. When R is rotated about the line $y = 2$, this area element sweeps out a thin, washer-shaped volume element of thickness dx and radius $2 - x^2$, having a **PLE 4** A ring-shaped solid is generated by rotating the finite plane region R bounded by the curve $y = x^2$ and the line $y = 1$ about the line ind its volume.
 PERE 4 A ring-shaped solid is generated by rotating the fin **EXAMPLE 4** A ring-shaped solid is generated by rotating the finite plane region R bounded by the curve $y = x^2$ and the line $y = 1$ about the line $y = 2$. Find its volume.
 Solution First, we solve the pair of equations ne region
tut the line
of x. The
ward from
nt sweeps
, having a
rea of this
a circle of **EXAMPLE 4** A ring-shaped solid is generated by rotating the finite plane region
 $y = 2$. Find its volume.
 Solution First, we solve the pair of equations $y = x^2$ and $y = 1$ to obtain the

intersections at $x = -1$ and **EXAMPLE 4** A ring-snaped solid is generated by rotating the lime plane region $y = 2$. Find its volume.
 Solution First, we solve the pair of equations $y = x^2$ and $y = 1$ to obtain the intersections at $x = -1$ and $x = 1$ **R** bounded by the curve $y = y = 2$. Find its volume.
 Solution First, we solve the pair of equations intersections at $x = -1$ and $x = 1$. The solid lies area element of R at position x is a vertical strip $y = x^2$ to $y = 1$ be the line $y = 1$ about the line
 $y = 1$ about the line
 $y = 1$ to obtain the
 $x = 1$ about the line
 $y = 1$ to obtain the
 $y = 1$ to obtain the
 $y = 1$ to obtain the
 $y = 1$ about the line
 $y = 1$ about the line
 $y = 1$ area element of R at position x is a vertical strip of width dx extending upward from $y = x^2$ to $y = 1$. When R is rotated about the line $y = 2$, this area element sweeps out a thin, washer-shaped volume element of thick = 1. The solid lies between these two values is a vertical strip of width dx extending
rotated about the line $y = 2$, this area element of thickness dx and radius 2-
iddle. (See Figure 7.7.) The cross-section
of radius 2 hese two values of x. The
x extending upward from
this area element sweeps
d radius $2 - x^2$, having a
ross-sectional area of this
ea of the hole, a circle of
x.
x.

$$
dV = (\pi(2 - x^2)^2 - \pi(1)^2) dx = \pi(3 - 4x^2 + x^4) dx.
$$

The sum of the first term is the area of a circle of radius 2⁻¹ times the area of the force of radius 1. Thus,
\n
$$
dV = (\pi(2 - x^2)^2 - \pi(1)^2) dx = \pi(3 - 4x^2 + x^4) dx.
$$
\nSince the solid extends from $x = -1$ to $x = 1$, its volume is
\n
$$
V = \pi \int_{-1}^{1} (3 - 4x^2 + x^4) dx = 2\pi \int_{0}^{1} (3 - 4x^2 + x^4) dx
$$
\n
$$
= 2\pi \left(3x - \frac{4x^3}{3} + \frac{x^5}{5}\right)\Big|_{0}^{1} = 2\pi \left(3 - \frac{4}{3} + \frac{1}{5}\right) = \frac{56\pi}{15}
$$
 cubic units.
\nSometimes we want to rotate a region bounded by curves with equations of the form
\n $x = g(y)$ about the y-axis. In this case, the roles of x and y are reversed, and we use
\nhorizontal slices instead of vertical ones.
\n**EXAMPLE 5** Find the volume of the solid generated by rotating the region to the
\nright of the y-axis and to the left of the curve $x = 2y - y^2$ about

 $\left(\frac{4x^3}{3} + \frac{x^5}{5}\right)\Big|_0^1 = 2\pi \left(3 - \frac{4}{3} + \frac{1}{5}\right) = \frac{56\pi}{15}$ cubic units.

to rotate a region bounded by curves with equations of the form y-axis. In this case, the roles of x and y are reversed, and we use tead - right of the y-axis and to the left of the curve $x = 2y - y^2$ about the y-axis. Sometimes we want to rotate a region bounded by curves with equa
 $x = g(y)$ about the y-axis. In this case, the roles of x and y are rever-

horizontal slices instead of vertical ones.
 EXAMPLE 5 Find the volume of the so ed by curves with equations of the form
oles of x and y are reversed, and we use
lid generated by rotating the region to the
the left of the curve $x = 2y - y^2$ about
and $x = 0$, we have
 $y = 2$.

$$
2y - y^2 = 0 \qquad \Longrightarrow \quad y = 0 \quad \text{or} \quad y = 2.
$$

The solid lies between the horizontal planes at $y = 0$ and $y = 2$. A horizontal area
element at height y and having thickness dy rotates about the y-axis to generate a thin
disk-shaped volume element of radius $2y - y^2$ an The solid lies between the horizontal planes at $y = 0$ and $y = 2$. A horizontal area element at height y and having thickness dy rotates about the y-axis to generate a thin disk-shaped volume element of radius $2y - y^2$ an The solid lies between the horizontal planes at $y = 0$ and $y = 2$. A helement at height y and having thickness dy rotates about the y-axis to g disk-shaped volume element of radius $2y - y^2$ and thickness dy. (See I volume $y = 0$ and $y = 2$. A horizontal area
ates about the y-axis to generate a thin
and thickness dy. (See Figure 7.8.) Its
 y^4) dy. 7 Applications of Integration

The solid lies between the horizontal planes at

element at height y and having thickness dy rotations

disk-shaped volume element of radius $2y - y^2$

volume is
 $x = 2y - y^2$
 $dV = \pi (2y - y^2)^2$ = 0 and $y = 2$. A horizontal area
s about the y-axis to generate a thin
d thickness dy. (See Figure 7.8.) Its
d dy. The solid lies between the horizontal planes at $y = 0$ and $y = 2$. A
element at height y and having thickness dy rotates about the y-axis to
disk-shaped volume element of radius $2y - y^2$ and thickness dy. (See
volume is
ontal planes at $y = 0$ and $y = 2$. A horisponcickness dy rotates about the y-axis to gendius $2y - y^2$ and thickness dy. (See Fig.
 $(4y^2 - 4y^3 + y^4) dy$.

$$
dV = \pi (2y - y^2)^2 dy = \pi (4y^2 - 4y^3 + y^4) dy.
$$

$$
dV = \pi(2y - y^2)^2 dy = \pi(4y^2 - 4y^3 + y^3) dy.
$$

\nThus, the volume of the solid is
\n
$$
V = \pi \int_0^2 (4y^2 - 4y^3 + y^4) dy
$$
\n
$$
= \pi \left(\frac{4y^3}{3} - y^4 + \frac{y^5}{5}\right) \Big|_0^2
$$
\n
$$
\pi \int_0^2 (4y^3 - 16y^3 + y^4) dy
$$
\n
$$
= \pi \left(\frac{32}{3} - 16 + \frac{32}{5}\right) = \frac{16\pi}{15} \text{ cubic units.}
$$
\nCylindrical Shells

\nSuppose that the region *R* bounded by $y = f(x) \ge 0$, $y = 0$, $x = a \ge 0$, and $x = b > a$ is rotated about the *y*-axis to generate a solid of revolution. In order

Thus, the volume of the solid is
 $V = \pi \int_0^2 (4y^2 - 4y^3 + y^4) dy$
 $= \pi \left(\frac{4y^3}{3} - y^4 + \frac{y^5}{5}\right)\Big|_0^2$
 $= \pi \left(\frac{32}{3} - 16 + \frac{32}{5}\right) = \frac{16\pi}{15}$ cubic units.
 Cylindrical Shells

Suppose that the region R bounded by $V = \pi \int_0^2 (4y^2 - 4y^3 + y^4) dy$
 $= \pi \left(\frac{4y^3}{3} - y^4 + \frac{y^5}{5}\right)\Big|_0^2$
 $= \pi \left(\frac{32}{3} - 16 + \frac{32}{5}\right) = \frac{16\pi}{15}$ cubic units.
 Cylindrical Shells

Suppose that the region *R* bounded by $y = f(x) \ge 0$, $y = 0$, $x = a \ge 0$, = $\pi \left(\frac{32}{3} - y^2 + \frac{1}{5} \right) \Big|_0$

= $\pi \left(\frac{32}{3} - 16 + \frac{32}{5} \right) = \frac{16\pi}{15}$ cubic units.
 Cylindrical Shells

Suppose that the region *R* bounded by $y = f(x) \ge 0$, $y = 0$, $x = a \ge 0$, and
 $x = b > a$ is rotated about **Cylindrical Shells**

Suppose that the region R bounded by $y = f(x) \ge 0$, $y = 0$, $x = a \ge 0$, and $x = b > a$ is rotated about the y-axis to generate a solid of revolution. In order to find the volume of the solid using (plane) = $\pi \left(\frac{4y^3}{3} - y^4 + \frac{y^5}{5} \right) \Big|_0^2$

= $\pi \left(\frac{32}{3} - 16 + \frac{32}{5} \right) = \frac{16\pi}{15}$ cubic units.
 Cylindrical Shells

Suppose that the region *R* bounded by $y = f(x) \ge 0$, $y = 0$, $x = a \ge 0$, and
 $x = b > a$ is rotated **Cylindrical Shells**
Suppose that the region R bounded by $y = f(x) \ge x = b > a$ is rotated about the y-axis to generate a to find the volume of the solid using (plane) slices, we v sectional area $A(y)$ in each plane of height y, enerate a solid of revolution. In order
enerate a solid of revolution. In order
lices, we would need to know the cross-
nd this would entail solving the equation
porm $x = g(y)$. In practice this can be
inference $2\pi x$
 $y = f$

The standard area element of R at position x is a vertical strip of width dx, height $f(x)$, and area $dA = f(x) dx$. When R is rotated about the y-axis, this strip sweeps out a volume element in the shape of a circular **cylind** The standard area element of *R* at position *x* is a vertical strifulned that $f(x)$, and area $dA = f(x) dx$. When *R* is rotated about the y-a out a volume element in the shape of a circular **cylindrical shell** h: $f(x)$, and

$$
dV = 2\pi x f(x) dx.
$$

SECTION 7.1: Volumes by Slicing—Solids of Revolution 399
The volume of the solid obtained by rotating the plane region
 $y \le f(x)$, $0 \le a < x < b$ about the y-axis is SECTION 7.1: Volumes by Slicing—Solids of Revolution 399

The volume of the solid obtained by rotating the plane region
 $0 \le y \le f(x)$, $0 \le a < x < b$ about the y-axis is
 $V = 2\pi \int_{a}^{b} x f(x) dx$. SECTION 7.1: Volumes by Slicing—Solids of

of the solid obtained by rotating the plane region
 $\le a < x < b$ about the y-axis is
 $x f(x) dx$.

$$
V = 2\pi \int_{a}^{b} x f(x) dx.
$$

The volume of the solid obtained by rotating the plane region
 $0 \le y \le f(x)$, $0 \le a < x < b$ about the y-axis is
 $V = 2\pi \int_a^b x f(x) dx$.
 EXAMPLE 6 (The volume of a torus) A disk of radius a has centre at the point
 $(b, 0)$, whe be the solid obtained by rotating the plane region
 $0 \le a < x < b$ about the y-axis is
 $\int_a^b x f(x) dx$.

(The volume of a torus) A disk of radius *a* has centre at the point
 $(b, 0)$, where $b > a > 0$. The disk is rotated about t The volume of the solid obtained by rotating the plane region
 $0 \le y \le f(x)$, $0 \le a < x < b$ about the y-axis is
 $V = 2\pi \int_a^b x f(x) dx$.
 EXAMPLE 6 (The volume of a torus) A disk of radius a has centre at the point
 $\begin{array}{l}\n\textbf$ The volume of the solid obtained by rotating the plane region
 $0 \le y \le f(x)$, $0 \le a < x < b$ about the y-axis is
 $V = 2\pi \int_{a}^{b} x f(x) dx$.
 EXAMPLE 6 (The volume of a torus) A disk of radius *a* has centre at the point

generat the **the volume of a torus**) A disk of radius a has centre at the point b , 0), where $b > a > 0$. The disk is rotated about the *y*-axis to oughnut-shaped solid), illustrated in Figure 7.10. Find its volume.

with centre a $0 \le y \le f(x)$, $0 \le a < x < b$ about the y-ax
 $V = 2\pi \int_{a}^{b} x f(x) dx$.
 AMPLE 6 (The volume of a torus) A di

(b, 0), where $b > a > 0$. The

rate a torus (a doughnut-shaped solid), illustriciant

(h, 0) and hav
 $b)^{2} + y^{2} = a^{2}$ **EXAMPLE 6** (The volume of a torus) A disk of radius *a* has centre at the point $(b, 0)$, where $b > a > 0$. The disk is rotated about the *y*-axis to generate a torus (a doughnut-shaped solid), illustrated in Figure 7.10. F **EXAMPLE 6** (The volume of a torus) A disk of radiu

(b, 0), where $b > a > 0$. The disk is rc

generate a torus (a doughnut-shaped solid), illustrated in Fig
 Solution The circle with centre at (b, 0) and having radius
 ^a² .x b/², ^b ^a ^x ^b ^C ^a about the ^y-axis. The

 $(x - b)^2 + y^2 = a^2$, so its upper semicircle is the graph of the function

$$
f(x) = \sqrt{a^2 - (x - b)^2}.
$$

EXAMPLE 6 (The volume of a torus) A disk of radius a has compared a torus (b, 0), where $b > a > 0$. The disk is rotated abogenerate a torus (a doughnut-shaped solid), illustrated in Figure 7.10.
 Solution The circle wit

volume of the complete torus is
\n
$$
V = 2 \times 2\pi \int_{b-a}^{b+a} x \sqrt{a^2 - (x - b)^2} dx
$$
\nLet $u = x - b$,
\n
$$
du = dx
$$
\n
$$
= 4\pi \int_{-a}^{a} (u + b) \sqrt{a^2 - u^2} du
$$
\n
$$
= 4\pi \int_{-a}^{a} u \sqrt{a^2 - u^2} du + 4\pi b \int_{-a}^{a} \sqrt{a^2 - u^2} du
$$
\n
$$
= 0 + 4\pi b \frac{\pi a^2}{2} = 2\pi^2 a^2 b
$$
 cubic units.
\n(The first of the final two integrals is 0 because the integrand is odd and the interval is
\nsymmetric about 0; the second is the area of a semicircle of radius a.) Note that the
\nvolume of the torus is $(\pi a^2)(2\pi b)$, that is, the area of the disk being rotated times the
\ndistance travelled by the centre of that disk as it rotates about the y-axis. This result
\nwill be generalized by Pappus's Theorem in Section 7.5.

 $\frac{du}{dt} = u \times \frac{du}{dt}$
 $\frac{du^2}{dt^2} du + 4\pi b \int_{-a}^{a} \sqrt{a^2 - u^2} du$
 $2\pi^2 a^2 b$ cubic units.

tegrals is 0 because the integrand is odd and the interval is

ond is the area of a semicircle of radius *a*.) Note that the
 $(2\$ = $4\pi \int_{-a}^{a} (u + b)\sqrt{a^2 - u^2} du$

= $4\pi \int_{-a}^{a} u \sqrt{a^2 - u^2} du + 4\pi b \int_{-a}^{a} \sqrt{a^2 - u^2} du$

= $0 + 4\pi b \frac{\pi a^2}{2} = 2\pi^2 a^2 b$ cubic units.

(The first of the final two integrals is 0 because the integrand is odd and the in $\int_{-a}^{0} u \sqrt{a^2 - u^2} du + 4\pi b \int_{-a}^{a} \sqrt{a^2 - u^2} du$
 $= 0 + 4\pi b \frac{\pi a^2}{2} = 2\pi^2 a^2 b$ cubic units.

(The first of the final two integrals is 0 because the integrand is odd and the interval is

symmetric about 0; the secon

 $y = x^2$, $0 \le x \le 1$ about the y-axis

Solution The interior of the bowl corresponds to revolving the region given by $x^2 \le y \le 1$, $0 \le x \le 1$ about the *y*-axis. The area element at position *x* has height $1 - x^2$ and generates a cylindrical shell of volume **Solution** The interior of the bowl corresponds to revolving the region given by $x^2 \le y \le 1$, $0 \le x \le 1$ about the y-axis. The area element at position x has height $1 - x^2$ and generates a cylindrical shell of volume dV **Solution** The interior of the bowl corresponds to revolving the region given by $x^2 \le y \le 1$, $0 \le x \le 1$ about the y-axis. The area element at position x has height $1 - x^2$ and generates a cylindrical shell of volume dV e region given by $x^2 \le$
on x has height $1 - x^2$
) dx. (See Figure 7.11.) **Solution** The interior of the bowl corresponds to revolving the regio $y \le 1$, $0 \le x \le 1$ about the y-axis. The area element at position x hand generates a cylindrical shell of volume $dV = 2\pi x(1 - x^2) dx$. (Starting the bo and generates a cylindrical shell of volume $dV = 2\pi x(1 - x^2) dx$. (See Figure 7.11.) the bowl corresponds to revolving the regional vertex
 y -axis. The area element at position x

shell of volume $dV = 2\pi x(1 - x^2) dx$.

where $dV = 2\pi x(1 - x^2) dx$. l corresponds to revolving the region given by
s. The area element at position x has height 1
volume $dV = 2\pi x(1 - x^2) dx$. (See Figure
cubic units.

$$
V = 2\pi \int_0^1 x(1 - x^2) dx
$$

= $2\pi \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}$ cubic units.

and generates a cylindrical shell of volume $dV = 2\pi x(1 - x^2) dx$. (See Figure 7.11.)

Thus, the volume of the bowl is
 $V = 2\pi \int_0^1 x(1 - x^2) dx$
 $= 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}$ cubic units.

We have described two Thus, the volume of the bowl is
 $V = 2\pi \int_0^1 x(1 - x^2) dx$
 $= 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}$ cubic units.

We have described two methods for determining the volume of a solid of revolution,

slicing and cylindrica $V = 2\pi \int_0^1 x(1 - x^2) dx$
 $= 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}$ cubic units.

We have described two methods for determining the volume of a solid of revolution,

slicing and cylindrical shells. The choice of method fo $V = 2\pi \int_0^1 x(1 - x^2) dx$
 $= 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}$ cubic units.

We have described two methods for determining the volume of a solid of revolution,

slicing and cylindrical shells. The choice of method fo $V = 2\pi \int_0^{\infty} x(1 - x^2) dx$
 $= 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}$ cubic units.

We have described two methods for determining the volume of a solid of revolution,

slicing and cylindrical shells. The choice of method $V = 2\pi \int_0^1 x(1-x^2) dx$
 $= 2\pi \left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big|_0^1 = \frac{\pi}{2}$ cubic units.

We have described two methods for determining the volume of a solid of revolution,

slicing and cylindrical shells. The choice of method for $= 2\pi \left(\frac{x^2}{2} - \frac{x^3}{4}\right)\Big|_0 = \frac{\pi}{2}$ cubic units.
We have described two methods for determining the volume of a solid of revolution,
slicing and cylindrical shells. The choice of method for a particular solid is usual (2 4 $J|_0$ 2
We have described two methods for determining the volume of a solid of revolution,
slicing and cylindrical shells. The choice of method for a particular solid is usually
dictated by the form of the equations We have described two methods for determining the volume of a solid of revolution,
slicing and cylindrical shells. The choice of method for a particular solid is usually
dictated by the form of the equations defining the We have described two methods for determining the volume of a solid of revolution,
slicing and cylindrical shells. The choice of method for a particular solid is usually
dictated by the form of the equations defining the For more concreaned with the equivalent in the region of the form of the equided by the form of the equations defining the region being rotated and by the axis of rotation. The volume element dV can always be determined b structured by the form of the equations defining the region being rotated and by the axis of rotation. The volume element dV can always be determined by rotating a suitable area element dA about the axis of rotation. If t about a vertical ine shell into the value of the region of rotation. The volume element d V can always be determined by rotating a suitable area element d A about the axis of rotation. If the region is bounded by vertical so consumed a demond of the axis of rotation. If the region is bounded by vertical lines
area element dA about the axis of rotation. If the region is bounded by vertical lines
and one or more graphs of the form $y = f(x)$, t and one or more graphs of the form $y = f$
vertical strip of width dx. If the rotation is ab-
line, this strip generates a disk- or washer-shape
is about the y-axis or any other vertical line, the
thickness dx. On the other

SECTION 7.1: Volumes by Sticing—Solids of Revolution 401
\nof the solid is
\n
$$
V = 2\pi \int_0^a (b-x) x dx = 2\pi \left(\frac{bx^2}{2} - \frac{x^3}{3}\right)\Big|_0^a = \pi \left(a^2b - \frac{2a^3}{3}\right)
$$
 cubic units.
\n
\n
\nFigure 7.12 The volume element for
\nExample 8

Figure 7.12 The volume element for Theory of the Scample 8

EXERCISES 7.1

The volume element for

EXERCISES 7.1

That the volume of each solid S in Exercises 1–4 in two ways,

ting the method of slicing and the method of cylindrical shells

- Figure 7.12 The volume element for

EXERCISES 7.1

Find the volume of each solid S in Exercises 1–4 in two ways,

using the method of slicing and the method of cylindrical shells.

1. S is generated by rotating about the Figure 7.12 The volume element for

Example 8

EXERCISES 7.1

Find the volume of each solid *S* in Exercises 1-4 in two ways,

using the method of slicing and the method of cylindrical shells.

1. *S* is generated by rota EXERCISES 7.1

Individual about the volume of each solid S in Exercises 1-4 in two ways,

is generated by rotating about the x-axis the region bounded

2. S is generated by rotating the region of Exercise 1 about the

2. by $y = x^2$, $y = 0$, and $x = 1$.
- y-axis.
- **EXERCISES 7.1**

ind the volume effect solid S in Exercises 1-4 in two ways,

sing the method of slicing and the method of cylindrical shells. axis of the hole lies along

1. S is generated by rotating about the x-axis th **EXERCISES 7.1**

ind the volume of each solid *S* in Exercises 1-4 in two ways,

is generated by rotating about the *x*-axis the region bounded

1. *S* is generated by rotating about the *x*-axis the region bounded

2. *S* by $y = x^2$ and $y = \sqrt{x}$ between $x = 0$ and $x = 1$. find the v **EXERCISE 1.4**

in the volume of each solid S in Exercises 1–4 in two ways,

1. S is generated by rotating about the x-axis the region bounded

by $y = x^2$, $y = 0$, and $x = 1$.

2. S is generated by rotating the region of
- y-axis.

Find the volumes of the solids obtained if the plane region bounded

5. At is generated by rotating about the *x*-axis the region bounded

5. At is generated by rotating the region of Exercise 1 about the

16. Find the vo 1. *S* is generated by rotating about the *x*-axis the region bounded

2. *S* is generated by rotating the region of Exercise 1 about the
 y-axis.

3. *S* is generated by rotating about the *x*-axis the region bounded
 the y-axis. 2. S is generated by rotating the region of Exercise 1 about the

3. S is generated by rotating about the x-axis the region bounded

4. S is generated by rotating the region of Exercise 3 about the

4. S is generated by r by $y = x^2$ and $y = \sqrt{x}$ between $x = 0$ and $x = 1$.
 4. S is generated by rotating the region of Exercise 3 about the

ind the volumes of the solids obtained if the plane regions R

secribed in Exercises 5–10 are rotated 4. S is generated by rotating the region of Exercise 3 about the
 y -axis.

ind the volumes of the solids obtained if the plane regions R

of water is in

the welvest is bounded by $y = x(2-x)$ and $y = 0$ between $x = 0$ and

-
- .
- .
-
-
-
- Find the volumes of the solids obtained if the plane regions R

the y-axis and (b)

the y-axis momed by $y = x(2 x)$ and $y = 0$ between $x = 0$ and
 $x = 2$.
 6. R is the finite region bounded by $y = x$ and $y = x^2$.
 8. R *x* = 2.
 R is the finite region bounded by $y = x$ and $y = x^2$.
 R is the finite region bounded by $y = 1$ sand $x = 4y - y^2$.
 R is bounded by $y = 1 + \sin x$ and $y = 1$ from $x = 0$ to the volume of the solid solid solid so generated. 12. Find the volume of the solid generated by $y = 1$, $x = 1$ from $x = 0$ to $x = \pi$.

13. What percentage of the volume of a ball of radius 2 is removed.

13. What percentage of the volume of a ball of radius 2 is removed. **8.** *R* is bounded by $y = 1 + \sin x$ and $y = 1$ from $x = 0$ to $x = \pi$.
 9. *R* is bounded by $y = 1/(1 + x^2)$, $y = 2$, $x = 0$, and $x = 1$.
 10. *R* is the finite region bounded by $y = 1/x$ and $3x + 3y = 10$.
 11. The triang *R* is bounded by $y = 1/(1 + x^2)$, $y = 2$, $x = 0$, and $x = 1$.
 R is the finite region bounded by $y = 1/x$ and $3x + 3y = 10$.

The triangular region with vertices $(0, -1)$, $(1, 0)$, and $(0, 1)$ is
 23. Repeat Exercise 2
- $0 \le y \le 1 x^2$ about the line $y = 1$.
-
- 10. *R* is the finite region bounded by $y = 1/x$ and $3x + 3y = 10$.

11. The triangular region with vertices $(0, -1)$, $(1, 0)$, and $(0, 1)$ is

12. Find the volume of the solid generated by rotating the region
 $0 \le y \le 1 -$ The triangular region with vertices $(0, -1)$, $(1, 0)$, and $(0, 1)$ is
 23 . Repeat Exercise 22 wind about the line $x = 2$. Find the volume of the solid so 24 . Early editions of this to

generated.

Find the volume of 11. The triangular region with vertices $(0, -1)$, $(1, 0)$, and $(0, 1)$ is

totated about the line $x = 2$. Find the volume of the solid generated by rotating the region

12. Find the volume of the solid generated by rota
-

right-circular cone of height h and base radius $b > a$. If the axis of the hole lies along that of the cone, find the volume of the remaining part of the cone. A

A

A
 $\frac{b}{x}$

axis of the hole lies along that of the cone, find the volume of

the remaining part of the cone.

Find the volume of the solid obtained by rotating a circular

disk about one of its tangent lines. **Example 18 The remaining part of the cone.** The remaining part of the cone. Since $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ and $\frac{1}{2}$ are remaining part of the cone. Find the volume of the remaining part of the cone.
Fin 16. Find the volume of height *h* and base radius $b > a$. If the values of the hole lies along that of the cone, find the volume of the remaining part of the cone.
 16. Find the volume of the solid obtained by rotating a

-
- Fight-circular cone of height h and base radius $b > a$. If the axis of the hole lies along that of the cone, find the volume of the remaining part of the cone.
Find the volume of the solid obtained by rotating a circular d 19. The slice set of height *h* and base radius $b > a$. If the axis of the hole lies along that of the cone, find the volume of the remaining part of the cone.

16. Find the volume of the solid obtained by rotating a circu parameterizative of height *h* and base radius $b > a$. If the axis of the hole lies along that of the cone, find the volume of the remaining part of the cone.
Find the volume of the solid obtained by rotating a circular di Fight-circular cone of height h and base radius $b > a$. If the axis of the hole lies along that of the cone, find the volume of the remaining part of the cone.
Find the volume of the solid obtained by rotating a circular d 18. The same is the same of height *h* and base radius $b > a$. If the axis of the hole lies along that of the cone, find the volume of the remaining part of the cone.

16. Find the volume of the solid obtained by rotating right-circular cone of height *h* and base radius $b > a$. If the axis of the hole lies along that of the cone, find the volume of the remaining part of the cone.
Find the volume of the solid obtained by rotating a circular Figure the bological bond of the cone, find the volume of
the remaining part of the cone.
Find the volume of the solid obtained by rotating a circular
disk about one of its tangent lines.
A plane slices a ball of radius a the remaining part of the cone.
 16. Find the volume of the solid obtained by rotating a circular

disk about one of its tangent lines.
 17. A plane slices a ball of radius *a* into two pieces. If the plane

passes *b* Find the volume of the solid obtained by rotating a circu
disk about one of its tangent lines.
A plane slices a ball of radius *a* into two pieces. If the pl
passes *b* units away from the centre of the ball (where *b*
fi by rotating a circular
wo pieces. If the plane
the ball (where $b < a$),
we did if the same solom so
s 20 cm. What volume
olution obtained by
 $0 = 1$ about the x-axis.
Example 6 by slicing
using cylindrical disk about one of its tangent lines.

17. A plane slices a ball of radius *a* into two pieces. If the plane

passes *b* units away from the centre of the ball (where *b* < *a*),

find the volume of the smaller piece.

18. A plane slices a ball of radius *a* into two pieces. If the plane
passes *b* units away from the centre of the ball (where $b < a$),
find the volume of the smaller piece.
Water partially fills a hemispherical bowl of radius
-
- $(a^2) + (y^2/b^2) = 1$ about the *x*-axis.
- shells.
- 18. Water partially fills a hemispherical bowl of radius 30 cm so
that the maximum depth of the water is 20 cm. What volume
of water is in the bowl?
19. Find the volume of the ellipsoid of revolution obtained by
rotating bowl of radius 30 cm so

r is 20 cm. What volume

evolution obtained by
 b^2) = 1 about the x-axis.

of Example 6 by slicing

aan using cylindrical

and $y = 0$ and lying to the

ex-axis and (b) about the

of revolution Water partially fills a hemispherical bowl of radius 30 cm so
that the maximum depth of the water is 20 cm. What volume
of water is in the bowl?
Find the volume of the ellipsoid of revolution obtained by
rotating the elli wact partially ints a neimspherical bowl of radius 50 cm so
that the maximum depth of the water is 20 cm. What volume
of water is in the bowl?
Find the volume of the ellipsoid of revolution obtained by
rotating the ellips that the maximum depth of the water is 20 cm. W
of water is in the bowl?
Find the volume of the ellipsoid of revolution obt
rotating the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ abou
Recalculate the volume of the torus of Example 6
 19. Find the volume of the ellipsoid of revolution obtained by
rotating the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ about the x-axis.
20. Recalculate the volume of the torus of Example 6 by slicing
perpendicular to the y-axis rathe evolution obtained by
 b^2) = 1 about the x-axis.

of Example 6 by slicing

aan using cylindrical

and $y = 0$ and lying to the

ex-axis and (b) about the

of revolution generated in

and $y = 0$ and lying to the

-axis. Find and volation of the chipsolon of Levolution obtained by
rotating the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ about the x-axis.
Recalculate the volume of the torus of Example 6 by slicing
perpendicular to the y-axis rather than otating the empty $(x / u) + (y / v) = 1$ about the x-axis.
Recalculate the volume of the torus of Example 6 by slicing
perpendicular to the y-axis rather than using cylindrical
shells.
The region R bounded by $y = e^{-x}$ and $y = 0$ 23. Repeat Exercise 22 with rotation about the y-axis.

21. The region *R* bounded by $y = e^{-x}$ and $y = 0$ and lying to the right of $x = 0$ is rotated (a) about the x-axis and (b) about the y-axis. Find the volume of the s
-
-
- 21. The region *R* bounded by $y = e^{-x}$ and $y = 0$ and lying to the right of $x = 0$ is rotated (a) about the *x*-axis and (b) about the *y*-axis. Find the volume of the solid of revolution generated in each case.
22. The r The region *R* bounded by $y = e^{-x}$ and $y = 0$ and lying to the right of $x = 0$ is rotated (a) about the *x*-axis and (b) about the *y*-axis. Find the volume of the solid of revolution generated in each case.
The region *R* The region *R* bounded by $y = e^x$ and $y = 0$ and rying to the right of $x = 0$ is rotated (a) about the *x*-axis and (b) about the *y*-axis. Find the volume of the solid of revolution generated in each case.
The region *R* Figure of $x = 0$ is folded (a) about the x-axis and (b) about the y-axis. Find the volume of the solid of revolution generated in each case.
The region R bounded by $y = x^{-k}$ and $y = 0$ and lying to the right of $x = 1$ is y-axis. That the volume of the solid of revolution generated in
each case.
The region R bounded by $y = x^{-k}$ and $y = 0$ and lying to the
right of $x = 1$ is rotated about the x-axis. Find all real values
of k for which the Cach case.
The region *R* bounded by $y = x^{-k}$ and $y = 0$ and lying to the right of $x = 1$ is rotated about the *x*-axis. Find all real values of *k* for which the solid so generated has finite volume.
Repeat Exercise 22 w 22. The region R bounded by $y = x^2$ and $y = 0$ and sying to the right of $x = 1$ is rotated about the x-axis. Find all real values of k for which the solid so generated has finite volume.

23. Repeat Exercise 22 with rotat Figure of $x = 1$ is folded about the x -axis. Find an Icar values
of k for which the solid so generated has finite volume.
Repeat Exercise 22 with rotation about the y -axis.
Early editions of this text incorrectly de Repeat Exercise 22 with rotation about the y-axis.

Early editions of this text incorrectly defined a prism or

cylinder as being a solid for which cross-sections parallel to

the base were congruent to the base. Does thi Early editions of this text incorrectly defined a prism or cylinder as being a solid for which cross-sections parallel to the base were congruent to the base. Does this define a larger or smaller set of solids than the de
-

CHAPTER 7 Applications of Integration
prism according to the definition given in early editions? Is it
a prism according to the definition in this edition? If the
height of S is b cm, what is the volume of S?
Find the vol CHAPTER 7 Applications of Integration

prism according to the definition given in early editions? Is it

a prism according to the definition in this edition? If the

height of S is b cm, what is the volume of S?

Find the

- CHAPTER 7 Applications of Integration
prism according to the definition given in early editions? Is it Use Simplear
prism according to the definition in this edition? If the
height of S is b cm, what is the volume of S?
F **402** CHAPTER 7 Applications of Integration

prism according to the definition given in early editions? Is it

a prism according to the definition in this edition? If the

height of S is b cm, what is the volume of S?
 1 CHAPTER 7 Applications of Integration

prism according to the definition given in early editions? Is it

a prism according to the definition in this edition? If the

height of S is b cm, what is the volume of S?

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prism according to the definition given in early edition? If sit

a prism according to the definition in this edition? If the

height of S is b cm, what is the volume of S?

Find the CHAPTER 7 Applications of Integration

prism according to the definition given in early editions? Is it

a prism according to the definition in this edition? If the

height of S is b cm, what is the volume of S?

Find the
- **Example 127.** Given the surface area of a sphere of radius r and vertical surface area of a sphere of radius r is a set of radius r is kr², **Exerce** area of a sphere of radius r is kr², where k is a constant independ , prism according to the definition given in early editions? Is it
a prism according to the definition in this edition? If the
height of S is b cm, what is the volume of S?
Find the volume of the solid generated by rotating prism according to the definition given in early editions? Is it

a prism according to the definition in this edition? If the

height of S is b cm, what is the volume of S?

Find the volume of the solid generated by rotat prism according to the definition given in early editions? Is it

a prism according to the definition in this edition? If the

height of S is b cm, what is the volume of S?

Find the cultures of the solid generated by rot a prism according to the definition in this edition? If the height of *S* is *b* cm, what is the volume of *S*?
Find the volume of the solid generated by rotating the finite region in the first quadrant bounded by the coo

- The region shaded in Figure 7.13 is rotated about the *x*-axis.

The region shaded in Figure 7.13 is rotated about the *x*-axis.

The region shaded in Figure 7.13 is rotated about the *y*-axis.

The region shaded in Figure
-

Use Simpson's Rule to find the volume of the resulting solid.
The region shaded in Figure 7.13 is rotated about the line $x = -1$. Use Simpson's Rule to find the volume of the resulting solid. Use Simpson's Rule to find the volume of the resulting solid.
 30. The region shaded in Figure 7.13 is rotated about the line $x = -1$. Use Simpson's Rule to find the volume of the resulting solid.

The following problems Use Simpson's Rule to find the volume of the resulting
The region shaded in Figure 7.13 is rotated about the $1x = -1$. Use Simpson's Rule to find the volume of th
resulting solid.
following problems are *very difficult*. Y Use Simpson's Rule to find the volume of the resulting solid.
 30. The region shaded in Figure 7.13 is rotated about the line
 $x = -1$. Use Simpson's Rule to find the volume of the

resulting solid.

The following proble Use Simpson's Rule to find the volume of the resulting solid.
 30. The region shaded in Figure 7.13 is rotated about the line
 $x = -1$. Use Simpson's Rule to find the volume of the

resulting solid.

The following proble Use Simpson's Rule to find the volume of the resulting s
 30. The region shaded in Figure 7.13 is rotated about the line
 $x = -1$. Use Simpson's Rule to find the volume of the

resulting solid.

The following problems ar

Use Simpson's Rule to find the volume of the resulting solid.
 31. The region shaded in Figure 7.13 is rotated about the line
 $x = -1$. Use Simpson's Rule to find the volume of the

resulting solid.

The following probl Use Simpson's Rule to find the volume of the resulting solid.
The region shaded in Figure 7.13 is rotated about the line
 $x = -1$. Use Simpson's Rule to find the volume of the
resulting solid.
following problems are *very d* Use Simpson's Rule to find the volume of the resulting solid.
The region shaded in Figure 7.13 is rotated about the line
 $x = -1$. Use Simpson's Rule to find the volume of the
resulting solid.
following problems are *very d* Solution is Kind to find the volume of the resulting solution.
The region shaded in Figure 7.13 is rotated about the line
resulting solid.
following problems are *very difficult*. You will need some
nuity and a lot of har The region shaded in Figure 7.13 is rotated about the line $x = -1$. Use Simpson's Rule to find the volume of the resulting solid.
following problems are *very difficult*. You will need some nuity and a lot of hard work to glass.

Figure 7.14

Figure 7.14

The finite plane region bounded by the curve $xy = 1$ and the

straight line $2x + 2y = 5$ is rotated about that line to generate

a solid of revolution. Find the volume of that solid.

EXECUTE: EXECUTE: The finite plane region bounded by the curve $xy = 1$ and the resulting solid.

straight line $2x + 2y = 5$ is rotated about that line to generate

a solid of revolution. Find the volume of that solid.
 solut the x-axis.
 Solution is that are not straight line $2x + 2y = 5$ is rotated about that line to generate

a solid of revolution. Find the volume of that solid.
 Solution is the view of the solids of revolution. Al of the solid in solid in Section. Find the volume of that solid.

Solid in the v-axis.

Solid in every plane perpendicular to some fixed axis.

Solid in the value of the solid in Section 7.1 can be used to determine volum about the x-axis.
 i132. The finite plane region bounded by the curve $xy = 1$ and the resulting solid.
 Solut the y-axis.
 a solid of revolution. Find the volume of that solid.
 Solution
 Solution
 Properate a Slicing
The method of slicing introduced in Section 7.1 can be used to determine volumes of
solids that are not solids of revolution. All we need to know is the area of cross-section
of the solid in every plane perpendi g

od of slicing introduced in Section 7.1 can be use

are not solids of revolution. All we need to know

d in every plane perpendicular to some fixed axis

d lies between the planes at $x = a$ and $x = b > a$

e plane at x is solids that are not solids of revolution. All we need to know is the area of cross-section
of the solid in every plane perpendicular to some fixed axis. If that axis is the x-axis,
if the solid lies between the planes at

$$
V = \int_{a}^{b} A(x) \, dx.
$$

is exist and cones are solids of evolution.

Pyramids and cones are solid lies between the planes at $x = a$ and $x = b > a$, and if the cross-sectional

in the plane at x is the continuous (or even piecewise continuous) functi It the solid lies between the planes at $x = a$ and $x = b > a$, and if the cross-sectional
area in the plane at x is the continuous (or even piecewise continuous) function $A(x)$,
then the volume of the solid is
 $V = \int_a^b A(x) dx$.
 area in the plane at x is the continuous (or even piecewise continuous) function $A(x)$,
then the volume of the solid is
 $V = \int_a^b A(x) dx$.
In this section we consider some examples that are not solids of revolution.
Pyramids then the volume of the solid is
 $V = \int_a^b A(x) dx$.

In this section we consider some examples that are not solids of revolution.
 Pyramids and **cones** are solids consisting of all points on line segments that join

a fixed $V = \int_{a}^{b} A(x) dx$.

In this section we consider some examples that are not solids of revolution.
 Pyramids and **cones** are solids consisting of all points on line segments that join

a fixed point, the **vertex**, to all t $V = \int_a^b A(x) dx$.

In this section we consider some examples that are not s
 Pyramids and **cones** are solids consisting of all po

a fixed point, the **vertex**, to all the points in a region lyin

vertex. The region is call

$$
V = \frac{1}{3} Ah,
$$

SECTION 7.2: More Volumes by Slicing **403**
where A is the area of the base region, and h is the height from the vertex to the plane
of the base, measured in the direction perpendicular to that plane. We will give a very
s SECTION 7.2: More Volumes by Slicing **403**
where *A* is the area of the base region, and *h* is the height from the vertex to the plane
of the base, measured in the direction perpendicular to that plane. We will give a ve SECTION 7.2: More Volumes by Slicing **403**
where *A* is the area of the base region, and *h* is the height from the vertex to the plane
of the base, measured in the direction perpendicular to that plane. We will give a ve SECTION 7.2

where *A* is the area of the base region, and *h* is the height

of the base, measured in the direction perpendicular to the

simple proof of this fact in Section 16.4. For the time b

of a rectangular base.

-
-

Solution Cross-sections of the pyramid in planes parallel to the base are similar rectangles. If the origin is at the vertex of the pyramid and the *x*-axis is perpendicular to the base, then the cross-section at positi **Solution** Cross-sections of the pyramid in planes parallel to the base angles. If the origin is at the vertex of the pyramid and the *x*-axis is to the base, then the cross-section at position *x* is a rectangle whose x (a)

tions of the pyramid in planes parallel to the is at the vertex of the pyramid and the α :

cross-section at position x is a rectangle

sponding dimensions of the base. For exa

let h times the length PQ, as can be angles. If the origin is at the vertex of the pyramid and the *x*-axis is perpendicular
to the base, then the cross-section at position *x* is a rectangle whose dimensions are
 x/h times the corresponding dimensions of th

$$
A(x) = \left(\frac{x}{h}\right)^2 A.
$$

$$
V = \int_0^h \left(\frac{x}{h}\right)^2 A \, dx = \frac{A}{h^2} \frac{x^3}{3} \bigg|_0^h = \frac{1}{3} Ah
$$
 cubic units.

the length LM is x/h times the length PQ, as can be seen from the similar triangles OLM and OPQ . Thus, the area of the rectangular cross-section at x is $A(x) = \left(\frac{x}{h}\right)^2 A$.
The volume of the pyramid is therefore $V = \int_$ *Figure 7.16(b)*. Thus, the area of the rectangular cross-section at *x* is $A(x) = \left(\frac{x}{h}\right)^2 A$.
The volume of the pyramid is therefore
 $V = \int_0^h \left(\frac{x}{h}\right)^2 A dx = \frac{A}{h^2} \frac{x^3}{3} \Big|_0^h = \frac{1}{3} Ah$ cubic units.
A similar argum $A(x) = \left(\frac{x}{h}\right)^2 A$.
The volume of the pyramid is therefore
 $V = \int_0^h \left(\frac{x}{h}\right)^2 A dx = \frac{A}{h^2} \frac{x^3}{3} \Big|_0^h = \frac{1}{3} Ah$ cubic units.
A similar argument, resulting in the same formula for the
that is, a pyramid with a more ge that $\left(\frac{x^3}{3}\right)_0^h = \frac{1}{3}Ah$ cubic units.

the same formula for the volume, holds for a cone,

general (curved) shape to its base, such as that in

t as obvious as in the case of the pyramid, the cross-

times that of The volume of the pyramid is therefore
 $V = \int_0^h \left(\frac{x}{h}\right)^2 A dx = \frac{A}{h^2} \frac{x^3}{3} \Big|_0^h = \frac{1}{3} Ah$ cubic units.

A similar argument, resulting in the same formula for the volume, holds for a cone,

that is, a pyramid with

EXAMPLE 2 A tent has a circular base of radius a metres and is supported by a horizontal ridge bar held at height b metres above a diameter of in the base by vertical supports at each end of the diameter. The materi A tent has a circular base of radius a metres and is supported by
a horizontal ridge bar held at height b metres above a diameter of
supports at each end of the diameter. The material of the tent is
at each cross-sect **EXAMPLE 2** A tent has a circular base of radius a metres and is supported by a horizontal ridge bar held at height b metres above a diameter of the base by vertical supports at each end of the diameter. The material **EXAMPLE 2** A tent has a circular base of radius *a* metres and is supported by a horizontal ridge bar held at height *b* metres above a diameter of the base by vertical supports at each end of the diameter. The material **EXAMPLE 2** A tent has a circular base of radius *a* metres and is supported by a horizontal ridge bar held at height *b* metres above a diameter of the base by vertical supports at each end of the diameter. The material **EXAMPLE 2** A tent has a circular base of radius *a* metres and is supported by a horizontal ridge bar held at height *b* metres above a diameter of the base by vertical supports at each end of the diameter. The material **EXAMPLE 2** A tent has a circular base of radius *a* metres and is s
a horizontal ridge bar held at height *b* metres above a
the base by vertical supports at each end of the diameter. The material c
stretched tight so th the base of radius *a* metres and is supported by the held at height *b* metres above a diameter of d of the diameter. The material of the tent is perpendicular to the ridge bar is an isosceles me of the tent.

<u>Exercised</u> a horizontal ridge bar held at height *b* metres abouthe base by vertical supports at each end of the diameter. The mater stretched tight so that each cross-section perpendicular to the ridge batriangle. (See Figure 7.17.

$$
A(x) = \frac{1}{2} \left(2\sqrt{a^2 - x^2} \right) b = b\sqrt{a^2 - x^2}.
$$

Solution Let the *x*-axis be the diameter of the base under the ridge bar. The cross-section at position *x* has base length
$$
2\sqrt{a^2 - x^2}
$$
, so its area is\n
$$
A(x) = \frac{1}{2} (2\sqrt{a^2 - x^2})b = b\sqrt{a^2 - x^2}.
$$
\nThus, the volume of the solid is\n
$$
V = \int_{-a}^{a} b\sqrt{a^2 - x^2} dx = b \int_{-a}^{a} \sqrt{a^2 - x^2} dx = b \frac{\pi a^2}{2} = \frac{\pi}{2} a^2 b \text{ m}^3.
$$
\nNote that we evaluated the last integral by inspection. It is the area of a half-disk of radius *a*.

radius a.

EXAMPLE 3 Two circular cylinders, each having radius *a*, intersect so that their axes meet at right angles. Find the volume of the region lying inside both cylinders.
 Solution We represent the cylinders in a three-d **EXAMPLE 3** Two circular cylinders, each having radius *a*, intersect so that their axes meet at right angles. Find the volume of the region lying inside both cylinders.
 Solution We represent the cylinders in a three-d **EXAMPLE 3** Two circular cylinders, each having radius *a*, intersect so that their axes meet at right angles. Find the volume of the region lying inside both cylinders.
 Solution We represent the cylinders in a three-d **EXAMPLE 3** Two circular cylinders, each having radius a, intersect so that their axes meet at right angles. Find the volume of the region lying inside both cylinders.
 Solution We represent the cylinders in a three-dim **EXAMPLE 3** Two circular cylinders, each having radius *a*, intersect so that their axis are more are at right angles. Find the volume of the region lying inside both cylinders.
 Solution We represent the cylinders in a **EXAMPLE 3** Two circular cylinders, each having radius a, intersect so that their axes meet at right angles. Find the volume of the region lying inside both cylinders.
 Solution We represent the cylinders in a three-dim **Solution** We represent the cylinders in a three-dimensional Cartesian coordinate system where the plane containing the x- and y-axes is horizontal and the z-axis is vertical. One-eighth of the solid is represented in Fig **Solution** We represent the cylinders in a three-dimer
tem where the plane containing the x- and y-axes is h
cal. One-eighth of the solid is represented in Figure 7.
three coordinates being positive. The two cylinders ha
 sent the cylinders in a three-dimensional Cartesian coordinate
containing the x- and y-axes is horizontal and the z-axis is
ne solid is represented in Figure 7.18, that part corresponding
ing positive. The two cylinders h mal Cartesian coordinate sys-

zontal and the *z*-axis is verti-

that part corresponding to all

zxes along the *x*- and *y*-axes,

rsects the plane of the *y*- and
 x- and *z*-axes in a circle of

ylinders (and having tem where the plane containing the x- and y-axes is horizontal and the z-axis is verti-
cal. One-eighth of the solid is represented in Figure 7.18, that part corresponding to all
three coordinates being positive. The two the cylinders in a three-dimensional Cart
taining the x- and y-axes is horizontal are bidid is represented in Figure 7.18, that part
positive. The two cylinders have axes alor
er with axis along the x-axis intersects the

cubic subsets the plane of the y-axes,
ects the plane of the y-and
c- and z-axes in a circle of
linders (and having $x \ge 0$,
neight z above the xy-plane
The volume V of the whole
cubic units.

$$
V = 8 \int_0^a (a^2 - z^2) dz = 8 \left(a^2 z - \frac{z^3}{3} \right) \Big|_0^a = \frac{16}{3} a^3
$$
 cubic units.

- **EXERCISES 7.2**
 1. A solid is 2 m high. The cross-section of the solid at height x
 1. A solid is 2 m high. The cross-section of the solid at height x
 11. A solid has a circular

above its base has area 3x square **ERCISES 7.2**

A solid is 2 m high. The cross-section of the solid at height x

above its base has area 3x square metres. Find the volume of

the solid is 2 m high. The cross-section of the solid at height x

the solid is
- **XERCISES 7.2**
A solid is 2 m high. The cross-section of the solabove its base has area 3x square metres. Find the solid.
The cross-section at height z of a solid of height rectangle with dimensions z and $h z$. Find the **EXERCISES 7.2**
 2. 1. A solid is 2 m high. The cross-section of the solid at height x

above its base has area 3x square metres. Find the volume of
 2. The cross-section at height z of a solid of height h is a
 2. XERCISES 7.2
A solid is 2 m high. The cross-section of the solid at height x
above its base has area 3x square metres. Find the volume of
the solid of height h is a
Find the volume of the solid of height h is a
solid.
F solid. **EXERCISES 7.2**
 1. A solid is 2 m high. The cross-section of the solid at height x

above its base has area 3x square metres. Find the volume of

the solid.

2. The cross-section at height z of a solid of height h is a **XERCISES 7.2**

A solid is 2 m high. The cross-section of the solid at height x

above its base has area 3x square metres. Find the volume of

the solid.

The cross-section at height z of a solid of height h is a

rectang
-
- 1. A solid is 2 m high. The cross-section of the solid at height x

2. The cross-section at height z of a solid of height h is a

2. The cross-section at height z of a solid of height h is a

3. The cross-section at heigh A solid is 2 m high. The cross-section of the solid at height x

above its base has area 3x square metres. Find the volume of

the solid.

The cross-section at height z of a solid of height h is a

solid.

Find the volume A solid is 2 in light. The close-section of the solid at height x

above its base has area 3x square metres. Find the volume of

the solid.

The cross-section at height z of a solid of height h is a

solid.

13. The recta **12.** The cross-section at height z of a solid of height h is a

Find the volume of

rectangle with dimensions z and $h - z$. Find the volume of the

solid.
 3. Find the volume of a solid of height 1 whose cross-section a The cross-section at height z of a solid of height h is a

rectangle with dimensions z and $h - z$. Find the volume of the

solid.

Find the volume of a solid of height 1 whose cross-section at

height z is an ellipse with The cross-section at height 2 of a solid of height *n* is a
rectangle with dimensions z and $h - z$. Find the volume of the
solid.
13. T
Find the volume of a solid of height 1 whose cross-section at
height z is an ellipse solid.
 13. Find the volume of a solid of height 1 whose cross-section at
 13. Find the volume of a solid of height 1 whose cross-section at
 4. A solid extends from $x = 1$ to $x = 3$. The cross-section of the

solid Find the volume of a solid of height 1 whose cross-section at

height z is an ellipse with semi-axes z and $\sqrt{1-z^2}$.

A solid extends from $x = 1$ to $x = 3$. The cross-section of the

solid in the plane perpendicular to is an ellipse with semi-axes z and $\sqrt{1-z^2}$, to or

stends from $x = 1$ to $x = 3$. The cross-section of the

he plane perpendicular to the x-axis at x is a square

Find the volume of the solid. The values of the solid.

-
- cross-section at position x is an equilateral triangle with edge
length \sqrt{x} . Find the volume of the solid.
- **4.** A solid extends from $x = 1$ to $x = 3$. The cross-section of the solid.

solid in the plane perpendicular to the *x*-axis at *x* is a square

of side *x*. Find the volume of the solid.
 5. A solid is 6 ft high. Its A solid in the plane perpendicular to the *x*-axis at *x* is a square

solid in the plane perpendicular to the *x*-axis at *x* is a square

of side *x*. Find the volume of the solid.

A solid is 6 ft high. Its horizontal of side x. Find the volume of the solid.
A solid is 6 ft high. Its horizontal cross-section at height z ft above its base is a rectangle with length $2 + z$ ft and width $8 - z$ ft. Find the volume of the solid.
A solid exten $(1 - (y/h))$ radians.
- 5. A solid is 6 ft high. Its horizontal cross-section at height z ft

above its base is a rectangle with length $2 + z$ ft and width
 $8 z$ ft. Find the volume of the solid.

6. A solid extends along the x-axis from $x = 1$ above its base is a rectangle with length $2 + z$ ft and width $8 - z$ ft. Find the volume of the solid.
A solid extends along the x-axis from $x = 1$ to $x = 4$. Its
cross-section at position x is an equilateral triangle with volume of the solid.

Sum g the x-axis from $x = 1$ to $x = 4$. Its

sition x is an equilateral triangle with edge

ne volume of the solid.

Fa solid that is h cm high if its horizontal

y height y above its base is a circu 6. A solid extends along the *x*-axis from $x = 1$ to $x = 4$. Its

cross-section at position *x* is an equilateral triangle with edge

length \sqrt{x} . Find the volume of the solid.

7. Find the volume of a solid that is *h* Length \sqrt{x} . Find the volume of the solid.

Find the volume of a solid that is h cm high if its horizontal

cross-section at any height y above its base is a circular sector

having radius a cm and angle $2\pi (1 - (y/h))$ ra Find the volume of a solid that is h cm high if its horizontal
cross-section at any height y above its base is a circular sector
having radius a cm and angle $2\pi (1 - (y/h))$ radians.
The opposite ends of a solid are at $x = 0$ *h* cm high if its horizontal

we its base is a circular sector
 $(1 - (y/h))$ radians.

tt $x = 0$ and $x = 2$. The area

plane perpendicular to the

the volume of the solid is

solid in any horizontal plane

solid in any hori
-
- cross-section at any height y above its base is a circular sector

having radius a cm and angle $2\pi (1 (y/h))$ radians.

8. The opposite ends of a solid are at $x = 0$ and $x = 2$. The area

of cross-section of the solid in a having radius a cm and angle $2\pi (1 - (y/h))$ radians.

The opposite ends of a solid are at $x = 0$ and $x = 2$. The area

of cross-section of the solid in a plane perpendicular to the

x-axis at x is kx^3 square units. The v Find the volume of a solid that is h cm high if its horizontal

cross-section at any height y above its base is a circular sector

having radius a cm and angle $2\pi (1 - (y/h))$ radians.

The opposite ends of a solid are at x The opposite ends of a solid are at $x = 0$ and $x = 2$. The area
of cross-section of the solid in a plane perpendicular to the
x-axis at x is kx^3 square units. The volume of the solid is
4 cubic units. Find k.
Find the having radius a cm and angle $2\pi (1 - (y/h))$ radians.
The opposite ends of a solid are at $x = 0$ and $x = 2$. The area
of cross-section of the solid in a plane perpendicular to the
or-xxis at x is kx^3 square units. The vol
- 11. A solid has a circular base of radius r . All sections of the solid perpendicular to a particular diameter of the base are squares.
Find the volume of the solid.
12. Repeat Exercise 11 but with sections that are equi A solid has a circular base of radius r. All sections of the solid
perpendicular to a particular diameter of the base are squares.
Find the volume of the solid.
Repeat Exercise 11 but with sections that are equilateral
tri A solid has a circular base of radius r. All sections of the solid
perpendicular to a particular diameter of the base are squares.
Find the volume of the solid.
Repeat Exercise 11 but with sections that are equilateral
tri 11. A solid has a circular base of radius r . All sections of the solid
perpendicular to a particular diameter of the base are squares.
Find the volume of the solid.
12. Repeat Exercise 11 but with sections that are equi
-
- height z is an ellipse with semi-axes z and $\sqrt{1-z^2}$.

to one of these legs is half of a circular disk. Find the volume
 Δ solid extends from $x = 1$ to $x = 3$. The cross section of the of the solid. A solid has a circular base of radius r. All sections of the solid
perpendicular to a particular diameter of the base are squares.
Find the volume of the solid.
Repeat Exercise 11 but with sections that are equilateral
tri 11. A solid has a circular base of radius r . All sections of the solid perpendicular to a particular diameter of the base are squares.
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perpendicular to a particular diameter of the base are squares.
Find the volume of the solid.
Repeat Exercise 11 but with sections that are equilateral
tri A solid has a circular base of radius r. All sections of the solid
perpendicular to a particular diameter of the base are squares.
Find the volume of the solid.
Repeat Exercise 11 but with sections that are equilateral
tri A solid has a circular base of radius r . All sections
perpendicular to a particular diameter of the base a
Find the volume of the solid.
Repeat Exercise 11 but with sections that are equila
triangles instead of squares. 11. A solid has a circular base of radius r . All sections of the solid perpendicular to a particular diameter of the base are squares.
Find the volume of the solid.
12. Repeat Exercise 11 but with sections that are equi A solid has a chedial base of fadius *r*. All sections of the solid perpendicular to a particular diameter of the base are squares.
Find the volume of the solid.
Repeat Exercise 11 but with sections that are equilateral
tr Find the volume of the solid.

Repeat Exercise 11 but with sections that are equilateral

triangles instead of squares.

The base of a solid is an isosceles right-angled triangle with

equal legs measuring 12 cm. Each cros
	-

15. The top of a circular cylinder of radius r is a plane inclined at
an angle to the horizontal. (See Figure 7.19)
above the base, find the volume of the cylinder. (Note that Figure 7.19

Figure 7.19

The top of a circular cylinder of radius r is a plane inclined at

an angle to the horizontal. (See Figure 7.19.) If the lowest and

highest points on the top are at heights a and b, respectively Figure 7.19

Figure 7.19

The top of a circular cylinder of radius r is a plane inclined at

an angle to the horizontal. (See Figure 7.19.) If the lowest and

highest points on the top are at heights a and b, respectively above the base, find the volume of the cylinder. (Note that Figure 7.19

The top of a circular cylinder of radius r is a plane inclined at

an angle to the horizontal. (See Figure 7.19.) If the lowest and

highest points on the top are at heights a and b, respectively,

above the Figure 7.19

Figure 7.19

The top of a circular cylinder of radius r is a plane inclined at

an angle to the horizontal. (See Figure 7.19.) If the lowest and

highest points on the top are at heights a and b, respectively Figure 7.19

The top of a circular cylinder of radius r is a plane inclined at

an angle to the horizontal. (See Figure 7.19.) If the lowest and

highest points on the top are at heights a and b, respectively,

above the

406 CHAPTER 7 Applications of Integration
 16. (Volume of an ellipsoid) Find the volume enclosed by the

ellipsoid
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{a^2} = 1.$
 18. (A smaller notch) Repe ellipsoid

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.
$$

CHAPTER 7 Applications of Integration

COLUME **of an ellipsoid** Find the volume enclosed by the

ellipsoid
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
 18. (A smaller notch) Repeat
 $Hintur:$ This is not a solid of revolutio CHAPTER 7 Applications of Integration

(Volume of an ellipsoid) Find the volume enclosed by the

ellipsoid
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Hint: This is not a solid of revolution. As in Example 3, the
 Hint: (Volume of an ellipsoid) Find the volume enclosed by the

ellipsoid
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
 Hint: This is not a solid of revolution. As in Example 3, the
 z-axis is perpendicular to the plane of the closed by the

log. What volum

cutting the notcomple 3, the

and y-axes. Each

the ellipsoid

Thus,

of the ellipse
 Example 3, the
 I9. What volume of

circular hole of

circular hole of

the hole tilted a

circular (Volume of an ellipsoid) Find the volume enclosed by the

ellipsoid
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

Hint: This is not a solid of revolution. As in Example 3, the

raxis is perpendicular to the plane of the *x*- $(x/a)^2 + (y/b)^2 = 1$ is πab .

One plane face of the notch is perpendicular to the axis of the
log. What volume of wood was removed from the log by
cutting the notch?
(A smaller notch) Repeat Exercise 17, but assume that the One plane face of the notch is perpendicular to the axis of the
log. What volume of wood was removed from the log by
cutting the notch?
(A smaller notch) Repeat Exercise 17, but assume that the
notch penetrates only one qu One plane face of the notch is perpendicular to the axis
log. What volume of wood was removed from the log by
cutting the notch?
(A smaller notch) Repeat Exercise 17, but assume that
notch penetrates only one quarter way (18. One plane face of the notch is perpendicular to the axis of the

19. What volume of wood was removed from the log by

18. (A smaller notch) Repeat Exercise 17, but assume that the

19. What volume of wood is removed fr One plane face of the notch is perpendicular to the axis of the
log. What volume of wood was removed from the log by
cutting the notch?
(A smaller notch) Repeat Exercise 17, but assume that the
notch penetrates only one q One plane face of the notch is perpendicular to the axis of the log. What volume of wood was removed from the log by cutting the notch?

(A smaller notch) Repeat Exercise 17, but assume that the notch penetrates only one

-
- 19. One plane face of the notch is perpendicular to the axis of the log. What volume of wood was removed from the log by cutting the notch?
 18. (A smaller notch) Repeat Exercise 17, but assume that the notch penetrates
- One plane face of the notch is perpendicular to the axis of the
log. What volume of wood was removed from the log by
cutting the notch?
(**A smaller notch**) Repeat Exercise 17, but assume that the
notch penetrates only one rpendicular to the axis of the
emoved from the log by
ise 17, but assume that the
r way (10 cm) into the log.
d from a 3-in-thick board if a
led through it with the axis of
to board?
he axes of two circular
i. If the radii One plane face of the notch is perpendicular to the axis of the log. What volume of wood was removed from the log by cutting the notch?
 18. (A smaller notch) Repeat Exercise 17, but assume that the notch penetrates onl One plane face of the notch is perpendicular to the axis of the log. What volume of wood was removed from the log by cutting the notch?

(A smaller notch) Repeat Exercise 17, but assume that the notch penetrates only one both cylinders has volume
of $V = 8$ $\int_{b}^{b} \sqrt{b^2 - z^2} \sqrt{a^2 - z^2} dz$.
What volume of wood was ferroved from the log by
cutting the notch) Repeat Exercise 17, but assume that the
notch penetrates only one quarter way (10 (More intersecting cylinders) The axes of two circular

cylinders intersect at right angles. If the radii of the cylinders

are *a* and *b* ($a > b > 0$), show that the region lying inside

both cylinders has volume
 $V = 8 \int$ (More intersecting cylinders) The axes of two circular
cylinders intersect at right angles. If the radii of the cylinders
are *a* and *b* ($a > b > 0$), show that the region lying inside
both cylinders has volume
 $V = 8 \int_0^b$

$$
V = 8 \int_0^b \sqrt{b^2 - z^2} \sqrt{a^2 - z^2} \, dz.
$$

cylinders intersect at right angles. If the radii of the
are a and b $(a > b > 0)$, show that the region lying
both cylinders has volume
 $V = 8 \int_0^b \sqrt{b^2 - z^2} \sqrt{a^2 - z^2} dz$.
Hint: Review Example 3. Try to make a similar dia
s $V = 8 \int_0^b \sqrt{b^2 - z^2} \sqrt{a^2 - z^2} dz$.

Hint: Review Example 3. Try to make a similar diagram,

showing only one-eighth of the region. The integral is not

easily evaluated.
 21. A circular hole of radius 2 cm is drilled $V = 8 \int_0^b \sqrt{b^2 - z^2} \sqrt{a^2 - z^2} dz$.

Hint: Review Example 3. Try to make a similar diagram,

showing only one-eighth of the region. The integral is not

easily evaluated.

A circular hole of radius 2 cm is drilled throug $V = 8 \int_0^b \sqrt{b^2 - z^2} \sqrt{a^2 - z^2} dz$.

Hint: Review Example 3. Try to make a similar diagram,

showing only one-eighth of the region. The integral is not

easily evaluated.

A circular hole of radius 2 cm is drilled throug $V = 8 \int_0^b \sqrt{b^2 - z^2} \sqrt{a^2 - z^2} dz$.

Hint: Review Example 3. Try to make a similar diagram,

showing only one-eighth of the region. The integral is not

easily evaluated.

A circular hole of radius 2 cm is drilled throug $V = 8 \int_0 \sqrt{b^2 - z^2} \sqrt{a^2 - z^2} dz$.

Hint: Review Example 3. Try to make a similar diagram, showing only one-eighth of the region. The integral is not easily evaluated.

A circular hole of radius 2 cm is drilled through th *Hint:* Review Example 3. Try to make a similar diagram, showing only one-eighth of the region. The integral is not easily evaluated. A circular hole of radius 2 cm is drilled through the middle of a circular log of radius Figure 7.20

The arc Length and Surface Area

The arc Len

In this section we consider how integrals can be used to find the lengths of curves and

In this section we consider how integrals can be used to find the lengths of curves and

In this section we consider how integrals ca <p>Now in Figure 7.20.</p>\n<p>with a numerical integration function to get the answer.</p>\n<p>But face Area of the number of numbers, we consider how integrals can be used to find the lengths of the areas of the surfaces of solids of the region.</p>\n<p>Arct Length</p>\n<p>If A and B are two points in the plane, let $|AB|$ denote the distance between the distance. The number of numbers are the same, we can use the distance between the distance between the direction of the region.</p> Surface Area

In this section we consider how integrals can

the areas of the surfaces of solids of revolution

Arc Length

If A and B are two points in the plane, let |A

that is, the length of the straight line segment

SECTION 7.3: Arc Length and Surface Area 407
points with straight line segments forms a *polygonal approximation* to
$$
C
$$
, having length

$$
L_n = |P_0 P_1| + |P_1 P_2| + \dots + |P_{n-1} P_n| = \sum_{i=1}^n |P_{i-1} P_i|.
$$

SECTION 7.3: Arc Length and 3

ts with straight line segments forms a *polygonal approximation* to
 $L_n = |P_0 P_1| + |P_1 P_2| + \cdots + |P_{n-1} P_n| = \sum_{i=1}^n |P_{i-1} P_i|$.

tion tells us that the shortest curve joining two points is a section 7.3: Arc Length and Surra

is forms a polygonal approximation to \mathcal{C} ,
 $\cdots + |P_{n-1}P_n| = \sum_{i=1}^n |P_{i-1}P_i|$.

it curve joining two points is a straight

lygonal approximation to \mathcal{C} cannot exce

is more SECTION 7.5: Arc Length and Surface
forms a *polygonal approximation* to C , h
 $\cdot + |P_{n-1}P_n| = \sum_{i=1}^n |P_{i-1}P_i|$.
curve joining two points is a straight li
ygonal approximation to C cannot excee
more vertices to the SECTION 7.5: Arc Length and Surrace Area 467

orms a *polygonal approximation* to C , having length
 $+ |P_{n-1}P_n| = \sum_{i=1}^{n} |P_{i-1}P_i|$.

curve joining two points is a straight line segment,

gonal approximation to C c SECTION 7.3: Arc Length and Surface Area **407**
points with straight line segments forms a *polygonal approximation* to C , having length
 $L_n = |P_0 P_1| + |P_1 P_2| + \cdots + |P_{n-1} P_n| = \sum_{i=1}^n |P_{i-1} P_i|$.
Intuition tells us that SECTION 7.3: Arc Length and Surface Area **407**
points with straight line segments forms a *polygonal approximation* to C , having length
 $L_n = |P_0 P_1| + |P_1 P_2| + \cdots + |P_{n-1} P_n| = \sum_{i=1}^n |P_{i-1} P_i|$.
Intuition tells us that points with straight line segments forms a *polygonal approximation* to \mathcal{C} , having length
 $L_n = |P_0 P_1| + |P_1 P_2| + \cdots + |P_{n-1} P_n| = \sum_{i=1}^n |P_{i-1} P_i|$.

Intuition tells us that the shortest curve joining two points is points with straight line segments forms a *polygonal approximation* to \mathcal{C} , having length
 $L_n = |P_0 P_1| + |P_1 P_2| + \cdots + |P_{n-1} P_n| = \sum_{i=1}^n |P_{i-1} P_i|$.

Intuition tells us that the shortest curve joining two points is points with straight line segments forms a *polygonal approximation* to C , having length
 $L_n = |P_0 P_1| + |P_1 P_2| + \cdots + |P_{n-1} P_n| = \sum_{i=1}^n |P_{i-1} P_i|$.

Intuition tells us that the shortest curve joining two points is a st points with straight line segments forms a *potygonal approximation* to C, having rength
 $L_n = |P_0 P_1| + |P_1 P_2| + \cdots + |P_{n-1} P_n| = \sum_{i=1}^n |P_{i-1} P_i|$.

Intuition tells us that the shortest curve joining two points is a stra $L_n = |P_0 P_1| + |P_1 P_2| + \cdots + |P_{n-1} P_n| =$
Intuition tells us that the shortest curve joining two to the length L_n of any such polygonal approxims of \mathcal{C} . If we increase *n* by adding more vertices to vertices, L_n ca tion tells us that the shortest curve joining two points is a straight line segment,
he length L_n of any such polygonal approximation to C cannot exceed the length
or exp. L_n cannot get smaller and may increase. If Intuition tells us that the shortest curve joining two points is a straight line segment,
so the length L_n of any such polygonal approximation to C cannot exceed the length
of C. If we increase *n* by adding more vertic or C. If we increase *n* by adding more vertices to the polygonal line between existing
vertices, L_n cannot get smaller and may increase. If there exists a finite number *K* such
that $L_n \le K$ for every polygonal approxi vertices, L_n cannot get smaller and may increase. If there exert
that $L_n \le K$ for every polygonal approximation to \mathcal{C} , then th
number K (by the completeness of the real numbers), and v
arc length of \mathcal{C} .
The

that the length L_n of every polygonal approximation to $\mathcal C$ satisfies $L_n \leq s$.
A curve with a finite arc length is said to be **rectifiable**. Its arc length s is the limit

 $g_n \rvert_2$ is not over y polygonal approximation to c, then there will be a sinulest setch
that the completeness of the real numbers), and we call this smallest *K* the
ength of *C*.
The arc length of the curve *C* from *A* The **arc length of the curve** C **from A to B** is the smallest real number s such that the length L_n of every polygonal approximation to C satisfies $L_n \le s$.
A curve with a finite arc length is said to be **rectifiable** The **arc length of the curve** \mathcal{C} **from A to B** is the smallest real number s such that the length L_n of every polygonal approximation to \mathcal{C} satisfies $L_n \leq s$.
A curve with a finite arc length is said to be **r** The **arc length** of the curve \mathcal{C} from *A* to *B* is the smallest real num
that the length L_n of every polygonal approximation to \mathcal{C} satisfies *L*
A curve with a finite arc length is said to be **rectifiable**. that the length L_n of every polygonal approximation to C satisfies $L_n \le s$.
A curve with a finite arc length is said to be **rectifiable**. Its arc length s is the 1
of the lengths L_n of polygonal approximations as n The **arc length** of the curve \mathbb{C} from *A* to *B* is the smallest real number *s* such that the length L_n of every polygonal approximation to \mathbb{C} satisfies $L_n \le s$.
A curve with a finite arc length is said to that the length L_n of every polygonal approximation to C satisfies $L_n \le s$.
A curve with a finite arc length is said to be **rectifiable**. Its arc length s is the limit
of the lengths L_n of polygonal approximations as he lengths L_n of polygonal approximations as $n \to \infty$
imum segment length $|P_{i-1}P_i| \to 0$.
It is possible to construct continuous curves that are bour
finity anywhere) but are not rectifiable; they have infinit
ologica all approximations as $n \to \infty$ in such a v
 ${}_1P_i| \to 0$.

continuous curves that are bounded (they contrinuous curves that are bounded (they contrect

contrinuous curves that are **smooth**; they will

show is derivatives.
 struct continuous curves that are bounded
struct continuous curves that are bounded
it are not rectifiable; they have infinite leng
we will assume that our curves are **smooth**; it
inuous derivatives.
he Graph of a Functi It is possible to construct commutation curves that
to infinity anywhere) but are not rectifiable; they hapathological examples, we will assume that our curve
by functions having continuous derivatives.
The Arc Length of

derivative f' there. If $\mathcal C$ is the gr pathological examples, we will assume that our curves a
by functions having continuous derivatives.
 The Arc Length of the Graph of a Function

Let f be a function defined on a closed, finite interval

derivative f' the ssume that our curves are **smooth**; they will be defined
derivatives.
ph of a Function
a closed, finite interval [a, b] and having a continuous
raph of f, that is, the graph of the equation $y = f(x)$,
des a polygonal appr **on**
erval [*a*, *b*] and having a continuous
he graph of the equation $y = f(x)$,
proximation to \mathcal{C} . For the partition
The length of the polygonal line
 $(f(x_i) - f(x_{i-1}))^2$

let P_i be the point $(x_i, f(x_i))$, $(0 \le i \le n)$. The length of the polygonal line

The Arc Length of the Graph of a Function
Let *f* be a function defined on a closed, finite interval [*a*, *b*] and having a continuous
derivative *f'* there. If *C* is the graph of *f*, that is, the graph of the equation
$$
y = f(x)
$$
,
then any partition of [*a*, *b*] provides a polygonal approximation to *C*. For the partition

$$
\{a = x_0 < x_1 < x_2 < \cdots < x_n = b\},
$$

let *P_i* be the point $(x_i, f(x_i))$, $(0 \le i \le n)$. The length of the polygonal line
 $P_0 P_1 P_2 \dots P_{n-1} P_n$ is

$$
L_n = \sum_{i=1}^n |P_{i-1} P_i| = \sum_{i=1}^n \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}
$$

$$
= \sum_{i=1}^n \sqrt{1 + (\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}})^2} \Delta x_i,
$$

where $\Delta x_i = x_i - x_{i-1}$. By the Mean-Value Theorem there exists a number *c_i* in the interval [x_{i-1} , x_i] such that

$$
\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(c_i),
$$

1 line
in the

$$
= \sum_{i=1}^{n} \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}
$$

where $\Delta x_i = x_i - x_{i-1}$. By the Mean-Value Theorem there
interval $[x_{i-1}, x_i]$ such that

$$
\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(c_i),
$$

so we have $L_n = \sum_{i=1}^{n} \sqrt{1 + (f'(c_i))^2} \Delta x_i$.
Thus, L_n is a Riemann sum for $\int_a^b \sqrt{1 + (f'(x))^2} dx$. Bei
mann sums as $n \to \infty$ in such a way that $\max(\Delta x_i) \to 0$, the

where $\Delta x_i = x_i - x_{i-1}$. By the Mean-Value Theorem there exists a num
interval $[x_{i-1}, x_i]$ such that
 $\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(c_i),$
so we have $L_n = \sum_{i=1}^n \sqrt{1 + (f'(c_i))^2} \Delta x_i$.
Thus, L_n is a Riemann sum for $\int_a^b \sqrt{1 +$ 1-Value Theorem there exists a number c_i in the
 Δx_i .
 $\frac{1 + (f'(x))^2}{x} dx$. Being the limit of such Rie-
 $\tan x(\Delta x_i) \to 0$, that integral is the length of where $\Delta x_i = x_i - x_{i-1}$. By the Mean-Value Theorem there exists a number c_i in the interval $[x_{i-1}, x_i]$ such that
 $\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(c_i)$,

so we have $L_n = \sum_{i=1}^n \sqrt{1 + (f'(c_i))^2} \Delta x_i$.

Thus, L_n is a Riemann where $\Delta x_i = x_i - x_{i-1}$. By the Mean-Value Theorem there exists a number c_i in the
interval $[x_{i-1}, x_i]$ such that
 $\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}} = f'(c_i)$,
so we have $L_n = \sum_{i=1}^n \sqrt{1 + (f'(c_i))^2} \Delta x_i$.
Thus, L_n is a Riemann sum

.

$$
s = \int_a^b \sqrt{1 + \left(f'(x)\right)^2} \, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx.
$$

 $i=1$

You can regard the integral formula above as giving the arc length s of $\mathcal C$ as a "sum"
of **arc length elements:**

You can regard the integral formula above as giving the arc length *s* of *C* as a ' of **arc length elements**:
\n
$$
s = \int_{x=a}^{x=b} ds, \qquad \text{where} \qquad ds = \sqrt{1 + (f'(x))^2} dx.
$$

Figure 7.22 provides a convenient way to remember this; it also suggests how we can arrive at similar formulas for arc length elements of other kinds of curves. The *differential triangle* in the figure suggests hat You can regard the integral formula above as giving the arc length s of \mathcal{C} as a "sum"
of **arc length elements**:
 $s = \int_{x=a}^{x=b} ds$, where $ds = \sqrt{1 + (f'(x))^2} dx$.
Figure 7.22 provides a convenient way to remember this; it al You can regard the integral formula above as giving the arc length s of \mathcal{C} as
of **arc length elements**:
 $s = \int_{x=a}^{x=b} ds$, where $ds = \sqrt{1 + (f'(x))^2} dx$.
Figure 7.22 provides a convenient way to remember this; it also sugge $s = \int_{x=a}^{x=b} ds$, where $ds = \sqrt{1 + (f'(x))^{2}} dx$.

Figure 7.22 provides a convenient way to remember this; it also

arrive at similar formulas for arc length elements of other kind
 ential triangle in the figure suggests that
 $ds = \sqrt{1 + (f'(x))^2} dx$.

at way to remember this; it also suggests how we can

length elements of other kinds of curves. The *differ*ents that

and taking the square root, we get

$$
(ds)^2 = (dx)^2 + (dy)^2.
$$

Dividing this equation by
$$
(dx)^2
$$
 and taking the square root, we get
\n
$$
\left(\frac{ds}{dx}\right)^2 = 1 + \left(\frac{dy}{dx}\right)^2
$$
\n
$$
\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}
$$
\n
$$
ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + (f'(x))^2} dx.
$$
\nA similar argument shows that for a curve specified by an equation of the form $x = g(y)$, $(c \le y \le d)$, the arc length element is
\n
$$
ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \sqrt{1 + (g'(y))^2} dy.
$$

 $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + (f'(x))^2} dx$.

A similar argument shows that for a curve specified by an equation of the form $x = g(y)$, $(c \le y \le d)$, the arc length element is
 $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \sqrt{1 + (g'(y))^2} dy$.
 EXAMPLE 1 Fi

$$
ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \sqrt{1 + \left(g'(y)\right)^2} dy.
$$

 3^{λ} is commuted in for a curve specified by an equation of the form $x =$
 $\sqrt{1 + (g'(y))^2} dy$.

agth of the curve $y = x^{2/3}$ from $x = 1$ to $x = 8$.
 $x^{-1/3}$ is continuous between $x = 1$ and $x = 8$ and

e curve is given by $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + (f'(x))^2} dx$.

A similar argument shows that for a curve specified by an equation of the form $x = g(y)$, $(c \le y \le d)$, the arc length element is
 $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} dy = \sqrt{1 + (g'(y))^2} dy$.
 EXAMPLE 1 Fi

$$
s = \int_{1}^{8} \sqrt{1 + \frac{4}{9}x^{-2/3}} dx = \int_{1}^{8} \sqrt{\frac{9x^{2/3} + 4}{9x^{2/3}}} dx
$$

\n
$$
= \int_{1}^{8} \frac{\sqrt{9x^{2/3} + 4}}{3x^{1/3}} dx \qquad \text{Let } u = 9x^{2/3} + 4,
$$

\n
$$
du = 6x^{-1/3} dx
$$

\n
$$
= \frac{1}{18} \int_{13}^{40} u^{1/2} du = \frac{1}{27} u^{3/2} \Big|_{13}^{40} = \frac{40\sqrt{40} - 13\sqrt{13}}{27} \text{ units.}
$$

\n**EXAMPLE 2** Find the length of the curve $y = x^{4} + \frac{1}{32x^{2}}$ from $x = 1$ to $x = 2$.
\n**olution** Here $\frac{dy}{dx} = 4x^{3} - \frac{1}{16x^{3}}$ and

1 $\frac{1}{32x^2}$ from $x = 1$ to $x = 2$. **Solution** Here $\frac{dy}{dx} = 4x^3 - \frac{1}{16x^3}$ and $\frac{1}{16x^3}$ and $1 + \left(\frac{dy}{dx}\right)^2 = 1 + \left(4x^3 - \frac{1}{16x^3}\right)^2$ $= 1 + (4x^3)^2 - \frac{1}{2} + \left(\frac{1}{16x^3}\right)^2$ $\frac{1}{2} + \left(\frac{1}{16x^3} \right)$ $\left(\frac{1}{16x^3}\right)^2$ $=(4x^3)^2+\frac{1}{2}+\left(\frac{1}{16x^3}\right)^2=\left(4x^3+\frac{1}{16}\right)$ $\frac{1}{2} + \left(\frac{1}{16x^3} \right) = \left(4x^3 + \frac{1}{16x^3} \right)$ $\left(\frac{1}{16x^3}\right)^2 = \left(4x^3 + \frac{1}{16x^3}\right)^2$.

SECTION 7.3: Arc Length and Surface Area 409
The expression in the last set of parentheses is positive for $1 \le x \le 2$, so the length of
the curve is
 $s = \int_0^2 \left(4x^3 + \frac{1}{16x^3}\right) dx = \left(x^4 - \frac{1}{32x^2}\right)\Big|^2$

SECTION 7.3: Arc Length and Surface Area 409
\nThe expression in the last set of parentheses is positive for
$$
1 \le x \le 2
$$
, so the length of
\nthe curve is
\n
$$
s = \int_1^2 \left(4x^3 + \frac{1}{16x^3}\right) dx = \left(x^4 - \frac{1}{32x^2}\right)\Big|_1^2
$$
\n
$$
= 16 - \frac{1}{128} - \left(1 - \frac{1}{32}\right) = 15 + \frac{3}{128}
$$
 units.
\nThe examples above are deceptively simple; the curves were chosen so that the arc
\nlength integrals could be easily evaluated. For instance, the number 32 in the curve in
\nExample 2 was chosen so the expression $1 + (dy/dx)^2$ would turn out to be a perfect
\nsquare and its square root would cause no problems. Because of the square root in

The expression in the last set of parentheses is positive for $1 \le x \le 2$, so the length of

the curve is
 $s = \int_1^2 \left(4x^3 + \frac{1}{16x^3}\right) dx = \left(x^4 - \frac{1}{32x^2}\right)\Big|_1^2$
 $= 16 - \frac{1}{128} - \left(1 - \frac{1}{32}\right) = 15 + \frac{3}{128}$ units.
 The examples above are deceptively simple; the curves were chosen so that the are examples above are deceptively simple; the curves were chosen so that the arc length integrals could be easily evaluated. For instance, the the curve is
 $s = \int_1^2 \left(4x^3 + \frac{1}{16x^3}\right) dx = \left(x^4 - \frac{1}{32x^2}\right)\Big|_1^2$
 $= 16 - \frac{1}{128} - \left(1 - \frac{1}{32}\right) = 15 + \frac{3}{128}$ units.

The examples above are deceptively simple; the curves were chosen so that the arc

length in the formula, arc length problems for most curves lead to integrals that are difficult or $s = \int_1 (\frac{4\lambda}{16\lambda^3}) dx = (\frac{\lambda}{128} - \frac{1}{32\lambda^2})\Big|_1$
= $16 - \frac{1}{128} - (1 - \frac{1}{32}) = 15 + \frac{3}{128}$ units.
The examples above are deceptively simple; the curves were chosen so that the arc
length integrals could be easily eval $= 16 - \frac{1}{128} - (1 - \frac{1}{32}) = 15 + \frac{1}{128}$ units.

he examples above are deceptively simple; the curves were chosen so that the arc

mapth integrals could be easily evaluated. For instance, the number 32 in the curve in
 e are deceptively simple; the curves were chosen so that the arc
ld be easily evaluated. For instance, the number 32 in the curve in
sen so the expression $1 + (dy/dx)^2$ would turn out to be a perfect
re root would cause no p The examples above are deceptively simple; the curves were chosen so that the arc length integrals could be easily evaluated. For instance, the number 32 in the curve in Example 2 was chosen so the expression $1 + (dy/dx)^2$ w length integrals could be easily evaluated. For instance, the number 32 in the curve in Example 2 was chosen so the expression $1 + (dy/dx)^2$ would turn out to be a perfect square and its square root would cause no problems.

Example 2 was chosen so the expression $1 + (dy/dx)^2$ would turn out to be a perfect
square and its square root would cause no problems. Because of the square root in
the formula, arc length problems for most curves lead to i erical techniques.
 Tugated panels) Flat rectangular sheets

to be formed into corrugated roofing panelal shape shown in Figure 7.23. The per

Its amplitude is 5 cm, so the panel is 10

cut if the resulting panels must b

cross-section

Solution One period of the sinusoidal cross-section is shown in Figure 7.24. The distances are all in metres; the 5 cm amplitude is shown as 1/20 m, and the 20 cm period is shown as 2/10 m. The curve has equation $y = \frac{$ **Solution** One period of the sinusoidal cross-section is shown in Figure 7.24. The distances are all in metres; the 5 cm amplitude is shown as 1/20 m, and the 20 cm period is shown as 2/10 m. The curve has equation
 $y = \frac{$ Figure 7.24. The
n, and the 20 cm
length of the flat
 $\frac{1}{2}$
 $\frac{1}{t}$
 $\frac{1}{t}$
 $\frac{1}{t}$
 $\frac{1}{t}$

$$
y = \frac{1}{20} \sin(10\pi x).
$$

 x sheet required is 25 times the length of one period of the sine curve: 2/10 Note that 25 periods are required to produce a 5 m long panel. The length of the flat

Solution One period of the sinusoidal cross-section is shown in Figure 7.24. The distances are all in metres; the 5 cm amplitude is shown as 1/20 m, and the 20 cm period is shown as 2/10 m. The curve has equation
\n
$$
y = \frac{1}{20} \sin(10\pi x).
$$
\nNote that 25 periods are required to produce a 5 m long panel. The length of the flat sheet required is 25 times the length of one period of the sine curve:
\n
$$
s = 25 \int_0^{2/10} \sqrt{1 + (\frac{\pi}{2} \cos(10\pi x))^2} dx
$$
\nLet $t = 10\pi x$,
\n
$$
dt = 10\pi dx
$$
\n
$$
= \frac{5}{2\pi} \int_0^{2\pi} \sqrt{1 + \frac{\pi^2}{4} \cos^2 t} dt = \frac{10}{\pi} \int_0^{\pi/2} \sqrt{1 + \frac{\pi^2}{4} \cos^2 t} dt.
$$
\nThe integral can be evaluated numerically using the techniques of the previous chapter or by using the definite integral function on an advanced scientific calculator or a computer. The value is $s \approx 7.32$. The flat metal sheet should be about 7.32 m long to yield a 5 m long finished panel.

 $=\frac{5}{2\pi}\int_0^{2\pi}\sqrt{1+\frac{\pi^2}{4}}\cos^2 t dt = \frac{10}{\pi}\int_0^{\pi/2}\sqrt{1+\frac{\pi^2}{4}}\cos^2 t dt$.
The integral can be evaluated numerically using the techniques of the previous chap-
ter or by using the definite integral function on an advance $= \frac{5}{2\pi} \int_0^{2\pi} \sqrt{1 + \frac{\pi^2}{4}} \cos^2 t \, dt = \frac{10}{\pi} \int_0^{2\pi/2} \sqrt{1 + \frac{\pi^2}{4}} \cos^2 t \, dt$.
The integral can be evaluated numerically using the techniques of the previous chapter or by using the definite integral function on 2 π J₀ V 4 π J₀ V 4 π J₀ V 4
The integral can be evaluated numerically using the techniques of the previous chap-
ter or by using the definite integral function on an advanced scientific calculator or a
comp The integral can be evaluated numerically using the techniques of the previous ter or by using the definite integral function on an advanced scientific calculat computer. The value is $s \approx 7.32$. The flat metal sheet shou

EXAMPLE 4 (The circumference of an ellipse) Find the circumference of the ellipse
 $\frac{x^2}{\sqrt{1-x^2}} + \frac{y^2}{\sqrt{1-x^2}} = 1$. **EXAMPLE 4** (The circumference of an ellipse) Find the circumference of the ellipse

$$
\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,
$$

EXAMPLE 4 (The circumference of an ellipse) Find the circumpose ellipse
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

where $a \ge b > 0$. See Figure 7.25.
 Solution The upper half of the ellipse has equation $y = b\sqrt{1 - \frac{x^2}{a^2}}$ **EXAMPLE 4** (The circumference of an ellipse) Find the circumference of the
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$,

where $a \ge b > 0$. See Figure 7.25.
 Solution The upper half of the ellipse has equation $y = b\sqrt{1 - \frac{x^2}{a^2}} = \frac{b}{$ $\sqrt{1-\frac{x^2}{a^2}} = \frac{b}{a}\sqrt{a^2-x^2}.$ $\overline{a^2} = -\sqrt{a^2 - x^2}.$ $b \sim$ a^{\dagger} v a^{\dagger} - λ . $\sqrt{a^2 - x^2}$. Hence,

;
;

$$
\frac{dy}{dx} = -\frac{b}{a} \frac{x}{\sqrt{a^2 - x^2}},
$$

so

$$
\frac{dy}{dx} = -\frac{b}{a} \frac{x}{\sqrt{a^2 - x^2}},
$$

\nso
\n
$$
1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{b^2}{a^2} \frac{x^2}{a^2 - x^2}
$$
\n
$$
= \frac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)}.
$$
\nThe circumference of the ellipse is four times the arc length of the part lying in the first quadrant, so
\n
$$
\frac{x^2}{a^2 + \frac{y^2}{b^2} = 1}
$$
\n
$$
s = 4 \int_0^a \frac{\sqrt{a^4 - (a^2 - b^2)x^2}}{a\sqrt{a^2 - x^2}} dx
$$
\nLet $x = a \sin t$,
\n
$$
\frac{dx}{dx} = a \sin t
$$

 $\frac{dx^2}{dx}$.

uur times the arc length of the part lying in the first
 $\frac{dx}{dx}$ Let $x = a \sin t$,
 $\frac{dx}{dx} = a \cos t \, dt$

$$
1 + \left(\frac{dy}{dx}\right)^2 = 1 + \frac{b^2}{a^2} \frac{x^2}{a^2 - x^2}
$$

\n
$$
= \frac{a^4 - (a^2 - b^2)x^2}{a^2(a^2 - x^2)}.
$$

\nThe circumference of the ellipse is four times the arc length of the part lying in the first
\nquadrant, so
\n
$$
s = 4 \int_0^a \frac{\sqrt{a^4 - (a^2 - b^2)x^2}}{a\sqrt{a^2 - x^2}} dx
$$
 Let $x = a \sin t$,
\n
$$
dx = a \cos t dt
$$

\n
$$
= 4 \int_0^{\pi/2} \frac{\sqrt{a^4 - (a^2 - b^2)a^2 \sin^2 t}}{a(a \cos t)}
$$
 $a \cos t dt$
\n
$$
= 4 \int_0^{\pi/2} \sqrt{a^2 - (a^2 - b^2) \sin^2 t} dt
$$

\n
$$
= 4a \int_0^{\pi/2} \sqrt{1 - \frac{a^2 - b^2}{a^2} \sin^2 t} dt
$$

\n
$$
= 4a \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2 t} dt
$$
 units,
\nwhere $\varepsilon = (\sqrt{a^2 - b^2})/a$ is the *eccentricity* of the ellipse. (See Section 8.1 for a
\ndiscussion of ellipses.) Note that $0 \le \varepsilon < 1$. The function $E(\varepsilon)$, defined by
\n
$$
E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2 t} dt,
$$

where $\varepsilon = (\sqrt{a^2 - b^2})/a$ is the *eccentricity* of the ellipse. (See Section 8.1 for a discussion of ellipses.) Note that $0 \le \varepsilon < 1$. The function $E(\varepsilon)$, defined by

$$
E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2 t} \, dt,
$$

 $= 4a \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2 t} dt$ units,

where $\varepsilon = (\sqrt{a^2 - b^2})/a$ is the *eccentricity* of the ellipse. (See Section 8.1 for a

discussion of ellipses.) Note that $0 \le \varepsilon < 1$. The function $E(\varepsilon)$, defined by
 $E(\varepsilon$ where $\varepsilon = (\sqrt{a^2 - b^2})/a$ is the *eccentricity* of the ellipse. (See Section 8.1 for a discussion of ellipses.) Note that $0 \le \varepsilon < 1$. The function $E(\varepsilon)$, defined by $E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2 t} dt$, is called th where $\varepsilon = (\sqrt{a^2 - b^2})/a$ is the *eccentricity* of the ellipse. (See Section 8.1 for a discussion of ellipses.) Note that $0 \le \varepsilon < 1$. The function $E(\varepsilon)$, defined by $E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2 t} dt$,
is called th where $\varepsilon = (\sqrt{a^2 - b^2})/a$ is the *eccentricity* of the ellipse. (See Section 8.1 for a discussion of ellipses.) Note that $0 \le \varepsilon < 1$. The function $E(\varepsilon)$, defined by $E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2 t} dt$,
is called th discussion of ellipses.) Note that $0 \le \varepsilon < 1$. The function $E(\varepsilon)$, defined by
 $E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2 t} dt$,

is called the **complete elliptic integral of the second kind**. The integral cannot be

evaluated $E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2 t} dt$,

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evaluated by elementary techniques for general ε , although numerical methods can be

appli units. is called the **complete elliptic integral of the second kind**. The evaluated by elementary techniques for general ε , although numeriapplied to find approximate values for any given value of ε . Table for various va evaluated by elementary techniques for general ε , although numerical methods can be applied to find approximate values for any given value of ε . Tables of values of $E(\varepsilon)$ for various values of ε can be foun applied to find approximate values for any given value of ε . Tables of values of $E(\varepsilon)$ for various values of ε can be found in collections of mathematical tables. As shown above, the circumference of the ellips for various values of ε can be found in collections of mathematical tables. As shown
above, the circumference of the ellipse is given by $4aE(\varepsilon)$. Note that for $a = b$ we have
 $\varepsilon = 0$, and the formula returns the ci

SECTION 7.3: Arc Length and Surface Area 411
The area of a surface of revolution can be found by integrating an area element dS
constructed by rotating the arc length element ds of the curve about the given line. If
the r SECTION 7.3: Arc Length and Surface Area **411**
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constructed by rotating the arc length element ds of the curve about the given line. If
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constructed by rotating the arc length element ds of the curve about the given line. If
the SECTION 7.3:

area of a surface of revolution can be found by in

tructed by rotating the arc length element ds of the

radius of rotation of the element ds is r, then it get

of width ds and length (circumference) $2\pi r$ The area of a surface of revolution can be found by integrating an area element dS constructed by rotating the arc length element ds of the curve about the given line. If the radius of rotation of the element ds is r The area of a surface of revolution can be found by integrating an area element dS constructed by rotating the arc length element ds of the curve about the given line. If the radius of rotation of the element ds is r The area of a surface of revolution can be four
constructed by rotating the arc length element *ds*
the radius of rotation of the element *ds* is *r*, th
band of width *ds* and length (circumference) 2*1*
 $dS = 2\pi r ds$,
as ed by rotating the arc length element ds of the curve about the given line. If
s of rotation of the element ds is r, then it generates, on rotation, a circular
width ds and length (circumference) $2\pi r$. The area of this

axis

adius of rotation of the element ds is r, then it generates, or

l of width ds and length (circumference) $2\pi r$. The area of this
 $dS = 2\pi r ds$,

nown in Figure 7.26. The areas of surfaces of revolution arou

btained by i $dS = 2\pi r \, ds$,

nown in Figure 7.26. The areas of surfaces of revolution around various lines can

btained by integrating dS with appropriate choices of r. Here are some important

ial cases:
 Area of a surface of revolu faces of revolution around various lines carpriate choices of r. Here are some importan

the curve $y = f(x)$ is rotated about the

blution so generated is
 $|f(x)|\sqrt{1 + (f'(x))^2} dx$.

If $f'(x)$ is continuous on [a, b] and the curve $y = f(x)$ is rotated about the

and cases:

\nArea of a surface of revolution

\nIf
$$
f'(x)
$$
 is continuous on [a, b] and the curve $y = f(x)$ is rotated about the *x*-axis, the area of the surface of revolution so generated is

\n
$$
S = 2\pi \int_{x=a}^{x=b} |y| \, ds = 2\pi \int_{a}^{b} |f(x)| \sqrt{1 + (f'(x))^2} \, dx.
$$

\nIf the rotation is about the *y*-axis, the surface area is

\n
$$
S = 2\pi \int_{x=a}^{x=b} |x| \, ds = 2\pi \int_{a}^{b} |x| \sqrt{1 + (f'(x))^2} \, dx.
$$

\nIf *g'(y)* is continuous on [*c*, *d*] and the curve *x* = *g(y)* is rotated about the *x*-axis, the area of the surface of revolution so generated is

\n
$$
S = 2\pi \int_{x=a}^{y=d} |y| \, ds = 2\pi \int_{a}^{d} |y| \sqrt{1 + (g'(y))^2} \, dy.
$$

$$
S = 2\pi \int_{x=a}^{x=b} |x| \, ds = 2\pi \int_{a}^{b} |x| \sqrt{1 + (f'(x))^2} \, dx.
$$

If $g'(y)$ is continuous on [c, d] and the curve $x = g(y)$ is rotated about the

$$
S = 2\pi \int_{x=a}^{x=b} |x| ds = 2\pi \int_{a}^{b} |x| \sqrt{1 + (f'(x))^{2}} dx.
$$

If $g'(y)$ is continuous on $[c, d]$ and the curve $x = g(y)$ is rotated about the
x-axis, the area of the surface of revolution so generated is

$$
S = 2\pi \int_{y=c}^{y=d} |y| ds = 2\pi \int_{c}^{d} |y| \sqrt{1 + (g'(y))^{2}} dy.
$$
If the rotation is about the *y*-axis, the surface area is

$$
S = 2\pi \int_{y=a}^{y=d} |x| ds = 2\pi \int_{c}^{d} |g(y)| \sqrt{1 + (g'(y))^{2}} dy.
$$

$$
S = 2\pi \int_{y=c}^{y=d} |x| ds = 2\pi \int_{c}^{d} |g(y)| \sqrt{1 + (g'(y))^{2}} dy.
$$

If the rotation is about the y-axis, the surface area is
 $S = 2\pi \int_{y=c}^{y=d} |x| ds = 2\pi \int_{c}^{d} |g(y)| \sqrt{1 + (g'(y))^2} dy$.
 Remark Students sometimes wonder whether such complicated formulas are actu-

ally necessary. Why not just $S = 2\pi \int_{y=c}^{y=d} |x| ds = 2\pi \int_{c}^{d} |g(y)| \sqrt{1 + (g'(y))^2} dy$.
 Remark Students sometimes wonder whether such complicated formulas are actu-
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ally n *f* $s = 2\pi \int_{y=c}^{y=d} |x| ds = 2\pi \int_{c}^{d} |g(y)| \sqrt{1 + (g'(y))^2} dy$.
 Remark Students sometimes wonder whether such complicated formulas are actually necessary. Why not just use $dS = 2\pi |y| dx$ for the area element when $y = f(x)$ is r $S = 2\pi \int_{y=c} |x| ds = 2\pi \int_{c} |g(y)| \sqrt{1 + (g'(y))^2} dy$.
 Remark Students sometimes wonder whether such complicated formulas are actually necessary. Why not just use $dS = 2\pi |y| dx$ for the area element when $y = 2\pi |y| ds$? After all **Remark** Students sometimes wonder whether such complicated formulas are actually necessary. Why not just use $dS = 2\pi |y| dx$ for the area element when $y = f(x)$ is rotated about the x-axis instead of the more complicated are $S = 2\pi \int_{y=c}^{y=d} |x| ds = 2\pi \int_{c}^{d} |g(y)| \sqrt{1 + (g'(y))^2} dy$.
 Remark Students sometimes wonder whether such complicated formulas are actually necessary. Why not just use $dS = 2\pi |y| dx$ for the area element when $y = f(x)$ is rotat **Remark** Students sometimes wonder whether such complicated formulas are actually necessary. Why not just use $dS = 2\pi |y| dx$ for the area element when $y = f(x)$ is rotated about the x-axis instead of the more complicated are **Remark** Students sometimes wonder whether such complicated formulas are actually necessary. Why not just use $dS = 2\pi |y| dx$ for the area element when $y = f(x)$ is rotated about the x-axis instead of the more complicated are **Remark** Students sometimes wonder whether such complicated formulas are actually necessary. Why not just use $dS = 2\pi |y| dx$ for the area element when $y = f(x)$ is rotated about the x-axis instead of the more complicated are **Remark** Students sometimes wonder whether such complicated formulas are actually necessary. Why not just use $dS = 2\pi |y| dx$ for the area element when $y = f(x)$ is rotated about the x-axis instead of the more complicated are where the symmetry will be a streamed when the error sum completed formulas are actually necessary. Why not just use $dS = 2\pi |y| dx$ for the area element when $y = f(x)$ is rotated about the x-axis instead of the more complica **Remark** Students sometimes wonder whether such complicated formulas are actually necessary. Why not just use $dS = 2\pi |y| dx$ for the area element when $y = f(x)$ is rotated about the *x*-axis instead of the more complicated a $f(x)$ is columed to the area of the band in the solution of the area of the area of the area of the regarding dx and ds as both being infinitely small, and we certainly used dx for the width of the disk-shaped volume elem all ds as both being infinitely small, and we
sk-shaped volume element when we rotated
axis to generate a solid of revolution. The
kness Δx , the volume of a slice of the solid
 Δx , but the error is *small compared to* and the curve are length element along the curve when you rotate a curve to find the area of a surface of revolution. The reason is somewhat subtle. For small thickness Δx , the volume of a slice of the solid of revolut because the region under $y = f(x)$ about the x-axis to gener
reason is somewhat subtle. For small thickness Δx , the
of revolution is only approximately $\pi y^2 \Delta x$, but the
volume of this slice. On the other hand, if we u EXAMPLE 5 (Surface area of a sphere) Find the area of the surface of a sphere of radius *a*.

EXAMPLE 5 (Surface area of a sphere) Find the area of that band. If, for instance, the surface of a thin band of the surface of On the other hand, if we use $2\pi |y| \Delta x$
and of the surface of revolution corresponds is *not small compared to the area of that bos* slope 1 at *x*, then the width of the band is $\Delta S = 2\pi \sqrt{2}|y| \Delta x$, not just 2π th e

Solution Such a sphere can be generated by rotating the semicircle with equation $y = \sqrt{a^2 - x^2}$, $(-a \le x \le a)$, about the *x*-axis. (See Figure 7.27.) Since $\frac{dy}{dx} = \frac{x}{a} = \frac{x}{a}$ **Solution** Such a sphere can be generated by rotating the semicircle with equation $y = \sqrt{a^2 - x^2}$, $(-a \le x \le a)$, about the *x*-axis. (See Figure 7.27.) Since **a2** $\frac{2\pi}{a^2 - x^2}$, $(-a \le x \le a)$, about the x-axis. (See Figure 7.27.) Since $\frac{x}{a^2 - x^2} = -\frac{x}{\sqrt{a^2 - x^2}} = -\frac{x}{y}$, **Solution** Such a sphere can be generated by rotating the semicircle $y = \sqrt{a^2 - x^2}$, $(-a \le x \le a)$, about the x-axis. (See Figure 7.27.) Si
 $\frac{dy}{dx} = -\frac{x}{\sqrt{a^2 - x^2}} = -\frac{x}{y}$,

the area of the sphere is given by
 $S = 2\pi \$

$$
\frac{dy}{dx} = -\frac{x}{\sqrt{a^2 - x^2}} = -\frac{x}{y},
$$

$$
dx = \sqrt{a^2 - x^2} \quad y
$$

\nthe area of the sphere is given by
\n
$$
S = 2\pi \int_{-a}^{a} y \sqrt{1 + \left(\frac{x}{y}\right)^2} dx
$$
\n
$$
= 4\pi \int_{0}^{a} \sqrt{y^2 + x^2} dx
$$
\n
$$
= 4\pi \int_{0}^{a} \sqrt{a^2} dx = 4\pi a x \Big|_{0}^{a} = 4\pi a^2 \text{ square units.}
$$
\narea element on a sphere
\n**EXAMPLE 6** (Surface area of a parabolic dish) Find the surface area of a parabolic reflector whose shape is obtained by rotating the parabolic arc $y = x^2$, $(0 \le x \le 1)$, about the y-axis, as illustrated in Figure 7.28.
\n**Solution** The arc length element for the parabola $y = x^2$ is $ds = \sqrt{1 + 4x^2} dx$, so the required surface area is

EXAMPLE 6 parabolic reflector whose shape is obtained by rotating the parabolic arc $y = x^2$, $(0 \le x \le 1)$, about the *y*-axis, as illustrated in Figure 7.28. \int_0^{1}
= $4\pi \int_0^a \sqrt{a^2} dx = 4\pi a x \Big|_0^a = 4\pi a^2$ square units.

EXAMPLE 6 (Surface area of a parabolic dish) Find the surface area of a parabolic reflector whose shape is obtained by rotating the parabolic arc $y = x$ Trace area of a

ing the parabolic
 $\frac{1}{1 + 4x^2} dx$, so = $4\pi \int_0^a \sqrt{a^2} dx = 4\pi a x \Big|_0^a = 4\pi a^2$ square units.

 EXAMPLE 6 (Surface area of a parabolic dish) Find to parabolic reflector whose shape is obtained by are $y = x^2$, $(0 \le x \le 1)$, about the y-axis, as illustrate **a** square units.
 a square units.
 a se shape is obtained by rotating the parabolic

as illustrated in Figure 7.28.
 b arabola $y = x^2$ is $ds = \sqrt{1 + 4x^2} dx$, so

Let $u = 1 + 4x^2$,
 $du = 8x dx$

Solution The arc length element for the parabola $y = x^2$ is $ds = \sqrt{1 + 4x^2} dx$, so

Solution The arc length element for the parabola
$$
y = x^2
$$
 is $ds = \sqrt{1 + 4x^2} dx$, so
\nthe required surface area is
\n
$$
S = 2\pi \int_0^1 x \sqrt{1 + 4x^2} dx
$$
\nLet $u = 1 + 4x^2$,
\n $du = 8x dx$
\n
$$
= \frac{\pi}{6} u^{3/2} \Big|_1^5 = \frac{\pi}{6} (5\sqrt{5} - 1)
$$
 square units.
\n**EXERCISES 7.3**
\nExercises 1–16, find the lengths of the given curves.
\n $y = 2x - 1$ from $x = 1$ to $x = 3$
\n $y = ax + b$ from $x = A$ to $x = B$
\n**13.** $y = \ln \cos x$ from $x = \pi/6$ to $x = \pi/4$
\n**14.** $y = x^2$ from $x = 0$ to $x = 2$
\n**15.** $y = \ln \frac{e^x - 1}{e^x + 1}$ from $x = 2$ to $x = 4$

EXERCISES 7.3

In Exercises 1–16, find the lengths of the given curves.

1. $y = 2x - 1$ from $x = 1$ to $x = 3$

1. $y = ax + b$ from $x = A$ to $x = B$

1. $y = 1$ to $x = 3$

1. $y = 1$ to $x = 3$

1. $y = x^2$ from $x = 1$ to $x = 3$

1. 1. $y = 2x - 1$ from $x = 1$ to $x = 3$ 2. $y = ax + b$ from $x = A$ to $x = B$ 3. $y = \frac{2}{3}x^{3/2}$ from $x = 0$ to $x = 8$ 4. $y^2 = (x - 1)^3$ from $(1, 0)$ to $(2, 1)$ **S** 7.3

ind the lengths of the given curves.
 $mx = 1$ to $x = 3$
 $mx = A$ to $x = B$
 $mx = 0$ to $x = 8$
 E 16. y
 $m x = 0$ to $x = 8$
 E 16. y
 F 16. y

17. Fi 5. $y^3 = x^2$ from $(-1, 1)$ to $(1, 1)$ **SES 7.3**

-16, find the lengths of the given curves.

-1 from $x = 1$ to $x = 3$

-b from $x = A$ to $x = B$
 $h^{3/2}$ from $x = 0$ to $x = 8$

-1)³ from (1, 0) to (2, 1)

from (-1, 1) to (1, 1) **6.** $2(x + 1)^3 = 3(y - 1)^2$ from $(-1, 1)$ to $(0, 1 + \sqrt{2/3})$ mgths of the given curves.
 13. $y = \ln \cos x$ f

to $x = 3$
 14. $y = x^2$ from
 14. $y = x^2$ from

to $x = 8$
 15. $y = \ln \frac{e^x - 1}{e^x + 1}$
 16. $y = \ln x$ from

(0) to (2, 1)
 17. Find the circu
 $x^{2/3} + y^{2/3}$

both co 7. $y = \frac{x^3}{12} + \frac{1}{x}$ from $x = 1$ to $x = 4$ $\frac{x^3}{12} + \frac{1}{x}$ from $x = 1$ to $x = 4$ 8. $y = \frac{x^3}{3} + \frac{1}{4x}$ from $x = 1$ to $x = 2$ $\frac{1}{3} + \frac{1}{4x}$ from $x = 1$ to $x = 2$ 1 $\frac{1}{4x}$ from $x = 1$ to $x = 2$ 9. $4y = 2 \ln x - x^2$ from $x = 1$ to $x = e$ **10.** $y = x^2 - \frac{\ln x}{8}$ from $x = 1$ to $x = 2$ $\frac{11}{8}$ from $x = 1$ to $x = 2$ 11. $y = \frac{e^x + e^{-x}}{2}$ (= cosh x) from $x = 0$ to x = $\frac{e}{2}$ (= cosh x) from $x = 0$ to $x = a$ and (1. **12.** $y = \ln(1 - x^2)$ from $x = -(1/2)$ to $x = 1/2$

14.
$$
y = x^2
$$
 from $x = 0$ to $x = 2$

15.
$$
y = \ln \frac{e^x - 1}{e^x + 1}
$$
 from $x = 2$ to $x = 4$

I 16. $y = \ln x$ from $x = 1$ to $x = e$
17. Find the circumference of the closed curve 13. $y = \ln \cos x$ from $x = \pi/6$ to $x = \pi/4$

14. $y = x^2$ from $x = 0$ to $x = 2$

15. $y = \ln \frac{e^x - 1}{e^x + 1}$ from $x = 2$ to $x = 4$

16. $y = \ln x$ from $x = 1$ to $x = e$

17. Find the circumference of the closed curve $x^{2/3} + y^{2/3$ = $\pi/6$ to $x = \pi/4$
to $x = 2$
 $x = 2$ to $x = 4$
to $x = e$
ce of the closed curve
Hint: The curve is symmetric about
(why?), so one-quarter of it lies in the
or a calculator with an integration $y = \ln \cos x$ from $x = \pi/6$ to $x = \pi/4$
 $y = x^2$ from $x = 0$ to $x = 2$
 $y = \ln \frac{e^x - 1}{e^x + 1}$ from $x = 2$ to $x = 4$
 $y = \ln x$ from $x = 1$ to $x = e$

Find the circumference of the closed curve
 $x^{2/3} + y^{2/3} = a^{2/3}$. *Hint*: $y = \ln \cos x$ from $x = \pi/6$ to $x = \pi/4$
 $y = x^2$ from $x = 0$ to $x = 2$
 $y = \ln \frac{e^x - 1}{e^x + 1}$ from $x = 2$ to $x = 4$
 $y = \ln x$ from $x = 1$ to $x = e$

Find the circumference of the closed curve
 $x^{2/3} + y^{2/3} = a^{2/3}$. Hint: Th 13. $y = \ln \cos x$ from $x = \pi/6$ to $x = \pi/4$

14. $y = x^2$ from $x = 0$ to $x = 2$

15. $y = \ln \frac{e^x - 1}{e^x + 1}$ from $x = 2$ to $x = 4$

16. $y = \ln x$ from $x = 1$ to $x = e$

17. Find the circumference of the closed curve
 $x^{2/3} + y^{2/$ 14. $y = x^2$ from $x = 0$ to $x = 2$

15. $y = \ln \frac{e^x - 1}{e^x + 1}$ from $x = 2$ to $x = 4$

16. $y = \ln x$ from $x = 1$ to $x = e$

17. Find the circumference of the closed curve
 $x^{2/3} + y^{2/3} = a^{2/3}$. Hint: The curve is symmetric 15. $y = \ln \frac{e^x - 1}{e^x + 1}$ from $x = 2$ to $x = 4$

16. $y = \ln x$ from $x = 1$ to $x = e$

17. Find the circumference of the closed curve
 $x^{2/3} + y^{2/3} = a^{2/3}$. Hint: The curve is symmetric about

both coordinate axes (why?), 17. Find the circumference of the closed curve
 $x^{2/3} + y^{2/3} = a^{2/3}$. Hint: The curve is symmetric about

both coordinate axes (why?), so one-quarter of it lies in the

first quadrant.

Use numerical methods (or a calcu **16.** $y = \ln x$ from $x = 1$ to $x = e$
 17. Find the circumference of the closed curve
 $x^{2/3} + y^{2/3} = a^{2/3}$. *Hint*: The curve is symmetric about

both coordinate axes (why?), so one-quarter of it lies in the

first quad $y = \ln x$ from $x = 1$ to $x = e$
Find the circumference of the closed curve
 $x^{2/3} + y^{2/3} = a^{2/3}$. *Hint*: The curve is symmetric
both coordinate axes (why?), so one-quarter of it lie
first quadrant.
numerical methods (or a

- **E** 18. $y = x^4$ from $x = 0$ to $x = 1$
-
- **C 19.** $y = x^{1/3}$ from $x = 1$ to $x = 2$
C 20. The circumference of the ellipse $3x^2 + y^2 = 3$
- **21.** The shorter arc of the ellipse $x^2 + 2y^2 = 2$ between (0, 1) and $(1, 1/\sqrt{2})$

In Exercises 22–29, find the areas of the surfaces obtained by

rotating the given curve about the indicated lines.

22. $y = x^2$, ($0 \le x \le 2$), about the y-axis the surface example of the surface example of the surface ex 22–29, find the areas of the surfaces obtained by
given curve about the indicated lines.
 $(0 \le x \le 2)$, about the y-axis
 $(0 \le x \le 1)$, about the x-axis
 $\frac{1}{2}$, $(0 \le x \le 1)$, about the x-axis 22–29, find the areas of the surfaces obtained by
given curve about the indicated lines.
, ($0 \le x \le 2$), about the *y*-axis
, ($0 \le x \le 1$), about the *x*-axis
 $\binom{2}{2}$, ($0 \le x \le 1$), about the *y*-axis In Exercises 22–29, find the areas of the surfaces obtained by

21. y = x^2 , ($0 \le x \le 2$), about the y-axis

23. $y = x^3$, ($0 \le x \le 1$), about the x-axis

24. $y = x^{3/2}$, ($0 \le x \le 1$), about the x-axis

25. $y = x^{3/2}$, (In Exercises 22–29, find the areas of the surfaces obtained by
 22. $y = x^2$, ($0 \le x \le 2$), about the y-axis
 23. $y = x^3$, ($0 \le x \le 1$), about the x-axis
 24. $y = x^{3/2}$, ($0 \le x \le 1$), about the x-axis
 25. $y = x^{3/$. 22–29, find the areas of the surfaces obtained by
given curve about the indicated lines.
, ($0 \le x \le 2$), about the *x*-axis
, ($0 \le x \le 1$), about the *x*-axis
 $\binom{2}{2}$, ($0 \le x \le 1$), about the *x*-axis
, ($0 \le x \le 1$),

- 22. $y = x^2$, $(0 \le x \le 2)$, about the y-axis
- **23.** $y = x^3$, $(0 \le x \le 1)$, about the *x*-axis
24. $y = x^{3/2}$, $(0 \le x \le 1)$, about the *x*-axis
-
-
- **26.** $y = e^x$, $(0 \le x \le 1)$, about the *x*-axis
27. $y = \sin x$, $(0 \le x \le \pi)$, about the *x*-axis
-
- In Exercises 22–29, find the areas of the surfaces obtained by
 22. $y = x^2$, ($0 \le x \le 2$), about the y-axis

23. $y = x^3$, ($0 \le x \le 1$), about the x-axis

24. $y = x^{3/2}$, ($0 \le x \le 1$), about the x-axis

25. $y = x^{3/2}$, (**28.** $y = \frac{x^3}{12} + \frac{1}{x}$, $(1 \le x \le 4)$, about the x $\frac{x^3}{12} + \frac{1}{x}$, (1 $\leq x \leq 4$), about the *x*-axis 9, find the areas of the surfaces obtained by

curve about the indicated lines.
 $x \le 2$), about the y-axis the
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 $\le x \le 1$), about the x-axis
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 $x \le 1$), abo **29.** $y = \frac{x^3}{12} + \frac{1}{x}$, $(1 \le x \le 4)$, about the y $\frac{x^3}{12} + \frac{1}{x}$, (1 $\leq x \leq 4$), about the y-axis
- $x \le 2$), about the y-axis the
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 $(1 \le x \le 4)$, ab 23. $y = x^3$, $(0 \le x \le 1)$, about the x-axis

24. $y = x^{3/2}$, $(0 \le x \le 1)$, about the x-axis

25. $y = x^{3/2}$, $(0 \le x \le 1)$, about the y-axis

26. $y = e^x$, $(0 \le x \le 1)$, about the x-axis

27. $y = \sin x$, $(0 \le x \le \pi)$, about th $y = x^{3/2}$, $(0 \le x \le 1)$, about the *x*-axis
 $y = x^{3/2}$, $(0 \le x \le 1)$, about the *y*-axis
 $y = e^x$, $(0 \le x \le 1)$, about the *x*-axis
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 $y = x^3$, $(0 \le x \le 1)$, about the *x*-axis
 $y = x^{3/2}$, $(0 \le x \le 1)$, about the *x*-axis
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 $y = \frac{x^3}{12} + \frac{1}{x}$, $(1 \le x \le 4)$, about the the y-axis. 27. $y = \sin x$, $0 \le x \le \pi$, about the x-axis

28. $y = \frac{x^3}{12} + \frac{1}{x}$, $(1 \le x \le 4)$, about the x-axis

29. $y = \frac{x^3}{12} + \frac{1}{x}$, $(1 \le x \le 4)$, about the y-axis

31. **Gurface area of a cone**) Find the area of the curved su $y = \frac{x^3}{12} + \frac{1}{x}$, $(1 \le x \le 4)$, about the x-axis supplement the summarized by $x = \frac{x^3}{12} + \frac{1}{x}$, $(1 \le x \le 4)$, about the y-axis radius of the summarized by reduces (Surface of a right-circular cone of base radius \leq 4), about the *x*-axis
 \leq 4), about the *y*-axis
 ane) Find the area of the curved surface

e of base radius *r* and height *h* by

ne segment from (0, 0) to (*r*, *h*) about
 a doughnut?) Find the surface 23. $y = \frac{x^3}{12} + \frac{1}{x}$, $(1 \le x \le 4)$, about the y-axis 37. For what real value
30. (Surface area of a cone) Find the area of the curved surface rotating the curve of a right-circular cone of base radius r and height h $y = \frac{x^3}{12} + \frac{1}{x}$, $(1 \le x \le 4)$, about the y-axis
 Surface
 COLEG 138. 33. (Area of an oblate spheroid) Find the area of the surface of a right-circular cone of base radius r and height h by the surface area rotating the straight line segment from (0, 0) to (r, h) about **138.** The curve $y = \$
- $(x b)^2 + y^2 = a^2$ about the y-axis.
32. (Area of a prolate spheroid) Find the area of the surface or a riguing the straight line segment from (0, 0) to (r, h) about
the y-axis. Find the area of the horse can doughnut?) Find the surface area of the horse (doughnut) obtained by rotating the circle $(x - b)^2 + y^2 = a^2$ about
-
- y-axis. The Strains.

Contained on the set of the strate of the surface the surface obtained by rotating the ellipse $x^2 + 4y^2 = 4$ about the *s* infinite amount of paint to cover

y-axis.
 Mass, Moments, and Centre of Mass

Ma
- In Exercises 22–29, find the areas of the surfaces obtained by

I 34. The ellipse of Example

rotating the given curve about the indicated lines.

22. $y = x^2$, $(0 \le x \le 2)$, about the y-axis

23. $y = x^3$. $(0 \le x \le 1)$ abo **EXECTION 7.4:** Mass, Moments, and Centre of Mass **413**
 13. The ellipse of Example 4 is rotated about the line $y = c > b$ to generate a doughnut with elliptical cross-sections. Express the surface area of this doughnut in SECTION 7.4: Mass, Moments, and Centre of Mass 413
The ellipse of Example 4 is rotated about the line $y = c > b$
to generate a doughnut with elliptical cross-sections. Express
the surface area of this doughnut in terms of th SECTION 7.4: Mass, Moments, and Centre of Mass 413
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to generate a doughnut with elliptical cross-sections. Express
the surface area of this doughnut in terms of th **13.** SECTION 7.4: Mass, Moments, and Centre of Mass **413.**
 13. 14. The ellipse of Example 4 is rotated about the line $y = c > b$ to generate a doughnut with elliptical cross-sections. Express the surface area of this d SECTION 7.4: Mass, Moments, and Centre of Mass 413
The ellipse of Example 4 is rotated about the line $y = c > b$
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The ellipse of Example 4 is rotated about the line $y = c > b$
to generate a doughnut with elliptical cross-sections. Express
the surface area of this doughnut in terms of th
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	- 34. The ellipse of Example 4 is rotated about the line $y = c > b$
to generate a doughnut with elliptical cross-sections. Express
the surface area of this doughnut in terms of the complete
elliptic integral function $E(\varepsilon)$ The ellipse of Example 4 is rotated about the line $y = c > b$
to generate a doughnut with elliptical cross-sections. Express
the surface area of this doughnut in terms of the complete
elliptic integral function $E(\varepsilon)$ intr The ellipse of Example 4 is rotated about the line $y = c > b$
to generate a doughnut with elliptical cross-sections. Express
the surface area of this doughnut in terms of the complete
elliptic integral function $E(\varepsilon)$ intr The ellipse of Example 4 is rotated about the line $y = c > b$
to generate a doughnut with elliptical cross-sections. Express
the surface area of this doughnut in terms of the complete
elliptic integral function $E(\varepsilon)$ intr to generate a doughnut with elliptical cross-sections. Express
the surface area of this doughnut in terms of the complete
elliptic integral function $E(\varepsilon)$ introduced in that example.
Express the integral formula obtain 35. Express the integral function $E(\varepsilon)$ introduced in that example.

	35. Express the integral formula obtained for the length of the

	metal sheet in Example 3 in terms of the complete elliptic

	integral function $E(\varepsilon$ Express the integral function $E(e)$ intoduced in that example.
Express the integral formula obtained for the length of the
metal sheet in Example 3 in terms of the complete elliptic
integral function $E(\epsilon)$ introduced in is rotated about the line $y = c > b$
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coduced in Express the integral formula obtained for the length of the
metal sheet in Example 3 in terms of the complete elliptic
integral function $E(\epsilon)$ introduced in Example 4.
(An interesting property of spheres) If two parallel **I** 35. Express the integral function $E(\varepsilon)$ introduced in that example.
 I 35. Express the integral formula obtained for the length of the metal sheet in Example 3 in terms of the complete elliptic integral function (An interesting property of spheres) If two parallel planes
intersect a sphere, show that the surface area of that part of the
sphere lying between the two planes depends only on the
radius of the sphere and the distance 39. A hollow container in the shape of an infinitely long horn is general of the sphere, show that the surface area of that part of the sphere and the shadius of the sphere and the distance between the planes, and not on integral function $E(\epsilon)$ introduced in Example 4.
(An interesting property of spheres) If two parallel planes
intersect a sphere, show that the surface area of that part of the
sphere lying between the two planes depends
		-
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	- the x -axis. not on the position of the planes.

	For what real values of k does the surface generated by

	rotating the curve $y = x^k$, $(0 < x \le 1)$, about the y-axis have

	a finite surface area?

	The curve $y = \ln x$, $(0 < x \le 1)$, is rotat For what real values of k does the surface generated by
rotating the curve $y = x^k$, $(0 < x \le 1)$, about the y-axis have
a finite surface area?
The curve $y = \ln x$, $(0 < x \le 1)$, is rotated about the y-axis.
Find the area of t rotating the curve $y = x^k$, $(0 < x \le 1)$, about the y-axis have
a finite surface area?
The curve $y = \ln x$, $(0 < x \le 1)$, is rotated about the y-axis.
Find the area of the horn-shaped surface so generated.
A hollow container ite surface area?
curve $y = \ln x$, $(0 < x \le 1)$, is rotated about the y-axis.
the area of the horn-shaped surface so generated.
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	the area of the horn-shaped surface so generated.

	bllow container in the shape of an infinitely long horn is

	rated by rotating the curve $y = 1/x$, $(1 \le x < \infty)$, a
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and Centre of Mass
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plines are described in terms of densities over regions of space, the plane, or even the
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Many quantities of interest in physics, mechanics, ecology, finance, and other disci-
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Mass and Density
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If a solid object is made

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If a solid object is mad

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By "density at a point *P*" of a

solid object, we mean the limit
 $\rho(P)$ of mass/volume for the

part of the solid lying in sma **Mass and Density**

If a solid object is made of the solid that have the

this homogeneity by say

By "density at a point P" of a

example, a rectangular by

solid object, we mean the limit

solid object, we mean the limi plines are described in terms of densities over regions of space, the plane, or even the real line. To determine the total value of such a quantity we must add up (integrate) the contributions from the various places wher volume $V = 20 \times 10 \times 8 = 1,600 \text{ cm}^3$, and if it was made of material having constant of such a quantity we must add up (integrate)
es where the quantity is distributed.
hecous material, we would expect different parts
e to have the same mass as well. We express
bbject has constant density, that density be density $\rho = 3$ g/cm³, it would have mass $m = \rho V = 3 \times 1,600 = 4,800$ g. (We will use the lowercase Greek letter rho (ρ) to represent density.) m the various places where the quantity is distributed.
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ade of a homogeneous material, we would expect different parts

e the same volume to have the same mass as well. We express

saying that the object has co **Mass and Density**
If a solid object is made of a homogeneous material, we would expect different parts
of the solid that have the same volume to have the same mass as well. We express
this homogeneity by saying that the **IS AND DENSITY**

Solid object is made of a homogeneous material, we would expect different parts

le solid that have the same volume to have the same mass as well. We express

homogeneity by saying that the object has co If a solid object is made of a homogeneous material, we would expect different parts
of the solid that have the same volume to have the same mass as well. We express
this homogeneity by saying that the object has constant **Mass and Density**
If a solid object is made of a homogeneous material, we would expect different parts
of the solid that have the same volume to have the same mass as well. We express
this homogeneity by saying that the

this homogeneity by saying that the object has constant density, that density being
the mass divided by the volume for the whole object or for any part of it. Thus, for
example, a rectangular brick with dimensions 20 cm, the mass divided by the volume for the whole object or for any part of it. Thus, for example, a rectangular brick with dimensions 20 cm, 10 cm, and 8 cm would have volume $V = 20 \times 10 \times 8 = 1,600 \text{ cm}^3$, and if it was made example, a rectangular brick with dimensions 20 cm, 10 cm, and 8 cm would have
volume $V = 20 \times 10 \times 8 = 1,600 \text{ cm}^3$, and if it was made of material having constant
density $\rho = 3$ g/cm³, it would have mass $m = \rho V = 3 \times$ volume $V = 20 \times 10 \times 8 = 1,600 \text{ cm}^3$, and if it was made of material
density $\rho = 3$ g/cm³, it would have mass $m = \rho V = 3 \times 1,600 = 4$,
use the lowercase Greek letter rho (ρ) to represent density.)
If the density of t mass divided by the volume for the whole object of
mple, a rectangular brick with dimensions 20 cm, 1
me $V = 20 \times 10 \times 8 = 1,600 \text{ cm}^3$, and if it was mad
ity $\rho = 3$ g/cm³, it would have mass $m = \rho V = 3$
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Example 3 and **DUISI**

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By "density at a point *P*" of a

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solid object, we mean the limit
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solid object, we mean the limit volume $V = 20 \times 10 \times 8 =$
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of the regions approach to P .

so the mass *m* of the solid can be approximated:
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m = \sum \Delta m \approx \sum \rho(P) \Delta V.
$$

so the mass *m* of the solid can be approximated:
 $m = \sum \Delta m \approx \sum \rho(P) \Delta V$.

Such approximations become exact as we pass to the limit of differential mass and

volume elements, $dm = \rho(P) dV$, so we expect to be able to calculat so the mass m of the solid can be approximated:
 $m = \sum \Delta m \approx \sum \rho(P) \Delta V$.

Such approximations become exact as we pass to the limit of differential mass and

volume elements, $dm = \rho(P) dV$, so we expect to be able to calculate so the mass *m* of the solid can be approximated:
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Such approximations become exact as we pass to the limit of differential n

volume elements, $dm = \rho(P) dV$, so we expect to be able to calculate m

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uch approximations become exact as we pass to the limit of differential mass and

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 EXAMPLE 1 The densi it of differential mass and
ble to calculate masses as
 α ; of height H cm and base
 α , where h is the height in
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ight h above the base and Such approximations become exact as we pass to the limit of differential mass and
volume elements, $dm = \rho(P) dV$, so we expect to be able to calculate masses as
integrals, that is, as the limits of such sums:
 $m = \int dm = \int \rho(P) dV$

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m = \int dm = \int \rho(P) dV.
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area A cm² is $\rho = \rho_0(1+h)$ g/cm³, where h is the height

volume elements, $dm = \rho(P) dV$, so we expect to be able to calculate masses as
integrals, that is, as the limits of such sums:
 $m = \int dm = \int \rho(P) dV$.
EXAMPLE 1 The density of a solid vertical cylinder of height *H* cm and base
 Such approximations become exact as we pass to the limit of differential mass and
volume elements, $dm = \rho(P) dV$, so we expect to be able to calculate masses as
integrals, that is, as the limits of such sums:
 $m = \int dm = \int \rho(P) dV$ $m = \int dm = \int \rho(P) dV.$
 EXAMPLE 1 The density of a solid vertical cylinder of height *H* cm and base

area *A* cm² is $\rho = \rho_0(1 + h) g/cm^3$, where *h* is the height in

centimetres above the base and ρ_0 is a constant. Fin grads, that is, as the limits of such sums:
 $m = \int dm = \int \rho(P) dV$.
 AMPLE 1 The density of a solid vertical cylinder of height *H* area *A* cm² is $\rho = \rho_0(1 + h)$ g/cm³, where *h* is timetres above the base and ρ_0 is **EXAMPLE 1** The density of a solid vertical cylinder of height *H* cm and l area *A* cm² is $\rho = \rho_0(1 + h)$ g/cm³, where *h* is the heigh centimetres above the base and ρ_0 is a constant. Find the mass of the cylind The density of a solid vertical cylinder of heig
area A cm² is $\rho = \rho_0(1 + h)$ g/cm³, where
ove the base and ρ_0 is a constant. Find the mass of th
Figure 7.29(a). A slice of the solid at height h a
ss dh is a circu

$$
dm = \rho dV = \rho_0(1+h) A dh.
$$

$$
m = \int_0^H \rho_0 A (1 + h) \, dh = \rho_0 A \left(H + \frac{H^2}{2} \right) \, g.
$$

- Figure 7.29

(a) A solid cylinder whose density varies

with height
- centre

$$
\text{The formula}
$$

$$
\rho = \frac{\rho_0}{1+r^2} \text{ kg/m}^3.
$$

EXAMPLE 2 (Using spherical shells) The density of a certain spherical planet
the formula
 $\rho = \frac{\rho_0}{1 + r^2}$ kg/m³.
Find the mass of the planet.
Solution Recall that the surface area of a sphere of radius r is $4\pi r^$ spherical planet
tre according to
.
The planet can
radii between 0
7.29(b)) is equal
its density: **EXAMPLE 2** (Using spherical shells) The density of a certain spherical planet
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 $\rho = \frac{\rho_0}{1 + r^2}$ kg/m³.
Find the mass of the planet.
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of radius R m varies with distance r from the centre according to
 $\rho = \frac{\rho_0}{1 + r^2}$ kg/m³.

Find the mass of the planet.
 Solution Reca of radius R m varies with distance r from the centre according to
 $\rho = \frac{\rho_0}{1+r^2}$ kg/m³.

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be of radius R m varies with distance r from the centre accompanies of radius R m varies with distance r from the centre accompanies kg/m^3 .

the planet.

that the surface area of a sphere of radius r is $4\pi r^2$. The p

$$
dV = 4\pi r^2 dr; \qquad dm = \rho \, dV = 4\pi \rho_0 \frac{r^2}{1+r^2} dr.
$$

SECTION 7.4: Mass, Moments, and Centre of Mass 415
\nWe add the masses of these shells to find the mass of the whole planet:
\n
$$
m = 4\pi \rho_0 \int_0^R \frac{r^2}{1+r^2} dr = 4\pi \rho_0 \int_0^R \left(1 - \frac{1}{1+r^2}\right) dr
$$
\n
$$
= 4\pi \rho_0 (r - \tan^{-1} r) \Big|_0^R = 4\pi \rho_0 (R - \tan^{-1} R) \text{ kg.}
$$
\nSimilar techniques can be applied to find masses of one- and two-dimensional objects, such as wires and thin plates, that have variable densities of the forms mass/unit length (line density, which we will usually denote by δ) and σ = mass/unit area (areal density, which we will denote by σ).
\n**EXAMPLE 3** A wire of variable composition is stretched along the x-axis from $x = 0$ to $x = L$ cm. Find the mass of the wire if the line density

We add the masses of these shells to find the mass of the whole planet:
 $m = 4\pi \rho_0 \int_0^R \frac{r^2}{1 + r^2} dr = 4\pi \rho_0 \int_0^R \left(1 - \frac{1}{1 + r^2}\right) dr$
 $= 4\pi \rho_0 (r - \tan^{-1} r) \Big|_0^R = 4\pi \rho_0 (R - \tan^{-1} R)$ kg.

Similar techniques can be a We add the masses of these shells to find the mass of the whole planet:
 $m = 4\pi \rho_0 \int_0^R \frac{r^2}{1 + r^2} dr = 4\pi \rho_0 \int_0^R \left(1 - \frac{1}{1 + r^2}\right) dr$
 $= 4\pi \rho_0 (r - \tan^{-1} r) \Big|_0^R = 4\pi \rho_0 (R - \tan^{-1} R)$ kg.

Similar techniques can be a $m = 4\pi \rho_0 \int_0^R \frac{r^2}{1 + r^2} dr = 4\pi \rho_0 \int_0^R \left(1 - \frac{1}{1 + r^2}\right) dr$
 $= 4\pi \rho_0 (r - \tan^{-1} r)\Big|_0^R = 4\pi \rho_0 (R - \tan^{-1} R)$ kg.

Similar techniques can be applied to find masses of one- and two-dimensional objects,

such as wires = $4\pi \rho_0 (r - \tan^{-1} r) \Big|_0^R$ = $4\pi \rho_0 (R - \tan^{-1} R)$ kg.

imilar techniques can be applied to find masses of one- and two-dimensional objects,

ch as wires and thin plates, that have variable densities of the forms mass/uni ^x ^D ⁰ to ^x ^D ^L cm. Find the mass of the wire if the line density Similar techniques can be applied to find masses of one- and two-dimensional objects,
such as wires and thin plates, that have variable densities of the forms mass/unit length
(line density, which we will usually denote b Similar techniques can be applied to find masses of one- and two-dimensional objects,
such as wires and thin plates, that have variable densities of the forms mass/unit length
(line density, which we will usually denote b Similar techniques can be applied to find masses of one- and two-dimensional objects,
such as wires and thin plates, that have variable densities of the forms mass/unit length
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$$
m = \int_0^L kx \, dx = \left(\frac{kx^2}{2}\right)\Big|_0^L = \frac{kL^2}{2} \, \text{g}.
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olution The mass of a length element dx of the wire located at position x is given
 $y dm = \delta(x) dx = kx dx$. Thus, the mass of the wire is
 $m = \int_0^L kx dx = \left(\frac{kx^2}{2}\right)\Big|_0^L = \frac{kL^2}{2}$ g.
 EXAMPLE 4 Find the mass of a disk of ra $x = 0$ to $x = L$ cm. Find the mass of the wire if the line density
 $= kx$ g/cm, where k is a positive constant.

s of a length element dx of the wire located at position x is given
 kx dx. Thus, the mass of the wire is
 $\sigma = k(2a + x)$ g/cm². Here k is a constant. $\left[\frac{kx^2}{2}\right]_0^L = \frac{kL^2}{2}$ g.

and the mass of a disk of radius a cm whose centrigin in the *xy*-plane if the areal density at position.

Here k is a constant.

ensity depends only on the horizontal coordinate x, s hes $y = \sqrt{a^2 - x^2}$ **EXAMPLE 4**

 $m = \int_0^L kx \, dx = \left(\frac{kx^2}{2}\right)\Big|_0^L = \frac{kL^2}{2} \, g.$
 EXAMPLE 4 Find the mass of a disk of radius *a* cm whose centre is at the origin in the *xy*-plane if the areal density at position (x, y) is $\sigma = k(2a + x)$ g/cm². Here $m = \int_0^{\infty} kx \, dx = \left(\frac{kx}{2}\right)\Big|_0 = \frac{kL}{2}$ g.
 EXAMPLE 4 Find the mass of a disk of radius *a* cm whose centre is at the origin in the *xy*-plane if the areal density at position (x, y) is $\sigma = k(2a + x)$ g/cm². Here *k* **EXAMPLE 4** Find the mass of a disk of radius *a* cm whose centre is at the origin in the *xy*-plane if the areal density at position (x, y) is $\sigma = k(2a + x)$ g/cm². Here *k* is a constant.
Solution The areal density de \overrightarrow{x} $2\sqrt{a^2 - x^2} dx$ (see Figure 7.30); its **EXAMPLE 4** Find the mass of a disk of radius a cm whose centre is at the origin in the xy-plane if the areal density at position $(x, y) = k(2a + x) g/cm^2$. Here k is a constant.
 lution The areal density depends only on the h **Solution** The areal density depends only on the horizontal coordinate x, so it is con- $\sigma = k(2a + x)$ g/cm². Here *k* is a constant.
 Solution The areal density depends only on the horizontal coordin

stant along vertical lines on the disk. This suggests that thin vert

be used as area elements. A vertica

$$
dm = \sigma dA = 2k(2a + x)\sqrt{a^2 - x^2} dx.
$$

Hence, the mass of the disk is
\n
$$
m = \int_{x=a}^{x=a} dm = 2k \int_{-a}^{a} (2a + x)\sqrt{a^2 - x^2} dx
$$
\n
$$
= 4ak \int_{-a}^{a} \sqrt{a^2 - x^2} dx + 2k \int_{-a}^{a} x\sqrt{a^2 - x^2} dx
$$
\n
$$
= 4ak \frac{\pi a^2}{2} + 0 = 2\pi ka^3 g.
$$
\nWe used the area of a semicircle to evaluate the first integral. The second integral is zero because the integrand is odd and the interval is symmetric about $x = 0$.
\nDistributions of mass along one-dimensional structures (lines or curves) necessarily lead to integrals of functions of one variable, but distributions of mass on a surface or in space can lead to integrals involving functions of more than one variable. Such integrals are studied in multivariable calculus. (See, for example, Section 14.7.) In

= $4ak \int_{-a}^{a} \sqrt{a^2 - x^2} dx + 2k \int_{-a}^{a} x\sqrt{a^2 - x^2} dx$

= $4ak \frac{\pi a^2}{2} + 0 = 2\pi ka^3$ g.

We used the area of a semicircle to evaluate the first integral. The second integral is

zero because the integrand is odd and the int \int_{-a}^{+a} $= 4ak \frac{\pi a^2}{2} + 0 = 2\pi ka^3$ g.
We used the area of a semicircle to evaluate the first integral. The second integral is
zero because the integrand is odd and the interval is symmetric about $x = 0$.
Distributions of mass al = $4ak - \frac{1}{2} + 0 = 2\pi k a^3$ g.
We used the area of a semicircle to evaluate the first integral. The second integral is
zero because the integrand is odd and the interval is symmetric about $x = 0$.
Distributions of mass alo We used the area of a semicircle to evaluate the first integral. The second integral is zero because the integrand is odd and the interval is symmetric about $x = 0$.
Distributions of mass along one-dimensional structures We used the area of a semicircle to evaluate the first integral. The second integral is zero because the integrand is odd and the interval is symmetric about $x = 0$.

Distributions of mass along one-dimensional structures

Moments and Centres of Mass
The **moment** about the point $x = x_0$ of a mass m located at position x is the product $m(x - x_0)$ of the mass and its (signed) distance from x_0 **Moments and Centres of Mass**
The **moment** about the point $x = x_0$ of a mass m located at position x on the x-axis
is the product $m(x - x_0)$ of the mass and its (signed) distance from x_0 . If the x-axis
is a horizontal a **Moments and Centres of Mass**
The **moment** about the point $x = x_0$ of a mass m located at position x on the x-axis
is the product $m(x - x_0)$ of the mass and its (signed) distance from x_0 . If the x-axis
is a horizontal a **Moments and Centres of Mass**
The **moment** about the point $x = x_0$ of a mass m located at position x on the x-axis
is the product $m(x - x_0)$ of the mass and its (signed) distance from x_0 . If the x-axis
is a horizontal a **Moments and Centres of Mass**
The **moment** about the point $x = x_0$ of a mass m located at position x on the x-axis
is the product $m(x - x_0)$ of the mass and its (signed) distance from x_0 . If the x-axis
is a horizontal a **Moments and Centres of Mass**
The **moment** about the point $x = x_0$ of a mass m located at position x on the x-axis
is the product $m(x - x_0)$ of the mass and its (signed) distance from x_0 . If the x-axis
is a horizontal a **Moments and Centres of Mass**
The **moment** about the point $x = x_0$ of a mass m l
is the product $m(x - x_0)$ of the mass and its (signe
is a horizontal arm hinged at x_0 , the moment abou
weight of the mass m to cause the a

weight of the mass *m* to cause the arm to rotate. If several masses
$$
m_1
$$
, m_2 , m_3 , ...,
\n m_n are located at the points x_1 , x_2 , x_3 , ..., x_n , respectively, then the total moment
\nof the system of masses about the point $x = x_0$ is the sum of the individual moments
\n(see Figure 7.31):
\n
$$
M_{x=x_0} = (x_1 - x_0)m_1 + (x_2 - x_0)m_2 + \dots + (x_n - x_0)m_n = \sum_{j=1}^n (x_j - x_0)m_j.
$$
\nFigure 7.31 A system of discrete masses
\n
$$
\begin{array}{c}\nm_2 \nm_1 \nm_3 \nm_5 \nm_4 \nm_5 \nm_4 \nm_6 \nm_7 \nm_8 \nm_9 \nm_1 \nm_1 \nm_3 \nm_5 \nm_4 \nm_7 \nm_8 \nm_9 \nm_1 \nm_1 \nm_1 \nm_3 \nm_5 \nm_4 \nm_7 \nm_8 \nm_9 \nm_1 \nm_1 \nm_1 \nm_2 \nm_3 \nm_4 \nm_5 \nm_6 \nm_7 \nm_8 \nm_9 \nm_1 \nm_1 \nm_2 \nm_3 \nm_1 \nm_3 \nm_4 \nm_5 \nm_6 \nm_7 \nm_8 \nm_9 \nm_1 \nm_1 \nm_2 \nm_3 \nm_1 \nm_3 \nm_4 \nm_5 \nm_6 \nm_7 \nm_8 \nm_9 \nm_1 \nm_1 \nm_1 \nm_2 \nm_3 \nm_4 \nm_5 \nm_6 \nm_7 \nm_8 \nm_9 \nm_1 \nm_1 \nm_2 \nm_3 \nm_4 \nm_5 \nm_7 \nm_8 \nm_9 \nm_1 \nm_1 \nm_2 \nm_3 \nm_4 \nm_5 \nm_7 \nm_8 \nm_9 \nm_9 \nm_1 \nm_1 \nm_1 \nm_2 \nm_3 \nm_4 \nm_5 \nm_7 \nm_8 \nm_9 \nm_9 \nm_1 \nm_1 \nm_1 \nm_2 \nm_3 \nm_5 \nm_4 \nm_7 \nm_8 \nm_9 \nm_9 \nm_1 \nm_1 \nm_1 \nm_2 \nm_3 \nm_5 \nm_4 \nm_7 \nm_8 \nm_9 \nm_9 \nm_1 \nm_1 \nm_1 \nm_2 \nm_3 \
$$

The **centre of mass** of the system of masses is the point \bar{x} about which the total
moment of the system is zero. Thus,
 $0 = \sum_{j=1}^{n} (x_j - \bar{x})m_j = \sum_{j=1}^{n} x_j m_j - \bar{x} \sum_{j=1}^{n} m_j$.
The centre of mass of the system is ther

$$
0 = \sum_{j=1}^{n} (x_j - \bar{x}) m_j = \sum_{j=1}^{n} x_j m_j - \bar{x} \sum_{j=1}^{n} m_j.
$$

$$
\bar{x} = \frac{\sum_{j=1}^{n} x_j m_j}{\sum_{j=1}^{n} m_j} = \frac{M_{x=0}}{m},
$$

The centre of mass of the system is therefore given by
 $\bar{x} = \frac{\sum_{j=1}^{n} x_j m_j}{\sum_{j=1}^{n} m_j} = \frac{M_{x=0}}{m}$,

where *m* is the total mass of the system and $M_{x=0}$ is the total moment about $x = 0$.

If you think of the x- $\bar{x} = \frac{\sum_{j=1}^{n} x_j m_j}{n} = \frac{M_{x=0}}{m}$,
 $\sum_{j=1}^{n} m_j$

where *m* is the total mass of the system and $M_{x=0}$ is the total moment about $x = 0$.

If you think of the x-axis as being a weightless wire supporting the mass $\bar{x} = \frac{\sum_{j=1}^{n} x_j m_j}{n} = \frac{M_{x=0}}{m}$,

Where *m* is the total mass of the system and $M_{x=0}$ is the total moment about $x = 0$.

If you think of the *x*-axis as being a weightless wire supporting the masses, then \bar{x} $\bar{x} = \frac{\bar{y} - \bar{y}}{n}$ $= \frac{M_{x=0}}{m}$,

Where *m* is the total mass of the system and $M_{x=0}$ is the total moment about $x = 0$.

If you think of the *x*-axis as being a weightless wire supporting the masses, then \bar{x} $x = \frac{n}{\sum m_j} m_j$
 $y = \frac{m}{m}$,

where *m* is the total mass of the system and $M_{x=0}$ is the total moment about $x = 0$.

If you think of the *x*-axis as being a weightless wire supporting the masses, then \bar{x} is the po $\sum_{j=1}^{m_j} m_j$
 $\sum_{j=1}^{m_j}$

where *m* is the total mass of the system and $M_{x=0}$ is the total moment about $x = 0$.

If you think of the *x*-axis as being a weightless wire supporting the masses, then \bar{x} is

the $j=1$

ne *m* is the total mass of the system and $M_{x=0}$ is the total moment about $x = 0$.

u think of the *x*-axis as being a weightless wire supporting the masses, then \bar{x} is

boint at which the wire could be supp $\sum_{j=1}^{n} m_j$

where *m* is the total mass of the system and $M_{x=0}$ is the total moment about $x = 0$.

If you think of the *x*-axis as being a weightless wire supporting the masses, then \bar{x} is

the point at which t $\sum_{j=1}^{m_j} m_j$
where *m* is the total mass of the system and $M_{x=0}$ is the total moment about $x = 0$.
If you think of the *x*-axis as being a weightless wire supporting the masses, then \bar{x} is
the point at which the $x = f(x)$

where *m* is the total mass of the system and $M_{x=0}$ is the total moment about $x = 0$.

If you think of the *x*-axis as being a weightless wire supporting the masses, then \bar{x} is

the point at which the wire the point at which the wire could be supported
librium), not tipping either way. Even if the axis
say a seesaw, supported at $x = \bar{x}$, it will remain
provided it was balanced beforehand. For many
as though its total mass x-axis as being a weightiess wire supporting
the wire could be supported and remain in p
ng either way. Even if the axis represents a no
orted at $x = \bar{x}$, it will remain balanced after t
lanced beforehand. For many purpo

Now suppose that a one-dimensional distribution of mass
able line density $\delta(x)$ lies along the interval [a, b] of the x-a
dx at position x contains mass $dm = \delta(x) dx$, so its mome
 $x\delta(x) dx$ about $x = 0$. The total moment abou boxer that a one-dimensional distribution of m

ity $\delta(x)$ lies along the interval [a, b] of the x-

x contains mass $dm = \delta(x) dx$, so its mon

at $x = 0$. The total moment about $x = 0$ is

ents:
 $\int_a^b x \delta(x) dx$.

mass is
 $\delta(x$

When the elements:

\n
$$
M_{x=0} = \int_{a}^{b} x \delta(x) \, dx.
$$
\nSince the total mass is

\n
$$
m = \int_{a}^{b} \delta(x) \, dx,
$$
\nwe obtain the following formula for the centre of mass:

$$
m = \int_a^b \delta(x) \, dx,
$$

SECTION 7.4: Mass, Moments, and Centre of Mass **417**
The centre of mass of a distribution of mass with line density $\delta(x)$ on the interval [a, b] is given by SECTION 7.4: Mass, Moments, and

The centre of mass of a distribution of mass with line density

interval [a, b] is given by
 $\int_a^b x \delta(x) dx$ SECTION 7.4: Mass, Moments, and Centre of M

a distribution of mass with line density $\delta(x)$
 $x \delta(x) dx$ SECTION 7.4: Mass, Moments, and Centre of
distribution of mass with line density $\delta(x)$
 $\delta(x) dx$
 $\delta(x) dx$.

11.
$$
\bar{x} = \frac{M_{x=0}}{m} = \frac{\int_{a}^{b} x \delta(x) \, dx}{\int_{a}^{b} \delta(x) \, dx}.
$$

\n12.
$$
\text{12. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n13.
$$
\text{14. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n14.
$$
\text{15. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n15.
$$
\text{16. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n16.
$$
\text{17. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n17.
$$
\text{18. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n18.
$$
\text{19. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n19.
$$
\text{10. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n10.
$$
\text{11. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n11.
$$
\text{12. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n12.
$$
\text{13. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n13.
$$
\text{14. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n14.
$$
\text{15. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n15.
$$
\text{16. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n16.
$$
\text{17. } \frac{1}{2} \int_{a}^{b} \delta(x) \, dx
$$

\n17.
$$
\text{18. } \frac{1
$$

The centre of mass of a distribution of mass with line density $\delta(x)$ on the
interval [a, b] is given by
 $\bar{x} = \frac{M_{x=0}}{m} = \frac{\int_a^b x \delta(x) dx}{\int_a^b \delta(x) dx}$.
EXAMPLE 5 At what point can the wire of Example 3 be suspended so tha

 $\bar{x} = \frac{M_{x=0}}{m} = \frac{\int_a^b x \delta(x) dx}{\int_a^b \delta(x) dx}$.
 EXAMPLE 5 At what point can the wire of Example 3 be suspended so that it will balance?
 Solution In Example 3 we evaluated the mass of the wire to be $kL^2/2$ g. Its mo pended so that it

/2 g. Its moment about $x = 0$ is

Solution In Example 3 we evaluated the mass of the wire to be
$$
kL^2/2
$$
 g. Its moment
about $x = 0$ is

$$
M_{x=0} = \int_0^L x \delta(x) dx
$$

$$
= \int_0^L kx^2 dx = \left(\frac{kx^3}{3}\right)\Big|_0^L = \frac{kL^3}{3}
$$
g.cm.
(Note that the appropriate units for the moment are units of mass times units of distance: in this case gram-centimetres.) The centre of mass of the wire is

$$
\bar{x} = \frac{kL^3/3}{kL^2/2} = \frac{2L}{3}.
$$
The wire will be balanced if suspended at position $x = 2L/3$ cm.
Two- and Three-Dimensional Examples

$$
\bar{x} = \frac{kL^3/3}{kL^2/2} = \frac{2L}{3}.
$$

 $\left(\frac{1}{2} \right)^2 = \frac{1}{2} \left(\frac{1}{2} \right)^2 = \frac{1}{3}$ g.c.m.

(Note that the appropriate units for the moment are units of mass times unitance: in this case gram-centimetres.) The centre of mass of the wire is $\bar{x} = \frac{kL^3/3}{kL^$ (Note that the appropriate units for the moment are units of mass times units of dis-
tance: in this case gram-centimetres.) The centre of mass of the wire is
 $\bar{x} = \frac{kL^3/3}{kL^2/2} = \frac{2L}{3}$.
The wire will be balanced if tance: in this case gram-centimetres.) The centre of mass of the wire is
 $\bar{x} = \frac{kL^3/3}{kL^2/2} = \frac{2L}{3}$.

The wire will be balanced if suspended at position $x = 2L/3$ cm.
 Two- and Three-Dimensional Example 5 is one-(Note that the appropriate units for the moment are units of mass times units of dis-

tance: in this case gram-centimetres.) The centre of mass of the wire is
 $\bar{x} = \frac{kL^3/3}{kL^2/2} = \frac{2L}{3}$.

The wire will be balanced $\bar{x} = \frac{\kappa L^2/3}{kL^2/2} = \frac{2L}{3}$.

The wire will be balanced if suspended at position *n*
 Two- and Three-Dimensional Examples

The system of mass considered in Example 5 is of

straight line. If mass is distributed i in Example 5 is one-dimensional and
ed in a plane or in space, similar consid
at (x_1, y_1) , m_2 at (x_2, y_2) , ..., m_n at
 $\cdots + x_n m_n = \sum_{j=1}^n x_j m_j$,
 $\cdots + y_n m_n = \sum_{j=1}^n y_j m_j$. in Example 5 is one-dimensional and \vec{a} d in a plane or in space, similar conside
at (x_1, y_1) , m_2 at (x_2, y_2) , ..., m_n at \cdot
 $\cdot + x_n m_n = \sum_{j=1}^n x_j m_j$,
 $\cdot + y_n m_n = \sum_{j=1}^n y_j m_j$. **Two- and Three-Dimensional Examples**
The system of mass considered in Example 5 is one-dimensional and list
raight line. If mass is distributed in a plane or in space, similar considers
vail. For a system of masses m_1

moment about
$$
x = 0
$$
 is

\n
$$
M_{x=0} = x_1 m_1 + x_2 m_2 + \dots + x_n m_n = \sum_{j=1}^n x_j m_j,
$$
\nand the moment about $y = 0$ is

\n
$$
M_{y=0} = y_1 m_1 + y_2 m_2 + \dots + y_n m_n = \sum_{j=1}^n y_j m_j.
$$
\nThe centre of mass is the point (\bar{x}, \bar{y}) where

\n
$$
\sum_{j=1}^n x_j m_j
$$
\n
$$
\sum_{j=1}^n y_j m_j
$$

$$
M_{y=0} = y_1 m_1 + y_2 m_2 + \dots + y_n m_n = \sum_{j=1}^n y_j m_j.
$$

$$
M_{x=0} = x_1m_1 + x_2m_2 + \dots + x_nm_n = \sum_{j=1}^{n} x_jm_j,
$$

and the moment about $y = 0$ is

$$
M_{y=0} = y_1m_1 + y_2m_2 + \dots + y_nm_n = \sum_{j=1}^{n} y_jm_j.
$$
The centre of mass is the point (\bar{x}, \bar{y}) where

$$
\bar{x} = \frac{M_{x=0}}{m} = \frac{\sum_{j=1}^{n} x_jm_j}{\sum_{j=1}^{n} m_j} \quad \text{and} \quad \bar{y} = \frac{M_{y=0}}{m} = \frac{\sum_{j=1}^{n} y_jm_j}{\sum_{j=1}^{n} m_j}.
$$

For continuous distributions of mass, the sums become appropriate integrals.
EXAMPLE 6 Find the centre of mass of a rectangular plate that occupies the region $0 \le x \le a, 0 \le y \le b$, if the areal density of the material
in the plate at position (x, y) is $\sigma = ky$.

height

Solution Since the areal density is independent of x and the rectangle is symmetric
about the line $x = a/2$, the x-coordinate of the centre of mass must be $\bar{x} = a/2$. A
thin horizontal strip of width dy at height y (see F **Solution** Since the areal density is independent of x and the rectangle is symmetric about the line $x = a/2$, the x-coordinate of the centre of mass must be $\bar{x} = a/2$. A thin horizontal strip of width dy at height y (see **Solution** Since the areal density is independent of x and the rectangle is symmetric about the line $x = a/2$, the x-coordinate of the centre of mass must be $\bar{x} = a/2$. A thin horizontal strip of width dy at height y (see gle is symmetric
be $\bar{x} = a/2$. A
 $dm = aky dy$.
 dy . Hence, the **Solution** Since the areal density is independent of x and the rectangle is symmetric about the line $x = a/2$, the x-coordinate of the centre of mass must be $\bar{x} = a/2$. A thin horizontal strip of width dy at height y (see areal density is independent of x and the r

(2, the x-coordinate of the centre of mass

i width dy at height y (see Figure 7.32) has

trip about $y = 0$ is $dM_{y=0} = y dm =$

out $y = 0$ of the whole plate are
 $y dy = \frac{kab^2}{2}$, **Solution** Since the areal density is independent of x and the rectangle is symmetric
about the line $x = a/2$, the x-coordinate of the centre of mass must be $\bar{x} = a/2$. A
thin horizontal strip of width dy at height y (see

$$
m = ka \int_0^b y \, dy = \frac{kab^2}{2},
$$

$$
M_{y=0} = ka \int_0^b y^2 \, dy = \frac{kab^3}{3}.
$$

 $m = ka \int_0^b y \, dy = \frac{ka b^2}{2}$,
 $M_{y=0} = ka \int_0^b y^2 \, dy = \frac{kab^3}{3}$.

Therefore, $\bar{y} = M_{y=0}/m = 2b/3$, and the centre of mass of the plate is $(a/2, 2b/3)$.

The plate would be balanced if supported at this point.

For distribu $m = ka \int_0^b y \, dy = \frac{\kappa a b^3}{2}$,
 $M_{y=0} = ka \int_0^b y^2 \, dy = \frac{k a b^3}{3}$.

Therefore, $\bar{y} = M_{y=0}/m = 2b/3$, and the centre of mass of the plate is $(a/2, 2b/3)$.

The plate would be balanced if supported at this point.

For distr The plate would be balanced if supported at this point.

For distributions of mass in three-dimensional space one defines, analogo

ments $M_{x=0}$, $M_{y=0}$, and $M_{z=0}$ of the system of mass about the planes x

and $z =$ This point.

al space one defines, analogously, the mo-

a of mass about the planes $x = 0$, $y = 0$,
 $(\bar{x}, \bar{y}, \bar{z})$ where
 $\bar{z} = \frac{M_{z=0}}{m}$,
 $+m_n$. Again, the sums are replaced with
 \bar{z} . For distributions of mass in three-dimensional space one defines, analogously, the ments $M_{x=0}$, $M_{y=0}$, and $M_{z=0}$ of the system of mass about the planes $x = 0$, y and $z = 0$, respectively. The centre of mass is $(\$

$$
\bar{x} = \frac{M_{x=0}}{m}, \qquad \bar{y} = \frac{M_{y=0}}{m}, \qquad \text{and} \qquad \bar{z} = \frac{M_{z=0}}{m},
$$

em of mass about the planes $x = 0$, $y =$
s is $(\bar{x}, \bar{y}, \bar{z})$ where
d $\bar{z} = \frac{M_{z=0}}{m}$,
 $\cdots + m_n$. Again, the sums are replaced wiss.
s of a solid hemisphere of radius R ft if
above the base plane of the hemisphere
e m of mass about the planes $x = 0$, $y =$
is $(\bar{x}, \bar{y}, \bar{z})$ where
 $\frac{1}{\bar{z}} = \frac{M_{\bar{z}=0}}{m}$,
 $\cdot + m_n$. Again, the sums are replaced wiss.
of a solid hemisphere of radius R ft if the bove the base plane of the hemisphere of distributions of mass in three-differential space one defines, analogously, the mo-
entrs $M_{x=0}$, $M_{y=0}$, and $M_{z=0}$ of the system of mass about the planes $x = 0$, $y = 0$,
 $\bar{x} = \frac{M_{x=0}}{m}$, $\bar{y} = \frac{M_{y=0}}{m}$ density. The centre of mass is $(\bar{x}, \bar{y}, \bar{z})$ where
 $\bar{y} = \frac{M_{y=0}}{m}$, and $\bar{z} = \frac{M_{z=0}}{m}$,

ass: $m = m_1 + m_2 + \cdots + m_n$. Again, the sums are replaced with

cous distributions of mass.

Find the centre of mass of a .

noment about the base plane $z = 0$ is $dM_{z=0} = z dm = \rho_0 \pi (R^2 z^2 - z^4) dz$. The $\bar{x} = \frac{mx=0}{m}$, $\bar{y} = \frac{my=0}{m}$, and $\bar{z} = \frac{m_z=0}{m}$,
 m being the total mass: $m = m_1 + m_2 + \cdots + m_n$. Again, the sums are replaced with

integrals for continuous distributions of mass.
 EXAMPLE 7 Find the centre o *m* m
 m being the total mass: $m = m_1 + m_2 + \cdots + m_n$. Again, the sums are replaced with

integrals for continuous distributions of mass.
 EXAMPLE 7 Find the centre of mass of a solid hemisphere of radius *R* ft if its
 m being the total mass: $m = m_1 + m_2 + \cdots + m_n$. Again, the sums are replaced with
integrals for continuous distributions of mass.

EXAMPLE 7 Find the centre of mass of a solid hemisphere of radius R ft if its
density at hei integrals for continuous distributions of mass.
 EXAMPLE 7 Find the centre of mass of a solid hemisphere of radius R f density at height z ft above the base plane of the hemisphere of radius R f density at height z ft a a solid hemisphere of radius R ft if its
we the base plane of the hemisphere is
rtical axis (let us call it the z-axis), and
r to this axis. Therefore, the centre of
of the solid at height z above the base,
 $\frac{R^2 - z^2}{R^$ is $dV = \pi(R^2 - z^2) dz$, and its mass is $dm = \rho_0 z dV = \rho_0 \pi (R^2 z - z^3) dz$. Its / dz, and its mass is dm ^D 0z dV ^D 0.R² ced with
 R ft if its

phere is

xis), and

centre of

the base,
 $\frac{1}{2}$ volume
 $\frac{1}{2}$. Its
 dz . The m being the total mass: $m = m_1 + m_2 + \cdots + m_n$. Again, the sums are replaced with
integrals for continuous distributions of mass.
 EXAMPLE 7 Find the centre of mass of a solid hemisphere of radius R ft if its
 $\rho_0 z$ lb/ft *R* ft if its
isphere is
isphere is
axis), and
centre of
the base,
ts volume
 $3) dz$. Its
 $) dz$. The **EXAMPLE 7** Find the centre of mass of a solid interparational density at height z ft above the base $\rho_0 z$ lb/ft³.
 Solution The solid is symmetric about the vertical axis the density is constant in planes perpendic centre of mass of a solid hemisphere of r
t height z ft above the base plane of the
terric about the vertical axis (let us call it
nes perpendicular to this axis. Therefore
this axis. A slice of the solid at height z
disk mass must lie somewhere on this axis. A slice of the solid at height z above the bas
and having thickness dz, is a disk of radius $\sqrt{R^2 - z^2}$. (See Figure 7.33.) Its volum
is $dV = \pi (R^2 - z^2) dz$, and its mass is $dm = \rho_0 z d$ exis. A slice of the solid at height z above

of radius $\sqrt{R^2 - z^2}$. (See Figure 7.33.) 1

mass is $dm = \rho_0 z dV = \rho_0 \pi (R^2 z - z^2)$

0 is $dM_{z=0} = z dm = \rho_0 \pi (R^2 z^2 - z^4)$
 $= \rho_0 \pi \left(\frac{R^2 z^2}{2} - \frac{z^4}{4}\right)\Big|_0^R = \frac{\pi}{4} \rho_0$

$$
m = \rho_0 \pi \int_0^R (R^2 z - z^3) dz = \rho_0 \pi \left(\frac{R^2 z^2}{2} - \frac{z^4}{4}\right)\Big|_0^R = \frac{\pi}{4} \rho_0 R^4 \text{ lb.}
$$

$$
m = \rho_0 \pi \int_0^R (R^2 z - z^3) dz = \rho_0 \pi \left(\frac{R^2 z^2}{2} - \frac{z^4}{4}\right) \Big|_0^R = \frac{\pi}{4} \rho_0 R^4 \text{ lb.}
$$

The moment of the hemisphere about the plane $z = 0$ is

$$
M_{z=0} = \rho_0 \pi \int_0^R (R^2 z^2 - z^4) dz = \rho_0 \pi \left(\frac{R^2 z^3}{3} - \frac{z^5}{5}\right) \Big|_0^R = \frac{2\pi}{15} \rho_0 R^5 \text{ lb·ft.}
$$

The centre of mass therefore lies along the axis of symmetry of the hemisphere at
height $\bar{z} = M_{z=0}/m = 8R/15$ ft above the base of the hemisphere.
Thus, $\bar{z} = \frac{2\pi}{15}$ Find the centre of mass of a plate that occupies the region

be moment of the hemisphere about the plane $z = 0$ is
 $M_{z=0} = \rho_0 \pi \int_0^R (R^2 z^2 - z^4) dz = \rho_0 \pi \left(\frac{R^2 z^3}{3} - \frac{z^5}{5} \right) \Big|_0^R = \frac{2\pi}{15} \rho_0 R^5$ lb·ft.

the centre of mass therefore lies along the axis of symmetry of hemisphere about the plane $z = 0$ is
 $\int_0^R (R^2 z^2 - z^4) dz = \rho_0 \pi \left(\frac{R^2 z^3}{3} - \frac{z^5}{5} \right) \Big|_0^R = \frac{2\pi}{15} \rho_0 R^5$ lb-ft.

therefore lies along the axis of symmetry of the hemisphere at
 $n = 8R/15$ ft above the base is $\sigma(x)$. $M_{z=0} = \rho_0 \pi \int_0^R (R^2 z^2 - z^4) dz = \rho_0 \pi \left(\frac{R^2 z^3}{3} - \frac{z^5}{5}\right)\Big|_0^R = \frac{2\pi}{15} \rho_0 R^5$ lb-ft.

The centre of mass therefore lies along the axis of symmetry of the hemisphere at

height $\bar{z} = M_{z=0}/m = 8R/15$ ft above centre of mass therefore lies along the axis of symmetric that $\overline{z} = M_{z=0}/m = 8R/15$ ft above the base of the hemiss $\overline{AMPLE 8}$ Find the centre of mass of a plate the $a \le x \le b$, $0 \le y \le f(x)$, if the den $\overline{(x)}$.
AMPL

$$
dm = \sigma(x) f(x) dx
$$

SECTION 7.4: Mass, Moments, and C
and moment about
$$
x = 0
$$

$$
dM_{x=0} = x\sigma(x)f(x) dx.
$$

SECTION 7.4: Mass, Moments, and Centre of Mass 419
and moment about $x = 0$
 $dM_{x=0} = x\sigma(x) f(x) dx$.
Since the density depends only on x, the mass element dm has constant density, so the
y-coordinate of *its* centre of mass is SECTION 7.4: Mass, Moments, and Centre of Mass 419

and moment about $x = 0$
 $dM_{x=0} = x\sigma(x) f(x) dx$.

Since the density depends only on x, the mass element dm has constant density, so the

y-coordinate of *its* centre of mas y-coordinate of *its* centre of mass is at its midpoint: $\bar{y}_{dm} = \frac{1}{2}f(x)$. Therefore, the and Centre of Mass **419**
 419
 (b)
 f (x). Therefore, the SECTION 7.4: Mass, Moments, and Centre of N
and moment about $x = 0$
 $dM_{x=0} = x\sigma(x) f(x) dx$.
Since the density depends only on x, the mass element dm has constant den
y-coordinate of *its* centre of mass is at its midpoint: e the density depends only on x, the mass element dm
ordinate of its centre of mass is at its midpoint: \bar{y}_{dn}
hent of the mass element dm about $y = 0$ is
 $dM_{y=0} = \bar{y}_{dm} dm = \frac{1}{2} \sigma(x) (f(x))^2 dx$.
coordinates of the centre SECTION 7.4: Mass, Moments, and Centre of Mas

x,

the mass element dm has constant densit

mass is at its midpoint: $\bar{y}_{dm} = \frac{1}{2} f(x)$. There:
 dm about $y = 0$ is
 $\sigma(x) (f(x))^2 dx$. $dM_{x=0} = x\sigma(x) f(x) dx$.

Since the density depends only on x, the mass element dm has constant density, so the y-coordinate of *its* centre of mass is at its midpoint: $\bar{y}_{dm} = \frac{1}{2} f(x)$. Therefore, the moment of the mass epends only on x, the mass element dm has constaint that the centre of mass is at its midpoint: $\bar{y}_{dm} = \frac{1}{2} f(x)$

subsement dm about $y = 0$ is $m dm = \frac{1}{2} \sigma(x) (f(x))^2 dx$.
 \therefore the centre of mass of the plate are $\bar{x} = \$

$$
dM_{y=0} = \bar{y}_{dm} \, dm = \frac{1}{2} \sigma(x) (f(x))^2 \, dx.
$$

 $\frac{x=0}{m}$ and $\bar{y} = \frac{M_{y=0}}{m}$, , where centre of mass is at its midpoint: $\bar{y}_{dm} = \frac{1}{2} f(x)$.

ss element dm about $y = 0$ is
 $m dm = \frac{1}{2} \sigma(x) (f(x))^2 dx$.
 \vdots the centre of mass of the plate are $\bar{x} = \frac{M_{x=0}}{m}$ and
 $\sigma(x) f(x) dx$,
 $x\sigma(x) f(x) dx$,

$$
dM_{y=0} = \bar{y}_{dm} dm = \frac{1}{2} \sigma(x) (f(x))^{2} dx.
$$

coordinates of the centre of mass of the plate are $\bar{x} = \frac{M_{x=0}}{m}$

$$
m = \int_{a}^{b} \sigma(x) f(x) dx,
$$

$$
M_{x=0} = \int_{a}^{b} x \sigma(x) f(x) dx,
$$

$$
M_{y=0} = \frac{1}{2} \int_{a}^{b} \sigma(x) (f(x))^{2} dx.
$$

 $m = \int_a^b x\sigma(x)f(x) dx$,
 $M_{x=0} = \int_a^b x\sigma(x)(f(x))^2 dx$.
 Remark Similar formulas can be obtained if the density depends on y instead of

x, provided that the region admits a suitable horizontal area element (e.g., the region

mig $M_{x=0} = \int_{a}^{b} x\sigma(x) f(x) dx$,
 $M_{y=0} = \frac{1}{2} \int_{a}^{b} \sigma(x) (f(x))^{2} dx$.
 Remark Similar formulas can be obtained if the density depends on y instead of x, provided that the region admits a suitable horizontal area element (e $M_{x=0} = \int_{a}^{b} x\sigma(x) f(x) dx$,
 $M_{y=0} = \frac{1}{2} \int_{a}^{b} \sigma(x) (f(x))^{2} dx$.
 Remark Similar formulas can be obtained if the density depends on y instead of

x, provided that the region admits a suitable horizontal area element (**Remark** Similar formulas can be obtained if the density depends on y instead of x, provided that the region admits a suitable horizontal area element (e.g., the region might be specified by $c \le y \le d$, $0 \le x \le g(y)$). Findi **Remark** Similar formulas can be obtained if the density depends on y inst x, provided that the region admits a suitable horizontal area element (e.g., the might be specified by $c \le y \le d$, $0 \le x \le g(y)$). Finding centres of **EXERCISES** 7.4

Network of the specified that the might be specified that occupy region generally requires

in multivariable cannot in the mass and centre of mass for the systems in Exercises *x*, provided that the region admits a suitable

might be specified by $c \le y \le d$, $0 \le x \le$

that occupy regions specified by functions o

generally requires the use of "double integra-

in multivariable calculus. (See Sec that occupy regions specthed by functions of x,

generally requires the use of "double integrals."

in multivariable calculus. (See Section 14.7.)
 EXERCISES 7.4

in multivariable calculus. (See Section 14.7.)
 EXERCIS lmits a suitable horizontal area element (e.g., the region $\le d$, $0 \le x \le g(y)$). Finding centres of mass for plates by functions of x, but where the density depends on y, "double integrals." Such problems are therefore stu $\le d$, $0 \le x \le g(y)$). Finding centres of mass for plate
by functions of x, but where the density depends on y
"double integrals." Such problems are therefore studie
e Section 14.7.)
where r is the distance (in centimetres) admits a suitable horizontal area element (e.g., the region $y \le d$, $0 \le x \le g(y)$). Finding centres of mass for plates ed by functions of x, but where the density depends on y, of "double integrals." Such problems are there

- 1. Be alert for symmetries.

1. A straight wire along the x-axis from $x = 0$ to $x = L$; if the system of the systems in Exercises

1–16. Be alert for symmetries.

1. A straight wire of length L cm, where the density at dis
- generally requires the use of "

in multivariable calculus. (See
 XERCISES 7.4

the mass and centre of mass for the systems in Exercises

i. Be alert for symmetries.

A straight wire of length L cm, where the density at **EXERCISES 7.4**

in multivariable calculus. (See Section 14.7.)

in multivari **XERCISES 7.4**

the mass and centre of mass for the systems in Exercises

i. Be alert for symmetries.

A straight wire of length L cm, where the density at distance

sectional radius of the square

of areal density

secti **XERCISES 7.4**
the mass and centre of mass for the systems in Exercises
i. Be alert for symmetries.
A straight wire of length L cm, where the density at distance
s cm from one end is $\delta(s) = \sin \pi s/L$ g/cm
A straight wire alo **EXERCISES 7.4**

ind the mass and centre of mass for the systems in Exercises
 -16 . Be alert for symmetries.

1. A straight wire of length *L* cm, where the density at distance

3. A quarter-circular plate having radius 4. A quarter-circular plate of radius a occupying the radius of $x^2 + y^2 \le a^2$, $x \ge 0$, $y \ge 0$, having a region $\sigma(x) = \sigma_0 x$

5. A plate occupying the region $0 \le y \le 4 - x^2$ if the areal of region of the square of the re 6. Be alert for symmetries.

A straight wire of length L cm, where the density at distance
 $x = 0$ or $x = L$ if the

A straight wire along the x-axis from $x = 0$ to $x = L$ if the

density is constant δ_0 , but the cross-s s cm from one end is $\delta(s) = \sin \pi s/L$ g/cm

2. A straight wire along the x-axis from $x = 0$ to $x = L$ if the

density is constant δ_0 , but the cross-sectional radius of the

wire varies so that its value at x is $a + bx$

3.
-
-
-
- i. Be alert for symmetries.

A straight wire of length L cm, where the density at distar

s cm from one end is $\delta(s) = \sin \pi s/L$ g/cm

A straight wire along the x-axis from $x = 0$ to $x = L$ if th

density is constant δ_0 , b density is constant δ_0 , but the cross-sectional radius of the

3. A quarter-circular plate having radius a, constant areal density

5. A plater-circular plate of radius a occupying the region

4. A quarter-circular pl wire varies so that its value at x is $a + bx$
A quarter-circular plate having radius a, constant areal dens
 σ_0 , and occupying the region $x^2 + y^2 \le a^2$, $x \ge 0$, $y \ge 0$
A quarter-circular plate of radius a occupying th A quarter-circular plate having radius a, constant areal densit
 σ_0 , and occupying the region $x^2 + y^2 \le a^2$, $x \ge 0$, $y \ge 0$

A quarter-circular plate of radius a occupying the region
 $x^2 + y^2 \le a^2$, $x \ge 0$, $y \ge$ *σ*₀, and occupying the region $x^2 + y^2 \le a^2$, $x \ge 0$, $y \ge 0$ where *z* is the distance of radius *a* occupying the region
 4. A quarter-circular plate of radius *a* occupying the region from the centre c $x^2 + y^2$ **8.** $\sigma(x) = \sigma_0 x$ (be denoted by $\sigma(x) = \sigma_0 x$) (be denoted by $\sigma(x) = \sigma_0 x$) (c. A right-triangular plate with legs 2 m and 3 m if the areal density at any point *P* is 5*h* kg/m², *h* being the distance of *P* (be a cm
- $g/cm²$, where x is the distance from P to one edge of the square
- ,

-
- (See Section 14.7.)

where *r* is the distance (in centimetres) from *P* to one of the

diagonals of the square

9. A plate of areal density $\sigma(x)$ occupying the region $a \le x \le b$,
 $f(x) \le y \le g(x)$

10. A rectangular brick w where *r* is the distance (in centimetres) from *P* to one of the
diagonals of the square
A plate of areal density $\sigma(x)$ occupying the region $a \le x \le b$,
 $f(x) \le y \le g(x)$
A rectangular brick with dimensions 20 cm, 10 cm, and from *P* to one of the

ing the region $a \le x \le b$,

20 cm, 10 cm, and

where *x* is the distance

sity at *P* is *z* kg/m³,

plane at distance 2*R* m where *r* is the distance (in centimetres) from *P* to one of the
diagonals of the square
A plate of areal density $\sigma(x)$ occupying the region $a \le x \le b$
 $f(x) \le y \le g(x)$
A rectangular brick with dimensions 20 cm, 10 cm, and
 where *r* is the distance (in centimetres) from *P* to one of the diagonals of the square

9. A plate of areal density $\sigma(x)$ occupying the region $a \le x \le b$,
 $f(x) \le y \le g(x)$

10. A rectangular brick with dimensions 20 cm, where *r* is the distance (in centimetres) from *P* to one of the diagonals of the square
A plate of areal density $\sigma(x)$ occupying the region $a \le x \le b$, $f(x) \le y \le g(x)$
A rectangular brick with dimensions 20 cm, 10 cm, and A plate of areal density $\sigma(x)$ occupying the region $a \le x \le f(x) \le y \le g(x)$
A rectangular brick with dimensions 20 cm, 10 cm, and
5 cm if the density at P is kx g/cm³, where x is the distance
from P to one of the 10×5
- σ_0 , and occupying the region $x^2 + y^2 \le a^2$, $x \ge 0$, $y \ge 0$ where z is the distance from P to a plane at distance $2R$ m
A querter circular plate of redius a occupying the region from the centre of the ball ,
- where *r* is the distance (in centimetres) from *P* to one of the
diagonals of the square
A plate of areal density $\sigma(x)$ occupying the region $a \le x \le b$,
 $f(x) \le y \le g(x)$
A rectangular brick with dimensions 20 cm, 10 cm, and 12. **9.** A plate of areal density *σ*(*x*) occupying the region *a* ≤ *x* ≤ *b*, $f(x) ≤ y ≤ g(x)$

10. A rectangular brick with dimensions 20 cm, 10 cm, and 5 cm if the density at *P* is *kx* g/cm³, where *x* is the distan pying the region $a \le x \le b$,

ns 20 cm, 10 cm, and

n³, where x is the distance

ensity at P is z kg/m³,

a plane at distance 2R m

s a cm and height b cm if

, where z is the distance of

i a ball of radius a centred
- $f(x) \le y \le g(x)$
 $f(x) \le y \le g(x)$

adius of the

ant areal density
 $\therefore \ge 0, y \ge 0$

the region
 $f(x) \le y \le g(x)$

5 cm if the density at

from *P* to one of the

strep of the density
 $\therefore \ge 0, y \ge 0$

the region

from the centre A plate of acal density $\sigma(x)$ occupying the region $a \le x \le b$,
 $f(x) \le y \le g(x)$

A rectangular brick with dimensions 20 cm, 10 cm, and

5 cm if the density at P is kx g/cm³, where x is the distance

from P to one of the 1 10. A rectangular brick with dimensions 20 cm, 10 cm, and
5 cm if the density at *P* is *kx* g/cm³, where *x* is the distance
from *P* to one of the 10×5 faces
11. A solid ball of radius *R* m if the density at *P* The original state with dimensions 20 cm, to em, and

5 cm if the density at P is k x g/cm³, where x is the distance

from P to one of the 10×5 faces

A solid ball of radius R m if the density at P is z kg/m³,

wh the origin having as base the region $x^2 + y^2 \le a^2$, $x \ge 0$ in
the *xy*-plane, if the density at height *z* above the base is $\rho_0 z$
- $x = k$ and 3 m if the areal
and density
and the distance of the 10 x
and the density
 $y^2 \le a^2$, $x \ge 0$, $y \ge 0$
in the centre of the ball
and density
 $y \le 4 x^2$ if the areal
and 3 m if the areal
and 3 m if the areal
and the circular plate of radius *a* occupying the region
 $y^2 \le a^2$, $x \ge 0$, $y \ge 0$, having areal density
 $y^2 \le a^2$, $x \ge 0$, $y \ge 0$, having areal density
 y at (x, y) is ky
 y at (x, y) is ky
 y at (x, y) is From the density at P is $x \neq 0$ is $x \neq 0$ is $x \neq 0$ is $x \neq 0$.

11. A solid ball of radius R m if the density at P is z kg/m³, where z is the distance from P to a plane at distance $2R$ m from the ce kx g/cm³, where x is the distance of P from the axis of in or iadius *K* in *n* the density at *Y* is *z* kg/in,
s the distance from *P* to a plane at distance 2*R* m
centre of the ball
rcular cone of base radius *a* cm and height *b* cm if
y at point *P* is kz g/cm³, whe where 2 is the distance from *Y* to a plante at distance 2*K* in
from the centre of the ball
A right-circular cone of base radius *a* cm and height *b* cm if
the density at point *P* is kz g/cm³, where *z* is the dist A right-circular cone of base radius *a* cm and height *b* cm if
the density at point *P* is kz g/cm³, where *z* is the distance of
P from the base of the cone
The solid occupying the quarter of a ball of radius *a* A right-circular cone of base radius a cm and height b cm if
the density at point P is kz g/cm³, where z is the distance of
P from the base of the cone
The solid occupying the quarter of a ball of radius a centred at
 the density at point *P* is kz g/cm³, where z is the distance of *P* from the base of the cone
The solid occupying the quarter of a ball of radius *a* centred at
the origin having as base the region $x^2 + y^2 \le a^2$, x

- **420** CHAPTER 7 Applications of Integration
 15. A semicircular plate occupying the region $x^2 + y^2 \le a^2$, the star. T
 $y \ge 0$, if the density at distance *s* from the origin is taken to b
 ks g/cm²
 16. The wire CHAPTER 7 Applications of Integration

A semicircular plate occupying the region $x^2 + y^2 \le a^2$, the star. The radius
 $y \ge 0$, if the density at distance s from the origin is
 $ks g/cm^2$

The wire in Exercise 1 if it is ben ks g/cm²
-
- **16. II** 15. A semicircular plate occupying the region $x^2 + y^2 \le a^2$, the star.
 $y \ge 0$, if the density at distance *s* from the origin is taken to *ks* g/cm² *r*. Find
 16. The wire in Exercise 1 if it is bent in **17.** It is estimated that the density of matter in the neighbourhood
 17. It is estimated that the density of matter in the neighbourhood
 17. It is estimated that the density of matter in the neighbourhood

of a gas CHAPTER 7 Applications of Integration

A semicircular plate occupying the region $x^2 + y^2 \le a^2$,
 $y \ge 0$, if the density at distance *s* from the origin is
 $ks \text{ g/cm}^2$

The wire in Exercise 1 if it is bent in a semicirc CHAPTER 7 Applications of Integration

A semicircular plate occupying the region $x^2 + y^2 \le a^2$,
 $y \ge 0$, if the density at distance *s* from the origin is
 $k s$ g/cm²
 r . Find the approximate mass

The wire in Exerc

the star. The radius of the star is indeterminate but can be taken to be infinite since $\rho(r)$ decreases very rapidly for large r. Find the approximate mass of the star in terms of C and k. the star. The radius of the star is indeterminate but can be taken to be infinite since $\rho(r)$ decreases very rapidly for large r . Find the approximate mass of the star in terms of C and k . Find the average distance te but can be
rapidly for large
rrms of C and k.
tar of Exercise 17
r $dml/\int_0^\infty dm$,

 $y^2 \le a^2$, the star. The racking the star. The racking the star. The racking the star taken to be infinited that the approximate the centre of the star. the star. The radius of the star is indeterminate but can be
taken to be infinite since $\rho(r)$ decreases very rapidly for large
r. Find the average distance \bar{r} of matter in the star of Exercise 17
from the centre o the star. The radius of the star is indeterminate but can be
taken to be infinite since $\rho(r)$ decreases very rapidly for large
r. Find the approximate mass of the star in terms of C and k.
Find the average distance $\bar{r$ $\int_0^\infty r \, dm / \int_0^\infty dm$, the star. The radius of the star is indeterminate but can be
taken to be infinite since $\rho(r)$ decreases very rapidly for large
r. Find the approximate mass of the star in terms of C and k.
Find the average distance $\bar{r$ the star. The radius of the star is indeterminate b
taken to be infinite since $\rho(r)$ decreases very rap
r. Find the approximate mass of the star in term
Find the average distance \bar{r} of matter in the star
from the

7.5 Centroids

If matter is distributed uniformly in a system so that the density δ is constant, then that
If matter is distributed uniformly in a system so that the density δ is constant, then that
density cancels out of the nume If matter is distributed uniformly in a system so that the density δ is constant, then that
density cancels out of the numerator and denominator in sum or integral expressions
for coordinates of the centre of mass. In If matter is distributed uniformly in a system so that the density δ is constant, then that
density cancels out of the numerator and denominator in sum or integral expressions
for coordinates of the centre of mass. In If matter is distributed uniformly in a system so that the density δ is constant, then that density cancels out of the numerator and denominator in sum or integral expressions for coordinates of the centre of mass. In If matter is distributed uniformly in a system so that the density δ is constant, the density cancels out of the numerator and denominator in sum or integral expres for coordinates of the centre of mass. In such cases atter is distributed uniformly in a system so that the density δ is constant, then that ity cancels out of the numerator and denominator in sum or integral expressions ordinates of the centre of mass. In such cases the

If matter is distributed uniformly in a system so that the density δ is constant, then that density cancels out of the numerator and denominator in sum or integral expressions for coordinates of the centre of mass. In If matter is distributed uniformly in a system so that the density δ is constant, then that density cancels out of the numerator and denominator in sum or integral expressions for coordinates of the centre of mass. In If matter is distributed uniformly in a system so that the density δ is constant, then that density cancels out of the numerator and denominator in sum or integral expressions for coordinates of the centre of mass. In If matter is distributed uniformly in a system so that the density δ is constant, then that
density cancels out of the numerator and denominator in sum or integral expressions
for coordinates of the centre of mass. In *ns* cancels out of the namedator and denominator in sum of integral expressed cordinates of the centre of mass. In such cases the centre of mass depends of *hape* of the object, that is, on geometric properties of the re atter is distributed uniformly in a system so that the density δ is constant, then that
ity cancels out of the numerator and denominator in sum or integral expressions
oordinates of the centre of mass. In such cases th calculated using the same formulas as those used for
nsity (being constant) is taken to be unity, so the
ume of the region, and the moments are referred to
than of any mass occupying the region. If we set
in Example 8 of Exercise the unity, so the mass is just the
taken to be unity, so the mass is just the
he moments are referred to as **moments of**
ying the region. If we set $\sigma(x) = 1$ in the
7.4, we obtain the following result:
zion
 $x \$ centres of mass,
mass is just the
as **moments of**
 $r(x) = 1$ in the
g result:
, \bar{y}), where
 $(f(x))^2 dx$.

The centroid of a standard plane region
\nThe centroid of the plane region
$$
a \le x \le b
$$
, $0 \le y \le f(x)$, is (\bar{x}, \bar{y}) , where
\n
$$
\bar{x} = \frac{M_{x=0}}{A}, \quad \bar{y} = \frac{M_{y=0}}{A}, \text{ and}
$$
\n
$$
A = \int_{a}^{b} f(x) dx, \quad M_{x=0} = \int_{a}^{b} xf(x) dx, \quad M_{y=0} = \frac{1}{2} \int_{a}^{b} (f(x))^{2} dx.
$$
\nThus, for example, \bar{x} is the *average value* of the function *x* over the region.
\nThe centroids of some regions are obvious by symmetry. The centroid of a circular
\ndisk or an elliptical disk is at the centre of the disk. The centroid of a rectangle is at
\nthe centre also; the centre is the point of intersection of the diagonals. The centroid of
\nany region lies on any axes of symmetry of the region.

 $A = \int_{a}^{b} f(x) dx$, $M_{x=0} = \int_{a}^{b} xf(x) dx$, $M_{y=0} = \frac{1}{2} \int_{a}^{b} (f(x))^{2} dx$.

Thus, for example, \bar{x} is the *average value* of the function x over the region.

The centroids of some regions are obvious by symmetry. The EXAMPLE 1
 $\int_a^b f(x) dx$, $m_x = 0 - \int_a^b x f(x) dx$, $m_y = 0 - \frac{1}{2} \int_a^b (f(x)) dx$.

Hus, for example, \bar{x} is the *average value* of the function *x* over the region.

The centroids of some regions are obvious by symmetry. The cent age value of the function x over the region.

ons are obvious by symmetry. The centroid of a circular

ne centre of the disk. The centroid of a rectangle is at

point of intersection of the diagonals. The centroid of

ymm Thus, for example, \bar{x} is the *average value* of the function *x* over the region.

The centroids of some regions are obvious by symmetry. The centroid of a circular

disk or an elliptical disk is at the centre of the be centroid of a circular

id of a rectangle is at

onals. The centroid of

if-disk $-a \le x \le a$,

be half-disk.

c-coordinate is $\bar{x} = 0$,

the average value of The centroids of some regions are obvious by symmetry. The centroid of a circle disk or an elliptical disk is at the centre of the disk. The centroid of a rectangle the centre also; the centre is the point of intersection

Solution By symmetry, the centroid lies on the y-axis, so its x-coordinate is $\bar{x} = 0$.
(See Figure 7.35.) Since the area of the half-disk is $A = \frac{1}{2}\pi a^2$, the average value of **EXAMPLE 1** $0 \le y \le \sqrt{a^2 - x^2}$? Find the centroid of the half-disk.
 Solution By symmetry, the centroid lies on the y-axis, so its x-coordinate

(See Figure 7.35.) Since the area of the half-disk is $A = \frac{1}{2} \pi a^2$, t See Figure 7.35.) Since the area of the half-disk is $A = \frac{1}{2}\pi a^2$, the average value of
over the half-disk is
 $\bar{y} = \frac{M_{y=0}}{A} = \frac{2}{\pi a^2} \frac{1}{2} \int_{-a}^{a} (a^2 - x^2) dx = \frac{2}{\pi a^2} \frac{2a^3}{3} = \frac{4a}{3\pi}$.

the centroid o

$$
\bar{y} = \frac{M_{y=0}}{A} = \frac{2}{\pi a^2} \frac{1}{2} \int_{-a}^{a} (a^2 - x^2) \, dx = \frac{2}{\pi a^2} \frac{2a^3}{3} = \frac{4a}{3\pi}.
$$

 $\left(0, \frac{4a}{3\pi}\right)$.

Solution Here, the "region" is a one-dimensional curve, having length rather than
area. Again $\bar{x} = 0$ by symmetry. A short arc of length ds at height y on the semicircle
has moment $dM_{y=0} = y$ ds about $y = 0$. (See Figu $y = \sqrt{a^2 - x^2}$ area. Again $\bar{x} = 0$ **Solution** Here, the "region" is a one-dimensional curve, having length rather than area. Again $\bar{x} = 0$ by symmetry. A short arc of length ds at height y on the semicircle has moment $dM_{y=0} = y$ ds about $y = 0$. (See Fi **Solution** Here, the "region" is a one-dimensional curve, having length rather than area. Again $\bar{x} = 0$ by symmetry. A short arc of length ds at height y on the semicircle has moment $dM_{y=0} = y ds$ about $y = 0$. (See Figu SECTION 7.5: Centroids **421**

2, having length rather than

at height y on the semicircle

6.) Since

a dx **Solution** Here, the "region" is a one-dimensional curve, having length rat
area. Again $\bar{x} = 0$ by symmetry. A short arc of length ds at height y on the se
has moment $dM_{y=0} = y ds$ about $y = 0$. (See Figure 7.36.) Since
 the "region" is a one-dimensional curve, having length rather than

0 by symmetry. A short arc of length ds at height y on the semicircle
 $\left(\frac{dy}{dx}\right)^2 dx = \sqrt{1 + \frac{x^2}{a^2 - x^2}} dx = \frac{a dx}{\sqrt{a^2 - x^2}}$,
 $\left(\frac{a^2y}{a^2 - x^2}\right)^2$ o SECTION 7.5: Cen

a one-dimensional curve, having length

A short arc of length ds at height y on th

but $y = 0$. (See Figure 7.36.) Since
 $\sqrt{1 + \frac{x^2}{a^2 - x^2}} dx = \frac{a dx}{\sqrt{a^2 - x^2}}$,

semicircle, we have
 $a dx = \int_a^a dx = 2x^2$

$$
ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \frac{x^2}{a^2 - x^2}} dx = \frac{a \, dx}{\sqrt{a^2 - x^2}},
$$

and since $y = \sqrt{a^2 - x^2}$ on the semicircle, we have

$$
M_{y=0} = \int_{-a}^{a} \sqrt{a^2 - x^2} \frac{a \, dx}{\sqrt{a^2 - x^2}} = a \int_{-a}^{a} dx = 2a^2.
$$

has moment $dM_{y=0} = y ds$ about $y = 0$. (See Figure 7.36.) Since
 $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \frac{x^2}{a^2 - x^2}} dx = \frac{a dx}{\sqrt{a^2 - x^2}}$,

and since $y = \sqrt{a^2 - x^2}$ on the semicircle, we have
 $M_{y=0} = \int_{-a}^{a} \sqrt{a^2 - x^2} \frac{a dx}{\sqrt{a$ $\frac{1}{\pi a} = \frac{1}{\pi}$, and the centroid $2a$ $\frac{\pi}{\pi}$, and the centrol %, and the centroid of radius a is not of the semicircle $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx = \sqrt{1 + \frac{x^2}{a^2 - x^2}} dx = \frac{x^2}{\sqrt{a^2 - x^2}}$
and since $y = \sqrt{a^2 - x^2}$ on the semicircle, we have
 $M_{y=0} = \int_{-a}^{a} \sqrt{a^2 - x^2} \frac{a dx}{\sqrt{a^2 - x^2}} = a \int_{-a}^{a} dx = 2$
Since the length of the semicircle is πa $(0, \frac{2a}{\pi})$. Note that the centroid of = $\sqrt{1 + \frac{x^2}{a^2 - x^2}} dx = \frac{a dx}{\sqrt{a^2 - x^2}}$,
the semicircle, we have
 $\frac{a dx}{\sqrt{a^2 - x^2}} = a \int_{-a}^{a} dx = 2a^2$.
ircle is πa , we have $\bar{y} = \frac{M_{y=0}}{\pi a} = \frac{2a}{\pi}$, and the centroid
. Note that the centroid of a semicircle $y = \sqrt{a^2 - x^2}$ on the semicircle, we have
 $M_{y=0} = \int_{-a}^{a} \sqrt{a^2 - x^2} \frac{a dx}{\sqrt{a^2 - x^2}} = a \int_{-a}^{a} dx = 2a^2$.

Since the length of the semicircle is πa , we have $\bar{y} = \frac{M_{y=0}}{\pi a} = \frac{2a}{\pi}$, and the centroid

of the and since $y = \sqrt{a^2 - x^2}$ on the semicircle, we have
 $M_{y=0} = \int_{-a}^{a} \sqrt{a^2 - x^2} \frac{a dx}{\sqrt{a^2 - x^2}} = a \int_{-a}^{a} dx = 2a^2$.

Since the length of the semicircle is πa , we have $\bar{y} = \frac{M_{y=0}}{\pi a} = \frac{2a}{\pi}$,

of the semici $M_y = 0 = \int_{-a}^{\infty} \sqrt{a^2 - x^2} dx = a \int_{-a}^{\infty} dx = 2a$

Since the length of the semicircle is πa , we have $\bar{y} = \frac{M_y}{\pi a}$

of the semicircle is $\left(0, \frac{2a}{\pi}\right)$. Note that the centroid of a s

the same as that of half-Since the length of the semicircle is πa , we have $\bar{y} = \frac{M y = 0}{\pi a} = \frac{2a}{\pi}$, and the centroid of the semicircle is $\left(0, \frac{2a}{\pi}\right)$. Note that the centroid of a semicircle of radius a is not the same as that of

1 intersect.

The centroid of a triangle
The centroid of a triangle
The centroid of a triangle is the point at which all three medians of the triangle
intersect.
PROOF Recall that a median of a triangle is a straight line joinin **The centroid of a triangle**

The centroid of a triangle is the point at which all three medians of the triangle

intersect.
 PROOF Recall that a median of a triangle is a straight line joining one vertex of the

triang **Show that the centroid of a triangle is the point at which all three medians of the triangle intersect.**
PROOF Recall that a median of a triangle is a straight line joining one vertex of the triangle to the midpoint of medians. % as of the triangle

one vertex of the

a triangle, we will

st lie on all three
 $(a, m + c)$

Adopt a coordinate system where the median in question lies along the y-axis and such that a vertex of the triangle is at the origin. (See Figure 7.37.) Let the midpoint of the opposite side be $(0, m)$. Then the other two Adopt a coordinate system where the median in question lies along the y-axis and such that a vertex of the triangle is at the origin. (See Figure 7.37.) Let the midpoint of the opposite side be $(0, m)$. Then the other two Adopt a coordinate system where the median in question lies along the y-axis and
such that a vertex of the triangle is at the origin. (See Figure 7.37.) Let the midpoint
of the opposite side be $(0, m)$. Then the other two Adopt a coordinate system where the median in question
such that a vertex of the triangle is at the origin. (See Figu
of the opposite side be $(0, m)$. Then the other two vertices
coordinates of the form $(-a, m - c)$ and $(a, m$ Adopt a coordinate system where the median in question lies along the y-a

that a vertex of the triangle is at the origin. (See Figure 7.37.) Let the m

e opposite side be $(0, m)$. Then the other two vertices of the trian

$$
dM_{x=0} = -xh(-x) dx + xh(x) dx = 0,
$$

so the moment of the whole triangle about
$$
x = 0
$$
 is
\n
$$
M_{x=0} = \int_{x=-a}^{x=a} dM_{x=0} = 0.
$$

so the moment of the whole triangle about $x = 0$ is
 $M_{x=0} = \int_{x=-a}^{x=a} dM_{x=0} = 0.$

Therefore, the centroid of the triangle lies on the y-axis.
 Remark By simultaneously solving the equations of any two medians of a t so the moment of the whole triangle about $x = 0$ is
 $M_{x=0} = \int_{x=-a}^{x=a} dM_{x=0} = 0.$

Therefore, the centroid of the triangle lies on the y-axis.
 Remark By simultaneously solving the equations of any two medians of a t so the moment of the whole triangle about $x = 0$ is
 $M_{x=0} = \int_{x=-a}^{x=a} dM_{x=0} = 0.$

Therefore, the centroid of the triangle lies on the y-axis.
 Remark By simultaneously solving the equations of any two med

we can ve

 $M_{x=0} = \int_{x=-a}^{x=a} dM_{x=0} = 0.$

efore, the centroid of the triangle lies on the y-axis.
 aark By simultaneously solving the equations of any two medians of a triangle are verify the following formula:
 Coordinates of $M_{x=0} = \int_{x=-a} dM_{x=0} = 0.$

efore, the centroid of the triangle lies on the y-axis.
 Alark By simultaneously solving the equations of any two medians of a triangle,

an verify the following formula:
 Coordinates of th spanned that the triangle lies on the y-axis.
 Spanned Coordinates of the triangle
 Spanned Coordinates of the centroid of a triangle
 Coordinates of the centroid of a triangle

The coordinates of the centroid of a $M_{x=0} = \int_{x=-a}^{x=a} dM_{x=0} = 0.$

efore, the centroid of the triangle lies on the *y*-axis.
 aark By simultaneously solving the equations of any two medians of a triangle,

an verify the following formula:
 Coordinates Remark By simultan

we can verify the follow
 Coordinates of th

The coordinates of sponding coordinates of (x_1, y_1) , (
 $(\bar{x}, \bar{y}) = \left(\frac{x_1}{x_1}\right)$

$$
(\bar{x}, \bar{y}) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right).
$$

Coordinates of the centroid of a triangle

The coordinates of the centroid of a triangle are the averages of the corre-

sponding coordinates of the three vertices of the triangle. The triangle with

vertices $(x_1, y_1), ($ The coordinates of the centroid of a triangle are the averages of the corre-
sponding coordinates of the three vertices of the triangle. The triangle with
vertices $(x_1, y_1), (x_2, y_2),$ and (x_3, y_3) has centroid
 $(\bar{x}, \bar{y$ sponding coordinates of the three vertices of the triangle. The triangle with
vertices (x_1, y_1) , (x_2, y_2) , and (x_3, y_3) has centroid
 $(\bar{x}, \bar{y}) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.
If a region is a union of nonov subregions. sponding coordinates of the three vertices of the triangle. The triangle with
vertices $(x_1, y_1), (x_2, y_2),$ and (x_3, y_3) has centroid
 $(\bar{x}, \bar{y}) = \left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}\right)$.
If a region is a union of nonoverla If a region is a union of nonoverlapping subregions, then any moment of the region
is the sum of the corresponding moments of the subregions. This fact enables us
to calculate the centroid of the region if we know the cen egion
es us
II the
1, 2),
le, as
, and
there is the sum of the corresponding moments of the subregions. This fact ena
to calculate the centroid of the region if we know the centroids and areas of
subregions.
EXAMPLE 3 Find the centroid of the trapezoid with vertic is e subregions. This fact enables us
w the centroids and areas of all the
oid with vertices $(0, 0)$, $(1, 0)$, $(1, 2)$,
and a (nonoverlapping) triangle, as
nas centroid $(\bar{x}_S, \bar{y}_S) = (\frac{1}{2}, \frac{1}{2})$, and
, and its cen

 $\left(\frac{1}{2},\frac{1}{2}\right)$, and its area is $A_S = 1$. The triangle has area $A_T = \frac{1}{2}$, and its centroid is (\bar{x}_T, \bar{y}_T) , where **Solution** The trapezoid is the union of a square and a (nonoverlapping) triangle, as **EXAMPLE 3** Find the centroid of the trapezoid with vertices (0, 0), (1, 0), (1, 2),
 Solution The trapezoid is the union of a square and a (nonoverlapping) triangle, as

shown in Figure 7.38. By symmetry, the square ha and (0, 1).
 Solution The trapezoid is the union of a

shown in Figure 7.38. By symmetry, the so

its area is $A_S = 1$. The triangle has area A_i
 $\bar{x}_T = \frac{0+1+1}{3} = \frac{2}{3}$ and \bar{y}_T

Continuing to use subscripts

$$
\bar{x}_T = \frac{0+1+1}{3} = \frac{2}{3}
$$
 and $\bar{y}_T = \frac{1+1+2}{3} = \frac{4}{3}$.

$$
x_T = \frac{1}{3} \text{ and } y_T = \frac{1}{3} = \frac{1}{3}.
$$

Continuing to use subscripts *S* and *T* to denote the square and triangle, respectively,
we calculate

$$
M_{x=0} = M_{S;x=0} + M_{T;x=0} = A_S \bar{x}_S + A_T \bar{x}_T = 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} = \frac{5}{6},
$$

$$
M_{y=0} = M_{S;y=0} + M_{T;y=0} = A_S \bar{y}_S + A_T \bar{y}_T = 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{4}{3} = \frac{7}{6}.
$$

Since the area of the trapezoid is $A = A_S + A_T = \frac{3}{2}$, its centroid is

$$
(\bar{x}, \bar{y}) = \left(\frac{5}{6} \right) \frac{3}{2}, \frac{7}{6} \left(\frac{3}{2}\right) = \left(\frac{5}{9}, \frac{7}{9}\right).
$$

 2 ², it's control is

$$
(\bar{x}, \bar{y}) = \left(\frac{5}{6}\right)\frac{3}{2}, \frac{7}{6}\right)\left(\frac{3}{2}\right) = \left(\frac{5}{9}, \frac{7}{9}\right).
$$

 $M_{y=0} = M_{S;y=0} + M_{T;y=0} = A_S \bar{y}_S + A_T \bar{y}_T = 1 \times \frac{1}{2} + \frac{1}{2} \times \frac{4}{3} = \frac{7}{6}$.

ince the area of the trapezoid is $A = A_S + A_T = \frac{3}{2}$, its centroid is
 $(\bar{x}, \bar{y}) = \left(\frac{5}{6}\right)\left(\frac{3}{2}, \frac{7}{6}\right)\left(\frac{3}{2}\right) = \left(\frac{5}{9}, \frac{7}{9}\right)$ $y=0$ + M_1 ; $y=0$ - M_2 y_3 + M_1 y_1 - 1×2 + 2×3 - 6 .

Extrapezoid is $A = A_S + A_T = \frac{3}{2}$, its centroid is
 $\left(\frac{3}{2}, \frac{7}{6}\right) \left(\frac{3}{2}\right) = \left(\frac{5}{9}, \frac{7}{9}\right)$.

Find the centroid of the solid region ob parabola $y = 4 - x^2$. . $\frac{5}{6} \left(\frac{3}{2}, \frac{7}{6} \right) \left(\frac{3}{2}, \frac{7}{2} \right) = \left(\frac{5}{9}, \frac{7}{9} \right).$

Time in the centroid of the solid region obtained by rota

Time y-axis the first quadrant region lying between the x
 $-x^2$.

"
"
"
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"

 $(\bar{x}, \bar{y}) = \left(\frac{5}{6}\right)\left(\frac{3}{2}, \frac{7}{6}\right)\left(\frac{3}{2}\right) = \left(\frac{5}{9}, \frac{7}{9}\right).$
 EXAMPLE 4 Find the centroid of the solid region obtained by rotating about the y-axis the first quadrant region lying between the x-axis and the par $(\bar{x}, \bar{y}) = \left(\frac{5}{6}\right)\left(\frac{3}{2}, \frac{7}{6}\right)\left(\frac{3}{2}\right) = \left(\frac{5}{9}, \frac{7}{9}\right).$
 EXAMPLE 4 Find the centroid of the solid region obtained by rotating about the y-axis the first quadrant region lying between the x-axis and the par **EXAMPLE 4** Find the centroid of the solid region obtained
 EXAMPLE 4 Find the centroid of the solid region obtained

prabola $y = 4 - x^2$.
 Solution By symmetry, the centroid of the parabolic solid will metry, the y-ax

$$
dV = \pi x^2 dy = \pi (4 - y) dy
$$

SECTION 7.

\nand moment about the base plane

\n
$$
dM_{y=0} = y \, dV = \pi (4y - y^2) \, dy.
$$
\nHence, the volume of the solid is

and moment about the base plane

\n
$$
dM_{y=0} = y \, dV = \pi (4y - y^2) \, dy.
$$
\nHence, the volume of the solid is

\n
$$
V = \pi \int_0^4 (4 - y) \, dy = \pi \left(4y - \frac{y^2}{2} \right) \Big|_0^4 = \pi (16 - 8) = 8\pi,
$$
\nand its moment about $y = 0$ is

\n
$$
M_{y=0} = \pi \int_0^4 (4y - y^2) \, dy = \pi \left(2y^2 - \frac{y^3}{3} \right) \Big|_0^4 = \pi \left(32 - \frac{64}{3} \right)
$$

and moment about the base plane
\n
$$
dM_{y=0} = y \, dV = \pi (4y - y^2) \, dy.
$$
\nHence, the volume of the solid is
\n
$$
V = \pi \int_0^4 (4 - y) \, dy = \pi \left(4y - \frac{y^2}{2} \right) \Big|_0^4 = \pi (16 - 8) = 8\pi,
$$
\nand its moment about $y = 0$ is
\n
$$
M_{y=0} = \pi \int_0^4 (4y - y^2) \, dy = \pi \left(2y^2 - \frac{y^3}{3} \right) \Big|_0^4 = \pi \left(32 - \frac{64}{3} \right) = \frac{32}{3} \pi.
$$
\nHence, the centroid is located at $\bar{y} = \frac{32\pi}{3} \times \frac{1}{8\pi} = \frac{4}{3}.$
\n**Pappus's Theorem**
\nThe following theorem relates volumes or surface areas of revolution to the centroid of the region or curve being rotated.
\n**Pappus's Theorem**

 32π 1 4 $\frac{1}{3} \times \frac{1}{8\pi} = \frac{1}{3}$. 1 $\frac{1}{8\pi} = \frac{1}{3}.$ $\overline{3}$

 $m_{y=0} = n \int_0^{(4y-5y)}$

Hence, the centroid is located
 Pappus's Theorem

The following theorem relates

the region or curve being rotat
 T H E O R E M
 Pappus's Theorem

(a) If a plane region R lies of France, the centroid is located at $\bar{y} = \frac{32\pi}{3} \times \frac{1}{8\pi} =$
 Pappus's Theorem

The following theorem relates volumes or surface a

the region or curve being rotated.
 Pappus's Theorem

(a) If a plane region *R*

(a) If a plane region R lies on one side of a line L in that plane and is rotated about L to generate a solid of revolution, then the volume V of that solid is the product **Example 18 S 2018**
 Pappus's Theorem
 Compare region or curve being rotated.
 Pappus's Theorem

(a) If a plane region R lies on one side of a line L in that plane and is rotated about L to generate a solid of revol **FORMOTE 1989 THEOTEM**
 CONSTREMENT SOLUTE A PLACE ASSOCIATE A solid of revolution, then the volume V of that solid is the product of the area of R pus's Ineorem
following theorem relates volumes or surface areas of revolution to the centroid of
egion or curve being rotated.
Dus's Theorem
If a plane region R lies on one side of a line L in that plane and is rotat following theorem relates volumes or surface
egion or curve being rotated.
pus's Theorem
If a plane region R lies on one side of a line
L to generate a solid of revolution, then the
of the area of R and the distance trave **If a plane region R** lies on one side of a line L in that plane and is rotated about L to generate a solid of revolution, then the volume V of that solid is the product of the area of R and the distance travelled by the (a) If a plane region *R* lies on one side of a line *L* in that plane and is rotated about
 L to generate a solid of revolution, then the volume *V* of that solid is the product

of the area of *R* and the distance tra L to generate a solid of revolution, then the volume V of that solid is the product
of the area of R and the distance travelled by the centroid of R under the rotation;
that is,
 $V = 2\pi \bar{r} A$,
where A is the area of R, a

 $V = 2\pi \bar{r} A$

of the area of R and the distance travelled by the centroid of R under the rotation;
that is,
 $V = 2\pi \bar{r} A$,
where A is the area of R, and \bar{r} is the perpendicular distance from the centroid of
R to L.
If a plane cur where A is the area of R, and \bar{r} is the perpendicular distance from the centroid of R to L.
If a plane curve C lies on one side of a line L in that plane and is rotated about that line to generate a surface of revolu where *A* is the area of *R*, and \bar{r} is the perpendicular distance from the centroid of
 R to *L*.

(b) If a plane curve $\mathcal C$ lies on one side of a line *L* in that plane and is rotated about

that line to genera

 $S = 2\pi \bar{r} s$.

Length of \mathcal{C} times the distance travelled by the centroid of \mathcal{C} :
 $S = 2\pi \bar{r}s$,

where s is the length of the curve \mathcal{C} , and \bar{r} is the perpendicular distance from the

centroid of \mathcal{C} to the li (b) If a plane curve $\mathcal C$ lies on one side of a line L in that plane and is rotated about
that line to generate a surface of revolution, then the area S of that surface is the
length of $\mathcal C$ times the distance travell $S = 2\pi \bar{r}s$,
where s is the length of the curve \mathcal{C} , and \bar{r} is the perpendicular distance from the
centroid of \mathcal{C} to the line L.
PROOF We prove part (a). The proof of (b) is similar and is left as an e length of \mathcal{C} times the distance travelled by the centroid of \mathcal{C} :
 $S = 2\pi \bar{r}s$,

where s is the length of the curve \mathcal{C} , and \bar{r} is the perpendicular distance from the

centroid of \mathcal{C} to the li where *s* is the length of the curve \mathcal{C} , and \bar{r} is the perpendicular distance
centroid of \mathcal{C} to the line *L*.
PROOF We prove part (a). The proof of (b) is similar and is left as an exerc
Let us take *L* ength of the curve \mathcal{C} , and \bar{r} is the perpendicular distance from
the line L.
part (a). The proof of (b) is similar and is left as an exercise.
the y-axis and suppose that R lies between $x = a$ and x :
Thus $\bar{r$ the area of a thin strip of R at position x and having width dx. (See Figure 7.40.) This strip generates, on rotation about L, a cylindrical shell of volume $dV = 2\pi x dA$, so Let us take L to be the y-axis and suppose that R lies between $x = a$ and $x = b$
where $0 \le a < b$. Thus $\bar{r} = \bar{x}$, the x-coordinate of the centroid of R. Let dA denote Examples in the following examples interaction $\lambda = u$ and $\lambda = v$ and $\lambda = v$ and $\lambda = v$ and $\lambda = v$. Thus $\bar{r} = \bar{x}$, the *x*-coordinate of the centroid of *R*. Let *dA* denote the area of a thin strip of *R* at position

$$
V = 2\pi \int_{x=a}^{x=b} x \, dA = 2\pi M_{x=0} = 2\pi \bar{x}A = 2\pi \bar{r}A.
$$

where $0 \le u \le v$. Thus $t = x$, the x coordinate of the centroid of n . Eet u is the approach of α th strip of R at position x and having width dx . (See Figure 7.40.) This strip generates, on rotation about L , strip generates, on rotation about L, a cylindrical shell of volume $dV = 2\pi x dA$, so
strip generates, on rotation about L, a cylindrical shell of volume $dV = 2\pi x dA$, so
the volume of the solid of revolution is
 $V = 2\pi \int_{$ strip generates, on fold
about E, a cylindrical state is the volume of the solid of revolution is
 $V = 2\pi \int_{x=a}^{x=b} x dA = 2\pi M_{x=0} = 2\pi \bar{x}A$

As the following examples illustrate, Pappus's Theo

the centroid can be deter

EXAMPLE 5 Use Pappus's Theorem to find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$. $y = \sqrt{a^2 - x^2}.$

EXAMPLE 5 Use Pappus's Theorem to find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$.
 Solution The centroid of the semicircle lies on its axis of symmetry, the y-axis, so it is located at a point with coordinates $($ **EXAMPLE 5** Use Pappus's Theorem to find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$.
 Solution The centroid of the semicircle lies on its axis of symmetry, the y-axis, so it is located at a point with coordinates $($ **EXAMPLE 5** Use Pappus's Theorem to find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$.
 Solution The centroid of the semicircle lies on its axis of symmetry, the y-axis, so it is located at a point with coordinates (0 specifier
the set of the set of the sequare units,
square units, **EXAMPLE 5** Use Pappus's Theorem to find the centroid of the semicine $y = \sqrt{a^2 - x^2}$.
 Solution The centroid of the semicircle lies on its axis of symmetry, the y-z is located at a point with coordinates $(0, \bar{y})$. Si **Solution** The centroid of the semicircle lies on its axis of symmetry, the *y*-axis, so it is located at a point with coordinates (0, \bar{y}). Since the semicircle has length πa units and generates, on rotation about Iocated at a point with coordinates (0, \bar{y}). Since the semicircle has length πa units
nd generates, on rotation about the *x*-axis, a sphere having area $4\pi a^2$ square units,
e obtain, using part (b) of Pappus's **Solution** The centroid of the semicircle lies on its axis of symmetry, the *y*-axis, so it
is located at a point with coordinates $(0, \bar{y})$. Since the semicircle has length πa units
and generates, on rotation about t

$$
4\pi a^2 = 2\pi (\pi a)\bar{y}.
$$

obtation about the *x*-axis, a sphere having area $4\pi a^2$ square units,
 $x(t)$ of Pappus's Theorem,

(*i*) \bar{y} .

shown previously in Example 2.

Use Pappus's Theorem to find the volume and surface area of the

torus e 2.

d the volume and surface area of the

rotating the disk $(x - b)^2 + y^2 \le a^2$

10 in Section 7.1.)

which is at distance $\bar{r} = b$ units from

square units, the volume of the torus

and generates, on rotation about the x-axis, a sphere having area $4\pi a^2$ square units,
we obtain, using part (b) of Pappus's Theorem,
 $4\pi a^2 = 2\pi (\pi a)\bar{y}$.
Thus $\bar{y} = 2a/\pi$, as shown previously in Example 2.
EXAMPLE Thus $\bar{y} = 2a/\pi$, as shown previously in Example 2.
 EXAMPLE 6 Use Pappus's Theorem to find the volume and surface area

torus (doughnut) obtained by rotating the disk $(x-b)^2 + y^2$

about the y-axis. Here $0 < a < b$. (See is

 $V = 2\pi b (\pi a^2) = 2\pi^2 a^2 b$ cubic units.

EXAMPLE 6 Use Pappus's Theorem to find the volume and surface area of the torus (doughnut) obtained by rotating the disk $(x - b)^2 + y^2 \le a^2$ about the y-axis. Here $0 < a < b$. (See Figure 7.10 in Section 7.1.)
Solution The Form the circular boundary of the disk is at $(b, 0)$, which is at distance $\bar{r} = b$ units from
the axis of rotation. Since the disk is at $(b, 0)$, which is at distance $\bar{r} = b$ units from
the axis of rotation. Since the obtain be axis of rotation. Since the disk has area πa^2 square units, the volume of the axis of rotation. Since the disk has area πa^2 square units, the volume of the is $V = 2\pi b(\pi a^2) = 2\pi^2 a^2 b$ cubic units.
To find the Fo find the startice area 5 of the totals
rotate the circular boundary of the di
obtain
 $S = 2\pi b(2\pi a) = 4\pi^2 ab$ square

structures in Exercises 1–21.
15. The quantities to use Pappus's
16. The solid about the $x \ge 0, y$ 15. The quarter-circle arc $x^2 + y^2 = r^2$, $x \ge 0$, $y \ge 0$
16. The solution of the solution of the gradient b ; x  0; y  ⁰

$$
S = 2\pi b (2\pi a) = 4\pi^2 ab
$$
 square units.

Forma the staract area 3 of the torus (in case you
rotate the circular boundary of the disk, which ha
obtain
 $S = 2\pi b(2\pi a) = 4\pi^2 ab$ square units.
Find the centroids of the geometric structures in Exercises 1–21.
Be alert EXERCISES 7.5

Find the centroids of the geometric structures in Exercises 1–21.

Be alert for symmetries and opportunities to use Pappus's

16. The guarter-center.

16. The guarter-cline when the y-a

17. The quarter-dis Theorem. **EXERCISES 7.5**

ind the centroids of the geometric structures in Exercises 1–21.

ie alert for symmetries and opportunities to use Pappus's

heorem.
 1. The quarter-disk $x^2 + y^2 \le r^2$, $x \ge 0$, $y \ge 0$
 2. The regio **EXERCISES 7.5**

ind the centroids of the geometric structures in Exercises

ie alert for symmetries and opportunities to use Pappus's

heorem.

1. The quarter-disk $x^2 + y^2 \le r^2, x \ge 0, y \ge 0$

2. The region $0 \le y \le 9 - x^2$ **EXERCISES 7.5**

ind the centroids of the geometric structures in Exercises 1–21.

ie alert for symmetries and opportunities to use Pappus's

heorem.

1. The quarter-disk $x^2 + y^2 \le r^2, x \ge 0, y \ge 0$

2. The region $0 \le y \le$ **EXERCISES 7.5**

ind the centroids of the geometric structures in Exercises 1–21.

ie alert for symmetries and opportunities to use Pappus's

heorem.

1. The quarter-disk $x^2 + y^2 \le r^2$, $x \ge 0$, $y \ge 0$

2. The region 0 ind the centroids of the geometric structures in Exercises 1–21.

i.e alert for symmetries and opportunities to use Pappus's

16.

theorem.

1. The quarter-disk $x^2 + y^2 \le r^2$, $x \ge 0$, $y \ge 0$

2. The region $0 \le y \le 9 - x^$ ind the centroids of the geometric structures in Exercises 1–21.

ie alert for symmetries and opportunities to use Pappus's

16.

heorem.

1. The quarter-disk $x^2 + y^2 \le r^2$, $x \ge 0$, $y \ge 0$

2. The region $0 \le y \le 9 - x^2$

-
-
- 1 $\sqrt{1+x^2}$
-
- 5. The circular disk segment $0 \le y \le \sqrt{4 x^2} 1$
- **6.** The semi-elliptic disk $0 \le y \le b\sqrt{1-(x/a)^2}$
- 2. The region $0 \le y \le 9 x^2$

3. The region $0 \le x \le 1$, $0 \le y \le \frac{1}{\sqrt{1 + x^2}}$

4. The circular disk sector $x^2 + y^2 \le r^2$, $0 \le y \le x$

5. The circular disk segment $0 \le y \le \sqrt{4 x^2} 1$

6. The semi-elliptic disk $0 \le y \le b\$ 3. The region $0 \le x \le 1, 0 \le y \le \frac{1}{\sqrt{1 + x^2}}$

4. The circular disk sector $x^2 + y^2 \le r^2, 0 \le y \le x$

5. The circular disk segment $0 \le y \le \sqrt{4 - x^2} - 1$

6. The semi-elliptic disk $0 \le y \le b\sqrt{1 - (x/a)^2}$

7. The quadrilateral 4. The circular disk sector $x^2 + y^2 \le r^2$, $0 \le y \le x$

5. The circular disk segment $0 \le y \le \sqrt{4 - x^2} - 1$

6. The semi-elliptic disk $0 \le y \le b\sqrt{1 - (x/a)^2}$ —

7. The quadrilateral with vertices (in clockwise order) (0, 0),
 5. The circular disk segment $0 \le y \le \sqrt{4 - x^2} - 1$

6. The semi-elliptic disk $0 \le y \le b\sqrt{1 - (x/a)^2}$

7. The quadrilateral with vertices (in clockwise order) (0, 0),

(3, 1), (4, 0), and (2, -2)

8. The region bounded by the
- $y = \sqrt{1 (x 1)^2}$, the y-axis, and the line $y = x 2$
-
-
-
-
-
-
-
-

-
-
- 17. The region in Figure 7.41(a)
18. The region in Figure 7.41(b)
19. The region in Figure 7.41(c)
20. The region in Figure 7.41(d)
- 17. The region in Figure 7.41(a)

18. The region in Figure 7.41(b)

19. The region in Figure 7.41(c)

20. The region in Figure 7.41(d)

21. The region in Figure 7.41(d)
- 17. The region in Figure 7.41(a)

18. The region in Figure 7.41(b)

19. The region in Figure 7.41(c)

20. The region in Figure 7.41(d)

21. The solid obtained by rotating the plane region
 $0 \le y \le 2x x^2$ about the line 17. The region in Figure 7.41(a)

18. The region in Figure 7.41(b)

19. The region in Figure 7.41(c)

20. The region in Figure 7.41(d)

21. The solid obtained by rotating the plane region
 $0 \le y \le 2x - x^2$ about the line $0 \le y \le 2x - x^2$ about the line $y = -2$
- 17. The region in Figure 7.41(a)

18. The region in Figure 7.41(b)

19. The region in Figure 7.41(c)

20. The region in Figure 7.41(d)

21. The solid obtained by rotating the plane region
 $0 \le y \le 2x x^2$ about the line about the 1.41(a)

about the line y 2.41(c)

about the line y = -2

from (1, 0) to (0, 1) is rotated about the line

part of a conical surface. Find the area of The region in Figure 7.41(a)

The region in Figure 7.41(b)

The region in Figure 7.41(c)

The region in Figure 7.41(d)

The solid obtained by rotating the plane region
 $0 \le y \le 2x - x^2$ about the line $y = -2$

The line segm The region in Figure 7.41(c)

The region in Figure 7.41(d)

The solid obtained by rotating the plane region
 $0 \le y \le 2x - x^2$ about the line $y = -2$

The line segment from (1, 0) to (0, 1) is rotated about the li
 $x = 2$ to
-
- 20. The region in Figure 7.41(d)

21. The solid obtained by rotating the plane region
 $0 \le y \le 2x x^2$ about the line $y = -2$

22. The line segment from (1, 0) to (0, 1) is rotated about the line
 $x = 2$ to generate part o The solid obtained by folding the plane region
 $0 \le y \le 2x - x^2$ about the line $y = -2$

The line segment from (1, 0) to (0, 1) is rotated a
 $x = 2$ to generate part of a conical surface. Find

that surface.

The triangle w
- region $0 \le x \le \pi/2$, $0 \le y \le \sqrt{x} \cos x$.
-
- volume of that solid. It is below

An equilateral triangle of edge *s* cm is rotated about one of its

the right

edges to generate a solid. Find the volume and surface area of

find to 5 decimal places the coordinates of An equilateral triangle of edge *s* cm is rotated about one
edges to generate a solid. Find the volume and surface a
that solid.
Find to 5 decimal places the coordinates of the centroid
region $0 \le x \le \pi/2$, $0 \le y \le \sqrt{x} \cos x$
- edges to generate a solid. Find the volume and surface area
that solid.
25. Find to 5 decimal places the coordinates of the centroid of t
region $0 \le x \le \pi/2$, $0 \le y \le \sqrt{x} \cos x$.
26. Find to 5 decimal places the coordinates that solid.

figure. Observe that t

Find to 5 decimal places the coordinates of the centroid of the

region $0 \le x \le \pi/2$, $0 \le y \le \sqrt{x} \cos x$.

Find to 5 decimal places the coordinates of the centroid of the

region $0 < x \le \pi$ Find to 5 decimal places the coordinates of the centroid of the

region $0 \le x \le \pi/2$, $0 \le y \le \sqrt{x} \cos x$.

Find to 5 decimal places the coordinates of the centroid of the

region $0 < x \le \pi/2$, $\ln(\sin x) \le y \le 0$.

Find the centr include $0 \le x \le \pi/2$, $0 \le y \le \sqrt{x} \cos x$.
Find to 5 decimal places the coordinates of the centroid of
region $0 \le x \le \pi/2$, $\ln(\sin x) \le y \le 0$.
Find the centroid of the infinitely long spike-shaped region
lying between the x-axi 26. Find to 5 decimal places the coordinates of the centroid of the

region $0 < x \le \pi/2$, $\ln(\sin x) \le y \le 0$.

27. Find the centroid of the infinitely long spike-shaped region

27. Find the centroid of the infinitely long spi **EE 25.** Find to 5 decimal places the coordinates of the centroid of the stores **W** and **B**
 EE 26. Find to 5 decimal places the coordinates of the centroid of the transform of $\langle x \rangle \leq y \leq \sqrt{x} \cos x$.
 EE 26. Find the
	-
-
- Using between the *x*-axis and the curve $y = (x + 1)^{-3}$ and to
the right of the *y*-axis.
928. Show that the curve $y = e^{-x^2}(-\infty < x < \infty)$ generates a
surface of finite area when rotated about the *x*-axis. What
does this the right of the y-axis.

Show that the curve $y = e^{-x^2} (-\infty < x < \infty)$ generates a

submerged. (See F

surface of finite area when rotated about the x-axis. What

does this imply about the location of the centroid of this

s Show that the curve $y = e^{-x^2}(-\infty < x < \infty)$ generates a
surface of finite area when rotated about the x-axis. What
does this imply about the location of the centroid of this
infinitely long curve?
Dotain formulas for the co stability of a floating object $y \le x \le y$, y given that the boat time states of finite area when rotated about the x-axis. What does this imply about the location of the centroid of this somewhat infinitely long curve?
 someon time are when to due to the centroid of this square cross-section with infinitely long curve?

Obtain formulas for the coordinates of the centroid of this square cross-section with infinitely long curve?

Obtain fo the infinitely long curve?

In particular, will the

infinitely long curve?

Obtain formulas for the coordinates of the centroid of the

edge upward? Prove

plane region $c \le y \le d$, $0 < f(y) \le x \le g(y)$.

(Stability of a floati by the coordinates of the centroid of the

plane region $c \le y \le d$, $0 < f(y) \le x \le g(y)$.

Prove part (b) of Pappus's Theorem (Theorem 2).
 (Stability of a floating object) Determining the orientation

that a floating object Obtain formulas for the coordinates of the centroid of the

plane region $c \le y \le d$, $0 < f(y) \le x \le g(y)$. another symbol

Prove part (b) of Pappus's Theorem (Theorem 2).
 (Stability of a floating object) Determining the orie plane region $c \le y \le a, 0 < f(y) \le x \le g(y)$.

Trove part (b) of Pappus's Theorem (Theorem 2).

(Stability of a floating object) Determining the orientation

that a floating object will assume is a problem of critical

importanc Prove part (b) of Pappus's Theorem (Theorem 2).

(**Stability of a floating object**) Determining the orientation

that a floating object will assume is a problem of critical

importance to ship designers. Boats must be des (**Stability of a floating object**) Determining the orientation
that a floating object will assume is a problem of critical
importance to ship designers. Boats must be designed to float
stably in an upright position; if th that a floating object will assume is a problem of critical
importance to ship designers. Boats must be designed to float
stably in an upright position; if the boat tilts somewhat from
upright, the forces on it must be su importance to ship designers. Boats must be designed to float
stably in an upright position; if the boat tilts somewhat from
upright, the forces on it must be such as to right it again. The
two forces on a floating object It is an upright position; if the boat tilts somewhat from

ght, the forces on it must be such as to right it again. The

forces on a floating object that need to be taken into

unt are its weight W and the balancing buoy upright, the forces on it must be such as to right it again. The
two forces on a floating object that need to be taken into
account are its weight **W** and the balancing buoyant force
 $\mathbf{B} = -\mathbf{W}$. The weight **W** must two forces on a floating object that need to be taken into

account are its weight **W** and the balancing buoyant force

purposes as being applied at the centre of mass (CM) of the

object. The buoyant force, however, acts account are its weight **W** and the balancing buoyant force
 B = -**W**. The weight **W** must be treated for mechanical

purposes as being applied at the centre of mass (CM) of the

object. The buoyant force, however, acts $\mathbf{B} = -\mathbf{W}$. The weight **W** must be treated for mechanical
purposes as being applied at the centre of mass (CM) of the
object. The buoyant force, however, acts at the *centre of*
buoyancy (CB), which is the centre

The anti-tect of a hemispherical hull surmounted by a conical tower
supporting a navigation light. The buoy has a vertical axis of
symmetry. If it is upright, both the CM and the CB lie on this
line, as shown in the left h

The solid obtained by rotating the plane region
 $0 \le y \le 2x - x^2$ about the line $y = -2$

The line segment from (1, 0) to (0, 1) is rotated about the line
 $x = 2$ to generate part of a conical surface. Find the area of

tha 22. The line segment from (1, 0) to (0, 1) is rotated about the line
 $x = 2$ to generate part of a conical surface. Find the area of

that surface.

23. The triangle with vertices (0, 0), (1, 0), and (0, 1) is rotated

ab that surface.

23. The triangle with vertices (0, 0), (1, 0), and (0, 1) is rotated

about the line $x = 2$ to generate a certain solid. Find the

volume of that solid.

24. An equilateral triangle of edge s cm is rotated 22. The line segment from (1, 0) to (0, 1) is rotated about the line
 $x = 2$ to generate part of a conical surface. Find the area of

that surface.

a The triangle with vertices (0, 0), (1, 0), and (0, 1) is rotated

abou and to
allow the centre O of the right half of the figure. Thereas of
the right half of the figure. Thereas of
slightly from the vertical as
figure. Observe that the CM
of the
of the buoy, but the CB lies c
forces **W** and e a certain solid. Find the

lies below the centre O of the

interval about one of its

the right half of the figure. The

the volume and surface area of

sightly from the vertical as

figure. Observe that the CM

ordinat Figure 7.42

Is this upright flotation of the buoy stable? It is if the CM

lies below the centre O of the hemispherical hull, as shown in

the right half of the figure. To see why, imagine the buoy tilted

slightly from Figure 7.42
Is this upright flotation of the buoy stable? It is if the CM
lies below the centre O of the hemispherical hull, as shown in
the right half of the figure. To see why, imagine the buoy tilted
slightly from the v Figure 7.42
Is this upright flotation of the buoy stable? It is if the CM
lies below the centre O of the hemispherical hull, as shown in
the right half of the figure. To see why, imagine the buoy tilted
slightly from the v Figure 7.42
Is this upright flotation of the buoy stable? It is if the CM
lies below the centre O of the hemispherical hull, as shown in
the right half of the figure. To see why, imagine the buoy tilted
slightly from the v Figure 7.42
Is this upright flotation of the buoy stable? It is if the CM
lies below the centre O of the hemispherical hull, as shown in
the right half of the figure. To see why, imagine the buoy tilted
slightly from the v Figure 7.42
Is this upright flotation of the buoy stable? It is if the CM
lies below the centre O of the hemispherical hull, as shown in
the right half of the figure. To see why, imagine the buoy tilted
slightly from the v Figure 1.42
Is this upright flotation of the buoy stable? It is if the CM
lies below the centre O of the hemispherical hull, as shown in
the right half of the figure. To see why, imagine the buoy tilted
slightly from the Is this upright flotation of the buoy stable? It is if the CM
lies below the centre O of the hemispherical hull, as shown in
the right half of the figure. To see why, imagine the buoy tilted
slightly from the vertical as s below the centre O of the hemispherical hull, as shown in
ight half of the figure. To see why, imagine the buoy tilted
tily from the vertical as shown in the right half of the
re. Observe that the CM still lies on the axis the right half of the figure. To see why, imagine the buoy tilted
slightly from the vertical as shown in the right half of the
figure. Observe that the CM still lies on the axis of symmetry
of the buoy, but the CB lies on slightly from the vertical as shown in the right half of the
figure. Observe that the CM still lies on the axis of symmetry
of the buoy, but the CB lies on the vertical line through O. The
forces **W** and **B** no longer act

figure. Observe that the CM still lies on the axis of symmetry
of the buoy, but the CB lies on the vertical line through O. The
forces **W** and **B** no longer act along the same line, but their
torques are such as to rotate of the buoy, but the CB lies on the vertical line through O. The forces **W** and **B** no longer act along the same line, but their torques are such as to rotate the buoy back to a vertical upright position. If CM had been ab forces **W** and **B** no longer act along the same line, but their torques are such as to rotate the buoy back to a vertical upright position. If CM had been above O in the left figure, the torques would have been such as to torques are such as to rotate the buoy back to a vertical upright
position. If CM had been above O in the left figure, the
torques would have been such as to tip the buoy over once it
was displaced even slightly from the v position. If CM had been above O in the left figure, the torques would have been such as to tip the buoy over once it was displaced even slightly from the vertical. A wooden beam has a square cross-section and specific gra

In this section we present some examples of the use of integration to calculate quanti-
In this section we present some examples of the use of integration to calculate quanti-
ties arising in physics and mechanics. Figure 7.43

Supplications

The this section we present some examples of the use of integration

ties arising in physics and mechanics.

trough

Hydrostatic Pressure
The **pressure** p at depth h beneath the surface of a liexerted on a horizontal plane surface at that depth due to **Hydrostatic Pressure**
The **pressure** p at depth h beneath the surface of a liquid is the *force per unit area* exerted on a horizontal plane surface at that depth due to the weight of the liquid above it. Hence, p is giv **Hydrostatic Pressure**
The **pressure** p at depth h beneath the surface of a liquid is the *force per unit area* exerted on a horizontal plane surface at that depth due to the weight of the liquid above it. Hence, p is giv **Hydrostatic Pressure**
The **pressure** p at depth h beneath the surface of a liquid i
exerted on a horizontal plane surface at that depth due to the v
it. Hence, p is given by
 $p = \rho gh$,

 $p = \rho gh$,

Hydrostatic Pressure
The **pressure** p at depth h beneath the surface of a liquid is the force per unit area
exerted on a horizontal plane surface at that depth due to the weight of the liquid above
it. Hence, p is given **Hydrostatic Pressure**
The **pressure** p at depth h beneath the surface of a liquid is the *force per unit area*
exerted on a horizontal plane surface at that depth due to the weight of the liquid above
it. Hence, p is giv approximately, $\rho = 1,000 \text{ kg/m}^3$ and $g = 9.8 \text{ m/s}^2$, so the pressure at dep a liquid is the *force per unit area*
ue to the weight of the liquid above
leration produced by gravity where
t the surface of the earth we have,
, so the pressure at depth h m is
= 1 kg·m/s², the force that imparts it. Hence, p is given by
 $p = \rho gh$,

where ρ is the density of the liquid, and g is the acceleration produced by gravity where

the fluid is located. (See Figure 7.44.) For water at the surface of the earth we have,

ap duced by gravity where
 e of the earth we have,

sure at depth h m is
 h , the force that imparts

pressure at any depth

face is the same as that $p = \rho gh$,
where ρ is the density of the liquid, and g is the acceleration p
the fluid is located. (See Figure 7.44.) For water at the surf
approximately, $\rho = 1,000 \text{ kg/m}^3$ and $g = 9.8 \text{ m/s}^2$, so the p
 $p = 9,800h \text{ N$ the liquid, and g is the acceleration produced by gravity v
Figure 7.44.) For water at the surface of the earth we
kg/m³ and $g = 9.8$ m/s², so the pressure at depth h m
re is the newton (N); 1 N = 1 kg·m/s², the for

$$
p = 9,800h \, \text{N/m}^2
$$
.

The unit of force used here is the newton (N); $1 N = 1 kg·m/s²$, the force that imparts

 $p = \rho gh$,
 $p = \rho gh$,
 $p = \rho sh$ a density of the liquid, and g is the acceleration produced by gravity where

duid is located. (See Figure 7.44.) For water at the surface of the earth we have,
 αx oximately, $\rho = 1,000 \text{ kg$ where ρ is the density of the liquid, and g is the acceleration produced by gravity where
the fluid is located. (See Figure 7.44.) For water at the surface of the earth we have,
approximately, $\rho = 1,000 \text{ kg/m}^3$ and

where ρ is the density of the liquid, and g is the acceleration produced by gravity where
the fluid is located. (See Figure 7.44.) For water at the surface of the earth we have,
approximately, $\rho = 1,000 \text{ kg/m}^3$ and approximately, $\rho = 1,000 \text{ kg/m}^3$ and $g = 9.8 \text{ m/s}^2$, so the pressure at depth h m is
 $p = 9,800h \text{ N/m}^2$.

The unit of force used here is the newton (N); 1 N = 1 kg·m/s², the force that imparts

an acceleration of $p = 9,800h \text{ N/m}^2$.
The unit of force used here is the newton (N); $1 \text{ N} = 1 \text{ kg} \cdot \text{m/s}^2$, the force that imparts
an acceleration of 1 m/s^2 to a mass of 1 kg .
The molecules in a liquid interact in such a way The unit of force used here is the newton (N); $1 N = 1 kg \cdot m/s^2$, the force that imparts
an acceleration of $1 m/s^2$ to a mass of $1 kg$.
The molecules in a liquid interact in such a way that the pressure at any depth
acts equa The unit of force used here is the newton (N); $1 N = 1 kg \cdot m/s^2$, the force that imparts
an acceleration of $1 m/s^2$ to a mass of $1 kg$.
The molecules in a liquid interact in such a way that the pressure at any depth
acts equa an acceleration of 1 m/s² to a mass of 1 kg.

The molecules in a liquid interact in such a way that the pressure at any depth

acts equally in all directions; the pressure against a vertical surface is the same as that
 The unit of force used here is the newton (N); $1 N = 1 kg \cdot m/s^2$, the force that im
an acceleration of $1 m/s^2$ to a mass of $1 kg$.
The molecules in a liquid interact in such a way that the pressure at any of
acts equally in a gainst a horizontal surface at the same depth. This is **Pascal's principle**.

The total force exerted by a liquid on a horizontal surface (say, the bottom of a nk holding the liquid) is found by multiplying the area of th exerted by a liquid on a horizontal surface (say, the bottom of a
iid) is found by multiplying the area of that surface by the pressure
surface below the top of the liquid. For nonhorizontal surfaces,
re is not constant o tank holding the liquid) is found by multiplying the area of that surface by the pressure
at the depth of the surface below the top of the liquid. For nonhorizontal surfaces,
however, the pressure is not constant over the Solution A horizontal strip of the surface of the plate at depth h m and having width dh m (see Equal strip of the surface of the surface cannot be determined so easily. In this case we divide the surface into are a elem However, the pleasare is not constant over the whole surface, and the total force to be determined so easily. In this case we divide the surface into area elements dA at some particular depth h, and we then sum (i.e., i The surface into area elements dA , each
i.e., integrate) the corresponding force
r trough is a semicircular plate of radius
nward. If the trough is full, so that the
total force of the water on the plate.
of the plate a ever, the pressure is not constant over the whole surface, and
etermined so easily. In this case we divide the surface into a
me particular depth h, and we then sum (i.e., integrate) th
hents $dF = \rho gh dA$ to find the total

 $2\sqrt{R^2 - h^2} dh$ m². The force of the wate depth h, and we then sum (i.e., integrate) the corresponding force
depth h, and we then sum (i.e., integrate) the corresponding force
h dA to find the total force.
One vertical wall of a water trough is a semicircular pla **EXAMPLE 1** One vertical wall of a water trough is a semicircular p

water comes up to the top of the plate, find the total force of the water on
 Solution A horizontal strip of the surface of the plate at depth h m

wi semicircular plate of radius
ne trough is full, so that the
f the water on the plate.
at depth h m and having
i, hence, its area is $dA =$
Let $u = R^2 - h^2$,
 $du = -2h dh$

$$
dF = \rho gh \, dA = 2\rho gh \sqrt{R^2 - h^2} \, dh.
$$

$$
F = \int_{h=0}^{h=R} dF = 2\rho g \int_{0}^{R} h\sqrt{R^2 - h^2} dh
$$
 Let $u = R^2 - h^2$,
\n
$$
du = -2h dh
$$

\n
$$
\approx \frac{2}{3} \times 9,800R^3 \approx 6,533R^3 \text{ N}.
$$

\n**EXAMPLE 2** (Force on a dam) Find the total force on a section of a dam 100 m
\nlong and having a vertical height of 10 m, if the surface holding
\nback the water is inclined at an angle of 30° to the vertical and the water comes up to
\nthe top of the dam.
\n**Solution** The water in a horizontal layer of thickness dh m at depth h m makes
\ncontact with the dam along a slanted strip of width dh sec 30° = $(2/\sqrt{3}) dh$ m. (See

²

o

force on a section of a dam 100 m

t of 10 m, if the surface holding

ertical and the water comes up to

ness dh m at depth h m makes

h sec 30° = $(2/\sqrt{3}) dh$ m. (See

3) dh m², and the force of water

 $\approx \frac{2}{3} \times 9,800R^3 \approx 6,533R^3$ N.
 EXAMPLE 2 (Force on a dam) Find the total force on a section of a dam 100 m

long and having a vertical height of 10 m, if the surface holding

back the water is inclined at an angl **EXAMPLE 2** (Force on a dam) Find the total force on a section of a dam 100 m
long and having a vertical height of 10 m, if the surface holding
back the water is inclined at an angle of 30° to the vertical and the water c f a dam 100 m
urface holding
er comes up to
 $\frac{h}{3}$ h m. (See
force of water **EXAMPLE 2** (Force on a dam) Find the total force on a section of a dam 100 m
long and having a vertical height of 10 m, if the surface holding
back the water is inclined at an angle of 30° to the vertical and the water c section of a dam 100 m

if the surface holding

d the water comes up to

m at depth h m makes
 $= (2/\sqrt{3}) dh$ m. (See

, and the force of water

th N. **EXAMPLE 2** (Force on a dam) Find the total force long and having a vertical height of back the water is inclined at an angle of 30° to the vertic the top of the dam.
Solution The water in a horizontal layer of thicknes **AMPLE 2** (Force on a dam) Find the total form
long and having a vertical height of
the water is inclined at an angle of 30° to the vertic
op of the dam.
thion The water in a horizontal layer of thicknes
act with the da **on a dam**) Find the total force on a section of a dam 100 m
d having a vertical height of 10 m, if the surface holding
t an angle of 30° to the vertical and the water comes up to
horizontal layer of thickness dh m at dep

$$
dF = \rho g h \, dA = \frac{200}{\sqrt{3}} \times 1,000 \times 9.8h \, dh \approx 1,131,600h \, dh \, N.
$$

Work

Work
When a force acts on an object to move that object, it is said to have done work on the
object. The amount of work done by a constant force is measured by the product of the
force and the distance through which it **Work**
When a force acts on an object to move that object, it is said to have done work on the
object. The amount of work done by a constant force is measured by the product of the
force and the distance through which it WUI **K**
When a force acts on an object to move that object,
object. The amount of work done by a constant forc
force and the distance through which it moves the c
is in the direction of the motion.
work = force × distance

work $=$ force \times distance

in a force acts on an object to move that object, it is said to have done **work** on the et. The amount of work done by a constant force is measured by the product of the e and the distance through which it moves the objec object. The amount of work done by a constant force is measured by the product of the
force and the distance through which it moves the object. This assumes that the force
is in the direction of the motion.
work = force **EVALUATE:**

We head a force acts on an object to move that object, it is said to have done **work** on the object. The amount of work done by a constant force is measured by the product of the force and the distance throug is in the direction of the motion.

work = force × distance

Work is always related to a particular force. If other forces acting on an object cause

it to move in a direction opposite to the force F, then work is said to depect. The amount of work done by a constant force is measured by the product of the
force and the distance through which it moves the object. This assumes that the force
is in the direction of the motion.
work = force × a particular force. If other forces acting on an obpposite to the force *F*, then work is said to have
in the direction of the *x*-axis moves an object fr
nd that the force varies continuously with the po
 $F(x)$ is a conti b x = b on that axis and that the force varies continuously with the position x of

e object; that is, $F = F(x)$ is a continuous function. The element of work done

y the force in moving the object through a very short dist force in the direction of the *x*-axis moves an object from $x = a$
xis and that the force varies continuously with the position *x* of
 $F = F(x)$ is a continuous function. The element of work done
ing the object through a ver bbject; that is, $F = F(x)$ is a continuou
ne force in moving the object through a v
 $F(x) dx$, so the total work done by the
 $W = \int_{x=a}^{x=b} dW = \int_{a}^{b} F(x) dx$.
AMPLE 3 (**Stretching or compress**
force $F(x)$ required to ex
its lon

$$
W = \int_{x=a}^{x=b} dW = \int_{a}^{b} F(x) dx.
$$

by the force in moving the object through a very short distance from x to $x + dx$ is
 $dW = F(x) dx$, so the total work done by the force is
 $W = \int_{x=a}^{x=b} dW = \int_{a}^{b} F(x) dx$.
 EXAMPLE 3 (Stretching or compressing a spring) By Ho $W = \int_{x=a}^{b} dW = \int_{a}^{b} F(x) dx.$
 EXAMPLE 3 (Stretching or compressing a spring) By Hooke's Law, the force $F(x)$ required to extend (or compress) an elastic spring to x units longer (or shorter) than its natural length is **EXAMPLE 3** (Stretching or compressing a spring) By Hooke's Law, the force $F(x)$ required to extend (or compress) an elastic spring to x units longer (or shorter) than its natural length is proportional to x:
 $F(x) = kx$, w **EXAMPLE 3** (Stretching or compressing a spring) By Hooke's
force $F(x)$ required to extend (or compress) an elastic
x units longer (or shorter) than its natural length is proportional to x:
 $F(x) = kx$,
where k is the **sprin EXAMPLE 3** (Stretching or compressing a spring) By Hooke's Law, the
force $F(x)$ required to extend (or compress) an elastic spring to
x units longer (or shorter) than its natural length is proportional to x:
 $F(x) = kx$,
wh **3** (Stretching or compressing a sprint force $F(x)$ required to extend (or cot (or shorter) than its natural length is propor x,

e spring constant for the particular spring end a certain spring to 4 cm longer than it

do

$$
F(x) = kx,
$$

EXAMPLE 3 force $F(x)$ required to extend (or compress) an elastic spring to
x units longer (or shorter) than its natural length is proportional to x:
 $F(x) = kx$,
where k is the **spring constant** for the particular spring where k is the **spring constant** for the particular spring. If a force of 2,000 N is
required to extend a certain spring to 4 cm longer than its natural length, how much
work must be done to extend it that far?
Solution

$$
W = \int_0^4 kx \, dx = k \frac{x^2}{2} \bigg|_0^4 = 500 \frac{\text{N}}{\text{cm}} \times \frac{4^2 \text{ cm}^2}{2} = 4,000 \text{ N} \cdot \text{cm} = 40 \text{ N} \cdot \text{m}.
$$

Figure 7.47 Pumping water out of a
conical tank

EXAMPLE 4 (Work done to pump out a tank) Water fills a tank in the shape
of a right-circular cone with top radius 3 m and depth 4 m. How
much work must be done (against gravity) to pump all the water out of the tank ove **EXAMPLE 4** (Work done to pump out a tank) Water fills a tank
of a right-circular cone with top radius 3 m and dep
much work must be done (against gravity) to pump all the water out of
the top edge of the tank?
Solution ut a tank) Water fills a tank in the shape
with top radius 3 m and depth 4 m. How
co pump all the water out of the tank over
er at height h above the vertex of the tank
h by similar triangles. The volume of this **EXAMPLE 4** (Work done to pump ou

of a right-circular cone v

much work must be done (against gravity) to

the top edge of the tank?
 Solution A thin, disk-shaped slice of water

has radius r (see Figure 7.47), where **AMPLE 4** (work done to pamp out a tank)
of a right-circular cone with top range of the tank?
of a right-circular cone with top range of the tank?
tion A thin, disk-shaped slice of water at height
radius r (see Figure 7

has radius r (see Figure 7.47), where $r = \frac{3}{4}h$ by similar triangles. The volume of this For a right-circular cone with top radius 3 m and depth 4 m. How
much work must be done (against gravity) to pump all the water out of the tank over
the top edge of the tank?
Solution A thin, disk-shaped slice of water has radius r (see Figure 7.47), where $r = \frac{3}{4} h$ by similar triangles. The volume of this
slice is
 $dV = \pi r^2 dh = \frac{9}{16} \pi h^2 dh$,
and its weight (the force of gravity on the mass of water in the slice) is
 $dF = \rho g dV = \frac{9}{16$

$$
dV = \pi r^2 dh = \frac{9}{16} \pi h^2 dh,
$$

$$
dF = \rho g \, dV = \frac{9}{16} \rho g \, \pi h^2 \, dh.
$$

slice is
 $dV = \pi r^2 dh = \frac{9}{16} \pi h^2 dh$,

and its *weight* (the force of gravity on the mass of water in the slice) is
 $dF = \rho g dV = \frac{9}{16} \rho g \pi h^2 dh$.

The water in this disk must be raised (against gravity) a distance $(4 - h)$ and its weight (the force of gravity on the mass of water in the slice) is
 $dF = \rho g dV = \frac{9}{16} \rho g \pi h^2 dh$.

The water in this disk must be raised (against gravity) a distance $(4 - h)$ m by the

pump. The work required to do $dF = \rho g dV = \frac{9}{16} \rho g \pi h^2 dh.$
The water in this disk must be raised (against gravity) a distance $(4 - h)$ m by the
pump. The work required to do this is
 $dW = \frac{9}{16} \rho g \pi (4 - h) h^2 dh.$
The total work that must be done to empt *dh*.
aised (against gravity) a distance $(4 - h)$
is is
h.
2. to empty the tank is the sum (integral)
ths between 0 and 4 m:
dh

$$
dW = \frac{9}{16} \rho g \pi (4 - h) h^2 dh.
$$

$$
W = \int_0^4 \frac{9}{16} \rho g \pi (4h^2 - h^3) dh
$$

= $\frac{9}{16} \rho g \pi \left(\frac{4h^3}{3} - \frac{h^4}{4}\right)\Big|_0^4$
= $\frac{9\pi}{16} \times 1,000 \times 9.8 \times \frac{64}{3} \approx 3.69 \times 10^5$ N-m.

EXAMPLE 5 (Work to raise material into orbit) The gravitational force of
earth is given by

$$
F(h) = \frac{Km}{(R + h)^2},
$$

$$
F(h) = \frac{Km}{(R+h)^2},
$$

;

SECTION 7.6: Other Physical Applications **429**
where R is the radius of the earth and K is a constant that is independent of m and h.
Determine, in terms of K and R, the work that must be done against gravity to raise
an SECTION 7.6: Other Physical Applications
where R is the radius of the earth and K is a constant that is independent of m
Determine, in terms of K and R , the work that must be done against gravity t
an object fro SECTION 7.6: Other Physical Applications **429**
where *R* is the radius of the earth and *K* is a constant that is independent of *m* and *h*.
Determine, in terms of *K* and *R*, the work that must be done against gravity SECTION 7.6: Other Physical Applications 429
where *R* is the radius of the earth and *K* is a constant that is independent of *m* and *h*.
Determine, in terms of *K* and *R*, the work that must be done against gravity to SECTION 7.6: Other Physical Applications 429
Where *R* is the radius of the earth and *K* is a constant that is independent of *m* and *h*.
Determine, in terms of *K* and *R*, the work that must be done against gravity to Determine, in terms of *K* and *R*, the work that must be done against gravity to raise
an object from the surface of the earth to:
(a) a height *H* above the surface of the earth, and
(b) an infinite height above the sur

-
-

$$
dW = \frac{Km}{(R+h)^2} dh.
$$

Hint: The work done to raise the mass *m* from height *h* to height *h* + *dh* is

\n
$$
dW = \frac{Km}{(R+h)^2} dh.
$$
\nThe total work to raise it from height *h* = 0 to height *h* = *H* is

\n
$$
W = \int_0^H \frac{Km}{(R+h)^2} dh = \frac{-Km}{R+h} \Big|_0^H = Km \left(\frac{1}{R} - \frac{1}{R+H} \right).
$$
\nIf *R* and *H* are measured in metres and *F* is measured in newtons, then *W* is measured in newton-meters (N-m), or joules.

\nThe total work necessary to raise the mass *m* to an infinite height is

\n
$$
W = \int_0^\infty \frac{Km}{(R+h)^2} dh = \lim_{h \to 0} Km \left(\frac{1}{h} - \frac{1}{h} \right) = \frac{Km}{(R+h)^2}.
$$

$$
dW = \frac{Km}{(R+h)^2} dh.
$$
\n(a) The total work to raise it from height $h = 0$ to height $h = H$ is\n
$$
W = \int_0^H \frac{Km}{(R+h)^2} dh = \frac{-Km}{R+h} \Big|_0^H = Km \left(\frac{1}{R} - \frac{1}{R+H} \right).
$$
\nIf R and H are measured in metres and F is measured in newtons, then W is measured in newton-meters (N-m), or joules.\n(b) The total work necessary to raise the mass m to an infinite height is\n
$$
W = \int_0^\infty \frac{Km}{(R+h)^2} dh = \lim_{H \to \infty} Km \left(\frac{1}{R} - \frac{1}{R+H} \right) = \frac{Km}{R}.
$$

 $W = \int_0^{\infty} \frac{R + h}{(R + h)^2} dh = \frac{1}{R + h} \Big|_0^{\infty} = K m \Big(\frac{1}{R} - \frac{1}{R + H} \Big).$

If R and H are measured in metres and F is measured in newtons, then W is measured in newton-metres (N-m), or joules.

(b) The total work necessa e measured in metres and F is measured in newtons, then W is
wton-metres (N·m), or joules.
necessary to raise the mass m to an infinite height is
 $\frac{Km}{(R+h)^2} dh = \lim_{H \to \infty} Km \left(\frac{1}{R} - \frac{1}{R+H} \right) = \frac{Km}{R}$.
One end of a to travel without friction along the length of the tank. Between the piston as (See Figure 7.48.)

(a) When the denth of the water is y metres ($x < y < 1$), what force does it exert on

(a) When the denth of the water i (b) The total work necessary to raise the mass m to an infinite height is
 $W = \int_0^\infty \frac{Km}{(R+h)^2} dh = \lim_{H \to \infty} Km \left(\frac{1}{R} - \frac{1}{R+H} \right) = \frac{Km}{R}$.
 EXAMPLE 6 One end of a horizontal tank with cross-section a square of ed (b) The total work necessary to raise the mass m to
 $W = \int_0^\infty \frac{Km}{(R+h)^2} dh = \lim_{H \to \infty} Km \left(\frac{1}{R}\right)$
 EXAMPLE 6 One end of a horizontal tank w

length *L* metres is fixed while the

to travel without friction along the $W = \int_0^\infty \frac{Km}{(R+h)^2} dh = \lim_{H \to \infty} Km \left(\frac{1}{R} - \frac{1}{R+H} \right) = \frac{Km}{R}$.
 EXAMPLE 6 One end of a horizontal tank with cross-section a square of edge length *L* metres is fixed while the other end is a square piston free t $N = \int_0^{\infty} (R + h)^2$ and $\frac{H}{H + \infty}$
 AMPLE 6 One end of a horizontal tan

length *L* metres is fixed whis

avel without friction along the length of the

there is some water in the tank; its depth c

Figure 7.48.)

Wh **EXAMPLE 6** One end of a horizontal tank with cross-section a square of edge
to travel without friction along the length of the tank. Between the piston and the fixed
end there is some water in the tank; its depth depends **AMPLE 6** One end of a horizontal tank with cross-section a square of edge
length L metres is fixed while the other end is a square piston free
avel without friction along the length of the tank. Between the piston and th **AMPLE 6** One end of a horizontal tank with cross-section a square of edge
length *L* metres is fixed while the other end is a square piston free
avel without friction along the length of the tank. Between the piston and (See Figure 7.48.)
(a) When the depth of the water is y metres $(0 \le y \le L)$, what force does it exert on

- the piston?
- Assume no water leaks out but that trapped air can escape from the face of the tank out friction along the length of the tank. Between the piston and the fixed there is some water in the tank; its depth depends on the pos (b) If the piston is X metres from the fixed end of the tank when the water depth is

Solution
(a) When the depth of water in the tank is y m, a horizontal strip on the face of the (see Figure 7.48.)

(a) When the depth of the water is y metres $(0 \le y \le L)$, what force does it exert on

the piston?

(b) If the piston is X metres from the fixed end of the tank when the water depth is
 $L/2$ metres, ho When the depth of the water is y metres ($0 \le y \le L$), what force does it exert on
the piston?
If the piston is X metres from the fixed end of the tank when the water depth is
 $L/2$ metres, how much work must be done to for there is some water in the tank; its depth depends on the position of the piston.
Figure 7.48.)
When the depth of the water is y metres $(0 \le y \le L)$, what force does it exert on
the piston?
If the piston is X metres from t Figure 7.48.)

When the depth of the water is y metres ($0 \le y \le L$), what force does it exert on

the piston?

If the piston is X metres from the fixed end of the tank when the water depth is
 $L/2$ metres, how much work m L/2 metres, how much work must be done to
L/2 metres, how much work must be done to
that distance and hence cause the water level to
Assume no water leaks out but that trapped air
of
tion
When the depth of water in the 8 X metres from the fixed end of the tank when the water depth is
ow much work must be done to force the piston in further to halve
nd hence cause the water level to increase to fill the available space?
ter leaks out but **Solution**

(a) When the depth of water in the tank is y m, a horizontal strip on the face of the

piston at depth z below the surface of the water $(0 \le z \le y)$ and having height

dz has area $dA = L dz$. Since the pressure at When the depth of water in the tank is y m, a horizontal strip on the face of the
piston at depth z below the surface of the water $(0 \le z \le y)$ and having height
 dz has area $dA = L dz$. Since the pressure at depth z is $\rho gz =$ e face of the
aving height
,800z N/m^2 ,
force on the
ten the water
. But we are
ae in moving piston at depth z below the surface of the water $(0 \le z \le y)$ and having height
dz has area $dA = L dz$. Since the pressure at depth z is $\rho gz = 9,800z$ N/m²,
the force of the water on the strip is $dF = 9,800Lz dz$ N. Thus, th on at depth z below the surface of the water $(0 \le z \le$
as area $dA = L dz$. Since the pressure at depth z is
orce of the water on the strip is $dF = 9,800 Lz dz$ N
on is
 $F = \int_0^y 9,800 Lz dz = 4,900 L y^2 N$, where $0 \le$
e distance fro w the surface of the water $(0 \le z \le y)$ and having heigh

lz. Since the pressure at depth z is $\rho gz = 9,800z \text{ N/m}^2$

on the strip is $dF = 9,800 \text{ L}z dz \text{ N}$. Thus, the force on the

z $dz = 4,900 \text{ L} y^2 \text{ N}$, where $0 \le y \$

$$
F = \int_0^y 9,800 L z dz = 4,900 L y^2 N, \text{ where } 0 \le y \le L.
$$

dz has area $dA = L dz$. Since the pressure at depthe force of the water on the strip is $dF = 9,800 L$
piston is
 $F = \int_0^y 9,800 L z dz = 4,900 L y^2 N$, whe
If the distance from the fixed end of the tank to the
depth is y m, then the $F = \int_0^y 9,800 L z dz = 4,900 L y^2 N$, where $0 \le y \le L$.

If the distance from the fixed end of the tank to the piston is x m when the water

depth is y m, then the volume of water in the tank is $V = Lxy$ m³. But we are

given t

$$
dW = 4,900 \, Ly^2(-dx) = -4,900 \, L \, \frac{L^2 X^2}{4x^2} \, dx.
$$

$$
W = -\int_{X}^{X/2} 4,900 \frac{L^3 X^2}{4} \frac{dx}{x^2}
$$

= 4,900 $\frac{L^3 X^2}{4} \left(\frac{2}{X} - \frac{1}{X}\right) = 1,225 \text{ N} \cdot \text{m}.$

Potential Energy and Kinetic Energy
The units of energy are the same as those of work (force \times distance). We against a force may be regarded as storing up energy for future use or for co **Potential Energy and Kinetic Energy**
The units of energy are the same as those of work (force \times distance). Work done
against a force may be regarded as storing up energy for future use or for conversion
to other forms **Potential Energy and Kinetic Energy**
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energy are the same as those of work (force \times e
e may be regarded as storing up energy for future
s. Such stored energy is called **potential energy**
or compressing an elastic spring, we are d **Potential Energy and Kinetic Energy**

The units of energy are the same as those of work (force \times distance). Work done

against a force may be regarded as storing up energy for future use or for conversion

to other fo to other forms. Such stored energy is called **potential energy** (P.E.). For instance,
in extending or compressing an elastic spring, we are doing work against the tension
in the spring and hence storing energy in the spri

$$
P.E. = -\int_{a}^{b} F(x) dx.
$$

in extending or compressing an elastic spring, we are doing wo
in the spring and hence storing energy in the spring. When w
(variable) force $F(x)$ to move an object from $x = a$ to $x = b$, th
P.E. $= -\int_a^b F(x) dx$.
Since the wo

D. Example and nence storing energy in the spring. When work is done against a able) force $F(x)$ to move an object from $x = a$ to $x = b$, the energy stored is
P.E. $= -\int_a^b F(x) dx$.

e the work is being done against F, the si P.E. $= -\int_a^b F(x) dx$.

Since the work is being done against F, the signs of $F(x)$ and $b - a$ are opposite,

so the integral is negative; the explicit negative sign is included so that the calculated

potential energy will be P.E. $= -\int_{a}^{b} F(x) dx$.

Since the work is being done against *F*, the signs of *s*

so the integral is negative; the explicit negative sign is i

potential energy will be positive.

One of the forms of energy into which p $\int_{a}^{b} F(x) dx$.

is being done against *F*, the signs of *F*

is negative; the explicit negative sign is in

y will be positive.

forms of energy into which potential energ

the energy of motion. If an object of mas
 \therefore

$$
K.E. = \frac{1}{2}m v^2.
$$

so the integral is negative; the explicit negative sign is included so that the calculated
potential energy will be positive.

One of the forms of energy into which potential energy can be converted is **kinetic**

energy (gravitation are positive.

One of the forms of energy into which potential energy can be converted is **kinetic**
 energy (K.E.), the energy of motion. If an object of mass m is moving with velocity

v, it has kinetic ene One of the forms of energy into which potential energy cornergy (K.E.), the energy of motion. If an object of mass *n*, *v*, it has kinetic energy
 $K.E. = \frac{1}{2} m v^2$.

For example, if an object is raised and then dropped, **EVALUATE:** The energy of motion. It an object of mass m is moving with velocity has kinetic energy
 K.E. = $\frac{1}{2}m v^2$.

example, if an object is raised and then dropped, it accelerates downward under

ity as more an One of the forms of energy into which potential energy can be converted is **kinetic**
 energy (K.E.), the energy of motion. If an object of mass m is moving with velocity

v, it has kinetic energy

K.E. = $\frac{1}{2}m v^2$.
 So in motion. It an object of mass *m* is moving with
s raised and then dropped, it accelerates downward of the potential energy stored in it when it was
potential energy stored in a mass *m* as it moves are influence of For example, if an object is raised and then dropped, it accelerates downward under
gravity as more and more of the potential energy stored in it when it was raised is
converted to kinetic energy.
Consider the change in p K.E. $= \frac{1}{2}mv^2$.

For example, if an object is raised and then dropped, it accelerates downward under

gravity as more and more of the potential energy stored in it when it was raised is

converted to kinetic energy.
 example, if an object is raised and then c
ity as more and more of the potential en
erted to kinetic energy.
Consider the change in potential energy s
is from a to b under the influence of a for
P.E.(b) – P.E.(a) = $-\int_a^b$

$$
P.E.(b) - P.E.(a) = -\int_{a}^{b} F(x) dx.
$$

For example, it all solvents that then displayed in it when if
gravity as more and more of the potential energy stored in it when if
converted to kinetic energy.
Consider the change in potential energy stored in a mass m x-axis from *a* to *b* under the influence of a force $F(x)$ depending only on *x*:

P.E.(*b*) – P.E.(*a*) = $-\int_{a}^{b} F(x) dx$.

(The change in P.E. is negative if *m* is moving in the direction of *F*.) According to

Newton'

$$
F(x) = m \frac{dv}{dt}
$$
 (force = mass × acceleration).

P.E.(b) – P.E.(a) =
$$
-\int_{a}^{b} F(x) dx
$$
.
\n(The change in P.E. is negative if *m* is moving in the direction of *F*.) Accor
\nNewton's Second Law of Motion, the force $F(x)$ causes the mass *m* to acc
\nwith acceleration dv/dt given by
\n
$$
F(x) = m\frac{dv}{dt} \qquad \text{(force = mass × acceleration).}
$$
\nBy the Chain Rule we can rewrite dv/dt in the form
\n
$$
\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx},
$$
\nso $F(x) = mv\frac{dv}{dx}$. Hence,
\nP.E.(b) – P.E.(a) = $-\int_{a}^{b} mv\frac{dv}{dx} dx$
\n $= -m\int_{x=a}^{x=b} v dv$
\n $= -\frac{1}{2}mv^2\Big|_{x=a}^{x=b}$
\n $= K.E.(a) - K.E.(b).$
\nIt follows that
\nP.E.(b) + K.E.(b) = P.E.(a) + K.E.(a).
\nThis shows that the total energy (potential + kinetic) remains constant as the

 $P.E.(b) + K.E.(b) = P.E.(a) + K.E.(a).$

 $= -m \int_{x=a}^{x=b} v dv$
 $= -\frac{1}{2} m v^2 \Big|_{x=a}^{x=b}$
 $=$ K.E.(a) – K.E.(b).

It follows that

P.E.(b) + K.E.(b) = P.E.(a) + K.E.(a).

This shows that the total energy (potential + kinetic) remains constant as the mass m

moves = $-m \int_{x=a} v dv$

= $-\frac{1}{2} m v^2 \Big|_{x=a}^{x=b}$

= K.E.(a) - K.E.(b).

It follows that

P.E.(b) + K.E.(b) = P.E.(a) + K.E.(a).

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moves under $J_{x=a}$
 $= -\frac{1}{2} m v^2 \Big|_{x=a}^{x=b}$
 $= K.E.(a) - K.E.(b).$

It follows that
 $P.E.(b) + K.E.(b) = P.E.(a) + K.E.(a).$

This shows that the total energy (potential + kinetic) remains constant as the mass m

moves under the influence of a force $= -\frac{1}{2}mv^2\Big|_{x=a}^{x=b}$
 $=$ K.E.(a) - K.E.(b).

It follows that

P.E.(b) + K.E.(b) = P.E.(a) + K.E.(a).

This shows that the total energy (potential + kinetic) remains constant as the mass m

moves under the influence

EXAMPLE 7 (Escape velocity) Use the result of Example 5 together with the
following known values,
a) the radius R of the earth is about 6,400 km, or 6.4×10^6 m, SECTION 7.6: Other Physical Application
(**Escape velocity**) Use the result of Example 5 together
following known values,
the earth is about 6,400 km, or 6.4×10^6 m,
of gravity g at the surface of the earth is about 9. **EXAMPLE 7** (Escape velocity) Use the result of Example 5 together with the following known values,

(a) the radius R of the earth is about 6;400 km, or 6:4 \times 10⁶ m,

(b) the acceleration of gravity g at the surface

(**EXAMPLE 7** (**Escape velocity**) Use the result of Example 5 together with the following known values,

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(a) the radius R of the earth is about 6,400 km, or 6.4×10^6 m, (b) the acceleration of gravity g at the surface of the earth is a Sollowing known values,

(a) the radius R of the earth is about 6,400 km, or 6.4×10^6 m,

(b) the acceleration of gravity g at the surface of the earth is about 9.8 m/s²,

to determine the constant K in the gravitat (a) the radius *R* of the earth is about 6,400 km, or 6.4 × 10⁶ m,

(b) the acceleration of gravity *g* at the surface of the earth is about

to determine the constant *K* in the gravitational force formula of Ex

this the surface of the earth. The **escape velocity** is the (minimum) speed that such a
projectile must have at firing to ensure that it will continue to move farther and farther
away from the earth and not fall back.
Solutio

$$
F = \frac{Km}{(R+0)^2} = \frac{Km}{R^2}.
$$

projectile must have at firing to ensure that it will continue to move farther and farther
away from the earth and not fall back.
 Solution According to the formula of Example 5, the force of gravity on a mass
 m kg a

$$
\frac{Km}{R^2} = mg \qquad \text{and} \quad K = gR^2.
$$

:

 $F = \frac{Km}{(R+0)^2} = \frac{Km}{R^2}$.
According to Newton's Second Law of Motion, this force is related to the acceleration
of gravity (g) there by the equation $F = mg$. Thus,
 $\frac{Km}{R^2} = mg$ and $K = gR^2$.
According to the Law of Co $F = \frac{Km}{(R+0)^2} = \frac{Km}{R^2}$.
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of gravity (g) there by the equation $F = mg$. Thus,
 $\frac{Km}{R^2} = mg$ and $K = gR^2$.
According to the Law of Conservation of Energy, the proj According to Newton's Second Law of Motion, this force i
of gravity (g) there by the equation $F = mg$. Thus,
 $\frac{Km}{R^2} = mg$ and $K = gR^2$.
According to the Law of Conservation of Energy, the pro-
kinetic energy at firing to $\frac{R^2}{R^2} - m_g$ and $K = gK$.

According to the Law of Conservation of Energy, the

kinetic energy at firing to do the work necessary to ra

By the result of Example 5, this required energy is Kn

projectile is v, we wan

$$
\frac{1}{2}mv^2 \ge \frac{Km}{R}.
$$

$$
v \ge \sqrt{\frac{2K}{R}} = \sqrt{2gR} \approx \sqrt{2 \times 9.8 \times 6.4 \times 10^6} \approx 1.12 \times 10^4
$$
 m/s.

projectile is v, we want
 $\frac{1}{2}mv^2 \ge \frac{Km}{R}$.

Thus, v must satisfy
 $v \ge \sqrt{\frac{2K}{R}} = \sqrt{2gR} \approx \sqrt{2 \times 9.8 \times 6.4 \times 10^6} \approx 1.12 \times 10^4$ m/s.

Thus, the escape velocity is approximately 11.2 km/s and is independent of the $\frac{1}{2}mv^2 \ge \frac{Km}{R}$.

Thus, v must satisfy
 $v \ge \sqrt{\frac{2K}{R}} = \sqrt{2gR} \approx \sqrt{2 \times 9.8 \times 6.4 \times 10^6} \approx 1.12 \times 10^4$ m/s.

Thus, the escape velocity is approximately 11.2 km/s and is independent of the mass

m. In this calculat $\frac{1}{2}mv^2 \ge \frac{\Delta m}{R}$.

Thus, v must satisfy
 $v \ge \sqrt{\frac{2K}{R}} = \sqrt{2gR} \approx \sqrt{2 \times 9.8 \times 6.4 \times 10^6} \approx 1.12 \times 10^4$ m/s.

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m. In this calculat Thus, v must satisfy
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Thus, the escape velocity is approximately 11.2 km/s and is independent of the mas
 m. In this calculation we have neglected an $v \ge \sqrt{\frac{R}{R}} =$
Thus, the escape v
m. In this calculati
Such resistance de
force. The effect c
initial kinetic energ

EXERCISES 7.6
A tank has a square base 2 m on each side and vertical sides
6 m high. If the tank is 1. Thus, the escape velocity is approximately 11.2 km
 m . In this calculation we have neglected any air resi

Such resistance depends on velocity rather than on

force. The effect of such resistance would be to us

init *m*. In this calculation we have neglected any air ress

Such resistance depends on velocity rather than on

force. The effect of such resistance would be to u

initial kinetic energy and so raise the escape veloci
 XERC

- Such resistance depends on velocity rather than on po
force. The effect of such resistance would be to use initial kinetic energy and so raise the escape velocity.

XERCISES 7.6
A tank has a square base 2 m on each side force. The effect of such resistant
initial kinetic energy and so raise
 $XERC1SES 7.6$
A tank has a square base 2 m on each side and vertical sides
6 m high. If the tank is filled with water, find the total force
exerted by
- **EXERCISES 7.6**
 EXERCISES 7.6
 EXERCISES 7.6
 EXERCISES 7.6
 EXERCISES 7.6
 EXERCISES 7.6
 ORDINATELATE ASSES more of the tank is filled with water, find the total force

exerted by the water (a) on the botto **XERCISES 7.6**
A tank has a square base 2 m on each side and vertical sides
6 m high. If the tank is filled with water, find the total force
exerted by the water (a) on the bottom of the tank and (b) on
one of the four ve
-

- 32 CHAPTER 7 Applications of Integration
4. A pyramid with a square base, 4 m on each side and four
equilateral triangular faces, sits on the level bottom of a lake at
a place where the lake is 10 m deep. Find the total f CHAPTER 7 Applications of Integration
A pyramid with a square base, 4 m on each side and four
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a place where the lake is 10 m deep. Find the total force of CHAPTER 7 Applications of Integration

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- **32** CHAPTER 7 Applications of Integration
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equilateral triangular faces, sits on the level bottom of a lake at
a place where the lake is 10 m deep. Find the total force of the
hock on a canal has a gate in th A pyramid with a square base, 4 m on each side and four
equilateral triangular faces, sits on the level bottom of a lake at
a place where the lake is 10 m deep. Find the total force of the
water on each of the triangular f 4. A pyramid with a square base, 4 m on each side an equilateral triangular faces, sits on the level bottom a place where the lake is 10 m deep. Find the total water on each of the triangular faces.
5. A lock on a canal h id with a square base, 4 m on each side and four
al triangular faces, sits on the level bottom of a lake at
there the lake is 10 m deep. Find the total force of the
each of the triangular faces.
an a canal has a gate in t A pyramd with a square base, 4 m on each side and tour
equilateral triangular faces, sits on the level bottom of a lake at
a place where the lake is 10 m deep. Find the total force of the
water on each of the triangular f equilateral triangular faces, sits on the level bottom of a lake at
a place where the lake is 10 m deep. Find the total force of the
water on each of the triangular faces.
A lock on a canal has a gate in the shape of a ve **EXECUTE THE CONDUCE THE CONDUCE CONDENSATE C** A lock on a canal has a gate in the shape of a vertical rectangle

5 m wide and 20 m high. If the water on one side of the gate

comes up to the top of the gate, and the water on the other side

comes only 6 m up the gate 3. Find the total work that must be done to pump all the water in

8. Find the total Departual work that must be exerted to hold the gate in place.

6. If 100 N-cm of work must be done to compress an elastic

spring to 3
-
-
- pool.
-
- It foo N-ein of work mast be done to compress an easas spring to 3 cm shorter than its natural length, how muce must be done to compress it 1 cm further?
Find the total work that must be done to pump all the v
the tank of some water in the tank between the piston and the fixed end;

the child work that must be done to pump all the water in

Find the work that must be done to pump all the water in a full

fried the work that must be done to Find the total work that must be done to pump all the water in

the swimming pool of Exercise 2 out over the top edge of the

Find the work that must be done to pump all the water in a full

to reduce the distance

pool.
 the swimming pool of Exercise 2 out over the top edge of the

pool.

Find the work that must be done to pump all the water in a full

the mispherical bowl of radius a m to a height h m above the

air can escape. Hint: T
 pool.

Find the work that must be done to pump all the water in a full

tend the mispherical bowl of radius a m to a height h m above the

tend is a fixed interval cylindrical tank has radius R m. One end of the

tend of Find the work that must be done to pump all the water in a full
hemispherical bowl of radius a m to a height h m above the
top of the bowl.
A horizontal cylindrical tank has radius R m. One end of the
tank is a fixed disk Some water in the tank between the piston and the fixed end;

State domestics and the piston of the piston. What force

does the water exert on the piston when the surface of the

water is y m $(-R \le y \le R)$ above the centre

- Find the total 20 out over the top of the swimming pool of Exercise 2 out over the top edge of the top edge of the top edge of the tank of Exercise 2 out over the top edge of the swimming pool of Exercise 2 out over the t **6.** If 100 N·cm of work must be done to compress an elastic

spring to 3 cm shorter than its natural length, how much work

must be done to compress it 1 cm further?
 11. Continuing the previous profile that of Exercis If 100 N cm of work must be done to compress an elastic

spring to 3 cm shorter than its natural length, how much work

must be done to compress it 1 cm further?

Find the total work that must be done to pump all the wate **10. 11.** A horizontal cylindrical tank has radius R m free to travel along the lank be the periods that must be done to pump all the water in the term of the point of Exercise 2 out over the top edge of the swimming po Find the total work that must be done to pump all the water in

Find the total work that must be done to pump all the water in

the swimming pool of Exercise 2 out over the top edge of the

pool.

Find the work that must Find the total work that must be done to pump all the water in

the tank of Exercise 1 out over the top of the tank.

Find the total work that must be done to pump all the water in

the swimming pool of Exercise 2 out ove Figure 7.50
 I 11. Continuing the previous problem, suppose that when the piston is X m from the fixed end of the tank the water level is at the centre of the piston face. How much work must be done to reduce the dist Figure 7.50

Continuing the previous problem, suppose that when the

piston is X m from the fixed end of the tank the water level is

at the centre of the piston face. How much work must be done

to reduce the distance fr Figure 7.50

Continuing the previous problem, suppose that when the

piston is X m from the fixed end of the tank the water level is

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piston is X m from the fixed end of the tank the water level is

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to reduce the distance fr Figure 7.50
Continuing the previous problem, suppose that when the
piston is X m from the fixed end of the tank the water level is
at the centre of the piston face. How much work must be done
to reduce the distance from Figure 7.50
Continuing the previous problem, suppose that when the
piston is X m from the fixed end of the tank the water level is
at the centre of the piston face. How much work must be done
to reduce the distance from t Figure 7.50
Continuing the previous problem, suppose that when the
piston is X m from the fixed end of the tank the water level is
at the centre of the piston face. How much work must be done
to reduce the distance from Figure 7.50
Continuing the previous problem, suppose that when the
piston is X m from the fixed end of the tank the water level is
at the centre of the piston face. How much work must be done
to reduce the distance from t Continuing the previous problem, suppose that when the piston is X m from the fixed end of the tank the water level is at the centre of the piston face. How much work must be done to reduce the distance from the piston Continuing the previous problem, suppose that when the
piston is X m from the fixed end of the tank the water level is
at the centre of the piston face. How much work must be done
to reduce the distance from the piston length $X/2$. at the centre of the piston face. How much work must be done
to reduce the distance from the piston to the fixed end to
 $X/2$ m, and thus cause the water to fill the volume between the
piston and the fixed end of the tank to reduce the distance from the piston to the fixed end to $X/2$ m, and thus cause the water to fill the volume between the piston and the fixed end of the tank? As in Example 6, you can assume the piston can move without $X/2$ m, and thus cause the water to fill the volume between the piston and the fixed end of the tank? As in Example 6, you can assume the piston can move without friction and that trapped air can escape. *Hint*: The tech piston and the fixed end of the tank? As in Example 6, you can assume the piston can move without friction and that trapped air can escape. *Hint*: The technique used to solve part (b) of Example 6 is very difficult to ap assume the piston can move without friction and that trapped
air can escape. *Hint*: The technique used to solve part (b) of
Example 6 is very difficult to apply here. Instead, calculate the
work done to raise the water i
	- **12.** A bucket is raised vertically from ground level at a constant speed of 2 m/min by a winch. If the bucket weighs 1 kg and contains 15 kg of water when it starts up but loses water by leakage at a rate of 1 kg/min the eighs 1 kg and
loses water by
much work must
height of 10 m?
 $\frac{1}{2}$
in value of the
' over [a, b]:

water is y m ($-R \le y \le R$) above the centre of the piston
face? (See Figure 7.50.)
leakage at a rate of 1 kg/min thereafter, how much work must
be done by the winch to raise the bucket to a height of 10 m?
leading be done b be done by the winch to raise the bucket to a height of 10 m?
 SUSINESS, FINANCE, AND ECOLOGY

If the rate of change $f'(x)$ of a function $f(x)$ is known, the change in value of the

function over an interval from $x = a$ be done by the winch to rain

iness, Finance, and Ecology

e rate of change $f'(x)$ of a function $f(x)$ is knot

tion over an interval from $x = a$ to $x = b$ is just t
 $f(b) - f(a) = \int^b f'(x) dx$. be done by the winch to raise the bucket to a h
 a. and Ecology
 b. o f a function $f(x)$ is known, the change is
 $\lim x = a$ to $x = b$ is just the integral of f'
 $f'(x) dx$.

$$
f(b) - f(a) = \int_a^b f'(x) dx.
$$

Business, Finance, and Ecology

If the rate of change $f'(x)$ of a function $f(x)$ is known, the change in value of the

function over an interval from $x = a$ to $x = b$ is just the integral of f' over [a, b]:
 $f(b) - f(a) = \int$ **SUSINESS, FINANCE, and Ecology**

If the rate of change $f'(x)$ of a function $f(x)$ is known, the change in value of the

function over an interval from $x = a$ to $x = b$ is just the integral of f' over $[a, b]$:
 $f(b) - f(a) = \int$ travelled by the car during the time interval [0, T] (hours) is $\int_0^T v(t) dt$ km. thange in value of the

d of f' over $[a, b]$:

then the distance
 $v(t) dt$ km. e rate of change $f'(x)$ of a function $f(x)$ is known, the change in value of the
tion over an interval from $x = a$ to $x = b$ is just the integral of f' over $[a, b]$:
 $f(b) - f(a) = \int_a^b f'(x) dx$.
example, if the speed of a moving function over an interval from $x = a$ to $x = b$ is just the integral
function over an interval from $x = a$ to $x = b$ is just the integral
 $f(b) - f(a) = \int_a^b f'(x) dx$.
For example, if the speed of a moving car at time t is $v(t)$ kis

 $f(b) - f(a) = \int_a^b f'(x) dx$.

or example, if the speed of a moving car at time t is $v(t)$ km/h, then the distance

avelled by the car during the time interval [0, T] (hours) is $\int_0^T v(t) dt$ km.

Similar situations arise naturall speed of a moving car at time *t* is $v(t)$ km/h, then the distance
during the time interval [0, *T*] (hours) is $\int_0^T v(t) dt$ km.
ms arise naturally in business and economics, where the rates of
led marginals.
(**Finding to** *Ja*

For example, if the speed of a moving car at time *t* is $v(t)$ km/h, then the distance

travelled by the car during the time interval [0, *T*] (hours) is $\int_0^T v(t) dt$ km.

Similar situations arise naturally in busin For example, if the speed of a moving car at
travelled by the car during the time interval [0,
Similar situations arise naturally in busine
change are often called marginals.
 $\overline{EXAMPLE 1}$ (Finding total revenue fr
calcula travelled by the car during the time interval [0, T] (hours) is $\int_0^x v(t) dt$ km.

Similar situations arise naturally in business and economics, where the rates of

change are often called marginals.
 EXAMPLE 1 (**Finding** Similar situations arise naturally in business and economics, where the rates of change are often called marginals.
 EXAMPLE 1 (Finding total revenue from marginal revenue) A supplier of calculators realizes a marginal change are often called marginals.
 EXAMPLE 1 (Finding total revenue from marginal revelue of sculator when she has sold x calculators. What will be her total of 100 calculators?
 Solution The marginal revenue is the Similar situations arise naturally in business and economics,
ge are often called marginals.
AMPLE 1 (Finding total revenue from marginal revenue
calculators realizes a marginal revenue of \$15
for when she has sold x ca

$$
dR = (15 - 5e^{-x/50}) dx
$$
 dollars

SECTION 7.7: Applications in Business, Finance, and Ecology 433
\nThe total revenue from the sale of the first 100 calculators is \$R, where
\n
$$
R = \int_{x=0}^{x=100} dR = \int_{0}^{100} (15 - 5e^{-x/50}) dx
$$
\n
$$
= (15x + 250e^{-x/50})\Big|_{0}^{100}
$$
\n
$$
= 1,500 + 250e^{-2} - 250 \approx 1,283.83,
$$
\nthat is, about \$1,284.
\n**The Present Value of a Stream of Payments**
\nSuppose that you have a business that generates income continuously at a variable rate
\n $P(t)$ dollars per year at time *t* and that you expect this income to continue for the next
\nT years. How much is the business worth today?
\nThe answer surely depends on interest rates. One dollar to be received *t* years from
\nnow is worth less than one dollar received today which could be invested at interest to

= $(15x + 250e^{-x/30})\Big|_0$
= 1, 500 + 250 $e^{-2} - 250 \approx 1,283.83$,
that is, about \$1,284.
The Present Value of a Stream of Payments
Suppose that you have a business that generates income continuously at a variable
 $P(t)$ do

= $1,500 + 250e^{-2} - 250 \approx 1,283.83$,

is, about \$1,284.
 Present Value of a Stream of Payments

oose that you have a business that generates income continuously at a variable rate

dollars per year at time *t* and that y **The Present Value of a Stream of Payments**

Suppose that you have a business that generates income continuously at a variable rate $P(t)$ dollars per year at time t and that you expect this income to continue for the next **The Present Value of a Stream of Payments**
 Suppose that you have a business that generates income continuously at a variable rate $P(t)$ dollars per year at time t and that you expect this income to continue for the ne **The Present Value of a Stream of Payments**
Suppose that you have a business that generates income continuously at a variable rate $P(t)$ dollars per year at time t and that you expect this income to continue for the next **Present Value of a Stream of Payments**
oose that you have a business that generates income continuously at a variable rate
dollars per year at time t and that you expect this income to continue for the next
ars. How much **EXECT VALUE OF A SUREM VALUE OF A SUREM CONTAINS**

Suppose that you have a business that generates income continuously at a variable rate $P(t)$ dollars per year at time t and that you expect this income to continue for t Suppose that you have a business that generates income continuous!
 $P(t)$ dollars per year at time t and that you expect this income to co

T years. How much is the business worth today?

The answer surely depends on inte year at time *t* and that you expect this income to couch is the business worth today?
surely depends on interest rates. One dollar to be recost than one dollar received today, which could be invorse dollar *t* years from

$$
\lim_{n \to \infty} \left(1 + \frac{\delta}{n} \right)^{nt} = e^{\delta t} \quad \text{dollars}
$$

and the was such a payment of states. Therefore, a payment of \$1 after t years must be worth only so with less than one dollar received today, which could be invested at interest to yield more than one dollar t years from yield more than one dollar *I* years from now. The ingner the interest rate, the lower the value today of a payment that is not due until sometime in the future.

To analyze this situation, suppose that the nominal intere value today of a payment that is not due until sometime in the future.

To analyze this situation, suppose that the nominal interest rate is $r\%$ per annum,

but is compounded continuously. Let $\delta = r/100$. As shown in Se To analyze this situation, suppose that the nominal inter
but is compounded continuously. Let $\delta = r/100$. As shown
ment of \$1 today will grow to
 $\lim_{n\to\infty} \left(1 + \frac{\delta}{n}\right)^{nt} = e^{\delta t}$ dollars
after t years. Therefore, a pay $\lim_{n\to\infty} \left(1 + \frac{\delta}{n}\right)^{nt} = e^{\delta t}$ dollars
 $\lim_{n\to\infty} \left(1 + \frac{\delta}{n}\right)^{nt} = e^{\delta t}$ dollars
 t years. Therefore, a payment of \$1 after *t* years must be worth only \$ $e^{-\delta t}$ today.

is called the *present value* of the f To analyze this situation, suppose that the nominal interest rate is $r\%$ per annum,
but is compounded continuously. Let $\delta = r/100$. As shown in Section 3.4, an invest-
ment of \$1 today will grow to
 $\lim_{n\to\infty} \left(1 + \frac{\delta}{$ $\lim_{n\to\infty} \left(1 + \frac{1}{n}\right) = e^{\theta t}$ dollars
after t years. Therefore, a payment of \$1 after t years must
This is called the *present value* of the future payment. V
interest rate δ is frequently called a *discount rate*; $\left(\frac{\delta}{n}\right)^2 = e^{\delta t}$ dollars

Therefore, a payment of \$1 after *t* years must be welve

the *present value* of the future payment. When

is frequently called a *discount rate*; it represents

ts are discounted.

to the b

Returning to the business income problem, in the short time interval from *t* to *t* + *t*, the business produces income \$*P*(*t*) *dt*, of which the present value is \$*e*^{-8*t*} *P*(*t*) *dt*.
herefore, the present value

$$
V = \int_0^T e^{-\delta t} P(t) dt.
$$

duces income \$*P*(*t*) *dt*, of which the present value is \$*e*^{$-8t$}*P*(*t*) *dt*.
thent value \$*V* of the income stream over the time interval [0, *T*] is
contributions:
 $P(t) dt$.
What is the present value of a constant, Therefore, the present value \$*V* of the income stream over the time interval [0, *T*] is
the "sum" of these contributions:
 $V = \int_0^T e^{-\delta t} P(t) dt$.
EXAMPLE 2 What is the present value of a constant, continual stream of pa $V = \int_0^T e^{-\delta t} P(t) dt.$
 EXAMPLE 2 What is the present value of a constant, continual stream

ments at a rate of \$10,000 per year, to continue forever,

now? Assume an interest rate of 6% per annum, compounded continuousl $e^{-\delta t} P(t) dt$.

What is the present value of a constant, continual streaments at a rate of \$10,000 per year, to continue foreversion interest rate of 6% per annum, compounded continuously.

required present value is
 $e^{-0.$ **EXAMPLE 2** What is the present value of a constant, continual stream of pay-
mow? Assume an interest rate of 6% per annum, compounded continuously.
Solution The required present value is
 $V = \int_0^\infty e^{-0.06t} 10,000 dt = 10,$

now? Assume an interest rate of 6% per annum, compounded continuously.
\n**Solution** The required present value is
\n
$$
V = \int_0^\infty e^{-0.06t} 10,000 dt = 10,000 \lim_{R \to \infty} \frac{e^{-0.06t}}{-0.06} \Big|_0^R \approx $166,667.
$$
\nThe **Economics of Exploiting Renewable Resources**
\nAs noted in Section 3.4, the rate of increase of a biological population sometimes
\nconforms to a logistic model¹
\n
$$
\frac{dx}{dt} = kx \left(1 - \frac{x}{L}\right).
$$

1.1 This example was suggested by Professor C. W. Clark, of the University of British Columbia.

1.1 This example was suggested by Professor C. W. Clark, of the University of British Columbia.

$$
\frac{dx}{dt} = kx\left(1 - \frac{x}{L}\right).
$$

Here, $x = x(t)$ is the size (or biomass) of the population at time t, k is the natural rate at which the population would grow if its food supply were unlimited, and L is the natural limiting size of the population—the carr Here, $x = x(t)$ is the size (or biomass) of the population at time t, k is the natural rate at which the population would grow if its food supply were unlimited, and L is the natural limiting size of the population—the carr Here, $x = x(t)$ is the size (or biomass) of the population at time t, k is the natural rate at which the population would grow if its food supply were unlimited, and L is the natural limiting size of the population—the carr Here, $x = x(t)$ is the size (or biomass) of the population at time t, k is the natural
rate at which the population would grow if its food supply were unlimited, and L is the
natural limiting size of the population—the carr Here, $x = x(t)$ is the size (or biomass) of the population at time t, k is the natural
rate at which the population would grow if its food supply were unlimited, and L is the
natural limiting size of the population—the carr Here, $x = x(t)$ is the size (or biomass) of the population at time t, k is the natural
rate at which the population would grow if its food supply were unlimited, and L is the
natural limiting size of the population—the carr ize (or biomass) of the population at time t , k is the natural
ion would grow if its food supply were unlimited, and L is the
e population—the carrying capacity of its environment. Such
ply, for example, to the Anta Here, $x = x(t)$ is the size (or biomass) of the population at time t, k is the natural
rate at which the population would grow if its food supply were unlimited, and L is the
natural limiting size of the population—the carr matural infitm g size of the population—the carrying capacity of its environment. Such models are thought to apply, for example, to the Antarctic blue whale and to several species of fish and trees. If the resource is har

$$
\frac{dx}{dt} = kx\left(1 - \frac{x}{L}\right) - h(t).
$$
\n^(*)

$$
h(t) = kx\left(1 - \frac{x}{L}\right),
$$

models are thought to apply, for example, to the Antarctic blue whale and to several
species of fish and trees. If the resource is harvested (say, the fish are caught) at a rate
 $h(t)$ units per year at time t, then the po species or fish and trees. If the resource is narvested (say, the fish are caught) at a rate $h(t)$ units per year at time t, then the population grows at a slower rate:
 $\frac{dx}{dt} = kx\left(1 - \frac{x}{L}\right) - h(t)$. (*)

In particular, els are thought to apply, for example, to the Anta
ies of fish and trees. If the resource is harvested (s
units per year at time t, then the population grows
 $\frac{dx}{dt} = kx \left(1 - \frac{x}{L}\right) - h(t)$.
triticular, if we harvest the po

$$
T = ph(t) = pkx \left(1 - \frac{x}{L}\right).
$$

In particular, if we harvest the population at its current rate of growth,
 $h(t) = kx \left(1 - \frac{x}{L}\right)$,

then $dx/dt = 0$, and the population will maintain a constant size. Assume that each

unit of harvest produces an income of (*) and $\frac{dx}{dt} = kx\left(1 - \frac{x}{L}\right) - h(t)$.

(*)

In particular, if we harvest the population at its current rate of growth,
 $h(t) = kx\left(1 - \frac{x}{L}\right)$,

then $dx/dt = 0$, and the population will maintain a constant size. Assume that $h(t) = kx igg(1 - \frac{x}{L}\bigg)$,
then $dx/dt = 0$, and the population will maintain a constant size. Assume that each
unit of harvest produces an income of $\$p$ for the fishing industry. The total annual
income from harvesting the re then $dx/dt = 0$, and the population will maintain a constant size. Assume that early then $dx/dt = 0$, and the population will maintain a constant size. Assume that early to find the population sincome from harvesting the reso

 $dx/dt = 0$, and the population will maintain a constant size. Assume that each
of harvest produces an income of $\$p$ for the fishing industry. The total annual
me from harvesting the resource at its current rate of growth unit of harvest produces an income of \$p for the fishing industry. The total annual
income from harvesting the resource at its current rate of growth will be
 $T = ph(t) = pkx \left(1 - \frac{x}{L}\right)$.
Considered as a function of x, this to then $dx/dt = 0$, and the population will maintain a constant size. Assume that each
unit of harvest produces an income of $\$p$ for the fishing industry. The total annual
income from harvesting the resource at its current r arvesting the resource at its current rate of growth $= pkx\left(1 - \frac{x}{L}\right)$.

a function of x, this total annual income is quadratic $= L/2$, the value that ensures $dT/dx = 0$. The ir um annual income by ensuring that the popu $H = pnt(t) = p\kappa x (1 - \frac{1}{L})$.
Considered as a function of x, this total annual income is quadratic and has a maximum
value when $x = L/2$, the value that ensures $dT/dx = 0$. The industry can maintain
a stable maximum annual incom the maximal size of the population with no harvesting.

The analysis above, however, does not take into account the discounted value of fu-

ture harvests. If the discount rate is δ , compounded continuously, then the p is above, however, does not take into account t

f the discount rate is δ , compounded continuous
 $\frac{\delta}{p}ph(t) dt$ due between t and $t + dt$ years from the fishery in futu
 $e^{-\delta t}ph(t) dt$.

rategy will maximize T? If we subst $t + dt$ years from now is $e^{-\delta t} ph(t) dt$.

Externe in future years is

we substitute for $h(t)$ from equation (*)

we get
 dt
 $pe^{-\delta t} \frac{dx}{dt} dt$.

where $U = pe^{-\delta t}$ and $dV = \frac{dx}{dt} dt$.

$$
T = \int_0^\infty e^{-\delta t} \, ph(t) \, dt.
$$

of the income
$$
\$ph(t) dt
$$
 due between *t* and *t* + *dt* years from now is $e^{-\delta t}ph(t) dt$.
\nThe total present value of all income from the fishery in future years is
\n
$$
T = \int_0^\infty e^{-\delta t} ph(t) dt.
$$
\nWhat fishing strategy will maximize *T*? If we substitute for *h(t)* from equation (*)
\ngoverning the growth rate of the population, we get
\n
$$
T = \int_0^\infty p e^{-\delta t} \left[kx \left(1 - \frac{x}{L} \right) - \frac{dx}{dt} \right] dt
$$
\n
$$
= \int_0^\infty k p e^{-\delta t} x \left(1 - \frac{x}{L} \right) dt - \int_0^\infty p e^{-\delta t} \frac{dx}{dt} dt.
$$
\nIntegrate by parts in the last integral above, taking $U = p e^{-\delta t}$ and $dV = \frac{dx}{dt} dt$:
\n
$$
T = \int_0^\infty k p e^{-\delta t} x \left(1 - \frac{x}{L} \right) dt - \left[p e^{-\delta t} x \right]_0^\infty + \int_0^\infty p \delta e^{-\delta t} x dt
$$

 $\frac{dx}{dt} dt$:

$$
T = \int_0^\infty e^{-\delta t} ph(t) dt.
$$

\nWhat fishing strategy will maximize T? If we substitute for $h(t)$ from equation (*)
\ngoverning the growth rate of the population, we get
\n
$$
T = \int_0^\infty pe^{-\delta t} \left[kx \left(1 - \frac{x}{L} \right) - \frac{dx}{dt} \right] dt
$$
\n
$$
= \int_0^\infty kpe^{-\delta t} x \left(1 - \frac{x}{L} \right) dt - \int_0^\infty pe^{-\delta t} \frac{dx}{dt} dt.
$$

\nIntegrate by parts in the last integral above, taking $U = pe^{-\delta t}$ and $dV = \frac{dx}{dt} dt$:
\n
$$
T = \int_0^\infty kpe^{-\delta t} x \left(1 - \frac{x}{L} \right) dt - \left[pe^{-\delta t} x \right]_0^\infty + \int_0^\infty p\delta e^{-\delta t} x dt \right]
$$
\n
$$
= px(0) + \int_0^\infty pe^{-\delta t} \left[kx \left(1 - \frac{x}{L} \right) - \delta x \right] dt.
$$

\nTo make this expression as large as possible, we should choose the population size x
\nto maximize the quadratic expression
\n
$$
Q(x) = kx \left(1 - \frac{x}{L} \right) - \delta x
$$
\nat as early a time t as possible, and keep the population size constant at that level
\nthereafter. The maximum occurs where $Q'(x) = k - (2kx/L) - \delta = 0$, that is, where
\n
$$
x = \frac{L}{2} - \frac{\delta L}{2k} = (k - \delta) \frac{L}{2k}.
$$

$$
Q(x) = kx\left(1 - \frac{x}{L}\right) - \delta x
$$

$$
x = \frac{L}{2} - \frac{\delta L}{2k} = (k - \delta) \frac{L}{2k}.
$$

To make this expression as large as possible, we should choose the population size x
to maximize the quadratic expression
 $Q(x) = kx \left(1 - \frac{x}{L}\right) - \delta x$
at as early a time t as possible, and keep the population size constant a 10 make this expression as large as possible, we should choose the population size x

to maximize the quadratic expression
 $Q(x) = kx \left(1 - \frac{x}{L}\right) - \delta x$

at as early a time t as possible, and keep the population size constan to maximize the quadratic expression
 $Q(x) = kx ig(1 - \frac{x}{L}\big) - \delta x$

at as early a time t as possible, and keep the population size constant at that level

thereafter. The maximum occurs where $Q'(x) = k - (2kx/L) - \delta = 0$, that is, w $Q(x) = kx ig(1 - \frac{x}{L}\big) - \delta x$
at as early a time t as possible, and keep the population size constant at that level
thereafter. The maximum occurs where $Q'(x) = k - (2kx/L) - \delta = 0$, that is, where
 $x = \frac{L}{2} - \frac{\delta L}{2k} = (k - \delta)\frac{L}{2k}$ at as early a time t as possible, and keep the population size constant at that level
thereafter. The maximum occurs where $Q'(x) = k - (2kx/L) - \delta = 0$, that is, where
 $x = \frac{L}{2} - \frac{\delta L}{2k} = (k - \delta) \frac{L}{2k}$.
The maximum present va at as early a time t as possible, and keep the pop
thereafter. The maximum occurs where $Q'(x) = k \cdot x = \frac{L}{2} - \frac{\delta L}{2k} = (k - \delta) \frac{L}{2k}$.
The maximum present value of the fishery is realize
at this value. Note that this popul

If $\frac{L}{x} = (k - \delta)L/(2k)$

If extinction $\frac{1}{x} = (k - \delta)L/(2k)$

Of course, this model fails to take into consideration other factors that may affect

the fishing strategy, such as the increased cost of harvesting when the po Nextinction $\overline{x} = (k - \delta)L/(2k)$

Of course, this model fails to take into consideration other factors that may affect

the fishing strategy, such as the increased cost of harvesting when the population level

is small and For a restriction $\overline{f}_x = (k - \delta)L/(2k)$ L_x
 \overline{f}_x of course, this model fails to take into consideration other factors that may affect

the fishing strategy, such as the increased cost of harvesting when the populati **Example 19** $\left\{\frac{L}{2} \right\}$ $\left\{\frac{L}{2} \right\}$ $\left\{\frac{L}{2} \right\}$ $\left\{\frac{L}{2} \right\}$ of course, this model fails to take into consideration other factors that may affect the fishing strategy, such as the increased cost of har **For the case of the case of whales and most trees.** The case of allow example it is small and the effect of competition among various parts of the fishing industry.
Nevertheless, it does explain the regrettable fact that Extinction $x = (k - \delta)L/(2k)$ $L = x$

Of course, this model fails to take into consideration other factors that may affect

the fishing strategy, such as the increased cost of harvesting when the population level

is small and the fishing strategy

is small and the e

Nevertheless, it do

industry based on

resource. This is

resource is low, as

to allow economic

EXERCISES 7.7

. (Cost of production) The marginal cost of production in a

coal Nevertheless, it does explain the regrettable fact that
industry based on a renewable resource can find it i
resource. This is especially likely to happen when
resource is low, as it is for the case of whales and mos
to a dustry based on a renewable resource can find it if
source. This is especially likely to happen when the
source is low, as it is for the case of whales and mos
allow economics alone to dictate the management of
allow econo Find the regrettable fact that, under some circumstances, and

wable resource can find it in its best interest to destroy the

Ily likely to happen when the natural growth rate k of the

the case of whales and most tree de resource can find it in its best interest to destroy the
likely to happen when the natural growth rate k of the
case of whales and most trees. There is good reason not
lictate the management of the resource.
Find the

- industry base
resource. Th
resource is lo
to allow econ
to allow econ
to allow econ
to allow econ
to allow econ
 $\mathbf{X} \in \mathbb{R} \cup \mathbb{R} \subseteq \mathbb{R} \cup \mathbb{R}$
(Cost of production) The marginal cost of production in
coal mine is $x + 6 \times 10^{-6} x^2$ per ton after the First x tons are produced in a to allow economics alone to dictate the management
 XERCISES 7.7
 XERCISES 7.7
 Cost of production) The marginal cost of production in a

coal mine is $$6 - 2 \times 10^{-3}x + 6 \times 10^{-6}x^2$ pe **ERCISES 7.1**
 Cost of production) The marginal cost of production in a
 Cost of production) The marginal cost of production in a
 Cost of production) The marginal cost of production in a

first x tons are produced **ERCISES 7.7**
 Cost of production) The marginal cost of production in a

coal mine is $$6 - 2 \times 10^{-3}x + 6 \times 10^{-6}x^2$ per ton after the

first x tons are produced each day. In addition, there is a fixed

cost of \$4,000 p **EXERCISES 7.7**

2. (Cost of production) The marginal cost of production in a

coal mine is $$6 - 2 \times 10^{-3}x + 6 \times 10^{-6}x^2$ per ton after the

first x tons are produced each day. In addition, there is a fixed

cost of \$4,0 **ERCISES 7.7**
 ERCISES 7.7
 ERCISES 7.7
 ERCISES 7.7
 ERCISES 7.7
 ERCISES 7.7
 EXERCISES 7 XERCISES 7.7

(Cost of production) The marginal cost of production in a

coal mine is $$6 - 2 \times 10^{-3}x + 6 \times 10^{-6}x^2$ per ton after the

first x tons are produced each day. In addition, there is a fixed now and increasin **XERCISES 1.1**

(Cost of production) The marginal cost of production in a

coal mine is $$6 - 2 \times 10^{-3}x + 6 \times 10^{-6}x^2$ per ton after the

first x tons are produced each day. In addition, there is a fixed

cost of \$4,000 p 1. (Cost of production) The marginal cost of production in a

coal mine is $$6 - 2 \times 10^{-3}x + 6 \times 10^{-6}x^2$ per ton after the

first x tons are produced each day. In addition, there is a fixed

cost of \$4,000 per day to ope **SES 7.7**
 oduction) The marginal cost of production in a

s \$6 - 2 × 10⁻³x + 6 × 10⁻⁶x² per ton after the

over a 10-year period

are produced each day. In addition, there is a fixed

on a day when 1,000 tons are
-
- **(Cost of production)** The marginal cost of production in a

cost of medicine, it is $86 2 \times 10^{-3} x + 6 \times 10^{-6} x^2$ per ton affer the

first x tons are produced each day. In addition, there is a fixed

cost of \$4,000 per charges clients at a continuously decreasing marginal rate of $\frac{4}{1 + \sqrt{t}}$ per hour when the client has already used t hours during a month. How much will be billed to a client the total sales) The price of the matter in galaxies and the present all cost of $$4,000$ per day to open the mine. Find the total cost of the trate is 5%.

production on a day when 1,000 tons are produced.
 (Total sales cost of \$4,000 per day to open the mine. Find the total cost of the s 5%.

production on a day when 1,000 tons are produced.
 (Total sales) The sales of a new computer chip are modelled over a 10-year per

by $s(t) = te^{-t/1$ 2. (**Total sales)** The sales of a new computer chip are modelled by $s(t) = te^{-t/10}$, where $s(t)$ is the number of thousands of chips sold per week, *t* weeks after the chip was introduced to the market. How many chips were
- (**Iotal sales**) The sales of a new computer chip are modelled

by $s(t) = te^{-t/10}$, where $s(t)$ is the number of thousands of

the market. How many chips were sold in the first year?
 (Internet connection rates) An interne by $s(t) = te^{-t/10}$, where $s(t)$ is the number of thousands of

the mark. How and increasing steadily

the market. How many chips were sold in the first year?

(**Internet connection rates**) An internet service provider

char chips sold per week, t weeks after the chip was introduced to

the market. How many chips were sold in the first year?
 (Internet connection rates) An internet service provider

compounded continuo

charges clients at a (**Total sales**) The sales of a new computer chip are modelled over a 10-year period by $s(t) = te^{-t/10}$, where $s(t)$ is the number of thousands of the enties after the chip was introduced to the market. How many chips source (**Internet connection rates**) An internet service provider
 $$4/(1 + \sqrt{t})$ per hour when the client has already used
 $$4/(1 + \sqrt{t})$ per hour when the client has already used
 $$4/(1 + \sqrt{t})$ per hour when the client as already charges clients at a continuously decreasing marginal ra

\$4/(1 + \sqrt{t}) per hour when the client has already used

t hours during a month. How much will be billed to a c

who uses x hours in a month? (x need not be an $\frac{\$4}{1} + \sqrt{t}$ per hour when the client has already used
 t hours during a month. How much will be billed to a client
 4. (Total revenue from declining sales) The price per kilogram
 4. (Total revenue from declini *continuous stream of payment problems*) Find the present value of a

discount rate is computed to a stream of maple syrup in a store rises at a constant rate from \$10 at

price per kilogram

price time the periods of the who uses x hours in a month? (x need not be an integer.) of \$5,000 per year an

4. (Total revenue from declining sales) The price per kilogram

of maple syrup in a store rises at a constant rate from \$10 at

or more rises 4. (**Total revenue from declining sales**) The price per kilogram
of maple syrup in a store rises at a constant rate from \$10 at
the beginning of the year to \$15 at the end of the year. As the
price rises, the quantity sol of maple syrup in a store rises at a constant rate from \$10 at
the beginning of the year to \$15 at the end of the year. As the
price rises, the quantity sold decreases; the sales rate is
 $400/(1 + 0.1t)$ kg/year at time t y There rises, the quantity sold decreases; the sales rate is
 $400/(1 + 0.1t)$ kg/year at time t years, $(0 \le t \le 1)$. What

total revenue does the store obtain from sales of the syrup

during the year?
 Stream of payment pr

400/(1 + 0.1*t*) kg/year at time *t* years, $(0 \le t \le 1)$. What

total revenue does the store obtain from sales of the syrup

during the year?
 37. 37. 37. 37. 38. 37. 48. 48. 48. 48. 48. 48. 48. total revenue does the store obtain from sales of the syrup

during the year?

Stream of payment problems) Find the present value of a

ontinuous stream of payments of \$1,000 per year for the periods

and discount rates g during the year?
 eam of payment problems) Find the present vinuous stream of payments of \$1,000 per year foliscount rates given in Exercises 5–10. In each

bunt rate is compounded continuously.

10 years at a discount **Stream of payment problems)** Find the present value of a

ontinuous stream of payments of \$1,000 per year for the periods

and discount rates given in Exercises 5–10. In each case the

iscount rate is compounded continuo 10. Beginning in 10 years and continuing forever after at a

discount rate of 5%

20. In each case the

discount rate is compounded continuously.

5. 10 years at a discount rate of 2%

7. 10 years beginning 2 years from n

-
-
-
- discount rates given in Exercises 5–10. In each case the
bunt rate is compounded continuously.
10 years at a discount rate of 2%
10 years at a discount rate of 5%
10 years beginning 2 years from now at a discount ra
-
-
- likely to happen when the natural growth rate k of the
case of whales and most trees. There is good reason not
lictate the management of the resource.
Find the present value of a continuous stream of payments
over a 10case of whales and most trees. There is good
lictate the management of the resource.
Find the present value of a continuous stream of pa
over a 10-year period beginning at a rate of \$1,000
now and increasing steadily at \$
- 11. Find the present value of a continuous stream of payments
over a 10-year period beginning at a rate of \$1,000 per year
now and increasing steadily at \$100 per year. The discount
rate is 5%.
12. Find the present value Find the present value of a continuous stream of payments
over a 10-year period beginning at a rate of \$1,000 per year
now and increasing steadily at \$100 per year. The discount
rate is 5%.
Find the present value of a con Find the present value of a continuous stream of payments
over a 10-year period beginning at a rate of \$1,000 per year
now and increasing steadily at \$100 per year. The discount
rate is 5%.
Find the present value of a con Find the present value of a continuous stream over a 10-year period beginning at a rate of \$1, now and increasing steadily at \$100 per year. There is 5% .
Find the present value of a continuous stream over a 10-year per 11. Find the present value of a continuous stream of payments
over a 10-year period beginning at a rate of \$1,000 per year
now and increasing steadily at \$100 per year. The discount
rate is 5%.
12. Find the present value Find the present value of a continuous stream of payments
over a 10-year period beginning at a rate of \$1,000 per year
now and increasing steadily at \$100 per year. The discount
rate is 5%.
Find the present value of a con Find the present value of a continuous stream of payments
over a 10-year period beginning at a rate of \$1,000 per year
now and increasing steadily at \$100 per year. The discount
rate is 5%.
Find the present value of a con over a 10-year period beginning at a rate of \$1,000 p
now and increasing steadily at \$100 per year. The dis
rate is 5%.
Find the present value of a continuous stream of pay-
over a 10-year period beginning at a rate of \$1
-
- **12.** Find the present value of a continuous stream of payments
over a 10-year period beginning at a rate of \$1,000 per year
now and increasing steadily at 10% per year. The discount rate
is 5%.
13. Money flows continuo Find the present value of a continuous stream of payments
over a 10-year period beginning at a rate of \$1,000 per year
now and increasing steadily at 10% per year. The discount rate
is 5%.
Money flows continuously into an Find the present value of a continuous stream of payments
over a 10-year period beginning at a rate of \$1,000 per year
now and increasing steadily at 10% per year. The discount rate
is 5%.
Money flows continuously into an over a 10-year period beginning at a rate of \$1,000 pe
now and increasing steadily at 10% per year. The disc
is 5%.
Money flows continuously into an account at a rate of
per year. If the account earns interest at a rate o eriod beginning at a rate of \$1,000 per year
ing steadily at 10% per year. The discount rate
intinuously into an account at a rate of \$5,000
ccount earns interest at a rate of 5%
intinuously, how much will be in the accou now and increasing steadily at 10% per year. The discount rate
is 5%.
Money flows continuously into an account at a rate of \$5,000
per year. If the account earns interest at a rate of 5%
compounded continuously, how much 12. Find the present value of a continuous stream of payments
over a 10-year period beginning at a rate of \$1,000 per year
now and increasing steadily at 10% per year. The discount rate
is 5%.
13. Money flows continuously Money flows continuously into an account at a rate of \$5,000
per year. If the account earns interest at a rate of 5%
compounded continuously, how much will be in the account
after 10 years?
Money flows continuously into a now and increasing steadily at 10% per year. The discots is 5%.

Money flows continuously into an account at a rate of \$

per year. If the account earns interest at a rate of 5%

compounded continuously, how much will be per year. If the account earns interest at a rate of 5%
compounded continuously, how much will be in the account
after 10 years?
Money flows continuously into an account beginning at a rate
of \$5.000 per year and increasi
	-

$$
\lambda(t) = \int_0^t \delta(\tau) \, d\,\tau.
$$

balance in the account to reach \$1,000,000?

If the discount rate δ varies with time, say $\delta = \delta(t)$, show that

the present value of a payment of \$*P* due *t* years from now is
 $\delta P e^{-\lambda(t)}$, where
 $\lambda(t) = \int_0^t \delta(\tau) d\tau$ 15. If the discount rate δ varies with time, say $\delta = \delta(t)$, show that
the present value of a payment of \$*P* due *t* years from now is
\$ $Pe^{-\lambda(t)}$, where
 $\lambda(t) = \int_0^t \delta(\tau) d\tau$.
What is the value of a stream of payments d the present value of a payment of \$*P* due *t* years from now is
 $$Pe^{-\lambda(t)}$, where$
 $\lambda(t) = \int_0^t \delta(\tau) d\tau.$

What is the value of a stream of payments due at a rate \$*P*(*t*) at time *t*, from *t* = 0 to *t* = *T*?
 (Discount balance in the account to reach \$1,000,000?

If the discount rate δ varies with time, say $\delta = \delta(t)$, show that

the present value of a payment of \$*P* due *t* years from now is
 $\Re e^{-\lambda(t)}$, where
 $\lambda(t) = \int_0^t \delta(\tau) d\tau$ If the discount rate δ varies with time, say $\delta = \delta(t)$, show that
the present value of a payment of \$*P* due *t* years from now is
 $$Pe^{-\lambda(t)}$$, where
 $\lambda(t) = \int_0^t \delta(\tau) d\tau$.
What is the value of a stream of payments due the present value of a payment of \$*P* due *t* years from now is
 $$Pe^{-\lambda(t)}$, where$
 $\lambda(t) = \int_0^t \delta(\tau) d\tau.$

What is the value of a stream of payments due at a rate \$*P*(*t*) at time *t*, from *t* = 0 to *t* = *T*?
 (Discount $\lambda(t) = \int_0^t \delta(\tau) d\tau.$
at is the value of a stream of payments due at a rate \$
me *t*, from $t = 0$ to $t = T$?
scount rates and population models) Suppose that t
wth rate of a population is a function of the populatio
:

$$
\frac{dx}{dt} = F(x) - h(t).
$$

CHAPTER 7 Applications of Integration

Show that the value of x that maximizes the present value of

all future harvests satisfies $F'(x) = \delta$, where δ is the

(continuously compounded) discount rate. *Hint*: Mimic the
 CHAPTER 7 Applications of Integration
Show that the value of x that maximizes the present value of
all future harvests satisfies $F'(x) = \delta$, where δ is the
(continuously compounded) discount rate. *Hint:* Mimic the
argu of Integration

maximizes the present value of
 $f(x) = \delta$, where δ is the

logistic model, and

discount rate. *Hint:* Mimic the

logistic model, and

(a) The maximum

logistic case.

(b) The annual rev

sustainable hal CHAPTER 7 Applications of Integration

Show that the value of x that maximizes the present value of

all future harvests satisfies $F'(x) = \delta$, where δ is the

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 CHAPTER 7 Applications of Integration

Show that the value of x that maximizes the present value of

all future harvests satisfies $F'(x) = \delta$, where δ is the

(continuously compounded) discount rate. *Hint*: Mimic the

- 17. (Managing a fishery) The carrying capacity of a certain lake

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17. (Managing a fishery CHAPTER 7 Applications of Integration

Show that the value of x that maximizes the present value of

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 CHAPTER 7 Applications of Integration

Show that the value of x that maximizes the present value of

all future harvests satisfies $F'(x) = \delta$, where δ is the

(continuously compunded) discount rate. *Hint:* Mimic the

a Show that the value of x that maximizes the present value of

all future harvests satisfies $F'(x) = \delta$, where δ is the

continuously compounded) discount rate. *Hint:* Mimic the

argument used above for the logistic cas Show that the value of x that maximizes the present value of

all future harvests satisfies $F'(x) = \delta$, where δ is the

(continuously compounded) discount rate. *Hint:* Mimic the

argument used above for the logistic ca Show that the value of *x* that maximizes the present value of

all future harvests satisfies $F'(x) = \delta$, where δ is the

(continuously compounded) discount rate. *Hint*: Mimic the

argument used above for the logistic Show that the value of x that maximizes the present value of

all future harvests satisfies $F'(x) = \delta$, where δ is the

(continuously compounded) discount rate. *Hint:* Mimic the

argument used above for the logistic ca level? (continuously compounded) discount rate. *Hint:* Mimic the (a) The maximum sustaina argument used above for the logistic case. (b) The annual revenue res is $L = 80,000$ of a certain species of ths. The natural growth rate denote the significant used above for the logistic case.

(Managing a fishery) The carrying capacity of a certain lake

is $L = 80,000$ of a certain species of fish. The natural growth

antarctic of this species is 12% per (Managing a fishery) The carrying capacity of a certain lake

is $L = 80,000$ of a certain species of fish. The natural growth

are of this species is 12% per year ($k = 0.12$). Each fish is

should be maintained in the lak about $L = 150,000$. One blue whale is worth the carrain species of fish. The natural growth

arta of this species is 12% per year ($k = 0.12$). Each fish is

should be maintained in the lake to maximize the present

value
-

\$10,000. Assuming that the blue whale population satisfies a
logistic model, and using the data above, find the following:
(a) The maximum sustainable annual harvest of blue whales.
(b) The annual revenue resulting from th \$10,000. Assuming that the blue whale population satisfies a
logistic model, and using the data above, find the following:
(a) The maximum sustainable annual harvest of blue whales.
(b) The annual revenue resulting from th

-
- $$10,000$. Assuming that the blue whale population satisfies a logistic model, and using the data above, find the following:
(a) The maximum sustainable annual harvest of blue whales.
(b) The annual revenue resulting from
- (\$10,000. Assuming that the blue whale population satisfies a logistic model, and using the data above, find the following:

(a) The maximum sustainable annual harvest of blue whales.

(b) The annual revenue resulting fro 000. Assuming that the blue whale population satisfies a
stic model, and using the data above, find the following:
The maximum sustainable annual harvest of blue whale:
The annual revenue resulting from the maximum annual \$10,000. Assuming that the blue whale population satisfies a logistic model, and using the data above, find the following:
(a) The maximum sustainable annual harvest of blue whales.
(b) The annual revenue resulting from t (a) 000. Assuming that the blue whale population satisfies a
stic model, and using the data above, find the following:
The maximum sustainable annual harvest of blue whales.
The annual revenue resulting from the maximum a 000. Assuming that the blue whale population satisfies a
stic model, and using the data above, find the following:
The maximum sustainable annual harvest of blue whales.
The annual revenue resulting from the maximum annua 000. Assuming that the blue whale population satisfies a
stic model, and using the data above, find the following:
The maximum sustainable annual harvest of blue whales.
The annual revenue resulting from the maximum annua \$10,000. Assuming that the blue whale population satisfies a
logistic model, and using the data above, find the following:
(a) The maximum sustainable annual harvest of blue whales.
(b) The annual revenue resulting from t stic model, and using the data above, find the following:
The maximum sustainable annual harvest of blue whales.
The annual revenue resulting from the maximum annual
sustainable harvest.
The annual interest generated if t The maximum sustainable annual harvest of blue w
The annual revenue resulting from the maximum an
sustainable harvest.
The annual interest generated if the whale populatio
(assumed to be at the level $L/2$ supporting the
-
- (b) The annual revenue resulting from the maximum annual sustainable harvest.

(c) The annual interest generated if the whale population

(assumed to be at the level $L/2$ supporting the maximum

sustainable harvest) is e (c) The annual revente resulting from the maximum annual sustainable harvest.

(c) The annual interest generated if the whale population

(assumed to be at the level $L/2$ supporting the maximum

sustainable harvest) is e is distainable harvest.

(c) The annual interest generated if the whale population

(assumed to be at the level $L/2$ supporting the maximum

sustainable harvest) is exterminated and the proceeds

invested at 2%. (d) at 5 (c) The annual interest generated if the whale population

(assumed to be at the level $L/2$ supporting the maximum

sustainable harvest) is exterminated and the proceeds

invested at 2%. (d) at 5%.

(e) The total present

7.8 Probability

Probability theory is a very important field of application of calculus. This subject

Theory is a very important field of application of calculus. This subject

Probability theory is a very important field of application Frobability theory is a very important field of application of calculus. This subject

Probability theory is a very important field of application of calculus. This subject

cannot, of course, be developed thoroughly here— Probability theory is a very important field of application of calculus. This subject
cannot, of course, be developed thoroughly here—an adequate presentation requires
one or more whole courses—but we can give a brief intr Probability theory is a very important field of application of calculus. This subject
cannot, of course, be developed thoroughly here—an adequate presentation requires
one or more whole courses—but we can give a brief intr

ability theory is a very important field of application of calculus. This subject
oot, of course, be developed thoroughly here—an adequate presentation requires
or more whole courses—but we can give a brief introduction th Probability theory is a very important field of application of calculus. This subject
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cannot, of course, be developed thoroughly here—an adequate presentation requires
one or more whole courses—but we can give a brief int cannot, of course, be developed thoroughly here—an adequate presentation requires
one or more whole courses—but we can give a brief introduction that suggests some
of the ways sums and integrals are used in probability th Probability theory is a very important field of application of calculus. This subject
cannot, of course, be developed thoroughly here—an adequate presentation requires
one or more whole courses—but we can give a brief int of the ways sums and integrals are used in probability theory.

In the context of probability theory the term **experiment** is used to denote a pro-

cess that can result in different **outcomes**. A particular outcome is al In the context of probability theory the term **experiment** is used to denote a pro-
cess that can result in different **outcomes**. A particular outcome is also called a **real-**
ization. The set of all possible outcomes i any one to the coin to the coin the coincil in the coincil in the coincil and the sample space is also called a real-
 ization. The set of all possible outcomes is called the **sample space** for the experiment.

For exam **ization**. The set of all possible outcomes is called the
ization. The set of all possible outcomes is called the
For example, the process might be the tossing of a c
possible outcomes: H (the coin lands horizontal wi
c example, the process might be the tossing of a coin for which we could have three
tible outcomes: H (the coin lands horizontal with "heads" showing on top), T (the
lands horizontal with "tails" showing on top), or E (the possible outcomes: *H* (the coin lands horizontal with "heads" showing on top), *T* (the coin lands horizontal with "tails" showing on top), or *E* (the coin lands and remains standing on its edge). Of course, outcome *E* coin lands horizontal with "tails" showing on top), or *E* (the coin lands and remains standing on its edge). Of course, outcome *E* is not very likely unless the coin is quite thick, but it can happen. So our sample spac

standing on its edge). Of course, outcome *E* is not very likely unless the coin is quite thick, but it can happen. So our sample space is $S = \{H, T, E\}$. Suppose we were to toss the coin a great many times, and observe th thick, but it can happen. So our sample space is $S = \{H, T, E\}$. Suppose we were
to toss the coin a great many times, and observe that the outcomes H and T each oc-
cur on 49% of the tosses while E occurs only 2% of to toss the coin a great many times, and observe that the outcomes *H* and *T* each oc-
cur on 49% of the tosses while *E* occurs only 2% of the time. We would say that on
any one toss of the coin the outcomes *H* and *T* cur on 49% of the tosses while *E* occurs only 2% of the time. We
any one toss of the coin the outcomes *H* and *T* each have probabili
probability 0.02.
An **event** is any subset of the sample space. The **probability**
rea parameter that the outcomes *H* and *T* each oc-
only 2% of the time. We would say that on
and *T* each have probability 0.49 and *E* has
e space. The **probability** of an event is a
es the proportion of times the outcome both 2% of the time. We would say that on

and *T* each have probability 0.49 and *E* has

be space. The **probability** of an event is a

res the proportion of times the outcome of

to that event if the experiment is repea Example 3 and T each oc-
is H and T each oc-
ie would say that on
ility 0.49 and E has
ity of an event is a
mes the outcome of
periment is repeated
ce is certain, and its
ssibly occur, and its
possible events; w Experiment can be expected to belong to that event if the experiment is repeated by times. If the event is the whole sample space, its occurrence is certain, and its ability is 1; if the event is the empty set $\emptyset = \{\}$,

 $Pr(\emptyset) = 0,$ $Pr({T}) = 0.49,$ $Pr({H}) = 0.49, Pr({E}) = 0.02, Pr({H, E}) = 0.51,$ $Pr({H, T}) = 0.98,$ $Pr(S) = 1.$

many times. If the event is the whole sample space, its occurrence is certain, and its
probability is 1; if the event is the empty set $\emptyset = \{\}$, it cannot possibly occur, and its
probability is 0. For the coin-tossing ex may three vent sate whote sample space, its occurring to probability is 1; if the event is the empty set $\theta = \{\}$, it cannot possibly occur, and its probability is 1; if the event is the empty set $\theta = \{\}$, it cannot poss probability is 1, it includes the empty set $b - \{f\}$, it cannot possibly occal, and its
probability is 0. For the coin-tossing experiment, there are eight possible events; we
record their probabilities as follows:
 $Pr(\{\theta\$ ble events; we
 $\langle E \rangle = 0.51$,
 $= 1$.
 r intersection

netimes called

me can belong

, consisting of

events *A* and

to at least one procedurity is 0. 1 of the cont-tossing experiment, there are eight possible events, we
record their probabilities as follows:
 $\Pr(\theta) = 0$, $\Pr({T}) = 0.49$, $\Pr({H, T}) = 0.98$, $\Pr({T, E}) = 0.51$,
 $\Pr(S) = 1$.
Given any two events A Provide the event $\text{Pr}(\{T\}) = 0.49$, Pr($\{H, T\}$) = 0.98, Pr($\{T, E\}$) = 0.51,
Pr($\{H\}$) = 0.49, Pr($\{E\}$) = 0.02, Pr($\{H, E\}$) = 0.51, Pr(S) = 1.
Given any two events A and B (subsets of sample space S), their $Pr(\emptyset) = 0$, $Pr({T}) = 0.49$, $Pr({H, T}) = 0.98$,
 $Pr({H}) = 0.49$, $Pr({E}) = 0.02$, $Pr({H, E}) = 0.51$,

Given any two events *A* and *B* (subsets of sample space *A* \cap *B* consists of those outcomes belonging to both *A* and *B*;

the

SECTION 7.8: Probability **437**
We summarize the basic rules governing probability as follows: if S is a sample
e, Ø is the empty subset of S, and A and B are any events, then
(a) $0 \leq Pr(A) \leq 1$, SECTION 7.8: Probability **437**
We summarize the basic rules governing probability as follows: if S is a sample
space, \emptyset is the empty subset of S, and A and B are any events, then
(a) $0 \leq Pr(A) \leq 1$,
(b) $Pr(\emptyset) = 0$ and SECTION 7.8:

We summarize the basic rules governing probability as follows:
 θ , \emptyset is the empty subset of *S*, and *A* and *B* are any events, then

(a) $0 \leq Pr(A) \leq 1$,

(b) $Pr(\emptyset) = 0$ and $Pr(S) = 1$,

(c) $Pr(A^c) = 1 -$ We summarize the basic rules governing proles, \emptyset is the empty subset of S, and A and B are

(a) $0 \leq Pr(A) \leq 1$,

(b) $Pr(\emptyset) = 0$ and $Pr(S) = 1$,

(c) $Pr(A^c) = 1 - Pr(A)$,

(d) $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap$

that just adding $Pr(A) + Pr(B)$

- (a) $0 \leq Pr(A) \leq 1$,
-
- (c) $Pr(A^c) = 1 Pr(A)$,
-

We summarize the basic rules governing probability as follows: if S is a, Ø is the empty subset of S, and A and B are any events, then

(a) $0 \leq Pr(A) \leq 1$,

(b) $Pr(\emptyset) = 0$ and $Pr(S) = 1$,

(c) $Pr(A^c) = 1 - Pr(A)$,

(d) $Pr(A \cup B) = Pr$ We summarize the basic rules governing probability as follows: if *S* is a sample
space, \emptyset is the empty subset of *S*, and *A* and *B* are any events, then
(a) $0 \le Pr(A) \le 1$,
(b) $Pr(\emptyset) = 0$ and $Pr(S) = 1$,
(c) $Pr(A^c) = 1 -$ We summarize the basic rules governing probability as follows: if *S* is a sample
space, θ is the empty subset of *S*, and *A* and *B* are any events, then
(a) $0 \leq Pr(A) \leq 1$,
(b) $Pr(\theta) = 0$ and $Pr(S) = 1$,
(c) $Pr(A^c) = 1 -$ We summarize the basic rules governing probability as follows: if *S* is a sample
space, θ is the empty subset of *S*, and *A* and *B* are any events, then
(a) $0 \leq Pr(A) \leq 1$,
(b) $Pr(\theta) = 0$ and $Pr(S) = 1$,
(c) $Pr(A^c) = 1$ arize the basic rules governing probability as follows: if *S* is a sample
empty subset of *S*, and *A* and *B* are any events, then
 $Pr(A) \le 1$,
 $P = 0$ and $Pr(S) = 1$,
 $P^c = 1 - Pr(A)$,
 $P = Pr(A) + Pr(B) - Pr(A \cap B)$.
adding $Pr(A) + Pr(B)$ would e, \emptyset is the empty subset of S , and A and B are any events, then

(a) $0 \leq Pr(A) \leq 1$,

(b) $Pr(\emptyset) = 0$ and $Pr(S) = 1$,

(c) $Pr(A^c) = 1 - Pr(A)$,

(d) $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$.

that just adding $Pr(A) + Pr(B)$ would count outco

$$
Pr(A^c) = Pr(\lbrace E \rbrace) = 0.02 = 1 - 0.98 = 1 - Pr(\lbrace H, T \rbrace) = 1 - Pr(A)
$$

$$
Pr(A \cup B) = Pr(S) = 1 = 0.51 + 0.51 - 0.02 = Pr(A) + Pr(B) - Pr(A \cap B).
$$

(c) $P(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$.

Note that just adding $Pr(A) + Pr(B)$ would count outcomes in $A \cap B$ twice. As

an example, in our coin-tossing experiment if $A = \{H, T\}$ and $B = \{H, E\}$, then
 $A^c = \{E\}, A \cup B = \{H, T, E\} = S$, and $A \$ (a) $Pr(A \cup B) = Pr(A) + Pr(B) - Pr(A \cap B)$.

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 $A^c = \{E\}, A \cup B = \{H, T, E\} = S$, and $A \$ Note that just adding $Pr(A) + Pr(B)$ would count outcomes in $A \cap B$ twice. As
an example, in our coin-tossing experiment if $A = \{H, T\}$ and $B = \{H, E\}$, then
 $A^c = \{E\}, A \cup B = \{H, T, E\} = S$, and $A \cap B = \{H\}$. We have
 $Pr(A^c) = Pr(\{E\})$ From theory and paramogeneous matrix $A = \{H, T\}$ and $B = \{H, E\}$, then $A^c = \{E\}, A \cup B = \{H, T, E\} = S$, and $A \cap B = \{H, T\}$ we have $\Pr(A^c) = \Pr(\{E\}) = 0.02 = 1 - 0.98 = 1 - \Pr(\{H, T\}) = 1 - \Pr(A)$
 $\Pr(A \cup B) = \Pr(S) = 1 = 0.51 + 0.51 - 0.02 = \Pr(A) + \$ an example, in our conn-tossing experiment if $A = \{H, T, F\}$ and $B = \{H, Y, F\}$ and $B = \{H, T, E\}$, then $A^c = \{E\}, A \cup B = \{H, T, E\} = S$, and $A \cap B = \{H\},$ We have $\Pr(A \cup B) = \Pr(S) = 1 = 0.51 + 0.51 - 0.02 = \Pr(A) + \Pr(B) - \Pr(A \cap B)$.
Remark T $Pr(A^c) = Pr(\{E\}) = 0.02 = 1 - 0.98 = 1 - Pr(\{H, T\}) = 1 - Pr(A)$
 $Pr(A \cup B) = Pr(S) = 1 = 0.51 + 0.51 - 0.02 = Pr(A) + Pr(B) - Pr(A \cap B)$.
 Remark The generality of these rules of probability can be misleading. Probability

only has meaning in terms of a giv $Pr(A^c) = Pr({E}) = 0.02 = 1 - 0.98 = 1 - Pr({H, T}) = 1 - Pr(A)$
 $Pr(A \cup B) = Pr(S) = 1 = 0.51 + 0.51 - 0.02 = Pr(A) + Pr(B) - Pr(A \cap B)$.
 Remark The generality of these rules of probability can be misleading. Probability

only has meaning in terms of a given s $Pr(A \cup B) = Pr(S) = 1 = 0.51 + 0.51 - 0.02 = Pr(A) + Pr(B) - Pr(A \cap B)$.
Remark The generality of these rules of probability can be misleading. Probability only has meaning in terms of a given sample space or measure. In popular culture proba FI($A \cup B$) = FI(B) = 1 = 0.31 + 0.31 - 0.02 = FI(A) + FI(B) - FI(A 1+ B).
 Remark The generality of these rules of probability can be misleading. Probability

only has meaning in terms of a given sample space or **Remark** The generality of these rules of probability only has meaning in terms of a given sample space probability is sometimes cited in the absence of a stheory also has infamous paradoxes and jokes that a probabilities only has meaning in terms of a given sample space or measure probability is sometimes cited in the absence of a sample sp
theory also has infamous paradoxes and jokes that arise from
probabilities across more than one samp theory also has infamous paradoxes and jokes that arise from attempting to compute
probabilities across more than one sample space or computing them inadvertently from
a different sample space than a user had in mind. Misu probabilities across more than one sample space or computing them inadvertently from
a different sample space than a user had in mind. Misunderstandings do arise in an
overlooked shift in a question about a probability, im

a different sample space than a user had in mind. Misunderstandings do arise in an
overlooked shift in a question about a probability, implying an unnoticed change in the
sample space, or lack of precision about what the s overlooked shift in a question about a probability, implying an unnoticed change in the
sample space, or lack of precision about what the sample space actually is. Infamous
disputes about the "correct" probability have ari sample space, or lack of precision about what the sample space actually is. Infamous
disputes about the "correct" probability have arisen as a result. These are beyond the
scope of this section.
Discrete Random Variables disputes about the "correct" probability have arisen as a result. These are beyond the
scope of this section.
Discrete Random Variables
A **random variable** is a function defined on a sample space. We will denote random
v **Discrete Random Variables**
 A random variable is a function defined on a sample space. We will denote random

variables by using uppercase letters such as X and Y . If the sample space contains only

discrete outcom **CHETE KANIOOM VATIADIES**
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the sample space. We will denote random

the by using uppercase letters such as X and Y. If the sample space contains only

tete outcomes (l A **random variable** is a function defined on a sample space. We will denote random variables by using uppercase letters such as *X* and *Y*. If the sample space contains only discrete outcomes (like the sample space for t **Discrete Random Variables**
A **random variable** is a function defined on a sample space. We will denote random
variables by using uppercase letters such as X and Y. If the sample space contains only
discrete outcomes (lik **EXECTE RANDOM VATIADIES**
A **random variable** is a function defined on a sample space. We will denote random
discrete outcomes (like the sample space for the coin-tossing experiment), a random
variable on it will have onl

variable. If, on the other hand, the sample
of, say, heights of trees, then a random variable
of, say, heights of trees, then a random variable
and will
We will study both types in this section.
Most discrete random var on it will have only discrete values and will be called a **discr**

2. If, on the other hand, the sample space contains all possible m

heights of trees, then a random variable equal to that measureme

a continuum of real Most discrete random variables have only finitely many values, but some can
have infinitely many values if, say, the sample space consisted of the positive integers
{1, 2, 3, ...}. A discrete random variable X has an asso

$$
\sum_{x} f(x) = \sum_{x} \Pr(X = x) = 1,
$$

f defined on the range of X by $f(x) = Pr(X = x)$ for each possible value x of X.
Typically, f is represented by a bar graph; the sum of the heights of all the bars must
be 1,
 $\sum_{x} f(x) = \sum_{x} Pr(X = x) = 1$,
since it is certain that t Example 1 of the sum of the heights of all the bars must

Fr($X = x$) = 1,

at the experiment must produce an outcome, and therefore a value

A single fair die is rolled so that it will show one of the numbers

1 to 6 on to be 1,
 $\sum_{x} f(x) = \sum_{x} Pr(X = x) = 1$,

since it is certain that the experiment must produce an outcome, and therefore a value

of X.
 EXAMPLE 1 A single fair die is rolled so that it will show one of the numbers

any roll, t $\sum_{x} f(x) = \sum_{x} Pr(X = x) = 1$,
since it is certain that the experiment must produce an outcome, and therefore a value
of X.
EXAMPLE 1 A single fair die is rolled so that it will show one of the numbers
any roll, then X is a $\sum_{x} J(x) = \sum_{x} F1(A = x) = 1$,
since it is certain that the experiment must produce an outcome, and therefore a value
of X.
EXAMPLE 1 A single fair die is rolled so that it will show one of the numbers
any roll, then X is a Since it is certain that the experiment must produce an out
of X.
 EXAMPLE 1 A single fair die is rolled so that it will be a formulate the original of the damy roll, then X is a discrete random variable with 6 poss

fa For must produce an outcome, and therefore a value
ie is rolled so that it will show one of the numbers
when it stops. If X denotes the number showing on
m variable with 6 possible values. Since the die is
re likely than

$$
f(n) = \Pr(X = n) = \frac{1}{6}
$$
 for each *n* in {1, 2, 3, 4, 5, 6}.

The discrete random variable X is therefore said to be distributed **uniformly**. All the bars in the graph of its probability function f have the same height. (See Figure 7.52.) The discrete random variable *X* is therefore said to be distributed **uniformly**. All the bars in the graph of its probability function *f* have the same height. (See Figure 7.52.) Note that The discrete random variable X is therefore s
bars in the graph of its probability function f
Note that
 $\sum_{n=1}^{6} Pr(X = n) = 1,$ The discrete random variable *X* is therefore said to be distributed **uniformly**. All the bars in the graph of its probability function *f* have the same height. (See Figure 7.52.) Note that
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reflec The discrete random variable *X* is therefore said to be distributed **uniformly**. All the bars in the graph of its probability function *f* have the same height. (See Figure 7.52.) Note that
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reflec

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\sum_{n=1}^{6} \Pr(X = n) = 1,
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$$
Pr(1 \le X \le 4) = \sum_{n=1}^{4} Pr(X = n) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}.
$$

Frame that the rolled die must certainly give one of the six possible out-

Summers. The probability that a roll will produce a value from 1 to 4 is
 $Pr(1 \le X \le 4) = \sum_{n=1}^{4} Pr(X = n) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$.
 that the rolled die must certainly give one of the six possible out-
lity that a roll will produce a value from 1 to 4 is
 $= \sum_{n=1}^{4} Pr(X = n) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$.
What is the sample space for the number Probability that a roll will produce a value from 1 to 4 is
 $Pr(1 \le X \le 4) = \sum_{n=1}^{4} Pr(X = n) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$.
 EXAMPLE 2 What is the sample space for the numbers showing on top when

two fair dice $Pr(1 \le X \le 4) = \sum_{n=1}^{4} Pr(X = n) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$.
 EXAMPLE 2 What is the sample space for the numbers showing on top when two fair dice are rolled? What is the probability that a 4 and a 2 will be s Pr(1 ≤ X ≤ 4) = $\sum_{n=1}^{4} Pr(X = n) = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$
 EXAMPLE 2 What is the sample space for t

two fair dice are rolled? What

will be showing? Find the probability function for

the sum of the two numbers showin EXAMPLE 2 What is the sample space for the numbers showing on top when
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EXAMPLE 2 What is the sample space for the numbers showing on top when two fair dice are rolled? What is the probability that a 4 and a 2 will be showing? Find the probability function for the random variable X that giv Pr(1 ≤ X ≤ 4) = $\sum_{n=1}$ Pr(X = n) = $\frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{2}{3}$.
 EXAMPLE 2 What is the sample space for the numbers showing on top when will be showing? Find the probability function for the random var **EXAMPLE 2** What is the sample space for the numbers showing on top when
wo fair dice are rolled? What is the probability that a 4 and a 2
will be showing? Find the probability function for the random variable X that give **EXAMPLE 2** two fair dice are rolled? What is the probability that a 4 and a 2 will be showing? Find the probability function for the random variable X that gives the sum of the two numbers showing on the dice. What is th **EXAMPLE 2** What is the sample space for the numbers showing on top when will be showing? Find the probability function for the random variable X that gives the sum of the two numbers showing on the dice. What is the prob whil be showing? Find the probability function for the random variable X that gives
the sum of the two numbers showing on the dice. What is the probability that that sum
is less than 10?
Solution The sample space consis **then** 10?
 then The sample space consists of all pairs of integers (m, n) satisfying $1 \le$

6 and $1 \le n \le 6$. There are 36 such pairs, so the probability of any one of them

36. Two of the pairs, (4, 2) and (2, 4), cor **Dotation** of the two numbers showing on the dice. What is the probability that that sum
han 10?
Dotation 10?
Dotation 10.
Dotation 1 $\le n \le 6$. There are 36 such pairs, so the probability of any one of them
. Two o The sample space consists of all pairs of integers (m, n) satisfying $1 \le$
 $1 \le n \le 6$. There are 36 such pairs, so the probability of any one of them

b of the pairs, (4, 2) and (2, 4), correspond to a 4 and a 2 showing, The sample space consists of all pairs of integers (m, n) satisfying $1 \le$
 $1 \le n \le 6$. There are 36 such pairs, so the probability of any one of them

of the pairs, (4, 2) and (2, 4), correspond to a 4 and a 2 showing, s 1 ≤ *n* ≤ 6. There are 36 such pairs, so the probability of any one of them

o of the pairs, (4, 2) and (2, 4), correspond to a 4 and a 2 showing, so the

of that event is (1/36) + (1/36) = 1/18. The random variable *X* $m \leq 6$ and
is 1/36. Ty
probability
by $X(m, n)$
following t
of that valu
 $k =$
 $k =$

	Probability function for the sum of two dice Table 2.						
	$k = m + n$	outcomes for which $X = k$	$f(k) = Pr(X = k)$				
	2	(1, 1)	1/36				
	3	(1, 2), (2, 1)	$2/36 = 1/18$				
	4	(1, 3), (2, 2), (3, 1)	$3/36 = 1/12$				
$f(k) = Pr(X = k)$	5	(1, 4), (2, 3), (3, 2), (4, 1)	$4/36 = 1/9$				
	6	$(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)$	5/36				
		$(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)$	$6/36 = 1/6$				
	8	$(2,6)$, $(3,5)$, $(4,4)$, $(5,3)$, $(6,2)$	5/36				
	9	(3, 6), (4, 5), (5, 4), (6, 3)	$4/36 = 1/9$				
	10	$(4,6)$, $(5,5)$, $(6,4)$	$3/36 = 1/12$				
	11	(5,6), (6,5)	$2/36 = 1/18$				
	12	(6, 6)	1/36				

$$
Pr(X < 10) = 1 - Pr(X \ge 10) = 1 - \left(\frac{1}{12} + \frac{1}{18} + \frac{1}{36}\right) = \frac{5}{6}.
$$

The bar graph of the probability function f is shown in Figure 7.53. We have
 $Pr(X < 10) = 1 - Pr(X \ge 10) = 1 - \left(\frac{1}{12} + \frac{1}{18} + \frac{1}{36}\right) = \frac{5}{6}$.
 Expectation, Mean, Variance, and Standard Deviation

Consider a simple gamb The bar graph of the probability function f is shown in Figure 7.53. We have
 $Pr(X < 10) = 1 - Pr(X \ge 10) = 1 - \left(\frac{1}{12} + \frac{1}{18} + \frac{1}{36}\right) = \frac{5}{6}$.
 Expectation, Mean, Variance, and Standard Deviation

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 Expectation, Mean, Variance, and Standard Deviation

Consider a simple gambling game in which the player pays the house C dollars for the p $Pr(X < 10) = 1 - Pr(X \ge 10) = 1 - \left(\frac{1}{12} + \frac{1}{18} + \frac{1}{36}\right) = \frac{5}{6}.$
 Expectation, Mean, Variance, and Standard Deviation

Consider a simple gambling game in which the player pays the house *C* dollars for the privilege of

SECTION 7.8: Probability **439**

or 6 dollars, each with probability 1/6. In *n* games the player can expect to win about $n/6 + 2n/6 + 3n/6 + 4n/6 + 5n/6 + 6n/6 = 21n/6 = 7n/2$ dollars, so that his

expected *average winnings pe* SECTION 7.8: Probability **439**

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expected *average winnings pe* sECTION 7.8: Probability **439**
or 6 dollars, each with probability 1/6. In *n* games the player can expect to win about
 $n/6 + 2n/6 + 3n/6 + 4n/6 + 5n/6 + 6n/6 = 21n/6 = 7n/2$ dollars, so that his
expected *average winnings per* section 7.8: Probability **439**

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 $n/6 + 2n/6 + 3n/6 + 4n/6 + 5n/6 + 6n/6 = 21n/6 = 7n/2$ dollars, so that his

expected *average winnings p* SECTION 7.8: Pr

or 6 dollars, each with probability 1/6. In *n* games the player can expected $n/6 + 2n/6 + 3n/6 + 4n/6 + 5n/6 + 6n/6 = 21n/6 = 7n/2$ dollars

expected *average winnings per game* are 7/2 dollars, that is, \$3.50 or 6 dollars, each with probabili
 $n/6 + 2n/6 + 3n/6 + 4n/6 +$

expected *average winnings per*

player can expect, on average, to

or **mean**, of the discrete randon

Greek letter "mu" (pronounced "
 DEFINITION"
 Mean or dollars, each with probability 1/6. In *n* games the $+ 2n/6 + 3n/6 + 4n/6 + 5n/6 + 6n/6 = 21r$
cted *average winnings per game* are 7/2 dollars,
er can expect, on average, to lose money. The amou
ean, of the discrete random dollars, each with probability 1/6. In *n* games the player can expect to win about $+ 2n/6 + 3n/6 + 4n/6 + 5n/6 + 6n/6 = 21n/6 = 7n/2$ dollars, so that his cred *average winnings per game* are 7/2 dollars, that is, \$3.50. If section *i.s:* Probability 1/6. In *n* games the player can expect to win about $+ 2n/6 + 3n/6 + 4n/6 + 5n/6 + 6n/6 = 21n/6 = 7n/2$ dollars, so that his cted *average winnings per game are 7/2* dollars, that is, \$3.50. If *C* > ars, each with probability 1/6. In *n* games the player can expect to win $n/6 + 3n/6 + 4n/6 + 5n/6 + 6n/6 = 21n/6 = 7n/2$ dollars, so th *average winnings per game* are 7/2 dollars, that is, \$3.50. If $C > 3$.
n expect, on aver ability 1/6. In *n* games the player can expect of $6 + 5n/6 + 6n/6 = 21n/6 = 7n/2$ dollar per game are 7/2 dollars, that is, \$3.50. If e, to lose money. The amount 3.5 is called the ndom variable *X*. The mean is usually den For an expect, on average, to lose money. The amount 3.5 is called the **expectation**,
 Ream of the discrete random variable X. The mean is usually denoted by μ , the
 Mean or expectation

If X is a discrete random v

 $2 \frac{1}{2}$

is ter "mu" (pronounced "mew").
 n or expectation

is a discrete random variable with range of values R and probabil

tion f, then the **mean** (denoted μ), or expectation of X (denoted $E(X$
 $\mu = E(X) = \sum_{x \in R} x f(x)$.

, the incommode that with range of values R and produce the values R and proportion (denoted μ), or **expectation** of X (denoted x f (denoted x f (x).
 $\sum_{R} x f(x)$.

$$
\mu = E(X) = \sum_{x \in R} x f(x).
$$

$$
E(g(X)) = \sum_{x \in R} g(x) f(x).
$$

If X is a discrete random variable with range of values R and probability
function f, then the **mean** (denoted μ), or **expectation** of X (denoted $E(X)$),
is
 $\mu = E(X) = \sum_{x \in R} x f(x)$.
Also, the **expectation** of any functio function f, then the **mean** (denoted μ), or **expectation** of X (denoted $E(X)$),

is
 $\mu = E(X) = \sum_{x \in R} x f(x)$.

Also, the **expectation** of any function $g(X)$ of the random variable X is
 $E(g(X)) = \sum_{x \in R} g(x) f(x)$.

Note that Also, the **expectation** of any function $g(X)$ of the random variable X is
 $E(g(X)) = \sum_{x \in R} g(x) f(x)$.

Note that in this usage $E(X)$ does not define a function of X but a constant (parameter)

associated with the random varia Also, the **expectation** of any function $g(X)$ of the random variable
 $E(g(X)) = \sum_{x \in R} g(x) f(x)$.

Note that in this usage $E(X)$ does not define a function of X but a consta

associated with the random variable X. Note also th x).

y function $g(X)$ of the random variable X is

s not define a function of X but a constant (parameter)

ble X. Note also that if $f(x)$ were a mass density such

en μ would be the moment of the mass about 0 and,
 x Also, the **expectation** of any function $g(X)$ of the random variable X is
 $E(g(X)) = \sum_{x \in R} g(x) f(x)$.

that in this usage $E(X)$ does not define a function of X but a constant (parameter)

ciated with the random variable X. No $E(g(X)) = \sum_{x \in R} g(x) f(x)$.

Note that in this usage $E(X)$ does not define a function of X but a constant

associated with the random variable X. Note also that if $f(x)$ were a mass

as that studied in Section 7.4, then μ w $E(g(X)) = \sum_{x \in R} g(x)$

Note that in this usage $E(X)$ doe

associated with the random varia

as that studied in Section 7.4, th

since the total mass would be \sum

Another parameter used to d

variable is the variable's stan **Example 1.1** $x \in R$

that in this usage $E(X)$ does not define a function of X but a consciated with the random variable X. Note also that if $f(x)$ were a nat studied in Section 7.4, then μ would be the moment of the m that in this usage $E(X)$ does not define a function of X but a constant (parameter)
ciated with the random variable X. Note also that if $f(x)$ were a mass density such
at studied in Section 7.4, then μ would be the mom that in this usage $E(X)$ does not define a function of X but a constant (parameter)
ciated with the random variable X. Note also that if $f(x)$ were a mass density such
at studied in Section 7.4, then μ would be the mom ciated with the random variable X. Note also that if
aat studied in Section 7.4, then μ would be the mon
e the total mass would be $\sum_{x \in R} f(x) = 1$, μ would
Another parameter used to describe the way probabi
able is variable X. Note also that if $f(x)$ were a ma
4, then μ would be the moment of the ma
e $\sum_{x \in R} f(x) = 1$, μ would in fact be the
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ndard deviation.
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om varia X but a constant (parameter)
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X from its mean μ . The
 $f(x)$.

Variance and standard deviation
 Variance and standard deviation

The **variance** of a random variable X with range R and probability function

f is the expectation of the square of the distance of X from its mean μ

$$
\sigma^{2} = \text{Var}(X) = E((X - \mu)^{2}) = \sum_{x \in R} (x - \mu)^{2} f(x).
$$

f is the expectation of the square of the distance of X from its mean μ . The
variance is denoted σ^2 or Var(X).
 $\sigma^2 = \text{Var}(X) = E((X - \mu)^2) = \sum_{x \in R} (x - \mu)^2 f(x)$.
The **standard deviation** of X is the square root of the va variance is denoted σ^2 or Var(X).
 $\sigma^2 = \text{Var}(X) = E((X - \mu)^2) = \sum_{x \in R} (x - \mu)^2 f(x)$.

The **standard deviation** of X is the square root of the variance and therefore

is denoted σ .

The symbol σ is the lowercase Greek $\sigma^2 = \text{Var}(X) = E((X - \mu)^2) = \sum_{x \in R} (x - \mu)^2 f(x)$.

The **standard deviation** of X is the square root of the variance and therefore

is denoted σ .

The symbol σ is the lowercase Greek letter "sigma." (The symbol Σ used $\sigma^2 = \text{Var}(X) = E((X - \mu)^2) = \sum_{x \in R} (x - \mu)^2 f(x)$.

The **standard deviation** of *X* is the square root of the variance and therefore

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The symbol σ is the lowercase Greek letter "sigma." (The symbol Σ use variance is denoted σ^2 or Var(X).
 $\sigma^2 = \text{Var}(X) = E((X - \mu)^2) = \sum_{x \in R} (x - \mu)^2 f(x)$.

The **standard deviation** of X is the square root of the variance and therefore

is denoted σ .

The symbol σ is the lowercase Greek The **standard deviation** of X is the square root of the variance and therefore is denoted σ .
The symbol σ is the lowercase Greek letter "sigma." (The symbol Σ used for summation is an uppercase sigma.) The standa The **standard deviation** of *X* is the square root of the variance and therefore
is denoted σ .
The symbol σ is the lowercase Greek letter "sigma." (The symbol Σ used for sum-
mation is an uppercase sigma.) The st is denoted σ .
The symbol σ is the lowercase Greek letter "sigma." (The symbol Σ used for sum-
mation is an uppercase sigma.) The standard deviation gives a measure of how spread
out the probability distribution o The **standard deviation** of *X* is the square root of the variance
is denoted σ .
The symbol σ is the lowercase Greek letter "sigma." (The symbol
mation is an uppercase sigma.) The standard deviation gives a mea
out **d** σ .
 d σ .
 z is the lowercase Greek letter "sigma." (The symbol Σ used for sum-

phpercase sigma.) The standard deviation gives a measure of how spread

bility distribution of X is. The smaller the standar out the probability distribution of *X* is. The smaller
the probability is concentrated at values of *X* clos
Figure 7.55 illustrate the probability functions of tv
space {1, 2, ..., 9}, one having small σ and one wit
 Example 7.53 and χ (The symbol 2) used for sum-
rd deviation gives a measure of how spread
te smaller the standard deviation, the more
of X close to the mean. Figure 7.54 and
ons of two random variables with sample
d o be dat values of X close to the mean. Figure 7.54 and bability functions of two random variables with sample ag small σ and one with large σ . Note how a significant ty lies between $\mu - \sigma$ and $\mu + \sigma$ in each case. x close to the mean. Figure 7.54 and
ix close to the mean. Figure 7.54 and
ns of two random variables with sample
one with large σ . Note how a significant
 $\mu - \sigma$ and $\mu + \sigma$ in each case. Note also
7.54 is symmetric, mean. Figure 7.54 and
n variables with sample
Note how a significant
in each case. Note also
, resulting in $\mu = 5$, the
7.55 is skewed a bit to
ition of variance can be
 $f(x)$

e {1, 2, ..., 9}, one having small
$$
\sigma
$$
 and one with large σ . Note how a significant
ion of the total probability lies between $\mu - \sigma$ and $\mu + \sigma$ in each case. Note also
the distribution of probability in Figure 7.54 is symmetric, resulting in $\mu = 5$, the
point of the sample space, while the distribution in Figure 7.55 is skewed a bit to
ight, resulting in $\mu > 5$.
Since $\sum_{x \in R} f(x) = 1$, the expression given in the definition of variance can be
itten as follows:

$$
\sigma^2 = \text{Var}(X) = \sum_{x \in R} (x^2 - 2\mu x + \mu^2) f(x)
$$

$$
= \sum_{x \in R} x^2 f(x) - 2\mu \sum_{x \in R} x f(x) + \mu^2 \sum_{x \in R} f(x)
$$

$$
= \sum_{x \in R} x^2 f(x) - 2\mu^2 + \mu^2 = E(X^2) - \mu^2,
$$

that is,
\n
$$
\sigma^2 = \text{Var}(X) = E(X^2) - \mu^2 = E(X^2) - (E(X))^2.
$$
\nTherefore, the standard deviation of *X* is given by
\n
$$
\sigma = \sqrt{E(X^2) - \mu^2}.
$$

$$
\sigma = \sqrt{E(X^2) - \mu^2}.
$$

at is,
 $\sigma^2 = \text{Var}(X) = E(X^2) - \mu^2 = E(X^2) - (E(X))^2$.

herefore, the standard deviation of X is given by
 $\sigma = \sqrt{E(X^2) - \mu^2}$.
 EXAMPLE 3 Find the mean of the random variable X of Example 2. Also find

the expectation of X² = $E(X^2) - \mu^2 = E(X^2) - (E(X))^2$.

and deviation of X is given by
 $-\mu^2$.

Find the mean of the random variable X of Example 2.

the expectation of X^2 and the standard deviation of X.
 $2 \times \frac{1}{2} + 3 \times \frac{2}{2} + 4 \times \frac{3}{2} + 5$ S,
 $\sigma^2 = \text{Var}(X) = E(X^2) - \mu^2 = E(X^2) - (E(X))$

efore, the standard deviation of X is given by
 $\sigma = \sqrt{E(X^2) - \mu^2}$.

AMPLE 3 Find the mean of the random varia

the expectation of X^2 and the stand

tion We have
 $\mu = E(X) = 2 \times$

that is,
\n
$$
\sigma^2 = \text{Var}(X) = E(X^2) - \mu^2 = E(X^2) - (E(X))^2.
$$
\nTherefore, the standard deviation of *X* is given by
\n
$$
\sigma = \sqrt{E(X^2) - \mu^2}.
$$
\n**EXAMPLE 3** Find the mean of the random variable *X* of Example 2. Also find the expectation of X^2 and the standard deviation of *X*.
\n**Solution** We have
\n
$$
\mu = E(X) = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + 6 \times \frac{5}{36} + 7 \times \frac{6}{36}
$$
\n
$$
+ 8 \times \frac{5}{36} + 9 \times \frac{4}{36} + 10 \times \frac{3}{36} + 11 \times \frac{2}{36} + 12 \times \frac{1}{36} = 7,
$$
\na fact that is fairly obvious from the symmetry of the graph of the probability function in Figure 7.53. Also,
\n
$$
E(X^2) = 2^2 \times \frac{1}{36} + 3^2 \times \frac{2}{36} + 4^2 \times \frac{3}{36} + 5^2 \times \frac{4}{36} + 6^2 \times \frac{5}{36}
$$

36 36 36 36 36
\na fact that is fairly obvious from the symmetry of the graph of the probability function
\nin Figure 7.53. Also,
\n
$$
E(X^2) = 2^2 \times \frac{1}{36} + 3^2 \times \frac{2}{36} + 4^2 \times \frac{3}{36} + 5^2 \times \frac{4}{36} + 6^2 \times \frac{5}{36}
$$
\n
$$
+ 7^2 \times \frac{6}{36} + 8^2 \times \frac{5}{36} + 9^2 \times \frac{4}{36} + 10^2 \times \frac{3}{36}
$$
\n
$$
+ 11^2 \times \frac{2}{36} + 12^2 \times \frac{1}{36} = \frac{1,974}{36} \approx 54.8333.
$$
\nThe variance of *X* is $\sigma^2 = E(X^2) - \mu^2 \approx 54.8333 - 49 = 5.8333$, so the standard deviation of *X* is $\sigma \approx 2.4152$.
\n**Continuous Random Variables**
\nNow we consider an example with a continuous range of possible outcomes.
\n**EXAMPLE 4** Suppose that a needle is dropped at random on a flat table with a straight line drawn on it. For each drop, let *X* be the acute an-

The variance of X is $\sigma^2 = E(X^2) - \mu^2 \approx 54.8333 - 49 = 5.8333$, so the standard

 $+11^2 \times \frac{2}{36} + 12^2 \times \frac{1}{36} = \frac{1,974}{36} \approx 54.8333$.

the variance of X is $\sigma^2 = E(X^2) - \mu^2 \approx 54.8333 - 49 = 5.8333$, so the standard
 ontinuous Random Variables
 ontinuous Random Variables

tow we consider an exam 36 36 36
 $s \sigma^2 = E(X^2) - \mu^2 \approx 54.8333 - 49 = 5.8333$, so the standard ≈ 2.4152 .
 dom Variables

example with a continuous range of possible outcomes.

Suppose that a needle is dropped at random on a flat table with a The variance of X is $\sigma^2 = E(X^2) - \mu^2 \approx 54.8333 - 49 = 5.8333$, so the standard deviation of X is $\sigma \approx 2.4152$.
 Continuous Random Variables

Now we consider an example with a continuous range of possible outcomes.
 EXA $+11^2 \times \frac{2}{36} + 12^2 \times \frac{1}{36} = \frac{1.974}{36} \approx 54.8333$.
The variance of X is $\sigma^2 = E(X^2) - \mu^2 \approx 54.8333 - 49 = 5.8333$, so the standard deviation of X is $\sigma \approx 2.4152$.
Continuous Random Variables
Now we consider an exa **Continuous Random Variables**
Now we consider an example with a continuous range of possible outcomes.
 EXAMPLE 4 Suppose that a needle is dropped at random on a flat table with

gle, measured in degrees, that the needl The variance of *X* is $\sigma^2 = E(X^2) - \mu^2 \approx 54.8333 - 49 = 5.8333$, so the standard deviation of *X* is $\sigma \approx 2.4152$.
 Continuous Random Variables

Now we consider an example with a continuous range of possible outcomes.
 Continuous Random Variables
 Continuous Random Variables

Now we consider an example with a continuous range of possible outcomes.
 EXAMPLE 4 Suppose that a needle makes with the line. (See Figure 7.56(a).

Evidentl Now we consider an example with a continuous range of possible outcomes.
 EXAMPLE 4 Suppose that a needle is dropped at random on a flat table with

a straight line drawn on it. For each drop, let X be the acute an-

gl **EXAMPLE 4** Suppose that a needle is dropped at random on a flat table with a straight line drawn on it. For each drop, let X be the acute angle, measured in degrees, that the needle makes with the line. (See Figure 7.56(**EXAMPLE 4** Suppose that a needle is dropped a
gle, measured in degrees, that the needle makes with Evidently, X can take any real value in the interval [0
continuous **random variable**. The probability that X
value is 0. Evidently, *X* can take any real value in the interval [0, 90]; therefore, *X* is **continuous random variable**. The probability that *X* takes on any particular value is 0. (There are infinitely many real numbers in [0, 9 Evidently, *X* can take any real value in the interval [0, 90]; inerefore, *X* is called a
continuous **random variable**. The probability that *X* takes on any particular real
value is 0. (There are infinitely many real nu **CONTIFINIONS TRANGON VARTABLE.** The probability that X takes on any particular real value is 0. (There are infinitely many real numbers in [0, 90], and none is more likely the same as the probability that it lies in any any other.) However, the probability that
ame as the probability that it lies in any of
nerval has length 10 and the interval of
probability is
 $Pr(10 \le X \le 20) = \frac{10}{90} = \frac{1}{9}$.
e generally, if $0 \le x_1 \le x_2 \le 90$, then
However, the probability that *X* lies in some
probability that it lies in any other interval of
length 10 and the interval of all possible valu
is
 ≤ 20) = $\frac{10}{90} = \frac{1}{9}$.
if $0 \leq x_1 \leq x_2 \leq 90$, then
 $\leq x_2$)

$$
Pr(10 \le X \le 20) = \frac{10}{90} = \frac{1}{9}.
$$

$$
Pr(x_1 \le X \le x_2) = \frac{1}{90}(x_2 - x_1).
$$

$$
f(x) = \frac{1}{90}, \qquad 0 \le x \le 90.
$$

Pr($10 \le X \le 20$) = $\frac{10}{90} = \frac{1}{9}$.

More generally, if $0 \le x_1 \le x_2 \le 90$, then

Pr($x_1 \le X \le x_2$) = $\frac{1}{90}(x_2 - x_1)$.

This situation can be conveniently represented as follows: Let $f(x)$ be defined on the interva this probability is
 $Pr(10 \le X \le 20) = \frac{10}{90} = \frac{1}{9}$.

More generally, if $0 \le x_1 \le x_2 \le 90$, then
 $Pr(x_1 \le X \le x_2) = \frac{1}{90}(x_2 - x_1)$.

This situation can be conveniently represented as follows: Let $f(x)$ be defined on t Friend $f(x) \leq X \leq 20$ = $\frac{10}{90} = \frac{1}{9}$.

More generally, if $0 \leq x_1 \leq x_2 \leq 90$, then

Friend $f(x_1 \leq X \leq x_2) = \frac{1}{90}(x_2 - x_1)$.

This situation can be conveniently represented as follows: Let $f(x)$ be defined on Pr(10 ≤ X ≤ 20) = $\frac{1}{90} = \frac{1}{9}$.

More generally, if 0 ≤ x₁ ≤ x₂ ≤ 90, then

Pr(x₁ ≤ X ≤ x₂) = $\frac{1}{90}(x_2 - x_1)$.

This situation can be conveniently represented as follows: Let $f(x)$ be defined on the

int

- Figure 7.56

(a) X is the acute angle, measured in

degrees, that the needle makes with
-

A

a continuous random v

satisfy $a \le x_1 \le x_2 \le$
 $Pr(x_1 \le X \le x_2)$

which is the area abov

vided $f(x) \ge 0$. In or

satisfy two conditions:

Note that this definition of

probability density function

generalizes the prob a commutation state
satisfy $a \le x_1 \le x_2$
which is the area in the satisfy $a \le x_1 \le x_2$
which is the area in the satisfy two conditions of
probability density function
generalizes the probability
function used in the dis Sausiy $a \le x_1$
 $Pr(x_1 \le X$

which is the armodology of the probability

Note that this definition of

probability density function

generalizes the probability

function used in the discrete case

if we regard the bar gra **Function 19** Propose in the discrete case

function used in the discrete case

function used in the discrete case

if we regard the bar graphs there are allows the probability

time of the discrete case

if we regard the $ext{Prove that this definition of probability density function is the area above to be given by the formula $y = 0$. In order, $y = 0$, and $y = 0$, and $y = 0$. In order, $y = 0$, and $y = 0$, and $y = 0$, and $y = 0$. In order, $y = 0$ and $y = 0$, and $y = 0$, and $y = 0$. In order, $y = 0$, and $y =$$ which is the area at

vided $f(x) \ge 0$. In

satisfy two condition

probability density function

generalizes the probability

function used in the discrete case

if we regard the bar graphs there

as the graphs of step fun which is the a

vided $f(x) \ge$

Satisfy two comprobability density function

generalizes the probability

function used in the discrete case

if we regard the bar graphs there

as the graphs of step functions

with unit ba

 $\frac{1}{4}$ A function defined on an interval [a, b] is a probal
a continuous random variable X distributed on [a, is
satisfy $a \le x_1 \le x_2 \le b$, we have
 $Pr(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x) dx$,
which is the area above the interval [x_1, x_2

$$
\Pr(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x) \, dx,
$$

Probability density functions

A function defined on an interval [a, b] is a probability density function for

a continuous random variable X distributed on [a, b] if, whenever x_1 and x_2

attisfy $a \le x_1 \le x_2 \le b$, ion defined on an interval [a, b] is a probability density function for

uous random variable X distributed on [a, b] if, whenever x_1 and x_2
 $\{x_1 \le x_2 \le b$, we have
 $x_1 \le X \le x_2$ = $\int_{x_1}^{x_2} f(x) dx$,

s the area

-
- (b) $\int_a^b f(x) dx = 1$ (Pr($a \le X \le b$) = 1).

 $Pr(x_1 \le X \le x_2) = \int_{x_1}^{x_2} f(x) dx$,
which is the area above the interval $[x_1, x_2]$ and under the graph of f, pro-
vided $f(x) \ge 0$. In order to be such a probability density function, f must
satisfy two conditions:
(a) $f(x)$ which is the area above the interval $[x_1, x_2]$ and under the graph of f, provided $f(x) \ge 0$. In order to be such a probability density function, f must satisfy two conditions:

(a) $f(x) \ge 0$ on $[a, b]$ (probability cannot which is the area above the interval $[x_1, x_2]$ and under the graph of f, provided $f(x) \ge 0$. In order to be such a probability density function, f must satisfy two conditions:

(a) $f(x) \ge 0$ on $[a, b]$ (probability cannot which is the area above the interval $[x_1, x_2]$ and under the vided $f(x) \ge 0$. In order to be such a probability density satisfy two conditions:

(a) $f(x) \ge 0$ on [a, b] (probability cannot be negative (b) $\int_a^b f(x) dx = 1$ vided $f(x) \ge 0$. In order to be such a probability density function, f must
satisfy two conditions:
(a) $f(x) \ge 0$ on [a, b] (probability cannot be negative) and
(b) $\int_a^b f(x) dx = 1$ (Pr($a \le X \le b$) = 1).
e ideas extend to ra stats if we conditions:

(a) $f(x) \ge 0$ on [a, b] (probability cannot be negative) and

(b) $\int_a^b f(x) dx = 1$ (Pr($a \le X \le b$) = 1).

These ideas extend to random variables distributed on semi-infinite or infinite intervals,

b where the ace above the metrod [x₁, x₂] and under the graph or *f*, provided $f(x) \ge 0$. In order to be such a probability density function, *f* must satisfy two conditions:

(a) $f(x) \ge 0$ on [a, b] (probability cannot (a) $f(x) \ge 0$ on [a, b] (probability c

(b) $\int_a^b f(x) dx = 1$ (Pr($a \le X \le$

be ideas extend to random variables distribute

the integrals appearing will be improper

ed by sums in the analysis of discrete random

ontinuous r

but the integrals appearing will be improper in those cases. In any event, the role
played by sums in the analysis of discrete random variables is taken over by integrals
for continuous random variables.
In the example of played by sums in the analysis of discrete rando
for continuous random variables.
In the example of the dropping needle, the j
izontal straight line graph, and we termed such a
uniform probability density function on the

$$
f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise.} \end{cases}
$$

In the example of the dropping needle, the probability density function has a hor-
ontal straight line graph, and we termed such a probability distribution uniform. The
inform probability density function on the interval graph, and we termed such a probability distribution uniform. The
density function on the interval [a, b] is
 $\frac{1}{a}$ if $a \le x \le b$
otherwise.
as are commonly encountered as density functions for continuous
(**The exponent** uniform probability density function on the interval [a, b] is
 $f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$

Many other functions are commonly encountered as density functions for continuous

random variables.
 EXAMP $f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise.} \end{cases}$

Many other functions are commonly encountered as density functions for continuous

random variables.
 EXAMPLE 5 (The exponential distribution) The length of time *T* that $f(x) = \begin{cases} \frac{1}{b-a} & \text{if } a \leq x \leq b \\ 0 & \text{otherwise.} \end{cases}$
y other functions are commonly encountered as density fun
om variables.
AMPLE 5 (The exponential distribution) The lengt
particular atom in a radioactive sample sur-
ran Many other functions are commonly encountered as density functions for continuous
random variables.
EXAMPLE 5 (The exponential distribution) The length of time *T* that any
particular atom in a radioactive sample surviv (x) = $\begin{pmatrix} b-a \\ 0 \end{pmatrix}$ otherwise.

other functions are commonly encountered as density functions for contin

m variables.
 IMPLE 5 (**The exponential distribution**) The length of time *T* than

particular atom in a ra **EXAMPLE 5** (The exponential distribution) The length of time *T* that any particular atom in a radioactive sample survives before decaying is a random variable taking values in $[0, \infty)$. It has been observed that the pr random variables.
 EXAMPLE 5 (The exponential distribution) The length of time *T* that any

is a random variable taking values in $[0, \infty)$. It has been observed that the proportion

of atoms that survive to time *t* b **The exponential distribution**) The length of time *T* the articular atom in a radioactive sample survives before defaking values in $[0, \infty)$. It has been observed that the proto to time *t* becomes small exponentially a

$$
Pr(T \ge t) = Ce^{-t}
$$

:

$$
\int_t^{\infty} f(x) dx = \Pr(T \ge t) = Ce^{-kt}.
$$

urvives before decaying

stread that the proportion

st increases; thus,

uble T. Then

undamental Theorem of

c. C is determined by the

m $(e^{-kR} - 1) = C$. is a random variable taking values in $[0, \infty)$. It has been observ
of atoms that survive to time *t* becomes small exponentially as $Pr(T \ge t) = Ce^{-kt}$.
Let *f* be the probability density function for the random variab
 \int_t^{\in $0 \quad y \in I$ at $x = 1$. We find particular atom in a radioactive sample

random variable taking values in $[0, \infty)$. It has been obs

oms that survive to time t becomes small exponentially

Pr($T \ge t$) = Ce^{-kt} .

f be the probability density function for ive to time *t* becomes small exponentially as
 Ce^{-kt} .

ability density function for the random variat
 $= Pr(T \ge t) = Ce^{-kt}$.

is equation with respect to *t* (using the Fun
 $\sin - f(t) = -Cke^{-kt}$, so $f(t) = Cke^{-kt}$.
 $\cos^2 f(t) dt = 1$. We i in a radioactive sample survives before \sin [0, ∞). It has been observed that the pr
mes small exponentially as *t* increases; th
nction for the random variable *T*. Then
 Ce^{-kt} .
respect to *t* (using the Fundament of atoms that survive to time *t* becomes small exponentially as *t* increases; thus,
 $Pr(T \ge t) = Ce^{-kt}$.

Let *f* be the probability density function for the random variable *T*. Then
 $\int_{t}^{\infty} f(x) dx = Pr(T \ge t) = Ce^{-kt}$.

Differen

$$
1 = Ck \int_0^\infty e^{-kt} dt = \lim_{R \to \infty} Ck \int_0^R e^{-kt} dt = -C \lim_{R \to \infty} (e^{-kR} - 1) = C.
$$

EXAMPLE 6 For what value of C is $f(x) = C(1 - x^2)$ a probability density function on [-1, 1]? If X is a random variable with this density, that is the probability that $X \le 1/2$? (a) a probability density
able with this density, For what value of *C* is $f(x) = C(1 - x^2)$ a probability density
function on $[-1, 1]$? If *X* is a random variable with this density,
ity that $X \le 1/2$?
that $f(x) \ge 0$ on $[-1, 1]$ if $C \ge 0$. Since **EXAMPLE 6** For what value of *C* is $f(x) = C(1 - x^2)$ function on $[-1, 1]^2$ If *X* is a random var what is the probability that $X \le 1/2$?
 Solution Observe that $f(x) \ge 0$ on $[-1, 1]$ if $C \ge 0$. Since **EXAMPLE 6** For what value of *C* is $f(x) = C(1 - x^2)$ a probability density function on $[-1, 1]$? If *X* is a random variable with this density, what is the probability that $X \le 1/2$?
Solution Observe that $f(x) \ge 0$ on **PLE 6** For what value of *C* is $f(x) = C$
function on $[-1, 1]$? If *X* is a ran
he probability that $X \le 1/2$?
(b) Observe that $f(x) \ge 0$ on $[-1, 1]$ if $C \ge 0$
 $f(x) dx = C \int_{-1}^{1} (1 - x^2) dx = 2C \left(x - \frac{x}{2}\right)$ of *C* is $f(x) = C(1 - x^2)$ a probability
1, 1]? If *X* is a random variable with this
2?
00 [-1, 1] if *C* \geq 0. Since
0) $dx = 2C\left(x - \frac{x^3}{3}\right)\Big|^{1} = \frac{4C}{3}$, **EXAMPLE 6** For what value of *C* is $f(x) = C(1 - x^2)$ a probability density
what is the probability that $X \le 1/2$?
Solution Observe that $f(x) \ge 0$ on [-1, 1] if $C \ge 0$. Since
 $\int_{-1}^{1} f(x) dx = C \int_{-1}^{1} (1 - x^2) dx = 2C \left(x - \frac{x$ of *C* is $f(x) = C(1 - x^2)$ a probability

1]? If *X* is a random variable with this

?
 $\left[-1, 1 \right]$ if $C \ge 0$. Since
 $dx = 2C\left(x - \frac{x^3}{3}\right)\Big|_0^1 = \frac{4C}{3}$,

notion if $C = 3/4$. In this case
 $\left(x - \frac{x^3}{3} \right)\Big|_0^{1/2}$

$$
\int_{-1}^{1} f(x) dx = C \int_{-1}^{1} (1 - x^2) dx = 2C \left(x - \frac{x^3}{3} \right) \Big|_{0}^{1} = \frac{4C}{3},
$$

$$
\int_{-1}^{1} f(x) dx = C \int_{-1}^{1} (1 - x^2) dx = 2C \left(x - \frac{x^3}{3} \right) \Big|_{0}^{1} = \frac{4C}{3},
$$

\n $f(x)$ will be a probability density function if $C = 3/4$. In this case
\n
$$
\Pr\left(X \le \frac{1}{2}\right) = \frac{3}{4} \int_{-1}^{1/2} (1 - x^2) dx = \frac{3}{4} \left(x - \frac{x^3}{3}\right) \Big|_{-1}^{1/2}
$$
\n
$$
= \frac{3}{4} \left(\frac{1}{2} - \frac{1}{24} - (-1) + \frac{-1}{3}\right) = \frac{27}{32}.
$$
\nBy analogy with the discrete case, we formulate definitions for the mean (or expectation), variance, and standard deviation of a continuous random variable as follows:
\n**DEFINITIONS**
\nIf X is a continuous random variable on [a, b] with probability density function $f(x)$, the mean μ , (or expectation $E(X)$) of X is
\n
$$
\mu = E(X) = \int_{-1}^{b} x f(x) dx.
$$

with the discrete case, we formulate definitiance, and standard deviation of a continuous
i.e. a continuous random variable on [a, b] with
(x), the **mean** μ , (or **expectation** $E(X)$) of X
= $E(X) = \int_a^b x f(x) dx$.
pectatio rete case, we formulate definitions for the mean (order dard deviation of a continuous random variable as
andom variable on [a, b] with probability density f
 u , (or **expectation** $E(X)$) of X is
b
 $x f(x) dx$.
iunction g of

5 malogy with the discrete case, we formulate definitions for the
n), variance, and standard deviation of a continuous random var
If X is a continuous random variable on [a, b] with probability d
tion $f(x)$, the **mean** μ

$$
u = E(X) = \int_{a}^{b} x f(x) dx.
$$

$$
E(g(X)) = \int_{a}^{b} g(x) f(x) dx.
$$

 $\mu = E(X) = \int_{a}^{b} x f(x) dx.$
The expectation of a function g of X is
 $E(g(X)) = \int_{a}^{b} g(x) f(x) dx.$
Similarly, the **variance** σ^2 of X is the mean of the from its mean:
 $\sigma^2 = \text{Var}(X) = E((X - x)^2) = \int_{a}^{b} (x - y) dx.$ c) dx.

of X is the mean of the squared deviation of X
 $y^{2} = \int_{0}^{b} (x - y)^{2} f(x) dx$ Example 3 squared deviation of X
 $f(x) dx$, $E(g(X)) = \int_a^b g(x) f(x) dx$.

Similarly, the **variance** σ^2 of X is the mean of the squared deviation of X

from its mean:
 $\sigma^2 = \text{Var}(X) = E((X - \mu)^2) = \int_a^b (x - \mu)^2 f(x) dx$,

and the **standard deviation** is the square root of the var Similarly, the **variance** σ^2 of X is the mean of the squared deviation of X
from its mean:
 $\sigma^2 = \text{Var}(X) = E((X - \mu)^2) = \int_a^b (x - \mu)^2 f(x) dx$,
and the **standard deviation** is the square root of the variance.
As was the case fo

$$
\mu = E(X) = \int_{a}^{b} x f(x) dx.
$$

The expectation of a function g of X is

$$
E(g(X)) = \int_{a}^{b} g(x) f(x) dx.
$$

Similarly, the **variance** σ^{2} of X is the mean of the squared deviation of
from its mean:

$$
\sigma^{2} = \text{Var}(X) = E((X - \mu)^{2}) = \int_{a}^{b} (x - \mu)^{2} f(x) dx,
$$

$$
\sigma^2 = E(X^2) - \mu^2
$$
, $\sigma = \sqrt{E(X^2) - \mu^2}$.

 $\sigma^2 = \text{Var}(X) = E((X - \mu)^2) = \int_a^{\infty} (x - \mu)^2 f(x) dx$,

and the **standard deviation** is the square root of the variance.

As was the case for a discrete random variable, it is easily shown that
 $\sigma^2 = E(X^2) - \mu^2$, $\sigma = \sqrt{E(X^2) - \mu^$ **but in the standard deviation** is the square root of the variance.

As was the case for a discrete random variable, it is easily shown that
 $\sigma^2 = E(X^2) - \mu^2$, $\sigma = \sqrt{E(X^2) - \mu^2}$.

Again the standard deviation gives a m and the **standard deviation** is the square root of the variance.

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 $\sigma^2 = E(X^2) - \mu^2$, $\sigma = \sqrt{E(X^2) - \mu^2}$.

Again the standard deviation gives a meas As was the case for a discrete random variable, it is easily shown that
 $\sigma^2 = E(X^2) - \mu^2$, $\sigma = \sqrt{E(X^2) - \mu^2}$.

Again the standard deviation gives a measure of how spread out the probability distri-

bution of X is. The $\sigma^2 = E(X^2) - \mu^2$, $\sigma = \sqrt{E(X^2) - \mu^2}$.

gain the standard deviation gives a measure of how spread out the probability distri-

tution of X is. The smaller the standard deviation, the more concentrated is the area

nder t μ^2 , $\sigma = \sqrt{E(X^2) - \mu^2}$.

deviation gives a measure of how spread out the probability distri-

smaller the standard deviation, the more concentrated is the area

arrow around the mean, and so the smaller is the probab $\sigma^2 = E(X^2) - \mu^2$, $\sigma = \sqrt{E(X^2) - \mu^2}$.

Again the standard deviation gives a measure of how spread out the probability distribution of *X* is. The smaller the standard deviation, the more concentrated is the area under t Example 1. The smaller the standard deviation gives a measure of
bution of X is. The smaller the standard deviation
under the density curve around the mean, and s
value of X will be far away from the mean. (See
EXAMPLE 7 n the standard deviation gives a measure of how s
on of X is. The smaller the standard deviation, there the density curve around the mean, and so the s
e of X will be far away from the mean. (See Figure
 $\overline{AMPLE 7}$ Find

 $X \leq \mu + \sigma$).

$$
\mu = E(X) = \int_{a}^{b} \frac{x}{b-a} dx = \frac{1}{b-a} \frac{x^2}{2} \bigg|_{a}^{b} = \frac{1}{2} \frac{b^2 - a^2}{b-a} = \frac{b+a}{2}.
$$

$$
E(X^{2}) = \int_{a}^{b} \frac{x^{2}}{b-a} dx = \frac{1}{b-a} \frac{x^{3}}{3} \Big|_{a}^{b} = \frac{1}{3} \frac{b^{3} - a^{3}}{b-a} = \frac{b^{2} + ab + a^{2}}{3}.
$$

\nHence, the variance is
\n
$$
\sigma^{2} = E(X^{2}) - \mu^{2} = \frac{b^{2} + ab + a^{2}}{3} - \frac{b^{2} + 2ab + a^{2}}{4} = \frac{(b-a)^{2}}{12},
$$

\nand the standard deviation is
\n
$$
\sigma = \frac{b-a}{2\sqrt{3}} \approx 0.29(b-a).
$$

$$
\sigma^{2} = E(X^{2}) - \mu^{2} = \frac{b^{2} + ab + a^{2}}{3} - \frac{b^{2} + 2ab + a^{2}}{4} = \frac{(b-a)^{2}}{12},
$$

$$
\sigma = \frac{b-a}{2\sqrt{3}} \approx 0.29(b-a).
$$

Finally,

$$
\Pr(\mu - \sigma \le X \le \mu + \sigma) = \int_{\mu - \sigma}^{\mu + \sigma} \frac{dx}{b - a} = \frac{1}{b - a} \frac{2(b - a)}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \approx 0.577.
$$

 $\sigma = \frac{b-a}{2\sqrt{3}} \approx 0.29(b-a).$

inally,
 $\Pr(\mu - \sigma \le X \le \mu + \sigma) = \int_{\mu-\sigma}^{\mu+\sigma} \frac{dx}{b-a} = \frac{1}{b-a} \frac{2(b-a)}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \approx 0.577.$
 EXAMPLE 8 Find the mean μ and the standard deviation σ of a random variable

x distribu viation is
 $\leq \mu + \sigma$) = $\int_{\mu-\sigma}^{\mu+\sigma} \frac{dx}{b-a} = \frac{1}{b-a} \frac{2(b-a)}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \approx 0.577.$

Find the mean μ and the standard deviation σ of a random variable *X* distributed exponentially with density function f Finally,
 $Pr(\mu - \sigma \le X \le \mu + \sigma) = \int_{\mu - \sigma}^{\mu + \sigma} \frac{dx}{b - a} = \frac{1}{b - a} \frac{2(b - a)}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \approx 0.577.$
 EXAMPLE 8 Find the mean μ and the standard deviation σ of a random variable X distributed exponentially with

on the interval
$$
[0, \infty)
$$
. Find $Pr(\mu - \sigma \le X \le \mu + \sigma)$.
\n**Solution** We use integration by parts to find the mean:
\n
$$
\mu = E(X) = k \int_0^\infty xe^{-kx} dx
$$
\n
$$
= \lim_{R \to \infty} k \int_0^R xe^{-kx} dx
$$
\nLet $U = x$, $dV = e^{-kx} dx$.
\nThen $dU = dx$, $V = -e^{-kx}/k$.
\n
$$
= \lim_{R \to \infty} \left(-xe^{-kx} \Big|_0^R + \int_0^R e^{-kx} dx \right)
$$
\n
$$
= \lim_{R \to \infty} \left(-Re^{-kR} - \frac{1}{k} (e^{-kR} - 1) \right) = \frac{1}{k}, \text{ since } k > 0.
$$
\nThus, the mean of the exponential distribution is $1/k$. This fact can be quite useful in determining the value of k for an exponentially distributed random variable. A similar integration by parts enables us to evaluate
\n
$$
E(X^2) = k \int_0^\infty x^2 e^{-kx} dx = 2 \int_0^\infty xe^{-kx} dx = \frac{2}{k^2}.
$$

;

$$
E(X^{2}) = k \int_{0}^{\infty} x^{2} e^{-kx} dx = 2 \int_{0}^{\infty} x e^{-kx} dx = \frac{2}{k^{2}},
$$

so the variance of the exponential distribution is
\n
$$
\sigma^2 = E(X^2) - \mu^2 = \frac{1}{k^2},
$$
\nand the standard deviation is equal to the mean
\n
$$
\sigma = \mu = \frac{1}{k}.
$$

$$
\sigma=\mu=\frac{1}{k}.
$$

so the variance of the exponential distribution is
\n
$$
\sigma^2 = E(X^2) - \mu^2 = \frac{1}{k^2},
$$
\nand the standard deviation is equal to the mean
\n
$$
\sigma = \mu = \frac{1}{k}.
$$
\nNow we have
\n
$$
Pr(\mu - \sigma \le X \le \mu + \sigma) = Pr(0 \le X \le 2/k)
$$
\n
$$
= k \int_0^{2/k} e^{-kx} dx
$$
\n
$$
= -e^{-kx} \Big|_0^{2/k}
$$
\n
$$
= 1 - e^{-2} \approx 0.86,
$$
\nwhich is independent of the value of k. Exponential densities for small and large values
\nof k are graphed in Figure 7.58.

Example 1.1 The **Normal Distribution**
 $\frac{1}{k}$
 The Normal Distribution

The most important probability distributions are the so-called **normal** or **Gaussian**

distributions. Such distributions govern the behaviour **is a family of normal Distribution**
 i $\frac{1}{k}$
 is a family of normal distributions govern the behaviour of many interesting **called the standard normal probability density**
 called the standard normal Distribution

The most important probability distributions are the so-called **normal** or **Gaussian**

distributions. Such distributions govern function: **The Normal Distribution**

The most important probability

distributions. Such distribution

variables, in particular, those as

is a family of normal distribution

called the **standard normal dist**

function:
 DEFINITIO Normal Distribution
most important probability distributions are the so-called **normal** or **Gau**
ibutions. Such distributions govern the behaviour of many interesting ra
ables, in particular, those associated with rando ¹/_{*k*} $\frac{1}{x}$ $\frac{1}{x}$ $\frac{1}{x}$ **Contains 10 introduced to the important probability distributions are tons. Such distributions govern the behad, in particular, those associated with rand ly of normal distribution The Normal Distribution**
 The most important probability distributions are the so-called **normal** or **Gaussian** distributions, Such distributions govern the behaviour of many interesting random variables, in particula variables, in particular, those associated with random errors in measurements. There
is a family of normal distributions, all related to the particular normal distribution
called the **standard normal distribution**, which

$$
f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad -\infty < z < \infty.
$$

graph of the standard normal distribution, which has the following probability density function:
 The standard normal distribution, which has the following probability density function:
 The standard normal probability The standard normal probability density
 $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, $-\infty < z < \infty$.

common to use z to denote the random variable in the standard normal distributions are obtained from this one by a change of varial

h of is the standard normal distribution;
one by a change of variable. The
ell shape, as shown in Figure 7.59.
has no elementary antiderivative, The standard normal probability density
 $f(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \qquad -\infty < z < \infty.$

It is common to use z to denote the random variable in the

the other normal distributions are obtained from this one

graph of the standa

$$
I = \int_{-\infty}^{\infty} e^{-z^2/2} dz
$$

functions

 $1 \qquad \qquad 210$ $\frac{1}{\sqrt{2\pi}}e^{-z^2/2}$ Since $z e^{-z}$

ECTION 7.8: Probability 445
cannot be evaluated using the Fundamental Theorem of Calculus, although it is a con-
vergent improper integral. The integral can be evaluated using techniques of multi-
variable calculus involv section 7.8: Probability **445**
cannot be evaluated using the Fundamental Theorem of Calculus, although it is a con-
vergent improper integral. The integral can be evaluated using techniques of multi-
variable calculus inv sECTION 7.8: Probability **445**
cannot be evaluated using the Fundamental Theorem of Calculus, although it is a con-
vergent improper integral. The integral can be evaluated using techniques of multi-
variable calculus inv sECTION 7.8: Probability **44!**
cannot be evaluated using the Fundamental Theorem of Calculus, although it is a con
vergent improper integral. The integral can be evaluated using techniques of multi
variable calculus invol SECTION 7.8: Probability **445**

2. of Calculus, although it is a con-

1. understanding techniques of multi-

2. times that the above-

2. of two variables. (We do so

2. of the above-

2. of the above-

2. of the above-
 section 7.8: Probability **445**
cannot be evaluated using the Fundamental Theorem of Calculus, although it is a con-
vergent improper integral. The integral can be evaluated using techniques of multi-
variable calculus inv evaluated using the Fundamental Theorem of
mproper integral. The integral can be evalua
alculus involving double integrals of function
le 4 of Section 14.4.) The value is $I = \sqrt{2\pi}$
andard normal density $f(z)$ is indeed cannot be evaluated using the Fundamental Theorem of Calculus, although it is a convergent improper integral. The integral can be evaluated using techniques of multi-
variable calculus involving double integrals of functi vergent improper integral. The integral can be evaluated using techniques c
variable calculus involving double integrals of functions of two variables. (V
in Example 4 of Section 14.4.) The value is $I = \sqrt{2\pi}$, which ens

$$
\int_{-\infty}^{\infty} f(z) dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1.
$$

 $\int_{-\infty}^{\infty} f(z) dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz = 1.$

Since $ze^{-z^2/2}$ is an odd function of z and its integral on $(-\infty, \infty)$ converges, the mean

of the standard normal distribution is 0:
 $\mu = E(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty$

$$
\int_{-\infty} f(z) dz = \frac{1}{\sqrt{2\pi}} \int_{-\infty} e^{-z^2/2} dz = 1.
$$

\nSince $ze^{-z^2/2}$ is an odd function of z and its integral on $(-\infty, \infty)$ converges
\nof the standard normal distribution is 0:
\n
$$
\mu = E(Z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} ze^{-z^2/2} dz = 0.
$$

\nWe calculate the variance of the standard normal distribution using integration
\nas follows:
\n
$$
\sigma^2 = E(Z^2)
$$

$$
\sigma^2 = E(Z^2)
$$

\n
$$
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} dz
$$

\n
$$
= \frac{1}{\sqrt{2\pi}} \lim_{R \to \infty} \int_{-R}^{R} z^2 e^{-z^2/2} dz
$$
 Let $U = z$, $dV = ze^{-z^2/2} dz$.
\nThen $dU = dz$, $V = -e^{-z^2/2}$.
\n
$$
= \frac{1}{\sqrt{2\pi}} \lim_{R \to \infty} \left(-ze^{-z^2/2} \Big|_{-R}^{R} + \int_{-R}^{R} e^{-z^2/2} dz \right)
$$

\n
$$
= \frac{1}{\sqrt{2\pi}} \lim_{R \to \infty} (-2Re^{-R^2/2}) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz
$$

\n
$$
= 0 + 1 = 1.
$$

\nHence, the standard deviation of the standard normal distribution is 1.
\nOther normal distributions are obtained from the standard normal distribution by a change of variable.
\n**DEFINITION**
\n**The general normal distribution**
\nA random variable X on $(-\infty, \infty)$ is said to be *normally distributed with mean μ and standard deviation σ (where μ is any real number and $\sigma > 0$) if its probability density function $f_{\mu,\sigma}$ is given in terms of the standard normal density f by*

= $\frac{1}{\sqrt{2\pi}} \lim_{R \to \infty} (-2Re^{-R^2/2}) + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} dz$

= 0 + 1 = 1.

ce, the standard deviation of the standard normal distribution is 1.

Other normal distributions are obtained from the standard normal if θ is equivalently that the standard normal distribution is 1.

Other normal distributions are obtained from the standard normal distribution by
 if the general normal distribution

A random variable X on $(-\infty, \infty$ its probability density function $f_{\mu,\sigma}$ is given in terms of the standard normal density f by The general normal distribution

A random variable X on $(-\infty, \infty)$ is said to be *normally distributed with*
 mean μ and *standard deviation* σ (where μ is any real number and $\sigma > 0$) if

its probability dens A random variable X on $(-\infty, \infty)$ is sa
 mean μ and *standard deviation* σ (where

its probability density function $f_{\mu,\sigma}$ is giv

density f by
 $f_{\mu,\sigma}(x) = \frac{1}{\sigma} f\left(\frac{x - \mu}{\sigma}\right) = \frac{1}{\sigma \sqrt{2\pi}}$

(See Figure Mom variable *X* on $(-\infty, \infty)$ is said to be *no*
 μ and *standard deviation* σ (where μ is any reachologility density function $f_{\mu,\sigma}$ is given in terms

ty *f* by
 $f_{\mu,\sigma}(x) = \frac{1}{\sigma} f\left(\frac{x - \mu}{\sigma}\right) = \frac{1}{\sigma \$ $f(\cos \alpha)$ is said to be *normally distribute*
 iation σ (where μ is any real number and σ :

action $f_{\mu,\sigma}$ is given in terms of the standard r
 $\left(\frac{-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$.

change of vari

$$
f_{\mu,\sigma}(x) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}.
$$

Figure 7.60.) Using the change of variable $z = (x-\mu)/\sigma$, $dz = dx/\sigma$
by that

$$
\int_{-\infty}^{\infty} f_{\mu,\sigma}(x) dx = \int_{-\infty}^{\infty} f(z) dz = 1,
$$

$$
\int_{\mu,\sigma}^{\infty} f_{\mu,\sigma}(x) ds = \int_{-\infty}^{\infty} f(z) dz = 1,
$$

$$
\int_{\mu,\sigma}^{\infty} f_{\mu,\sigma}(x) ds = \int_{-\infty}^{\infty} f(z) dz = 1.
$$

$$
E(X) = \mu \quad \text{and} \quad E((X-\mu)^2) = \sigma^2.
$$

 $f_{\mu,\sigma}(x) = \frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$.

(See Figure 7.60.) Using the change of variable $z = (x - \mu)/\sigma$, $dz = dx/\sigma$, we can verify that
 $\int_{-\infty}^{\infty} f_{\mu,\sigma}(x) dx = \int_{-\infty}^{\infty} f(z) dz = 1$,

so $f_{\mu,\sigma}(x)$ i $y = f_{\mu,\sigma}(x)$ (See Figure 7.60.) Using the change of variable $z = (x - \mu)/\sigma$, $dz = dx/\sigma$, we can

$$
J_{\mu,\sigma}(x) = \frac{\sigma}{\sigma} \int \left(\frac{\sigma}{\sigma}\right)^2 = \frac{\sigma}{\sigma \sqrt{2\pi}} e^{-\frac{\sigma}{\sigma} \sqrt{2\pi}}.
$$

(See Figure 7.60.) Using the change of variable $z = (x - \mu)/\sigma$, $dz = \mu$ verify that

$$
\int_{-\infty}^{\infty} f_{\mu,\sigma}(x) dx = \int_{-\infty}^{\infty} f(z) dz = 1,
$$
so $f_{\mu,\sigma}(x)$ is indeed a probability density function. Using the same change
we can show that

$$
E(X) = \mu \quad \text{and} \quad E((X - \mu)^2) = \sigma^2.
$$

 $\int_{-\infty}^{\infty} f_{\mu,\sigma}(x) dx = \int_{-\infty}^{\infty} f(z) dz = 1,$
so $f_{\mu,\sigma}(x)$ is indeed a probability density function. Using the same change of variable,
we can show that
 $E(X) = \mu$ and $E((X - \mu)^2) = \sigma^2$.
Hence, the density $f_{\mu,\sigma}$ does i

$$
E(X) = \mu
$$
 and $E((X - \mu)^2) = \sigma^2$.

To find the probability $Pr(Z \le z)$ we compute what is called the **cumulative**
ibution function of a random variable with standard normal distribution,
 $F(z) = \frac{1}{z} \int_{z}^{z} e^{-x^2/2} dx = Pr(Z \le z)$ To find the probability $Pr(Z \le z)$ we compute what is called the **cumulative distribution function** of a random variable with standard normal distribution,
 $F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx = Pr(Z \le z),$ To find the probability $Pr(Z \le z)$ we consider the violation function of a random variable with $F(z) = \frac{1}{\sqrt{2\pi}} \int^{z} e^{-x^2/2} dx = Pr(Z \le z)$

$$
F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx = \Pr(Z \le z),
$$

To find the probability $Pr(Z \le z)$ we compute what is called the **cumulative**
distribution function of a random variable with standard normal distribution,
 $F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx = Pr(Z \le z)$,
which represents the To find the probability $Pr(Z \le z)$ we compute what is called the **cumulative**
distribution function of a random variable with standard normal distribution,
 $F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx = Pr(Z \le z)$,
which represents the To find the probability $Pr(Z \le z)$ we compute what is called t

distribution function of a random variable with standard normal distri
 $F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx = Pr(Z \le z),$

which represents the area under the stand z, as shown in Figure 7.61. According to the definition of the error function in Section 6.4, an antiderivative of $e^{-z^2/2}$ is $\sqrt{2/\pi}$ erf($z/\sqrt{2}$). Thus, To find the probability $Pr(Z \le z)$ we c
 ibution function of a random variable with
 $F(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-x^2/2} dx = Pr(Z \le \theta)$

th represents the area under the standard n

shown in Figure 7.61. According to the d

an

$n - \infty$ to z				\boldsymbol{z}		λ				
										For convenience in the following examples and exercises, we include an abbreviated lookup table for this expression. Alternatively, $F(z)$ is easily defined in Maple to cal- culate to any desired number of decimal places, say 10, using the known error function:
			> $F := x$ -> $(1/2) * (erf(z/sqrt(2)) + 1);$							
		which can then be used to calculate values of F . See the following examples.								
Table 3.		Values of the standard normal distribution function $F(z)$ (rounded to 3 decimal places)								
\mathcal{Z}	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
-3.0	0.001	0.001	0.001	0.000	0.000	0.000	0.000	0.000	0.000	0.000
-2.0	0.023	0.018	0.014	0.011	0.008	0.006	0.005	0.003	0.003	0.002
-1.0	0.159	0.136	0.115	0.097	0.081	0.067	0.055	0.045	0.036	0.029
-0.0	0.500	0.460	0.421	0.382	0.345	0.309	0.274	0.242	0.212	0.184
0.0	0.500	0.540	0.579	0.618	0.655	0.691	0.726	0.758	0.788	0.816
1.0	0.841	0.864	0.885	0.903	0.919	0.933	0.945	0.955	0.964	0.971
2.0	0.977	0.982	0.986	0.989	0.992	0.994	0.995	0.997	0.997	0.998
3.0	0.999	0.999	0.999	1.000	1.000	1.000	1.000	1.000	1.000	1.000
		EXAMPLE 9 three decimal places or using Maple to 10 decimal places. Solution Using values from the table, we obtain	$Pr(-1.2 \le Z \le 2.0) = Pr(Z \le 2.0) - Pr(Z < -1.2)$				If Z is a standard normal random variable, find (a) Pr($-1.2 \le Z \le 2.0$), and (b) Pr($Z \ge 1.5$), using the table to $= F(2.0) - F(-1.2) \approx 0.977 - 0.115$			

0.982 0.986 0.989 0.992 0.994 0.995 0.997 0.997 0.998
\n0.999 0.999 1.000 1.000 1.000 1.000 1.000 1.000 1.000
\n**AMPLE 9** If Z is a standard normal random variable, find
\n(a) Pr(
$$
-1.2 \le Z \le 2.0
$$
), and (b) Pr(Z ≥ 1.5), using the table to
\ndecimal places or using Maple to 10 decimal places.
\n**tion** Using values from the table, we obtain
\n
$$
Pr(-1.2 \le Z \le 2.0) = Pr(Z \le 2.0) - Pr(Z < -1.2)
$$
\n
$$
= F(2.0) - F(-1.2) \approx 0.977 - 0.115
$$
\n
$$
= 0.862
$$
\n
$$
Pr(Z \ge 1.5) = 1 - Pr(Z < 1.5)
$$
\n
$$
= 1 - F(1.5) \approx 1 - 0.933 = 0.067.
$$

SECTION 7.8: Probability 447
After making the Maple definition shown above, we calculate $Pr(-1.2 \le Z \le 2.0)$ to
10 decimal places using
> evalf (F(2) - F(-1.2), 10) SECTION 7.8: Probabil

After making the Maple definition shown above, we calculate $Pr(-1.2 \le Z$

10 decimal places using
 $> \text{evalf}(F(2) - F(-1.2), 10)$ SECTION 7.8: F

After making the Maple definition shown above, we calculate Pr(-1.2

10 decimal places using
 $>$ eval f (F(2) - F(-1.2), 10)

0.8621801977

and for Pr(Z \geq 1.5)
 $>$ eval f(1 - (F(1.5), 10)

SI

After making the Maple definition shown above, we calcul

10 decimal places using
 $>$ eval f (F(2) - F(-1.2), 10)

0.8621801977 0:8621801977 r making the Maple definition shown above, we
ecimal places using
eval f (F(2) - F(-1.2), 10)
0.8621801977
and for Pr(Z \geq 1.5)
eval f(1 - (F(1.5), 10)
0.0668072012

```
0:0668072012
```
0.8621801977

and for $Pr(Z \ge 1.5)$
 \sim eval f (1 – (F (1.5), 10)

0.0668072012
 EXAMPLE 10 A certain random variable X is distributed normally with mean

2 and standard deviation 0.4. Find (a) $Pr(1.8 \le X \le 2.4)$, and

b 2 and standard deviation 0.4. Find (a) Pr(1.8 $\leq X \leq 2.4$), and
ting the table to three decimal places or using Maple to 10 decimal
ding the table to three decimal places or using Maple to 10 decimal 9 eval f (F (2) - F (-1.2), 10)

0.8621801977

and for Pr(Z \ge 1.5)

9 eval f (1 - (F (1.5), 10)

0.0668072012
 EXAMPLE 10 A certain random variable X is distributed normally with mean

2 and standard deviation 0.4. F places. **EXAMPLE 10** A certain random variable *X* is distributed normally with mean 2 and standard deviation 0.4, Find (a) Pr(1.8 \leq *X* \leq 2.4), and (b) Pr(*X* > 2.4), using the table to three decimal places or using Mapl **EXAMPLE 10** A certain random variable *X* is distributed normally with mean 2 and standard deviation 0.4. Find (a) Pr(1.8 \leq *X* \leq 2.4), and (b) Pr(*X* > 2.4), using the table to three decimal places or using Mapl **EXAMPLE 10** A certain random variable *X* is distributed normally with mean 2 and standard deviation 0.4. Find (a) Pr(1.8 $\leq X \leq$ 2.4), and (b) Pr(*X* > 2.4), using the table to three decimal places or using Maple to 1 **E 10** A certain random variable *X* is distributed normally with n

2 and standard deviation 0.4. Find (a) Pr(1.8 $\leq X \leq 2.4$),

2.4), using the table to three decimal places or using Maple to 10 dec

Since *X* is dist

EXAMPLE 10
\n2 and standard deviation 0.4. Find (a) Pr(1.8 ≤ X ≤ 2.4), and
\n(b) Pr(X > 2.4), using the table to three decimal places or using Maple to 10 decimal
\nplaces.
\n**Solution** Since X is distributed normally with mean 2 and standard deviation 0.4,
\nZ = (X – 2)/0.4 is distributed according to the standard normal distribution (with
\nmean 0 and standard deviation 1). Accordingly,
\nPr(1.8 ≤ X ≤ 2.4) = Pr(−0.5 ≤ Z ≤ 1)
\n= F(1) – F(−0.5) ≈ 0.841 – 0.309 = 0.532,
\nPr(X > 2.4) = Pr(Z > 1) = 1 – Pr(Z ≤ 1)
\n= 1 – F(1) ≈ 1 – 0.841 = 0.159.
\nAlternatively, using Maple with F defined as above, Pr(1.8 ≤ X ≤ 2.4) is
\n
$$
\approx
$$
 evalf (F (1) – F (−0.5), 10)
\n0.5328072072
\nFor Pr(X > 2.4)
\n
$$
\approx
$$
 evalf (1 – F (1), 10)
\n0.1586552540

0:5328072072

0:1586552540

Alternatively, using Maple with *F* defined as ab
 $>$ eval f (F (1) - F (-0.5), 10)

0.532807207

For Pr(*X* > 2.4)
 $>$ eval f (1 - F (1), 10)

0.158655254
 Heavy Tails

With continuous random variables over an infini 0.5328072072
 \times eval f (1 – F(1), 10)
 0.1586552540
 Heavy Tails

With continuous random variables over an infinite domain, complications of improper

integrals arise for certain established probability density f ^{0.5328072072}

For Pr(*X* > 2.4)

> eval f (1 - F (1), 10)

0.1586552540
 Heavy Tails

With continuous random variables over an infinite domain, complications of improper

integrals arise for certain established probab For $Pr(X > 2.4)$
 \ge eval $f(1 - F(1), 10)$
 0.1586552540
 Heavy Tails

With continuous random variables over an infinite domain, complications of improper

integrals arise for certain established probability density fun For F($\lambda > 2.4$)
 λ eval f (1 – F(1), 10)

0.1586552540
 Heavy Tails

With continuous random variables over an infinite domain, complications of improper

integrals arise for certain established probability density f Example 1 and 1586552540
 a contribution 1586552540
 a contribution arise over an infinite domain, complications of improper

integrals arise for certain established probability density functions that do not satisfy
 distribution: **CONTAINS**
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a continuous random variables over an infinite domain, complications

grads arise for certain established probability density functions the

e functions needed for the normal distribution to hold. For an imprime
 0.1586552540
 Heavy Tails

With continuous random variables over an infinite domain, complications of improper

integrals arise for certain established probability density functions that do not satisfy

the conditions to hold. For an important class of
tiance do not exist. For example,
n arising in physics is the **Cauchy**
 $\langle \times \infty,$
so see of the mean and standard devia-
is symmetric about the line $x = \mu$,
However, μ is not really a portant class of

For example,

is the **Cauchy**

is the **Cauchy**

standard devia-

the line $x = \mu$,

ally a mean and
 $\int_{-\infty}^{\infty} x^2 C(x) dx$

 $C(x) = \frac{1}{\pi} \frac{\gamma}{(x - \mu)^2 + \gamma^2}$ $-\infty$ $\pi (x - \mu)^2 + \nu^2$ $\mathcal V$

the conditions heeded for the normal distribution to flotd. Total important class of
these functions the integrals for the mean or variance do not exist. For example,
distribution:
The Cauchy probability density
 $C(x) = \frac$ distribution:
 The Cauchy probability density
 $C(x) = \frac{1}{\pi} \frac{\gamma}{(x - \mu)^2 + \gamma^2}$ $-\infty < x < \infty$.

Here the constants μ and γ play roles similar to those of the mean and standard deviation in the normal distribution. T **The Cauchy probability density**
 $C(x) = \frac{1}{\pi} \frac{\gamma}{(x - \mu)^2 + \gamma^2} \qquad -\infty < x < \infty$.

Here the constants μ and γ play roles similar to those of the mean and standard devia-

tion in the normal distribution. The graph of **Solution Control in the Cauchy probability density**
 $C(x) = \frac{1}{\pi} \frac{\gamma}{(x - \mu)^2 + \gamma^2}$ $-\infty < x < \infty$.

Here the constants μ and γ play roles similar to those of the mean and standard deviation in the normal distribu $C(x) = \frac{1}{\pi} \frac{\gamma}{(x - \mu)^2 + \gamma^2}$ $-\infty < x < \infty$.

Here the constants μ and γ play roles similar to those of the mean and standard devia-

tion in the normal distribution. The graph of $C(x)$ is symmetric about the line

 $y = \frac{\text{Cauchy Pr}(x < X < \infty)}{\text{Normal Pr}(x < X < \infty)}$

with $\mu = 0$ and $\gamma = \sqrt{2/\pi}$, the latter value being chosen so that both curves would peak at the same height. Observe the tails of these curves (i.e., the parts where $|x| > 2$, Figure 7.62 shows the graphs of the standard Normal density and the Cauchy density
Figure 7.62 shows the graphs of the standard Normal density and the Cauchy density
with $\mu = 0$ and $\gamma = \sqrt{2/\pi}$, the latter value being c / Figure 7.62 shows the graphs of the standard Normal density and the Cauchy density
Figure 7.62 shows the graphs of the standard Normal density and the Cauchy density
with $\mu = 0$ and $\gamma = \sqrt{2/\pi}$, the latter value being c Figure 7.62 shows the graphs of the standard Normal density and the Cauchy density
with $\mu = 0$ and $\gamma = \sqrt{2/\pi}$, the latter value being chosen so that both curves would
peak at the same height. Observe the tails of these Figure 7.62 shows the graphs of the standard Normal density and the Cauchy density
with $\mu = 0$ and $\gamma = \sqrt{2/\pi}$, the latter value being chosen so that both curves would
peak at the same height. Observe the tails of these Figure 7.62 shows the graphs of the standard Normal density and the Cauchy density
with $\mu = 0$ and $\gamma = \sqrt{2/\pi}$, the latter value being chosen so that both curves would
peak at the same height. Observe the tails of these distribution, re 7.62 shows the graphs of the standard Norma $\mu = 0$ and $\gamma = \sqrt{2/\pi}$, the latter value being c at the same height. Observe the tails of these cure While the normal curve is higher than the Catial factor in the normal d with $\mu = 0$ and $\gamma = \sqrt{2/\pi}$, the latter value being chosen so that both curves would
peak at the same height. Observe the tails of these curves (i.e., the parts where $|x| > 2$,
say). While the normal curve is higher than for every positive integer *n* as $|x| \to \infty$ while the Cauchy density is only $O(|x|^{-2})$.
Because of this polynomial asymptotic behaviour as $|x| \to \infty$, the Cauchy density is said to have **fat tails** or **heavy tails**. To und

$$
Pr(X > x) = \frac{1}{2} - \frac{1}{\pi} \tan^{-1} x,
$$

Because of this polynomial asymptotic behaviour as $|x| \to \infty$, the Cauchy density is
said to have **fat tails or heavy tails**. To understand the significance of this, we use
direct integration to find the tail probability f saad to have **tat tails** or **heavy tails**. To understand the significance of this, we use
direct integration to find the tail probability from some x to infinity for the Cauchy
distribution,
 $Pr(X > x) = \frac{1}{2} - \frac{1}{\pi} \tan^{-1} x$ For $(X > x) = \frac{1}{2} - \frac{1}{\pi} \tan^{-1} x$,

divide it by the corresponding tail probability, $Pr(X > x) = 1 - F(x)$, for the

dard normal distribution. This ratio is plotted in Figure 7.63 for $1 \le x \le 8$. From

t $x = 7$ on, the ratio gro

distribution,
 $Pr(X > x) = \frac{1}{2} - \frac{1}{\pi} \tan^{-1} x$,

and divide it by the corresponding tail probability, $Pr(X > x) = 1 - F(x)$, for the

standard normal distribution. This ratio is plotted in Figure 7.63 for $1 \le x \le 8$. From

about direct integration to find the tail probability from some x to infinity for the Cauchy distribution,
 $Pr(X > x) = \frac{1}{2} - \frac{1}{\pi} \tan^{-1} x$,

and divide it by the corresponding tail probability, $Pr(X > x) = 1 - F(x)$, for the

standard $Pr(X > x) = \frac{1}{2} - \frac{1}{\pi} \tan^{-1} x$,

and divide it by the corresponding tail probability, $Pr(X > x) = 1 - F(x)$, for the

standard normal distribution. This ratio is plotted in Figure 7.63 for $1 \le x \le 8$. From

about $x = 7$ on, the and divide it by the corresponding tail probability, $Pr(X > x) = 1 - F(x)$, for the standard normal distribution. This ratio is plotted in Figure 7.63 for $1 \le x \le 8$. From about $x = 7$ on, the ratio grows extremely rapidly! This and divide it by the corresponding tail probability, $Pr(X > x) = 1 - F(x)$, for the standard normal distribution. This ratio is plotted in Figure 7.63 for $1 \le x \le 8$. From about $x = 7$ on, the ratio grows extremely rapidly! This standard normal distribution. This ratio is plotted in Figure 7.63 for $1 \le x \le 8$. From
about $x = 7$ on, the ratio grows extremely rapidly! This ratio shows that the amount
of probability in the tail of $C(x)$ is very "hea about $x = 7$ on, the ratio grows extremely rapidly! This ratio shows that the amount
of probability in the tail of $C(x)$ is very "heavy" relative to the normal.
 $C(x)$ is far from the only heavy-tailed probability density of probability in the tail of $C(x)$ is very "heavy" relative to the normal.
 $C(x)$ is far from the only heavy-tailed probability density function. One important

class of heavy-tailed probability distributions are the **Lé** $C(x)$ is far from the only heavy-tailed probability density function. O
class of heavy-tailed probability distributions are the **Lévy stable distril**
densities $S_{\alpha}(x)$ for $0 < \alpha \le 2$. Except for $\alpha = 1$ (the Cauchy case class of neavy-tanea probability distributions are the **Levy stable distributions** with densities $S_{\alpha}(x)$ for $0 < \alpha \le 2$. Except for $\alpha = 1$ (the Cauchy case) and $\alpha = 2$ (the densities $S_{\alpha}(x)$ are not elementary func densities $S_{\alpha}(x)$ for $0 < \alpha \leq 2$. Except for $\alpha = 1$ (the Cattery case) and $\alpha = 2$ (the normal case), the densities $S_{\alpha}(x)$ are not elementary functions, and providing exact descriptions of them is beyond the socpe normal case), the densities $S_{\alpha}(x)$ are not elementary functions, and providing exact
descriptions of them is beyond the scope of this section. They can be represented
explicitly as integral transforms (see Section 18.7

standard physics examples given, normality does not hold. $C(x)$ arises theoretically in descriptions of them is beyond the scope of this section. They can be represented
explicitly as integral transforms (see Section 18.7), but must be computed numerically
to get specific values, as is the case for any other explicitly as integral transforms (see section 18.7), out must be computed intimericany
to get specific values, as is the case for any other nonalgebraic function. The graphs
densities. The definitive differences are foun likely to be symmetric values, as is the case for any other nonlargebrate function. The graphs of the symmetric versions of $S_{\alpha}(x)$ are similar to those of the normal and Cauchy densities. The definitive differences are of the symmetric versions of $S_{\alpha}(x)$ are similar to those of the normal and Cauchy
densities. The definitive differences are found in the specifics of the tail behaviours,
which are discussed in Exercises 23–24.
Remark densities. The definitive differences are found in t
which are discussed in Exercises $23-24$.
Remark Poincaré's telling remark, in the quotatic
humorously warns us that a presumption of norms
standard physics examples **Remark** Poincaré's telling remark, in the quotations at the beginning of this chapter, humorously warns us that a presumption of normality is common. However, in the standard physics examples given, normality does not ho **NEIFFORT SETTS** FORCIDE TO THE TRACTATE TO THE STANDARY WARRED STANDARY WARRED STANDARY WARRED STANDARY SURVEX STANDARY IS CONTINUO TO THE THE STANDARY OF THE THE MENTIME THE THE THE THE THE THE THE THE WARRED WITH THE T numorously warns us unat a presumption of normality is common. However, in the standard physics examples given, normality does not hold. $C(x)$ arises theoretically in them. Without a theoretical basis, any presumption of

standard physics examples given, normanty does not nota. $C(x)$ arises theoretically in them. Without a theoretical basis, any presumption of normality can only be confirmed with data alone. But the tail represents the lea mem. wunout a theoretical basis, any presumpton of normanty can only be confirmed with data alone. But the tail represents the least probable events, so tail data are least likely to be observed over any finite time. Thus, whin data alone. But the tall represents the least probable events, so tall data are least
likely to be observed over any finite time. Thus, the greatest deviations from normal
distributions are the least likely to be obse the sharp growth of the ratio in Figure 7.63 shows that the probability of such the sharp growth of the ratio in Figure 7.63 shows that the probability of normal distributions, we mistakenly assume that the outcomes of a h

SECTION 7.8: Probability 449
events can be seriously underestimated in this circumstance. People adhering to the
assumption of normality can thus experience costly surprises. A surprise of this type
is named a **Black Swan** SECTION 7.8: Probability **449**
events can be seriously underestimated in this circumstance. People adhering to the
assumption of normality can thus experience costly surprises. A surprise of this type
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assumption of normality can thus experience costly surprises. A surprise of this type
is named a **Black Swa** events can be seriously underestimated in this circumstance. People adhering to the assumption of normality can thus experience costly surprises. A surprise of this type is named a **Black Swan** by Nassim Taleb, who authore events can be seriously underestimated in this circumstance. People adhering to the assumption of normality can thus experience costly surprises. A surprise of this type is named a **Black Swan** by Nassim Taleb, who authore events can be seriously underestimated in this circumstan assumption of normality can thus experience costly surprisine a **Black Swan** by Nassim Taleb, who authored topic, entitled *The Black Swan: The Impact of the Highly* EXERCISES 7.8

EXERCISES 7.8

EXERCISES 7.8

EXERCISES 7.8 **1.** How much should you be willing to pay to play a game where the probability that it consists the conditional of the pay to play a game where where where where when the conditional should you be willing to pay to play the probability that it is red is 0.6.

(a) If you reach in and pull out two balls, what is the

proba s, always exist for finite data. Thus statistical uncer-
e moot. Caution should be taken for questions that
probability that it is red is 0.6.
If you reach in and pull out two balls, what is the
probability they are both b

- discretely sampled. While the mean and variance may

tail process, they will, nonetheless, always exist for fin

tainty based on normality may be moot. Caution sho

depend heavily on normality.
 XERCISES 7.8

How much sh tail process, they will, nonetheless, always exist for finite tainty based on normality may be moot. Caution should be depend heavily on normality.
 XERCISES 7.8

How much should you be willing to pay to play a game wher tainty based on normality may be moot. Caution

depend heavily on normality.
 XERCISES 7.8

How much should you be willing to pay to play a game where

you toss the coin discussed at the beginning of this section and

w **ERCISES 7.8**

How much should you be willing to pay to play a game where

you toss the coin discussed at the beginning of this section and

win \$1 if it comes up heads, \$2 if it comes up tails, and \$50 if

it remains sta **EXERCISES 7.8**

2. A die is weighted so that if X represents the number showing that is well also that if X represents the number showing

2. A die is weighted so that if X represents the number showing
 $n \in \{1, 2, 3, 4$ depend heavily on normality.
 XERCISES 7.8

How much should you be willing to pay to play a game where

you toss the coin discussed at the beginning of this section and

win \$1 if it comes up heads, \$2 if it comes up ta How much should you be willing to pay to play a game where

the probability that it is

you toss the coin discussed at the beginning of this section and

win \$1 if it comes up heads, \$2 if it comes up tails, and \$50 if

i 3. Find the mean and standard deviation of the random variable

2. A find the mean and standard deviation of the random variable

3. Find the mean and standard deviation of the random variable

3. Find the mean and standa win \$1 if it comes up heads, \$2 if it comes up tails, and
it remains standing on its edge? Assume you will play th
game many times and would like to at least break even.
A die is weighted so that if X represents the numbe
- 1. It remains standing on its edge? Assume you will play the

2. A die is weighted so that if X represents the number showing

2. A die is weighted so that if X represents the number showing

2. A die is weighted so that
	-
	-
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-
- game many times and would like to at least break even.

A die is weighted so that if X represents the number showi

on top when the die is rolled, then $Pr(X = n) = Kn$ for
 $n \in \{1, 2, 3, 4, 5, 6\}$.

(a) Find the value of the **2.** A die is weighted so that if X represents the number showing

on top when the die is rolled, then $Pr(X = n) = Kn$ for
 $n \in \{1, 2, 3, 4, 5, 6\}$.

(a) Find the value of the constant K.

(b) Find the probability that $X \le 3$ on top when the die is rolled, then $Pr(X = n) = Kn$ for
 $n \in \{1, 2, 3, 4, 5, 6\}$.

(a) Find the value of the constant K.

(b) Find the probability that $X \le 3$ on any roll of the die.

Find the standard deviation of your wini *n* $\in \{1, 2, 3, 4, 5, 6\}$.

(a) Find the value of the constant *K*.

(b) Find the probability that $X \le 3$ on any roll of the die.

Find the standard deviation of your winings on a roll of the die

in Exercise 1.

Find (a) Find the value of the constant K.

(b) Find the probability that $X \le 3$ on any roll of the die.

Find the standard deviation of your winings on a roll of the die

in Exercise 1.

Find the mean and standard deviation (a) the probability that $X \le 3$ on any roll of the die.

Find the standard deviation of your winings on a roll of the die

in Exercise 1.

Find the mean and standard deviation of the random variable

X in Exercise 2.

A 3. Find the standard deviation of your winings on a roll of the die

in Exercise 1.

4. Find the mean and standard deviation of the random variable

x in Exercise 2.

X in Exercise 2.

X in Exercise 2.

X in Exercise 2.
 Find the standard deviation of your winnings on a fort of the true (b) the mean μ , variance σ^2 ,

Find the mean and standard deviation of the random variable

X in Exercise 2.

A die is weighted so that the probabil Find the mean and standard deviation of the random variable

X in Exercise 2.

A die is weighted so that the probability of rolling each of the

numbers 2, 3, 4, and 5 is still 1/6, but the probability of rolling

1 is 9/ X in Exercise 2.

A die is weighted so that the probability of rolling each of the

uumbers 2, 3, 4, and 5 is still 1/6, but the probability of rolling

in is 9/60 and the probability of rolling 6 is 11/60. What are the
 A die is weighted so that the probability of rolling each of the

numbers 2, 3, 4, and 5 is still 1/6, but the probability of rolling

1 is 9/60 and the probability of rolling 6 is 11/60. What are the

mean and standard d Example 3.

Example 3.

Example 3.

Example 3.

Example 3.

Example 3.

In and standard deviation of the number X rolled using
 $\mathbf{F}_{\mu,\sigma}(x)$ defined in the probability that $X \leq 3$?

In and standard deviation of the n 9.62.8 2, 3, 4, and 3 is sum 170, bat all probability of rols
9/60 and the probability of rolling 6 is 11/60. What are
n and standard deviation of the number X rolled using
die? What is the probability that $X \le 3$?
dice,
- -
	-
- mean and standard deviation of the number X rolled using
 C. Two dice, each weighted like the one in Exercise 5, are

thrown. Let X be the random variable giving the sum of the

thrown. Let X be the random variable givi this die? What is the probability that $X \le 3$?

Two dice, each weighted like the one in Exercise 5, are

thrown. Let X be the random variable giving the sum of the

numbers showing on top of the two dice.

(a) Find the Two dice, each weighted like the one in Exercise 5, are

thrown. Let X be the random variable giving the sum of the

numbers showing on top of the two dice.

(a) Find the probability function for X.

(b) Determine the mea Number, can well with the three one in Exactise 3, are

interval and the three three three three three times is bowing on top of the two dice.

(a) Find the probability function for X.

(b) Determine the mean and standard places.) (a) Find the probability function for X.

(b) Determine the mean and standard deviation of X.

Compare them with those found for unweighted dice in

Example 3.

A thin but biased coin has probability 0.55 of landing heads Find the probability function for X .
Determine the mean and standard deviation of X .
Compare them with those found for unweighted d
Example 3.
in but biased coin has probability 0.55 of landing
0.45 of landing tails. (b) Determine the mean and standard deviation of A.

Compare them with those found for unweighted dice in

Example 3.

A thin but biased coin has probability 0.55 of landing heads

for this coin.) The coin is tossed three A thin but biased coin has probability 0.55 of landing heads

and 0.45 of landing tails. (Standing on its edge is not possible

for this coin.) The coin is tossed three times. (Determine all

and has mean μ and stand
 and halong the set of possible

is coin.) The coin is spoolanity 0.55 of landing reads

into 0.45 of landing tails. (Standing on its edge is not possible

erical answers to the following questions to six decimal

example and 0.45 of landing tails. (Standing on its edge is not possible
for this coin.) The coin is tossed three times. (Determine all
places.)
(a) What is the sample space of possible outcomes of the
three tosses?
(b) What is t
	-
	- outcomes?
	-
	- 1? (a) What is the sample space of possible outcomes of the

	three tosses?

	(b) What is the probability of each of these possible

	outcomes?

	(c) Find the probability function for the number X of times

	leads comes up during three tosses?

	(b) What is the probability of each of these possible

	(b) What is the probability function for the number X of times

	leads comes up during the three tosses.

	(d) What is the probability that the number of
		-
	-

-
- (a) the probability that it is red is 0.6.

(a) If you reach in and pull out two balls, what is the

probability they are both blue?

(b) Suppose you reach in the bag of 20 balls and pull out three

balls. Describe the sam balls. Balls and pull out two balls, what is the probability they are both blue?
Suppose you reach in the bag of 20 balls and pull out three balls. Describe the sample space of possible outcomes of this experiment. What i probability that it is red is 0.6.
If you reach in and pull out two balls, what is the
probability they are both blue?
Suppose you reach in the bag of 20 balls and pull out three
balls. Describe the sample space of possib probability that it is red is 0.6.
If you reach in and pull out two balls, what is the
probability they are both blue?
Suppose you reach in the bag of 20 balls and pull out three
balls. Describe the sample space of possib For each function funct the probability that it is red is 0.6.

(a) If you reach in and pull out two balls, what is the

probability they are both blue?

(b) Suppose you reach in the bag of 20 balls and pull out three

balls. Describe the sample the probability that it is red is 0.6.

(a) If you reach in and pull out two balls, what is

probability they are both blue?

(b) Suppose you reach in the bag of 20 balls and p

balls. Describe the sample space of possibl (a) If you reach in and pull out two balls, what is the
probability they are both blue?
(b) Suppose you reach in the bag of 20 balls and pull out
balls. Describe the sample space of possible outcome
this experiment. What both blue?

herefore the both blue?

herefore the bag of 20 balls and pull out three

sample space of possible outcomes of

hat is the expectation of the number of

three balls you pulled out?

ercises 9–15, find the fol probability they are both blue?

(b) Suppose you reach in the bag of 20 balls and pull ou

balls. Describe the sample space of possible outcon

this experiment. What is the expectation of the nun

red balls among the thre (b) Suppose you reach in the bag of 20 balls and pull out three
balls. Describe the sample space of possible outcomes of
this experiment. What is the expectation of the number of
red balls among the three balls you pulled balls. Describe the sample space of possible outcomes of
this experiment. What is the expectation of the number of
red balls among the three balls you pulled out?
each function $f(x)$ in Exercises 9–15, find the following: (b) Suppose you reach in the bag of 20 balls and pull out three
balls. Describe the sample space of possible outcomes of
this experiment. What is the expectation of the number of
red balls among the three balls you pulled

-
-
- this experiment. What is the expectation of the numbe
red balls among the three balls you pulled out?
each function $f(x)$ in Exercises 9–15, find the following:
the value of C for which f is a probability density on the
g band this experiment. What is the expectation of red balls among the three balls you pulled

For each function $f(x)$ in Exercises 9–15, find the

(a) the value of C for which f is a probability despite interval,

(b) the bethose the sample space of possible otations of
timent. What is the expectation of the number of
among the three balls you pulled out?
 $f(x)$ in Exercises 9–15, find the following:
' for which f is a probability density o red balls among the three balls you pulled
For each function $f(x)$ in Exercises 9–15, find the
(a) the value of C for which f is a probability den
given interval,
(b) the mean μ , variance σ^2 , and standard deviatie
 on the three balls you pulled out?

in Exercises 9–15, find the following:

which *f* is a probability density on the

nece σ^2 , and standard deviation σ of the
 f, and
 $u + \sigma$), that is, the probability that the For each function $f(x)$ in Exercises 9–15, find the following:

(a) the value of C for which f is a probability density on the

given interval,

(b) the mean μ , variance σ^2 , and standard deviation σ of the

prob (a) the value of *C* for which *f* is a probability density on the given interval,

(b) the mean μ , variance σ^2 , and standard deviation σ of the probability density *f*, and

(c) Pr($\mu - \sigma \le X \le \mu + \sigma$), that is, ty f , and
 $\leq \mu + \sigma$), that is, the probability that the
 X is no further than one standard deviation

aan.

0, 3] **10.** $f(x) = Cx$ on [1, 2]

[0, 1] **12.** $f(x) = C \sin x$ on [0, π]
 $f(x^2)$ on [0, 1]
 $f(x) = C \sin x$ on [0, Pr($\mu - \sigma \le X \le \mu + \sigma$), that is, the probability that the
random variable X is no further than one standard deviation
away from its mean.
 $f(x) = Cx$ on [0, 3] **10.** $f(x) = Cx$ on [1, 2]
 $f(x) = Cx^2$ on [0, 1] **12.** $f(x) = C \sin x$ on
-
-
-
-
- $\int_0^{\infty} e^{-x^2} dx = \sqrt{\pi}/2.$ 16. It is in the relation and deviation

16. If $f(x) = Cx$ on [0,3]

16. $f(x) = Cx$ on [1,2]

17. $f(x) = Cx^2$ on [0,1]

12. $f(x) = C \sin x$ on [0, π]

13. $f(x) = C(x - x^2)$ on [0,1]

14. $f(x) = C xe^{-kx}$ on [0, ∞), $(k > 0)$

15. $f(x) = C$
-
- away from its mean.
 $f(x) = Cx$ on [0, 3] 10. $f(x) = Cx$ on [1, 2]
 $f(x) = Cx^2$ on [0, 1] 12. $f(x) = C \sin x$ on [0, π]
 $f(x) = C(x x^2)$ on [0, 1]
 $f(x) = C xe^{-kx}$ on [0, ∞), $(k > 0)$
 $f(x) = C e^{-x^2}$ on [0, ∞). *Hint:* Use prop 9. $f(x) = Cx$ on [0, 3]

10. $f(x) = Cx$ on [1, 2]

11. $f(x) = Cx^2$ on [0, 1]

12. $f(x) = C \sin x$ on [0, π]

13. $f(x) = C(x - x^2)$ on [0, ∞), $(k > 0)$

15. $f(x) = C e^{-x^2}$ on [0, ∞). *Hint:* Use properties of the

standard norm $f(x) = Cx^2$ on [0, 1] **12.** $f(x) = C \sin x$ on [0, π]
 $f(x) = C(x - x^2)$ on [0, 1]
 $f(x) = C xe^{-kx}$ on [0, ∞), $(k > 0)$
 $f(x) = C e^{-x^2}$ on [0, ∞). *Hint:* Use properties of the

standard normal density to show that $\int_0^\infty e^{-x^2} dx = \sqrt$
- for this coin.) The coin is tossed three times. (Determine all
numerical answers to the following questions to six decimal
places.)
(a) What is the sample space of possible outcomes of the
three tosses?
(b) What is the pr 9. $f(x) = Cx$ on [0, 3]

10. $f(x) = Cx$ on [1, 2]

11. $f(x) = Cx^2$ on [0, 1]

12. $f(x) = C \sin x$ on [0, π]

13. $f(x) = C(x - x^2)$ on [0, 0, 1]

14. $f(x) = C \, xe^{-kx}$ on [0, ∞), Rix : Use properties of the

standard normal density 2 $\pi(1 + x^2)$ is a probability density on k > 0)

it: Use properties of the

w that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2$.

able to be uniformly distributed

why.

ow that the normal density

probability density function

eviation σ .

is a probability density on

a random v $f(x) = C e^{-x^2}$ on $[0, \infty)$. *Hint:* Use properties of the standard normal density to show that $\int_0^{\infty} e^{-x^2} dx = \sqrt{\pi}/2$.
Is it possible for a random variable to be uniformly distributed on the whole real line? Explain wh $f(x) = C e^{-x}$ on [0, ∞). *Hml*. Ose properties of the
standard normal density to show that $\int_0^{\infty} e^{-x^2} dx = \sqrt{\pi}/2$.
Is it possible for a random variable to be uniformly distributed
on the whole real line? Explain why. standard normal density to show that $j_0 e^x$ $dx = \sqrt{\pi}/2$.
Is it possible for a random variable to be uniformly distributed
on the whole real line? Explain why.
Carry out the calculations to show that the normal density
 f Is it possible for a random variable to be uniformly distributed
on the whole real line? Explain why.
Carry out the calculations to show that the normal density
 $f_{\mu,\sigma}(x)$ defined in the text is a probability density fun on the whole real line? Explain why.
Carry out the calculations to show that the normal density
 $f_{\mu,\sigma}(x)$ defined in the text is a probability density function
and has mean μ and standard deviation σ .
Show that f 17. Carry out the calculations to show that the normal dens $f_{\mu,\sigma}(x)$ defined in the text is a probability density funct and has mean μ and standard deviation σ .

18. Show that $f(x) = \frac{2}{\pi(1 + x^2)}$ is a probabilit (a) the uniform distribution of $f_{\mu,\sigma}(x)$ or the exponential distribution
Carry out the calculations to show that the normal density
Carry out the calculations to show that the normal density
Carry out the calculations w that $f(x) = \frac{2}{\pi(1 + x^2)}$ is a probability density on

c). Find the expectation of a random variable X having

density. If a machine generates values this random

ble X, how much would you be willing to pay, per gan

ay [0, ∞). Find the expectation of a random variable *X* having
this density. If a machine generates values this random
variable *X*, how much would you be willing to pay, per game,
to play a game in which you operate th variable X, how much would you be willing to pay, per game,
to play a game in which you operate the machine to produce a
value of X and win X dollars? Explain.
Calculate $Pr(|X - \mu| \ge 2\sigma)$ for
(a) the uniform distribution o
	- -
		-
		-
	- to play a game in which you operate the machine to produce a
value of X and win X dollars? Explain.
Calculate $Pr(|X \mu| \ge 2\sigma)$ for
(a) the uniform distribution on [a, b],
(b) the exponential distribution with density $f(x)$

CHAPTER 7 Applications of Integration

malfunctions is 20 hours, find the probability that the system,

having just had a malfunction corrected, will operate without

malfunction for at least 12 hours.

The number X of me CHAPTER 7 Applications of Integration

malfunctions is 20 hours, find the probability that the system,

having just had a malfunction corrected, will operate without

malfunction for at least 12 hours.

The number X of me CHAPTER 7 Applications of Integration
malfunctions is 20 hours, find the probability that the system,
having just had a malfunction corrected, will operate without
malfunction for at least 12 hours.
The number X of metres

- 21. The number X of metres of cable probability that the system,

22. Lévy stable probability having just had a malfunction corrected, will operate without malfunction for at least 12 hours.

21. The number X of metres of CHAPTER 7 Applications of Integration

malfunctions is 20 hours, find the probability that the system,

having just had a malfunction corrected, will operate without

malfunction for at least 12 hours.

The number X of me CHAPTER 7 Applications of Integration

malfunctions is 20 hours, find the probability that the system,

having just had a malfunction corrected, will operate without

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The number X of me CHAPTER 7 Applications of Integration

malfunctions is 20 hours, find the probability that the system,

having just had a malfunction corrected, will operate without

malfunction for a least 12 hours.

The number X of met malfunctions is 20 hours, find the probability that the system,

having just had a malfunction corrected, will operate without

following asy

malfunction for at least 12 hours.

The number X of metres of cable produced a malfunctions is 20 hours, find the probability that the system,

having just had a malfunction corrected, will operate without

following asymptotic be

malfunction for at least 12 hours.

21. The number X of metres of ca
- malfunctions is 20 hours, find the probability that the system,

having just had a malfunction corrected, will operate without

malfunction for at least 12 hours.

The number X of metres of cable produced any day by a

va maltunctions is 20 hours, find the probability that the system,

having just had a malfunction corrected, will operate without

malfunction for at least 12 hours.

The number X of metres of cable produced any day by a

ca malfunctions is 20 hours, find the probability that the system,

having just had a malfunction corrected, will operate without

following asymptotic bu

malfunction for a least 12 hours.

The number X of meters of cable p The number X of metres of cable produced any day by a

cable-making company is a normally distributed random

variable with mean 5,000 and standard deviation 200. On

what fraction of the days the company operates will th cable-making company is a normally distributed random
variable with mean 5,000 and standard deviation 200. On
what fraction of the days the company operates will the
number of metres of cable produced exceed 5,500?
A spin continuous.) Suppose it sticks at 1¹ had of the the and the ristry due in the interval [0, 1].

The synalises uniformly distribution to the spinner's

What is the mean and standard deviation of the spinner's

values? (Note: the rand
- 23. Lévy stable probability densities are known to have the following asymptotic behaviour as $x \to \infty$

$$
S_{\alpha}(x) = c_{\alpha}x^{-(1+\alpha)} + O\left(x^{-(1+2\alpha)}\right)
$$

Lévy stable probability densities are known to have the
following asymptotic behaviour as $x \to \infty$
 $S_{\alpha}(x) = c_{\alpha}x^{-(1+\alpha)} + O(x^{-(1+2\alpha)})$
for $0 < \alpha < 2$, and for simplicity $S_{\alpha}(x)$ is assumed symmetric
about $x = 0$. Note that Lévy stable probability densities are known to have the
following asymptotic behaviour as $x \to \infty$
 $S_{\alpha}(x) = c_{\alpha}x^{-(1+\alpha)} + O(x^{-(1+2\alpha)})$
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following asymptotic behaviour as $x \to \infty$
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ving asymptotic behaviour as $x \to \infty$
 $\chi_{\ell}(x) = c_{\alpha} x^{-(1+\alpha)} + O(x^{-(1+2\alpha)})$
 $\leq \alpha \leq 2$, and for simplicity $S_{\alpha}(x)$ is assumed symmetric
 $x = 0$. Note that the normal following asymptotic behaviour as $x \to \infty$
 $S_{\alpha}(x) = c_{\alpha}x^{-(1+\alpha)} + O(x^{-(1+2\alpha)})$

for $0 < \alpha < 2$, and for simplicity $S_{\alpha}(x)$ is assumed symmetric

about $x = 0$. Note that the normal case, $\alpha = 2$, is excluded.

(a) Under wh

- $\int_{-\infty}^{\infty} x^p S_{\alpha}(x) dx$ for some $p \ge 0$) exist?
-
- $S_{\alpha}(x) = c_{\alpha}x^{-(1+\alpha)} + O(x^{-(1+2\alpha)})$
 $0 < \alpha < 2$, and for simplicity $S_{\alpha}(x)$ is assumed symmetric

to $x = 0$. Note that the normal case, $\alpha = 2$, is excluded.

Under what conditions can moments (i.e.,
 $\int_{-\infty}^{\infty} x^p S_{\alpha}($ $S_{\alpha}(x) = c_{\alpha}x^{-(1+\alpha)} + O(x^{-(1+2\alpha)})$

for $0 < \alpha < 2$, and for simplicity $S_{\alpha}(x)$ is assumed symmetric

about $x = 0$. Note that the normal case, $\alpha = 2$, is excluded.

(a) Under what conditions can moments (i.e.,
 $\int_{-\infty}^{\$ $S_{\alpha}(x) = c_{\alpha}x^{-\alpha + \alpha + \beta + O\left(x^{-\alpha + \beta + O\left(x^{-\alpha + O\left(x^{\alpha + O$ for $0 < \alpha < 2$, and for simplicity $S_{\alpha}(x)$ is assumed symmetric
about $x = 0$. Note that the normal case, $\alpha = 2$, is excluded.
(a) Under what conditions can moments (i.e.,
 $\int_{-\infty}^{\infty} x^p S_{\alpha}(x) dx$ for some $p \ge 0$) exis

This final section on applications of integration concentrates on application of the in-
definite integral rather than of the definite integral. We can use the techniques of
integration concentrates on application of the definite integral rather than of symmetric Levy stable
distributions valid for large x.
Example 2
This final section on applications
of integral rather than of the definite integral. We can use the techniques of
diffe **Ential Equations of integration** concentrates on application of the in-
definite integral rather than of the definite integral. We can use the techniques of
integration developed in Chapters 5 and 6 to solve certain kind **Ential Equations**
This final section on applications of integration concentrates on application of the in-
definite integral rather than of the definite integral. We can use the techniques of
integration developed in Chap **Some Example 12**

Somethian Scription Concentrates on application of the in-

definite integral rather than of the definite integral. We can use the techniques of

integration developed in Chapters 5 and 6 to solve certai **Example 18.1.1 Consider the Consider the Section Section**
This final section on applications of integration concentrates of
definite integral rather than of the definite integral. We can
integration developed in Chapters **Example 16 Equations**

This final section on applications of integration conce

definite integral rather than of the definite integral.

integration developed in Chapters 5 and 6 to solve c

ential equations that arise in This final section on applications of integration concentrates on application of the in-
definite integral rather than of the definite integral. We can use the techniques of
integration developed in Chapters 5 and 6 to so This final section on applications of integration concentrates on application of
definite integral rather than of the definite integral. We can use the techni
integration developed in Chapters 5 and 6 to solve certain kin definite integral rather than of the definite integral. We can use the techniques of
integration developed in Chapters 5 and 6 to solve certain kinds of first-order differ-
ential equations that arise in a variety of mode

$$
\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right),
$$

Separable Equations

Consider the logistic equation introduced in Section 3.4 to model the growth of an

animal population with a limited food supply:
 $\frac{dy}{dt} = ky \left(1 - \frac{y}{L}\right)$,

where $y(t)$ is the size of the populati **Separable Equations**

Consider the logistic equation introduced in Section 3.4 to model the growth of an

animal population with a limited food supply:
 $\frac{dy}{dt} = ky ig(1 - \frac{y}{L}\big)$,

where $y(t)$ is the size of the populatio **Separable Equations**
Consider the logistic equation introduced in Section 3.4 to model the growth of an
animal population with a limited food supply:
 $\frac{dy}{dt} = ky ig(1 - \frac{y}{L}\big)$,
where $y(t)$ is the size of the population at sider the logistic equation introduced in Section 3.4 to model the growth of an
all population with a limited food supply:
 $\frac{dy}{dt} = ky ig(1 - \frac{y}{L}\big)$,
ev(t) is the size of the population at time t, k is a positive constant

consider the logistic equation infolduced in section 5.4 to model the growth of an animal population with a limited food supply:
 $\frac{dy}{dt} = ky \left(1 - \frac{y}{L}\right)$,

where $y(t)$ is the size of the population at time t, k is a pos $\frac{dy}{dt} = ky \left(1 - \frac{y}{L}\right)$,
where $y(t)$ is the size of the population at time t, k is a positive constant related to
the fertility of the population, and L is the steady-state population size that can be
sustained by the a $\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$,
where $y(t)$ is the size of the population at time t, k is a positive constant related to
the fertility of the population, and L is the steady-state population size that can be
sustained by the a form $\frac{y}{L} = ky\left(1 - \frac{y}{L}\right)$,
 $y(t)$ is the size of the population at time

ility of the population, and L is the stead

ed by the available food supply. This eq

and $y = L$, that are constant functions of the logistic equati $\frac{dy}{dt} = ky ig(1 - \frac{y}{L}\big)$,
 $e y(t)$ is the size of the population at time t, k is a pertility of the population, and L is the steady-state pointed by the available food supply. This equation has 0 and $y = L$, that are consta and solved by integrating both sides. Expanding the left side in partial fractions and
integrating both scalled **separable equation** is an example of a class of first-order differential equations
called **separable equatio** The logistic equation is an example of a class of ficalled **separable equations** because when they are writely can be separated with only the dependent variable of only the independent variable on the other. The logistic The logistic equation is an example of a class of inst-order directed
called **separable equations** because when they are written in terms of different
they can be separated with only the dependent variable on one side of

$$
\frac{L\,dy}{y(L-y)} = k\,dt
$$

$$
\int \left(\frac{1}{y} + \frac{1}{L - y}\right) dy = kt + C.
$$

$$
\ln y - \ln(L - y) = kt + C,
$$

$$
\ln \left(\frac{y}{L - y} \right) = kt + C.
$$

SECTION 7.9: First-Order Differential Equations 451
We can solve this equation for y by taking exponentials of both sides:
 $\frac{y}{L-y} = e^{kt+C} = C_1 e^{kt}$

SECTION 7.9: First-Order Differential Equations 451
\nWe can solve this equation for y by taking exponentials of both sides:
\n
$$
\frac{y}{L-y} = e^{kt+C} = C_1 e^{kt}
$$
\n
$$
y = (L-y)C_1 e^{kt}
$$
\n
$$
y = \frac{C_1 L e^{kt}}{1 + C_1 e^{kt}},
$$
\nwhere $C_1 = e^C$.
\nGenerally, separable equations are of the form
\n
$$
\frac{dy}{dx} = f(x)g(y).
$$
\nWe solve them by rewriting them in the form
\n
$$
\frac{dy}{g(y)} = f(x) dx
$$

$$
y = \frac{C_1 L e^{kt}}{1 + C_1 e^{kt}},
$$

where $C_1 = e^C$.
Generally, separable equations are of the form

$$
\frac{dy}{dx} = f(x)g(y).
$$

We solve them by rewriting them in the form

$$
\frac{dy}{g(y)} = f(x) dx
$$

$$
\frac{dy}{g(y)} = f(x) dx
$$

 $rac{dy}{dx} = f(x)g(y).$
We solve them by rewriting them in the form
 $rac{dy}{g(y)} = f(x) dx$
and integrating both sides. Note that the separable equation above will have a constant
solution $y(x) = C$ for any constant C satisfying $g(C) = 0$. where $C_1 = e^C$.

Generally, separable equations are of the form
 $\frac{dy}{dx} = f(x)g(y)$.

We solve them by rewriting them in the form
 $\frac{dy}{g(y)} = f(x) dx$

and integrating both sides. Note that the separable equation above will ha

EXAMPLE 1 Solve the equation
$$
\frac{dy}{dx} = \frac{x}{y}
$$
.

We solve them by rewriting them in the form
 $\frac{dy}{g(y)} = f(x) dx$

and integrating both sides. Note that the separable equation above will have a constant

solution $y(x) = C$ for any constant C satisfying $g(C) = 0$.
 EXAMPLE 1 $\frac{dy}{g(y)} = f(x) dx$
and integrating both sides. Note that the se
solution $y(x) = C$ for any constant C sati
EXAMPLE 1 Solve the equation $\frac{dy}{dx}$
Solution We rewrite the equation in the
to get
 $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1$, **Solution** We rewrite the equation in the form $y dy = x dx$ and integrate both sides

.

$$
\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1,
$$

EXAMPLE 1 Solve the equation $\frac{dy}{dx} = \frac{x}{y}$.
 Solution We rewrite the equation in the form $y dy = x dx$ and integrate both sides

to get
 $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1$,

or $y^2 - x^2 = C$, where $C = 2C_1$ is an arbitrary cons **EXAMPLE 1** Solve the equation $\frac{dy}{dx} = \frac{x}{y}$.
 Solution We rewrite the equation in the form $y dy = x dx$ and integrate both sides

to get
 $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1$,

or $y^2 - x^2 = C$, where $C = 2C_1$ is an arbitrary cons **Solution** We rewrite the equation in the form $y dy = x dx$ and intervalsed
to get
 $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1$,
or $y^2 - x^2 = C$, where $C = 2C_1$ is an arbitrary constant. The solutions rectangular hyperbolas. (See Figure 7.64.) $\frac{1}{2}y^2 = \frac{1}{2}x^2 + C_1,$
 $y^2 - x^2 = C$, where $C = 2C_1$ is an arbitrary constant. The solution curves

cctangular hyperbolas. (See Figure 7.64.) Their asymptotes $y = x$ and $y = -x$

so solutions corresponding to $C = 0$.

$$
\begin{cases} \frac{dy}{dx} = x^2 y^3\\ y(1) = 3. \end{cases}
$$

EXAMPLE 2 Solve the initial-value problem
 $\begin{cases} \frac{dy}{dx} = x^2 y^3 \\ y(1) = 3. \end{cases}$
 Solution Separating the differential equation gives $\frac{dy}{y^3} = x^2 dx$. Thus,
 $\int \frac{dy}{y^3} = \int x^2 dx$, so $\frac{-1}{x^3} = \frac{x^3}{x^3} + C$. $\frac{dy}{y^3} = x^2 dx$. Thus, dx . Thus,

$$
\int \frac{dy}{y^3} = \int x^2 dx, \qquad \text{so} \qquad \frac{-1}{2y^2} = \frac{x^3}{3} + C.
$$

Solution Separating the differential equation gives $\frac{dy}{y^3} = x^2 dx$. Thu
 $\int \frac{dy}{y^3} = \int x^2 dx$, so $\frac{-1}{2y^2} = \frac{x^3}{3} + C$.

Since $y = 3$ when $x = 1$, we have $-\frac{1}{18} = \frac{1}{3} + C$ and $C = -\frac{7}{18}$. So

value into the $\frac{1}{18} = \frac{1}{3} + C$ and $C = -\frac{7}{18}$. Substituting this c. Thus,
 $\frac{7}{18}$. Substituting this
tisfies $v(1) = 3$) **Solution** Separating the differential equation gives $\frac{dy}{y^3} = x^2 dx$. Thus,
 $\int \frac{dy}{y^3} = \int x^2 dx$, so $\frac{-1}{2y^2} = \frac{x^3}{3} + C$.

Since $y = 3$ when $x = 1$, we have $-\frac{1}{18} = \frac{1}{3} + C$ and $C = -\frac{7}{18}$. Substituting this

Solution Separating the differential equation gives
$$
\frac{dy}{y^3} = x^2 dx
$$
. Thus,
\n
$$
\int \frac{dy}{y^3} = \int x^2 dx
$$
, so $\frac{-1}{2y^2} = \frac{x^3}{3} + C$.
\nSince $y = 3$ when $x = 1$, we have $-\frac{1}{18} = \frac{1}{3} + C$ and $C = -\frac{7}{18}$. Substituting this value into the above solution and solving for y, we obtain
\n
$$
y(x) = \frac{3}{\sqrt{7 - 6x^3}}
$$
. (Only the positive square root of y^2 satisfies $y(1) = 3$.)
\nThis solution is valid for $x < (\frac{7}{6})^{1/3}$. (See Figure 7.65.)

 $(\frac{7}{6})^{1/3}$. (See Figure 7.65)

 $y^2 - x^2 = C$

EXAMPLE 3 Solve the **integral equation**
$$
y(x) = 3 + 2 \int_{1}^{x} ty(t) dt
$$
.
Solution Differentiating the integral equation with respect to x gives

$$
\frac{dy}{dx} = 2x y(x) \quad \text{or} \quad \frac{dy}{y} = 2x dx.
$$

$$
\frac{dy}{dx} = 2x y(x) \qquad \text{or} \qquad \frac{dy}{y} = 2x dx.
$$

EXAMPLE 3 Solve the **integral equation** $y(x) = 3 + 2 \int_1^x t y(t) dt$.
 Solution Differentiating the integral equation with respect to x gives
 $\frac{dy}{dx} = 2x y(x)$ or $\frac{dy}{y} = 2x dx$.

Thus, ln $|y(x)| = x^2 + C$, and solving for y, $\int_{1}^{x} ty(t) dt$.

x gives

. Putting $x = 1$ in the

so $C_1 = 3/e$ and **EXAMPLE 3** Solve the **integral equation** $y(x) = 3 + 2 \int_1^x t y(t) dt$.
 Solution Differentiating the integral equation with respect to x gives
 $\frac{dy}{dx} = 2x y(x)$ or $\frac{dy}{y} = 2x dx$.

Thus, $\ln |y(x)| = x^2 + C$, and solving for y, y

$$
y(x) = 3e^{x^2 - 1}.
$$

thus, $\ln |y(x)| = x^2 + C$, and solving for y, $y(x) = C_1 e^{x^2}$. Putting $x = 1$ in the tegral equation provides an initial value: $y(1) = 3 + 0 = 3$, so $C_1 = 3/e$ and $y(x) = 3e^{x^2-1}$.
 EXAMPLE 4 (A solution concentration problem $1,2^2 + C$, and solving for y , $y(x) = C_1 e^{x^2}$. Putting $x = 1$ in the ovides an initial value: $y(1) = 3 + 0 = 3$, so $C_1 = 3/e$ and

(A solution concentration problem) Initially a tank contains 1,000 L of brine with 50 kg o Thus, $\ln |y(x)| = x^2 + C$, and solving for y , $y(x) = C_1 e^{x^2}$. Putting $x = 1$ in the integral equation provides an initial value: $y(1) = 3 + 0 = 3$, so $C_1 = 3/e$ and $y(x) = 3e^{x^2-1}$.
 EXAMPLE 4 (A solution concentration pro integral equation provides an initial value: $y(1) = 3 + 0 = 3$, so $C_1 = 3/e$ and
 $y(x) = 3e^{x^2-1}$.
 EXAMPLE 4 (A solution concentration problem) Initially a tank contains

10 g of salt per litre is flowing into the tank $y(x) = 3e^{x^2-1}$.
 EXAMPLE 4 (A solution concentration problem) Initially a tank contains

10 g of salt per litre is flowing into the tank at a constant rate of 10 L/min. If the con-

tents of the tank are kept thorough integral equation provides an initial value: $y(1) = 3 + 0 = 3$, so $C_1 = 3/e$ and
 $y(x) = 3e^{x^2-1}$.
 EXAMPLE 4 (A solution concentration problem) Initially a tank contains

10 g of salt per litre is flowing into the tank

EXAMPLE 4 (A solution concentration problem) Initially a tank contains 10 g of salt per litre is flowing into the tank at a constant rate of 10 L/min. If the contents of the tank are kept thoroughly mixed at all times, **EXAMPLE 4** (A solution concentration problem) Initially a tank contains 1,000 L of brine with 50 kg of dissolved salt. Brine containing 10 g of salt per litre is flowing into the tank at a constant rate of 10 L/min. If t **EXAMPLE 4** (A solution concentration problem) Initially a tank contains
10 g of salt per litre is flowing into the tank at a constant rate of 10 L/min. If the con-
tents of the tank are kept thoroughly mixed at all times **EXAMPLE 4** (A solution concentration problem) imitally a talk contains
10 g of salt per litre is flowing into the tank at a constant rate of 10 L/min. If the con-
tents of the tank are kept thoroughly mixed at a constant of the tank are kept thoroughly mixed at all times, and if the t 10 L/min, how much salt remains in the tank at the end of 4 **tion** Let $x(t)$ be the number of kilograms of salt in solution. Thus, $x(0) = 50$. Salt is comin 10 g of salt per litre is flowing into the tank at a constant rate of 10 L/min. If the solution of the tank are kept thoroughly mixed at all times, and if the solution also flows out at 10 L/min, how much salt remains in **Solution** Let $x(t)$ be the initial condition of halos and in solution in the talk at the initial condition in the solution in the solution ± 100 g/ $K = 100$ g/min = 1/10 kg/min. At all times the tank at tarate of 10 g/

$$
\frac{dx}{dt} = \text{ rate in } -\text{ rate out } = \frac{1}{10} - \frac{x}{100} = \frac{10 - x}{100}.
$$

variables:

$$
\frac{dx}{10-x} = \frac{dt}{100}.
$$

$$
-\ln|10 - x| = \frac{t}{100} + C.
$$

 $\frac{dx}{dt}$ = rate in – rate out = $\frac{1}{10} - \frac{x}{100} = \frac{10 - x}{100}$.

Although $x(t) = 10$ is a constant solution of the differential equation, it does

satisfy the initial condition $x(0) = 50$, so we will find other solution Although $x(t) = 10$ is a constant solution of the differential equation, it does not
satisfy the initial condition $x(0) = 50$, so we will find other solutions by separating
variables:
 $\frac{dx}{10-x} = \frac{dt}{100}$.
Integrating bot Although $x(t) = 10$ is a constant solution of the differential equation, it does not
satisfy the initial condition $x(0) = 50$, so we will find other solutions by separating
variables:
 $\frac{dx}{10-x} = \frac{dt}{100}$.
Integrating bot Integrating both sides of this equation, we obtain
 $-\ln|10 - x| = \frac{t}{100} + C$.

Observe that $x(t) \neq 10$ for any finite time t (since ln 0 is not defined). Since $x(0) =$

50 > 10, it follows that $x(t) > 10$ for all $t > 0$. (Integrating both sides of this equation, we obtain
 $-\ln|10 - x| = \frac{t}{100} + C$.

Observe that $x(t) \neq 10$ for any finite time t (since ln 0 is not defined). Since $x(0) =$

50 > 10, it follows that $x(t) > 10$ for all $t > 0$. $(x$ Integrating both sides of this equation, we obtain
 $-\ln|10 - x| = \frac{t}{100} + C$.

Observe that $x(t) \neq 10$ for any finite time t (since ln 0 is n

50 > 10, it follows that $x(t) > 10$ for all $t > 0$. $(x(t)$

so it cannot take any Observe that $x(t) \neq 10$ for any finite time t (since ln 0 is not de 50 > 10, it follows that $x(t) > 10$ for all $t > 0$. ($x(t)$ is ne so it cannot take any value less than 10 without somewhere ta the Intermediate-Value The Example 100 $x_1 = 100$ and $x(t) \neq 10$ for any finite time t (since ln 0
 > 10 , it follows that $x(t) > 10$ for all $t > 0$. (x

cannot take any value less than 10 without some

intermediate-Value Theorem.) Hence, we can d the Intermediate-Value Theorem.) Hence, we can drop the absolute value from the solution above and obtain
 $\ln(x - 10) = -\frac{t}{100} - C$.

Since $x(0) = 50$, we have $-C = \ln 40$ and
 $x = x(t) = 10 + 40e^{-t/100}$.

After 40 min there wi

$$
\ln(x - 10) = -\frac{t}{100} - C.
$$

$$
x = x(t) = 10 + 40e^{-t/100}
$$

EXAMPLE 5 (A rate of reaction problem) In a chemical reaction that goes to completion in solution, one molecule of each of two reactants, A and B, combine to form each molecule of the product C. According to the law of **COMPTAGE SECTION 7.9:** First-Order Differential Equations **453**
(A rate of reaction problem) In a chemical reaction that goes to completion in solution, one molecule of each of two reactants, A orm each molecule of the **EXAMPLE 5** (A rate of reaction problem) In a chemical reaction that goes to completion in solution, one molecule of each of two reactants, A and B, combine to form each molecule of the product C. According to the law of **EXAMPLE 5** (A rate of reaction problem) In a chemical reaction that goes to completion in solution, one molecule of each of two reactants, A and B, combine to form each molecule of the product C. According to the law of **EXAMPLE 5** (A rate of reaction problem) In a chemical reaction that goes to completion in solution, one molecule of each of two reactants, A and B, combine to form each molecule of the product C. According to the law of **EXAMPLE 5** (A rate of reaction problem) In a completion in solution, one molecule of *A* combine to form each molecule of the product *C* action, the reaction proceeds at a rate proportional to the of *A* and *B* in the SECTION 7.9: First-Order Differential Equations **453**
 action problem) In a chemical reaction that goes to

n solution, one molecule of each of two reactants, *A*

lecule of the product *C*. According to the law of mass of A and $b > 0$ molecules/cm³ of B, then the number $x(t)$ of molecules/cm³ of C **EXAMPLE 5** (A rate of reaction problem) In a chemical reaction that goes to completion in solution, one molecule of each of two reactants, A and B, combine to form each molecule of the product C. According to the law of **EXAMPLE 5** (A rate of reaction problem) In a chemical reaction that goes to completion in solution, one molecule of each of two reactants, A and B, combine to form each molecule of the product C. According to the law of and *B*, combine to form each molecule of the product *C*. According to the law of mass
action, the reaction proceeds at a rate proportional to the product of the concentrations
of *A* and *B* in the solution. Thus, if th action, the reaction proceeds at a rate proportional to the product of the concentrations
of *A* and *B* in the solution. Thus, if there were initially present *a* > 0 molecules/cm³
of *A* and *b* > 0 molecules/cm³ of

$$
\frac{dx}{dt} = k(a-x)(b-x).
$$

that $b \neq a$: of variables and the technique of partial fraction decomposition under the assumptit

that $b \neq a$:
 $\int \frac{dx}{(a-x)(b-x)} = k \int dt = kt + C$.

Since
 $\frac{1}{(a-x)(b-x)} = \frac{1}{b-a} \left(\frac{1}{a-x} - \frac{1}{b-x} \right)$,

and since necessarily $x \leq a$ and x

$$
\int \frac{dx}{(a-x)(b-x)} = k \int dt = kt + C.
$$

Since

$$
\frac{1}{(a-x)(b-x)} = \frac{1}{b-a} \left(\frac{1}{a-x} - \frac{1}{b-x} \right),
$$

$$
\frac{1}{b-a}(-\ln(a-x) + \ln(b-x)) = k t + C,
$$

or

$$
\int \frac{ax}{(a-x)(b-x)} = k \int dt = kt + C.
$$

\nSince
\n
$$
\frac{1}{(a-x)(b-x)} = \frac{1}{b-a} \left(\frac{1}{a-x} - \frac{1}{b-x} \right),
$$

\nand since necessarily $x \le a$ and $x \le b$, we have
\n
$$
\frac{1}{b-a} \left(-\ln(a-x) + \ln(b-x) \right) = kt + C,
$$

\nor
\n
$$
\ln \left(\frac{b-x}{a-x} \right) = (b-a)kt + C_1, \text{ where } C_1 = (b-a)C.
$$

\nBy assumption, $x(0) = 0$, so $C_1 = \ln(b/a)$ and
\n
$$
\ln \frac{a(b-x)}{b(a-x)} = (b-a)kt.
$$

\nThis equation can be solved for x to yield $x = x(t) = \frac{ab(e^{(b-a)kt} - 1)}{be^{(b-a)kt} - a}.$

$$
\ln \frac{a(b-x)}{b(a-x)} = (b-a) kt.
$$

 $(a-b-a)C$.
 $\frac{ab(e^{(b-a)kt} - 1)}{be^{(b-a)kt} - a}$. $\frac{b(e^{(b-a)kt} - 1)}{be^{(b-a)kt} - a}$ In $\frac{a(b-x)}{b(a-x)} = (b-a)kt$.

This equation can be solved for x to yield $x = x(t) = \frac{ab(e^{(b-a)kt} - 1)}{be^{(b-a)kt} - a}$
 EXAMPLE 6 Find a family of curves, each of which intersects with equation of the form $y = Cx^2$ at right angles.

y assumption, $x(0) = 0$, so $C_1 = \ln(b/a)$ and
 $\ln \frac{a(b-x)}{b(a-x)} = (b-a)kt$.

his equation can be solved for x to yield $x = x(t) = \frac{ab(e^{(b-a)kt} - 1)}{be^{(b-a)kt} - a}$.
 EXAMPLE 6 Find a family of curves, each of which intersects every pa $y = 0$, so $C_1 = \ln(b/a)$ and
 $(b - a) kt$.
 $c = \ln(b/a)$ and
 $c = \ln(b/a)$ and
 $c = \ln(b/a)$ and
 $c = \frac{a b (e^{(b-a)kt} - 1)}{b e^{(b-a)kt} - a}$.

Find a family of curves, each of which intersects every para

with equation of the form $y = Cx^2$ at $\frac{b(e^{(b-a)kt} - 1)}{be^{(b-a)kt} - a}$.

aich intersects every parabola

at right angles.

e differential equation $s x(t) = \frac{ab(e^{(b-a)kt} - 1)}{be^{(b-a)kt} - a}$.
each of which intersects every parabola $y = Cx^2$ at right angles.
satisfies the differential equation **EXAMPLE 6** Find a family of curves.

with equation of the form
 Solution The family of parabolas $y = Cx$
 $\frac{d}{dx} \left(\frac{y}{x^2}\right) = \frac{d}{dx}C = 0;$

that is,
 $x^2 \frac{dy}{dx} - 2xy = 0$ or $\frac{dy}{dx} = \frac{2y}{x^2}$

$$
\frac{d}{dx}\left(\frac{y}{x^2}\right) = \frac{d}{dx}C = 0;
$$

$$
x^2 \frac{dy}{dx} - 2xy = 0 \qquad \text{or} \qquad \frac{dy}{dx} = \frac{2y}{x}.
$$

Solution The family of parabolas $y = Cx^2$ satisfies the differential example of $\frac{d}{dx} \left(\frac{y}{x^2}\right) = \frac{d}{dx}C = 0$;
that is,
 $x^2 \frac{dy}{dx} - 2xy = 0$ or $\frac{dy}{dx} = \frac{2y}{x}$.
Any curve that meets the parabolas $y = Cx^2$ at righ at and the intersects every parabola
 $y = Cx^2$ at right angles.

Satisfies the differential equation

at right angles must, at any point (x, y)

at of the slope of the particular parabola

must satisfy **Solution** The family of parabolas $y = Cx^2$ satisfies the differential equation
 $\frac{d}{dx} \left(\frac{y}{x^2}\right) = \frac{d}{dx}C = 0;$

that is,
 $x^2 \frac{dy}{dx} - 2xy = 0$ or $\frac{dy}{dx} = \frac{2y}{x}$.

Any curve that meets the parabolas $y = Cx^2$ at ri $\frac{d}{dx} \left(\frac{y}{x^2}\right) = \frac{d}{dx} C = 0;$

that is,
 $x^2 \frac{dy}{dx} - 2xy = 0$ or $\frac{dy}{dx} = \frac{2y}{x}$.

Any curve that meets the parabolas $y = Cx^2$ at right angles must, at any point (x, y)

on it, have slope equal to the negative rec

$$
\frac{dy}{dx} = -\frac{x}{2y}.
$$

 $\frac{1}{2}x^2 + C_1$ or $x^2 + 2y$ Separation of the variables leads to $2y dy = -x dx$, and integration of both sides
then yields $y^2 = -\frac{1}{2}x^2 + C_1$ or $x^2 + 2y^2 = C$, where $C = 2C_1$. This equation
represents a family of ellipses centred at the origin. Each Separation of the variables leads to $2y dy = -x dx$, and integration of both sides
then yields $y^2 = -\frac{1}{2}x^2 + C_1$ or $x^2 + 2y^2 = C$, where $C = 2C_1$. This equation
represents a family of ellipses centred at the origin. Each Separation of the variables leads to $2y dy = -x dx$, and integration of both sides
then yields $y^2 = -\frac{1}{2}x^2 + C_1$ or $x^2 + 2y^2 = C$, where $C = 2C_1$. This equation
represents a family of ellipses centred at the origin. Each Separation of the variables leads to $2y dy = -x dx$, and integrals then yields $y^2 = -\frac{1}{2}x^2 + C_1$ or $x^2 + 2y^2 = C$, where C = represents a family of ellipses centred at the origin. Each ellipse at right angles, as shown i Separation of the variables leads to $2y dy = -x dx$, and in then yields $y^2 = -\frac{1}{2}x^2 + C_1$ or $x^2 + 2y^2 = C$, where C represents a family of ellipses centred at the origin. Each ellips at right angles, as shown in Figure 7.6 Separation of the variables leads to $2y dy = -x dx$, and integration of both sides
then yields $y^2 = -\frac{1}{2}x^2 + C_1$ or $x^2 + 2y^2 = C$, where $C = 2C_1$. This equation
represents a family of ellipses centred at the origin. Each

$$
\frac{dy}{dx} + p(x)y = q(x), \qquad (*)
$$

First-Order Linear Equations

A first-order **linear differential equation** is one of the type
 $\frac{dy}{dx} + p(x)y = q(x)$, (*)

where $p(x)$ and $q(x)$ are given functions, which we assume to be continuous. The

equation is called First-Order Linear Equations

A first-order linear differential equation is one of the type
 $\frac{dy}{dx} + p(x)y = q(x)$, (*)

where $p(x)$ and $q(x)$ are given functions, which we assume to be continuous. The

equation is called **no** ing homogeneous equation, **EXECTE AND AND EXECT SOLUTE CONSTRAINT A** first-order linear differential equation is one of the type
 $\frac{dy}{dx} + p(x)y = q(x)$, (*)

where $p(x)$ and $q(x)$ are given functions, which we assume to be continuous. The

equation is $\frac{d}{dx} + p(x)y = q(x),$ (*)

where $p(x)$ and $q(x)$ are given functions, which we assume to b

equation is called **nonhomogeneous** unless $q(x)$ is identically zer

ing **homogeneous** equation,
 $\frac{dy}{dx} + p(x)y = 0$,

is separable and $f(x)y = q(x)$, (*)
and $q(x)$ are given functions, which we assume
alled **nonhomogeneous** unless $q(x)$ is identicall
eous equation,
 $(x)y = 0$,
and so is easily solved to give $y = Ke^{-\mu(x)}$, whe
mtiderivative of $p(x)$:
 $\int p(x) dx$ an

$$
\frac{dy}{dx} + p(x)y = 0,
$$

 $\frac{dy}{dx} + p(x)y = 0$,
parable and so is easily solved to give $y = Ke^{-\mu(x)}$, where K is any constant and
is any antiderivative of $p(x)$:
 $\mu(x) = \int p(x) dx$ and $\frac{d\mu}{dx} = p(x)$.
There are two methods for solving the nonhomogeneous equ

$$
\mu(x) = \int p(x) dx
$$
 and $\frac{d\mu}{dx} = p(x)$.

 $\frac{d}{dx} + p(x)y = 0$,

is separable and so is easily solved to give $y = Ke^{-\mu(x)}$, where K is any
 $\mu(x)$ is any antiderivative of $p(x)$:
 $\mu(x) = \int p(x) dx$ and $\frac{d\mu}{dx} = p(x)$.

There are two methods for solving the nonhomogeneous is separable and so is easily solved to give $y = Ke^{-\mu(x)}$, where K is any constant and $\mu(x)$ is any antiderivative of $p(x)$:
 $\mu(x) = \int p(x) dx$ and $\frac{d\mu}{dx} = p(x)$.

There are two methods for solving the nonhomogeneous equatio is separable and so is easily solved to give $y = K e^{-\mu x/\sigma}$, where K is any constant and $\mu(x)$ is any antiderivative of $p(x)$:
 $\mu(x) = \int p(x) dx$ and $\frac{d\mu}{dx} = p(x)$.

There are two methods for solving the nonhomogeneous equa $\mu(x)$ is any anticent
value of $p(x)$.

There are two methods for solving the nonhomogeneous equation (*). Bo

volve the function $\mu(x)$ defined above.
 METHOD I. Using an Integrating Factor. Multiply equation (*) by e for solving the nonhomogeneous equation (*). Both
ned above.
tegrating Factor. Multiply equation (*) by $e^{\mu(x)}$ (wh
tor for the equation) and observe that the left side is j
the Product Rule
 $\int \frac{dy}{dx} + e^{\mu(x)} \frac{d\mu}{dx} y(x)$

$$
\frac{d}{dx}(e^{\mu(x)}y(x)) = e^{\mu(x)}\frac{dy}{dx} + e^{\mu(x)}\frac{d\mu}{dx}y(x)
$$

$$
= e^{\mu(x)}\left(\frac{dy}{dx} + p(x)y\right) = e^{\mu(x)}q(x).
$$

Therefore, $e^{\mu(x)}$ $y(x) = \int e^{\mu(x)} q(x) dx$, or

SECTION 7.9: First-Order Differential Equ
\n
$$
y(x) = e^{-\mu(x)} \int e^{\mu(x)} q(x) dx.
$$

SECTION 7.9: First-Order Differential Equations **455**
 $y(x) = e^{-\mu(x)} \int e^{\mu(x)} q(x) dx$.
 METHOD II. Variation of the Parameter. Start with the solution of the corre-

sponding homogeneous equation, namely $y = Ke^{-\mu(x)}$, and repla SECTION 7.9: First-Order Differential Equations **455**
 $y(x) = e^{-\mu(x)} \int e^{\mu(x)} q(x) dx$.
 METHOD II. Variation of the Parameter. Start with the solution of the corre-

sponding homogeneous equation, namely $y = Ke^{-\mu(x)}$, and repla $p(x) = e^{-\mu(x)} \int e^{\mu(x)} q(x) dx$.
 METHOD II. Variation of the Parameter. Start with the solution of the corresponding homogeneous equation, namely $y = Ke^{-\mu(x)}$, and replace the constant (i.e., parameter) K by an as yet unknown f $y(x) = e^{-\mu(x)} \int e^{\mu(x)} q(x) dx$.
 METHOD II. Variation of the Parameter. Start with the solution of the corresponding homogeneous equation, namely $y = Ke^{-\mu(x)}$, and replace the constant (i.e., parameter) *K* by an as yet unknown **METHOD II.** Variation of the Parameter.
sponding homogeneous equation, namely $y = h$
parameter) K by an as yet unknown function k
substitute this expression for y into the differen
 $\frac{d}{dx} (k(x)e^{-\mu(x)}) + p(x)k(x)e^{-\mu(x)} = q$
 $k'(x)e^{-\$ **Variation of the Parameter.** Start with the solution of t
ogeneous equation, namely $y = Ke^{-\mu(x)}$, and replace the cons
by an as yet unknown function $k(x)$ of the independent variat
expression for y into the differential eq

$$
\frac{d}{dx} (k(x)e^{-\mu(x)}) + p(x)k(x)e^{-\mu(x)} = q(x)
$$

\n
$$
k'(x)e^{-\mu(x)} - \mu'(x)k(x)e^{-\mu(x)} + p(x)k(x)e^{-\mu(x)} = q(x),
$$

\nwhich, since $\mu'(x) = p(x)$, reduces to
\n
$$
k'(x) = e^{\mu(x)}q(x).
$$

\nIntegrating the right side leads to the solution for $k(x)$ and thereby to the solution y
\nfor (*).
\n**EXAMPLE 7** Solve $\frac{dy}{dx} + \frac{y}{x} = 1$ for $x > 0$. Use both methods for comparison.

$$
k'(x) = e^{\mu(x)}q(x).
$$

$$
\frac{d}{dx} (k(x)e^{-\mu(x)}) + p(x)k(x)e^{-\mu(x)} = q(x)
$$

\n $k'(x)e^{-\mu(x)} - \mu'(x)k(x)e^{-\mu(x)} + p(x)k(x)e^{-\mu(x)} = q(x),$
\nwhich, since $\mu'(x) = p(x)$, reduces to
\n $k'(x) = e^{\mu(x)}q(x).$
\nIntegrating the right side leads to the solution for $k(x)$ and thereby to the solution y
\nfor (*).
\n**EXAMPLE 7** Solve $\frac{dy}{dx} + \frac{y}{x} = 1$ for $x > 0$. Use both methods for comparison.
\n**Solution** Here, $p(x) = 1/x$, so $\mu(x) = \int p(x) dx = \ln x$ (for $x > 0$).
\nMETHOD I. The integrating factor is $e^{\mu(x)} = x$. We calculate
\n
$$
\frac{d}{dx}(xy) = x\frac{dy}{dx} + y = x\left(\frac{dy}{dx} + \frac{y}{x}\right) = x,
$$

\nand so
\n $xy = \int x dx = \frac{1}{2}x^2 + C.$

$$
\frac{d}{dx}(xy) = x\frac{dy}{dx} + y = x\left(\frac{dy}{dx} + \frac{y}{x}\right) = x,
$$

$$
xy = \int x \, dx = \frac{1}{2}x^2 + C.
$$

Finally,

$$
y = \frac{1}{x} \left(\frac{1}{2} x^2 + C \right) = \frac{x}{2} + \frac{C}{x}.
$$

and so
 $xy = \int x dx = \frac{1}{2}x^2 + C$.

Finally,
 $y = \frac{1}{x}(\frac{1}{2}x^2 + C) = \frac{x}{2} + \frac{C}{x}$.

This is a solution of the given equation for any value of the constant C.

METHOD II. The corresponding homogeneous equation, $\frac{dy}{dx} + \frac$ $xy = \int x dx = \frac{1}{2}x^2 + C$.
Finally,
 $y = \frac{1}{x}(\frac{1}{2}x^2 + C) = \frac{x}{2} + \frac{C}{x}$.
This is a solution of the given equation for any value of the constant C.
METHOD II. The corresponding homogeneous equation, $\frac{dy}{dx} + \frac{y}{x} = 0$, h $\frac{1}{dx} + \frac{1}{x} = 0$, has solution \mathcal{V} ont *C*.
 $\frac{y}{x} = 0$, has solution
 $x(x)$ and substituting $y = Ke^{-\mu(x)} = \frac{K}{r}$. Replacing the constant K wi $\frac{1}{x}$. Replacing the com $\begin{aligned} (C) &= \frac{x}{2} + \frac{C}{x}. \end{aligned}$
the given equation for any value of the constant C.
orresponding homogeneous equation, $\frac{dy}{dx} + \frac{y}{x} = 0$, has solution
. Replacing the constant K with the function $k(x)$ and substituti Finally,
 $y = \frac{1}{x} \left(\frac{1}{2} x^2 + C \right) = \frac{x}{2} + \frac{C}{x}$.

This is a solution of the given equation for any value of the constant *C*.

METHOD II. The corresponding homogeneous equation, $\frac{dy}{dx} + \frac{y}{x} = 0$, has $y = Ke^{-\mu(x)} =$ METHOD II. The corresponding homoger
 $y = Ke^{-\mu(x)} = \frac{K}{x}$. Replacing the constant

into the given differential equation we obta
 $\frac{1}{x}k'(x) - \frac{1}{x^2}k(x) + \frac{1}{x^2}k(x) = 1$,

so that $k'(x) = x$ and $k(x) = \frac{x^2}{2} + C$, wh
 $y = \frac$ ig homogeneous equation, $\frac{dy}{dx} + \frac{y}{x} = 0$, has solution
the constant K with the function $k(x)$ and substituting
on we obtain
 $(x) = 1$,
 $\frac{x^2}{2} + C$, where C is any constant. Therefore,

$$
\frac{1}{x}k'(x) - \frac{1}{x^2}k(x) + \frac{1}{x^2}k(x) = 1,
$$

 $Q'(x) = x$ and $k(x) = \frac{x^2}{2} + C$, where C is any const $\frac{1}{x}k'(x) - \frac{1}{x^2}k(x) + \frac{1}{x^2}k(x) = 1,$
so that $k'(x) = x$ and $k(x) = \frac{x^2}{2} + C$, where *C* is any constant. Therefore,
 $y = \frac{k(x)}{x} = \frac{x}{2} + \frac{C}{x}$,
the same solution obtained by METHOD I.

$$
y = \frac{k(x)}{x} = \frac{x}{2} + \frac{C}{x},
$$

Remark Both methods really amount to the same calculations expressed in different ways. Use whichever one you think is easiest to understand. The remaining examples in this section will be done by using integrating facto **Remark** Both methods really amount to the same calculations expressed in different ways. Use whichever one you think is easiest to understand. The remaining examples in this section will be done by using integrating facto **Remark** Both methods really amount to the same calculations expressed in different ways. Use whichever one you think is easiest to understand. The remaining examples in this section will be done by using integrating facto **Remark** Both methods really amount to the same calculations expressed in different ways. Use whichever one you think is easiest to understand. The remaining examples in this section will be done by using integrating fact **Remark** Both methods really amount to the same calculations express
ways. Use whichever one you think is easiest to understand. The remain this section will be done by using integrating factors, but variation of
prove us **Emark** Both methods really amount to the same cal
ays. Use whichever one you think is easiest to unders
this section will be done by using integrating factors,
rove useful later on (Section 18.6) to deal with nonh
quatio **Remark** Both methods really amount to the same calculations expres
ways. Use whichever one you think is easiest to understand. The remain this section will be done by using integrating factors, but variation of
prove use the same calculations expressed in different
est to understand. The remaining examples
ting factors, but variation of parameters will
all with nonhomogeneous linear differential
 $/2$ and $e^{\mu(x)} = e^{x^2/2}$. We calculate
 $=$

EXAMPLE 8 Solve
$$
\frac{dy}{dx} + xy = x^3
$$
.

Given later on (Section 16.0) to deal with nonnonlogeneous linear differential

\ntions of second or higher order.

\nWhen there,
$$
p(x) = x
$$
, so $\mu(x) = x^2/2$ and $e^{\mu(x)} = e^{x^2/2}$. We calculate

\n
$$
\frac{d}{dx}(e^{x^2/2}y) = e^{x^2/2}\frac{dy}{dx} + e^{x^2/2}xy = e^{x^2/2}\left(\frac{dy}{dx} + xy\right) = x^3e^{x^2/2}.
$$

\n
$$
e^{x^2/2}y = \int x^3e^{x^2/2}dx
$$
 Let $U = x^2$, $dV = x e^{x^2/2}dx$.

Thus,

EXAMPLE 8 Solve
$$
\frac{dy}{dx} + xy = x^3
$$
.
\n**Solution** Here, $p(x) = x$, so $\mu(x) = x^2/2$ and $e^{\mu(x)} = e^{x^2/2}$. We calculate
\n
$$
\frac{d}{dx}(e^{x^2/2}y) = e^{x^2/2}\frac{dy}{dx} + e^{x^2/2}xy = e^{x^2/2}\left(\frac{dy}{dx} + xy\right) = x^3e^{x^2/2}.
$$
\nThus,
\n
$$
e^{x^2/2}y = \int x^3e^{x^2/2} dx
$$
\nLet $U = x^2$, $dV = xe^{x^2/2} dx$.
\nThen $dU = 2x dx$, $V = e^{x^2/2}$.
\n
$$
= x^2e^{x^2/2} - 2\int xe^{x^2/2} dx
$$
\n
$$
= x^2e^{x^2/2} - 2e^{x^2/2} + C,
$$
\nand, finally, $y = x^2 - 2 + Ce^{-x^2/2}$.
\n**EXAMPLE 9** (An inductance-resistance circuit) An electric circuit (see Figure 7.67) contains a constant DC voltage source of V volts, a switch, a resistor of size R ohms, and an inductor of size L henrys. The circuit has no capacitance. The switch, initially open so that no current is flowing, is closed at time

circuit

 $\int_{x^2/2}^{x^2/2} - 2 \int x e^{x^2/2} dx$
 $\int_{x^2/2}^{x^2/2} - 2 e^{x^2/2} + C$,
 $-2 + Ce^{-x^2/2}$.

(An inductance-resistance circuit) An electric circuit (see Figure 7.67) contains a constant DC voltage source of *V* volts, a size *R* $= x^2 e^{x^2/2} - 2 \int x e^{x^2/2} dx$
 $= x^2 e^{x^2/2} - 2 e^{x^2/2} + C$,

and, finally, $y = x^2 - 2 + Ce^{-x^2/2}$.
 EXAMPLE 9 (An inductance-resistance circuit) An electric circuit (see

Figure 7.67) contains a constant DC voltage source $x^2 e^{x^2/2} - 2e^{x^2/2} + C$,

and, finally, $y = x^2 - 2 + Ce^{-x^2/2}$.
 EXAMPLE 9 (An inductance-resistance circuit) An electric circuit (see

switch, a resistor of size R ohms, and an inductor of size L henrys. The circuit ha $x^2 e^{x^2/2} - 2e^{x^2/2} + C$,

and, finally, $y = x^2 - 2 + Ce^{-x^2/2}$.
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= $x^2 e^{x^2/2} - 2 e^{x^2/2} + C$,

and, finally, $y = x^2 - 2 + C e^{-x^2/2}$.
 EXAMPLE 9 (An inductance-resistance circuit) An electric circuit (see

Figure 7.67) contains a constant DC voltage sour = $x^2 e^{x^2/2} - 2e^{x^2/2} + C$,
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EXAMPLE 9 (An inductance-resistance circuit) An electric circuit (see
switch, a resistor of size R ohms, and an inductor of size L henrys. The circuit has = $x^2 e^{x^2/2} - 2e^{x^2/2} + C$,
and, finally, $y = x^2 - 2 + Ce^{-x^2/2}$.
EXAMPLE 9 (An inductance-resistance circuit) An electric circuit (see
switch, a resistor of size R ohms, and an inductor of size L henrys. The circuit has and, finally, $y = x^2 - 2 + Ce^{-x^2/2}$.
 EXAMPLE 9 (An inductance-resistance circuit) An electric circuit

Figure 7.67) contains a constant DC voltage source of V vol

switch, a resistor of size R ohms, and an inductor of si **EXECUTE ANDEX** IN CONSULTABLE TO SIZE LATE INTIME SURFACT AND CONDUCT SIZE LATE AND CONDUCT SIZE LATE ON A CONDUCT AND ANDEX INTO EXECUTE ON SIZE A time are accorded to the current begins to flow at that time. If the ind current would state thy jump from 0 ampetes when $t > 0$ of $t = v/k$ ampetes
 $t > 0$. However, if $L > 0$ the current cannot change instantaneously, it will do

on time t. Let the current t seconds after the switch is closed

$$
\begin{cases}\nL\frac{dI}{dt} + RI = V \\
I(0) = 0.\n\end{cases}
$$

Solution The DE can be written in the form $\frac{dI}{dt}$ + $\frac{R}{L}$ = $\frac{V}{dt}$

Find $I(t)$. What is $\lim_{t\to\infty} I(t)$? How long does it take after the switch is closed

the current to rise to 90% of its limiting value?

Solu dI R V $\frac{d\tau}{dt} + \frac{d\tau}{L}I = \frac{d\tau}{L}$. It is linear and has R V $\frac{R}{L}I = \frac{V}{L}$. It is linear and has \overline{L} . It is filled and has Examples 2 and the set of the set o Example 1 and $I(t)$ sausnes the initial-value problem
 $\begin{cases} L \frac{dI}{dt} + RI = V \\ I(0) = 0. \end{cases}$

Find $I(t)$. What is $\lim_{t\to\infty} I(t)$? How long does it take after the

the current to rise to 90% of its limiting value?
 Solution It is the DE can be written in the form $\frac{dI}{dt} + \frac{R}{L}I = \frac{V}{L}$. It is linear and has

g factor $e^{\mu(t)}$, where
 $= \int \frac{R}{L} dt = \frac{Rt}{L}$.
 $e^{Rt/L}I$ $= e^{Rt/L} \left(\frac{dI}{dt} + \frac{R}{L}I \right) = e^{Rt/L} \frac{V}{L}$
 $e^{Rt/L}I = \frac{V}{L} \int e^{R$ between $\frac{dE}{dt} + \frac{1}{L}I = \frac{1}{L}$. It is linear and has
 $= e^{Rt/L} \frac{V}{L}$
 $e^{Rt/L} + C$

$$
\mu(t) = \int \frac{R}{L} dt = \frac{Rt}{L}.
$$

Therefore,

IDENTIFY The DE can be written in the form
$$
\frac{d}{dt} + \frac{1}{L}I = \frac{1}{L}
$$
. It is linear and
rating factor $e^{\mu(t)}$, where

$$
\mu(t) = \int \frac{R}{L} dt = \frac{Rt}{L}.
$$
Before,

$$
\frac{d}{dt} (e^{Rt/L}I) = e^{Rt/L} \left(\frac{dI}{dt} + \frac{R}{L}I\right) = e^{Rt/L} \frac{V}{L}
$$

$$
e^{Rt/L}I = \frac{V}{L} \int e^{Rt/L} dt = \frac{V}{R} e^{Rt/L} + C
$$

$$
I(t) = \frac{V}{R} + Ce^{-Rt/L}.
$$

:

SECTION 7.9: First-Order Differential Equations 457

Since $I(0) = 0$, we have $0 = (V/R) + C$, so $C = -V/R$. Thus, the current flowing

at any time $t > 0$ is
 $I(t) = \frac{V}{V} (1 - e^{-Rt/L})$. Since $I(0) = 0$, we have $0 = (V/R) + C$, so $C = -V$
at any time $t > 0$ is
 $I(t) = \frac{V}{R} \left(1 - e^{-Rt/L}\right)$. SECTION
 $P(I(0) = 0$, we have $0 = (V/R) + C$, so C
 $I(t) = \frac{V}{R} \left(1 - e^{-Rt/L}\right).$ SECTION 7.9: First-Order Differ

we have $0 = (V/R) + C$, so $C = -V/R$. Thus
 $1 - e^{-Rt/L}$.

his solution that $\lim_{x \to \infty} I(t) = V/R$: the stead e $I(0) = 0$, we have $0 = (V/R) + C$, so $C = -V/R$. Thus, the current flowity time $t > 0$ is
 $I(t) = \frac{V}{R} \left(1 - e^{-Rt/L}\right)$.

clear from this solution that $\lim_{t\to\infty} I(t) = V/R$; the *steady-state* current is then that would flow if t (1) = 0, we have $0 = (V/R) + C$, so $C = -V/I$

ne $t > 0$ is
 $= \frac{V}{R} \left(1 - e^{-Rt/L}\right)$.
 Γ from this solution that $\lim_{t\to\infty} I(t) = V/R$; then twould flow if the inductance were zero.

will be 90% of this limiting value when
 $1 -$

$$
I(t) = \frac{V}{R} \left(1 - e^{-Rt/L} \right).
$$

SECTION 7.9: First-Order Differential Equations **457**

Since $I(0) = 0$, we have $0 = (V/R) + C$, so $C = -V/R$. Thus, the current flowing

at any time $t > 0$ is
 $I(t) = \frac{V}{R} \left(1 - e^{-Rt/L}\right)$.

It is clear from this solution that $\$ Since $I(0) = 0$, we have $0 = (V/R) + C$, so $C = -V/R$. Thus, the current flowing
at any time $t > 0$ is
 $I(t) = \frac{V}{R} \left(1 - e^{-Rt/L}\right)$.
It is clear from this solution that $\lim_{t\to\infty} I(t) = V/R$; the *steady-state* current is the
curren

$$
\frac{V}{R}\left(1-e^{-Rt/L}\right) = \frac{90}{100}\frac{V}{R}.
$$

It is clear from this solution that $\lim_{t\to\infty} I(t) = V/R$; the steady-state current is the
current that would flow if the inductance were zero.
 $I(t)$ will be 90% of this limiting value when
 $\frac{V}{R} \left(1 - e^{-Rt/L}\right) = \frac{90}{100} \frac$ It is clear from this solution that $\lim_{t\to\infty} I(t) = V/R$; the *steady-stai*
current that would flow if the inductance were zero.
 $I(t)$ will be 90% of this limiting value when
 $\frac{V}{R} \left(1 - e^{-Rt/L}\right) = \frac{90}{100} \frac{V}{R}$.
This e

It is clear from this solution that $\lim_{t\to\infty} t(t) = v/t$, the steady-state current is the current that would flow if the inductance were zero.
 $I(t)$ will be 90% of this limiting value when
 $\frac{V}{R} \left(1 - e^{-Rt/L}\right) = \frac{90}{100}$ $\frac{V}{R}$ $\left(1 - e^{-Rt/L}\right) = \frac{90}{100} \frac{V}{R}$.
This equation implies that $e^{-Rt/L} = 1/10$, or $t = (L \ln 10)/R$. The current will grow to 90% of its limiting value in $(L \ln 10)/R$ seconds.
Our final example reviews a typical *stre I*(*t*) will be 90% of this limiting value when
 $\frac{V}{R} \left(1 - e^{-Rt/L}\right) = \frac{90}{100} \frac{V}{R}$.

This equation implies that $e^{-Rt/L} = 1/10$, or $t = (L \ln 10)/R$. The current will grow

to 90% of its limiting value in $(L \ln 10)/R$ s

 \overline{R} $(1 - e^{-Rt/L}) = \frac{1}{100} \overline{R}$.

This equation implies that $e^{-Rt/L} = 1/10$, or $t = (L \ln 10)/R$. The current will grow o 90% of its limiting value in $(L \ln 10)/R$ seconds.

Dur final example reviews a typical *stream of* the $e^{-Rt/L} = 1/10$, or $t = (L \ln 10)/R$. The current will grow
g value in $(L \ln 10)/R$ seconds.
eviews a typical *stream of payments* problem of the sort consid-
This time we treat the problem as an initial-value problem for a This equation implies that $e^{-Rt/L} = 1/10$, or $t = (L \ln 10)/R$. The current will grow
to 90% of its limiting value in $(L \ln 10)/R$ seconds.
Our final example reviews a typical *stream of payments* problem of the sort consid-
r to 90% of its limiting value in $(L \ln 10)/R$ seconds.

Our final example reviews a typical *stream of payments* problem of the sort considered in Section 7.7. This time we treat the problem as an initial-value problem for a Our final example reviews a typical *stream of payments* problem of the sort considered in Section 7.7. This time we treat the problem as an initial-value problem for a differential equation.
 EXAMPLE 10 A savings accou Our final example reviews a typical *stream of payments* problem of the sort considered in Section 7.7. This time we treat the problem as an initial-value problem for a differential equation.
 EXAMPLE 10 A savings accou **EXAMPLE 10**
 EXAMPLE 10 EXAMPLE 10 A savings account is opened with a deposit of *A* dollars. At any
time *t* years thereafter, money is being continually deposited into
the account at a rate of $(C + Dt)$ dollars per year. If interest is also bei **EXAMPLE 10** A savings account is opened with
time t years thereafter, money is b
the account at a rate of $(C + Dt)$ dollars per year. If
the account at a nominal rate of 100R percent per year,
the balance $B(t)$ dollars in t

 $A = 5,000, C = 1,000, D = 200, R = 0.13,$ and $t = 5$.
 Solution As noted in Section 3.4, continuous compounding of interest at a nominal

rate of 100*R* percent causes \$1.00 to grow to e^{Rt} dollars in *t* years. Without su **Solution** As noted in Section 3.4, continuous compounding
rate of 100*R* percent causes \$1.00 to grow to e^{Rt} dollars in t yea
deposits, the balance in the account would grow according to t
of exponential growth:
 $\frac{$ rate of 100*K* percent causes \$1.00 to grow to e²²³ dollars in *t* years. Without subsequent
deposits, the balance in the account would grow according to the differential equation
of exponential growth:
 $\frac{dB}{dt} = RB$.
All

$$
\frac{dB}{dt} = RB.
$$

$$
\frac{dB}{dt} = RB + (C + Dt)
$$

or exponential growth:
 $\frac{dB}{dt} = RB$.

Allowing for additional growth due to the continual deposits, we observe that *B* must

satisfy the differential equation
 $\frac{dB}{dt} = RB + (C + Dt)$

or, equivalently, $dB/dt - RB = C + Dt$. This is a $\frac{dB}{dt} = RB.$

wing for additional growth due to the continual deposity the differential equation
 $\frac{dB}{dt} = RB + (C + Dt)$

quivalently, $dB/dt - RB = C + Dt$. This is a li
 $= -R$. Hence, we may take $\mu(t) = -Rt$ and $e^{\mu(t)}$:
 $\frac{d}{dt}(e^{-Rt$ the due to the continual deposits, we observe that *B* must

n
 $B = C + Dt$. This is a linear equation for *B* having

ake $\mu(t) = -Rt$ and $e^{\mu(t)} = e^{-Rt}$. We now calculate
 $\frac{dB}{dt} - Re^{-Rt} B(t) = (C + Dt)e^{-Rt}$

$$
\frac{d}{dt}(e^{-Rt} B(t)) = e^{-Rt} \frac{dB}{dt} - Re^{-Rt} B(t) = (C + Dt) e^{-Rt}
$$

and

Using for additional growth due to the coninual deposits, we observe that *B* must
\n for the differential equation

\n
$$
\frac{dB}{dt} = RB + (C + Dt)
$$
\nquivalently,
$$
dB/dt - RB = C + Dt
$$
. This is a linear equation for *B* having
\n
$$
= -R
$$
. Hence, we may take
$$
\mu(t) = -Rt
$$
 and
$$
e^{\mu(t)} = e^{-Rt}
$$
. We now calculate

\n
$$
\frac{d}{dt}(e^{-Rt}B(t)) = e^{-Rt} \frac{dB}{dt} - Re^{-Rt}B(t) = (C + Dt)e^{-Rt}
$$

\n
$$
e^{-Rt}B(t) = \int (C + Dt)e^{-Rt} dt
$$
 Let
$$
U = C + Dt
$$
,
$$
dV = e^{-Rt} dt
$$
. Then
$$
dU = D dt
$$
,
$$
V = -e^{-Rt}/R
$$
.

\n
$$
= -\frac{C + Dt}{R}e^{-Rt} + \frac{D}{R}\int e^{-Rt}dt
$$

\n
$$
= -\frac{C + Dt}{R}e^{-Rt} - \frac{D}{R^2}e^{-Rt} + K
$$
,
$$
(K = \text{constant})
$$
.

\n
$$
B(t) = -\frac{C + Dt}{R} - \frac{D}{R^2} + Ke^{Rt}
$$
.

:

Hence,

$$
B(t) = -\frac{C+Dt}{R} - \frac{D}{R^2} + Ke^{Rt}.
$$

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\nCHAPTER 7 Applications of Integration
\nSince
$$
A = B(0) = -\frac{C}{R} - \frac{D}{R^2} + K
$$
, we have $K = A + \frac{C}{R} + \frac{D}{R^2}$ and
\n
$$
B(t) = \left(A + \frac{C}{R} + \frac{D}{R^2}\right)e^{Rt} - \frac{C + Dt}{R} - \frac{D}{R^2}.
$$
\nFor the illustration $A = 5,000$, $C = 1,000$, $D = 200$, $R = 0.13$, and $t = 5$, we obtain, using a calculator, $B(5) = 19,762.82$. The account will contain \$19,762.82, after five years, under these circumstances.
\n**EXERCISES 7.9**
\nSolve the separable equations in Exercises 1–10.
\n27. Why is the solution given in Example 5 not valid for $a = b$?

27. Why is the solution given in Example 5 not valid for $a = b$?

28. An object of mass *m* falling near the surface of the earth is retarded by air resistance proportional to its velocity so that, 5,000, $C = 1,000$, $D = 200$, $R = 0.13$, and $t = 5$, we
 $B(5) = 19,762.82$. The account will contain \$19,762.82,

se circumstances.

27. Why is the solution given in Example 5 not valid for $a = b$?

Find the solution for th

 $B(t) = \left(A + \frac{1}{R} + \frac{1}{R^2}\right)e^{-t}$

For the illustration $A = 5,000$, C

obtain, using a calculator, $B(5) =$

after five years, under these circumst
 EXERCISES 7.9

Solve the separable equations in Exercises 1–10.

2. $\frac{$ 1. $\frac{dy}{dx} = \frac{y}{2x}$ $\overline{dx} = \overline{2x}$ 2. $\overline{dx} = \mathcal{V}$ $rac{y}{2x}$ 2. $rac{dy}{dx} = \frac{3y-1}{x}$ 28. A 3. $\frac{dy}{dx} = \frac{x^2}{y^2}$ $\overline{dx} = \overline{y^2}$ 4. $\overline{dx} = x$ x^2 a $\frac{x^2}{y^2}$ **4.** $\frac{dy}{dx} = x^2 y^2$ 5. $\frac{d}{dt} = t$ dY $\frac{dY}{dt} = tY$ 6. $\frac{dx}{dt} = e^x \sin t$ 7. $\frac{dy}{dx} = 1 - y^2$ 8. $\frac{dy}{dx} = 1$ 8. $\frac{dy}{dx} = 1 + y^2$ where 9. $\frac{dy}{dt} = 2 + e^y$ dt ^D ² ^C ^e^y 10. dy dx ^D ^y² .1 y/ 3. $\frac{dy}{dx} = \frac{x^2}{y^2}$

4. $\frac{dy}{dx} = x^2y^2$

5. $\frac{dY}{dt} = tY$

6. $\frac{dx}{dt} = e^x \sin t$

7. $\frac{dy}{dx} = 1 - y^2$

8. $\frac{dy}{dx} = 1 + y^2$

9. $\frac{dy}{dt} = 2 + e^y$

10. $\frac{dy}{dx} = y^2(1 - y)$

Solve the linear equations in Exercises 11–16.

1

11. $\frac{dy}{dx} - \frac{2y}{x} = x^2$ 12. $\frac{dy}{dx}$ $rac{2y}{x} = x^2$ 12. $rac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^2}$ $2v \quad 1$ $\frac{x}{x} = \frac{1}{x^2}$ determ $\frac{1}{x^2}$ 13. $\frac{dy}{dx} + 2y = 3$
14. $\frac{dy}{dx} + y = e^x$ 15. $\frac{dy}{dx} + y = x$ $\frac{dy}{dx} + y = x$ **16.** $\frac{dy}{dx} + 2e^x y = e^x$ **16.** $\frac{dy}{dx} - y = 0$ 9. $\frac{dy}{dt} = 2 + e^y$

Solve the linear equations in Exercises 11–16.

Solve the linear equations in Exercises 11–16.

11. $\frac{dy}{dx} - \frac{2y}{x} = x^2$

12. $\frac{dy}{dx} + \frac{2y}{x} = \frac{1}{x^2}$

13. $\frac{dy}{dx} + 2y = 3$

14. $\frac{dy}{dx} + y = e^x$

15.
$$
\frac{dy}{dx} + y = x
$$

\nSolve the initial-value problems in Exercises 17–20.
\n17. $\begin{cases} \frac{dy}{dt} + 10y = 1 \\ \frac{dy}{dt} + 10y = 1 \end{cases}$
\n18. $\begin{cases} \frac{dy}{dx} + 3x^2y = x^2 \\ \frac{dy}{dx} + 3x^2y = x^2 \end{cases}$
\n19. $\begin{cases} x^2y' + y = x^2e^{1/x} \\ y(1) = 3e \end{cases}$
\n10. $\begin{cases} x^2y' + y = x^2e^{1/x} \\ y(1) = 3e \end{cases}$
\n11. $y(x) = 2 + \int_0^x \frac{t}{y(t)} dt$
\n12. $y(x) = 2 + \int_0^x \frac{t}{y(t)} dt$
\n13. $y(x) = 1 + \int_1^x \frac{y(t) dt}{t(t+1)}$
\n14. $y(x) = 3 + \int_0^x \frac{y(t)}{t(t+1)} dt$
\n15. If $a > b > 0$ in Example 5, find $\lim_{t \to \infty} x(t)$.
\n16. $\frac{dy}{dx} + 2e^{x}y = e^{x}$
\n17. $\begin{cases} \frac{dy}{dt} = mx + e^{x} \\ \frac{dy}{dx} = mx + e^{x} \end{cases}$
\n18. $\begin{cases} \frac{dy}{dx} + 3x^2y = x^2 \\ \frac{dy}{dx} = x^2 \end{cases}$
\n19. $\begin{cases} \frac{dy}{dt} = mx + e^{x} \\ y(1) = 2x e^{-x} \end{cases}$
\n10. $\begin{cases} \frac{dy}{dt} = mx + e^{x} \\ y(0) = 1 \end{cases}$
\n11. $\begin{cases} \frac{dy}{dt} + 10y = 1 \\ y(0) = 2x e^{-x} \end{cases}$
\n12. $\begin{cases} \frac{dy}{dt} = mx + e^{x} \\ y(0) = 1 \end{cases}$
\n13. Find the family hypothesis $xy = 3 + \int_0^x e^{-y(t)} dt$
\n14.

21.
$$
y(x) = 2 + \int_0^x \frac{t}{y(t)} dt
$$
 22. $y(x) = 1 + \int_0^x \frac{(y(t))^2}{1 + t^2} dt$ 31. Find
\n23. $y(x) = 1 + \int_1^x \frac{y(t) dt}{t(t+1)}$ 24. $y(x) = 3 + \int_0^x e^{-y(t)} dt$ 32. Re
\n25. If $a > b > 0$ in Example 5, find $\lim_{t \to \infty} x(t)$.
\n26. If $b > a > 0$ in Example 5, find $\lim_{t \to \infty} x(t)$.
\n28. If $a > b > 0$ in Example 5, find $\lim_{t \to \infty} x(t)$.
\n29. If $b > a > 0$ in Example 5, find $\lim_{t \to \infty} x(t)$.

-
- 5. If $a > b > 0$ in Example 5, find $\lim_{t\to\infty} x(t)$.

6. If $b > a > 0$ in Example 5, find $\lim_{t\to\infty} x(t)$.
 CHAPTER REVIEW

(ey Ideas

What do the following phrases mean?
 \circ a solid of revolution

• What do the following phrases mean?

-
- (5) = 19,762.82. The account will contain \$19,762.82,
reumstances.
Why is the solution given in Example 5 not valid for $a = b$?
Find the solution for the case $a = b$.
An object of mass m falling near the surface of the eart according to Newton's Second Law of Motion,

why is the solution given in Example 5 not valid for $a = b$?

Find the solution for the case $a = b$.

An object of mass m falling near the surface of the earth is

retarded by ai

$$
m\frac{dv}{dt} = mg - kv,
$$

Why is the solution given in Example 5 not valid for $a = b$?
Find the solution for the case $a = b$.
An object of mass m falling near the surface of the earth is
retarded by air resistance proportional to its velocity so tha Why is the solution given in Example 5 not valid for $a = b$?
Find the solution for the case $a = b$.
An object of mass m falling near the surface of the earth is
retarded by air resistance proportional to its velocity so tha Find the solution for the case $a = b$.
An object of mass m falling near the surface of the earth is
retarded by air resistance proportional to its velocity so that,
according to Newton's Second Law of Motion,
 $m \frac{dv}{dt} = mg - kv$ Why is the solution given in Example 5 not valid for $a = b$?
Find the solution for the case $a = b$.
An object of mass m falling near the surface of the earth is
retarded by air resistance proportional to its velocity so tha Why is the solution given in Example 5 not valid for $a = b$?
Find the solution for the case $a = b$.
An object of mass m falling near the surface of the earth is
retarded by air resistance proportional to its velocity so tha Find the solution for the case $a = b$.

An object of mass m falling near the surface of the earth is

retarded by air resistance proportional to its velocity so that,

according to Newton's Second Law of Motion,
 $m \frac{dv}{dt}$ $m \frac{dv}{dt} = mg - kv$,

where $v = v(t)$ is the velocity of the object at time t, and g is

the acceleration of gravity near the surface of the earth.

Assuming that the object falls from rest at time $t = 0$, that is,
 $v(0) = 0$, f $m \frac{dv}{dt} = mg - kv$,
where $v = v(t)$ is the velocity of the object at time t, and g is
the acceleration of gravity near the surface of the earth.
Assuming that the object falls from rest at time $t = 0$, that is,
 $v(0) = 0$, find $m \frac{d\tau}{dt} = mg - kv$,
where $v = v(t)$ is the velocity of the object at time t, and g is
the acceleration of gravity near the surface of the earth.
Assuming that the object falls from rest at time $t = 0$, that is,
 $v(0) = 0$, fin where $v = v(t)$ is the velocity of the object at time t, a
the acceleration of gravity near the surface of the earth
Assuming that the object falls from rest at time $t = 0$,
 $v(0) = 0$, find the velocity $v(t)$ for any $t > 0$

$$
m\frac{dv}{dt} = mg - kv^2.
$$

- **30.** Find the amount in a savings account after one year if the initial balance in the account was $$1,000$, if the interest is paid continuously into the account at a nominal rate of 10% per annum, compounded continuously, and if the account is being $(1 + (\cos x)y) = 2xe^{-\sin x}$ aimum, compounded commutatively, and if the account is being
continuously depleted (by taxes, say) at a rate of $y^2/1,000,000$ dollars per year, where $y = y(t)$ is the balance in the account
after t years. How large can the account grow? How long will $m \frac{dv}{dt} = mg - kv^2.$
 x^2

30. Find the amount in a save initial balance in the accordination of the accordination of the same in the accordination of the same of the same of the divideo and continuously depleted (b) dolla it take the account to grow to half this balance? Example 1 and the amount in a savingtial balance in the accomposition in a saving initial balance in the accomposition in the accomposition of the accomposition of the accomposition of the accomposition of the determinanc object strikes the ground). Show $v(t)$ approaches a limit as $t \to \infty$. Do you need the explicit formula for $v(t)$ to determine this limiting velocity?

29. Repeat Exercise 28 except assume that the air resistance is propo $t \rightarrow \infty$. Do you need the explicit formula for $v(t)$ to
determine this limiting velocity?
Repeat Exercise 28 except assume that the air resistance is
proportional to the square of the velocity so that the equation
of moti determine this limiting velocity?
Repeat Exercise 28 except assume that the air resistance is
proportional to the square of the velocity so that the equation
of motion is
 $m\frac{dv}{dt} = mg - kv^2$.
Find the amount in a savings acc Repeat Exercise 28 except assume that the air resistance is
proportional to the square of the velocity so that the equation
of motion is
 $m\frac{dv}{dt} = mg - kv^2$.
Find the amount in a savings account after one year if the
initial componentially also the square of the velocity so that the equation
of motion is
 $m\frac{dv}{dt} = mg - kv^2$.
Find the amount in a savings account after one year if the
initial balance in the account was \$1,000, if the interest is p determine this limiting velocity?

Repeat Exercise 28 except assume that the air resistance is

proportional to the square of the velocity so that the equation

of motion is
 $m \frac{dv}{dt} = mg - kv^2$.

Find the amount in a saving $m \frac{dv}{dt} = mg - kv^2$.

Find the amount in a savings account after one year if the

initial balance in the account was \$1,000, if the interest is paid

continuously into the account at a nominal rate of 10% per

amnum, compoun $m\frac{dv}{dt} = mg - kv^2$.

Find the amount in a savings account after one year if the

initial balance in the account was \$1,000, if the interest is paid

continuously into the account at a nominal rate of 10% per

annum, compoun 31. Find the family of curves each of which intersects all of the interpretionally into the account at a nominal rate of 10% per annum, compounded continuously, and if the account is being continuously depleted (by taxes, Find the amount in a savings account after one year if the
initial balance in the account was \$1,000, if the interest is paid
continuously into the account at a nominal rate of 10% per
annum, compounded continuously, and 30. Find the amount in a savings account after one year if the
initial balance in the account was \$1,000, if the interest is paid
continuously into the account at a nominal rate of 10% per
amnum, compounded continuously, initial balance in the account was \$1,000, if the interest is paid
continuously into the account at a nominal rate of 10% per
annum, compounded continuously, and if the account is being
continuously depleted (by taxes, continuously into the account at a nominal rate of 10% per
annum, compounded continuously, and if the account is being
continuously depleted (by taxes, say) at a rate of $y^2/1,000,000$
dollars per year, where $y = y(t)$
	- 31. Find the family of curves each of which intersects all of the hyperbolas $xy = C$ at right angles.
32. Repeat the solution concentration problem in Example 4, dt $\qquad \qquad$ \ldots
- annum, compounded continuously, and if the account is being
continuously depleted (by taxes, say) at a rate of $y^2/1,000,000$
dollars per year, where $y = y(t)$ is the balance in the account
after t years. How large can the continuously depleted (by taxes, say) at a rate of y^2 /
dollars per year, where $y = y(t)$ is the balance in th
after t years. How large can the account grow? How
it take the account to grow to half this balance?
Find the 2. Repeat the solution concentration problem in Example
changing the rate of inflow of brine into the tank to 12
but leaving all the other data as they were in that exam
Note that the volume of liquid in the tank is no lo changing the rate of inflow of brine into the tank to 12 L/min
but leaving all the other data as they were in that example.
Note that the volume of liquid in the tank is no longer constant
as time increases.

 \Diamond a volum out it aving an the onter trada as they were in that example.

Note that the volume of liquid in the tank is no longer constant

as time increases.
 \Diamond a volume element
 \Diamond the arc length of a curve
 \Diamond the moment o
	- -
		-
- \diamond the centre of mass of a distribution of mass
 \diamond the centroid of a plane region
 \diamond a first-order separable differential equation
 \diamond a first-order linear differential equation
 \bullet a first-order linear differ
-
-
-
- ∴ \circ the centre of mass of a distribution of mass
 \circ the centroid of a plane region
 \circ a first-order separable differential equation
 \circ a first-order linear differential equation
 Let *D* be the plane region ∴ \diamond the centre of mass of a distribution of mass
 \diamond the centroid of a plane region
 \diamond a first-order separable differential equation
 \diamond a first-order linear differential equation
 Let *D* be the plane region • Let D be the plane region $0 \le y \le f(x)$, $a \le x \le b$. Use the centre of mass of a distribution of mass

the centroid of a plane region
 $\begin{aligned}\n\text{a first-order separable differential equation} \\
\text{a first-order linear differential equation} \\
\text{Let } D \text{ be the plane region } 0 \le y \le f(x), a \le x \le b. \text{ Use} \\
\text{integrals to represent the following:} \\
\text{the volume generated by revolving } D \text{ about the } x\text{-axis}\n\end{aligned}$ $\begin{aligned}\n\text{a first-order linear differential equation} \\
\text{b. Find the area of the surface } \sqrt$ integrals to represent the following the velocity of a plane is the centroid of a plane region
 α first-order separable differential equation
 Let *D* be the plane region $0 \le y \le f(x)$, $a \le x \le b$. Use

integrals to rep the centre of mass of a distribution of mass

o the centroid of a plane region

o a first-order separable differential equation

o a first-order linear differential equation
 Let D be the plane region $0 \le y \le f(x)$, $a \$ \diamond the centre of mass of a distribution of mass
 \diamond the centroid of a plane region
 \diamond a first-order separable differential equation
 \diamond a first-order linear differential equation
 \diamond a first-order linear diff \diamond the centre of mass of a distribution of mass
 \diamond the centroid of a plane region
 \diamond a first-order separable differential equation
 \diamond a first-order linear differential equation
 \bigcup **Let** *D* be the plane reg \diamond the centroid of a plane region
 \diamond a first-order separable differential equation
 \diamond a first-order linear differential equation

Let *D* be the plane region $0 \le y \le f(x)$, $a \le x \le b$. Use

integrals to represent the \circ a first-order separable differential equation
 \circ a first-order linear differential equation

Let *D* be the plane region $0 \le y \le f(x)$, $a \le x \le b$.
 integrals to represent the following:
 \circ the volume generated A a first-order linear differential equation

Let *D* be the plane region $0 \le y \le f(x)$, $a \le x \le b$.

integrals to represent the following:
 \cdot the volume generated by revolving *D* about the *x*-axis
 \cdot the volume gene
	-
	-
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-
- Let C be the curve $y = f(x)$, $a \le x \le b$. Use integrals to Let *D* be the plane region $0 \le y \le f(x)$, $a \le x \le$
integrals to represent the following:
 \diamond the volume generated by revolving *D* about the *x*-axis
 \diamond the volume generated by revolving *D* about the *y*-axis
 \diamond the → the volume generated by revolving *D* about the *y*-a

→ the moment of *D* about the *y*-axis

→ the moment of *D* about the *x*-axis

→ the centroid of *D*

• Let *C* be the curve $y = f(x)$, $a \le x \le b$. Use i

represent
	-
	-
- y-axis

Let C be the curve $y = f(x)$, $a \le x \le b$. Use integrals to
represent the following:
 \diamond the length of C
 \diamond the area of the surface generated by revolving C about the
 x -axis
 \diamond the area of the surface generated by revo **Species the following:**

Spressent the following:

the length of C

the area of the surface generated by revolving C about the

x-axis

the area of the surface generated by revolving C about the

y-axis
 right Exercises From the length of C
the length of C
the area of the surface generated by revolving C about the
x-axis
the area of the surface generated by revolving C about the
y-axis
 \vec{v} with \vec{v} axis
Figure 7.68 shows cross-se the length of C
the area of the surface generated by revolving C about the
x-axis
the area of the surface generated by revolving C about the
y-axis
tiew Exercises
Figure 7.68 shows cross-sections along the axes of two c

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- **Example 1.68**
 Example 1.68 10. Suppose two functions
 \rightarrow \rightarrow \rightarrow 1 cm
 \rightarrow 1 cm

Figure 7.68

Water sitting in a bowl evaporates at a rate proportional to its

surface area. Show that the depth of water in the bowl decreases

at a constant r = 1 cm

figure 7.68

Figure 7.68

Figure 7.68

bout evaporates at a rate proportional to its

w that the depth of water in the bowl decreases

regardless of the shape of the bowl.

high and its volume is 16 cubic feet. It **Example 19** Figure 7.68

Water sitting in a bowl evaporates at a rate proportional to its

surface area. Show that the depth of water in the bowl decreases

at a constant rate, regardless of the shape of the bowl.

A bar stants *a* and *b*. 2. Water sitting in a bowl approximates at a rate proportional to its
surface area. Show that the depth of water in the bowl decreases
at a constant rate, regardless of the shape of the bowl.
3. A barrel is 4 ft high and Water sitting in a bowl evaporates at a rate proportional to its
surface area. Show that the depth of water in the bowl decreases
at a constant rate, regardless of the shape of the bowl.
A barrel is 4 ft high and its volu surface area. Show that the depth of water in the bowl
at a constant rate, regardless of the shape of the bowl
A barrel is 4 ft high and its volume is 16 cubic
top and bottom are circular disks of radius 1 f
side wall is
	-

- \diamond the centre of mass of a distribution of mass
 \diamond the centroid of a plane region
 \diamond a first-order separable differential equation
 \diamond a first-order separable differential equation
 \diamond 6. Find the linear diffe CHAPTER REVIEW 459

5. Find to 4 decimal places the value of the positive constant *a*

for which the curve $y = (1/a) \cosh ax$ has arc length 2 units

between $x = 0$ and $x = 1$.

6. Find the area of the surface obtained by rota CHAPTER REVIEW 459
Find to 4 decimal places the value of the positive constant *a*
for which the curve $y = (1/a) \cosh ax$ has arc length 2 units
between $x = 0$ and $x = 1$.
Find the area of the surface obtained by rotating the c between $x = 0$ and $x = 1$. **5.** Find to 4 decimal places the value of the positive constant *a* for which the curve $y = (1/a) \cosh ax$ has arc length 2 units between $x = 0$ and $x = 1$.
6. Find the area of the surface obtained by rotating the curve $y = \$ CHAPTER REVIEW 459

ond to 4 decimal places the value of the positive constant *a*

or which the curve $y = (1/a) \cosh ax$ has arc length 2 units

tween $x = 0$ and $x = 1$.

ond the area of the surface obtained by rotating the cu
	-
	-
- Example the centre of mass of a distribution of mass

the centroid of a plane region

in the curve $y = (1/$

a first-order separable differential equation

Let D be the plane region $0 \le y \le f(x)$, $a \le x \le b$. Use

the volume **5.** Find to 4 decimal places the value of the positive constant *a* for which the curve $y = (1/a) \cosh ax$ has arc length 2 units between $x = 0$ and $x = 1$.
 6. Find the area of the surface obtained by rotating the curve $y =$ Find to 4 decimal places the value of the positive constant *a* for which the curve $y = (1/a) \cosh ax$ has arc length 2 units between $x = 0$ and $x = 1$.
Find the area of the surface obtained by rotating the curve $y = \sqrt{x}$, $(0 \$ Find to 4 decimal places the value of the positive constant *a*
for which the curve $y = (1/a) \cosh ax$ has arc length 2 units
between $x = 0$ and $x = 1$.
Find the area of the surface obtained by rotating the curve $y = \sqrt{x}$, $(0 \$ Find to 4 decimal places the value of the positive constant a
for which the curve $y = (1/a) \cosh ax$ has arc length 2 units
between $x = 0$ and $x = 1$.
Find the area of the surface obtained by rotating the curve $y = \sqrt{x}$, $(0 \le x$

- of two circular

and of the same

and of the same
 $\begin{array}{c}\n\hline\n\text{end}\n\end{array}$
 1. Cm 5

 gas

gas

Figure 7.70

According to Boyle's Law, the product of the pressure and vol-

ume of a gas remains constant if the gas expands or is com-

pressed isothermally. The cylinder in Figure 7.70 is filled with

a gas th gas

Figure 7.70

According to Boyle's Law, the product of the pressure and vol-

ume of a gas remains constant if the gas expands or is com-

pressed isothermally. The cylinder in Figure 7.70 is filled with

a gas that ex Figure 7.70

According to Boyle's Law, the product of the pressure and vol-

ume of a gas remains constant if the gas expands or is com-

pressed isothermally. The cylinder in Figure 7.70 is filled with

a gas that exerts Figure 7.70
According to Boyle's Law, the product of the pressure and vol-
ume of a gas remains constant if the gas expands or is com-
pressed isothermally. The cylinder in Figure 7.70 is filled with
a gas that exerts a f **Example 19** Figure 7.70
 Example 3 According to Boyle's Law, the product of the pressure and volume of a gas remains constant if the gas expands or is com-

pressed isothermally. The cylinder in Figure 7.70 is filled w Figure 7.70
According to Boyle's Law, the product of the pressure and volume of a gas remains constant if the gas expands or is com-
pressed isothermally. The cylinder in Figure 7.70 is filled with
a gas that exerts a for 10. Suppose two functions f and g have the solid product of the server and volume of a gas remains constant if the gas expands or is compressed isothermally. The cylinder in Figure 7.70 is filled with a gas that exerts a **Figure 7.70**
**According to Boyle's Law, the product of the pressure and volume of a gas remains constant if the gas expands or is com-
pressed isothermally. The cylinder in Figure 7.70 is filled with
a gas that exerts a Example 11**
 Example 1.70
 Example 8 A a a start is the product of the pressure and volume of a gas remains constant if the gas expands or is compressed isothermally. The cylinder in Figure 7.70 is filled with a gas t According to Boyle's Law, the product of the pressure and volume of a gas remains constant if the gas expands or is compressed isothermally. The cylinder in Figure 7.70 is filled with a gas that exerts a force of 1,000 N ume of a gas remains constant if the gas expands or is com-
pressed isothermally. The cylinder in Figure 7.70 is filled with
a gas that exerts a force of 1,000 N on the piston when the pis-
ton is 20 cm above the base of 11. Find the equation of a curve that passes through the point of the pressure and volume of a gas remains constant if the gas expands or is compressed isothermally. The cylinder in Figure 7.70 is filled with a gas that e ume of a gas remains constant if the gas expands or is com-
pressed isothermally. The cylinder in Figure 7.70 is filled with
a gas that exerts a force of 1,000 N on the piston when the pis-
ton is 20 cm above the base of
	- about f and g ? 10. Suppose two functions f and g have the base?

	10. Suppose two functions f and g have the following property:

	for any $a > 0$, the solid produced by revolving the region of the

	xy-plane bounded by $y = f(x)$, $y = g(x)$, x Suppose two functions f and g have the following property:

	Suppose two functions f and g have the following property:

	for any $a > 0$, the solid produced by revolving the region of the
 xy -plane bounded by $y = f(x)$, $y = g$ 13. The income and expenses of a seasonal business result in de-
not a flux-plane bounded by $y = f(x)$, $y = g(x)$, $x = 0$, and $x = a$
about the *x*-axis has the same volume as the solid produced by
revolving the same region ab
	-
	-
	- be any $a > b$, the solid produced by Pevolving the region of the xy -plane bounded by $y = f(x)$, $y = g(x)$, $x = 0$, and $x = a$ about the x -axis has the same volume as the solid produced by revolving the same region about the Suppose two functions f and g have the base.

	Suppose two functions f and g have the following property:

	Suppose two functions f and g have the following the region of the

	for any $a > 0$, the solid produced by $y = f(x)$, Suppose two functions f and g have the following property:
for any $a > 0$, the solid produced by revolving the region of the
xy-plane bounded by $y = f(x)$, $y = g(x)$, $x = 0$, and $x = a$
about the *x*-axis has the same vo Find the equation of a curve that passes through the point (2, 4) and has slope $3y/(x - 1)$ at any point (x, y) on it.
Find a family of curves that intersect every ellipse of the form $3x^2 + 4y^2 = C$ at right angles.
The in Find the equation of a curve that passes through the point (2, 4)
and has slope $3y/(x - 1)$ at any point (x, y) on it.
Find a family of curves that intersect every ellipse of the form
 $3x^2 + 4y^2 = C$ at right angles.
The in revolving the same region about the *y*-axis. What can ye
about *f* and *g*?
11. Find the equation of a curve that passes through the point
and has slope $3y/(x - 1)$ at any point (x, y) on it.
12. Find a family of curves t 2. Find a family of curves that intersect every ellipse of the form
 $3x^2 + 4y^2 = C$ at right angles.

	3. The income and expenses of a seasonal business result in de-

	posits and withdrawals from its bank account that corr and has slope $3y/(x - 1)$ at any point (x, y) on it.

	Find a family of curves that intersect every ellipse of the form
 $3x^2 + 4y^2 = C$ at right angles.

	The income and expenses of a seasonal business result in de-

	posits to a flow rate into the account of $$P(t)/year$ at time t years,
where $P(t) = 10,000 \sin(2\pi t)$. If the account earns interest
at an instantaneous rate of 4% per year and has \$8,000 in it at
time $t = 0$, how much is in the acc

- **CONDETE:**

Intermet to the total volume of the account two years later?
 CONDETE:

The curve $y = e^{-kx} \sin x$, $(x \ge 0)$, is revolved about the

x-axis to generate a string of "beads" whose volumes decrease

to the right if **nging Problems**
curve $y = e^{-kx} \sin x$, $(x \ge 0)$, is revolved about the
is to generate a string of "beads" whose volumes dec
e right if $k > 0$.
Show that the ratio of the volume of the $(n + 1)$ st be
that of the *n*th bead dep
	-
	-
	-

30 CHAPTER 7 Applications of Integration
 2. (Conservation of earth) A landscaper wants to create on level

ground a ring-shaped pool having an outside radius of 10 m and

a maximum depth of 1 m surrounding a hill tha CHAPTER 7 Applications of Integration

(Conservation of earth) A landscaper wants to create on level

ground a ring-shaped pool having an outside radius of 10 m and

a maximum depth of 1 m surrounding a hill that will be **CHAPTER 7** Applications of Integration
 (Conservation of earth) A landscaper wants to create on level

ground a ring-shaped pool having an outside radius of 10 m and

a maximum depth of 1 m surrounding a hill that will CHAPTER 7 Applications of Integration

(Conservation of earth) A landscaper wants to create on level

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(Conservation of earth) A landscaper wants to create on level

ground a ring-shaped pool having an outside radius of 10 m and

a maximum depth of 1 m surrounding a hill that will be CHAPTER 7 Applications of Integration

ground a ring-shaped pool having an outside radius of 10 m and

a maximum depth of 1 m surrounding a hill that will be built up

using all the earth excavated from the pool. (See Fig (**Conservation of earth**) A landscaper wants to create on level
ground a ring-shaped pool having an outside radius of 10 m and
a maximum depth of 1 m surrounding a hill that will be built up
using all the earth excavated (**Conservation of earth**) A landscaper wants to create on level
ground a ring-shaped pool having an outside radius of 10 m and
a maximum depth of 1 m surrounding a hill that will be built up
using all the earth excavated a maximum depth of 1 m surrounding a hill that will be built up

solutions all the earth excavated from the pool. (See Figure 7.71.)

She decides to use a fourth-degree polynomial to determine the

eross-sectional shape o using all the earth excavated from the pool. (See Figure 7.71.) polygonal line *ABC*.
She decides to use a fourth-degree polynomial to determine the
metres sectional shape of the hill and pool bottom: at distance represen

$$
h(r) = a(r^2 - 100)(r^2 - k^2)
$$
 metres,

hill? ment the height above or

ground 1

Built on

netres,

r radius of the pool. Find

en above are all satisfied.

om the pool to build the

find the

concrete
 10 m

Figure 7.71
 CONTROVERTY:

Figure 7.71

(Rocket design) The nose of a rocket is a solid of revolution

of base radius r and height h that must join smoothly to the

cylindrical body of the rocket. (See Figure 7.72.) Tak Figure 7.71

Figure 7.71
 cocket design) The nose of a rocket is a solid of revolution

of base radius r and height h that must join smoothly to the

origin at the tip of the rocket. (See Figure 7.72.) Taking the

origin Figure 7.71

Figure 7.71
 (Rocket design) The nose of a rocket is a solid of revolution

of base radius r and height h that must join smoothly to the

evylindrical body of the rocket. (See Figure 7.72.) Taking the

orig Figure 7.71
 (Rocket design) The nose of a rocket is a solid of revolution

of base radius r and height h that must join smoothly to the

explindical body of the rocket. (See Figure 7.72.) Taking the

origin at the tip Figure 7.71
 (Rocket design) The nose of a rocket is a solid of re

of base radius r and height h that must join smooth

cylindrical body of the rocket. (See Figure 7.72.) Ta

origin at the tip of the nose and the x-axi Figure 7.71
 ket design) The nose of a rocket is a solid of revolution

see radius r and height h that must join smoothly to the

drical body of the rocket. (See Figure 7.72.) Taking the

n at the tip of the nose and th **(Rocket design)** The nose of a rocket is a solid of revolution

of base radius r and height h that must join smoothly to the

cylindrical body of the rocket. (See Figure 7.72.) Taking the

origin at the tip of the nose a **(Rocket design)** The nose of a rocket is a solid of revolution Figure of base radius r and height h that must join smoothly to the explination of the rocket. (See Figure 7.72.) Taking the space consists of points of the

$$
y = f(x) = ax + bx^2 + cx^3
$$

of base radius r and height h that must join smoothly to the

cylindrical body of the rocket. (See Figure 7.72.) Taking the

origin at the tip of the nose and the x-axis along the central axis

of the rocket, various nose Explination of the rocket. (See Figure 7.72.) Taking the

origin at the tip of the nose and the x-axis along the central axis

origin at the tip of the nose and the x-axis along the central axis

of the rocket, various no origin at the tip of the nose and the x-axis along the central axis space consists on points that satisfy the

of the rocket, various nose shapes can be obtained by revolving

the cubic curve
 $y = f(x) = ax + bx^2 + cx^3$

about the of the rocket, various nose shapes can be obtained by
the cubic curve
 $y = f(x) = ax + bx^2 + cx^3$
about the x-axis. The cubic curve must have slope
 h , and its slope must be positive for $0 < x < h$.
particular cubic curve that maximi be and the x-axis along the central axis
see shapes can be obtained by revolving
 $-bx^2 + cx^3$
cubic curve must have slope 0 at $x =$
ne positive for $0 < x < h$. Find the
hat maximizes the volume of the nose.
ce of the cubic mak

- $y = ax + bx + cx$
 $y = ax + bx + cx$
 $\begin{pmatrix} (h,r) \\ (h,r) \end{pmatrix}$
 $\begin{pmatrix} x \\ y \end{pmatrix}$
 $\begin{pmatrix} y \\ y \end{pmatrix}$
 \begin **Example 19 analogy with these form** and the $(n-1)$ -dimensional ball of radiu the $(n-1)$ -dimensional ball of r **A D .1)** $\begin{aligned}\n\mathbf{F} &\mathbf{F} \mathbf{F} \mathbf{$ **Example 19** By
 Example A; $n \geq 0$
 Example A; $n \geq 0$
	-
- (b) Find the value of *m* for which the length of the graph $y = f(x, m)$ between $x = 0$ and $x = 3$ is minimum. What is this minimum length? Compare it with the length of the polygonal line *ABC*. Find the value of *m* for which the length of the graph $y = f(x, m)$ between $x = 0$ and $x = 3$ is minimum. What is this minimum length? Compare it with the length of the polygonal line *ABC*. Find the value of *m* for which the length of the graph $y = f(x, m)$ between $x = 0$ and $x = 3$ is minimum. What is this minimum length? Compare it with the length of the polygonal line *ABC*.
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polygonal in the shape of a circular ring m
- (b) Find the value of *m* for which the length of the graph $y = f(x, m)$ between $x = 0$ and $x = 3$ is minimum. What is this minimum length? Compare it with the length of the polygonal line *ABC*.
 5. A concrete wall in the (b) Find the value of *m* for which the length of the graph $y = f(x, m)$ between $x = 0$ and $x = 3$ is minimum. What is this minimum length? Compare it with the length of the polygonal line *ABC*.
A concrete wall in the shape (b) Find the value of *m* for which the length of the graph $y = f(x, m)$ between $x = 0$ and $x = 3$ is minimum. What is this minimum length? Compare it with the length of the polygonal line *ABC*.
A concrete wall in the shape (b) Find the value of *m* for which the length of the graph $y = f(x, m)$ between $x = 0$ and $x = 3$ is minimum. What is this minimum length? Compare it with the length of the polygonal line *ABC*.
A concrete wall in the shape (b) Find the value of *m* for which the length of the graph $y = f(x, m)$ between $x = 0$ and $x = 3$ is minimum. What is this minimum length? Compare it with the length of the polygonal line *ABC*.
A concrete wall in the shape (b) Find the value of *m* for which the length of the graph $y = f(x, m)$ between $x = 0$ and $x = 3$ is minimum. What is this minimum length? Compare it with the length of the polygonal line *ABC*.
A concrete wall in the shape (b) Find the value of *m* for which the length of the graph $y = f(x, m)$ between $x = 0$ and $x = 3$ is minimum. What is this minimum length? Compare it with the length of the polygonal line *ABC*. A concrete wall in the shape A concrete wall in the shape of a circular ring must be built to
have maximum height 2 m, inner radius 15 m, and width 1 m at
ground level, so that its outer radius is 16 m. (See Figure 7.73.)
Built on level ground, the w

$$
f(x) = x(1-x)(ax+b)
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 m,

Figure 7.73

The volume of an *n*-dimensional ball) Euclidean *n*-dimensional

space consists of *points* $(x_1, x_2, ..., x_n)$ with *n* real coordi-

nates. By analogy with the 3-dimensional case, we call the set

of such point Figure 7.73

(The volume of an *n*-dimensional ball) Euclidean *n*-dimensional

space consists of *points* $(x_1, x_2, ..., x_n)$ with *n* real coordi-

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call the set
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ich has *vol*-
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(The volume of an *n*-dimensional ball) Euclidean *n*-dimensional

space consists of *points* $(x_1, x_2, ..., x_n)$ with *n* real coordi-

nates. By analogy with the 3-dimensional case, we call the set

of such poin **11-dimensional ball is the interval r Figure 7.73**
 11.dimensional ball) Euclidean *n*-dimensional space consists of *points* $(x_1, x_2, ..., x_n)$ with *n* real coordinates. By analogy with the 3-dimensional case, we call disk $x_1^2 + x_2^2 \le r^2$, which has *volume* (i.e., *area* Figure 7.73
 *n***-dimensional ball)** Euclidean *n*-dimensional
 ooints $(x_1, x_2, ..., x_n)$ with *n* real coordi-

with the 3-dimensional case, we call the set

aatisfy the inequality $x_1^2 + x_2^2 + \cdots + x_n^2 \le r^2$

ball centred Figure 7.73

volume of an *n*-dimensional ball) Euclidean *n*-dim

consists of *points* $(x_1, x_2, ..., x_n)$ with *n* real c.

By analogy with the 3-dimensional case, we call th

h points that satisfy the inequality $x_1^2 + x_2^$ of such points that satisfy the inequality $x_1^2 + x_2^2 + \cdots + x_n^2 \le r^2$
the *n*-dimensional *ball* centred at the origin. For example, the
1-dimensional ball is the interval $-r \le x_1 \le r$, which has *vol-*
ume (i.e., *leng* $x_2^2 + \cdots + x_n^2 \le r^2$
For example, the
 $\{r, \text{ which has } vol\}$
msional ball is the
area)
has volume
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th points that satisfy the inequality $x_1^2 + x_2^2 + \cdots +$
-dimensional *ball* centred at the origin. For exam-
nensional ball is the interval $-r \le x_1 \le r$, which
i.e., *leng*

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V_2(r) = \pi r^2 = \int_{-r}^r 2\sqrt{r^2 - x^2} \, dx
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$$
= \int_{-r}^r V_1\left(\sqrt{r^2 - x^2}\right) \, dx.
$$

 $x_1^2 + x_2^2 + x_3^2 \le r^2$ has volume

$$
V_3(r) = \frac{4}{3}\pi r^3 = \int_{-r}^r \pi \left(\sqrt{r^2 - x^2}\right)^2 dx
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\int_{-r}^r V_2 \left(\sqrt{r^2 - x^2}\right) dx.
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 $V_2(r) = \pi r^2 = \int_{-r}^{r} V_1 \left(\sqrt{r^2 - x^2}\right) dx.$

The 3-dimensional ball $x_1^2 + x_2^2 + x_3^2 \le r^2$ has volume $V_3(r) = \frac{4}{3}\pi r^3 = \int_{-r}^{r} \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$
 $= \int_{-r}^{r} V_2 \left(\sqrt{r^2 - x^2}\right) dx.$

By analogy with these formulas, t = $\int_{-r} V_1 (\sqrt{r^2 - x^2}) dx$.

The 3-dimensional ball $x_1^2 + x_2^2 + x_3^2 \le r^2$ has volume
 $V_3(r) = \frac{4}{3} \pi r^3 = \int_{-r}^r \pi (\sqrt{r^2 - x^2})^2 dx$

= $\int_{-r}^r V_2 (\sqrt{r^2 - x^2}) dx$.

By analogy with these formulas, the volume $V_n(r)$ of the the $(n-1)$ -dimensional ball of radius $\sqrt{r^2 - x^2}$ from $x = -r$ to $x = r$: = $\int_{-r}^{r} V_1(\sqrt{r^2 - x^2}) dx$.

3-dimensional ball $x_1^2 + x_2^2 + x_3^2 \le r^2$ has volum
 $V_3(r) = \frac{4}{3}\pi r^3 = \int_{-r}^{r} \pi(\sqrt{r^2 - x^2})^2 dx$

= $\int_{-r}^{r} V_2(\sqrt{r^2 - x^2}) dx$.

malogy with these formulas, the volume $V_n(r)$

mensional

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V_n(r) = \int_{-r}^r V_{n-1} \left(\sqrt{r^2 - x^2} \right) dx.
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The 3-dimensional ball $x_1^2 + x_2^2 + x_3^2 \le r^2$ has volume
 $V_3(r) = \frac{4}{3}\pi r^3 = \int_{-r}^r \pi \left(\sqrt{r^2 - x^2}\right)^2 dx$
 $= \int_{-r}^r V_2 \left(\sqrt{r^2 - x^2}\right) dx$.

By analogy with these formulas, the volume $V_n(r)$ of the
 n-dimensional ball balls) and V₂n^t ℓ ($\sqrt{r^2 - x^2}$) ax.

By analogy with these formulas, the volume $V_n(r)$ of the *n*-dimensional ball of radius *r* is the integral of the volume of the $(n - 1)$ -dimensional ball of radius $\sqrt{r^2 - x^2$ By analogy with these formulas, the volume $V_n(r)$ of the
 n-dimensional ball of radius *r* is the integral of the volume of

the $(n - 1)$ -dimensional ball of radius $\sqrt{r^2 - x^2}$ from $x = -r$

to $x = r$:
 $V_n(r) = \int_{-r}^r V_{n$ By analogy with these formulas, the volume $V_n(t)$ of the
n-dimensional ball of radius r is the integral of the volume of
the $(n - 1)$ -dimensional ball of radius $\sqrt{r^2 - x^2}$ from $x = -r$
to $x = r$:
 $V_n(r) = \int_{-r}^r V_{n-1}(\sqrt{r^$ $= \int_{-r}^{r} V_2 \left(\sqrt{r^2 - x^2}\right) dx$.
By analogy with these formulas, the volume $V_n(r)$ of the *n*-dimensional ball of radius *r* is the integral of the volume of the $(n - 1)$ -dimensional ball of radius $\sqrt{r^2 - x^2}$ from $x = -r$ By analogy with these formulas, the volume $V_n(r)$ of
 n-dimensional ball of radius *r* is the integral of the volume

the $(n - 1)$ -dimensional ball of radius $\sqrt{r^2 - x^2}$ from $x =$

to $x = r$:
 $V_n(r) = \int_{-r}^r V_{n-1}(\sqrt{r^2$

1 1. (Buffon's needle problem) A horizontal flat surface is ruled with parallel lines 10 cm apart, as shown in Figure 7.74. A nee in the first quadrant of the dle 5 cm long is dropped at random onto the surface. Find **(Buffon's needle problem)** A horizontal flat surface is ruled with parallel lines 10 cm apart, as shown in Figure 7.74. A nee-
dle 5 cm long is dropped at random onto the surface. Find the $(L, 0)$, and having the proposa **(Buffon's needle problem)** A horizontal flat surface is ruled with parallel lines 10 cm apart, as shown in Figure 7.74. A nee-
dle 5 cm long is dropped at random onto the surface. Find the $(L, 0)$, and having the properb **(Buffon's needle problem)** A horizontal flat surface is ruled with parallel lines 10 cm apart, as shown in Figure 7.74. A nee-
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probability **(Buffon's needle problem)** A horizontal flat surface is ruled

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with parallel lines 10 cm apart, as shown in Figure 7.74. A nee-

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dle 5 cm long is dropped at random onto the surface. Find the

the ' **(Buffon's needle problem)** A horizontal flat surface is ruled **El 8.** (The path of a trailer) Find with parallel lines 10 cm apart, as shown in Figure 7.74. A nee-
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with parallel lines 10 cm apart, as shown in Figure 7.74. A nee-

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 (Buffon's needle problem) A horizontal flat surface is ruled

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pr **(Buffon's needle problem)** A horizontal flat surface is ruled

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probability that the needle **(Buffon's needle problem)** A horizontal flat surface is ruled

with parallel lines 10 cm apart, as shown in Figure 7.74. A nee-

lines for long is dropped at random onto the surface. Find the

the rises of the relations de 5 cm long is dropped at random onto the surface. Find the $(2, 0)$, and having the probability that the needle intersects one of the lines. *Hint*: Let the "lower" end of the needle (the end further down the page const probability that the needle intersects one of the lines. *Hint*: Let

the "lower" end of the needle intersects one of the lines. *Hint*: Let

the "lower" end of the needle (the end further down the page

in the figure) be the "lower" end of the needle (the end further down the page

in the figure 7.75

in the figure 1.75

in the figure onsidered the reference point. (If both ends

are the same height, use the left end.) Let y be the distan in the figure) be considered the reference point. (If both ends
are the same height, use the left end.) Let y be the distance paint of the reference point to the nearest line above it, and let θ along the x-abe the ang

- **EI 8.** (The path of a trailer) Find the equation $y = f(x)$ of a curve in the first quadrant of the *xy*-plane, starting from the point (*L*, 0), and having the property that if the tangent line to the curve at *P* meets th **in the first quadrant of the xy-plane, starting from the point** (*L*, 0), and having the property that if the tangent line to the curve at *P* meets the y-axis at *Q*, then the length of *PQ* is the constant *L*. (See Fi **CHAPTER REVIEW 461**
 (The path of a trailer) Find the equation $y = f(x)$ of a curve

in the first quadrant of the xy-plane, starting from the point

(*L*, 0), and having the property that if the tangent line to the

curv CHAPTER REVIEW **461**
(The path of a trailer) Find the equation $y = f(x)$ of a curve
in the first quadrant of the xy-plane, starting from the point
(*L*, 0), and having the property that if the tangent line to the
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(The path of a trailer) Find the equation $y = f(x)$ of a curve
in the first quadrant of the *xy*-plane, starting from the point
(*L*, 0), and having the property that if the tangent line to the
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in the first quadrant of the *xy*-plane, starting from the point
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curve at *P* meets the (**The path of a trailer**) Find the equation $y = f(x)$ of a curve
in the first quadrant of the *xy*-plane, starting from the point
(*L*, 0), and having the property that if the tangent line to the
curve at *P* meets the *y*in the first quadrant of the *xy*-plane, starting from the point $(L, 0)$, and having the property that if the tangent line to the curve at *P* meets the *y*-axis at *Q*, then the length of *P Q* is the constant *L*. (See (*L*, 0), and having the property that if the tangent line to t
curve at *P* meets the *y*-axis at *Q*, then the length of *PQ* is t
constant *L*. (See Figure 7.75. This curve is called a **tract**
after the Latin participl
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\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.
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 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$

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These ellipsoids are

the earth) has its two

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A reasonable approx
 along the *x*-axis, as the trailer is pulled (dragged) by a tractor Q moving along the *y*-axis away from the origin.)
(**Approximating the surface area of an ellipsoid**) A physical geographer studying the flow of stream *Q* moving along the *y*-axis away from the origin.)

(Approximating the surface area of an ellipsoid) A physical

geographer studying the flow of streams around oval stones

needed to calculate the surface areas of many (Approximating the surface area of an ellipsoid) A physical
geographer studying the flow of streams around oval stones
needed to calculate the surface areas of many such stones that
he modelled as ellipsoids:
 $\frac{x^2}{a^2} +$ example the area of a general ellipsoid in terms and over the modelled to calculate the surface areas of many such stones that he modelled as ellipsoids:
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

He wanted a simple formu meeded to calculate the surface areas of many such stones that
needed to calculate the surface areas of many such stones that
he modelled as ellipsoids:
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
He wanted a simple formula some the modelled as ellipsoids:
 $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

He wanted a simple formula for the surface area so that he

could implement it in a spreadsheet containing the measure-

ments a, b, and c of the $rac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
He wanted a simple formula for the surface area so that he could implement it in a spreadsheet containing the measurements a, b, and c of the stones. Unfortunately, there is no exac $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.
He wanted a simple formula for the surface area so that he could implement it in a spreadsheet containing the measurements a, b, and c of the stones. Unfortunately, there is no exa $\frac{x}{a^2} + \frac{y}{b^2} + \frac{z}{c^2} = 1.$
He wanted a simple formula for the surface area so that he could implement it in a spreadsheet containing the measurements a, b, and c of the stones. Unfortunately, there is no exact for a^2 b^2 c^2
He wanted a simple formula for the surface area so that he could implement it in a spreadsheet containing the measure-
ments a, b, and c of the stones. Unfortunately, there is no
exact formula for the ar He wanted a simple formula for the surface area so that he could implement it in a spreadsheet containing the measure-
ments a, b, and c of the stones. Unfortunately, there is no exact formula for the area of a general el d implement it in a spreadsheet containing the measure-
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t formula for the area of a general ellipsoid in terms of ele-
tary functions. However, there are such formu He wanted a simple formula for the surface area so that he could implement it in a spreadsheet containing the measure-
ments a, b, and c of the stones. Unfortunately, there is no exact formula for the area of a general el could implement it in a spreadsheet containing the measure-
ments a, b, and c of the stones. Unfortunately, there is no
exact formula for the area of a general ellipsoid in terms of ele-
mentary functions. However, there means w, ϕ , that ψ corrects. entirely, there is no
exact formula for the area of a general ellipsoid in terms of ele-
mentary functions. However, there are such formulas for ellip-
soids of revolution, where two of interiors. However, there are such formulas for ellip-
tary functions. However, there are such formulas for ellip-
s of revolution, where two of the three semi-axes are equal.
se ellipsoids are called spheroids; an *oblat* (like an American football) has its two shorter semi-axes equal.
A reasonable approximation to the area of a general ellipsoid
can be obtained by linear interpolation between these two.
To be specific, assume the semi-axe

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- Calculate $S(a, a, c)$, the area of an obtate spheroid.
Calculate $S(a, c, c)$, the area of a prolate spheroid.
Construct an approximation for $S(a, b, c)$ that divides the
interval from $S(a, a, c)$ to $S(a, c, c)$ in the same ratio th
-

$$
\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1
$$