Computing Convex Hull (in 2D)

1389-2

Convex Set

Definition:

A subset S of the plane is called convex if and only if for any pair of points $p, q \in S$, the line segment $\overline{pq}$ is completely contained in S.

Convex Hull:

The convex hull $\mathcal{CH}(S)$ of a set S is the smallest convex set that contains S. To be more precise, it is the intersection of all convex sets that contain S.

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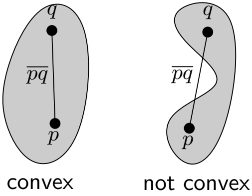

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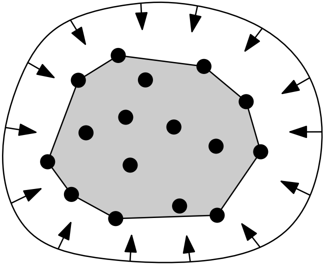

Computing CH:

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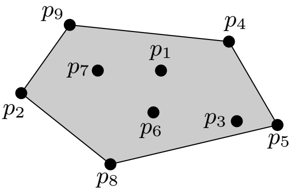
Input= set of points

$p_1, p_2, p_3, p_4, p_5, p_6, p_7, p_8, p_9$

Output= representation of the convex hull:

p_4, p_5, p_8, p_2, p_9



Geometry of the problem

Property:

If we direct the line through p and q such that $\mathcal{CH}(P)$ lies to the right, then all the points of P must lie to the right of this line. The reverse is also true: if all points of $P \setminus \{p, q\}$ lie to the right of the directed line through p and q, then pq is an edge of $\mathcal{CH}(P)$.

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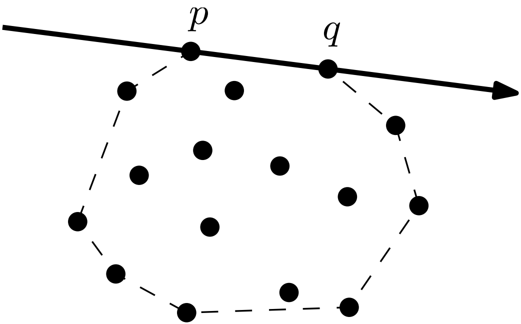

First algorithm

Algorithm SLOWCONVEXHULL(P)
Input. A set P of points in the plane.
Output. A list $\mathcal{L}$ containing the vertices of $\mathcal{CH}(P)$ in clockwise order.

```
E ← ∅.
for all ordered pairs (p,q) ∈ P × P with p not equal to q
    do valid ← true
        for all points r ∈ P not equal to p or q
            do if r lies to the left of the directed line from p to q
                then valid ← false.
        if valid then Add the directed edge pq→ to E.
From the set E of edges construct a list L of vertices of CH(P), sorted in clockwise order.
```

Clarify:

1. How do we test whether a point lies to the left or to the right of a directed line? (See Exercise 1.4)
2. How can we construct $\mathcal{L}$ from E?

First algorithm

Algorithm SlowConvexHull(P)
Input. A set P of points in the plane.
Output. A list $\mathcal{L}$ containing the vertices of $\mathcal{CH}(P)$ in clockwise order.

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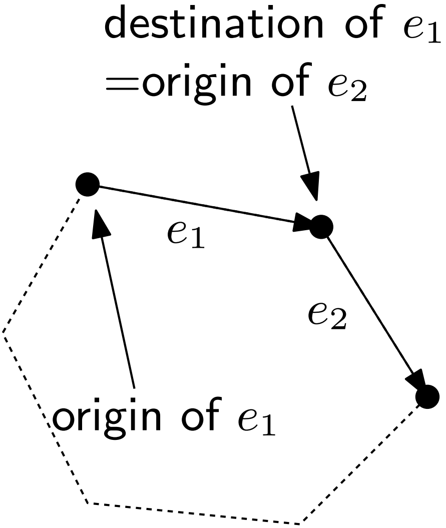
Computing $\mathcal{L}$:



Complexity of the algorithm

Algorithm SLOWCONVEXHULL(P)
Input. A set P of points in the plane.
Output. A list $\mathcal{L}$ containing the vertices of $\mathcal{CH}(P)$ in clockwise order.

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for all ordered pairs $(p,q) \in P \times P$ with p not equal to q
    do *valid* $\leftarrow$ **true**
        for all points $r \in P$ not equal to p or q
            do if r lies to the left of the directed line from p to q
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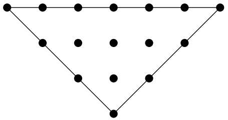
Degenerate case (or Degeneracy)

Degenerate Case:

A point r does not always lie to the right or to the left of the line through p and q, it can also happen that it **lies on** this line. What should we do then?



Solution:

A directed edge $\overrightarrow{pq}$ is an edge of $\mathcal{CH}(P)$ if and only if all other points $r \in P$ lie either strictly to the right of the directed line through p and q, or they lie on the open line segment $\overline{pq}$.

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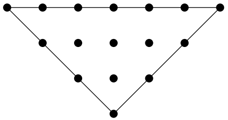

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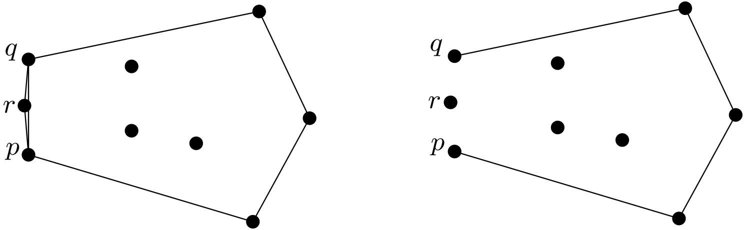

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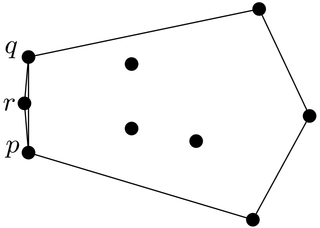

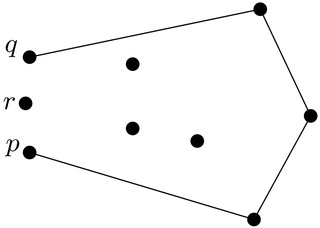

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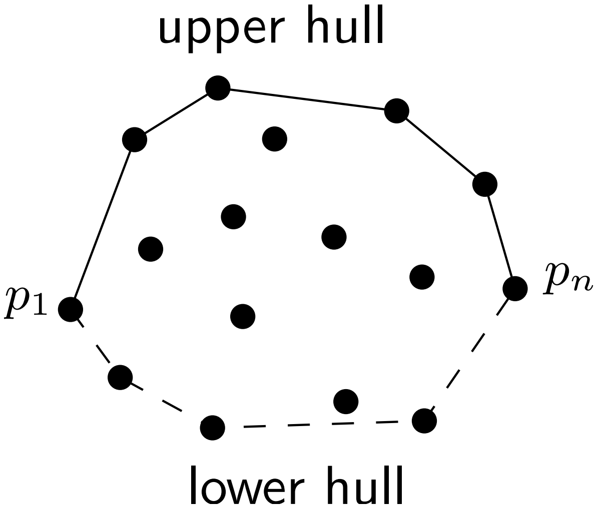
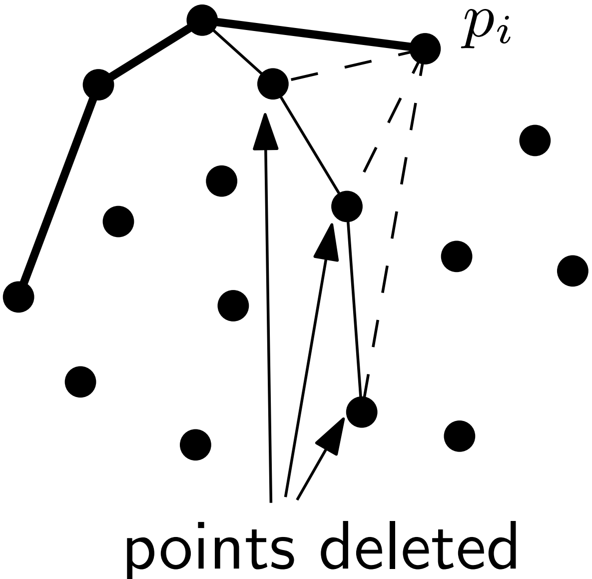
2nd algorithm: incremental




p_i
points deleted

2nd algorithm: incremental

Algorithm CONVEXHULL(P)
Input. A set P of points in the plane.
Output. A list containing the vertices of $\mathcal{CH}(P)$ in clockwise order.

1. Sort the points by x-coordinate, resulting in a sequence $p_1, \ldots, p_n$.
2. Put the points p_1 and p_2 in a list $\mathcal{L}_{\text{upper}}$, with p_1 as the first point.
3. **for** $i \leftarrow 3$ **to** n
4.   **do** Append p_i to $\mathcal{L}_{\text{upper}}$.
5.   **while** $\mathcal{L}_{\text{upper}}$ contains more than two points **and** the last three points in $\mathcal{L}_{\text{upper}}$ do not make a right turn
6.   **do** Delete the middle of the last three points from $\mathcal{L}_{\text{upper}}$.
7. Put the points p_n and p_{n-1} in a list $\mathcal{L}_{\text{lower}}$, with p_n as the first point.
8. **for** $i \leftarrow n-2$ **downto** 1
9.   **do** Append p_i to $\mathcal{L}_{\text{lower}}$.
10.   **while** $\mathcal{L}_{\text{lower}}$ contains more than 2 points **and** the last three points in $\mathcal{L}_{\text{lower}}$ do not make a right turn
11.   **do** Delete the middle of the last three points from $\mathcal{L}_{\text{lower}}$.
12. Remove the first and the last point from $\mathcal{L}_{\text{lower}}$ to avoid duplication of the points where the upper and lower hull meet.
13. Append $\mathcal{L}_{\text{lower}}$ to $\mathcal{L}_{\text{upper}}$, and call the resulting list $\mathcal{L}$.
14. **return** $\mathcal{L}$

2nd algorithm: incremental

Special cases:

1. Two points have same x-coordinate.
2. Three points on a line

Solution:

Use the lexicographic order.

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not a right turn

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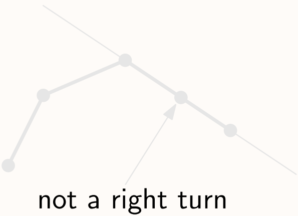

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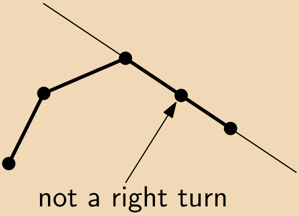

Robustness:

What does the algorithm do in the presence of rounding errors in the floating point arithmetic?

- When such errors occur, it can happen that a point is removed from the convex hull although it should be there, or that a point inside the real convex hull is not removed. But the structural integrity of the algorithm is unharmed: it will compute a closed polygonal chain.
- The only problem that can still occur is that, when three points lie very close together, a turn that is actually a sharp left turn can be interpreted as a right turn. This might result in a dent in the resulting polygon.

Proof of correctness:

Theorem: The convex hull of a set of n points in the plane can be computed in $\mathcal{O}(n \log n)$ time.

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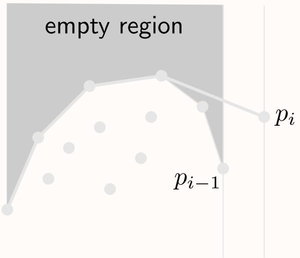

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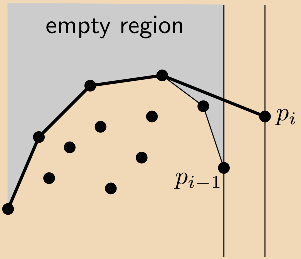

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Time Complexity:

- Sorting: $\mathcal{O}(n \log n)$.
- The for-loop is executed a linear number of times.
- For each execution of the for-loop the while-loop is executed at least once. For any extra execution a point is deleted from the current hull.
- So the time complexity for computing upper hull and lower hull is $\mathcal{O}(n)$.
- Total running time: $\mathcal{O}(n \log n)$.

Lower bound:

An $\Omega(n \log n)$ lower bound is known for the convex hull problem.

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Other algorithms:

Algorithm	Speed	Discovered By
Brute Force	$\mathcal{O}(n^4)$	[Anon, the dark ages]
Gift Wrapping	$\mathcal{O}(nh)$	[Chand & Kapur, 1970]
Graham Scan	$\mathcal{O}(n \log n)$	[Graham, 1972]
Jarvis March	$\mathcal{O}(nh)$	[Jarvis, 1973]
QuickHull	$\mathcal{O}(nh)$	[Eddy, 1977], [Bykat, 1978]
Divide-and-Conquer	$\mathcal{O}(n \log n)$	[Preparata & Hong, 1977]
Monotone Chain	$\mathcal{O}(n \log n)$	[Andrew, 1979]
Incremental	$\mathcal{O}(n \log n)$	[Kallay, 1984]
Marriage-before-Conquest	$\mathcal{O}(n \log h)$	[Kirkpatrick & Seidel, 1986]

n: number of points

h: number of points on the boundary of convex hull

Higher dimensions:

- The convex hull can be defined in any dimension.
- Convex hulls in 3-dimensional space can still be computed in $\mathcal{O}(n \log n)$ time (Chapter 11).
- For dimensions higher than 3, however, the complexity of the convex hull is no longer linear in the number of points.

END.