CHAPTER 6

Techniques of Integration

> “I'm very good at integral and differential calculus,
> I know the scientific names of beings animalculous;
> In short, in matters vegetable, animal, and mineral,
> I am the very model of a modern Major-General.”

William Schwenck Gilbert 1836–1911
from *The Pirates of Penzance*

Introduction

This chapter is completely concerned with how to evaluate integrals. The first four sections continue our search, begun in Section 5.6, for ways to find antiderivatives and, therefore, definite integrals by the Fundamental Theorem of Calculus. Section 6.5 deals with the problem of finding definite integrals of functions over infinite intervals, or over intervals where the functions are not bounded. The remaining three sections deal with techniques of *numerical integration* that can be used to find approximate values of definite integrals when an antiderivative cannot be found.

It is not necessary to cover the material of this chapter before proceeding to the various applications of integration discussed in Chapter 7, but some of the examples and exercises in that chapter do depend on techniques presented here.

6.1 Integration by Parts

Our next general method for antidifferentiation is called **integration by parts**. Just as the method of substitution can be regarded as inverse to the Chain Rule for differentiation, so the method for integration by parts is inverse to the Product Rule for differentiation.

Suppose that $U(x)$ and $V(x)$ are two differentiable functions. According to the Product Rule,

$$\frac{d}{dx}\big(U(x)V(x)\big) = U(x)\frac{dV}{dx} + V(x)\frac{dU}{dx}.$$

Integrating both sides of this equation and transposing terms, we obtain

$$\int U(x)\frac{dV}{dx}\,dx = U(x)V(x) - \int V(x)\frac{dU}{dx}\,dx$$

or, more simply,

$$\int U\,dV = UV - \int V\,dU.$$

The above formula serves as a *pattern* for carrying out integration by parts, as we will see in the examples below. In each application of the method, we break up the given integrand into a product of two pieces, U and V', where V' is readily integrated and where $\int VU'\,dx$ is usually (but not always) a *simpler* integral than $\int UV'\,dx$. The technique is called integration by parts because it replaces one integral with the sum of an integrated term and another integral that remains to be evaluated. That is, it accomplishes only *part* of the original integration.

EXAMPLE 1

$$\int xe^x\,dx \qquad \text{Let } U = x, \quad dV = e^x\,dx. \quad \text{Then } dU = dx, \quad V = e^x.$$

$$= xe^x - \int e^x\,dx \qquad (\text{i.e., } UV - \textstyle\int V\,dU)$$

$$= xe^x - e^x + C.$$

Note the form in which the integration by parts is carried out. We indicate at the side what choices we are making for U and dV and then calculate dU and V from these. However, we do not actually substitute U and V into the integral; instead, we use the formula $\int U\,dV = UV - \int V\,dU$ as a pattern or mnemonic device to replace the given integral by the equivalent partially integrated form on the second line.

Note also that had we included a constant of integration with V, for example, $V = e^x + K$, that constant would cancel out in the next step:

$$\int xe^x\,dx = x(e^x + K) - \int (e^x + K)\,dx$$
$$= xe^x + Kx - e^x - Kx + C = xe^x - e^x + C.$$

In general, do not include a constant of integration with V or on the right-hand side until the last integral has been evaluated.

Study the various parts of the following example carefully; they show the various ways in which integration by parts is used, and they give some insights into what choices should be made for U and dV in various situations. An improper choice can result in making an integral more difficult rather than easier. Look for a factor of the integrand that is easily integrated, and include dx with that factor to make up dV. Then U is the remaining factor of the integrand. Sometimes it is necessary to take $dV = dx$ only. When breaking up an integrand using integration by parts, choose U and dV so that, if possible, $V\,dU$ is "simpler" (easier to integrate) than $U\,dV$.

EXAMPLE 2 Use integration by parts to evaluate

(a) $\displaystyle\int \ln x\,dx$, (b) $\displaystyle\int x^2 \sin x\,dx$, (c) $\displaystyle\int x\tan^{-1} x\,dx$, (d) $\displaystyle\int \sin^{-1} x\,dx$.

Solution

(a)
$$\int \ln x\,dx \qquad \text{Let } U = \ln x, \quad dV = dx. \quad \text{Then } dU = dx/x, \quad V = x.$$

$$= x\ln x - \int x\frac{1}{x}\,dx$$

$$= x\ln x - x + C.$$

(b) We have to integrate by parts twice this time:

$$\int x^2 \sin x\,dx \qquad \text{Let } U = x^2, \quad dV = \sin x\,dx.$$
$$\text{Then } dU = 2x\,dx, \quad V = -\cos x.$$
$$= -x^2 \cos x + 2\int x \cos x\,dx \qquad \text{Let } U = x, \quad dV = \cos x\,dx.$$
$$\text{Then } dU = dx, \quad V = \sin x.$$
$$= -x^2 \cos x + 2\left(x \sin x - \int \sin x\,dx\right)$$
$$= -x^2 \cos x + 2x \sin x + 2\cos x + C.$$

(c)
$$\int x \tan^{-1} x\,dx \qquad \text{Let } U = \tan^{-1} x, \quad dV = x\,dx.$$
$$\text{Then } dU = dx/(1+x^2), \quad V = \tfrac{1}{2}x^2.$$
$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx$$
$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}\int \left(1 - \frac{1}{1+x^2}\right)dx$$
$$= \frac{1}{2}x^2 \tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C.$$

(d)
$$\int \sin^{-1} x\,dx \qquad \text{Let } U = \sin^{-1} x, \quad dV = dx.$$
$$\text{Then } dU = dx/\sqrt{1-x^2}, \quad V = x.$$
$$= x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx \qquad \text{Let } u = 1-x^2,$$
$$du = -2x\,dx$$
$$= x \sin^{-1} x + \frac{1}{2}\int u^{-1/2}\,du$$
$$= x \sin^{-1} x + u^{1/2} + C = x \sin^{-1} x + \sqrt{1-x^2} + C.$$

The following are two useful rules of thumb for choosing U and dV:

(i) If the integrand involves a polynomial multiplied by an exponential, a sine or a cosine, or some other readily integrable function, try U equals the polynomial and dV equals the rest.

(ii) If the integrand involves a logarithm, an inverse trigonometric function, or some other function that is not readily integrable but whose derivative is readily calculated, try that function for U and let dV equal the rest.

(Of course, these "rules" come with no guarantee. They may fail to be helpful if "the rest" is not of a suitable form. There remain many functions that cannot be antidifferentiated by any standard techniques; e.g., e^{x^2}.)

The following two examples illustrate a frequently occurring and very useful phenomenon. It may happen after one or two integrations by parts, with the possible application of some known identity, that the original integral reappears on the right-hand side. Unless its coefficient there is 1, we have an equation that can be solved for that integral.

EXAMPLE 3 Evaluate $I = \int \sec^3 x\,dx$.

Solution Start by integrating by parts:

$$I = \int \sec^3 x\,dx \qquad \text{Let } U = \sec x, \quad dV = \sec^2 x\,dx.$$
$$\text{Then } dU = \sec x \tan x\,dx, \quad V = \tan x.$$
$$= \sec x \tan x - \int \sec x \tan^2 x\,dx$$

$$= \sec x \tan x - \int \sec x(\sec^2 x - 1)\,dx$$
$$= \sec x \tan x - \int \sec^3 x\,dx + \int \sec x\,dx$$
$$= \sec x \tan x - I + \ln|\sec x + \tan x|.$$

This is an equation that can be solved for the desired integral I. Since $2I = \sec x \tan x + \ln|\sec x + \tan x|$, we have

$$\int \sec^3 x\,dx = I = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

This integral occurs frequently in applications and is worth remembering.

EXAMPLE 4 Find $I = \int e^{ax}\cos bx\,dx$.

Solution If either $a = 0$ or $b = 0$, the integral is easy to do, so let us assume $a \neq 0$ and $b \neq 0$. We have

$$I = \int e^{ax}\cos bx\,dx \qquad \text{Let } U = e^{ax}, \quad dV = \cos bx\,dx. \quad \text{Then } dU = a\,e^{ax}\,dx, \quad V = (1/b)\sin bx.$$
$$= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\int e^{ax}\sin bx\,dx$$
$$\text{Let } U = e^{ax}, \quad dV = \sin bx\,dx. \quad \text{Then } dU = ae^{ax}dx, \quad V = -(\cos bx)/b.$$
$$= \frac{1}{b}e^{ax}\sin bx - \frac{a}{b}\left(-\frac{1}{b}e^{ax}\cos bx + \frac{a}{b}\int e^{ax}\cos bx\,dx\right)$$
$$= \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx - \frac{a^2}{b^2}I.$$

Thus,

$$\left(1 + \frac{a^2}{b^2}\right)I = \frac{1}{b}e^{ax}\sin bx + \frac{a}{b^2}e^{ax}\cos bx + C_1$$

and

$$\int e^{ax}\cos bx\,dx = I = \frac{b\,e^{ax}\sin bx + a\,e^{ax}\cos bx}{b^2 + a^2} + C.$$

Observe that after the first integration by parts we had an integral that was different from, but no simpler than, the original integral. At this point we might have become discouraged and given up on this method. However, perseverance proved worthwhile; a second integration by parts returned the original integral I in an equation that could be solved for I. Having chosen to let U be the exponential in the first integration by parts (we could have let it be the cosine), we made the same choice for U in the second integration by parts. Had we switched horses in midstream and decided to let U be the trigonometric function the second time, we would have obtained

$$I = \frac{1}{b}e^{ax}\sin bx - \frac{1}{b}e^{ax}\sin bx + I,$$

that is, we would have *undone* what we accomplished in the first step.

If we want to evaluate a definite integral by the method of integration by parts, we must remember to include the appropriate evaluation symbol with the integrated term.

EXAMPLE 5 **(A definite integral)**

$$\int_1^e x^3 (\ln x)^2\,dx \qquad \text{Let } U = (\ln x)^2, \quad dV = x^3\,dx. \quad \text{Then } dU = 2\ln x\,(1/x)\,dx, \quad V = x^4/4.$$

$$= \frac{x^4}{4}(\ln x)^2\Big|_1^e - \frac{1}{2}\int_1^e x^3 \ln x\,dx \qquad \text{Let } U = \ln x, \quad dV = x^3\,dx. \quad \text{Then } dU = dx/x, \quad V = x^4/4.$$

$$= \frac{e^4}{4}(1^2) - 0 - \frac{1}{2}\left(\frac{x^4}{4}\ln x\Big|_1^e - \frac{1}{4}\int_1^e x^3\,dx\right)$$

$$= \frac{e^4}{4} - \frac{e^4}{8} + \frac{1}{8}\frac{x^4}{4}\Big|_1^e = \frac{e^4}{8} + \frac{e^4}{32} - \frac{1}{32} = \frac{5}{32}e^4 - \frac{1}{32}.$$

Reduction Formulas

Consider the problem of finding $\int x^4 e^{-x}\,dx$. We can, as in Example 1, proceed by using integration by parts four times. Each time will reduce the power of x by 1. Since this is repetitive and tedious, we prefer the following approach. For $n \geq 0$, let

$$I_n = \int x^n e^{-x}\,dx.$$

We want to find I_4. If we integrate by parts, we obtain a formula for I_n in terms of I_{n-1}:

$$I_n = \int x^n e^{-x}\,dx \qquad \text{Let } U = x^n, \quad dV = e^{-x}\,dx. \quad \text{Then } dU = nx^{n-1}\,dx, \quad V = -e^{-x}.$$

$$= -x^n e^{-x} + n\int x^{n-1} e^{-x}\,dx = -x^n e^{-x} + nI_{n-1}.$$

The formula

$$I_n = -x^n e^{-x} + nI_{n-1}$$

is called a **reduction formula** because it gives the value of the integral I_n in terms of I_{n-1}, an integral corresponding to a reduced value of the exponent n. Starting with

$$I_0 = \int x^0 e^{-x}\,dx = \int e^{-x}\,dx = -e^{-x} + C,$$

we can apply the reduction formula four times to get

$$\begin{aligned}
I_1 &= -xe^{-x} + I_0 = -e^{-x}(x+1) + C_1 \\
I_2 &= -x^2e^{-x} + 2I_1 = -e^{-x}(x^2 + 2x + 2) + C_2 \\
I_3 &= -x^3e^{-x} + 3I_2 = -e^{-x}(x^3 + 3x^2 + 6x + 6) + C_3 \\
I_4 &= -x^4e^{-x} + 4I_3 = -e^{-x}(x^4 + 4x^3 + 12x^2 + 24x + 24) + C_4.
\end{aligned}$$

EXAMPLE 6 Obtain and use a reduction formula to evaluate

$$I_n = \int_0^{\pi/2} \cos^n x\,dx \qquad (n = 0,\ 1,\ 2,\ 3,\ \ldots).$$

Solution Observe first that

$$I_0 = \int_0^{\pi/2} dx = \frac{\pi}{2} \quad \text{and} \quad I_1 = \int_0^{\pi/2} \cos x\, dx = \sin x \bigg|_0^{\pi/2} = 1.$$

Now let $n \ge 2$:

$$\begin{aligned} I_n &= \int_0^{\pi/2} \cos^n x\, dx = \int_0^{\pi/2} \cos^{n-1} x \cos x\, dx \\ &\qquad\qquad U = \cos^{n-1} x, \qquad dV = \cos x\, dx \\ &\qquad\qquad dU = -(n-1)\cos^{n-2} x \sin x\, dx, \quad V = \sin x \\ &= \sin x \cos^{n-1} x \bigg|_0^{\pi/2} + (n-1)\int_0^{\pi/2} \cos^{n-2} x \sin^2 x\, dx \\ &= 0 - 0 + (n-1)\int_0^{\pi/2} \cos^{n-2} x\,(1 - \cos^2 x)\, dx \\ &= (n-1) I_{n-2} - (n-1) I_n. \end{aligned}$$

Transposing the term $-(n-1)I_n$, we obtain $nI_n = (n-1)I_{n-2}$, or

$$I_n = \frac{n-1}{n} I_{n-2},$$

which is the required reduction formula. It is valid for $n \ge 2$, which was needed to ensure that $\cos^{n-1}(\pi/2) = 0$. If $n \ge 2$ is an *even integer*, we have

$$\begin{aligned} I_n &= \frac{n-1}{n} I_{n-2} = \frac{n-1}{n} \cdot \frac{n-3}{n-2} I_{n-4} = \cdots \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I_0 \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}. \end{aligned}$$

If $n \ge 3$ is an *odd* integer, we have

$$\begin{aligned} I_n &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_1 \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdot \cdots \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3}. \end{aligned}$$

See Exercise 38 for an interesting consequence of these formulas.

EXERCISES 6.1

Evaluate the integrals in Exercises 1–28.

1. $\int x \cos x\, dx$

2. $\int (x+3)e^{2x}\, dx$

3. $\int x^2 \cos \pi x\, dx$

4. $\int (x^2 - 2x)e^{kx}\, dx$

5. $\int x^3 \ln x\, dx$

6. $\int x(\ln x)^3\, dx$

7. $\int \tan^{-1} x\, dx$

8. $\int x^2 \tan^{-1} x\, dx$

9. $\int x \sin^{-1} x\, dx$

10. $\int x^5 e^{-x^2}\, dx$

11. $\int_0^{\pi/4} \sec^5 x\, dx$

12. $\int \tan^2 x \sec x\, dx$

13. $\int e^{2x} \sin 3x\, dx$

14. $\int x e^{\sqrt{x}}\, dx$

! **15.** $\int_{1/2}^{1} \frac{\sin^{-1} x}{x^2}\, dx$

16. $\int_0^1 \sqrt{x} \sin(\pi\sqrt{x})\, dx$

17. $\int x \sec^2 x\, dx$

18. $\int x \sin^2 x\, dx$

19. $\displaystyle\int \cos(\ln x)\,dx$

20. $\displaystyle\int_1^e \sin(\ln x)\,dx$

21. $\displaystyle\int \frac{\ln(\ln x)}{x}\,dx$

22. $\displaystyle\int_0^4 \sqrt{x}e^{\sqrt{x}}\,dx$

23. $\displaystyle\int \arccos x\,dx$

24. $\displaystyle\int x\sec^{-1} x\,dx$

25. $\displaystyle\int_1^2 \sec^{-1} x\,dx$

26. $\displaystyle\int (\sin^{-1} x)^2\,dx$

27. $\displaystyle\int x(\tan^{-1} x)^2\,dx$

28. $\displaystyle\int x\,e^x \cos x\,dx$

29. Find the area below $y = e^{-x}\sin x$ and above $y = 0$ from $x = 0$ to $x = \pi$.

30. Find the area of the finite plane region bounded by the curve $y = \ln x$, the line $y = 1$, and the tangent line to $y = \ln x$ at $x = 1$.

Reduction formulas

31. Obtain a reduction formula for $I_n = \int (\ln x)^n\,dx$, and use it to evaluate I_4.

32. Obtain a reduction formula for $I_n = \int_0^{\pi/2} x^n \sin x\,dx$, and use it to evaluate I_6.

33. Obtain a reduction formula for $I_n = \int \sin^n x\,dx$ (where $n \ge 2$), and use it to find I_6 and I_7.

34. Obtain a reduction formula for $I_n = \int \sec^n x\,dx$ (where $n \ge 3$), and use it to find I_6 and I_7.

35. By writing

$$\begin{aligned} I_n &= \int \frac{dx}{(x^2+a^2)^n} \\ &= \frac{1}{a^2}\int \frac{dx}{(x^2+a^2)^{n-1}} - \frac{1}{a^2}\int x\,\frac{x}{(x^2+a^2)^n}\,dx \end{aligned}$$

and integrating the last integral by parts, using $U = x$, obtain a reduction formula for I_n. Use this formula to find I_3.

36. If f is twice differentiable on $[a, b]$ and $f(a) = f(b) = 0$, show that

$$\int_a^b (x-a)(b-x)f''(x)\,dx = -2\int_a^b f(x)\,dx.$$

(*Hint:* Use integration by parts on the left-hand side twice.) This formula will be used in Section 6.6 to construct an error estimate for the Trapezoid Rule approximation formula.

37. If f and g are two functions having continuous second derivatives on the interval $[a, b]$, and if $f(a) = g(a) = f(b) = g(b) = 0$, show that

$$\int_a^b f(x)\,g''(x)\,dx = \int_a^b f''(x)\,g(x)\,dx.$$

What other assumptions about the values of f and g at a and b would give the same result?

38. **(The Wallis Product)** Let $I_n = \int_0^{\pi/2} \cos^n x\,dx$.

(a) Use the fact that $0 \le \cos x \le 1$ for $0 \le x \le \pi/2$ to show that $I_{2n+2} \le I_{2n+1} \le I_{2n}$, for $n = 0, 1, 2, \ldots$.

(b) Use the reduction formula $I_n = ((n-1)/n)I_{n-2}$ obtained in Example 6, together with the result of (a), to show that

$$\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1.$$

(c) Combine the result of (b) with the explicit formulas obtained for I_n (for even and odd n) in Example 6 to show that

$$\lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdots\frac{2n}{2n-1}\cdot\frac{2n}{2n+1} = \frac{\pi}{2}.$$

This interesting product formula for π is due to the seventeenth-century English mathematician John Wallis and is referred to as the Wallis Product.

6.2 Integrals of Rational Functions

In this section we are concerned with integrals of the form

$$\int \frac{P(x)}{Q(x)}\,dx,$$

where P and Q are polynomials. Recall that a **polynomial** is a function P of the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0,$$

where n is a nonnegative integer, $a_0, a_1, a_2, \ldots, a_n$ are constants, and $a_n \ne 0$. We call n the **degree** of P. A quotient $P(x)/Q(x)$ of two polynomials is called a **rational function**. (See Section P.6 for more discussion of polynomials and rational functions.) We need normally concern ourselves only with rational functions $P(x)/Q(x)$ where the degree of P is less than that of Q. If the degree of P equals or exceeds the degree of Q, then we can use division to express the fraction $P(x)/Q(x)$ as a polynomial plus another fraction $R(x)/Q(x)$, where R, the remainder in the division, has degree less than that of Q.

EXAMPLE 1 Evaluate $\displaystyle\int \frac{x^3+3x^2}{x^2+1}\,dx$.

Solution The numerator has degree 3 and the denominator has degree 2, so we need to divide. We use long division:

$$\begin{array}{r|l} & x \quad + 3 \\ \hline x^2+1 & x^3 + 3x^2 \\ & x^3 \qquad\quad + x \\ \hline & \quad 3x^2 - x \\ & \quad 3x^2 \qquad + 3 \\ \hline & \qquad\quad - x - 3 \end{array} \qquad \frac{x^3+3x^2}{x^2+1} = x+3-\frac{x+3}{x^2+1}.$$

Thus,

$$\begin{aligned}\int \frac{x^3+3x^2}{x^2+1}\,dx &= \int (x+3)\,dx - \int \frac{x}{x^2+1}\,dx - 3\int \frac{dx}{x^2+1} \\ &= \frac{1}{2}x^2 + 3x - \frac{1}{2}\ln(x^2+1) - 3\tan^{-1}x + C.\end{aligned}$$

EXAMPLE 2 Evaluate $\displaystyle\int \frac{x}{2x-1}\,dx$.

Solution The numerator and denominator have the same degree, 1, so division is again required. In this case the division can be carried out by manipulation of the integrand:

$$\frac{x}{2x-1} = \frac{1}{2}\,\frac{2x}{2x-1} = \frac{1}{2}\,\frac{2x-1+1}{2x-1} = \frac{1}{2}\left(1+\frac{1}{2x-1}\right),$$

a process that we call *short division* (see Section P.6). We have

$$\int \frac{x}{2x-1}\,dx = \frac{1}{2}\int \left(1+\frac{1}{2x-1}\right)dx = \frac{x}{2}+\frac{1}{4}\ln|2x-1|+C.$$

In the discussion that follows, we always assume that any necessary division has been performed and the quotient polynomial has been integrated. The remaining basic problem with which we will deal in this section is the following:

The basic problem

Evaluate $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$, where the degree of $P <$ the degree of Q.

The complexity of this problem depends on the degree of Q.

Linear and Quadratic Denominators

Suppose that $Q(x)$ has degree 1. Thus, $Q(x) = ax+b$, where $a \neq 0$. Then $P(x)$ must have degree 0 and be a constant c. We have $P(x)/Q(x) = c/(ax+b)$. The substitution $u = ax+b$ leads to

$$\int \frac{c}{ax+b}\,dx = \frac{c}{a}\int \frac{du}{u} = \frac{c}{a}\ln|u|+C,$$

so that for $c = 1$:

The case of a linear denominator

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + C.$$

Now suppose that $Q(x)$ is quadratic, that is, has degree 2. For purposes of this discussion we can assume that $Q(x)$ is either of the form $x^2 + a^2$ or of the form $x^2 - a^2$, since completing the square and making the appropriate change of variable can always reduce a quadratic denominator to this form, as shown in Section 6.2. Since $P(x)$ can be at most a linear function, $P(x) = Ax + B$, we are led to consider the following four integrals:

$$\int \frac{x\,dx}{x^2+a^2}, \qquad \int \frac{x\,dx}{x^2-a^2}, \qquad \int \frac{dx}{x^2+a^2}, \quad \text{and} \quad \int \frac{dx}{x^2-a^2}.$$

(If $a = 0$, there are only two integrals; each is easily evaluated.) The first two integrals yield to the substitution $u = x^2 \pm a^2$; the third is a known integral. The fourth integral will be evaluated by a different method below. The values of all four integrals are given in the following box:

The case of a quadratic denominator

$$\int \frac{x\,dx}{x^2+a^2} = \frac{1}{2}\ln(x^2+a^2) + C,$$
$$\int \frac{x\,dx}{x^2-a^2} = \frac{1}{2}\ln|x^2-a^2| + C,$$
$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C,$$
$$\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C.$$

To obtain the last formula in the box, let us try to write the integrand as a sum of two fractions with linear denominators:

$$\frac{1}{x^2-a^2} = \frac{1}{(x-a)(x+a)} = \frac{A}{x-a} + \frac{B}{x+a} = \frac{Ax + Aa + Bx - Ba}{x^2-a^2},$$

where we have added the two fractions together again in the last step. If this equation is to hold identically for all x (except $x = \pm a$), then the numerators on the left and right sides must be identical as polynomials in x. The equation $(A+B)x + (Aa - Ba) = 1 = 0x + 1$ can hold for all x only if

$$\begin{aligned} A + B &= 0 \qquad \text{(the coefficient of } x\text{)},\\ Aa - Ba &= 1 \qquad \text{(the constant term).} \end{aligned}$$

Solving this pair of linear equations for the unknowns A and B, we get $A = 1/(2a)$ and $B = -1/(2a)$. Therefore,

$$\begin{aligned} \int \frac{dx}{x^2-a^2} &= \frac{1}{2a}\int \frac{dx}{x-a} - \frac{1}{2a}\int \frac{dx}{x+a}\\ &= \frac{1}{2a}\ln|x-a| - \frac{1}{2a}\ln|x+a| + C\\ &= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C. \end{aligned}$$

Partial Fractions

The technique used above, involving the writing of a complicated fraction as a sum of simpler fractions, is called the **method of partial fractions**. Suppose that a polynomial $Q(x)$ is of degree n and that its highest degree term is x^n (with coefficient 1). Suppose also that Q factors into a product of n *distinct* linear (degree 1) factors, say,

$$Q(x) = (x - a_1)(x - a_2)\cdots(x - a_n),$$

where $a_i \neq a_j$ if $i \neq j$, $1 \leq i, j \leq n$. If $P(x)$ is a polynomial of degree smaller than n, then $P(x)/Q(x)$ has a **partial fraction decomposition** of the form

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x - a_1} + \frac{A_2}{x - a_2} + \cdots + \frac{A_n}{x - a_n}$$

for certain values of the constants $A_1, A_2, \ldots, A_n$. We do not attempt to give any formal proof of this assertion here; such a proof belongs in an algebra course. (See Theorem 1 below for the statement of a more general result.)

Given that $P(x)/Q(x)$ has a partial fraction decomposition as claimed above, there are two methods for determining the constants $A_1, A_2, \ldots, A_n$. The first of these methods, and one that generalizes most easily to the more complicated decompositions considered below, is to add up the fractions in the decomposition, obtaining a new fraction $S(x)/Q(x)$ with numerator $S(x)$, a polynomial of degree one less than that of $Q(x)$. This new fraction will be identical to the original fraction $P(x)/Q(x)$ if S and P are identical polynomials. The constants $A_1, A_2, \ldots, A_n$ are determined by solving the n linear equations resulting from equating the coefficients of like powers of x in the two polynomials S and P.

The second method depends on the following observation: if we multiply the partial fraction decomposition by $x - a_j$, we get

$$\begin{aligned}(x - a_j)&\frac{P(x)}{Q(x)} \\ &= A_1\frac{x - a_j}{x - a_1} + \cdots + A_{j-1}\frac{x - a_j}{x - a_{j-1}} + A_j + A_{j+1}\frac{x - a_j}{x - a_{j+1}} + \cdots + A_n\frac{x - a_j}{x - a_n}.\end{aligned}$$

All terms on the right side are 0 at $x = a_j$ except the jth term, A_j. Hence,

$$\begin{aligned}A_j &= \lim_{x \to a_j}(x - a_j)\frac{P(x)}{Q(x)} \\ &= \frac{P(a_j)}{(a_j - a_1)\cdots(a_j - a_{j-1})(a_j - a_{j+1})\cdots(a_j - a_n)},\end{aligned}$$

for $1 \leq j \leq n$. In practice, you can use this method to find each number A_j by cancelling the factor $x - a_j$ from the denominator of $P(x)/Q(x)$ and evaluating the resulting expression at $x = a_j$.

EXAMPLE 3 Evaluate $\displaystyle\int \frac{(x + 4)}{x^2 - 5x + 6}\,dx$.

Solution The partial fraction decomposition takes the form

$$\frac{x + 4}{x^2 - 5x + 6} = \frac{x + 4}{(x - 2)(x - 3)} = \frac{A}{x - 2} + \frac{B}{x - 3}.$$

We calculate A and B by both of the methods suggested above.

METHOD I. Add the partial fractions

$$\frac{x + 4}{x^2 - 5x + 6} = \frac{Ax - 3A + Bx - 2B}{(x - 2)(x - 3)},$$

and equate the coefficient of x and the constant terms in the numerators on both sides to obtain

$$A + B = 1 \qquad \text{and} \qquad -3A - 2B = 4.$$

Solve these equations to get $A = -6$ and $B = 7$.

METHOD II. To find A, cancel $x-2$ from the denominator of the expression $P(x)/Q(x)$ and evaluate the result at $x = 2$. Obtain B similarly.

$$A = \left.\frac{x+4}{x-3}\right|_{x=2} = -6 \qquad \text{and} \qquad B = \left.\frac{x+4}{x-2}\right|_{x=3} = 7.$$

In either case we have

$$\begin{aligned}\int \frac{(x+4)}{x^2-5x+6}\,dx &= -6\int \frac{1}{x-2}\,dx + 7\int \frac{1}{x-3}\,dx \\ &= -6\ln|x-2| + 7\ln|x-3| + C.\end{aligned}$$

EXAMPLE 4 Evaluate $I = \displaystyle\int \frac{x^3+2}{x^3-x}\,dx$.

Solution Since the numerator does not have degree smaller than the denominator, we must divide:

$$I = \int \frac{x^3-x+x+2}{x^3-x}\,dx = \int \left(1 + \frac{x+2}{x^3-x}\right) dx = x + \int \frac{x+2}{x^3-x}\,dx.$$

Now we can use the method of partial fractions.

$$\begin{aligned}\frac{x+2}{x^3-x} = \frac{x+2}{x(x-1)(x+1)} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+1} \\ &= \frac{A(x^2-1) + B(x^2+x) + C(x^2-x)}{x(x-1)(x+1)}\end{aligned}$$

We have

$$\begin{aligned} A + B + C &= 0 \qquad &\text{(coefficient of } x^2\text{)} \\ B - C &= 1 \qquad &\text{(coefficient of } x\text{)} \\ -A &= 2 \qquad &\text{(constant term).}\end{aligned}$$

It follows that $A = -2$, $B = 3/2$, and $C = 1/2$. We can also find these values using Method II of the previous example:

$$A = \left.\frac{x+2}{(x-1)(x+1)}\right|_{x=0} = -2, \qquad B = \left.\frac{x+2}{x(x+1)}\right|_{x=1} = \frac{3}{2}, \quad \text{and}$$

$$C = \left.\frac{x+2}{x(x-1)}\right|_{x=-1} = \frac{1}{2}.$$

Finally, we have

$$\begin{aligned} I &= x - 2\int \frac{1}{x}\,dx + \frac{3}{2}\int \frac{1}{x-1}\,dx + \frac{1}{2}\int \frac{1}{x+1}\,dx \\ &= x - 2\ln|x| + \frac{3}{2}\ln|x-1| + \frac{1}{2}\ln|x+1| + C.\end{aligned}$$

Next, we consider a rational function whose denominator has a quadratic factor that is equivalent to a sum of squares and cannot, therefore, be further factored into a product of real linear factors.

EXAMPLE 5 Evaluate $\displaystyle\int \frac{2+3x+x^2}{x(x^2+1)}\,dx$.

Solution Note that the numerator has degree 2 and the denominator degree 3, so no division is necessary. If we decompose the integrand as a sum of two simpler fractions, we want one with denominator x and one with denominator x^2+1. The appropriate form of the decomposition turns out to be

$$\frac{2+3x+x^2}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1} = \frac{A(x^2+1)+Bx^2+Cx}{x(x^2+1)}.$$

Note that corresponding to the quadratic (degree 2) denominator we use a linear (degree 1) numerator. Equating coefficients in the two numerators, we obtain

$$\begin{aligned} A + B &= 1 \qquad &&\text{(coefficient of } x^2)\\ C &= 3 &&\text{(coefficient of } x)\\ A &= 2 &&\text{(constant term).}\end{aligned}$$

Hence $A=2$, $B=-1$, and $C=3$. We have, therefore,

$$\begin{aligned}\int \frac{2+3x+x^2}{x(x^2+1)}\,dx &= 2\int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx + 3\int \frac{1}{x^2+1}\,dx\\ &= 2\ln|x| - \frac{1}{2}\ln(x^2+1) + 3\tan^{-1}x + C.\end{aligned}$$

We remark that addition of the fractions is the only reasonable real-variable method for determining the constants A, B, and C here. We could determine A by Method II of Example 3, but there is no simple equivalent way of finding B or C without using complex numbers.

Completing the Square

Quadratic expressions of the form Ax^2+Bx+C are often found in integrands. These can be written as sums or differences of squares using the procedure of completing the square, as was done to find the formula for the roots of quadratic equations in Section P.6. First factor out A so that the remaining expression begins with x^2+2bx, where $2b = B/A$. These are the first two terms of $(x+b)^2 = x^2+2bx+b^2$. Add the third term $b^2 = B^2/4A^2$ and then subtract it again:

$$\begin{aligned} Ax^2+Bx+C &= A\left(x^2+\frac{B}{A}x+\frac{C}{A}\right)\\ &= A\left(x^2+\frac{B}{A}x+\frac{B^2}{4A^2}+\frac{C}{A}-\frac{B^2}{4A^2}\right)\\ &= A\left(x+\frac{B}{2A}\right)^2+\frac{4AC-B^2}{4A}.\end{aligned}$$

The substitution $u = x + \dfrac{B}{2A}$ should then be made.

EXAMPLE 6 Evaluate $I = \displaystyle\int \frac{1}{x^3+1}\,dx$.

Solution Here $Q(x) = x^3+1 = (x+1)(x^2-x+1)$. The latter factor has no real roots, so it has no real linear subfactors. We have

$$\begin{aligned}\frac{1}{x^3+1} &= \frac{1}{(x+1)(x^2-x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}\\ &= \frac{A(x^2-x+1)+B(x^2+x)+C(x+1)}{(x+1)(x^2-x+1)}\end{aligned}$$

$$\begin{aligned} A + B &= 0 && \text{(coefficient of } x^2\text{)} \\ -A + B + C &= 0 && \text{(coefficient of } x\text{)} \\ A + C &= 1 && \text{(constant term).} \end{aligned}$$

Hence, $A = 1/3$, $B = -1/3$, and $C = 2/3$. We have

$$I = \frac{1}{3}\int \frac{dx}{x+1} - \frac{1}{3}\int \frac{x-2}{x^2-x+1}\,dx.$$

The first integral is easily evaluated; in the second we complete the square in the denominator: $x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$, and make a similar modification in the numerator.

$$\begin{aligned} I &= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{x - \frac{1}{2} - \frac{3}{2}}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}}\,dx \qquad \begin{array}{l}\text{Let } u = x - 1/2, \\ du = dx\end{array} \\ &= \frac{1}{3}\ln|x+1| - \frac{1}{3}\int \frac{u}{u^2 + \frac{3}{4}}\,du + \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\,du \\ &= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln\left(u^2 + \frac{3}{4}\right) + \frac{1}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2u}{\sqrt{3}}\right) + C \\ &= \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right) + C. \end{aligned}$$

Denominators with Repeated Factors

We require one final refinement of the method of partial fractions. If any of the linear or quadratic factors of $Q(x)$ is *repeated* (say, m times), then the partial fraction decomposition of $P(x)/Q(x)$ requires m distinct fractions corresponding to that factor. The denominators of these fractions have exponents increasing from 1 to m, and the numerators are all constants where the repeated factor is linear or linear where the repeated factor is quadratic. (See Theorem 1 below.)

EXAMPLE 7 Evaluate $\displaystyle\int \frac{1}{x(x-1)^2}\,dx$.

Solution The appropriate partial fraction decomposition here is

$$\begin{aligned} \frac{1}{x(x-1)^2} &= \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \\ &= \frac{A(x^2-2x+1) + B(x^2-x) + Cx}{x(x-1)^2}. \end{aligned}$$

Equating coefficients of x^2, x, and 1 in the numerators of both sides, we get

$$\begin{aligned} A + B &= 0 && \text{(coefficient of } x^2\text{)} \\ -2A - B + C &= 0 && \text{(coefficient of } x\text{)} \\ A &= 1 && \text{(constant term).} \end{aligned}$$

Hence, $A = 1$, $B = -1$, $C = 1$, and

$$\begin{aligned} \int \frac{1}{x(x-1)^2}\,dx &= \int \frac{1}{x}\,dx - \int \frac{1}{x-1}\,dx + \int \frac{1}{(x-1)^2}\,dx \\ &= \ln|x| - \ln|x-1| - \frac{1}{x-1} + C \\ &= \ln\left|\frac{x}{x-1}\right| - \frac{1}{x-1} + C. \end{aligned}$$

EXAMPLE 8 Evaluate $I = \displaystyle\int \frac{x^2+2}{4x^5+4x^3+x}\,dx$.

Solution The denominator factors to $x(2x^2+1)^2$, so the appropriate partial fraction decomposition is

$$\begin{aligned}\frac{x^2+2}{x(2x^2+1)^2} &= \frac{A}{x} + \frac{Bx+C}{2x^2+1} + \frac{Dx+E}{(2x^2+1)^2}\\ &= \frac{A(4x^4+4x^2+1)+B(2x^4+x^2)+C(2x^3+x)+Dx^2+Ex}{x(2x^2+1)^2}.\end{aligned}$$

Thus,

$$\begin{aligned} 4A + 2B &= 0 && \text{(coefficient of } x^4)\\ 2C &= 0 && \text{(coefficient of } x^3)\\ 4A + B + D &= 1 && \text{(coefficient of } x^2)\\ C + E &= 0 && \text{(coefficient of } x)\\ A &= 2 && \text{(constant term).}\end{aligned}$$

Solving these equations, we get $A = 2$, $B = -4$, $C = 0$, $D = -3$, and $E = 0$.

$$\begin{aligned} I &= 2\int\frac{dx}{x} - 4\int\frac{x\,dx}{2x^2+1} - 3\int\frac{x\,dx}{(2x^2+1)^2} \qquad \begin{array}{l}\text{Let } u = 2x^2+1,\\ du = 4x\,dx\end{array}\\ &= 2\ln|x| - \int\frac{du}{u} - \frac{3}{4}\int\frac{du}{u^2}\\ &= 2\ln|x| - \ln|u| + \frac{3}{4u} + C\\ &= \ln\left(\frac{x^2}{2x^2+1}\right) + \frac{3}{4}\frac{1}{2x^2+1} + C.\end{aligned}$$

The following theorem summarizes the various aspects of the method of partial fractions.

THEOREM 1

Partial fraction decompositions of rational functions

Let P and Q be polynomials with real coefficients, and suppose that the degree of P is less than the degree of Q. Then

(a) $Q(x)$ can be factored into the product of a constant K, real linear factors of the form $x - a_i$, and real quadratic factors of the form $x^2 + b_i x + c_i$ having no real roots. The linear and quadratic factors may be repeated:

$$\begin{aligned} Q(x) = K(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_j)^{m_j}(x^2+b_1x+c_1)^{n_1}\\ \cdots(x^2+b_kx+c_k)^{n_k}.\end{aligned}$$

The degree of Q is $m_1 + m_2 + \cdots + m_j + 2n_1 + 2n_2 + \cdots + 2n_k$.

(b) The rational function $P(x)/Q(x)$ can be expressed as a sum of partial fractions as follows:

(i) corresponding to each factor $(x-a)^m$ of $Q(x)$ the decomposition contains a sum of fractions of the form

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_m}{(x-a)^m};$$

(ii) corresponding to each factor $(x^2 + bx + c)^n$ of $Q(x)$ the decomposition contains a sum of fractions of the form

$$\frac{B_1x + C_1}{x^2 + bx + c} + \frac{B_2x + C_2}{(x^2 + bx + c)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + bx + c)^n}.$$

The constants $A_1, A_2, \ldots, A_m, B_1, B_2, \ldots, B_n, C_1, C_2, \ldots, C_n$ can be determined by adding up the fractions in the decomposition and equating the coefficients of like powers of x in the numerator of the sum with those in $P(x)$.

Part (a) of the above theorem is just a restatement of results discussed and proved in Section P.6 and Appendix II. The proof of part (b) is algebraic in nature and is beyond the scope of this text.

Note that part (a) does not tell us how to find the factors of $Q(x)$; it tells us only what form they have. We must know the factors of Q before we can make use of partial fractions to integrate the rational function $P(x)/Q(x)$. Partial fraction decompositions are also used in other mathematical situations, in particular, to solve certain problems involving differential equations.

EXERCISES 6.2

Evaluate the integrals in Exercises 1–28.

1. $\displaystyle\int \frac{2\,dx}{2x-3}$

2. $\displaystyle\int \frac{dx}{5-4x}$

3. $\displaystyle\int \frac{x\,dx}{\pi x+2}$

4. $\displaystyle\int \frac{x^2}{x-4}\,dx$

5. $\displaystyle\int \frac{1}{x^2-9}\,dx$

6. $\displaystyle\int \frac{dx}{5-x^2}$

7. $\displaystyle\int \frac{dx}{a^2-x^2}$

8. $\displaystyle\int \frac{dx}{b^2-a^2x^2}$

9. $\displaystyle\int \frac{x^2\,dx}{x^2+x-2}$

10. $\displaystyle\int \frac{x\,dx}{3x^2+8x-3}$

11. $\displaystyle\int \frac{x-2}{x^2+x}\,dx$

12. $\displaystyle\int \frac{dx}{x^3+9x}$

13. $\displaystyle\int \frac{dx}{1-6x+9x^2}$

14. $\displaystyle\int \frac{x\,dx}{2+6x+9x^2}$

15. $\displaystyle\int \frac{x^2+1}{6x-9x^2}\,dx$

16. $\displaystyle\int \frac{x^3+1}{12+7x+x^2}\,dx$

17. $\displaystyle\int \frac{dx}{x(x^2-a^2)}$

18. $\displaystyle\int \frac{dx}{x^4-a^4}$

19. (!) $\displaystyle\int \frac{x^3\,dx}{x^3-a^3}$

20. $\displaystyle\int \frac{dx}{x^3+2x^2+2x}$

21. $\displaystyle\int \frac{dx}{x^3-4x^2+3x}$

22. $\displaystyle\int \frac{x^2+1}{x^3+8}\,dx$

23. $\displaystyle\int \frac{dx}{(x^2-1)^2}$

24. $\displaystyle\int \frac{x^2\,dx}{(x^2-1)(x^2-4)}$

25. $\displaystyle\int \frac{dx}{x^4-3x^3}$

26. (!) $\displaystyle\int \frac{dt}{(t-1)(t^2-1)^2}$

27. (!) $\displaystyle\int \frac{dx}{e^{2x}-4e^x+4}$

28. (!) $\displaystyle\int \frac{d\theta}{\cos\theta(1+\sin\theta)}$

In Exercises 29–30 write the form that the partial fraction decomposition of the given rational function takes. Do not actually evaluate the constants you use in the decomposition.

29. $\displaystyle\frac{x^5+x^3+1}{(x-1)(x^2-1)(x^3-1)}$

30. $\displaystyle\frac{123-x^7}{(x^4-16)^2}$

31. Write $\displaystyle\frac{x^5}{(x^2-4)(x+2)^2}$ as the sum of a polynomial and a partial fraction decomposition (with constants left undetermined) of a rational function whose numerator has smaller degree than the denominator.

32. Show that $x^4 + 4x^2 + 16$ factors to $(x^2 + kx + 4)(x^2 - kx + 4)$ for a certain positive constant k. What is the value of k? Now repeat the previous exercise for the rational function $\displaystyle\frac{x^4}{x^4+4x^2+16}$.

33. (!) Suppose that P and Q are polynomials such that the degree of P is smaller than that of Q. If

$$Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n),$$

where $a_i \neq a_j$ if $i \neq j\,(1 \leq i, j \leq n)$, so that $P(x)/Q(x)$ has partial fraction decomposition

$$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n},$$

show that

$$A_j = \frac{P(a_j)}{Q'(a_j)} \qquad (1 \leq j \leq n).$$

This gives yet another method for computing the constants in a partial fraction decomposition if the denominator factors completely into distinct linear factors.

6.3 Inverse Substitutions

The substitutions considered in Section 5.6 were direct substitutions in the sense that we simplified an integrand by replacing an expression appearing in it with a single variable. In this section we consider the reverse approach: we replace the variable of integration with a function of a new variable. Such substitutions, called *inverse substitutions*, would appear on the surface to make the integral more complicated. That is, substituting $x = g(u)$ in the integral

$$\int_a^b f(x)\,dx$$

leads to the more "complicated" integral

$$\int_{x=a}^{x=b} f\big(g(u)\big)\,g'(u)\,du.$$

As we will see, however, sometimes such substitutions can actually simplify an integrand, transforming the integral into one that can be evaluated by inspection or to which other techniques can readily be applied. In any event, inverse substitutions can often be used to convert integrands to rational functions to which the methods of Section 6.2 can be applied.

The Inverse Trigonometric Substitutions

Three very useful inverse substitutions are

$$x = a\sin\theta, \qquad x = a\tan\theta, \qquad \text{and} \qquad x = a\sec\theta.$$

These correspond to the direct substitutions

$$\theta = \sin^{-1}\frac{x}{a}, \qquad \theta = \tan^{-1}\frac{x}{a}, \qquad \text{and} \qquad \theta = \sec^{-1}\frac{x}{a} = \cos^{-1}\frac{a}{x}.$$

The inverse sine substitution

Integrals involving $\sqrt{a^2 - x^2}$ (where $a > 0$) can frequently be reduced to a simpler form by means of the substitution

$$x = a\sin\theta \quad \text{or, equivalently,} \quad \theta = \sin^{-1}\frac{x}{a}.$$

Observe that $\sqrt{a^2 - x^2}$ makes sense only if $-a \le x \le a$, which corresponds to $-\pi/2 \le \theta \le \pi/2$. Since $\cos\theta \ge 0$ for such θ, we have

$$\sqrt{a^2 - x^2} = \sqrt{a^2(1 - \sin^2\theta)} = \sqrt{a^2\cos^2\theta} = a\cos\theta.$$

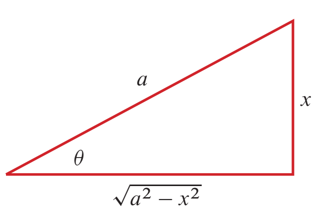

Figure 6.1

(If $\cos\theta$ were not nonnegative, we would have obtained $a|\cos\theta|$ instead.) If needed, the other trigonometric functions of θ can be recovered in terms of x by examining a right-angled triangle labelled to correspond to the substitution (see Figure 6.1)

$$\cos\theta = \frac{\sqrt{a^2 - x^2}}{a} \qquad \text{and} \qquad \tan\theta = \frac{x}{\sqrt{a^2 - x^2}}.$$

EXAMPLE 1 Evaluate $\displaystyle\int \frac{1}{(5 - x^2)^{3/2}}\,dx$.

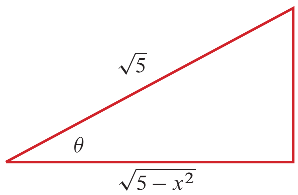

Figure 6.2

Solution Refer to Figure 6.2.

$$\int \frac{1}{(5-x^2)^{3/2}}\,dx \qquad \text{Let } x = \sqrt{5}\sin\theta,\ dx = \sqrt{5}\cos\theta\,d\theta$$

$$= \int \frac{\sqrt{5}\cos\theta\,d\theta}{5^{3/2}\cos^3\theta}$$

$$= \frac{1}{5}\int \sec^2\theta\,d\theta = \frac{1}{5}\tan\theta + C = \frac{1}{5}\frac{x}{\sqrt{5-x^2}} + C$$

EXAMPLE 2 Find the area of the circular segment shaded in Figure 6.3.

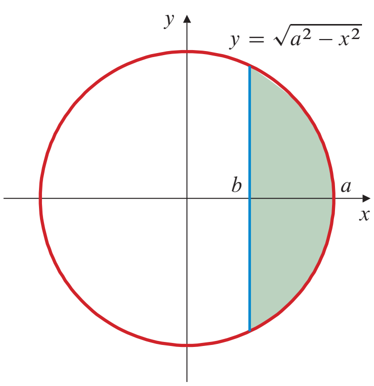

Figure 6.3

Solution The area is

$$A = 2\int_b^a \sqrt{a^2-x^2}\,dx \qquad \text{Let } x = a\sin\theta,\ dx = a\cos\theta\,d\theta$$

$$= 2\int_{x=b}^{x=a} a^2\cos^2\theta\,d\theta$$

$$= a^2\left(\theta + \sin\theta\cos\theta\right)\Big|_{x=b}^{x=a} \qquad \text{(as in Example 8 of Section 5.6)}$$

$$= a^2\left(\sin^{-1}\frac{x}{a} + \frac{x\sqrt{a^2-x^2}}{a^2}\right)\Bigg|_b^a \qquad \text{(See Figure 6.1.)}$$

$$= \frac{\pi}{2}a^2 - a^2\sin^{-1}\frac{b}{a} - b\sqrt{a^2-b^2} \text{ square units.}$$

The inverse tangent substitution

Integrals involving $\sqrt{a^2+x^2}$ or $\dfrac{1}{x^2+a^2}$ (where $a > 0$) are often simplified by the substitution

$$x = a\tan\theta \quad \text{or, equivalently,} \quad \theta = \tan^{-1}\frac{x}{a}.$$

Since x can take any real value, we have $-\pi/2 < \theta < \pi/2$, so $\sec\theta > 0$ and

$$\sqrt{a^2+x^2} = a\sqrt{1+\tan^2\theta} = a\sec\theta.$$

Other trigonometric functions of θ can be expressed in terms of x by referring to a right-angled triangle with legs a and x and hypotenuse $\sqrt{a^2+x^2}$ (see Figure 6.4):

$$\sin\theta = \frac{x}{\sqrt{a^2+x^2}} \quad \text{and} \quad \cos\theta = \frac{a}{\sqrt{a^2+x^2}}.$$

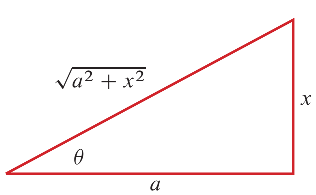

Figure 6.4

EXAMPLE 3 Evaluate (a) $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,dx$ and (b) $\displaystyle\int \frac{1}{(1+9x^2)^2}\,dx$.

Solution Figures 6.5 and 6.6 illustrate parts (a) and (b), respectively.

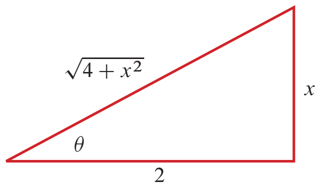

Figure 6.5

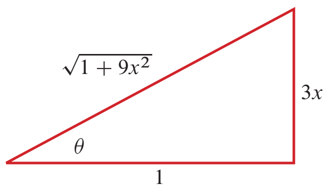

Figure 6.6

$$\text{(a)}\quad \int \frac{1}{\sqrt{4+x^2}}\,dx \qquad \begin{array}{l}\text{Let } x = 2\tan\theta,\\ dx = 2\sec^2\theta\,d\theta\end{array}$$
$$= \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta$$
$$= \int \sec\theta\,d\theta$$
$$= \ln|\sec\theta + \tan\theta| + C = \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$$
$$= \ln\left(\sqrt{4+x^2} + x\right) + C_1, \qquad \text{where } C_1 = C - \ln 2.$$

(Note that $\sqrt{4+x^2} + x > 0$ for all x, so we do not need an absolute value on it.)

$$\text{(b)}\quad \int \frac{1}{(1+9x^2)^2}\,dx \qquad \begin{array}{l}\text{Let } 3x = \tan\theta,\\ 3dx = \sec^2\theta\,d\theta,\\ 1+9x^2 = \sec^2\theta\end{array}$$
$$= \frac{1}{3}\int \frac{\sec^2\theta\,d\theta}{\sec^4\theta}$$
$$= \frac{1}{3}\int \cos^2\theta\,d\theta = \frac{1}{6}\left(\theta + \sin\theta\,\cos\theta\right) + C$$
$$= \frac{1}{6}\tan^{-1}(3x) + \frac{1}{6}\,\frac{3x}{\sqrt{1+9x^2}}\,\frac{1}{\sqrt{1+9x^2}} + C$$
$$= \frac{1}{6}\tan^{-1}(3x) + \frac{1}{2}\,\frac{x}{1+9x^2} + C$$

The inverse secant substitution

Integrals involving $\sqrt{x^2 - a^2}$ (where $a > 0$) can frequently be simplified by using the substitution

$$x = a\sec\theta \quad \text{or, equivalently,} \quad \theta = \sec^{-1}\frac{x}{a}.$$

We must be more careful with this substitution. Although

$$\sqrt{x^2-a^2} = a\sqrt{\sec^2\theta - 1} = a\sqrt{\tan^2\theta} = a|\tan\theta|,$$

we cannot always drop the absolute value from the tangent. Observe that $\sqrt{x^2-a^2}$ makes sense for $x \geq a$ and for $x \leq -a$.

If $x \geq a$, then $0 \leq \theta = \sec^{-1}\dfrac{x}{a} = \arccos\dfrac{a}{x} < \dfrac{\pi}{2}$, and $\tan\theta \geq 0$.

If $x \leq -a$, then $\dfrac{\pi}{2} < \theta = \sec^{-1}\dfrac{x}{a} = \arccos\dfrac{a}{x} \leq \pi$, and $\tan\theta \leq 0$.

In the first case $\sqrt{x^2-a^2} = a\tan\theta$; in the second case $\sqrt{x^2-a^2} = -a\tan\theta$.

EXAMPLE 4 Find $I = \displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$, where $a > 0$.

Solution For the moment, assume that $x \geq a$. If $x = a\sec\theta$, then $dx = a\sec\theta\,\tan\theta\,d\theta$ and $\sqrt{x^2-a^2} = a\tan\theta$. (See Figure 6.7). Thus,

$$I = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C$$
$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C = \ln\left|x + \sqrt{x^2-a^2}\right| + C_1,$$

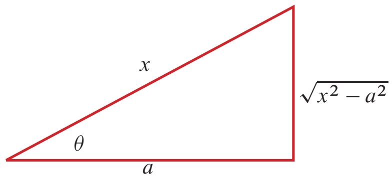

Figure 6.7

where $C_1 = C - \ln a$. If $x \le -a$, let $u = -x$ so that $u \ge a$ and $du = -dx$. We have

$$\begin{aligned} I &= -\int \frac{du}{\sqrt{u^2-a^2}} = -\ln|u + \sqrt{u^2-a^2}| + C_1 \\ &= \ln\left|\frac{1}{-x+\sqrt{x^2-a^2}}\,\frac{x+\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}}\right| + C_1 \\ &= \ln\left|\frac{x+\sqrt{x^2-a^2}}{-a^2}\right| + C_1 = \ln|x+\sqrt{x^2-a^2}| + C_2, \end{aligned}$$

where $C_2 = C_1 - 2\ln a$. Thus, in either case, we have

$$I = \ln|x + \sqrt{x^2-a^2}| + C.$$

The following example requires the technique of completing the square as presented in Section 6.2.

EXAMPLE 5 Evaluate (a) $\displaystyle\int \frac{1}{\sqrt{2x-x^2}}\,dx$ and (b) $\displaystyle\int \frac{x}{4x^2+12x+13}\,dx$.

Solution

(a)
$$\begin{aligned} \int \frac{1}{\sqrt{2x-x^2}}\,dx &= \int \frac{dx}{\sqrt{1-(1-2x+x^2)}} \\ &= \int \frac{dx}{\sqrt{1-(x-1)^2}} \qquad \text{Let } u = x-1,\ du = dx \\ &= \int \frac{du}{\sqrt{1-u^2}} \\ &= \sin^{-1} u + C = \sin^{-1}(x-1) + C. \end{aligned}$$

(b)
$$\begin{aligned} \int \frac{x}{4x^2+12x+13}\,dx &= \int \frac{x\,dx}{4\left(x^2+3x+\frac{9}{4}+1\right)} \\ &= \frac{1}{4}\int \frac{x\,dx}{\left(x+\frac{3}{2}\right)^2+1} \qquad \text{Let } u = x+(3/2),\ du = dx,\ x = u-(3/2) \\ &= \frac{1}{4}\int \frac{u\,du}{u^2+1} - \frac{3}{8}\int \frac{du}{u^2+1} \qquad \text{In the first integral let } v = u^2+1,\ dv = 2u\,du \\ &= \frac{1}{8}\int \frac{dv}{v} - \frac{3}{8}\tan^{-1} u \\ &= \frac{1}{8}\ln|v| - \frac{3}{8}\tan^{-1} u + C \\ &= \frac{1}{8}\ln(4x^2+12x+13) - \frac{3}{8}\tan^{-1}\left(x+\frac{3}{2}\right) + C_1, \end{aligned}$$

where $C_1 = C - (\ln 4)/8$.

Inverse Hyperbolic Substitutions

As an alternative to the inverse secant substitution $x = a\sec\theta$ to simplify integrals involving $\sqrt{x^2-a^2}$ (where $x \ge a > 0$), we can use the inverse hyperbolic cosine substitution $x = a\cosh u$. Since $\cosh^2 u - 1 = \sinh^2 u$, this substitution produces $\sqrt{x^2-a^2} = a\sinh u$. To express u in terms of x, we need the result, noted in Section 3.6,

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \ge 1.$$

To illustrate, we redo Example 4 using the inverse hyperbolic cosine substitution.

EXAMPLE 6 Find $I = \int \frac{dx}{\sqrt{x^2 - a^2}}$, where $a > 0$.

Solution Again we assume $x \geq a$. (The case where $x \leq -a$ can be handled similarly.) Using the substitution $x = a \cosh u$, so that $dx = a \sinh u \, du$, we have

$$\begin{aligned} I &= \int \frac{a \sinh u}{a \sinh u} \, du = \int du = U + C \\ &= \cosh^{-1} \frac{x}{a} + C = \ln \left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2} - 1} \right) + C \\ &= \ln \left(x + \sqrt{x^2 - a^2} \right) + C_1 \qquad (\text{where } C_1 = C - \ln a) \end{aligned}$$

Similarly, the inverse hyperbolic substitution $x = a \sinh u$ can be used instead of the inverse tangent substitution $x = a \tan \theta$ to simplify integrals involving $\sqrt{x^2 + a^2}$ or $\frac{1}{x^2 + a^2}$. In this case we have $dx = a \cosh u \, du$ and $x^2 + a^2 = a^2 \cosh^2 u$, and we may need the result

$$\sinh^{-1} x = \ln \left(x + \sqrt{x^2 + 1} \right)$$

valid for all x and proved in Section 3.6.

EXAMPLE 7 Evaluate $I = \int_0^4 \frac{dx}{(x^2 + 9)^{3/2}}$.

Solution We use the inverse substitution $x = 3 \sinh u$, so that $dx = 3 \cosh u \, du$ and $x^2 + 9 = 9 \cosh^2 u$. We have

$$\begin{aligned} I &= \int_{x=0}^{x=4} \frac{3 \cosh u}{27 \cosh^3 u} \, du = \frac{1}{9} \int_{x=0}^{x=4} \operatorname{sech}^2 u \, du = \frac{1}{9} \tanh u \Big|_{x=0}^{x=4} \\ &= \frac{1}{9} \frac{\sinh u}{\cosh u} \Big|_{x=0}^{x=4} = \frac{1}{9} \frac{x/3}{(\sqrt{x^2 + 9})/3} \Big|_0^4 = \frac{1}{9} \times \frac{4}{5} = \frac{4}{45}. \end{aligned}$$

Integrals involving $\sqrt{a^2 - x^2}$, where $|x| \leq a$, can be attempted with the aid of the inverse hyperbolic substitution $x = a \tanh u$, making use of the identity $1 - \tanh^2 u = \operatorname{sech}^2 u$. However, it is usually better to use the inverse sine substitution $x = a \sin \theta$ for such integrals. In general, it is better to avoid the inverse trigonometric substitutions unless you are very familiar with the identities satisfied by the hyperbolic functions as presented in Section 3.6.

Other Inverse Substitutions

Integrals involving $\sqrt{ax + b}$ can sometimes be made simpler with the substitution $ax + b = u^2$.

EXAMPLE 8

$$\int \frac{1}{1+\sqrt{2x}}\,dx \qquad \text{Let } 2x = u^2,\ 2\,dx = 2u\,du$$

$$= \int \frac{u}{1+u}\,du$$

$$= \int \frac{1+u-1}{1+u}\,du$$

$$= \int \left(1 - \frac{1}{1+u}\right) du \qquad \text{Let } v = 1+u,\ dv = du$$

$$= u - \int \frac{dv}{v} = u - \ln|v| + C$$

$$= \sqrt{2x} - \ln\left(1+\sqrt{2x}\right) + C$$

Sometimes integrals involving $\sqrt[n]{ax+b}$ will be much simplified by the hybrid substitution $ax+b = u^n$, $a\,dx = n\,u^{n-1}\,du$.

EXAMPLE 9

$$\int_{-1/3}^{2} \frac{x}{\sqrt[3]{3x+2}}\,dx \qquad \text{Let } 3x+2 = u^3,\ 3\,dx = 3u^2\,du$$

$$= \int_1^2 \frac{u^3-2}{3u}\,u^2\,du$$

$$= \frac{1}{3}\int_1^2 (u^4 - 2u)\,du = \frac{1}{3}\left(\frac{u^5}{5} - u^2\right)\bigg|_1^2 = \frac{16}{15}.$$

Note that the limits were changed in this definite integral: $u = 1$ when $x = -1/3$, and, coincidentally, $u = 2$ when $x = 2$.

If more than one fractional power is present, it may be possible to eliminate all of them at once.

EXAMPLE 10 Evaluate $\displaystyle\int \frac{1}{x^{1/2}(1+x^{1/3})}\,dx.$

Solution We can eliminate both the square root and the cube root by using the inverse substitution $x = u^6$. (The power 6 is chosen because 6 is the least common multiple of 2 and 3.)

$$\int \frac{dx}{x^{1/2}(1+x^{1/3})} \qquad \text{Let } x = u^6,\ dx = 6u^5\,du$$

$$= 6\int \frac{u^5\,du}{u^3(1+u^2)} = 6\int \frac{u^2}{1+u^2}\,du = 6\int \left(1 - \frac{1}{1+u^2}\right) du$$

$$= 6\,(u - \tan^{-1} u) + C = 6\,(x^{1/6} - \tan^{-1} x^{1/6}) + C.$$

The tan(θ/2) Substitution

There is a certain special substitution that can transform an integral whose integrand is a rational function of $\sin\theta$ and $\cos\theta$ (i.e., a quotient of polynomials in $\sin\theta$ and $\cos\theta$) into a rational function of x. The substitution is

$$x = \tan\frac{\theta}{2} \qquad \text{or, equivalently,} \qquad \theta = 2\tan^{-1} x.$$

Observe that

$$\cos^2\frac{\theta}{2} = \frac{1}{\sec^2\frac{\theta}{2}} = \frac{1}{1+\tan^2\frac{\theta}{2}} = \frac{1}{1+x^2},$$

so

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = \frac{2}{1+x^2} - 1 = \frac{1-x^2}{1+x^2}$$
$$\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2} = \frac{2x}{1+x^2}.$$

Also, $dx = \frac{1}{2}\sec^2\frac{\theta}{2}\,d\theta$, so

$$d\theta = 2\cos^2\frac{\theta}{2}\,dx = \frac{2\,dx}{1+x^2}.$$

In summary:

The tan(θ/2) substitution

If $x = \tan(\theta/2)$, then

$$\cos\theta = \frac{1-x^2}{1+x^2}, \qquad \sin\theta = \frac{2x}{1+x^2}, \qquad \text{and} \qquad d\theta = \frac{2\,dx}{1+x^2}.$$

Note that $\cos\theta$, $\sin\theta$, and $d\theta$ all involve only rational functions of x. We examined general techniques for integrating rational functions of x in Section 6.2.

EXAMPLE 11 $\displaystyle\int \frac{1}{2+\cos\theta}\,d\theta$

Let $x = \tan(\theta/2)$, so

$$\cos\theta = \frac{1-x^2}{1+x^2},$$
$$d\theta = \frac{2\,dx}{1+x^2}$$

$$= \int \frac{\frac{2\,dx}{1+x^2}}{2+\frac{1-x^2}{1+x^2}} = 2\int \frac{1}{3+x^2}\,dx$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\tan\frac{\theta}{2}\right) + C.$$

EXERCISES 6.3

Evaluate the integrals in Exercises 1–42.

1. $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$

2. $\displaystyle\int \frac{x^2\,dx}{\sqrt{1-4x^2}}$

3. $\displaystyle\int \frac{x^2\,dx}{\sqrt{9-x^2}}$

4. $\displaystyle\int \frac{dx}{x\sqrt{1-4x^2}}$

5. $\displaystyle\int \frac{dx}{x^2\sqrt{9-x^2}}$

6. $\displaystyle\int \frac{dx}{x\sqrt{9-x^2}}$

7. $\displaystyle\int \frac{x+1}{\sqrt{9-x^2}}\,dx$

8. $\displaystyle\int \frac{dx}{\sqrt{9+x^2}}$

9. $\displaystyle\int \frac{x^3\,dx}{\sqrt{9+x^2}}$

10. $\displaystyle\int \frac{\sqrt{9+x^2}}{x^4}\,dx$

11. $\displaystyle\int \frac{dx}{(a^2-x^2)^{3/2}}$

12. $\displaystyle\int \frac{dx}{(a^2+x^2)^{3/2}}$

13. $\displaystyle\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}}$

14. $\displaystyle\int \frac{dx}{(1+2x^2)^{5/2}}$

15. $\displaystyle\int \frac{dx}{x\sqrt{x^2-4}}, \quad (x>2)$

16. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}} \quad (x>a>0)$

17. $\displaystyle\int \frac{dx}{x^2+2x+10}$

18. $\displaystyle\int \frac{dx}{x^2+x+1}$

19. $\displaystyle\int \frac{dx}{(4x^2+4x+5)^2}$

20. $\displaystyle\int \frac{x\,dx}{x^2-2x+3}$

21. $\displaystyle\int \frac{x\,dx}{\sqrt{2ax-x^2}}$

22. $\displaystyle\int \frac{dx}{(4x-x^2)^{3/2}}$

23. $\displaystyle\int \frac{x\,dx}{(3-2x-x^2)^{3/2}}$

24. $\displaystyle\int \frac{dx}{(x^2+2x+2)^2}$

25. $\displaystyle\int \frac{dx}{(1+x^2)^3}$

26. $\displaystyle\int \frac{x^2\,dx}{(1+x^2)^2}$

27. $\displaystyle\int \frac{\sqrt{1-x^2}}{x^3}\,dx$

28. $\displaystyle\int \sqrt{9+x^2}\,dx$

29. $\displaystyle\int \frac{dx}{2+\sqrt{x}}$

30. $\displaystyle\int \frac{dx}{1+x^{1/3}}$

31. $\displaystyle\int \frac{1+x^{1/2}}{1+x^{1/3}}\,dx$

32. $\displaystyle\int \frac{x\sqrt{2-x^2}}{\sqrt{x^2+1}}\,dx$

33. $\displaystyle\int_{-\ln 2}^{0} e^x\sqrt{1-e^{2x}}\,dx$

34. $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx$

35. $\displaystyle\int_{-1}^{\sqrt{3}-1} \frac{dx}{x^2+2x+2}$

36. $\displaystyle\int_1^2 \frac{dx}{x^2\sqrt{9-x^2}}$

37. $\displaystyle\int \frac{t\,dt}{(t+1)(t^2+1)^2}$

38. $\displaystyle\int \frac{x\,dx}{(x^2-x+1)^2}$

39. $\displaystyle\int \frac{dx}{x(3+x^2)\sqrt{1-x^2}}$

40. $\displaystyle\int \frac{dx}{x^2(x^2-1)^{3/2}}$

41. $\displaystyle\int \frac{dx}{x(1+x^2)^{3/2}}$

42. $\displaystyle\int \frac{dx}{x(1-x^2)^{3/2}}$

In Exercises 43–45, evaluate the integral using the special substitution $x=\tan(\theta/2)$ as in Example 11.

43. $\displaystyle\int \frac{d\theta}{2+\sin\theta}$

44. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{1+\cos\theta+\sin\theta}$

45. $\displaystyle\int \frac{d\theta}{3+2\cos\theta}$

46. Find the area of the region bounded by $y=(2x-x^2)^{-1/2}$, $y=0$, $x=1/2$, and $x=1$.

47. Find the area of the region lying below $y=9/(x^4+4x^2+4)$ and above $y=1$.

48. Find the average value of the function $f(x)=(x^2-4x+8)^{-3/2}$ over the interval $[0,4]$.

49. Find the area inside the circle $x^2+y^2=a^2$ and above the line $y=b$, $(-a\le b\le a)$.

50. Find the area inside both of the circles $x^2+y^2=1$ and $(x-2)^2+y^2=4$.

51. Find the area in the first quadrant above the hyperbola $xy=12$ and inside the circle $x^2+y^2=25$.

52. Find the area to the left of $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and to the right of the line $x=c$, where $-a\le c\le a$.

53. Find the area of the region bounded by the x-axis, the hyperbola $x^2-y^2=1$, and the straight line from the origin to the point $\left(\sqrt{1+Y^2}, Y\right)$ on that hyperbola. (Assume $Y>0$.) In particular, show that the area is $t/2$ square units if $Y=\sinh t$.

54. Evaluate the integral $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-a^2}}$, for $x>a>0$, using the inverse hyperbolic cosine substitution $x=a\cosh u$.

6.4 Other Methods for Evaluating Integrals

Sections 5.6 and 6.1–6.3 explore some standard methods for evaluating both definite and indefinite integrals of functions belonging to several well-defined classes. There is another such method that is often used to solve certain kinds of differential equations but can also be helpful for evaluating integrals; after all, integrating $f(x)$ is equivalent to solving the DE $dy/dx=f(x)$. It goes by the name of the **Method of Undetermined Coefficients** or the **Method of Judicious Guessing**, and we will investigate it below.

Although anyone who uses calculus should be familiar with the basic techniques of integration, just as anyone who uses arithmetic should be familiar with the techniques of multiplication and division, technology is steadily eroding the necessity for being able to do long, complicated integrals by such methods. In fact, today there are several computer programs that can manipulate mathematical expressions symbolically (rather than just numerically) and that can carry out, with little or no assistance from us, the various algebraic steps and limit calculations that are required to calculate and simplify both derivatives and integrals. Much pain can be avoided and time saved by having the computer evaluate a complicated integral such as

$$\int \frac{1+x+x^2}{(x^4-1)(x^4-16)^2}\,dx$$

rather than doing it by hand using partial fractions. Even without the aid of a computer, we can use tables of standard integrals such as the ones in the back endpapers of this book to help us evaluate complicated integrals. Using computers or tables can nevertheless require that we perform some simplifications beforehand and can make demands on our ability to interpret the answers we get. We also examine some such situations in this section.

The Method of Undetermined Coefficients

The method consists of guessing a family of functions that may contain the integral, then using differentiation to select the member of the family with the derivative that matches the integrand. It should be stressed that both people and machines are able to calculate derivatives with fewer complications than are involved in calculating integrals.

The method of undetermined coefficients is not so much a method as a strategy, because the family might be chosen on little more than an informed guess. But other integration methods can involve guesswork too. There can be some guesswork, for example, in deciding which integration technique will work best. What technique is best can remain unclear even after considerable effort has been expended. For undetermined coefficients, matters are clear. If the wrong family is guessed, a contradiction quickly emerges. Moreover, because of its broad nature, it provides a general alternative to other integration techniques. Often the guess is easily made. For example, if the integrand belongs to a family that remains unchanged under differentiation, then a good first guess at the form of the antiderivative is that family. A few examples will illustrate the technique.

EXAMPLE 1 Evaluate $I = \int (x^2+x+1)\,e^x\,dx$ using the method of undetermined coefficients.

Solution Experience tells us that the derivative of a polynomial times an exponential is a different polynomial of the same degree times the exponential. Thus, we "guess" that

$$I = (a_0 + a_1 x + a_2 x^2)\,e^x + C.$$

We differentiate I and equate the result to the integrand to determine the actual values of the coeffieients a_0, a_1, and a_2.

$$\begin{aligned}\frac{dI}{dx} &= (a_1 + 2a_2 x)\,e^x + (a_0 + a_1 x + a_2 x^2)\,e^x \\ &= \big(a_2 x^2 + (a_1 + 2a_2)x + (a_0 + a_1)\big)e^x \\ &= (x^2 + x + 1)e^x,\end{aligned}$$

provided that $a_2 = 1$, $a_1 + 2a_2 = 1$, and $a_0 + a_1 = 1$. These equations imply that $a_2 = 1$, $a_1 = -1$, and $a_0 = 2$. Thus,

$$\int (x^2+x+1)e^x\,dx = I = (x^2 - x + 2)e^x + C.$$

EXAMPLE 2 Evaluate $y = \int x^3 \cos(3x)\,dx$ using the method of undetermined coefficients.

Solution The derivative of a sum of products of polynomials with sine or cosine functions is a sum of products of polynomials with sine or cosine functions. Thus, we try $y = P(x)\cos(3x) + Q(x)\sin(3x) + C$, where $P(x)$ and $Q(x)$ are polynomials of degrees m and n, respectively. The degrees m and n and the coefficients of the polynomials are determined by setting the derivative y' equal to the given integrand $x^3\cos(3x)$.

$$\begin{aligned} y' &= P'(x)\cos(3x) - 3P(x)\sin(3x) + Q'(x)\sin(3x) + 3Q'(x)\cos(3x) \\ &= x^3\cos 3x. \end{aligned}$$

Equating coefficients of like trigonometric functions, we find

$$P'(x) + 3Q(x) = x^3 \qquad \text{and} \qquad Q'(x) - 3P(x) = 0.$$

The second of these equations requires that $m = n - 1$. From the first we conclude that $n = 3$, which implies that $m = 2$. Thus, we let $P(x) = p_0 + p_1x + p_2x^2$ and $Q(x) = q_0 + q_1x + q_2x^2 + q_3x^3$ in these equations:

$$\begin{aligned} p_1 + 2p_2x + 3(q_0 + q_1x + q_2x^2 + q_3x^3) &= x^3 \\ q_1 + 2q_2x + 3q_3x^2 - 3(p_0 + p_1x + p_2x^2) &= 0. \end{aligned}$$

Comparison of coefficients with like powers yields:

$$\begin{array}{llll} p_1 + 3q_0 = 0 & 2p_2 + 3q_1 = 0 & 3q_2 = 0 & 3q_3 = 1 \\ q_1 - 3p_0 = 0 & 2q_2 - 3p_1 = 0 & 3q_3 - 3p_2 = 0, & \end{array}$$

which leads to $q_3 = 1/3$, $p_2 = 1/3$, $q_1 = -2/9$, and $p_0 = -2/27$, with $p_1 = q_0 = q_2 = 0$. Thus,

$$\int x^3\cos(3x)\,dx = y = \left(-\frac{2}{27} + \frac{x^2}{3}\right)\cos(3x) + \left(-\frac{2x}{9} + \frac{x^3}{3}\right)\sin(3x) + C.$$

EXAMPLE 3 Find the derivative of $f_{mn}(x) = x^m(\ln x)^n$ and use the result to suggest a trial formula for $I = \int x^3(\ln x)^2\,dx$. Thus, evaluate this integral.

Solution We have

$$f'_{mn}(x) = mx^{m-1}(\ln x)^n + nx^m(\ln x)^{n-1}\frac{1}{x} = mx^{m-1}(\ln x)^n + nx^{m-1}(\ln x)^{n-1}.$$

This suggests that we try

$$I = \int x^3(\ln x)^2\,dx = \int f_{32}(x)\,dx = Px^4(\ln x)^2 + Qx^4\ln x + Rx^4 + C$$

for constants P, Q, R, and C. Differentiating, we get

$$\frac{dI}{dx} = 4Px^3(\ln x)^2 + 2Px^3\ln x + 4Qx^3\ln x + Qx^3 + 4Rx^3 = x^3(\ln x)^2,$$

provided $4P = 1$, $2P + 4Q = 0$, and $Q + 4R = 0$. Thus, $P = 1/4$, $Q = -1/8$, and $R = 1/32$, and so

$$\int x^3(\ln x)^2\,dx = \frac{1}{4}x^4(\ln x)^2 - \frac{1}{8}x^4\ln x + \frac{1}{32}x^4 + C.$$

Remark These examples and most in the following exercises can also be done using integration by parts. Using undetermined coefficients does not replace other methods, but it does provide an alternative that gives insight into what types of functions will not work as guesses for the integral. This has implications for how computer algorithms can and cannot do antiderivatives. This issue is taken up in Exercise 21. Moreover, with access to a differentiation algorithm and a computer to manage details, this method can sometimes produce integrals more quickly and precisely than classical techniques alone.

Using Maple for Integration

Computer algebra systems are capable of evaluating both indefinite and definite integrals symbolically, as well as giving numerical approximations for those definite integrals that have numerical values. The following examples show how to use Maple to evaluate integrals.

We begin by calculating $\int 2^x\sqrt{1+4^x}\,dx$ and $\int_0^{\pi} 2^x\sqrt{1+4^x}\,dx$.

We use Maple's "int" command, specifying the function and the variable of integration:

```
> int(2^x*sqrt(1+4^x),x);
```

$$\frac{e^{(x\ln(2))}\sqrt{1+(e^{(x\ln(2))})^2}}{2\ln(2)}+\frac{\operatorname{arcsinh}(e^{(x\ln(2))})}{2\ln(2)}$$

If you don't like the inverse hyperbolic sine, you can convert it to a logarithm:

```
> convert(%,ln);
```

$$\frac{e^{(x\ln(2))}\sqrt{1+(e^{(x\ln(2))})^2}2\ln(2)+\ln\left(e^{(x\ln(2))}+\sqrt{1+(e^{(x\ln(2))})^2}\right)}{2\ln(2)}$$

The "%" there refers to the result of the previous calculation. Note how Maple prefers to use $e^{x\ln 2}$ in place of 2^x.

For the definite integral, you specify the interval of values of the variable of integration using two dots between the endpoints as follows:

```
> int(2^x*sqrt(1+4^x),x=0..Pi);
```

$$\frac{-\sqrt{2}-\ln(1+\sqrt{2})+2^{\pi}\sqrt{1+4^{\pi}}+\ln(2^{\pi}+\sqrt{1+4^{\pi}})}{2\ln(2)}$$

If you want a decimal approximation to this exact answer, you can ask Maple to evaluate the last result as a floating-point number:

```
> evalf(%);
```

$$56.955\,421\,55$$

Remark Maple defaults to giving 10 significant digits in its floating-point numbers unless you request a different precision by declaring a value for the variable "Digits":

```
> Digits := 20; evalf(Pi);
```

$$3.141\,592\,653\,589\,793\,238\,5$$

Suppose we ask Maple to do an integral that we know we can't do ourselves:

```
> int(exp(-x^2),x);
```

$$\frac{1}{2}\sqrt{\pi}\operatorname{erf}(x)$$

Maple expresses the answer in terms of the **error function** that is defined by

$$\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt.$$

But observe:

```
> Int(exp(-x^2),x=-infinity..infinity)
   = int(exp(-x^2), x=-infinity..infinity);
```

$$\int_{-\infty}^{\infty} e^{(-x^2)}\, dx = \sqrt{\pi}$$

Note the use of the *inert* Maple command "Int" on the left side to simply print the integral without any evaluation. The active command "int" performs the evaluation.

Computer algebra programs can be used to integrate symbolically many functions, but you may get some surprises when you use them, and you may have to do some of the work to get an answer useful in the context of the problem on which you are working. Such programs, and some of the more sophisticated scientific calculators, are able to evaluate definite integrals numerically to any desired degree of accuracy even if symbolic antiderivatives cannot be found. We will discuss techniques of numerical integration in Sections 6.6–6.8, but note here that Maple's `evalf(Int())` can always be used to get numerical values:

```
> evalf(Int(sin(cos(x)),x=0..1));
```

$$.738\,642\,998\,0$$

Using Integral Tables

You can get some help evaluating integrals by using an integral table, such as the one in the back endpapers of this book. Besides giving the values of the common elementary integrals that you likely remember while you are studying calculus, they also give many more complicated integrals, especially ones representing standard types that often arise in applications. Familiarize yourself with the main headings under which the integrals are classified. Using the tables usually means massaging your integral using simple substitutions until you get it into the form of one of the integrals in the table.

EXAMPLE 4 Use the table to evaluate $I = \displaystyle\int \frac{t^5}{\sqrt{3-2t^4}}\, dt$.

Solution This integral doesn't resemble any in the tables, but there are numerous integrals in the tables involving $\sqrt{a^2 - x^2}$. We can begin to put the integral into this form with the substitution $t^2 = u$, so that $2t\, dt = du$. Thus,

$$I = \frac{1}{2}\int \frac{u^2}{\sqrt{3-2u^2}}\, du.$$

This is not quite what we want yet; let us get rid of the 2 multiplying the u^2 under the square root. One way to do this is with the change of variable $\sqrt{2}u = x$, so that $du = dx/\sqrt{2}$:

$$I = \frac{1}{4\sqrt{2}}\int \frac{x^2}{\sqrt{3-x^2}}\, dx.$$

Now the denominator is of the form $\sqrt{a^2 - x^2}$ for $a = \sqrt{3}$. Looking through the part of the table (in the back endpapers) dealing with integrals involving $\sqrt{a^2 - x^2}$, we find the third one, which says that

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\, dx = -\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C.$$

Thus,

$$\begin{aligned} I &= \frac{1}{4\sqrt{2}}\left(-\frac{x}{2}\sqrt{3-x^2} + \frac{3}{2}\sin^{-1}\frac{x}{\sqrt{3}}\right) + C_1 \\ &= -\frac{t^2}{8}\sqrt{3-2t^4} + \frac{3}{8\sqrt{2}}\sin^{-1}\frac{\sqrt{2}\,t^2}{\sqrt{3}} + C_1. \end{aligned}$$

Many of the integrals in the table are reduction formulas. (An integral appears on both sides of the equation.) These can be iterated to simplify integrals as in some of the examples and exercises of Section 6.1.

EXAMPLE 5 Evaluate $I = \int_0^1 \frac{1}{(x^2+1)^3}\,dx$.

Solution The fourth integral in the table of Miscellaneous Algebraic Integrals says that if $n \neq 1$, then

$$\int \frac{dx}{(a^2 \pm x^2)^n} = \frac{1}{2a^2(n-1)}\left(\frac{x}{(a^2 \pm x^2)^{n-1}} + (2n-3)\int \frac{dx}{(a^2 \pm x^2)^{n-1}}\right).$$

Using $a = 1$ and the + signs, we have

$$\begin{aligned}\int_0^1 \frac{dx}{(1+x^2)^n} &= \frac{1}{2(n-1)}\left(\frac{x}{(1+x^2)^{n-1}}\bigg|_0^1 + (2n-3)\int_0^1 \frac{dx}{(1+x^2)^{n-1}}\right)\\ &= \frac{1}{2^n(n-1)} + \frac{2n-3}{2(n-1)}\int_0^1 \frac{dx}{(1+x^2)^{n-1}}.\end{aligned}$$

Thus, we have

$$\begin{aligned} I &= \frac{1}{16} + \frac{3}{4}\int_0^1 \frac{dx}{(1+x^2)^2}\\ &= \frac{1}{16} + \frac{3}{4}\left(\frac{1}{4} + \frac{1}{2}\int_0^1 \frac{dx}{1+x^2}\right)\\ &= \frac{1}{16} + \frac{3}{16} + \frac{3}{8}\tan^{-1}x\bigg|_0^1 = \frac{1}{4} + \frac{3\pi}{32}.\end{aligned}$$

Special Functions Arising from Integrals

The integrals

$$\int \frac{dx}{x} = \ln x + C \qquad \text{and} \int \frac{dx}{1+x^2} = \tan^{-1} x + C$$

both take algebraic functions to a function that is not produced by adding, subtracting, multiplying, or dividing. In the first case the integral expands the class of functions to include logarithms, and in the second case, trigonometric functions.

The functions we have dealt with so far have mostly come from a class called **Elementary Functions**, which consists of polynomials, logarithms, exponentials, trigonometric and hyperbolic functions, and their inverses, and also finite sums, differences, products, quotients, powers, and roots of such functions. The derivative of any differentiable elementary function is elementary, but an integral may or may not be elementary. This expands the class of functions to a wider class known, for historical reasons, as **Special Functions**. The subject of Special Functions is a large topic in applied mathematics. There are many standard special functions that are thoroughly studied and important for applications. For instance,

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x \sin t)\,dt$$

is a special function known as a **Bessel function** of the first kind of order zero. It is a solution of Bessel's equation (see Exercise 20), which is a differential equation. Traditionally, this function is introduced when series methods are used to solve differential equations (see Section 18.8), but it can be defined as a definite integral.

Another example is the **Error Function**, arising in the field of probability and statistics. It is encountered in connection with the integral of $f(x) = e^{-x^2}$, which does not have an elementary integral. If one did have such an integral, it would have to be of the form

$$\int e^{-x^2}\,dx = P(x)\,e^{-x^2}$$

for some polynomial P having finite degree. Such is not possible. See Exercise 21 below.

To deal with this situation we use the error function defined as

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt.$$

It follows that

$$\int e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}\operatorname{erf}(x) + C.$$

At first, this may seem like the integral is merely dressed up with a new name. In a way that is true, but it would be equally true for $\ln x$ or $\tan^{-1} x$ above if we knew nothing about them other than the integral definition. But we know more about $\ln x$, $\tan^{-1} x$, and $\operatorname{erf}(x)$ than simply that they are antiderivatives of simpler functions. Above all, we know that they are functions in their own right that are not algebraic in the case of the first two and not an elementary function in the latter case.

EXERCISES 6.4

In Exercises 1–4, use the method of undetermined coefficients to evaluate the given integrals.

1. $\int e^{3x} \sin(4x)\,dx$

2. $\int x\,e^{-x} \sin x\,dx$

3. $\int x^5\,e^{-x^2}\,dx$

4. $\int x^2(\ln x)^4\,dx$

5. Use Maple or another computer algebra program to check any of the integrals you have done in the exercises from Sections 5.6 and 6.1–6.3, as well as any of the integrals you have been unable to do.

6. Use Maple or another computer algebra program to evaluate the integral in the opening paragraph of this section.

7. Use Maple or another computer algebra program to re-evaluate the integral in Example 4.

8. Use Maple or another computer algebra program to re-evaluate the integral in Example 5.

Use the integral tables to help you find the integrals in Exercises 9–18.

9. $\int \frac{x^2}{\sqrt{x^2-2}}\,dx$

10. $\int \sqrt{(x^2+4)^3}\,dx$

11. $\int \frac{dt}{t^2\sqrt{3t^2+5}}$

12. $\int \frac{dt}{t\sqrt{3t-5}}$

13. $\int x^4(\ln x)^4\,dx$

14. $\int x^7 e^{x^2}\,dx$

15. $\int x\sqrt{2x-x^2}\,dx$

16. $\int \frac{\sqrt{2x-x^2}}{x^2}\,dx$

17. $\int \frac{dx}{(\sqrt{4x-x^2})^3}$

18. $\int \frac{dx}{(\sqrt{4x-x^2})^4}$

19. Use Maple or another computer algebra program to evaluate the integrals in Exercises 9–18.

20. Show that $y = J_0(x)$ satisfies the **Bessel equation of order zero**: $xy'' + y' + xy = 0$.

21. The Error Function erf(x)

(a) Express the integral $\int e^{-x^2}\,dx$ in terms of the Error Function.

(b) Given that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$ (which will be proved in Section 14.4), evaluate $\lim_{x\to\infty} \operatorname{erf}(x)$ and $\lim_{x\to-\infty} \operatorname{erf}(x)$.

(c) Show that $P(x)e^{-x^2}$ cannot be an antiderivative of $\operatorname{erf}(x)$ for any polynomial P.

(d) Use undetermined coefficients to evaluate $J = \int \operatorname{erf}(x)\,dx$.

6.5 Improper Integrals

Up to this point, we have considered definite integrals of the form

$$I = \int_a^b f(x)\,dx,$$

where the integrand f is *continuous* on the *closed, finite* interval $[a, b]$. Since such a function is necessarily *bounded*, the integral I is necessarily a finite number; for positive f it corresponds to the area of a **bounded region** of the plane, a region contained inside some disk of finite radius with centre at the origin. Such integrals are also called **proper integrals**. We are now going to generalize the definite integral to allow for two possibilities excluded in the situation described above:

(i) We may have $a = -\infty$ or $b = \infty$ or both.

(ii) f may be unbounded as x approaches a or b or both.

Integrals satisfying (i) are called **improper integrals of type I**; integrals satisfying (ii) are called **improper integrals of type II**. Either type of improper integral corresponds (for positive f) to the area of a region in the plane that "extends to infinity" in some direction and therefore is *unbounded*. As we will see, such integrals may or may not have finite values. The ideas involved are best introduced by examples.

Improper Integrals of Type I

EXAMPLE 1 Find the area of the region A lying under the curve $y = 1/x^2$ and above the x-axis to the right of $x = 1$. (See Figure 6.8(a).)

Solution We would like to calculate the area with an integral

$$A = \int_1^\infty \frac{dx}{x^2},$$

which is improper of type I, since its interval of integration is infinite. It is not immediately obvious whether the area is finite; the region has an infinitely long "spike" along the x-axis, but this spike becomes infinitely thin as x approaches ∞. In order to evaluate this improper integral, we interpret it as a limit of proper integrals over intervals $[1, R]$ as $R \to \infty$. (See Figure 6.8(b).)

$$A = \int_1^\infty \frac{dx}{x^2} = \lim_{R\to\infty} \int_1^R \frac{dx}{x^2} = \lim_{R\to\infty} \left(-\frac{1}{x}\right)\Bigg|_1^R$$
$$= \lim_{R\to\infty} \left(-\frac{1}{R} + 1\right) = 1$$

Since the limit exists (is finite), we say that the improper integral *converges*. The region has finite area $A = 1$ square unit.

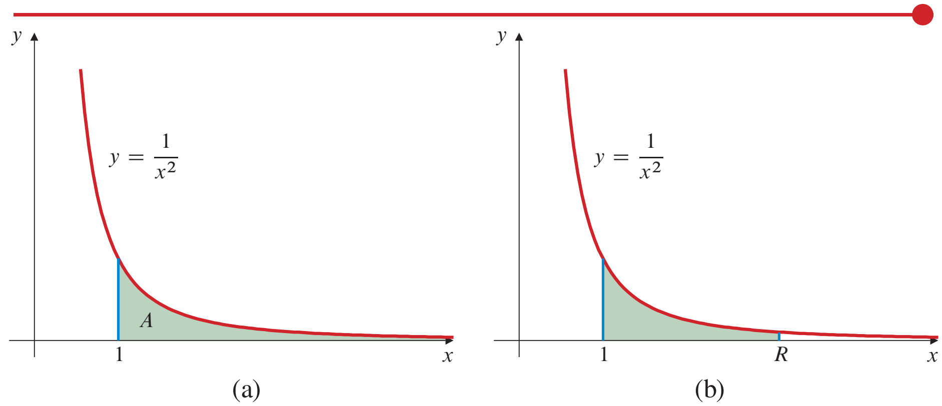

Figure 6.8

(a) $A = \int_1^\infty \frac{1}{x^2}\,dx$

(b) $A = \lim_{R\to\infty} \int_1^R \frac{1}{x^2}\,dx$

EXAMPLE 2 Find the area of the region under $y = 1/x$, above $y = 0$, and to the right of $x = 1$. (See Figure 6.9.)

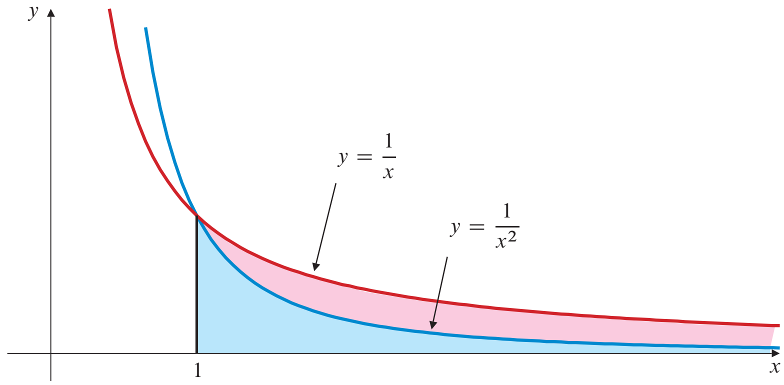

Figure 6.9 The area under the red curve is infinite. The area under the blue curve is finite.

Solution This area is given by the improper integral

$$A = \int_1^\infty \frac{dx}{x} = \lim_{R\to\infty} \int_1^R \frac{dx}{x} = \lim_{R\to\infty} \ln x \Big|_1^R = \lim_{R\to\infty} \ln R = \infty.$$

We say that this improper integral *diverges to infinity*. Observe that the region has a similar shape to the region under $y = 1/x^2$ considered in the above example, but its "spike" is somewhat thicker at each value of $x > 1$. Evidently, the extra thickness makes a big difference; this region has *infinite* area.

DEFINITION 1

Improper integrals of type I

If f is continuous on $[a, \infty)$, we define the improper integral of f over $[a, \infty)$ as a limit of proper integrals:

$$\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx.$$

Similarly, if f is continuous on $(-\infty, b]$, then we define

$$\int_{-\infty}^b f(x)\,dx = \lim_{R\to-\infty} \int_R^b f(x)\,dx.$$

In either case, if the limit exists (is a finite number), we say that the improper integral **converges**; if the limit does not exist, we say that the improper integral **diverges**. If the limit is ∞ (or $-\infty$), we say the improper integral **diverges to infinity** (or **diverges to negative infinity**).

The integral $\int_{-\infty}^\infty f(x)\,dx$ is, for f continuous on the real line, improper of type I at both endpoints. We break it into two separate integrals:

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^0 f(x)\,dx + \int_0^\infty f(x)\,dx.$$

The integral on the left converges if and only if *both* integrals on the right converge.

EXAMPLE 3 Evaluate $\displaystyle\int_{-\infty}^\infty \frac{1}{1+x^2}\,dx.$

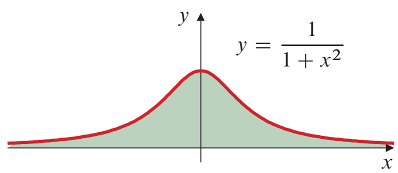

Figure 6.10

Solution By the (even) symmetry of the integrand (see Figure 6.10), we have

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{dx}{1+x^2} &= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2} \\ &= 2 \lim_{R\to\infty} \int_0^R \frac{dx}{1+x^2} \\ &= 2 \lim_{R\to\infty} \tan^{-1} R = 2\left(\frac{\pi}{2}\right) = \pi.\end{aligned}$$

The use of symmetry here requires some justification. At the time we used it we did not know whether each of the half-line integrals was finite or infinite. However, since both are positive, even if they are infinite, their sum would still be twice one of them. If one had been positive and the other negative, we would not have been justified in cancelling them to get 0 until we knew that they were finite. ($\infty + \infty = \infty$, but $\infty - \infty$ is not defined.) In any event, the given integral converges to π.

EXAMPLE 4 $\displaystyle\int_0^\infty \cos x\, dx = \lim_{R\to\infty} \int_0^R \cos x\, dx = \lim_{R\to\infty} \sin R.$

This limit does not exist (and it is not ∞ or $-\infty$), so all we can say is that the given integral diverges. (See Figure 6.11.) As R increases, the integral alternately adds and subtracts the areas of the hills and valleys but does not approach any unique limit.

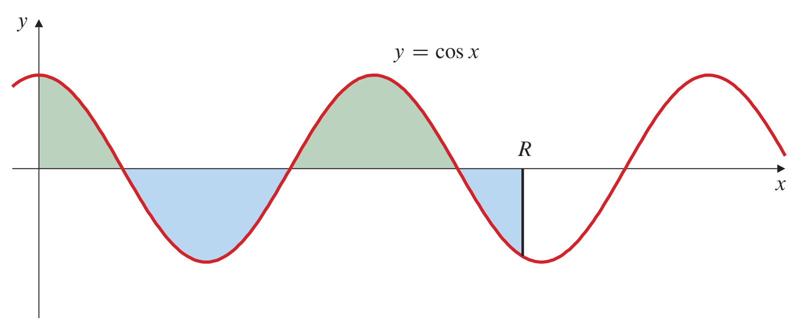

Figure 6.11 Not every divergent improper integral diverges to ∞ or $-\infty$

Improper Integrals of Type II

DEFINITION 2

Improper integrals of type II

If f is continuous on the interval $(a, b]$ and is possibly unbounded near a, we define the improper integral

$$\int_a^b f(x)\, dx = \lim_{c\to a+} \int_c^b f(x)\, dx.$$

Similarly, if f is continuous on $[a, b)$ and is possibly unbounded near b, we define

$$\int_a^b f(x)\, dx = \lim_{c\to b-} \int_a^c f(x)\, dx.$$

These improper integrals may converge, diverge, diverge to infinity, or diverge to negative infinity.

EXAMPLE 5 Find the area of the region S lying under $y = 1/\sqrt{x}$, above the x-axis, between $x = 0$ and $x = 1$.

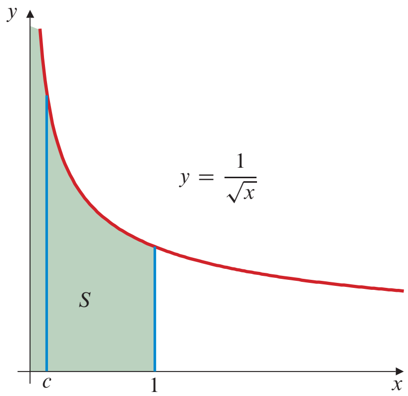

Figure 6.12 The shaded area is finite

Solution The area A is given by

$$A = \int_0^1 \frac{1}{\sqrt{x}}\,dx,$$

which is an improper integral of type II since the integrand is unbounded near $x = 0$. The region S has a "spike" extending to infinity along the y-axis, a vertical asymptote of the integrand, as shown in Figure 6.12. As we did for improper integrals of type I, we express such integrals as limits of proper integrals:

$$A = \lim_{c\to 0+} \int_c^1 x^{-1/2}\,dx = \lim_{c\to 0+} 2x^{1/2}\Big|_c^1 = \lim_{c\to 0+} (2 - 2\sqrt{c}) = 2.$$

This integral converges, and S has a finite area of 2 square units.

While improper integrals of type I are always easily recognized because of the infinite limits of integration, improper integrals of type II can be somewhat harder to spot. You should be alert for singularities of integrands and especially points where they have vertical asymptotes. It may be necessary to break an improper integral into several improper integrals if it is improper at both endpoints or at points inside the interval of integration. For example,

$$\int_{-1}^1 \frac{\ln|x|\,dx}{\sqrt{1-x}} = \int_{-1}^0 \frac{\ln|x|\,dx}{\sqrt{1-x}} + \int_0^{1/2} \frac{\ln|x|\,dx}{\sqrt{1-x}} + \int_{1/2}^1 \frac{\ln|x|\,dx}{\sqrt{1-x}}.$$

Each integral on the right is improper because of a singularity at one endpoint.

EXAMPLE 6 Evaluate each of the following integrals or show that it diverges:

$$\text{(a)}\ \int_0^1 \frac{1}{x}\,dx, \qquad \text{(b)}\ \int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx, \qquad \text{and} \qquad \text{(c)}\ \int_0^1 \ln x\,dx.$$

Solution

(a) $\displaystyle\int_0^1 \frac{1}{x}\,dx = \lim_{c\to 0+} \int_c^1 \frac{1}{x}\,dx = \lim_{c\to 0+} (\ln 1 - \ln c) = \infty.$

This integral diverges to infinity.

(b)
$$\begin{aligned}\int_0^2 \frac{1}{\sqrt{2x-x^2}}\,dx &= \int_0^2 \frac{1}{\sqrt{1-(x-1)^2}}\,dx && \text{Let } u = x-1,\ du = dx\\ &= \int_{-1}^1 \frac{1}{\sqrt{1-u^2}}\,du\\ &= 2\int_0^1 \frac{1}{\sqrt{1-u^2}}\,du && \text{(by symmetry)}\\ &= 2 \lim_{c\to 1-} \int_0^c \frac{1}{\sqrt{1-u^2}}\,du\\ &= 2 \lim_{c\to 1-} \sin^{-1} u\Big|_0^c = 2 \lim_{c\to 1-} \sin^{-1} c = \pi.\end{aligned}$$

This integral converges to π. Observe how a change of variable can be made even before an improper integral is expressed as a limit of proper integrals.

(c) $$\int_0^1 \ln x\,dx = \lim_{c\to 0+}\int_c^1 \ln x\,dx \qquad \text{(See Example 2(a) of Section 6.1.)}$$
$$= \lim_{c\to 0+}(x\ln x - x)\Big|_c^1$$
$$= \lim_{c\to 0+}(0 - 1 - c\ln c + c)$$
$$= -1 + 0 - \lim_{c\to 0+}\frac{\ln c}{1/c} \qquad \left[\frac{-\infty}{\infty}\right]$$
$$= -1 - \lim_{c\to 0+}\frac{1/c}{-(1/c^2)} \qquad \text{(by l'Hôpital's Rule)}$$
$$= -1 - \lim_{c\to 0+}(-c) = -1 + 0 = -1.$$

The integral converges to -1.

The following theorem summarizes the behaviour of improper integrals of types I and II for powers of x.

THEOREM 2

***p*-integrals**

If $0 < a < \infty$, then

(a) $$\int_a^\infty x^{-p}\,dx \quad \begin{cases} \text{converges to } \dfrac{a^{1-p}}{p-1} & \text{if } p > 1 \\ \text{diverges to } \infty & \text{if } p \le 1 \end{cases}$$

(b) $$\int_0^a x^{-p}\,dx \quad \begin{cases} \text{converges to } \dfrac{a^{1-p}}{1-p} & \text{if } p < 1 \\ \text{diverges to } \infty & \text{if } p \ge 1. \end{cases}$$

PROOF We prove part (b) only. The proof of part (a) is similar and is left as an exercise. Also, the case $p = 1$ of part (b) is similar to Example 6(a) above, so we need consider only the cases $p < 1$ and $p > 1$. If $p < 1$, then we have

$$\int_0^a x^{-p}\,dx = \lim_{c\to 0+}\int_c^a x^{-p}\,dx$$
$$= \lim_{c\to 0+}\frac{x^{-p+1}}{-p+1}\bigg|_c^a$$
$$= \lim_{c\to 0+}\frac{a^{1-p} - c^{1-p}}{1-p} = \frac{a^{1-p}}{1-p}$$

because $1 - p > 0$. If $p > 1$, then

$$\int_0^a x^{-p}\,dx = \lim_{c\to 0+}\int_c^a x^{-p}\,dx$$
$$= \lim_{c\to 0+}\frac{x^{-p+1}}{-p+1}\bigg|_c^a$$
$$= \lim_{c\to 0+}\frac{c^{-(p-1)} - a^{-(p-1)}}{p-1} = \infty.$$

The integrals in Theorem 2 are called ***p*-integrals**. It is very useful to know when they converge and diverge when you have to decide whether certain other improper integrals converge or not and you can't find the appropriate antiderivatives. (See the discussion of estimating convergence below.) Note that $\int_0^\infty x^{-p}\,dx$ does not converge for any value of p.

Remark If f is continuous on the interval $[a, b]$ so that $\int_a^b f(x)\,dx$ is a proper definite integral, then treating the integral as improper will lead to the same value:

$$\lim_{c\to a+}\int_c^b f(x)\,dx = \int_a^b f(x)\,dx = \lim_{c\to b-}\int_a^c f(x)\,dx.$$

This justifies the definition of the definite integral of a piecewise continuous function that was given in Section 5.4. To integrate a function defined to be different continuous functions on different intervals, we merely add the integrals of the various component functions over their respective intervals. Any of these integrals may be proper or improper; if any are improper, all must converge or the given integral will diverge.

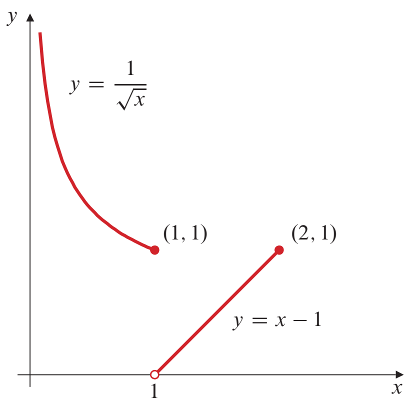

Figure 6.13 A discontinuous function

EXAMPLE 7 Evaluate $\int_0^2 f(x)\,dx$, where $f(x) = \begin{cases} 1/\sqrt{x} & \text{if } 0 < x \le 1 \\ x - 1 & \text{if } 1 < x \le 2. \end{cases}$

Solution The graph of f is shown in Figure 6.13. We have

$$\begin{aligned}\int_0^2 f(x)\,dx &= \int_0^1 \frac{dx}{\sqrt{x}} + \int_1^2 (x-1)\,dx \\ &= \lim_{c\to 0+}\int_c^1 \frac{dx}{\sqrt{x}} + \left(\frac{x^2}{2} - x\right)\bigg|_1^2 = 2 + \left(2 - 2 - \frac{1}{2} + 1\right) = \frac{5}{2};\end{aligned}$$

the first integral on the right is improper but convergent (see Example 5 above), and the second is proper.

Estimating Convergence and Divergence

When an improper integral cannot be evaluated by the Fundamental Theorem of Calculus because an antiderivative can't be found, we may still be able to determine whether the integral converges by comparing it with simpler integrals. The following theorem is central to this approach.

THEOREM 3

A comparison theorem for integrals

Let $-\infty \le a < b \le \infty$, and suppose that functions f and g are continuous on the interval (a, b) and satisfy $0 \le f(x) \le g(x)$. If $\int_a^b g(x)\,dx$ converges, then so does $\int_a^b f(x)\,dx$, and

$$\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx.$$

Equivalently, if $\int_a^b f(x)\,dx$ diverges to ∞, then so does $\int_a^b g(x)\,dx$.

PROOF Since both integrands are nonnegative, there are only two possibilities for each integral: it can either converge to a nonnegative number or diverge to ∞. Since $f(x) \le g(x)$ on (a, b), it follows by Theorem 3(e) of Section 5.4 that if $a < r < s < b$, then

$$\int_r^s f(x)\,dx \le \int_r^s g(x)\,dx.$$

This theorem now follows by taking limits as $r \to a+$ and $s \to b-$.

EXAMPLE 8 Show that $\int_0^\infty e^{-x^2}\,dx$ converges, and find an upper bound for its value.

Solution We can't integrate e^{-x^2}, but we can integrate e^{-x}. We would like to use the inequality $e^{-x^2} \le e^{-x}$, but this is only valid for $x \ge 1$. (See Figure 6.14.) Therefore, we break the integral into two parts.

On $[0, 1]$ we have $0 < e^{-x^2} \le 1$, so

$$0 < \int_0^1 e^{-x^2}\,dx \le \int_0^1 dx = 1.$$

On $[1, \infty)$ we have $x^2 \ge x$, so $-x^2 \le -x$ and $0 < e^{-x^2} \le e^{-x}$. Thus,

$$\begin{aligned} 0 < \int_1^\infty e^{-x^2}\,dx \le \int_1^\infty e^{-x}\,dx &= \lim_{R\to\infty} \left.\frac{e^{-x}}{-1}\right|_1^R \\ &= \lim_{R\to\infty}\left(\frac{1}{e} - \frac{1}{e^R}\right) = \frac{1}{e}. \end{aligned}$$

Hence, $\int_0^\infty e^{-x^2}\,dx$ converges and its value is less than $1 + (1/e)$.

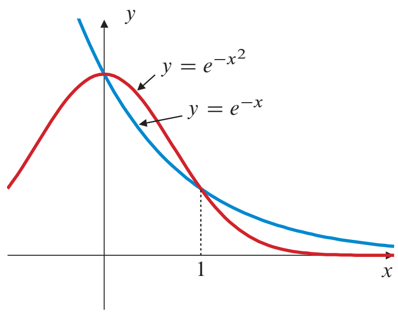

Figure 6.14 Comparing e^{-x^2} and e^{-x}

We remark that the above integral is, in fact, equal to $\frac{1}{2}\sqrt{\pi}$, although we cannot prove this now. See Section 14.4.

For large or small values of x many integrands behave like powers of x. If so, they can be compared with p-integrals.

EXAMPLE 9 Determine whether $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x + x^3}}$ converges.

Solution The integral is improper of both types, so we write

$$\int_0^\infty \frac{dx}{\sqrt{x+x^3}} = \int_0^1 \frac{dx}{\sqrt{x+x^3}} + \int_1^\infty \frac{dx}{\sqrt{x+x^3}} = I_1 + I_2.$$

On $(0, 1]$ we have $\sqrt{x + x^3} > \sqrt{x}$, so

$$I_1 < \int_0^1 \frac{dx}{\sqrt{x}} = 2 \qquad \text{(by Theorem 2).}$$

On $[1, \infty)$ we have $\sqrt{x + x^3} > \sqrt{x^3}$, so

$$I_2 < \int_1^\infty x^{-3/2}\,dx = 2 \qquad \text{(by Theorem 2).}$$

Hence, the given integral converges, and its value is less than 4.

In Section 4.10 we introduced big-O notation as a way of conveying growth-rate information in limit situations. We wrote $f(x) = O\big(g(x)\big)$ as $x \to a$ to mean the same thing as $|f(x)| \le K|g(x)|$ for some constant K on some open interval containing a. Similarly, we can say that $f(x) = O\big(g(x)\big)$ as $x \to \infty$ if for some constants a and K we have $|f(x)| \le K|g(x)|$ for all $x \ge a$.

EXAMPLE 10 $\displaystyle\frac{1+x^2}{1+x^4} = O\left(\frac{1}{x^2}\right)$ as $x \to \infty$ because, for $x \ge 1$ we have

$$\left|\frac{1+x^2}{1+x^4}\right| < \frac{2x^2}{x^4} = \frac{2}{x^2}.$$

EXAMPLE 11 Show that if $p > 1$ and f is continuous on $[1, \infty)$ and satisfies $f(x) = O(x^{-p})$, then $\int_1^\infty f(x)\,dx$ converges, and the error $E(R)$ in the approximation

$$\int_1^\infty f(x)\,dx \approx \int_1^R f(x)\,dx$$

satisfies $E(R) = O(R^{1-p})$ as $R \to \infty$.

Solution Since $f(x) = O(x^{-p})$ as $x \to \infty$, we have, for some $a \ge 1$ and some K, $f(x) \le K x^{-p}$ for all $x \ge a$. Thus,

$$\begin{aligned} |E(R)| &= \left| \int_R^\infty f(x)\,dx \right| \\ &\le K \int_R^\infty x^{-p}\,dx = K \left. \frac{x^{-p+1}}{-p+1} \right|_R^\infty = \frac{K}{p-1} R^{1-p}, \end{aligned}$$

so $E(R) = O(R^{1-p})$ as $R \to \infty$.

EXERCISES 6.5

In Exercises 1–22, evaluate the given integral or show that it diverges.

1. $\displaystyle\int_2^\infty \frac{1}{(x-1)^3}\,dx$
2. $\displaystyle\int_3^\infty \frac{1}{(2x-1)^{2/3}}\,dx$
3. $\displaystyle\int_0^\infty e^{-2x}\,dx$
4. $\displaystyle\int_{-\infty}^{-1} \frac{dx}{x^2+1}$
5. $\displaystyle\int_{-1}^1 \frac{dx}{(x+1)^{2/3}}$
6. $\displaystyle\int_0^a \frac{dx}{a^2-x^2}$
7. $\displaystyle\int_0^1 \frac{1}{(1-x)^{1/3}}\,dx$
8. $\displaystyle\int_0^1 \frac{1}{x\sqrt{1-x}}\,dx$
9. $\displaystyle\int_0^{\pi/2} \frac{\cos x\,dx}{(1-\sin x)^{2/3}}$
10. $\displaystyle\int_0^\infty x\,e^{-x}\,dx$
11. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$
12. $\displaystyle\int_0^\infty \frac{x}{1+2x^2}\,dx$
13. $\displaystyle\int_0^\infty \frac{x\,dx}{(1+2x^2)^{3/2}}$
14. $\displaystyle\int_0^{\pi/2} \sec x\,dx$
15. $\displaystyle\int_0^{\pi/2} \tan x\,dx$
16. $\displaystyle\int_e^\infty \frac{dx}{x\ln x}$
17. $\displaystyle\int_1^e \frac{dx}{x\sqrt{\ln x}}$
18. $\displaystyle\int_e^\infty \frac{dx}{x(\ln x)^2}$
19. $\displaystyle\int_{-\infty}^\infty \frac{x}{1+x^2}\,dx$
20. $\displaystyle\int_{-\infty}^\infty \frac{x}{1+x^4}\,dx$
21. $\displaystyle\int_{-\infty}^\infty x\,e^{-x^2}\,dx$
22. $\displaystyle\int_{-\infty}^\infty e^{-|x|}\,dx$
23. Find the area below $y = 0$, above $y = \ln x$, and to the right of $x = 0$.
24. Find the area below $y = e^{-x}$, above $y = e^{-2x}$, and to the right of $x = 0$.
25. Find the area of a region that lies above $y = 0$, to the right of $x = 1$, and under the curve $y = \dfrac{4}{2x+1} - \dfrac{2}{x+2}$.
26. Find the area of the plane region that lies under the graph of $y = x^{-2}e^{-1/x}$, above the x-axis, and to the right of the y-axis.
27. Prove Theorem 2(a) by directly evaluating the integrals involved.
28. Evaluate $\int_{-1}^1 (x \operatorname{sgn} x)/(x+2)\,dx$. Recall that $\operatorname{sgn} x = x/|x|$.
29. Evaluate $\int_0^2 x^2 \operatorname{sgn}(x-1)\,dx$.

In Exercises 30–41, state whether the given integral converges or diverges, and justify your claim.

30. $\displaystyle\int_0^\infty \frac{x^2}{x^5+1}\,dx$
31. $\displaystyle\int_0^\infty \frac{dx}{1+\sqrt{x}}$
32. $\displaystyle\int_2^\infty \frac{x\sqrt{x}\,dx}{x^2-1}$
33. $\displaystyle\int_0^\infty e^{-x^3}\,dx$
34. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}+x^2}$
35. $\displaystyle\int_{-1}^1 \frac{e^x}{x+1}\,dx$
36. $\displaystyle\int_0^\pi \frac{\sin x}{x}\,dx$
37. (!) $\displaystyle\int_0^\infty \frac{|\sin x|}{x^2}\,dx$
38. (!) $\displaystyle\int_0^{\pi^2} \frac{dx}{1-\cos\sqrt{x}}$
39. (!) $\displaystyle\int_{-\pi/2}^{\pi/2} \csc x\,dx$
40. (!) $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x}\ln x}$
41. (!) $\displaystyle\int_0^\infty \frac{dx}{xe^x}$

42. Given that $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$, evaluate

(a) $\displaystyle\int_0^\infty x^2 e^{-x^2}\,dx$ and (b) $\displaystyle\int_0^\infty x^4 e^{-x^2}\,dx$.

43. Suppose f is continuous on the interval $(0, 1]$ and satisfies $f(x) = O(x^p)$ as $x \to 0+$, where $p > -1$. Show that $\displaystyle\int_0^1 f(x)\,dx$ converges, and that if $0 < \epsilon < 1$, then the error $E(\epsilon)$ in the approximation

$$\int_0^1 f(x)\,dx \approx \int_\epsilon^1 f(x)\,dx$$

satisfies $E(\epsilon) = O(\epsilon^{p+1})$ as $\epsilon \to 0+$.

44. What is the largest value of k such that the error $E(\epsilon)$ in the approximation

$$\int_0^\infty \frac{dx}{\sqrt{x}+x^2} \approx \int_\epsilon^{1/\epsilon} \frac{dx}{\sqrt{x}+x^2},$$

where $0 < \epsilon < 1$, satisfies $E(\epsilon) = O(\epsilon^k)$ as $\epsilon \to 0+$.

45. If f is continuous on $[a, b]$, show that

$$\lim_{c\to a+} \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$$

Hint: A continuous function on a closed, finite interval is *bounded*: there exists a positive constant K such that $|f(x)| \le K$ for all x in $[a, b]$. Use this fact, together with parts (d) and (f) of Theorem 3 of Section 5.4, to show that

$$\lim_{c\to a+} \left(\int_a^b f(x)\,dx - \int_c^b f(x)\,dx \right) = 0.$$

Similarly, show that

$$\lim_{c\to b-} \int_a^c f(x)\,dx = \int_a^b f(x)\,dx.$$

46. **(The gamma function)** The gamma function $\Gamma(x)$ is defined by the improper integral

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt.$$

(Γ is the Greek capital letter gamma.)

(a) Show that the integral converges for $x > 0$.

(b) Use integration by parts to show that $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$.

(c) Show that $\Gamma(n+1) = n!$ for $n = 0, 1, 2, \ldots$.

(d) Given that $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$, show that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$ and $\Gamma(\frac{3}{2}) = \frac{1}{2}\sqrt{\pi}$.

In view of (c), $\Gamma(x+1)$ is often written $x!$ and regarded as a real-valued extension of the factorial function. Some scientific calculators (in particular, HP calculators) with the factorial function $n!$ built in actually calculate the gamma function rather than just the integral factorial. Check whether your calculator does this by asking it for 0.5!. If you get an error message, it's not using the gamma function.

6.6 The Trapezoid and Midpoint Rules

Most of the applications of integration, within and outside of mathematics, involve the definite integral

$$I = \int_a^b f(x)\,dx.$$

Thanks to the Fundamental Theorem of Calculus, we can evaluate such definite integrals by first finding an antiderivative of f. This is why we have spent considerable time developing techniques of integration. There are, however, two obstacles that can prevent our calculating I in this way:

(i) Finding an antiderivative of f in terms of familiar functions may be impossible, or at least very difficult.

(ii) We may not be given a formula for $f(x)$ as a function of x; for instance, $f(x)$ may be an unknown function whose values at certain points of the interval $[a, b]$ have been determined by experimental measurement.

In the next two sections we investigate the problem of approximating the value of the definite integral I using only the values of $f(x)$ at finitely many points of $[a, b]$. Obtaining such an approximation is called **numerical integration**. Upper and lower sums (or, indeed, any Riemann sum) can be used for this purpose, but these usually require much more calculation to yield a desired precision than the methods we will develop here. We will develop three methods for evaluating definite integrals numerically: the Trapezoid Rule, the Midpoint Rule, and Simpson's Rule (see Section 6.7). All of these methods can be easily implemented on a small computer or using a scientific calculator. The wide availability of these devices makes numerical integration a steadily more important tool for the user of mathematics. Some of the more advanced calculators have built-in routines for numerical integration.

All the techniques we consider require us to calculate the values of $f(x)$ at a set of equally spaced points in $[a, b]$. The computational "expense" involved in determining an approximate value for the integral I will be roughly proportional to the number of function values required, so that the fewer function evaluations needed to achieve a desired degree of accuracy for the integral, the better we will regard the technique. Time is money, even in the world of computers.

The Trapezoid Rule

We assume that $f(x)$ is continuous on $[a, b]$ and subdivide $[a, b]$ into n subintervals of equal length $h = (b - a)/n$ using the $n + 1$ points

$$x_0 = a, \quad x_1 = a + h, \quad x_2 = a + 2h, \quad \ldots, \quad x_n = a + nh = b.$$

We assume that the value of $f(x)$ at each of these points is known:

$$y_0 = f(x_0), \quad y_1 = f(x_1), \quad y_2 = f(x_2), \quad \ldots, \quad y_n = f(x_n).$$

The Trapezoid Rule approximates $\int_a^b f(x)\,dx$ by using straight line segments between the points (x_{j-1}, y_{j-1}) and (x_j, y_j), $(1 \le j \le n)$, to approximate the graph of f, as shown in Figure 6.15, and summing the areas of the resulting n *trapezoids*. A **trapezoid** is a four-sided polygon with one pair of parallel sides. (For our discussion we assume f is positive so we can talk about "areas," but the resulting formulas apply to any continuous function f.)

The first trapezoid has vertices $(x_0, 0)$, (x_0, y_0), (x_1, y_1), and $(x_1, 0)$. The two parallel sides are vertical and have lengths y_0 and y_1. The perpendicular distance between them is $h = x_1 - x_0$. The area of this trapezoid is h times the average of the parallel sides:

$$h\,\frac{y_0 + y_1}{2} \text{ square units.}$$

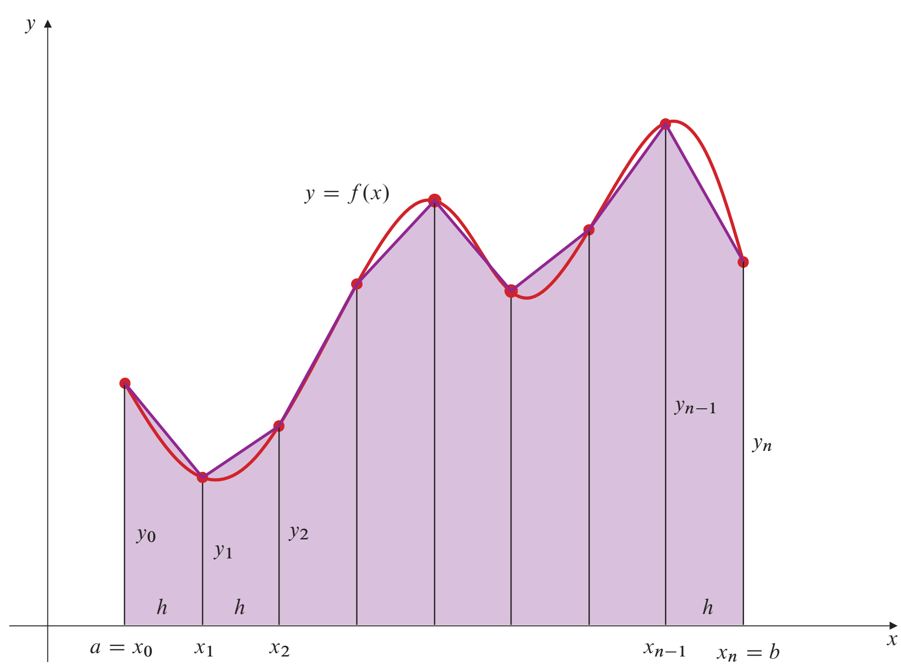

Figure 6.15 The area under $y = f(x)$ is approximated by the sum of the areas of n trapezoids

This can be seen geometrically by considering the trapezoid as the nonoverlapping union of a rectangle and a triangle; see Figure 6.16. We use this trapezoidal area to approximate the integral of f over the first subinterval $[x_0, x_1]$:

$$\int_{x_0}^{x_1} f(x)\,dx \approx h\,\frac{y_0 + y_1}{2}.$$

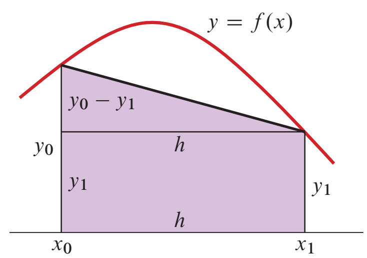

Figure 6.16 The trapezoid has area $y_1 h + \frac{1}{2}(y_0 - y_1)h = \frac{1}{2}h(y_0 + y_1)$

We can approximate the integral of f over any subinterval in the same way:

$$\int_{x_{j-1}}^{x_j} f(x)\,dx \approx h\,\frac{y_{j-1} + y_j}{2}, \qquad (1 \le j \le n).$$

It follows that the original integral I can be approximated by the sum of these trapezoidal areas:

$$\begin{aligned}\int_a^b f(x)\,dx &\approx h\left(\frac{y_0 + y_1}{2} + \frac{y_1 + y_2}{2} + \frac{y_2 + y_3}{2} + \cdots + \frac{y_{n-1} + y_n}{2}\right)\\ &= h\left(\frac{1}{2}\,y_0 + y_1 + y_2 + y_3 + \cdots + y_{n-1} + \frac{1}{2}\,y_n\right).\end{aligned}$$

DEFINITION 3

The Trapezoid Rule

The n-subinterval **Trapezoid Rule** approximation to $\int_a^b f(x)\,dx$, denoted T_n, is given by

$$T_n = h\left(\frac{1}{2}\,y_0 + y_1 + y_2 + y_3 + \cdots + y_{n-1} + \frac{1}{2}\,y_n\right).$$

We now illustrate the Trapezoid Rule by using it to approximate an integral whose value we already know:

$$I = \int_1^2 \frac{1}{x}\,dx = \ln 2 = 0.693\,147\,18\ldots.$$

(This value, and those of all the approximations quoted in these sections, were calculated using a scientific calculator.) We will use the same integral to illustrate other methods for approximating definite integrals later.

EXAMPLE 1 Calculate the Trapezoid Rule approximations T_4, T_8, and T_{16} for

$$I = \int_1^2 \frac{1}{x}\,dx.$$

Solution For $n = 4$ we have $h = (2-1)/4 = 1/4$; for $n = 8$ we have $h = 1/8$; for $n = 16$ we have $h = 1/16$. Therefore,

$$\begin{aligned}T_4 &= \frac{1}{4}\left[\frac{1}{2}(1) + \frac{4}{5} + \frac{2}{3} + \frac{4}{7} + \frac{1}{2}\left(\frac{1}{2}\right)\right] = 0.697\,023\,81\ldots\\ T_8 &= \frac{1}{8}\left[\frac{1}{2}(1) + \frac{8}{9} + \frac{4}{5} + \frac{8}{11} + \frac{2}{3} + \frac{8}{13} + \frac{4}{7} + \frac{8}{15} + \frac{1}{2}\left(\frac{1}{2}\right)\right]\\ &= \frac{1}{8}\left[4\,T_4 + \frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right] = 0.694\,121\,85\ldots\\ T_{16} &= \frac{1}{16}\left[8\,T_8 + \frac{16}{17} + \frac{16}{19} + \frac{16}{21} + \frac{16}{23} + \frac{16}{25} + \frac{16}{27} + \frac{16}{29} + \frac{16}{31}\right]\\ &= 0.693\,391\,20\ldots.\end{aligned}$$

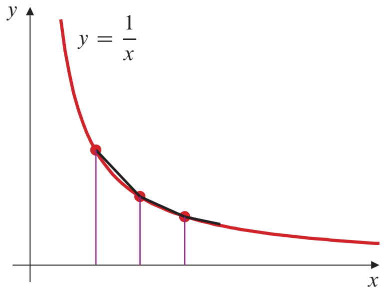

Figure 6.17 The trapezoid areas are greater than the area under the curve if the curve is concave upward

Note how the function values used to calculate T_4 were reused in the calculation of T_8, and similarly how those in T_8 were reused for T_{16}. When several approximations are needed, it is very useful to double the number of subintervals for each new calculation so that previously calculated values of f can be reused.

All Trapezoid Rule approximations to $I = \int_1^2 (1/x)\,dx$ are greater than the true value of I. This is because the graph of $y = 1/x$ is concave up on $[1, 2]$, and therefore the tops of the approximating trapezoids lie above the curve. (See Figure 6.17.)

We can calculate the exact errors in the three approximations since we know that $I = \ln 2 = 0.69314718\ldots$ (We always take the error in an approximation to be the true value minus the approximate value.)

$$\begin{aligned} I - T_4 &= 0.693\,147\,18\ldots - 0.697\,023\,81\ldots = -0.003\,876\,63\ldots \\ I - T_8 &= 0.693\,147\,18\ldots - 0.694\,121\,85\ldots = -0.000\,974\,67\ldots \\ I - T_{16} &= 0.693\,147\,18\ldots - 0.693\,391\,20\ldots = -0.000\,244\,02\ldots. \end{aligned}$$

Observe that the size of the error decreases to about a quarter of its previous value each time we double n. We will show below that this is to be expected for a "well-behaved" function like $1/x$.

Example 1 is somewhat artificial in the sense that we know the actual value of the integral so we really don't need an approximation. In practical applications of numerical integration we do not know the actual value. It is tempting to calculate several approximations for increasing values of n until the two most recent ones agree to within a prescribed error tolerance. For example, we might be inclined to claim that $\ln 2 \approx 0.69\ldots$ from a comparison of T_4 and T_8, and further comparison of T_{16} and T_8 suggests that the third decimal place is probably 3: $I \approx 0.693\ldots.$ Although this approach cannot be justified in general, it is frequently used in practice.

The Midpoint Rule

A somewhat simpler approximation to $\int_a^b f(x)\,dx$, based on the partition of $[a, b]$ into n equal subintervals, involves forming a Riemann sum of the areas of rectangles whose heights are taken at the midpoints of the n subintervals. (See Figure 6.18.)

DEFINITION 4

The Midpoint Rule

If $h = (b - a)/n$, let $m_j = a + \left(j - \frac{1}{2}\right) h$ for $1 \le j \le n$. The **Midpoint Rule** approximation to $\int_a^b f(x)\,dx$, denoted M_n, is given by

$$M_n = h\big(f(m_1) + f(m_2) + \cdots + f(m_n)\big) = h \sum_{j=1}^{n} f(m_j).$$

EXAMPLE 2 Find the Midpoint Rule approximations M_4 and M_8 for the integral $I = \displaystyle\int_1^2 \frac{1}{x}\,dx$, and compare their actual errors with those obtained for the Trapezoid Rule approximations above.

Solution To find M_4, the interval $[1, 2]$ is divided into four equal subintervals,

$$\left[1, \frac{5}{4}\right], \quad \left[\frac{5}{4}, \frac{3}{2}\right], \quad \left[\frac{3}{2}, \frac{7}{4}\right], \quad \text{and} \quad \left[\frac{7}{4}, 2\right].$$

The midpoints of these intervals are $9/8$, $11/8$, $13/8$, and $15/8$, respectively. The midpoints of the subintervals for M_8 are obtained in a similar way. The required Midpoint Rule approximations are

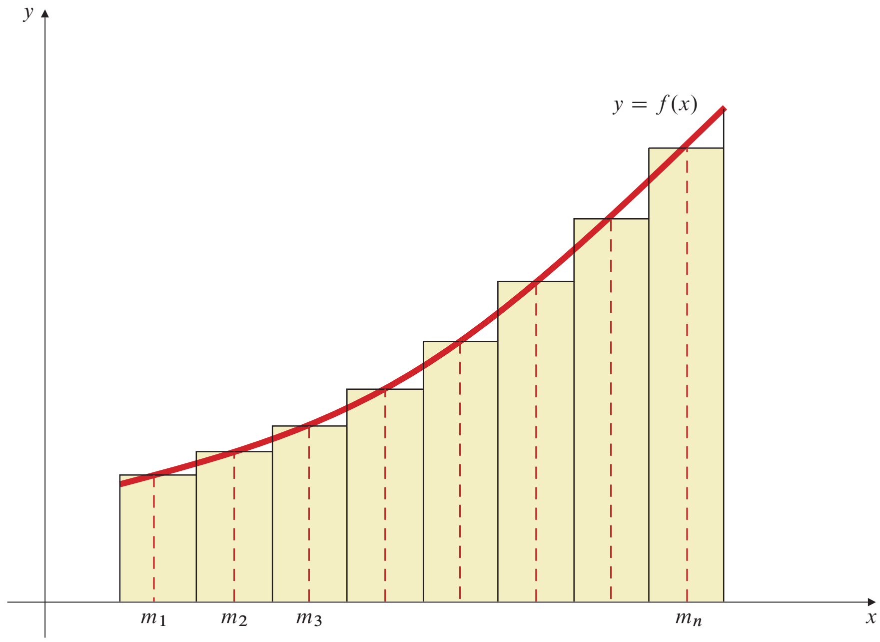

Figure 6.18 The Midpoint Rule approximation M_n to $\int_a^b f(x)\,dx$ is the Riemann sum based on the heights to the graph of f at the midpoints of the subintervals of the partition

$$M_4 = \frac{1}{4}\left[\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right] = 0.691\,219\,89\ldots$$

$$M_8 = \frac{1}{8}\left[\frac{16}{17} + \frac{16}{19} + \frac{16}{21} + \frac{16}{23} + \frac{16}{25} + \frac{16}{27} + \frac{16}{29} + \frac{16}{31}\right] = 0.692\,660\,55\ldots$$

The errors in these approximations are

$$I - M_4 = 0.693\,147\,18\ldots - 0.691\,219\,89\ldots = 0.001\,927\,29\ldots$$

$$I - M_8 = 0.693\,147\,18\ldots - 0.692\,660\,55\ldots = 0.000\,486\,63\ldots$$

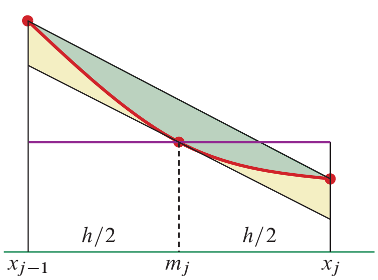

Figure 6.19 The Midpoint Rule error (the yellow area) is opposite in sign and about half the size of the Trapezoid Rule error (shaded in green)

These errors are of opposite sign and about *half the size* of the corresponding Trapezoid Rule errors $I - T_4$ and $I - T_8$. Figure 6.19 suggests the reason for this. The rectangular area $hf(m_j)$ is equal to the area of the trapezoid formed by the tangent line to $y = f(x)$ at $(m_j, f(m_j))$. The shaded region above the curve is the part of the Trapezoid Rule error due to the jth subinterval. The shaded area below the curve is the corresponding Midpoint Rule error.

One drawback of the Midpoint Rule is that we cannot reuse values of f calculated for M_n when we calculate M_{2n}. However, to calculate T_{2n} we can use the data values already calculated for T_n and M_n. Specifically,

$$T_{2n} = \tfrac{1}{2}(T_n + M_n).$$

A good strategy for using these methods to obtain a value for an integral I to a desired degree of accuracy is to calculate successively

$$T_n, \quad M_n, \quad T_{2n} = \frac{T_n + M_n}{2}, \quad M_{2n}, \quad T_{4n} = \frac{T_{2n} + M_{2n}}{2}, \quad M_{4n}, \quad \cdots$$

until two consecutive terms agree sufficiently closely. If a single quick approximation is needed, M_n is a better choice than T_n.

Error Estimates

The following theorem provides a bound for the error in the Trapezoid and Midpoint Rule approximations in terms of the second derivative of the integrand.

THEOREM 4

Error estimates for the Trapezoid and Midpoint Rules

If f has a continuous second derivative on $[a, b]$ and satisfies $|f''(x)| \le K$ there, then

$$\left| \int_a^b f(x)\,dx - T_n \right| \le \frac{K(b-a)}{12} h^2 = \frac{K(b-a)^3}{12n^2},$$
$$\left| \int_a^b f(x)\,dx - M_n \right| \le \frac{K(b-a)}{24} h^2 = \frac{K(b-a)^3}{24n^2},$$

where $h = (b-a)/n$. Note that these error bounds decrease like the square of the subinterval length as n increases.

PROOF We will prove only the Trapezoid Rule error estimate here. (The one for the Midpoint Rule is a little easier to prove; the method is suggested in Exercise 14 below.) The straight line approximating $y = f(x)$ in the first subinterval $[x_0, x_1] = [a, a+h]$ passes through the two points (x_0, y_0) and (x_1, y_1). Its equation is $y = A + B(x - x_0)$, where

$$A = y_0 \qquad \text{and} \qquad B = \frac{y_1 - y_0}{x_1 - x_0} = \frac{y_1 - y_0}{h}.$$

Let the function $g(x)$ be the vertical distance between the graph of f and this line:

$$g(x) = f(x) - A - B(x - x_0).$$

Since the integral of $A + B(x - x_0)$ over $[x_0, x_1]$ is the area of the first trapezoid, which is $h(y_0 + y_1)/2$ (see Figure 6.20), the integral of $g(x)$ over $[x_0, x_1]$ is the error in the approximation of $\int_{x_0}^{x_1} f(x)\,dx$ by the area of the trapezoid:

$$\int_{x_0}^{x_1} f(x)\,dx - h\frac{y_0 + y_1}{2} = \int_{x_0}^{x_1} g(x)\,dx.$$

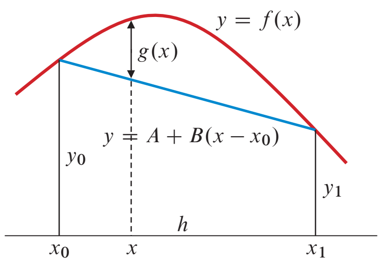

Figure 6.20 The error in approximating the area under the curve by that of the trapezoid is $\int_{x_0}^{x_1} g(x)\,dx$

Now g is twice differentiable, and $g''(x) = f''(x)$. Also $g(x_0) = g(x_1) = 0$. Two integrations by parts (see Exercise 36 of Section 6.1) show that

$$\int_{x_0}^{x_1} (x - x_0)(x_1 - x)\, f''(x)\,dx = \int_{x_0}^{x_1} (x - x_0)(x_1 - x)\, g''(x)\,dx$$
$$= -2\int_{x_0}^{x_1} g(x)\,dx.$$

By the triangle inequality for definite integrals (Theorem 3(f) of Section 5.4),

$$\left| \int_{x_0}^{x_1} f(x)\,dx - h\frac{y_0 + y_1}{2} \right| \le \frac{1}{2}\int_{x_0}^{x_1} (x - x_0)(x_1 - x)\,|f''(x)|\,dx$$
$$\le \frac{K}{2}\int_{x_0}^{x_1} \left(-x^2 + (x_0 + x_1)x - x_0x_1\right) dx$$
$$= \frac{K}{12}(x_1 - x_0)^3 = \frac{K}{12}h^3.$$

A similar estimate holds on each subinterval $[x_{j-1}, x_j]$ $(1 \le j \le n)$. Therefore,

$$\left| \int_a^b f(x)\,dx - T_n \right| = \left| \sum_{j=1}^{n} \left(\int_{x_{j-1}}^{x_j} f(x)\,dx - h\frac{y_{j-1} + y_j}{2} \right) \right|$$
$$\le \sum_{j=1}^{n} \left| \int_{x_{j-1}}^{x_j} f(x)\,dx - h\frac{y_{j-1} + y_j}{2} \right|$$
$$= \sum_{j=1}^{n} \frac{K}{12}h^3 = \frac{K}{12}nh^3 = \frac{K(b-a)}{12}h^2,$$

since $nh = b - a$. ■

We illustrate this error estimate for the approximations of Examples 1 and 2 above.

EXAMPLE 3 Obtain bounds for the errors for T_4, T_8, T_{16}, M_4, and M_8 for $I = \int_1^2 \frac{1}{x}\,dx$.

Solution If $f(x) = 1/x$, then $f'(x) = -1/x^2$ and $f''(x) = 2/x^3$. On $[1, 2]$ we have $|f''(x)| \le 2$, so we may take $K = 2$ in the estimate. Thus,

$$|I - T_4| \le \frac{2(2-1)}{12}\left(\frac{1}{4}\right)^2 = 0.0104\ldots,$$

$$|I - M_4| \le \frac{2(2-1)}{24}\left(\frac{1}{4}\right)^2 = 0.0052\ldots,$$

$$|I - T_8| \le \frac{2(2-1)}{12}\left(\frac{1}{8}\right)^2 = 0.0026\ldots,$$

$$|I - M_8| \le \frac{2(2-1)}{24}\left(\frac{1}{8}\right)^2 = 0.0013\ldots,$$

$$|I - T_{16}| \le \frac{2(2-1)}{12}\left(\frac{1}{16}\right)^2 = 0.00065\ldots.$$

The actual errors calculated earlier are considerably smaller than these bounds, because $|f''(x)|$ is rather smaller than $K = 2$ over most of the interval $[1, 2]$.

Remark Error bounds are not usually as easily obtained as they are in Example 3. In particular, if an exact formula for $f(x)$ is not known (as is usually the case if the values of f are obtained from experimental data), then we have no method of calculating $f''(x)$, so we can't determine K. Theorem 4 is of more theoretical than practical importance. It shows us that, for a "well-behaved" function f, the Midpoint Rule error is typically about half as large as the Trapezoid Rule error and that both the Trapezoid Rule and Midpoint Rule errors can be expected to decrease like $1/n^2$ as n increases; in terms of big-O notation,

$$I = T_n + O\left(\frac{1}{n^2}\right) \quad \text{and} \quad I = M_n + O\left(\frac{1}{n^2}\right) \qquad \text{as } n \to \infty.$$

Of course, actual errors are not equal to the error bounds, so they won't always be cut to exactly a quarter of their size when we double n.

EXERCISES 6.6

In Exercises 1–4, calculate the approximations T_4, M_4, T_8, M_8, and T_{16} for the given integrals. (Use a scientific calculator or computer spreadsheet program.) Also calculate the exact value of each integral, and so determine the exact error in each approximation. Compare these exact errors with the bounds for the size of the error supplied by Theorem 4.

1. $I = \int_0^2 (1 + x^2)\,dx$

2. $I = \int_0^1 e^{-x}\,dx$

3. $I = \int_0^{\pi/2} \sin x\,dx$

4. $I = \int_0^1 \frac{dx}{1 + x^2}$

5. Figure 6.21 shows the graph of a function f over the interval $[1, 9]$. Using values from the graph, find the Trapezoid Rule estimates T_4 and T_8 for $\int_1^9 f(x)\,dx$.

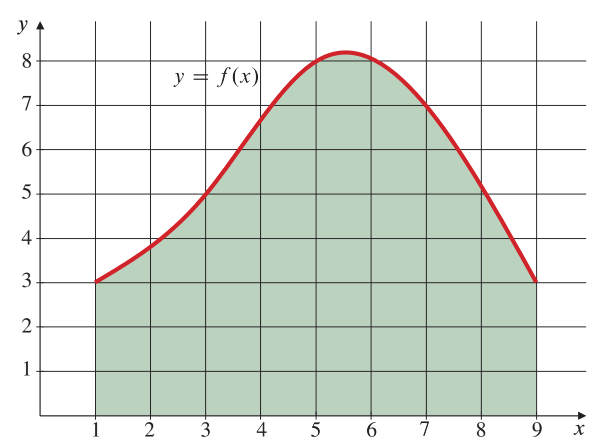

Figure 6.21

6. Obtain the best Midpoint Rule approximation that you can for $\int_1^9 f(x)\,dx$ from the data in Figure 6.21.

7. The map of a region is traced on the grid in Figure 6.22, where 1 unit in both the vertical and horizontal directions represents 10 km. Use the Trapezoid Rule to obtain two estimates for the area of the region.

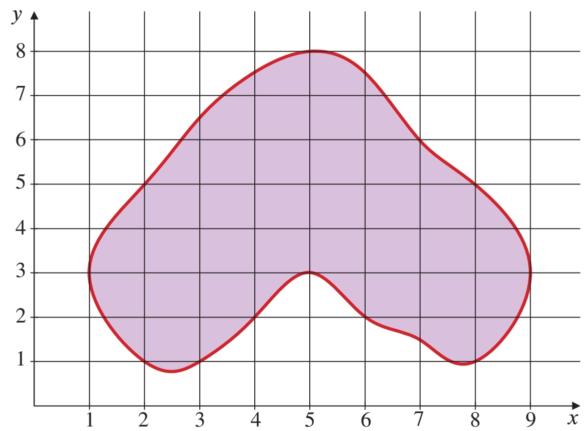

Figure 6.22

8. Find a Midpoint Rule estimate for the area of the region in Exercise 7.

9. Find T_4, M_4, T_8, M_8, and T_{16} for $\int_0^{1.6} f(x)\,dx$ for the function f whose values are given in Table 1.

Table 1.

x	$f(x)$	x	$f(x)$
0.0	1.4142	0.1	1.4124
0.2	1.4071	0.3	1.3983
0.4	1.3860	0.5	1.3702
0.6	1.3510	0.7	1.3285
0.8	1.3026	0.9	1.2734
1.0	1.2411	1.1	1.2057
1.2	1.1772	1.3	1.1258
1.4	1.0817	1.5	1.0348
1.6	0.9853		

10. Find the approximations M_8 and T_{16} for $\int_0^1 e^{-x^2}\,dx$. Quote a value for the integral to as many decimal places as you feel are justified.

11. Repeat Exercise 10 for $\int_0^{\pi/2} \frac{\sin x}{x}\,dx$. (Assume the integrand is 1 at $x = 0$.)

12. Compute the actual error in the approximation $\int_0^1 x^2\,dx \approx T_1$ and use it to show that the constant 12 in the estimate of Theorem 4 cannot be improved. That is, show that the absolute value of the actual error is as large as allowed by that estimate.

13. Repeat Exercise 12 for M_1.

14. Prove the error estimate for the Midpoint Rule in Theorem 4 as follows: If $x_1 - x_0 = h$ and m_1 is the midpoint of $[x_0, x_1]$, use the error estimate for the tangent line approximation (Theorem 11 of Section 4.9) to show that

$$|f(x) - f(m_1) - f'(m_1)(x - m_1)| \le \frac{K}{2}(x - m_1)^2.$$

Use this inequality to show that

$$\begin{aligned}
&\left|\int_{x_0}^{x_1} f(x)\,dx - f(m_1)h\right| \\
&= \left|\int_{x_0}^{x_1} \Big(f(x) - f(m_1) - f'(m_1)(x - m_1)\Big)dx\right| \\
&\le \frac{K}{24}h^3.
\end{aligned}$$

Complete the proof the same way used for the Trapezoid Rule estimate in Theorem 4.

6.7 Simpson's Rule

The Trapezoid Rule approximation to $\int_a^b f(x)\,dx$ results from approximating the graph of f by straight line segments through adjacent pairs of data points on the graph. Intuitively, we would expect to do better if we approximate the graph by more general curves. Since straight lines are the graphs of linear functions, the simplest obvious generalization is to use the class of quadratic functions, that is, to approximate the graph of f by segments of parabolas. This is the basis of Simpson's Rule.

Suppose that we are given three points in the plane, one on each of three equally spaced vertical lines, spaced, say, h units apart. If we choose the middle of these lines as the y-axis, then the coordinates of the three points will be, say, $(-h, y_L)$, $(0, y_M)$, and (h, y_R), as illustrated in Figure 6.23.

Constants A, B, and C can be chosen so that the parabola $y = A + Bx + Cx^2$ passes through these points; substituting the coordinates of the three points into the equation of the parabola, we get

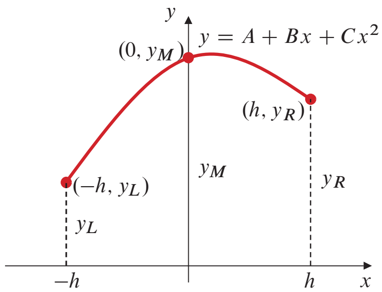

Figure 6.23 Fitting a quadratic graph through three points with equal horizontal spacing

$$\left.\begin{aligned} y_L &= A - Bh + Ch^2 \\ y_M &= A \\ y_R &= A + Bh + Ch^2 \end{aligned}\right\} \quad \Rightarrow \quad A = y_M \quad \text{and} \quad 2Ch^2 = y_L - 2y_M + y_R.$$

Now we have

$$\begin{aligned} \int_{-h}^{h} (A + Bx + Cx^2)\,dx &= \left(Ax + \frac{B}{2}x^2 + \frac{C}{3}x^3\right)\bigg|_{-h}^{h} = 2Ah + \frac{2}{3}Ch^3 \\ &= h\left(2y_M + \frac{1}{3}(y_L - 2y_M + y_R)\right) \\ &= \frac{h}{3}(y_L + 4y_M + y_R). \end{aligned}$$

Thus, the area of the plane region bounded by the parabolic arc, the interval of length $2h$ on the x-axis, and the left and right vertical lines is equal to $(h/3)$ times the sum of the heights of the region at the left and right edges and four times the height at the middle. (It is independent of the position of the y-axis.)

Now suppose that we are given the same data for f as we were given for the Trapezoid Rule; that is, we know the values $y_j = f(x_j)$ $(0 \le j \le n)$ at $n + 1$ equally spaced points

$$x_0 = a, \quad x_1 = a + h, \quad x_2 = a + 2h, \quad \ldots, \quad x_n = a + nh = b,$$

where $h = (b - a)/n$. We can approximate the graph of f over *pairs* of the subintervals $[x_{j-1}, x_j]$ using parabolic segments and use the integrals of the corresponding quadratic functions to approximate the integrals of f over these subintervals. Since we need to use the subintervals two at a time, we must assume that n is *even*. Using the integral computed for the parabolic segment above, we have

$$\begin{aligned} \int_{x_0}^{x_2} f(x)\,dx &\approx \frac{h}{3}(y_0 + 4y_1 + y_2) \\ \int_{x_2}^{x_4} f(x)\,dx &\approx \frac{h}{3}(y_2 + 4y_3 + y_4) \\ &\vdots \\ \int_{x_{n-2}}^{x_n} f(x)\,dx &\approx \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n). \end{aligned}$$

Adding these $n/2$ individual approximations, we get the Simpson's Rule approximation to the integral $\int_a^b f(x)\,dx$.

DEFINITION 5

Simpson's Rule

The **Simpson's Rule** approximation to $\int_a^b f(x)\,dx$ based on a subdivision of $[a, b]$ into an even number n of subintervals of equal length $h = (b - a)/n$ is denoted S_n and is given by:

$$\begin{aligned} \int_a^b f(x)\,dx &\approx S_n \\ &= \frac{h}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right) \\ &= \frac{h}{3}\left(\sum y_{\text{"ends"}} + 4\sum y_{\text{"odds"}} + 2\sum y_{\text{"evens"}}\right). \end{aligned}$$

Note that the Simpson's Rule approximation S_n requires no more data than does the Trapezoid Rule approximation T_n; both require the values of $f(x)$ at $n+1$ equally spaced points. However, Simpson's Rule treats the data differently, weighting successive values either $1/3$, $2/3$, or $4/3$. As we will see, this can produce a much better approximation to the integral of f.

EXAMPLE 1 Calculate the approximations S_4, S_8, and S_{16} for $I = \int_1^2 \frac{1}{x}\,dx$ and compare them with the actual value $I = \ln 2 = 0.693\,147\,18\ldots$, and with the values of T_4, T_8, and T_{16} obtained in Example 1 of Section 6.6.

Solution We calculate

$$S_4 = \frac{1}{12}\left[1 + 4\left(\frac{4}{5}\right) + 2\left(\frac{2}{3}\right) + 4\left(\frac{4}{7}\right) + \frac{1}{2}\right] = 0.693\,253\,97\ldots,$$

$$\begin{aligned} S_8 = \frac{1}{24}\bigg[1 + \frac{1}{2} &+ 4\left(\frac{8}{9} + \frac{8}{11} + \frac{8}{13} + \frac{8}{15}\right) \\ &+ 2\left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7}\right)\bigg] = 0.693\,154\,53\ldots, \end{aligned}$$

$$\begin{aligned} S_{16} = \frac{1}{48}\bigg[1 &+ \frac{1}{2} \\ &+ 4\left(\frac{16}{17} + \frac{16}{19} + \frac{16}{21} + \frac{16}{23} + \frac{16}{25} + \frac{16}{27} + \frac{16}{29} + \frac{16}{31}\right) \\ &+ 2\left(\frac{8}{9} + \frac{4}{5} + \frac{8}{11} + \frac{2}{3} + \frac{8}{13} + \frac{4}{7} + \frac{8}{15}\right)\bigg] = 0.693\,147\,65\ldots. \end{aligned}$$

The errors are

$$\begin{aligned} I - S_4 &= 0.693\,147\,18\ldots - 0.693\,253\,97\ldots = -0.000\,106\,79, \\ I - S_8 &= 0.693\,147\,18\ldots - 0.693\,154\,53\ldots = -0.000\,007\,35, \\ I - S_{16} &= 0.693\,147\,18\ldots - 0.693\,147\,65\ldots = -0.000\,000\,47. \end{aligned}$$

These errors are evidently much smaller than the corresponding errors for the Trapezoid or Midpoint Rule approximations.

Remark Simpson's Rule S_{2n} makes use of the same $2n+1$ data values that T_n and M_n together use. It is not difficult to verify that

$$S_{2n} = \frac{T_n + 2M_n}{3}, \qquad S_{2n} = \frac{2T_{2n} + M_n}{3}, \qquad \text{and} \qquad S_{2n} = \frac{4T_{2n} - T_n}{3}.$$

Figure 6.19 and Theorem 4 in Section 6.6 suggest why the first of these formulas ought to yield a particularly good approximation to I.

Obtaining an error estimate for Simpson's Rule is more difficult than for the Trapezoid Rule. We state the appropriate estimate in the following theorem, but we do not attempt any proof. Proofs can be found in textbooks on numerical analysis.

THEOREM 5

Error estimate for Simpson's Rule

If f has a continuous fourth derivative on the interval $[a, b]$, satisfying $|f^{(4)}(x)| \le K$ there, then

$$\left|\int_a^b f(x)\,dx - S_n\right| \le \frac{K(b-a)}{180}h^4 = \frac{K(b-a)^5}{180n^4},$$

where $h = (b-a)/n$.

Observe that, as n increases, the error decreases as the fourth power of h and, hence, as $1/n^4$. Using the big-O notation we have

$$\int_a^b f(x)\,dx = S_n + O\left(\frac{1}{n^4}\right) \qquad \text{as } n \to \infty.$$

This accounts for the fact that S_n is a much better approximation than is T_n, provided that h is small and $|f^{(4)}(x)|$ is not unduly large compared with $|f''(x)|$. Note also that for any (even) n, S_n gives the exact value of the integral of any *cubic* function $f(x) = A + Bx + Cx^2 + Dx^3$; $f^{(4)}(x) = 0$ identically for such f, so we can take $K = 0$ in the error estimate.

EXAMPLE 2 Obtain bounds for the absolute values of the errors in the approximations of Example 1.

Solution If $f(x) = 1/x$, then

$$f'(x) = -\frac{1}{x^2}, \qquad f''(x) = \frac{2}{x^3}, \qquad f^{(3)}(x) = -\frac{6}{x^4}, \qquad f^{(4)}(x) = \frac{24}{x^5}.$$

Clearly, $|f^{(4)}(x)| \le 24$ on $[1, 2]$, so we can take $K = 24$ in the estimate of Theorem 5. We have

$$|I - S_4| \le \frac{24(2-1)}{180}\left(\frac{1}{4}\right)^4 \approx 0.000\,520\,83,$$

$$|I - S_8| \le \frac{24(2-1)}{180}\left(\frac{1}{8}\right)^4 \approx 0.000\,032\,55,$$

$$|I - S_{16}| \le \frac{24(2-1)}{180}\left(\frac{1}{16}\right)^4 \approx 0.000\,002\,03.$$

Again we observe that the actual errors are well within these bounds.

EXAMPLE 3 A function f satisfies $|f^{(4)}(x)| \le 7$ on the interval $[1, 3]$, and the values $f(1.0) = 0.1860$, $f(1.5) = 0.9411$, $f(2.0) = 1.1550$, $f(2.5) = 1.4511$, and $f(3.0) = 1.2144$. Find the best possible Simpson's Rule approximation to $I = \int_1^3 f(x)\,dx$ based on these data. Give a bound for the size of the error, and specify the smallest interval you can that must contain the value of I.

Solution We take $n = 4$, so that $h = (3-1)/4 = 0.5$, and we obtain

$$\begin{aligned} I &= \int_1^3 f(x)\,dx \\ &\approx S_4 = \frac{0.5}{3}\big(0.1860 + 4(0.9411 + 1.4511) + 2(1.1550) + 1.2144\big) \\ &= 2.2132. \end{aligned}$$

Since $|f^{(4)}(x)| \le 7$ on $[1, 3]$, we have

$$|I - S_4| \le \frac{7(3-1)}{180}(0.5)^4 < 0.0049.$$

I must therefore satisfy

$$2.2132 - 0.0049 < I < 2.2132 + 0.0049 \quad \text{or} \quad 2.2083 < I < 2.2181.$$

EXERCISES 6.7

In Exercises 1–4, find Simpson's Rule approximations S_4 and S_8 for the given functions. Compare your results with the actual values of the integrals and with the corresponding Trapezoid Rule approximations obtained in Exercises 1–4 of Section 6.6.

1. $I = \int_0^2 (1 + x^2)\,dx$

2. $I = \int_0^1 e^{-x}\,dx$

3. $I = \int_0^{\pi/2} \sin x\,dx$

4. $I = \int_0^1 \frac{dx}{1 + x^2}$

5. Find the Simpson's Rule approximation S_8 for the integral in Exercise 5 of Section 6.6.

6. Find the best Simpson's Rule approximation that you can for the area of the region in Exercise 7 of Section 6.6.

7. Use Theorem 5 to obtain bounds for the errors in the approximations obtained in Exercises 2 and 3 above.

8. Verify that $S_{2n} = \dfrac{T_n + 2M_n}{3} = \dfrac{2T_{2n} + M_n}{3}$, where T_n and M_n refer to the appropriate Trapezoid and Midpoint Rule approximations. Deduce that $S_{2n} = \dfrac{4T_{2n} - T_n}{3}$.

9. Find S_4, S_8, and S_{16} for $\int_0^{1.6} f(x)\,dx$ for the function f whose values are tabulated in Exercise 9 of Section 6.6.

10. Find the Simpson's Rule approximations S_8 and S_{16} for $\int_0^1 e^{-x^2}\,dx$. Quote a value for the integral to the number of decimal places you feel is justified based on comparing the two approximations.

11. Compute the actual error in the approximation $\int_0^1 x^4\,dx \approx S_2$ and use it to show that the constant 180 in the estimate of Theorem 5 cannot be improved.

12. Since Simpson's Rule is based on quadratic approximation, it is not surprising that it should give an exact value for an integral of $A + Bx + Cx^2$. It is more surprising that it is exact for a cubic function as well. Verify by direct calculation that $\int_0^1 x^3\,dx = S_2$.

6.8 Other Aspects of Approximate Integration

The numerical methods described in Sections 6.6 and 6.7 are suitable for finding approximate values for integrals of the form

$$I = \int_a^b f(x)\,dx,$$

where $[a, b]$ is a finite interval and the integrand f is "well-behaved" on $[a, b]$. In particular, I must be a *proper* integral. There are many other methods for dealing with such integrals, some of which we mention later in this section. First, however, we consider what can be done if the function f isn't "well-behaved" on $[a, b]$. We mean by this that either the integral is improper or f doesn't have sufficiently many continuous derivatives on $[a, b]$ to justify whatever numerical methods we want to use.

The ideas of this section are best presented by means of concrete examples.

EXAMPLE 1 Describe how to would evaluate the integral $I = \int_0^1 \sqrt{x}\,e^x\,dx$ numerically?

Solution Although I is a proper integral, with integrand $f(x) = \sqrt{x}\,e^x$ satisfying $f(x) \to 0$ as $x \to 0+$, nevertheless, the standard numerical methods can be expected to perform poorly for I because the derivatives of f are not bounded near 0. This problem is easily remedied; just make the change of variable $x = t^2$ and rewrite I in the form

$$I = 2\int_0^1 t^2 e^{t^2}\,dt,$$

whose integrand $g(t) = t^2 e^{t^2}$ has bounded derivatives near 0. The latter integral can be efficiently approximated by the methods of Sections 6.6 and 6.7.

Approximating Improper Integrals

EXAMPLE 2 Describe how to evaluate $I = \int_0^1 \frac{\cos x}{\sqrt{x}}\,dx$ numerically.

Solution The integral is improper, but convergent because, on [0, 1],

$$0 < \frac{\cos x}{\sqrt{x}} \le \frac{1}{\sqrt{x}} \qquad \text{and} \qquad \int_0^1 \frac{dx}{\sqrt{x}} = 2.$$

However, since $\lim_{x\to 0+} \frac{\cos x}{\sqrt{x}} = \infty$, we cannot directly apply any of the techniques developed in Sections 6.6 and 6.7. (y_0 is infinite.) The substitution $x = t^2$ removes this difficulty:

$$I = \int_0^1 \frac{\cos t^2}{t}\,2t\,dt = 2\int_0^1 \cos t^2\,dt.$$

The latter integral is not improper and is well-behaved. Numerical techniques can be applied to evaluate it.

EXAMPLE 3 Show how to evaluate $I = \int_0^\infty \frac{dx}{\sqrt{2 + x^2 + x^4}}$ by numerical means.

Solution Here, the integral is improper of type I; the interval of integration is infinite. Although there is no singularity at $x = 0$, it is still useful to break the integral into two parts:

$$I = \int_0^1 \frac{dx}{\sqrt{2 + x^2 + x^4}} + \int_1^\infty \frac{dx}{\sqrt{2 + x^2 + x^4}} = I_1 + I_2.$$

I_1 is proper. In I_2 make the change of variable $x = 1/t$:

$$I_2 = \int_0^1 \frac{dt}{t^2\sqrt{2 + \frac{1}{t^2} + \frac{1}{t^4}}} = \int_0^1 \frac{dt}{\sqrt{2t^4 + t^2 + 1}}.$$

This is also a proper integral. If desired, I_1 and I_2 can be recombined into a single integral before numerical methods are applied:

$$I = \int_0^1 \left(\frac{1}{\sqrt{2 + x^2 + x^4}} + \frac{1}{\sqrt{2x^4 + x^2 + 1}}\right) dx.$$

Example 3 suggests that when an integral is taken over an infinite interval, a change of variable should be made to convert the integral to a finite interval.

Using Taylor's Formula

Taylor's formula (see Section 4.10) can sometimes be useful for evaluating integrals. Here is an example.

EXAMPLE 4 Use Taylor's formula for $f(x) = e^x$, obtained in Section 4.10, to evaluate the integral $\int_0^1 e^{x^2}\,dx$ to within an error of less than 10^{-4}.

Solution In Example 4 of Section 4.10 we showed that

$$f(x) = e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + E_n(x),$$

where

$$E_n(x) = \frac{e^X}{(n+1)!}\,x^{n+1}$$

for some X between 0 and x. If $0 \le x \le 1$, then $0 \le X \le 1$, so $e^X \le e < 3$. Therefore,

$$|E_n(x)| \le \frac{3}{(n+1)!}\,x^{n+1}.$$

Now replace x by x^2 in the formula for e^x above and integrate from 0 to 1:

$$\begin{aligned}\int_0^1 e^{x^2}\,dx &= \int_0^1 \left(1 + x^2 + \frac{x^4}{2!} + \cdots + \frac{x^{2n}}{n!}\right) dx + \int_0^1 E_n(x^2)\,dx \\ &= 1 + \frac{1}{3} + \frac{1}{5\times 2!} + \cdots + \frac{1}{(2n+1)n!} + \int_0^1 E_n(x^2)\,dx.\end{aligned}$$

We want the error to be less than 10^{-4}, so we estimate the remainder term:

$$\left|\int_0^1 E_n(x^2)\,dx\right| \le \frac{3}{(n+1)!}\int_0^1 x^{2(n+1)}\,dx = \frac{3}{(n+1)!(2n+3)} < 10^{-4},$$

provided $(2n+3)(n+1)! > 30{,}000$. Since $13 \times 6! = 9{,}360$ and $15 \times 7! = 75{,}600$, we need $n = 6$. Thus,

$$\begin{aligned}\int_0^1 e^{x^2}\,dx &\approx 1 + \frac{1}{3} + \frac{1}{5\times 2!} + \frac{1}{7\times 3!} + \frac{1}{9\times 4!} + \frac{1}{11\times 5!} + \frac{1}{13\times 6!} \\ &\approx 1.462\,64\end{aligned}$$

with error less than 10^{-4}.

Romberg Integration

Using Taylor's formula, it is possible to verify that for a function f having continuous derivatives up to order $2m+2$ on $[a,b]$ the error $E_n = I - T_n$ in the Trapezoid Rule approximation T_n to $I = \int_a^b f(x)\,dx$ satisfies

$$E_n = I - T_n = \frac{C_1}{n^2} + \frac{C_2}{n^4} + \frac{C_3}{n^6} + \cdots + \frac{C_m}{n^{2m}} + O\left(\frac{1}{n^{2m+2}}\right),$$

where the constants C_j depend on the $2j$th derivative of f. It is possible to use this formula to obtain higher-order approximations to I, starting with Trapezoid Rule approximations. The technique is known as **Romberg integration** or **Richardson extrapolation**.

To begin, suppose we have constructed Trapezoid Rule approximations for values of n that are powers of 2: $n = 1, 2, 4, 8, \ldots$. Accordingly, let us define

$$T_k^0 = T_{2^k}. \qquad \text{Thus,} \quad T_0^0 = T_1, \quad T_1^0 = T_2, \quad T_2^0 = T_4, \quad \ldots.$$

Using the formula for $T_{2^k} = I - E_{2^k}$ given above, we write

$$T_k^0 = I - \frac{C_1}{4^k} - \frac{C_2}{4^{2k}} - \cdots - \frac{C_m}{4^{mk}} + O\left(\frac{1}{4^{(m+1)k}}\right) \quad (\text{as } k \to \infty).$$

Similarly, replacing k by $k+1$, we get

$$T_{k+1}^0 = I - \frac{C_1}{4^{k+1}} - \frac{C_2}{4^{2(k+1)}} - \cdots - \frac{C_m}{4^{m(k+1)}} + O\left(\frac{1}{4^{(m+1)(k+1)}}\right).$$

If we multiply the formula for T_{k+1}^0 by 4 and subtract the formula for T_k^0, the terms involving C_1 will cancel out. The first term on the right will be $4I - I = 3I$, so let us also divide by 3 and define T_{k+1}^1 to be the result. Then as $k \to \infty$, we have

$$T_{k+1}^1 = \frac{4T_{k+1}^0 - T_k^0}{3} = I - \frac{C_2^1}{4^{2k}} - \frac{C_3^1}{4^{3k}} - \cdots - \frac{C_m^1}{4^{mk}} + O\left(\frac{1}{4^{(m+1)k}}\right).$$

(The C_i^1 are new constants.) Unless these constants are much larger than the previous ones, T_{k+1}^1 ought to be a better approximation to I than T_{k+1}^0 since we have eliminated the lowest order (and therefore the largest) of the error terms, $C_1/4^{k+1}$. In fact, Exercise 8 in Section 6.7 shows that $T_{k+1}^1 = S_{2^{k+1}}$, the Simpson's Rule approximation based on 2^{k+1} subintervals.

We can continue the process of eliminating error terms begun above. Replacing $k+1$ by $k+2$ in the expression for T_{k+1}^1, we obtain

$$T_{k+2}^1 = I - \frac{C_2^1}{4^{2(k+1)}} - \frac{C_3^1}{4^{3(k+1)}} - \cdots - \frac{C_m^1}{4^{m(k+1)}} + O\left(\frac{1}{4^{(m+1)(k+1)}}\right).$$

To eliminate C_2^1 we can multiply the second formula by 16, subtract the first formula, and divide by 15. Denoting the result T_{k+2}^2, we have, as $k \to \infty$,

$$T_{k+2}^2 = \frac{16T_{k+2}^1 - T_{k+1}^1}{15} = I - \frac{C_3^2}{4^{3k}} - \cdots - \frac{C_m^2}{4^{mk}} + O\left(\frac{1}{4^{(m+1)k}}\right).$$

We can proceed in this way, eliminating one error term after another. In general, for $j < m$ and $k \geq 0$,

$$T_{k+j}^j = \frac{4^j T_{k+j}^{j-1} - T_{k+j-1}^{j-1}}{4^j - 1} = I - \frac{C_{j+1}^j}{4^{(j+1)k}} - \cdots - \frac{C_m^j}{4^{mk}} + O\left(\frac{1}{4^{(m+1)k}}\right).$$

The big-O term refers to $k \to \infty$ for fixed j. All this looks very complicated, but it is not difficult to carry out in practice, especially with the aid of a computer spreadsheet. Let $R_j = T_j^j$, called a **Romberg approximation** to I, and calculate the entries in the following scheme in order from left to right and down each column when you come to it:

Scheme for calculating Romberg approximations

$$\begin{array}{ccccccc}
T_0^0 = T_1 = R_0 & \longrightarrow & T_1^0 = T_2 & \longrightarrow & T_2^0 = T_4 & \longrightarrow & T_3^0 = T_8 & \longrightarrow \\
 & & \downarrow & & \downarrow & & \downarrow & \\
 & & T_1^1 = S_2 = R_1 & & T_2^1 = S_4 & & T_3^1 = S_8 & \\
 & & & & \downarrow & & \downarrow & \\
 & & & & T_2^2 = R_2 & & T_3^2 & \\
 & & & & & & \downarrow & \\
 & & & & & & T_3^3 = R_3 &
\end{array}$$

Stop when T_j^{j-1} and R_j differ by less than the acceptable error, and quote R_j as the Romberg approximation to $\int_a^b f(x)\,dx$.

The top line in the scheme is made up of the Trapezoid Rule approximations T_1, T_2, T_4, T_8, ... Elements in subsequent rows are calculated by the following formulas:

Formulas for calculating Romberg approximations

$$T_1^1 = \frac{4T_1^0 - T_0^0}{3} \qquad T_2^1 = \frac{4T_2^0 - T_1^0}{3} \qquad T_3^1 = \frac{4T_3^0 - T_2^0}{3} \quad \cdots$$

$$T_2^2 = \frac{16T_2^1 - T_1^1}{15} \qquad T_3^2 = \frac{16T_3^1 - T_2^1}{15} \quad \cdots$$

$$T_3^3 = \frac{64T_3^2 - T_2^2}{63} \quad \cdots$$

In general, if $1 \le j \le k$, then $T_k^j = \dfrac{4^j T_k^{j-1} - T_{k-1}^{j-1}}{4^j - 1}$.

Each new entry is calculated from the one above and the one to the left of that one.

EXAMPLE 5 Calculate the Romberg approximations R_0, R_1, R_2, R_3, and R_4 for the integral $I = \displaystyle\int_1^2 \frac{1}{x}\,dx$.

Solution We will carry all calculations to 8 decimal places. Since we must obtain R_4, we will need to find all the entries in the first five columns of the scheme. First we calculate the first two Trapezoid Rule approximations:

$$R_0 = T_0^0 = T_1 = \frac{1}{2} + \frac{1}{4} = 0.750\,000\,00,$$

$$T_1^0 = T_2 = \frac{1}{2}\left[\frac{1}{2}(1) + \frac{2}{3} + \frac{1}{2}\left(\frac{1}{2}\right)\right] = 0.708\,333\,33.$$

The remaining required Trapezoid Rule approximations were calculated in Example 1 of Section 6.6, so we will just record them here:

$$T_2^0 = T_4 = 0.697\,023\,81,$$
$$T_3^0 = T_8 = 0.694\,121\,85,$$
$$T_4^0 = T_{16} = 0.693\,391\,20.$$

Now we calculate down the columns from left to right. For the second column:

$$R_1 = S_2 = T_1^1 = \frac{4T_1^0 - T_0^0}{3} = 0.694\,444\,44;$$

the third column:

$$S_4 = T_2^1 = \frac{4T_2^0 - T_1^0}{3} = 0.693\,253\,97,$$

$$R_2 = T_2^2 = \frac{16T_2^1 - T_1^1}{15} = 0.693\,174\,60;$$

the fourth column:

$$S_8 = T_3^1 = \frac{4T_3^0 - T_2^0}{3} = 0.693\,154\,53,$$

$$T_3^2 = \frac{16T_3^1 - T_2^1}{15} = 0.693\,147\,90,$$

$$R_3 = T_3^3 = \frac{64T_3^2 - T_2^2}{63} = 0.693\,147\,48;$$

and the fifth column:

$$S_{16} = T_4^1 = \frac{4T_4^0 - T_3^0}{3} = 0.693\,147\,65,$$

$$T_4^2 = \frac{16T_4^1 - T_3^1}{15} = 0.693\,147\,19,$$

$$T_4^3 = \frac{64T_4^2 - T_3^2}{63} = 0.693\,147\,18,$$

$$R_4 = T_4^4 = \frac{256T_4^3 - T_3^3}{255} = 0.693\,147\,18.$$

Since T_4^3 and R_4 agree to the 8 decimal places we are calculating, we expect that

$$I = \int_1^2 \frac{dx}{x} = \ln 2 \approx 0.693\,147\,18\ldots.$$

The various approximations calculated above suggest that for any given value of $n = 2^k$, the Romberg approximation R_n should give the best value obtainable for the integral based on the $n + 1$ data values $y_0, y_1, \ldots, y_n$. This is so only if the derivatives $f^{(n)}(x)$ do not grow too rapidly as n increases.

The Importance of Higher-Order Methods

Higher-order methods, such as Romberg, remove lower-order error by manipulating series. Removing lower-order error is of enormous importance for computation. Without it, even simple computations would be impossible for all practical purposes. For example, consider again the integral $I = \displaystyle\int_1^2 \frac{1}{x}\,dx$.

We can use Maple to compute this integral numerically to 16 digits (classical double precision),

```
> Digits=16:
> int(1/x, x = 1 .. 2.);
```

0.6931471805599453

Comparison with ln 2

```
> ln(2.);
```

0.6931471805599453

confirms the consistency of this calculation. Furthermore, we can compute the processor time for this calculation

```
> time(int(1/x, x = 1 .. 2.));
```

0.033

which indicates that, on the system used, 16 digits of accuracy is produced in hundredths of seconds of processor time.

Now let's consider what happens without removing lower-order error. If we were to estimate this integral using a simple end point Riemann sum, as we used in the original definition of a definite integral, the error is $O(h)$ or $O(1/n)$. Let the step size be 10^{-7}.

```
> 1e-7*add(1/(1+i/1e7), i = 1 .. 1e7);
```

0.6931471555599459

which has an error of 2.5×10^{-8}. The processor time used to do this sum computation is given by

```
> time(1e-7*add(1/(1+i/1e7), i = 1 .. 1e7));
                          175.777
```

that is, 175.577 seconds on the particular computer we used. (If you do the calculation on your machine your result will vary according to the speed of your system.) Note that we used the Maple "`add`" routine rather than "`sum`" in the calculations above. This was done to tell Maple to add the floating-point values of the terms one after another rather than to attempt a symbolic summation.

Because the computation time is proportional to the number n of rectangles used in the Riemann sum, and because the error is proportional to $1/n$, it follows that error times computation time is roughly constant. We can use this to estimate the time to compute the integral by this method to 16 digits of precision. Assuming an error of 10^{-16}, the time for the computation will be

$$175.777 \times 2.5 \times \frac{10^{-8}}{10^{-16}} \quad \text{seconds,}$$

or about 1,400 years.

Maple is not limited to 16 digits, of course. For each additional digit of precision, the Riemann sum method corresponds to a factor-of-ten increase in time because of lower-order error. The ability to compute such quantities is a powerful and important application of series expansions.

Other Methods

As developed above, the Trapezoid, Midpoint, Simpson, and Romberg methods all involved using equal subdivisions of the interval $[a, b]$. There are other methods that avoid this restriction. In particular, **Gaussian approximations** involve selecting evaluation points and weights in an optimal way so as to give the most accurate results for "well-behaved" functions. See Exercises 11–13 below. You can consult a text on numerical analysis to learn more about this method.

Finally, we note that even when you apply one of the methods of Sections 6.6 and 6.7, it may be advisable for you to break up the integral into two or more integrals over smaller intervals and then use different subinterval lengths h for each of the different integrals. You will want to evaluate the integrand at more points in an interval where its graph is changing direction erratically than in one where the graph is better behaved.

EXERCISES 6.8

Rewrite the integrals in Exercises 1–6 in a form to which numerical methods can be readily applied.

1. $\displaystyle\int_0^1 \frac{dx}{x^{1/3}(1+x)}$

2. $\displaystyle\int_0^1 \frac{e^x}{\sqrt{1-x}}\,dx$

3. $\displaystyle\int_{-1}^1 \frac{e^x}{\sqrt{1-x^2}}\,dx$

4. $\displaystyle\int_1^\infty \frac{dx}{x^2+\sqrt{x}+1}$

5. $\displaystyle\int_0^{\pi/2} \frac{dx}{\sqrt{\sin x}}$

6. $\displaystyle\int_0^\infty \frac{dx}{x^4+1}$

7. Find T_2, T_4, T_8, and T_{16} for $\int_0^1 \sqrt{x}\,dx$, and find the actual errors in these approximations. Do the errors decrease like $1/n^2$ as n increases? Why?

8. Transform the integral $I = \int_1^\infty e^{-x^2}\,dx$ using the substitution $x = 1/t$, and calculate the Simpson's Rule approximations S_2, S_4, and S_8 for the resulting integral (whose integrand has limit 0 as $t \to 0+$). Quote the value of I to the accuracy you feel is justified. Do the approximations converge as quickly as you might expect? Can you think of a reason why they might not?

9. Evaluate $I = \int_0^1 e^{-x^2}\,dx$, by the Taylor's formula method of Example 4, to within an error of 10^{-4}.

10. Recall that $\int_0^\infty e^{-x^2}\,dx = \frac{1}{2}\sqrt{\pi}$. Combine this fact with the result of Exercise 9 to evaluate $I = \displaystyle\int_1^\infty e^{-x^2}\,dx$ to 3 decimal places.

11. **(Gaussian approximation)** Find constants A and u, with u between 0 and 1, such that

$$\int_{-1}^1 f(x)\,dx = Af(-u) + Af(u)$$

holds for every cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$. For a general function $f(x)$

defined on $[-1, 1]$, the approximation

$$\int_{-1}^{1} f(x)\,dx \approx Af(-u) + Af(u)$$

is called a *Gaussian* approximation.

12. Use the method of Exercise 11 to approximate the integrals of (a) x^4, (b) $\cos x$, and (c) e^x, over the interval $[-1, 1]$, and find the error in each approximation.

13. **(Another Gaussian approximation)** Find constants A and B, and u between 0 and 1, such that

$$\int_{-1}^{1} f(x)\,dx = Af(-u) + Bf(0) + Af(u)$$

holds for every quintic polynomial $f(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$.

14. Use the Gaussian approximation

$$\int_{-1}^{1} f(x)\,dx \approx Af(-u) + Bf(0) + Af(u),$$

where A, B, and u are as determined in Exercise 13, to find approximations for the integrals of (a) x^6, (b) $\cos x$, and (c) e^x over the interval $[-1, 1]$, and find the error in each approximation.

15. Calculate sufficiently many Romberg approximations $R_1, R_2, R_3, \ldots$ for the integral

$$\int_{0}^{1} e^{-x^2}\,dx$$

to be confident you have evaluated the integral correctly to 6 decimal places.

16. Use the values of $f(x)$ given in the table accompanying Exercise 9 in Section 6.6 to calculate the Romberg approximations R_1, R_2, and R_3 for the integral

$$\int_{0}^{1.6} f(x)\,dx$$

in that exercise.

17. The Romberg approximation R_2 for $\int_a^b f(x)\,dx$ requires five values of f, $y_0 = f(a)$, $y_1 = f(a+h)$, $\ldots$, $y_4 = f(x+4h) = f(b)$, where $h = (b-a)/4$. Write the formula for R_2 explicitly in terms of these five values.

18. Explain why the change of variable $x = 1/t$ is not suitable for transforming the integral $\displaystyle\int_{\pi}^{\infty} \frac{\sin x}{1+x^2}\,dx$ into a form to which numerical methods can be applied. Try to devise a method whereby this integral could be approximated to any desired degree of accuracy.

19. If $f(x) = \dfrac{\sin x}{x}$ for $x \neq 0$ and $f(0) = 1$, show that $f''(x)$ has a finite limit as $x \to 0$. Hence, f'' is bounded on finite intervals $[0, a]$, and Trapezoid Rule approximations T_n to $\int_0^a \dfrac{\sin x}{x}\,dx$ converge suitably quickly as n increases. Higher derivatives are also bounded (Taylor's formula is useful for showing this), so Simpson's Rule and higher-order approximations can also be used effectively.

20. **(Estimating computation time)** With higher-order methods, the time to compute remains proportional to the number of intervals n used to numerically approximate an integral. But the error is reduced. For the Trapezoid Rule the error goes as $O(1/n^2)$. When $n = 1 \times 10^7$, the error turns out to be 6×10^{-16}. The computation time is approximately the same as that computed for the Riemann sum approximation to $\int_1^2 (1/x)\,dx$ discussed above (175.777 seconds for our computer), because we need essentially the same number of function evaluations. How long would it take our computer to get the trapezoid approximation to have quadruple (i.e., 32-digit) precession?

21. Repeat the previous exercise, but this time using Simpson's Rule, whose error is $O(1/n^4)$. Again use the same time, 175.777 s for $n = 1 \times 10^7$, but for Simpson's Rule the error for this calculation is 3.15×10^{-30}. How long would we expect our computer to take to achieve 32-digit accuracy (i.e., error 10^{-32})? Note, however, that Maple's integration package for the computer used took 0.134 seconds to achieve this precision. Will it have used a higher-order method than Simpson's Rule to achieve this time?

CHAPTER REVIEW

Key Ideas

- **What do the following terms and phrases mean?**
  - ◇ integration by parts
  - ◇ a reduction formula
  - ◇ an inverse substitution
  - ◇ a rational function
  - ◇ the method of partial fractions
  - ◇ a computer algebra system
  - ◇ an improper integral of type I
  - ◇ an improper integral of type II
  - ◇ a p-integral
  - ◇ the Trapezoid Rule
  - ◇ the Midpoint Rule
  - ◇ Simpson's Rule
- **Describe the inverse sine and inverse tangent substitutions.**
- **What is the significance of the comparison theorem for improper integrals?**
- **When is numerical integration necessary?**

Summary of Techniques of Integration

Students sometimes have difficulty deciding which method to use to evaluate a given integral. Often no one method will suffice to produce the whole solution, but one method may lead to a different, possibly simpler, integral that can then be dealt with on its own merits. Here are a few guidelines:

1. First, and always, be alert for simplifying substitutions. Even when these don't accomplish the whole integration, they can lead to integrals to which some other method can be applied.

2. If the integral involves a quadratic expression $Ax^2 + Bx + C$ with $A \neq 0$ and $B \neq 0$, complete the square. A simple substitution then reduces the quadratic expression to a sum or difference of squares.
3. Integrals of products of trigonometric functions can sometimes be evaluated or rendered simpler by the use of appropriate trigonometric identities such as:

$$\sin^2 x + \cos^2 x = 1$$
$$\sec^2 x = 1 + \tan^2 x$$
$$\csc^2 x = 1 + \cot^2 x$$
$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$
$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x)$$
$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x).$$

4. Integrals involving $(a^2 - x^2)^{1/2}$ can be transformed using $x = a\sin\theta$. Integrals involving $(a^2 + x^2)^{1/2}$ or $1/(a^2 + x^2)$ may yield to $x = a\tan\theta$. Integrals involving $(x^2 - a^2)^{1/2}$ can be transformed using $x = a\sec\theta$ or $x = a\cosh\theta$.
5. Use integration by parts for integrals of functions such as products of polynomials and transcendental functions, and for inverse trigonometric functions and logarithms. Be alert for ways of using integration by parts to obtain formulas representing complicated integrals in terms of simpler ones.
6. Use partial fractions to integrate rational functions whose denominators can be factored into real linear and quadratic factors. Remember to divide the polynomials first, if necessary, to reduce the fraction to one whose numerator has degree smaller than that of its denominator.
7. There is a table of integrals at the back of this book. If you can't do an integral directly, try to use the methods above to convert it to the form of one of the integrals in the table.
8. If you can't find any way to evaluate a definite integral for which you need a numerical value, consider using a computer or calculator and one of the numerical methods presented in Sections 6.6–6.8.

Review Exercises on Techniques of Integration

Here is an opportunity to get more practice evaluating integrals. Unlike the exercises in Sections 5.6 and 6.1–6.3, which used only the technique of the particular section, these exercises are grouped randomly, so you will have to decide which techniques to use.

1. $\displaystyle\int \frac{x\,dx}{2x^2 + 5x + 2}$

2. $\displaystyle\int \frac{x\,dx}{(x-1)^3}$

3. $\displaystyle\int \sin^3 x \cos^3 x\,dx$

4. $\displaystyle\int \frac{(1+\sqrt{x})^{1/3}}{\sqrt{x}}\,dx$

5. $\displaystyle\int \frac{3\,dx}{4x^2 - 1}$

6. $\displaystyle\int (x^2 + x - 2)\sin 3x\,dx$

7. $\displaystyle\int \frac{\sqrt{1-x^2}}{x^4}\,dx$

8. $\displaystyle\int x^3 \cos(x^2)\,dx$

9. $\displaystyle\int \frac{x^2\,dx}{(5x^3 - 2)^{2/3}}$

10. $\displaystyle\int \frac{dx}{x^2 + 2x - 15}$

11. $\displaystyle\int \frac{dx}{(4+x^2)^2}$

12. $\displaystyle\int (\sin x + \cos x)^2\,dx$

13. $\displaystyle\int 2^x \sqrt{1 + 4^x}\,dx$

14. $\displaystyle\int \frac{\cos x}{1 + \sin^2 x}\,dx$

15. $\displaystyle\int \frac{\sin^3 x}{\cos^7 x}\,dx$

16. $\displaystyle\int \frac{x^2\,dx}{(3 + 5x^2)^{3/2}}$

17. $\displaystyle\int e^{-x}\sin(2x)\,dx$

18. $\displaystyle\int \frac{2x^2 + 4x - 3}{x^2 + 5x}\,dx$

19. $\displaystyle\int \cos(3\ln x)\,dx$

20. $\displaystyle\int \frac{dx}{4x^3 + x}$

21. $\displaystyle\int \frac{x\ln(1+x^2)}{1+x^2}\,dx$

22. $\displaystyle\int \sin^2 x \cos^4 x\,dx$

23. $\displaystyle\int \frac{x^2}{\sqrt{2-x^2}}\,dx$

24. $\displaystyle\int \tan^4 x \sec x\,dx$

25. $\displaystyle\int \frac{x^2\,dx}{(4x+1)^{10}}$

26. $\displaystyle\int x\sin^{-1}\frac{x}{2}\,dx$

27. $\displaystyle\int \sin^5(4x)\,dx$

28. $\displaystyle\int \frac{dx}{x^5 - 2x^3 + x}$

29. $\displaystyle\int \frac{dx}{2 + e^x}$

30. $\displaystyle\int x^3 3^x\,dx$

31. $\displaystyle\int \frac{\sin^2 x \cos x}{2 - \sin x}\,dx$

32. $\displaystyle\int \frac{x^2 + 1}{x^2 + 2x + 2}\,dx$

33. $\displaystyle\int \frac{dx}{x^2\sqrt{1-x^2}}$

34. $\displaystyle\int x^3(\ln x)^2\,dx$

35. $\displaystyle\int \frac{x^3}{\sqrt{1-4x^2}}\,dx$

36. $\displaystyle\int \frac{e^{1/x}\,dx}{x^2}$

37. $\displaystyle\int \frac{x+1}{\sqrt{x^2+1}}\,dx$

38. $\displaystyle\int e^{(x^{1/3})}\,dx$

39. $\displaystyle\int \frac{x^3 - 3}{x^3 - 9x}\,dx$

40. $\displaystyle\int \frac{10^{\sqrt{x+2}}}{\sqrt{x+2}}\,dx$

41. $\displaystyle\int \sin^5 x \cos^9 x\,dx$

42. $\displaystyle\int \frac{x^2\,dx}{\sqrt{x^2-1}}$

43. $\displaystyle\int \frac{x\,dx}{x^2 + 2x - 1}$

44. $\displaystyle\int \frac{2x - 3}{\sqrt{4 - 3x + x^2}}\,dx$

45. $\displaystyle\int x^2\sin^{-1}(2x)\,dx$

46. $\displaystyle\int \frac{\sqrt{3x^2 - 1}}{x}\,dx$

47. $\displaystyle\int \cos^4 x \sin^4 x\,dx$

48. $\displaystyle\int \sqrt{x - x^2}\,dx$

49. $\displaystyle\int \frac{dx}{(4+x)\sqrt{x}}$

50. $\displaystyle\int x\tan^{-1}\frac{x}{3}\,dx$

51. $\displaystyle\int \frac{x^4 - 1}{x^3 + 2x^2}\,dx$

52. $\displaystyle\int \frac{dx}{x(x^2+4)^2}$

53. $\displaystyle\int \frac{\sin(2\ln x)}{x}\,dx$

54. $\displaystyle\int \frac{\sin(\ln x)}{x^2}\,dx$

55. $\displaystyle\int \frac{e^{2\tan^{-1}x}}{1+x^2}\,dx$

56. $\displaystyle\int \frac{x^3 + x - 2}{x^2 - 7}\,dx$

57. $\displaystyle\int \frac{\ln(3+x^2)}{3+x^2}x\,dx$

58. $\displaystyle\int \cos^7 x\,dx$

59. $\displaystyle\int \frac{\sin^{-1}(x/2)}{(4-x^2)^{1/2}}\,dx$

60. $\displaystyle\int \tan^4(\pi x)\,dx$

61. $\displaystyle\int \frac{(x+1)\,dx}{\sqrt{x^2+6x+10}}$

62. $\displaystyle\int e^x(1-e^{2x})^{5/2}\,dx$

63. $\displaystyle\int \frac{x^3\,dx}{(x^2+2)^{7/2}}$

64. $\displaystyle\int \frac{x^2}{2x^2-3}\,dx$

65. $\displaystyle\int \frac{x^{1/2}}{1+x^{1/3}}\,dx$

66. $\displaystyle\int \frac{dx}{x(x^2+x+1)^{1/2}}$

67. $\displaystyle\int \frac{1+x}{1+\sqrt{x}}\,dx$

68. $\displaystyle\int \frac{x\,dx}{4x^4+4x^2+5}$

69. $\displaystyle\int \frac{x\,dx}{(x^2-4)^2}$

70. $\displaystyle\int \frac{dx}{x^3+x^2+x}$

71. $\displaystyle\int x^2\tan^{-1}x\,dx$

72. $\displaystyle\int e^x\sec(e^x)\,dx$

73. $\displaystyle\int \frac{dx}{4\sin x-3\cos x}$

74. $\displaystyle\int \frac{dx}{x^{1/3}-1}$

75. $\displaystyle\int \frac{dx}{\tan x+\sin x}$

76. $\displaystyle\int \frac{x\,dx}{\sqrt{3-4x-4x^2}}$

77. $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx$

78. $\displaystyle\int \sqrt{1+e^x}\,dx$

79. $\displaystyle\int \frac{x^4\,dx}{x^3-8}$

80. $\displaystyle\int xe^x\cos x\,dx$

Other Review Exercises

1. Evaluate $I = \int xe^x\cos x\,dx$ and $J = \int xe^x\sin x\,dx$ by differentiating $e^x\big((ax+b)\cos x+(cx+d)\sin x\big)$ and examining coefficients.

2. For which real numbers r is the following reduction formula (obtained using integration by parts) valid?

$$\int_0^\infty x^r e^{-x}\,dx = r\int_0^\infty x^{r-1}e^{-x}\,dx$$

Evaluate the integrals in Exercises 3–6, or show that they diverge.

3. $\displaystyle\int_0^{\pi/2}\csc x\,dx$

4. $\displaystyle\int_1^\infty \frac{1}{x+x^3}\,dx$

5. $\displaystyle\int_0^1 \sqrt{x}\ln x\,dx$

6. $\displaystyle\int_{-1}^1 \frac{dx}{x\sqrt{1-x^2}}$

7. Show that the integral $I = \int_0^\infty (1/(\sqrt{x}\,e^x))\,dx$ converges and that its value satisfies $I < (2e+1)/e$.

8. By measuring the areas enclosed by contours on a topographic map, a geologist determines the cross-sectional areas A (m^2) through a 60 m high hill at various heights h (m) given in Table 2.

Table 2.

h	0	10	20	30	40	50	60
A	10,200	9,200	8,000	7,100	4,500	2,400	100

If she uses the Trapezoid Rule to estimate the volume of the hill (which is $V = \int_0^{60} A(h)\,dh$), what will be her estimate, to the nearest 1,000 m^3?

9. What will be the geologist's estimate of the volume of the hill in Exercise 8 if she uses Simpson's Rule instead of the Trapezoid Rule?

10. Find the Trapezoid Rule and Midpoint Rule approximations T_4 and M_4 for the integral $I = \int_0^1 \sqrt{2+\sin(\pi x)}\,dx$. Quote the results to 5 decimal places. Quote a value of I to as many decimal places as you feel are justified by these approximations.

11. Use the results of Exercise 10 to calculate the Trapezoid Rule approximation T_8 and the Simpson's Rule approximation S_8 for the integral I in that exercise. Quote a value of I to as many decimal places as you feel are justified by these approximations.

12. Devise a way to evaluate $I = \int_{1/2}^\infty x^2/(x^5+x^3+1)\,dx$ numerically, and use it to find I correct to 3 decimal places.

13. You want to approximate the integral $I = \int_0^4 f(x)\,dx$ of an unknown function $f(x)$, and you measure the following values of f:

Table 3.

x	0	1	2	3	4
$f(x)$	0.730	1.001	1.332	1.729	2.198

(a) What are the approximations T_4 and S_4 to I that you calculate with these data?

(b) You then decide to make more measurements in order to calculate T_8 and S_8. You obtain $T_8 = 5.5095$. What do you obtain for S_8?

(c) You have theoretical reasons to believe that $f(x)$ is, in fact, a polynomial of degree 3. Do your calculations support this theory? Why or why not?

Challenging Problems

1. (a) Some people think that $\pi = 22/7$. Prove that this is not so by showing that

$$\int_0^1 \frac{x^4(1-x)^4}{x^2+1}\,dx = \frac{22}{7}-\pi.$$

(b) If $I = \int_0^1 x^4(1-x)^4\,dx$, show that

$$\frac{22}{7}-I < \pi < \frac{22}{7}-\frac{I}{2}.$$

(c) Evaluate I and hence determine an explicit small interval containing π.

2. (a) Find a reduction formula for $\int (1-x^2)^n\,dx$.

(b) Show that if n is a positive integer, then

$$\int_0^1 (1-x^2)^n\,dx = \frac{2^{2n}(n!)^2}{(2n+1)!}.$$

(c) Use your reduction formula to evaluate $\int (1-x^2)^{-3/2}\,dx$.

3. (a) Show that x^4+x^2+1 factors into a product of two real quadratics, and evaluate $\int (x^2+1)/(x^4+x^2+1)\,dx$. *Hint:* $x^4+x^2+1 = (x^2+1)^2-x^2$.

(b) Use the same method to find $\int (x^2+1)/(x^4+1)\,dx$.

4. Let $I_{m,n} = \int_0^1 x^m(\ln x)^n\,dx$.

(a) Show that $I_{m,n} = (-1)^n\int_0^\infty x^n e^{-(m+1)x}\,dx$.

(b) Show that $I_{m,n} = \dfrac{(-1)^n n!}{(m+1)^{n+1}}$.

5. Let $I_n = \int_0^1 x^n e^{-x}\,dx$.

(a) Show that $0 < I_n < \dfrac{1}{n+1}$ and hence that $\lim_{n\to\infty} I_n = 0$.

(b) Show that $I_n = nI_{n-1} - \dfrac{1}{e}$ for $n \ge 1$, and $I_0 = 1 - \dfrac{1}{e}$.

(c) Verify by induction that $I_n = n!\left(1 - \dfrac{1}{e}\sum_{j=0}^{n} \dfrac{1}{j!}\right)$.

(d) Deduce from (a) and (c) that $\lim_{n\to\infty} \sum_{j=0}^{n} \dfrac{1}{j!} = e$.

6. If K is very large, which of the approximations T_{100} (Trapezoid Rule), M_{100} (Midpoint Rule), and S_{100} (Simpson's Rule) will be closest to the true value for $\int_0^1 e^{-Kx}\,dx$? Which will be farthest? Justify your answers. (*Caution:* This is trickier than it sounds!)

7. Simpson's Rule gives the exact definite integral for a cubic f. Suppose you want a numerical integration rule that gives the exact answer for a polynomial of degree 5. You might approximate the integral over the subinterval $[m-h, m+h]$ by something of the form $2h\left(af(m-h) + bf(m-\frac{h}{2}) + f(m) + bf(m+\frac{h}{2}) + af(m+h)\right)$ for some constants a, b, and c.

(a) Determine a, b, and c for which this will work. (*Hint:* Take $m = 0$ to make things simple.)

(b) Use this method to approximate $\int_0^1 e^{-x}\,dx$ using first one and then two of these intervals (thus evaluating the integrand at nine points).

8. The convergence of improper integrals can be a more delicate matter when the integrand changes sign. Here is one method that can be used to prove convergence in some cases where the comparison theorem fails.

(a) Suppose that $f(x)$ is differentiable on $[1, \infty)$, $f'(x)$ is continuous there, $f'(x) < 0$, and $\lim_{x\to\infty} f(x) = 0$. Show that $\int_1^\infty f'(x)\cos(x)\,dx$ converges. *Hint:* What is $\int_1^\infty |f'(x)|\,dx$?

(b) Under the same hypotheses, show that $\int_1^\infty f(x)\sin x\,dx$ converges. *Hint:* Integrate by parts and use (a).

(c) Show that $\int_1^\infty \dfrac{\sin x}{x}\,dx$ converges but $\int_1^\infty \dfrac{|\sin x|}{x}\,dx$ diverges. *Hint:* $|\sin x| \ge \sin^2 x = \dfrac{1-\cos(2x)}{2}$. Note that (b) would work just as well with $\sin x$ replaced by $\cos(2x)$.