LIGO-GW150914 (10 points)

In 2015, the gravitational-wave observatory LIGO detected, for the first time, the passing of gravitational waves (GW) through Earth. This event, named GW150914, was triggered by waves produced by two black holes that were orbiting on quasi-circular orbits. This problem will make you estimate some physical parameters of the system, from the properties of the detected signal.

Part A: Newtonian (conservative) orbits (3.0 points)

A.1 Consider a system of two stars with masses M_1, M_2, at locations $\vec{r}_1, \vec{r}_2$, respectively, with respect to the center-of-mass of the system, that is, **1.0pt**

$$M_1\vec{r}_1 + M_2\vec{r}_2 = 0\,. \tag{1}$$

The stars are isolated from the rest of the Universe and moving at non-relativistic velocities. Using Newton's laws, the acceleration vector of mass M_1 can be expressed as

$$\frac{\mathrm{d}^2\vec{r}_1}{\mathrm{d}t^2} = -\alpha\frac{\vec{r}_1}{r_1^n}\,, \tag{2}$$

where $r_1 = |\vec{r}_1|, r_2 = |\vec{r}_2|$. Find $n \in \mathbb{N}$ and $\alpha = \alpha(G, M_1, M_2)$, where G is Newton's constant [$G \simeq 6.67 \times 10^{-11}$ N m^2 kg^{-2}].

A.2 The total energy of the 2-mass system, in circular orbits, can be expressed as: **1.0pt**

$$E = A(\mu, \Omega, L) - G\frac{M\mu}{L}\,, \tag{3}$$

where

$$\mu \equiv \frac{M_1M_2}{M_1+M_2}\,, \qquad M \equiv M_1 + M_2\,, \tag{4}$$

are the *reduced mass* and *total mass* of the system, Ω is the angular velocity of each mass and L is the total separation $L = r_1 + r_2$. Obtain the explicit form of the term $A(\mu, \Omega, L)$.

A.3 Equation 3 can be simplified to $E = \beta G\frac{M\mu}{L}$. Determine the number β. **1.0pt**

Part B: Introducing relativistic dissipation (7.0 points)

The correct theory of gravity, *General Relativity*, was formulated by Einstein in 1915, and predicts that gravity travels with the speed of light. The messengers carrying information about the interaction are called GWs. GWs are emitted whenever masses are accelerated, making the system of masses lose energy.

Consider a system of two point-like particles, isolated from the rest of the Universe. Einstein proved that for small enough velocities the emitted GWs: 1) have a frequency which is twice as large as the orbital frequency; 2) can be characterized by a luminosity, i.e. emitted power $\mathcal{P}$, which is dominated by Einstein's quadrupole formula,

$$\mathcal{P} = \frac{G}{5c^5}\sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\right)\left(\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3}\right) . \tag{5}$$

Here, c is the velocity of light $c \simeq 3 \times 10^8$ m/s. For a system of 2 pointlike particles orbiting on the $x - y$ plane, Q_{ij} is the following table (i, j label the row/column number)

$$Q_{11} = \sum_{A=1}^{2} \frac{M_A}{3}\left(2x_A^2 - y_A^2\right) , \qquad Q_{22} = \sum_{A=1}^{2} \frac{M_A}{3}\left(2y_A^2 - x_A^2\right) , \qquad Q_{33} = -\sum_{A=1}^{2} \frac{M_A}{3}(x_A^2 + y_A^2) , \tag{6}$$

$$Q_{12} = Q_{21} = \sum_{A=1}^{2} M_A x_A y_A , \tag{7}$$

and $Q_{ij} = 0$ for all other possibilities. Here, (x_A, y_A) is the position of mass A in the center-of-mass frame.

B.1 For the circular orbits described in A.2 the components of Q_{ij} can be expressed as a function of time t as: 1.0pt

$$Q_{ii} = \frac{\mu L^2}{2}\left(a_i + b_i \cos kt\right) , \qquad Q_{ij} \overset{i \neq j}{=} \frac{\mu L^2}{2} c_{ij} \sin kt . \tag{8}$$

Determine k in terms of Ω and the numerical values of the constants a_i, b_i, c_{ij}.

B.2 Compute the power $\mathcal{P}$ emitted in gravitational waves for that system, and obtain: 1.0pt

$$\mathcal{P} = \xi \frac{G}{c^5}\mu^2 L^4 \Omega^6 . \tag{9}$$

What is the number ξ? [If you could not obtain ξ, use $\xi = 6.4$ in the following.]

B.3 In the absence of GW emission the two masses will orbit on a fixed circular orbit indefinitely. However, the emission of GWs causes the system to lose energy and to slowly evolve towards smaller circular orbits. Obtain that the rate of change $\frac{\mathrm{d}\Omega}{\mathrm{d}t}$ of the orbital angular velocity takes the form 1.0pt

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)^3 = (3\xi)^3 \frac{\Omega^{11}}{c^{15}}\left(GM_{\mathrm{c}}\right)^5 , \tag{10}$$

where M_{c} is called the *chirp mass*. Obtain M_{c} as a function of M and μ. This mass determines the increase in frequency during the orbital decay. [The name "chirp" is inspired by the high pitch sound (increasing frequency) produced by small birds.]

B.4	Using the information provided above, relate the orbital angular velocity Ω with the GW frequency f_{GW}. Knowing that, for any smooth function $F(t)$ and $a \neq 1$, $$\frac{\mathrm{d}F(t)}{\mathrm{d}t} = \chi F(t)^a \qquad \Rightarrow \qquad F(t)^{1-a} = \chi(1-a)(t-t_0)\,, \tag{11}$$ where χ is a constant and t_0 is an integration constant, show that (10) implies that the GW frequency is $$f_{\text{GW}}^{-8/3} = 8\pi^{8/3}\xi\left(\frac{GM_c}{c^3}\right)^{(2/3)+p}(t_0-t)^{2-p} \tag{12}$$ and determine the constant p.	2.0pt

On September 14, 2015 GW150914 was registered by the LIGO detectors, consisting of two L-shaped arms, each 4 km long. These arms changed by a relative length according to Fig. 1. The arms of the detector respond linearly to a passing gravitational wave, and the response pattern mimics the wave. This wave was created by two black holes on quasi-circular orbits; the loss of energy through gravitational radiation caused the orbit to shrink and the black holes to eventually collide. The collision point corresponds, roughly, to the peak of the signal after point D, in Fig. 1.

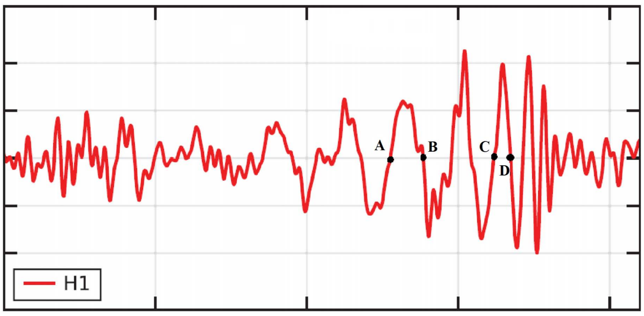

Figure 1. Strain, i.e. relative variation of the size of each arm, at the LIGO detector H1. The horizontal axis is time, and the points A, B, C, D correspond to $t = 0.000, 0.009, 0.034, 0.040$ seconds, respectively.

B.5	From the figure, estimate $f_{\text{GW}}(t)$ at $$t_{\overline{\text{AB}}} = \frac{t_{\text{B}}+t_{\text{A}}}{2} \quad \text{and} \quad t_{\overline{\text{CD}}} = \frac{t_{\text{D}}+t_{\text{C}}}{2}\,. \tag{13}$$ Assuming that (12) is valid all the way until the collision (which strictly speaking is not true) and that the two objects have equal mass, estimate the chirp mass, M_c, and total mass of the system, in terms of solar masses $M_\odot \simeq 2\times 10^{30}$ kg.	1.0pt

B.6	Estimate the minimal orbital separation between the two objects at $t_{\overline{\text{CD}}}$. Hence estimate a maximum size for each object, R_{max}. Obtain $R_\odot/R_{\text{max}}$ to compare this size with the radius of our Sun, $R_\odot \simeq 7\times 10^5$ km. Estimate also their orbital linear velocity at the same instant, v_{col}, comparing it with the speed of light, v_{col}/c.	1.0pt

Conclude that these are extremely fast moving, extremely compact objects indeed!

Where is the neutrino? (10 points)

When two protons collide with a very high energy at the Large Hadron Collider (LHC), several particles may be produced as a result of that collision, such as electrons, muons, neutrinos, quarks, and their respective anti-particles. Most of these particles can be detected by the particle detector surrounding the collision point. For example, quarks undergo a process called *hadronisation* in which they become a shower of subatomic particles, called "jet". In addition, the high magnetic field present in the detectors allows even very energetic charged particles to curve enough for their momentum to be determined. The ATLAS detector uses a superconducting solenoid system that produces a constant and uniform 2.00 Tesla magnetic field in the inner part of the detector, surrounding the collision point. Charged particles with momenta below a certain value will be curved so strongly that they will loop repeatedly in the field and most likely not be measured. Due to its nature, the neutrino is not detected at all, as it escapes through the detector without interacting.

Data: Electron rest mass, $m = 9.11 \times 10^{-31}$ kg; Elementary charge, $e = 1.60 \times 10^{-19}$ C;

Speed of light, $c = 3.00 \times 10^{8}$ m s^{-1}; Vacuum permittivity, $\epsilon_0 = 8.85 \times 10^{-12}$ F m^{-1}

Part A. ATLAS Detector physics (4.0 points)

A.1	Derive an expression for the cyclotron radius, r, of the circular trajectory of an electron acted upon by a magnetic force perpendicular to its velocity, and express that radius as a function of its kinetic energy, K; charge modulus, e; mass, m; and magnetic field, B. Assume that the electron is a non-relativistic classical particle.	0.5pt

Electrons produced inside the ATLAS detector must be treated relativistically. However, the formula for the cyclotron radius also holds for relativistic motion when the relativistic momentum is considered.

A.2	Calculate the minimum value of the momentum of an electron that allows it to escape the inner part of the detector in the radial direction. The inner part of the detector has a cylindrical shape with a radius of 1.1 meters, and the electron is produced in the collision point exactly in the center of the cylinder. Express your answer in MeV/c.	0.5pt

When accelerated perpendicularly to the velocity, relativistic particles of charge e and rest mass m emitt electromagnetic radiation, called synchrotron radiation. The emitted power is given by

$$P = \frac{e^2 a^2 \gamma^4}{6\pi\epsilon_0 c^3}$$

where a is the acceleration and $\gamma = [1 - (v/c)^2]^{-1/2}$.

A.3 A particle is called ultrarelativistic when its speed is very close to the speed of light. For an ultrarelativistic particle the emitted power can be expressed as: — 1.0pt

$$P = \xi \frac{e^4}{\epsilon_0 m^k c^n} E^2 B^2 \,,$$

where ξ is a real number, n, k are integers, E is the energy of the charged particle and B is the magnetic field. Find ξ, n and k.

A.4	In the ultrarelativistic limit, the energy of the electron as a function of time is: $$E(t) = \frac{E_0}{1 + \alpha E_0 t},$$ where E_0 is the initial energy of the electron. Find α as a function of e, c, B, ϵ_0 and m.	1.0pt

A.5	Consider an electron produced at the collision point along the radial direction with an energy of 100 GeV. Estimate the amount of energy that is lost due to synchrotron radiation until the electron escapes the inner part of the detector? Express your answer in MeV.	0.5pt

A.6	Find an expression for the cyclotron frequency of the electron as a function of time in the ultrarelativistic limit.	0.5pt

Part B. Finding the neutrino (6.0 points)

The collision between the two protons shown in Figure 1 leads to the production of a top quark (t) and an anti-top quark ($\bar{t}$), the heaviest elementary particles ever detected. The top quark decays into a W^+ boson and a bottom quark (b), while the anti-top quark decays into a W^- boson and an anti-bottom quark ($\bar{b}$). In the case depicted in Figure 1, the W^+ boson decays into a anti-muon (μ^+) and a neutrino (ν), and the W^- boson decays into a quark and an anti-quark. The task of this problem is to reconstruct the full momentum of the neutrino using the momenta of some detected particles. **For simplicity, all particles and jets in this problem will be considered massless, except for the top quark and W$^\pm$ bosons.**

The momenta of the top quark decay products can be determined from the experiment (see Table), except for the neutrino momentum component along the (z-axis). The total linear momentum of the final state particles caught by the detector is only zero on the transverse plane (xy plane), and not along the collision line (z-axis). As such, one can find the transverse momentum of the neutrino from the missing momentum in the transverse plane.

On June 4, 2015, the ATLAS experiment at the LHC recorded a proton-proton collision at 00:21:24 GMT+1 like the one represented in Figure 1.

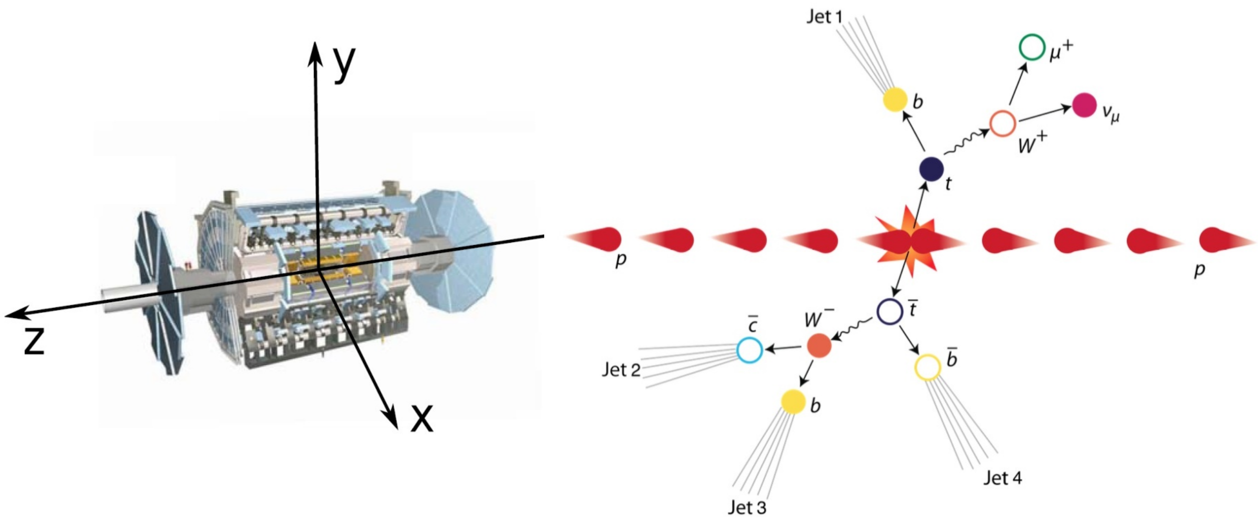

Figure 1. Schematic representation of the ATLAS detector coordinate system (left) and proton-proton collision (right).

The linear momenta of the three final-state particles coming from the top quark decay, including the neutrino, is presented below for each component.

Particle	p_x (GeV/c)	p_y (GeV/c)	p_z (GeV/c)
anti-muon (μ^+)	-24.7	-24.9	-12.4
jet 1 (j_1)	-14.2	$+50.1$	$+94.1$
neutrino (ν)	-104.1	$+5.3$	---

B.1 Find an equation which relates the square of the W^+ boson mass, m_{W}^2, with the neutrino and anti-muon momentum components presented in the table above. Express your answer in terms of the neutrino and anti-muon transverse momentum, **1.5pt**

$$\vec{p}_{\mathrm{T}}^{\,(\nu)} = p_x^{(\nu)}\,\hat{\imath} + p_y^{(\nu)}\,\hat{\jmath} \text{ and } \vec{p}_{\mathrm{T}}^{\,(\mu)} = p_x^{(\mu)}\,\hat{\imath} + p_y^{(\mu)}\,\hat{\jmath},$$

and their z-axis momentum components, $p_z^{(\mu)}$ and $p_z^{(\nu)}$.

B.2 Assuming a W^+ boson mass of $m_{\mathrm{W}} = 80.4$ GeV/c^2 calculate the two possible solutions for the neutrino momentum along the z-axis, $p_z^{(\nu)}$. Express your answer in GeV/c. **1.5pt**

B.3 Calculate the top quark mass for each one of the two previous solutions. Express your answer in GeV/c^2. **1.0pt**
[If you did not obtain the two solutions in B.2, use

$$p_z^{(\nu)} = 70 \text{ GeV}/c \quad\text{and}\quad p_z^{(\nu)} = -180 \text{ GeV}/c.]$$

The normalised number of collision events for the measurement of the top quark mass (as determined from the experiment), has two components: the so-called "signal" (corresponding to the decay of top quarks) and "background" (corresponding to events from other processes that do not include top quarks). Experimental data include both processes, see Fig. 2.

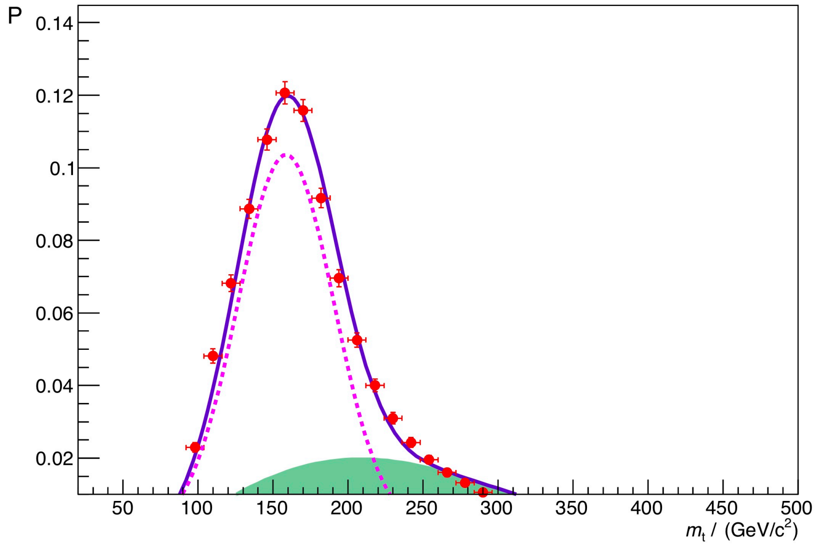

Figure 2. Top quark mass distribution as determined from the experiment, i.e. the normalised number of events plotted against the top quark mass. The dots correspond to the data. The dashed line corresponds to the "signal" and the shade to the "background".

B.4	According to the top quark mass distribution, which one of the two previous solutions is more likely to be the right one? Estimate the probability for the most likely solution.	1.0pt

B.5	Calculate the distance traveled by the top quark before decaying, using the most likely solution. Assume the top quark has a mean lifetime at rest of 5×10^{-25} s.	1.0pt

Physics of Live Systems (10 points)

Data: Normal atmospheric pressure, $P_0 = 1.013 \times 10^5$ Pa $= 760$ mmHg

Part A. The physics of blood flow (4.5 points)

In this part you will analyse two simplified models of blood flow in vessels.

Blood vessels are approximately cylindrical in shape, and it is known that for a steady, non turbulent flow of an incompressible fluid in a rigid cylinder, the difference in pressure of the fluid at the two ends of the cylinder is given by

$$\Delta P = \frac{8\ell\eta}{\pi r^4} Q \,, \tag{1}$$

where ℓ and r are the length and radius of the cylinder, η is the fluid viscosity and Q is the volumetric flow rate, i.e. the fluid volume that passes the cylinder cross section per unit time. This expression is often able to provide the correct order of magnitude for the pressure difference in a vessel, even without taking into account the pulsatile flow, the vessel's compressibility and irregular shape, and the fact that blood is not a simple fluid but a mixture of cells and plasma. Moreover, this expression has the same form as Ohm's law, with the volumetric flow rate being interpreted as a current, the difference in pressure as a voltage, and the factor $R = \frac{8\ell\eta}{\pi r^4}$ as a resistance.

Consider for example the symmetrical network of arterioles (small arteries) depicted in Figure 1 that delivers blood to the capillary bed of a tissue. In this network, at each bifurcation a vessel is divided in two identical vessels. However, the vessels of higher levels are thinner and shorter: consider that the radii and lengths of vessels in two consecutive levels, i and $i+1$, are related by $r_{i+1} = r_i/2^{1/3}$ and by $\ell_{i+1} = \ell_i/2^{1/3}$.

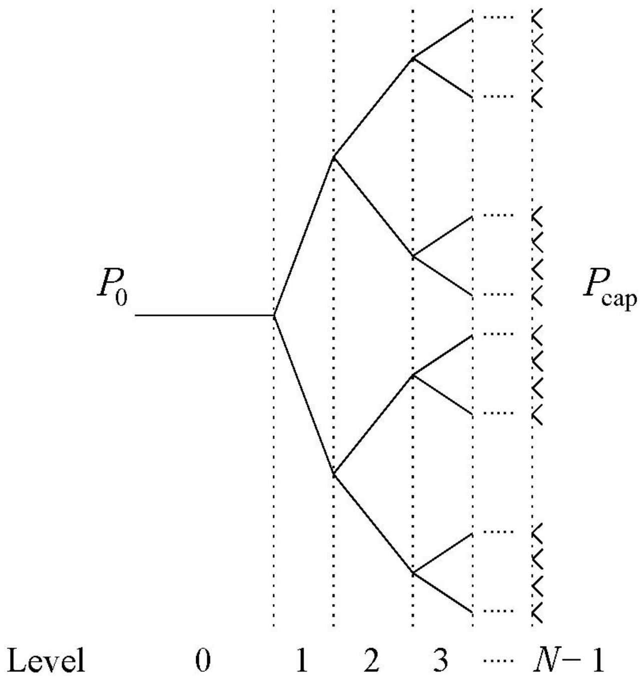

Figure 1. Network of arterioles.

A.1	Obtain an expression for the volumetric flow rate, Q_i, in a vessel at any level i, as a function of the total number of levels N, of the viscosity η, of the radius r_0 and length ℓ_0 of the first vessel, and of the difference $\Delta P = P_0 - P_{\text{cap}}$ between the pressure at the arteriole at level 0, P_0, and the pressure at the capillary bed, P_{cap}.	1.3pt

A.2	Calculate the numerical value of the volumetric flow rate Q_0 of the arteriole at level 0, if its radius is 6.0×10^{-5} m and its length is 2.0×10^{-3} m. Consider that the pressure at the arteriole inlet is 55 mmHg and the vessel network has $N = 6$ levels linking this arteriole to the capillary bed at the pressure 30 mmHg. Consider that the blood viscosity is $\eta = 3.5 \times 10^{-3}$ kg m^{-1} s^{-1}. Express your result in ml/h.	0.5pt

A blood vessel as an LCR circuit

The approximation of rigid cylindrical vessels falls short for several reasons. It is particularly important to include the time dependent flow and to take into account the change in vessel diameter that occurs when the pressure varies during a blood pumping cycle done by the heart. Moreover, it is observed that in the larger vessels the blood pressure varies significantly during a cycle, while in the smaller vessels the amplitude of the oscillations in pressure is much smaller, and the flow is almost time independent.

When the pressure increases in a single elastic vessel, there will be an increase in its diameter, thus permitting to store more fluid in the vessel, and to deliver it when the pressure drops. For this reason, the elastic behaviour of the vessel can be simulated by adding a capacitor to our initial description. Moreover, when taking into account the time dependent blood flow rate, one has to consider the inertia of the fluid, proportional to its density $\rho = 1.05 \times 10^3$ kg m^{-3}. This inertia can be described by an inductance in our model. In Figure 2 we represent the equivalent circuit for a single vessel in this model. The equivalent capacitance and inductance are given by

$$C = \frac{3\ell\pi r^3}{2Eh} \quad \text{and} \quad L = \frac{9\ell\rho}{4\pi r^2}\,, \tag{2}$$

respectively, where h is the width of the vessel wall and E is the artery Young's modulus, a coefficient that describes the alteration in size of the vessel tissue when a force is applied. The Young's modulus has units of pressure and is on the order of $E = 0.06$ MPa for arterioles.

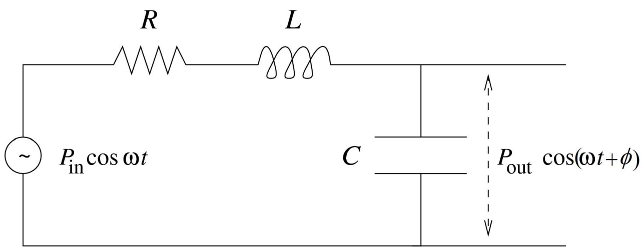

Figure 2. Equivalent electric circuit for a single vessel.

A.3	Obtain, in the stationary regime, the pressure amplitude at the vessel outlet, P_{out}, as a function of the pressure amplitude at the inlet, P_{in}, the equivalent resistance, R, inductance, L and capacitance, C, for a flow with angular frequency ω. Establish the condition between η, ρ, E, h, r and ℓ so that, for low frequencies, the pressure oscillation amplitude at the outlet is smaller than that of P_{in}.	2.0pt

A.4	For the vessel network in **A.2** estimate the maximum arteriole wall thickness h so that the condition established in **A.3** is satisfied (consider that h is level independent).	0.7pt

Part B. Tumour growth (5.5 points)

Tumour growth is a very complex process where biological mechanisms such as cell proliferation and natural selection are intertwined with physics. In this problem we will consider a simplified model of tumour growth that addresses the increase in pressure commonly observed in solid tumors.

Consider a group of normal cells forming a tissue surrounded by an inextensible basement membrane, which forces the tissue to maintain always the same form: a sphere of radius R (Figure 3).

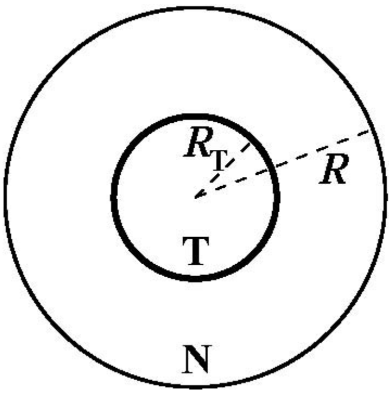

Figure 3. Simplified tumour.

Initially the tissue does not have residual stresses, i.e. the pressure at every point is equal to the atmospheric pressure.

At time $t = 0$, a tumour starts growing at the centre of this sphere and, as it grows, the pressure inside the tissue increases. Consider that both tissues (normal, N, and tumour, T) are compressible such that their densities, ρ_{N} and ρ_{T}, increase linearly with pressure:

$$\rho_{\text{N}} = \rho_0 \left(1 + \frac{p}{K_{\text{N}}}\right) , \quad \rho_{\text{T}} = \rho_0 \left(1 + \frac{p}{K_{\text{T}}}\right) , \tag{3}$$

where ρ_0 is the rest tissue density, p is the pressure difference to the atmospheric pressure and K_{N}, K_{T} are the compressibility moduli (bulk moduli) of the normal and tumour tissues, respectively. In general, tumours are stiffer and so they have a higher bulk modulus.

B.1	The mass of normal cells is not altered while the tumour is growing. Obtain the ratio between the tumour volume and the total tissue volume, $v = V_{\mathrm{T}}/V$, as a function of the ratio between the tumour mass (M_{T}) and the normal tissue mass (M_{N}), $\mu = M_{\mathrm{T}}/M_{\mathrm{N}}$ and the ratio of the bulk moduli, $\kappa = K_{\mathrm{N}}/K_{\mathrm{T}}$.	1.0pt

Hyperthermia is sometimes used together with chemotherapy and radiotherapy in the treatment of cancer. In hyperthermia the cancer cells are selectively heated from the normal human body temperature, 37 °C, to temperatures above 43 °C, inducing their death. Researchers are currently developing carbon nanotubes covered with special proteins capable of binding to tumour cells. When the tissue is irradiated with near-infrared radiation, the nanotubes absorb it in a much greater extent than the surrounding tissues and therefore can be selectively heated as well as the tumour cells to which they are attached.

Consider that the tumour, the normal cells and the surrounding tissue have a constant thermal conductivity k, i.e. in the geometry of this problem, the energy that crosses a spherical surface of radius r per unit time and per unit area is equal to k times the derivative of the temperature with respect to r. The nanotubes are uniformly distributed in the tumour volume and are able to deliver a power $\mathcal{P}$ of thermal energy per unit volume. Assume that the temperature is equal to the normal human body temperature very far away from the tumour.

B.2	Obtain, for the stationary state, the temperature at the centre of the tumour as a function of $\mathcal{P}$, k, the human body temperature and the tumour radius, R_{T}.	1.7pt

B.3	Obtain the minimum power per unit volume, $\mathcal{P}_{\mathrm{min}}$, needed to heat up all tumour cells in a tumour with 5.0 cm radius to a temperature larger than 43.0 °C. Take the thermal conductivity of the tissue to be equal to $k = 0.60$ W K^{-1}m^{-1}.	0.5pt

Consider that the tumour is irrigated by a vessel network with a branched structure like in question **A.1**. As the tumour grows, when its pressure p becomes larger than the pressure P_{cap} at the thinnest vessels, the radii of these vessels will decrease by a small amount δr. If this pressure reaches a critical value p_{c} (which would correspond to a radius decrease of δr_{c}), the thinnest vessels would collapse, compromising seriously the irrigation to the tumour. The pressure and the radius change can be related by the following phenomenological relation:

$$\frac{p}{P_{\mathrm{cap}}} - 1 = \left(\frac{p_{\mathrm{c}}}{P_{\mathrm{cap}}} - 1\right)\left(2 - \frac{\delta r}{\delta r_{\mathrm{c}}}\right)\frac{\delta r}{\delta r_{\mathrm{c}}} \,. \tag{4}$$

Consider that just the smallest vessels (of level $N-1$) have their radius altered when the tumour increases its pressure.

B.4	In the linear regime (i.e. consider that $p - P_{\mathrm{cap}}$ is very small), express the relative drop in the flow rate, $\frac{\delta Q_{N-1}}{Q_{N-1}}$, in these thinnest vessels, as a function of the tumour volume ratio $v = V_{\mathrm{T}}/V$ and K_N, N, p_{c}, δr_{c}, r_{N-1}, P_{cap}.	2.3pt

IPhO 2018
Lisbon, Portugal

Solutions to Theory Problem 1

LIGO-GW150914

(V. Cardoso, C. Herdeiro)

July 15, 2018

v6.0

GW150914 (10 points)

Part A. Newtonian (conservative) orbits (3.0 points)

A.1 Apply Newton's law to mass M_1:

$$M_1 \frac{\mathrm{d}^2\vec{r}_1}{\mathrm{d}t^2} = G\frac{M_1 M_2}{|\vec{r}_2 - \vec{r}_1|^2}\frac{\vec{r}_2 - \vec{r}_1}{|\vec{r}_2 - \vec{r}_1|} . \tag{1}$$

Use, from eq. (1) of the question sheet

$$\vec{r}_2 = -\frac{M_1}{M_2}\vec{r}_1 , \tag{2}$$

in eq. (1) above, to obtain

$$\frac{\mathrm{d}^2\vec{r}_1}{\mathrm{d}t^2} = -\frac{GM_2^3}{(M_1+M_2)^2 r_1^2}\frac{\vec{r}_1}{r_1} . \tag{3}$$

A.1 $$n = 3, \qquad \alpha = \frac{GM_2^3}{(M_1+M_2)^2} .$$	1.0pt

A.2 The total energy of the system is the sum of the two kinetic energies plus the gravitational potential energy. For circular motions, the linear velocity of each of the masses reads

$$|\vec{v}_1| = r_1\Omega , \qquad |\vec{v}_2| = r_2\Omega , \tag{4}$$

Thus, the total energy is

$$E = \frac{1}{2}(M_1 r_1^2 + M_2 r_2^2)\Omega^2 - \frac{GM_1M_2}{L} , \tag{5}$$

Now,

$$(M_1 r_1 - M_2 r_2)^2 = 0 \qquad \Rightarrow \qquad M_1 r_1^2 + M_2 r_2^2 = \mu L^2 . \tag{6}$$

Thus,

$$E = \frac{1}{2}\mu L^2\Omega^2 - G\frac{M\mu}{L} . \tag{7}$$

A.2 $$A(\mu,\Omega,L) = \frac{1}{2}\mu L^2\Omega^2 .$$	1.0pt

A.3 Energy (3) of the question sheet can be interpreted as describing a system of a mass μ in a circular orbit with angular velocity Ω, radius L, around a mass M (at rest). Equating the gravitational acceleration to the centripetal acceleration:

$$G\frac{M}{L^2} = \Omega^2 L . \tag{8}$$

This is indeed Kepler's third law (for circular orbits). Then, from (7),

$$E = -\frac{1}{2}G\frac{M\mu}{L} . \tag{9}$$

A.3 $$\beta = -\frac{1}{2} .$$	1.0pt

Part B - Introducing relativistic dissipation (7.0 points)

B.1 Some simple trigonometry for the x, y motion of the masses (in an appropriate Cartesian system) yields:

$$(x_1, y_1) = r_1(\cos(\Omega t), \sin(\Omega t)) , \qquad (x_2, y_2) = -r_2(\cos(\Omega t), \sin(\Omega t)) . \tag{10}$$

Then,

$$Q_{ij} = \frac{M_1 r_1^2 + M_2 r_2^2}{2} \begin{pmatrix} \frac{4}{3}\cos^2(\Omega t) - \frac{2}{3}\sin^2(\Omega t) & 2\sin(\Omega t)\cos(\Omega t) & 0 \\ 2\sin(\Omega t)\cos(\Omega t) & \frac{4}{3}\sin^2(\Omega t) - \frac{2}{3}\cos^2(\Omega t) & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} , \tag{11}$$

or, using some simple trigonometry and (6),

$$Q_{ij} = \frac{\mu L^2}{2} \begin{pmatrix} \frac{1}{3} + \cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & \frac{1}{3} - \cos 2\Omega t & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} . \tag{12}$$

B.1	1.0pt
$k = 2\Omega \, , \quad a_1 = a_2 = \frac{1}{3} \, , a_3 = -\frac{2}{3} , \quad b_1 = 1, b_2 = -1, b_3 = 0 \, , c_{12} = c_{21} = 1, c_{ij} \overset{\text{otherwise}}{=} 0 \, .$	

B.2 First take the derivatives:

$$\frac{\mathrm{d}^3 Q_{ij}}{\mathrm{d}t^3} = 4\Omega^3 \mu L^2 \begin{pmatrix} \sin 2\Omega t & -\cos 2\Omega t & 0 \\ -\cos 2\Omega t & -\sin 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{13}$$

Then perform the sum:

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5c^5}(4\Omega^3 \mu L^2)^2 [2\sin^2(2\Omega t) + 2\cos^2(2\Omega t)] = \frac{32}{5}\frac{G}{c^5}\mu^2 L^4 \Omega^6 . \tag{14}$$

B.2	1.0pt
$\xi = \frac{32}{5} \, .$	

B.3 Now we assume a sequency of Keplerian orbits, with decreasing energy, which is being taken from the system by the GWs.

First, from (9), differentiating with respect to time,

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{GM\mu}{2L^2}\frac{\mathrm{d}L}{\mathrm{d}t} , \tag{15}$$

Since this loss of energy is due to GWs, we equate it with (minus) the luminosity of GWs, given by (14)

$$\frac{GM\mu}{2L^2}\frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{32}{5}\frac{G}{c^5}\mu^2 L^4 \Omega^6 . \tag{16}$$

We can eliminate the L and $\mathrm{d}L/\mathrm{d}t$ dependence in this equation in terms of Ω and $\mathrm{d}\Omega/\mathrm{d}t$, by using Kepler's third law (8), which relates:

$$L^3 = G\frac{M}{\Omega^2} , \qquad \frac{\mathrm{d}L}{\mathrm{d}t} = -\frac{2}{3}\frac{L}{\Omega}\frac{\mathrm{d}\Omega}{\mathrm{d}t} . \tag{17}$$

Substituting in (16), we obtain:

$$\left(\frac{\mathrm{d}\Omega}{\mathrm{d}t}\right)^3 = \left(\frac{96}{5}\right)^3 \frac{\Omega^{11}}{c^{15}} G^5\mu^3 M^2 \equiv \left(\frac{96}{5}\right)^3 \frac{\Omega^{11}}{c^{15}} (GM_\mathsf{c})^5 \; . \tag{18}$$

B.3	1.0pt
$M_\mathsf{c} = (\mu^3 M^2)^{1/5}$.	

B.4 Angular and cycle frequencies are related as $\Omega = 2\pi f$. From the information provided above: *GWs have a frequency which is twice as large as the orbital frequency*, we have

$$\frac{\Omega}{2\pi} = \frac{f_\mathsf{GW}}{2} \; . \tag{19}$$

Formula (10) of the question sheet has the form

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \chi\Omega^{11/3} \; , \qquad \chi \equiv \frac{96}{5}\frac{(GM_\mathsf{c})^{5/3}}{c^5} \; . \tag{20}$$

Thus, from (11) of the question sheet

$$\Omega(t)^{-8/3} = \frac{8}{3}\chi(t_0 - t) \; , \tag{21}$$

or, using (20) and the definition of χ gives

$$f_\mathsf{GW}^{-8/3}(t) = \frac{(8\pi)^{8/3}}{5}\left(\frac{GM_\mathsf{c}}{c^3}\right)^{5/3} (t_0 - t) \; . \tag{22}$$

B.4	2.0pt
$p = 1$.	

B.5 From the figure, we consider the two Δt's as half periods. Thus, the (cycle) GW frequency is $f_\mathsf{GW} = 1/(2\Delta t)$. Then, the four given points allow us to compute the frequency at the mean time of the two intervals as

	$t_{\overline{\mathsf{AB}}}$	$t_{\overline{\mathsf{CD}}}$
t (s)	0.0045	0.037
f_GW (Hz)	$(2 \times 0.009)^{-1}$	$(2 \times 0.006)^{-1}$

Now, using (22) we have two pairs of (f_GW,t) values for two unknowns (t_0,M_c). Expressing (22) for both $t_{\overline{\mathsf{AB}}}$ and $t_{\overline{\mathsf{CD}}}$ and dividing the two equations we obtain:

$$t_0 = \frac{At_{\overline{\mathsf{CD}}} - t_{\overline{\mathsf{AB}}}}{A - 1} \; , \qquad A \equiv \left(\frac{f_\mathsf{GW}(t_{\overline{\mathsf{AB}}})}{f_\mathsf{GW}(t_{\overline{\mathsf{CD}}})}\right)^{-8/3} \; . \tag{23}$$

Replacing by the numerical values, $A \simeq 2.95$ and $t_0 \simeq 0.054$ s. Now we can use (22) for either of the two values $t_{\overline{\mathsf{AB}}}$ or $t_{\overline{\mathsf{CD}}}$ and determine M_c. One obtains for the chirp mass

$$M_\mathsf{c} \simeq 6 \times 10^{31}\ \mathsf{kg} \simeq 30 \times M_\odot \; . \tag{24}$$

Thus, the total mass M is

$$M = 4^{3/5} M_\mathsf{c} \simeq 69 \times M_\odot \; . \tag{25}$$

This result is actually remarkably close to the best estimates using the full theory of General Relativity! [Even though the actual objects do not have precisely equal masses and the theory we have just used is not valid very close to the collision.]

B.5 $$M_{\text{c}} \simeq 30 \times M_{\odot}\,, \qquad M \simeq 69 \times M_{\odot}\,.$$	1.0pt

B.6 From (8), Kepler's law states that $L = (GM/\Omega^2)^{1/3}$. The second pair of points highlighted in the plot correspond to the cycle prior to merger. Thus, we use (19) to obtain the orbital angular velocity at $t_{\overline{\text{CD}}}$:

$$\Omega_{t_{\overline{\text{CD}}}} \sim 2.6 \times 10^2 \text{ rad/s}\,. \tag{26}$$

Then, using the total mass (25) we find

$$L \sim 5 \times 10^2 \text{ km}\,. \tag{27}$$

Thus, these objects have a maximum radius of $R_{\text{max}} \sim 250$ km. Hence they have over 30 times more mass and,

$$\frac{R_{\odot}}{R_{\text{max}}} \sim 3 \times 10^3 \tag{28}$$

they are 3000 times smaller than the Sun and!

Their linear velocity is

$$v_{\text{col}} = \frac{L}{2}\Omega \simeq 7 \times 10^4 \text{ km/s}\,. \tag{29}$$

They are moving at over 20% of the velocity of light!

B.6 $$L_{\text{collision}} \sim 5 \times 10^2 \text{ km}\,, \qquad \frac{R_{\odot}}{R_{\text{max}}} \sim 3 \times 10^3\,, \qquad \frac{v_{\text{col}}}{c} \sim 0.2\,.$$	1.0pt

IPhO 2018
Lisbon, Portugal

Confidential

Solutions to Theory Problem 2

Where is the neutrino?

(Miguel C N Fiolhais and António Onofre)

July 24, 2018

v1.2

Where is the neutrino? (10 points)

Part A. ATLAS Detector physics (4.0 points)

A.1

The magnetic force is the centripetal force:

$$m\frac{v^2}{r} = evB \Rightarrow r = \frac{mv}{eB}\,.$$

First express the velocity in terms of the kinetic energy,

$$K = \frac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2K}{m}}\,,$$

and then insert it in the expression above for the radius to get

A.1	0.5pt
$$r = \frac{\sqrt{2Km}}{eB}\,.$$	

A.2

The radius of the circular motion of a charged particle in the presence of a uniform magnetic field is given by,

$$r = \frac{mv}{eB}\,.$$

This formula is valid in the relativistic scenario if the mass correction, $m \to \gamma m$ is included:

$$r = \frac{\gamma mv}{eB} = \frac{p}{eB} \Rightarrow p = reB\,.$$

Note that the radius of the circular motion is half the radius of the inner part of the detector. One obtains [1 MeV/$c = 5.34 \times 10^{-22}$ m kg s^{-1}]

A.2	0.5pt
$$p = 330 \text{ MeV}/c\,.$$	

A.3

The acceleration for the particle is $a = \frac{evB}{\gamma m} \sim \frac{ecB}{\gamma m}$, in the ultrarelativistic limit. Then,

$$P = \frac{e^4c^2\gamma^4B^2}{6\pi\epsilon_0 c^3\gamma^2 m^2} = \frac{e^4\gamma^2c^4B^2}{6\pi\epsilon_0 c^5 m^2}.$$

Since $E = \gamma mc^2$ we can obtain $\gamma^2 c^4 = \frac{E^2}{m^2}$ and, finally,

$$P = \frac{e^4}{6\pi\epsilon_0 m^4 c^5} E^2 B^2 .$$

Therefore,

A.3 1.0pt

$$\xi = \frac{1}{6\pi}\,, \quad n = 5 \ \text{ and } \ k = 4.$$

A.4

The power emitted by the particle is given by,

$$P = -\frac{\mathrm{d}E}{\mathrm{d}t} = \frac{e^4}{6\pi\epsilon_0 m^4 c^5} E^2 B^2 \,.$$

The energy of the particle as a function of time can be calculated from

$$\int_{E_0}^{E(t)} \frac{1}{E^2} \mathrm{d}E = -\int_0^t \frac{e^4}{6\pi\epsilon_0 m^4 c^5} B^2 \mathrm{d}t\,,$$

where $E(0) = E_0$. This leads to,

$$\frac{1}{E(t)} - \frac{1}{E_0} = \frac{e^4 B^2}{6\pi\epsilon_0 m^4 c^5}\, t \quad \Rightarrow \quad E(t) = \frac{E_0}{1 + \alpha E_0 t}\,,$$

with

A.4 1.0pt

$$\alpha = \frac{e^4 B^2}{6\pi\epsilon_0 m^4 c^5}\,.$$

A.5

If the initial energy of the electron is 100 GeV, the radius of curvature is extremely large ($r = \frac{E}{eBc} \approx 167$ m). Therefore, in approximation, one can consider the electron is moving in the inner part of the ATLAS detector along a straight line. The time of flight of the electron is $t = R/c$, where $R = 1.1$ m is the radius of the inner part of the detector. The total energy lost due to synchrotron radiation is,

$$\Delta E = E(R/c) - E_0 = \frac{E_0}{1 + \alpha E_0 \frac{R}{c}} - E_0 \approx -\alpha E_0^2 \frac{R}{c}$$

and

A.5 0.5pt

$$\Delta E = -56 \text{ MeV}\,.$$

A.6

In the ultrarelativistic limit, $v \approx c$ and $E \approx pc$. The cyclotron frequency is,

$$\omega(t) = \frac{c}{r(t)} = \frac{ecB}{p(t)} = \frac{ec^2B}{E(t)}$$

A.6 0.5pt

$$\omega(t) = \frac{ec^2B}{E_0}\left(1 + \frac{e^4B^2}{6\pi\epsilon_0 m^4 c^5}E_0 t\right) .$$

Part B. Finding the neutrino (6.0 points)

B.1

Since the W^+ boson decays into an anti-muon and a neutrino, one can use principles of conservation of energy and linear momentum to calculate the unknown $p_z^{(\nu)}$ of the neutrino. Moreover, the anti-muon and the neutrino can be considered massless, which implies that the magnitude of their momenta (times c) and their energies are the same. Therefore, the conservation of linear momentum can be expressed as

$$\vec{p}^{\,(W)} = \vec{p}^{\,(\mu)} + \vec{p}^{\,(\nu)} ,$$

and the conservation of energy as,

$$E^{(W)} = cp^{(\mu)} + cp^{(\nu)} .$$

In addition, one can also relate the energy and the momentum of the W^+ boson through its mass,

$$m_W^2 = (E^{(W)})^2/c^4 - (p^{(W)})^2/c^2$$

which leads to a quadratic equation on $p_z^{(\nu)}$,

$$\begin{aligned} m_W^2 &= \left[(p^{(\mu)} + p^{(\nu)})^2 - (\vec{p}^{\,(\mu)} + \vec{p}^{\,(\nu)})^2\right]/c^2 \\ &= \left(2p^{(\mu)}p^{(\nu)} - 2\vec{p}^{\,(\mu)} \cdot \vec{p}^{\,(\nu)}\right)/c^2 \end{aligned}$$

B.1 1.5pt

$$m_W^2 = \frac{1}{c^2}\left(2p^{(\mu)}\sqrt{(p_{\mathrm{T}}^{(\nu)})^2 + (p_z^{(\nu)})^2} - 2\vec{p}_{\mathrm{T}}^{\,(\mu)} \cdot \vec{p}_{\mathrm{T}}^{\,(\nu)} - 2p_z^{(\mu)}p_z^{(\nu)}\right) .$$

B.2

The numerical substitution directly in the answer of B.1, using

$$p^{(\mu)} = 37.2\,\text{GeV}/c \qquad m_W^2c^2 = 6464.2\,(\text{GeV}/c)^2 \qquad p_{\mathrm{T}}^{(\nu)\,2} = 10\,864.9\,(\text{GeV}/c)^2$$

$$\vec{p}_{\mathrm{T}}^{\,(\mu)} \cdot \vec{p}_{\mathrm{T}}^{\,(\nu)} = 2439.3(\text{GeV}/c)^2 \qquad p_z^{(\mu)} = -12.4\,\text{GeV}/c ,$$

leads to

$$6464.2 = 74.4\sqrt{10\,864.9 + p_z^{(\nu)\,2}} - 4878.6 + 24.8p^{(\nu)} .$$

This is a quadratic equation, equivalent to

$$0.88889\,p_z^{(\nu)\,2} + 101.64\,p_z^{(\nu)} - 12378 = 0$$

whose solutions are:

B.2 1.5pt

$$p_z^{(\nu)} = 74.0 \text{ GeV}/c \qquad \text{or} \qquad p_z^{(\nu)} = -188.3 \text{ GeV}/c.$$

The general solution of the equation above in B.1 leads to

$$\begin{aligned} p_z^{(\nu)} &= \frac{2\vec{p}_{\mathrm{T}}^{(\mu)} \cdot \vec{p}_{\mathrm{T}}^{(\nu)} p_z^{(\mu)} + m_W^2 c^2 p_z^{(\mu)}}{2(p_{\mathrm{T}}^{(\mu)})^2} \\ &\pm \frac{p^{(\mu)} \sqrt{-4(p_{\mathrm{T}}^{(\mu)})^2 (p_{\mathrm{T}}^{(\nu)})^2 + 4(\vec{p}_{\mathrm{T}}^{(\mu)} \cdot \vec{p}_{\mathrm{T}}^{(\nu)})^2 + 4\vec{p}_{\mathrm{T}}^{(\mu)} \cdot \vec{p}_{\mathrm{T}}^{(\nu)} m_W^2 c^2 + m_W^4 c^4}}{2(p_{\mathrm{T}}^{(\mu)})^2} \end{aligned}$$

Numerical substitution leads to the above mentioned values for $p_z^{(\nu)}$.

B.3

The final state particles of the top quark decay are the anti-muon, the neutrino and jet 1. Since the neutrino is now fully reconstructed the energy and linear momentum of the top quark can be calculated as,

$$\begin{aligned} E^{(\mathrm{t})} &= cp^{(\mu)} + cp^{(\nu)} + cp^{(j_1)} \\ \vec{p}^{\,(\mathrm{t})} &= \vec{p}^{\,(\mu)} + \vec{p}^{\,(\nu)} + \vec{p}^{\,(j_1)}\,. \end{aligned}$$

The top quark mass is,

$$\begin{aligned} m_{\mathrm{t}} &= \sqrt{(E^{(\mathrm{t})})^2/c^4 - (\vec{p}^{\,(\mathrm{t})})^2/c^2} \\ &= c^{-1}\sqrt{\left(p^{(\mu)} + p^{(\nu)} + p^{(j_1)}\right)^2 - \left(\vec{p}^{\,(\mu)} + \vec{p}^{\,(\nu)} + \vec{p}^{\,(j_1)}\right)^2}\,. \end{aligned}$$

The substitution of values leads to two possible masses:

B.3 1.0pt

$$m_{\mathrm{t}} = 169.3 \text{ GeV}/c^2 \qquad \text{or} \qquad m_{\mathrm{t}} = 311.2 \text{ GeV}/c^2$$

B.4

According to the frequency distribution for signal (dashed line), the probability of the $m_{\mathrm{t}} = 169.3$ GeV/c^2 solution is roughly 0.1 while the probability of the $m_{\mathrm{t}} = 311.2$ GeV/c^2 solution is below 0.01. Therefore,

B.4 The most likely candidate is the $m_{\mathrm{t}} = 169.3$ GeV/c^2 solution. 1.0pt

B.5

The top quark energy for the most likely candidate is $E^{(\mathrm{t})} = cp^{(\mu)} + cp^{(\nu)} + cp^{(j_1)} = 272.6$ GeV .

$$d = vt = v\gamma t_0 = \frac{p^{(\mathrm{t})}}{m_\mathrm{t}} t_0 = ct_0 \sqrt{\frac{{E^{(\mathrm{t})}}^2}{m_\mathrm{t}^2 c^4} - 1} \,.$$

B.5	1.0pt
$d = 2 \times 10^{-16}$ m .	

IPhO 2018
Lisbon, Portugal

Solutions to Theory Problem 3

Physics of Live Systems

(Rui Travasso, Lucília Brito)

July 24, 2018

v1.0

Physics of Live Systems (10 points)

Part A. The physics of blood flow (4.5 points)

A.1

Since the vessel network is symmetrical, the flow in a vessel of level $i+1$ is half the flow in a vessel of level i.

In this way, we can sum the pressure differences in all levels:

$$\Delta P = \sum_{i=0}^{N-1} Q_i R_i = Q_0 \sum_{i=0}^{N-1} \frac{R_i}{2^i}.$$

Introducing the radii dependences yields

$$\Delta P = Q_0 \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2^i \pi r_i^4} = Q_0 \frac{8\ell_0 \eta}{\pi r_0^4} \sum_{i=0}^{N-1} \frac{2^{4i/3}}{2^i 2^{i/3}} = Q_0 N \frac{8\ell_0 \eta}{\pi r_0^4}.$$

Therefore

$$Q_0 = \Delta P \frac{\pi r_0^4}{8N\ell_0 \eta}.$$

Hence, the flow rate for a vessel network in level i is

A.1 1.3pt

$$Q_i = \Delta P \frac{\pi r_0^4}{2^{i+3} N \ell_0 \eta}.$$

A.2

Replace values in the formula and change units appropriately

$$\begin{aligned} Q_0 &= \frac{\Delta P \pi r_0^4}{8N\ell_0 \eta} = \\ &= \frac{(55-30) \times 1.013 \times 10^5 \times 3.1415 \times (6.0 \times 10^{-5})^4}{760 \times 48 \times 2.0 \times 10^{-3} \times 3.5 \times 10^{-3}} = 4.0 \times 10^{-10}\ \mathrm{m^3/s} \end{aligned}$$

to obtain the final value in the requested unites:

A.2 0.5pt

$$Q_0 \simeq 1.5\ \mathrm{m}\ell/\mathrm{h}.$$

A.3

The current is given by

$$I = \frac{P_{\text{in}} \mathrm{e}^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}} \,.$$

The pressure difference in the capacitor is

$$P_{\text{out}} \mathrm{e}^{i(\omega t + \phi)} = \frac{P_{\text{in}} \mathrm{e}^{i\omega t}}{R + i\omega L + \frac{1}{i\omega C}} \, \frac{1}{i\omega C} = \frac{P_{\text{in}} \mathrm{e}^{i\omega t}}{i\omega C R - \omega^2 L C + 1} \,.$$

The amplitude is

$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}} \,.$$

To be smaller than P_{in}, for $\omega \to 0$:

$$(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2 > 1 \iff -2CL + C^2 R^2 > 0 \,.$$

Replacing the expressions for L, C, and R we get: $\frac{64\eta^2 \ell^2}{3Ehr^3\rho} > 1$.

A.3	2.0pt

$$P_{\text{out}} = \frac{P_{\text{in}}}{\sqrt{(1 - \omega^2 LC)^2 + \omega^2 C^2 R^2}} \,.$$

Condition:

$$\frac{64\eta^2 \ell^2}{3Ehr^3\rho} > 1 \,.$$

Alternative way to obtain P_{out}:

The amplitude of the current in the equivalent circuit is $I_0 = \frac{P_{\text{in}}}{Z}$, where

$$Z = \sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}$$

is the modulus of the impedance. Hence, the voltage amplitude in the capacitor is

$$P_{\text{out}} = \frac{1}{\omega C} \times I_0 = \frac{P_{\text{in}}}{\sqrt{\omega^2 C^2 R^2 + (\omega^2 LC - 1)^2}} .$$

A.4

The previous condition can also be expressed as

$$h < \frac{64\eta^2 \ell^2}{3Er^3\rho} \,.$$

For the network referred to in **A.2**

$$h < \frac{64\eta^2 \ell_0^2 \times 2^i}{3 \times 2^{2i/3} E r_0^3 \rho} = \frac{64 \times (3.5 \times 10^{-3})^2 \times (2.0 \times 10^{-3})^2}{3 \times 0.06 \times 10^6 \times (6.0 \times 10^{-5})^3 \times 1.05 \times 10^3} \times 2^{i/3} = 7.7 \times 10^{-5} \times 2^{i/3} \,.$$

For $i = 0$, in the worse case scenario,

$$h_{\text{max}} = 7.7 \times 10^{-5} \times 2^{0} = 7.7 \times 10^{-5} \text{ m}$$

This value is certainly observed in these vessels since their radius range from 18 μm to 60 μm. A wall width smaller than 80 μm is certainly reasonable.

A.4	Maximum $h = 8 \times 10^{-5}$ m	0.7pt

Part B. Tumor growth (5.5 points)

B.1

The expressions for the masses of tumour and normal tissue are written as:

$$\begin{cases} M_{\text{T}} = V_{\text{T}}\rho_{\text{T}} = V_{\text{T}}\rho_0(1 + \frac{p}{K_{\text{T}}}) \\ \\ M_{\text{N}} = V\rho_0 = (V - V_{\text{T}})\rho_0(1 + \frac{p}{K_{\text{N}}}) \end{cases}$$

The pressure, p, can be expressed as

$$p = \frac{M_{\text{T}}\,K_{\text{T}}}{V_{\text{T}}\,\rho_0} - K_{\text{T}}$$

and, then, used in the equation for M_{N}:

$$M_{\text{N}} = (V - V_{\text{T}})\,\frac{M_{\text{N}}}{V}\left[\left(1 - \frac{K_{\text{T}}}{K_{\text{N}}}\right) + \frac{M_{\text{T}}\,V K_{\text{T}}}{V_{\text{T}}\,M_{\text{N}}\,K_{\text{N}}}\right]$$

Simplifying and rearranging the terms, the equation for v becomes

$$(1 - \kappa)\;v^2 - (1 + \mu)\;v + \mu = 0\,,$$

for which the solution is (the other solution of the quadratic equation is not physically relevant since does not lead to $v = 0$ for $\mu = 0$)

B.1 1.0pt

$$v = \frac{1 + \mu - \sqrt{(1 + \mu)^2 - 4\mu\,(1 - \kappa)}}{2(1 - \kappa)}\,.$$

B.2

For $r < R_{\text{T}}$, the conservation of energy implies that

$$4\pi r^2(-k)\frac{\mathrm{d}T}{\mathrm{d}r} = \mathcal{P}\frac{4}{3}\pi r^3\,.$$

Therefore, the temperature difference to 37 °C = 310.15 K, $\Delta T(r)$, is given by

$$\Delta T(r) = -\frac{\mathcal{P} r^2}{6k} + C,$$

where C is a constant.

For $r > R_\mathrm{T}$, the conservation of energy implies that

$$4\pi r^2(-k)\frac{\mathrm{d}T}{\mathrm{d}r} = \mathcal{P}\frac{4}{3}\pi R_\mathrm{T}^3 .$$

Therefore, the temperature difference to 37 °C is

$$\Delta T(r) = \frac{\mathcal{P} R_\mathrm{T}^3}{3kr} .$$

In this case there is no constant, since very far away the increase in temperature is zero.

Matching the two solutions at $r = R_\mathrm{T}$ gives

$$C = \frac{\mathcal{P} R_\mathrm{T}^2}{2k} .$$

Therefore the temperature at the centre of the tumour, in SI units, is

B.2	Temperature: $310.15 + \frac{\mathcal{P} R_\mathrm{T}^2}{2k}$.	1.7pt

B.3

The increase in temperature at the tumour surface (the lower temperature in the tumour) is

$$\Delta T(R_\mathrm{T}) = \frac{\mathcal{P} R_\mathrm{T}^2}{3k} .$$

This increase should be equal to 6.0 K. Therefore,

$$\mathcal{P} = \frac{3\Delta T k}{R_\mathrm{T}^2} = \frac{3 \times 6 \times 0.6}{0.05^2} = 4.3\ \mathrm{kW/m^3}.$$

B.3	$\mathcal{P}_\mathrm{min} = 4.3\ \mathrm{kW/m^3}$.	0.5pt

B.4

We can relate δr with the pressure in the tumour, using the relation given in the text up to leading order in $p - P_\mathrm{cap}$: $\delta r = \frac{p - P_\mathrm{cap}}{2(p_\mathrm{c} - P_\mathrm{cap})}\,\delta r_\mathrm{c}$. Therefore, if $p - P_\mathrm{cap}$ is very small, also it is δr.

The pressure can be related with the volume. We know that

$$\frac{M_\mathrm{N}}{V_\mathrm{N}} = \frac{\rho_0 V}{V - V_\mathrm{T}} = \frac{\rho_0}{1 - v} = \rho_0\left(1 + \frac{p}{K_\mathrm{N}}\right) .$$

And so $p = \frac{K_\mathsf{N} v}{1-v}$.

When the thinner vessels are narrower, the flow rate in the main vessel is altered:

$$\Delta P = (Q_0 + \delta Q_0) \sum_{i=0}^{N-1} \frac{8\ell_i \eta}{2^i \pi r_i^4} = (Q_0 + \delta Q_0) \frac{8\ell_0 \eta}{\pi r_0^4} \left(\sum_{i=0}^{N-2} \frac{2^{4i/3}}{2^i 2^{i/3}} + \frac{2^{4(N-1)/3}}{2^{N-1} 2^{(N-1)/3} \left(1 - \frac{\delta r}{r_0/2^{(N-1)/3}}\right)^4} \right)$$

$$\implies \Delta P \simeq (Q_0 + \delta Q_0) \frac{\Delta P}{N Q_0} \left(N - 1 + 1 + \frac{4\,\delta r}{r_{N-1}} \right)$$

Noting that $\frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{\delta Q_0}{Q_0}$, we obtain

$$1 + \frac{\delta Q_{N-1}}{Q_{N-1}} = \frac{1}{1 + \frac{4\,\delta r}{N r_{N-1}}} \simeq 1 - \frac{4\,\delta r}{N r_{N-1}}\,.$$

And so:

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq -\frac{4}{N} \frac{\delta r}{r_{N-1}}\,.$$

Putting all together

B.4 2.3pt

$$\frac{\delta Q_{N-1}}{Q_{N-1}} \simeq -\frac{2}{N} \frac{K_\mathsf{N} v - (1-v) P_\mathsf{cap}}{(1-v)(p_\mathsf{c} - P_\mathsf{cap})} \frac{\delta r_\mathsf{c}}{r_{N-1}}\,.$$