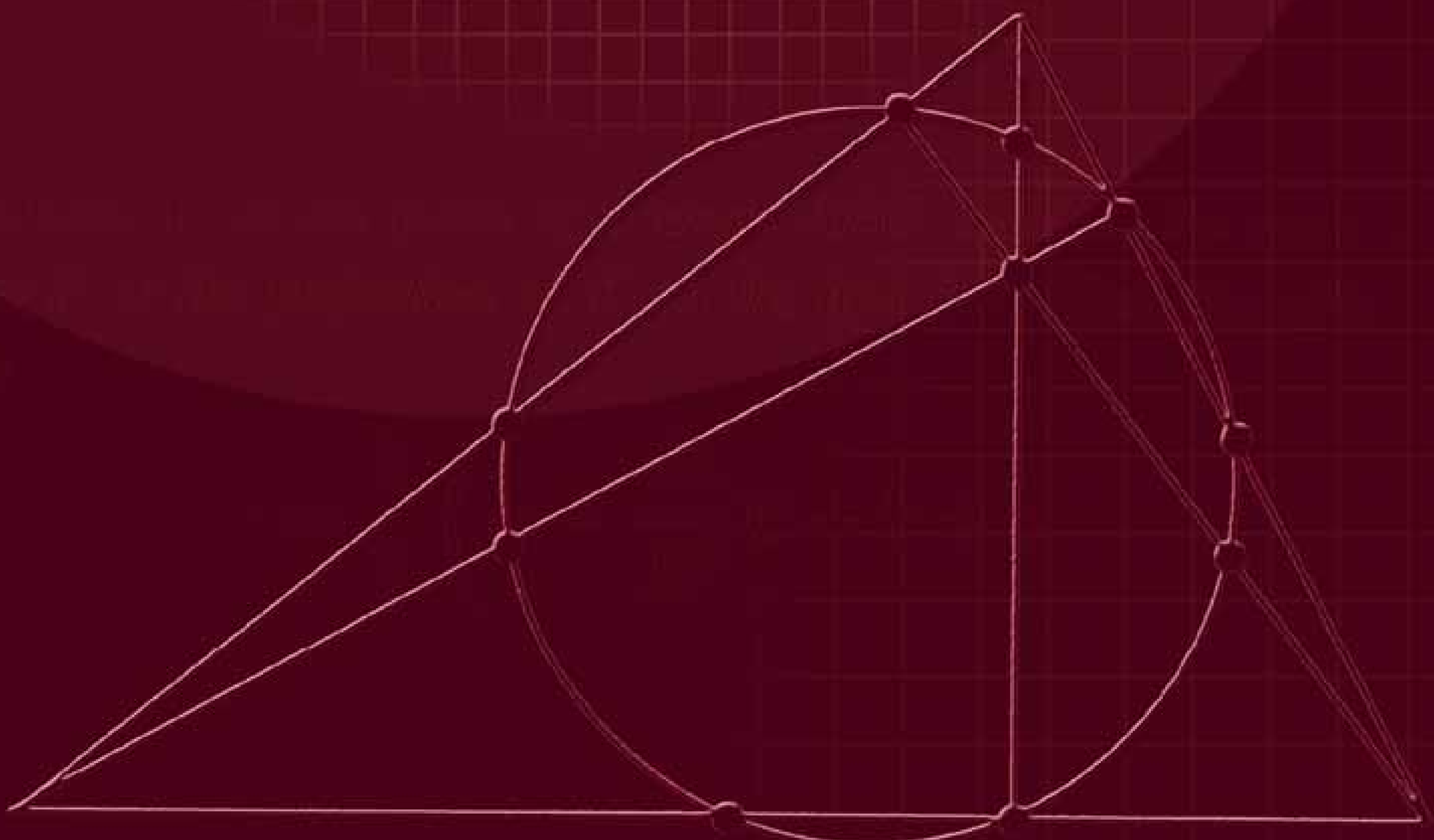
Complex Numbers from A to...Z

Titu Andreescu
Dorin Andrica



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About the Authors

Titu Andreescu received his BA, MS, and PhD from the West University of Timisoara, Romania. The topic of his doctoral dissertation was "Research on Diophantine Analysis and Applications." Professor Andreescu currently teaches at the University of Texas at Dallas. Titu is past chairman of the USA Mathematical Olympiad, served as director of the MAA American Mathematics Competitions (1998–2003), coach of the USA International Mathematical Olympiad Team (IMO) for 10 years (1993–2002), Director of the Mathematical Olympiad Summer Program (1995–2002) and leader of the USA IMO Team (1995–2002). In 2002 Titu was elected member of the IMO Advisory Board, the governing body of the world's most prestigious mathematics competition. Titu received the Edyth May Sliffe Award for Distinguished High School Mathematics Teaching from the MAA in 1994 and a "Certificate of Appreciation" from the president of the MAA in 1995 for his outstanding service as coach of the Mathematical Olympiad Summer Program in preparing the US team for its perfect performance in Hong Kong at the 1994 IMO. Titu's contributions to numerous textbooks and problem books are recognized worldwide.

Dorin Andrica received his PhD in 1992 from "Babeş-Bolyai" University in Cluj-Napoca, Romania, with a thesis on critical points and applications to the geometry of differentiable submanifolds. Professor Andrica has been chairman of the Department of Geometry at "Babeş-Bolyai" since 1995. Dorin has written and contributed to numerous mathematics textbooks, problem books, articles and scientific papers at various levels. Dorin is an invited lecturer at university conferences around the world—Austria, Bulgaria, Czech Republic, Egypt, France, Germany, Greece, the Netherlands, Serbia, Turkey, and USA. He is a member of the Romanian Committee for the Mathematics Olympiad and member of editorial boards of several international journals. Dorin has been a regular faculty member at the Canada–USA Mathcamps since 2001.

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The shortest path between two truths in the real domain passes through the complex domain.

Jacques Hadamard

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Contents

Preface

Solving algebraic equations has been historically one of the favorite topics of mathematicians. While linear equations are always solvable in real numbers, not all quadratic equations have this property. The simplest such equation is $x^2 + 1 = 0$. Until the 18th century, mathematicians avoided quadratic equations that were not solvable over $\mathbb{R}$. Leonhard Euler broke the ice introducing the "number" $\sqrt{-1}$ in his famous book *Elements of Algebra* as " ... *neither nothing, nor greater than nothing, nor less than nothing* ... " and observed " ... *notwithstanding this, these numbers present themselves to the mind; they exist in our imagination and we still have a sufficient idea of them; ... nothing prevents us from making use of these imaginary numbers, and employing them in calculation*". Euler denoted the number $\sqrt{-1}$ by i and called it the imaginary unit. This became one of the most useful symbols in mathematics. Using this symbol one defines complex numbers as $z = a + bi$, where a and b are real numbers. The study of complex numbers continues and has been enhanced in the last two and a half centuries; in fact, it is impossible to imagine modern mathematics without complex numbers. All mathematical domains make use of them in some way. This is true of other disciplines as well: for example, mechanics, theoretical physics, hydrodynamics, and chemistry.

Our main goal is to introduce the reader to this fascinating subject. The book runs smoothly between key concepts and elementary results concerning complex numbers. The reader has the opportunity to learn how complex numbers can be employed in solving algebraic equations, and to understand the geometric interpretation of complex numbers and the operations involving them. The theoretical part of the book is augmented by rich exercises and problems of various levels of difficulty. In Chapters 3 and 4 we cover important applications in Euclidean geometry. Many geometry problems may be solved efficiently and elegantly using complex numbers. The wealth of examples we provide, the presentation of many topics in a personal manner, the presence of numerous original problems, and the attention to detail in the solutions to selected exercises and problems are only some of the key features of this book.

Among the techniques presented, for example, are those for the real and the complex product of complex numbers. In complex number language, these are the analogues of the scalar and cross products, respectively. Employing these two products turns out to be efficient in solving numerous problems involving complex numbers. After covering this part, the reader will appreciate the use of these techniques.

A special feature of the book is Chapter 5, an outstanding selection of genuine Olympiad and other important mathematical contest problems solved using the methods already presented.

This work does not cover all aspects pertaining to complex numbers. It is not a complex analysis book, but rather a stepping stone in its study, which is why we have not used the standard notation e^{it} for $z = \cos t + i \sin t$, or the usual power series expansions.

The book reflects the unique experience of the authors. It distills a vast mathematical literature, most of which is unknown to the western public, capturing the essence of an abundant problem-solving culture.

Our work is partly based on a Romanian version, *Numere complexe de la A la ... Z*, authored by D. Andrica and N. Bişboacă and published by Millennium in 2001 (see our reference [10]). We are preserving the title of the Romanian edition and about 35% of the text. Even this 35% has been significantly improved and enhanced with up-to-date material.

The targeted audience includes high school students and their teachers, undergraduates, mathematics contestants such as those training for Olympiads or the W. L. Putnam Mathematical Competition, their coaches, and any person interested in essential mathematics.

This book might spawn courses such as Complex Numbers and Euclidean Geometry for prospective high school teachers, giving future educators ideas about things they could do with their brighter students or with a math club. This would be quite a welcome development.

Special thanks are given to Daniel Văcăreţu, Nicolae Bişboacă, Gabriel Dospinescu, and Ioan Şerdean for the careful proofreading of the final version of the manuscript. We

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Titu Andreescu
Dorin Andrica
October 2004

Notation

$\mathbb{Z}$	the set of integers
$\mathbb{N}$	the set of positive integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}^*$	the set of nonzero real numbers
$\mathbb{R}^2$	the set of pairs of real numbers
$\mathbb{C}$	the set of complex numbers
$\mathbb{C}^*$	the set of nonzero complex numbers
$[a, b]$	the set of real numbers x such that $a \leq x \leq b$
(a, b)	the set of real numbers x such that $a < x < b$
$\overline{z}$	the conjugate of the complex number z
$\lvert z \rvert$	the modulus or absolute value of complex number z
$\overrightarrow{AB}$	the vector AB
(AB)	the open segment determined by A and B
$[AB]$	the closed segment determined by A and B
$(AB$	the open ray of origin A that contains B
area$[F]$	the area of figure F
U_n	the set of n^{th} roots of unity
$\mathcal{C}(P; n)$	the circle centered at point P with radius n

1

Complex Numbers in Algebraic Form

1.1 Algebraic Representation of Complex Numbers

1.1.1 Definition of complex numbers

In what follows we assume that the definition and basic properties of the set of real numbers $\mathbb{R}$ are known.

Let us consider the set $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) |\ x, y \in \mathbb{R}\}$. Two elements (x_1, y_1) and (x_2, y_2) of $\mathbb{R}^2$ are equal if and only if $x_1 = x_2$ and $y_1 = y_2$. The operations of addition and multiplication are defined on the set $\mathbb{R}^2$ as follows:

$$z_1 + z_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in \mathbb{R}^2$$

and

$$z_1 \cdot z_2 = (x_1, y_1) \cdot (x_2, y_2) = (x_1x_2 - y_1y_2, x_1y_2 + x_2y_1) \in \mathbb{R}^2,$$

for all $z_1 = (x_1, y_1) \in \mathbb{R}^2$ and $z_2 = (x_2, y_2) \in \mathbb{R}^2$.

The element $z_1 + z_2 \in \mathbb{R}^2$ is called the *sum* of z_1, z_2 and the element $z_1 \cdot z_2 \in \mathbb{R}^2$ is called the *product* of z_1, z_2.

Remarks. 1) If $z_1 = (x_1, 0) \in \mathbb{R}^2$ and $z_2 = (x_2, 0) \in \mathbb{R}^2$, then $z_1 \cdot z_2 = (x_1x_2, 0)$.
(2) If $z_1 = (0, y_1) \in \mathbb{R}^2$ and $z_2 = (0, y_2) \in \mathbb{R}^2$, then $z_1 \cdot z_2 = (-y_1y_2, 0)$.

Examples. 1) Let $z_1 = (-5, 6)$ and $z_2 = (1, -2)$. Then

$$z_1 + z_2 = (-5, 6) + (1, -2) = (-4, 4)$$

and

$$z_1 z_2 = (-5, 6) \cdot (1, -2) = (-5 + 12, 10 + 6) = (7, 16).$$

(2) Let $z_1 = \left(-\frac{1}{2}, 1\right)$ and $z_2 = \left(-\frac{1}{3}, \frac{1}{2}\right)$. Then

$$z_1 + z_2 = \left(-\frac{1}{2} - \frac{1}{3}, 1 + \frac{1}{2}\right) = \left(-\frac{5}{6}, \frac{3}{2}\right)$$

and

$$z_1 z_2 = \left(\frac{1}{6} - \frac{1}{2}, -\frac{1}{4} - \frac{1}{3}\right) = \left(-\frac{1}{3}, -\frac{7}{12}\right).$$

Definition. The set $\mathbb{R}^2$, together with the addition and multiplication operations, is called the *set of complex numbers*, denoted by $\mathbb{C}$. Any element $z = (x, y) \in \mathbb{C}$ is called a *complex number*.

The notation $\mathbb{C}^*$ is used to indicate the set $\mathbb{C} \setminus \{(0, 0)\}$.

1.1.2 Properties concerning addition

The addition of complex numbers satisfies the following properties:

(a) **Commutative law**

$$z_1 + z_2 = z_2 + z_1 \text{ for all } z_1, z_2 \in \mathbb{C}.$$

(b) **Associative law**

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3) \text{ for all } z_1, z_2, z_3 \in \mathbb{C}.$$

Indeed, if $z_1 = (x_1, y_1) \in \mathbb{C}$, $z_2 = (x_2, y_2) \in \mathbb{C}$, $z_3 = (x_3, y_3) \in \mathbb{C}$, then

$$(z_1 + z_2) + z_3 = [(x_1, y_1) + (x_2, y_2)] + (x_3, y_3)$$

$$= (x_1 + x_2, y_1 + y_2) + (x_3, y_3) = ((x_1 + x_2) + x_3, (y_1 + y_2) + y_3),$$

and

$$z_1 + (z_2 + z_3) = (x_1, y_1) + [(x_2, y_2) + (x_3, y_3)]$$

$$= (x_1, y_1) + (x_2 + x_3, y_2 + y_3) = (x_1 + (x_2 + x_3), y_1 + (y_2 + y_3)).$$

The claim holds due to the associativity of the addition of real numbers.

(c) **Additive identity** There is a unique complex number $0 = (0, 0)$ such that

$$z + 0 = 0 + z = z \text{ for all } z = (x, y) \in \mathbb{C}.$$

(d) **Additive inverse** For any complex number $z = (x, y)$ there is a unique $-z = (-x, -y) \in \mathbb{C}$ such that

$$z + (-z) = (-z) + z = 0.$$

The reader can easily prove the claims (a), (c) and (d).

The number $z_1 - z_2 = z_1 + (-z_2)$ is called the *difference* of the numbers z_1 and z_2. The operation that assigns to the numbers z_1 and z_2 the number $z_1 - z_2$ is called *subtraction* and is defined by

$$z_1 - z_2 = (x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2) \in \mathbb{C}.$$

1.1.3 Properties concerning multiplication

The multiplication of complex numbers satisfies the following properties:

(a) **Commutative law**

$$z_1 \cdot z_2 = z_2 \cdot z_1 \text{ for all } z_1, z_2 \in \mathbb{C}.$$

(b) **Associative law**

$$(z_1 \cdot z_2) \cdot z_3 = z_1 \cdot (z_2 \cdot z_3) \text{ for all } z_1, z_2, z_3 \in \mathbb{C}.$$

(c) **Multiplicative identity** There is a unique complex number $1 = (1, 0) \in \mathbb{C}$ such that

$$z \cdot 1 = 1 \cdot z = z \text{ for all } z \in \mathbb{C}.$$

A simple algebraic manipulation is all that is needed to verify these equalities:

$$z \cdot 1 = (x, y) \cdot (1, 0) = (x \cdot 1 - y \cdot 0, x \cdot 0 + y \cdot 1) = (x, y) = z$$

and

$$1 \cdot z = (1, 0) \cdot (x, y) = (1 \cdot x - 0 \cdot y, 1 \cdot y + 0 \cdot x) = (x, y) = z.$$

(d) **Multiplicative inverse** For any complex number $z = (x, y) \in \mathbb{C}^*$ there is a unique number $z^{-1} = (x', y') \in \mathbb{C}$ such that

$$z \cdot z^{-1} = z^{-1} \cdot z = 1.$$

To find $z^{-1} = (x', y')$, observe that $(x, y) \neq (0, 0)$ implies $x \neq 0$ or $y \neq 0$ and consequently $x^2 + y^2 \neq 0$.

The relation $z \cdot z^{-1} = 1$ gives $(x, y) \cdot (x', y') = (1, 0)$, or equivalently

$$\begin{cases} xx' - yy' = 1 \\ yx' + xy' = 0. \end{cases}$$

Solving this system with respect to x' and y', one obtains

$$x' = \frac{x}{x^2 + y^2} \text{ and } y' = -\frac{y}{x^2 + y^2},$$

hence the multiplicative inverse of the complex number $z = (x, y) \in \mathbb{C}^*$ is

$$z^{-1} = \frac{1}{z} = \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2} \right) \in \mathbb{C}^*.$$

By the commutative law we also have $z^{-1} \cdot z = 1$.

Two complex numbers $z_1 = (z_1, y_1) \in \mathbb{C}$ and $z = (x, y) \in \mathbb{C}^*$ uniquely determine a third number called their *quotient*, denoted by $\frac{z_1}{z}$ and defined by

$$\frac{z_1}{z} = z_1 \cdot z^{-1} = (x_1, y_1) \cdot \left(\frac{x}{x^2 + y^2}, -\frac{y}{x^2 + y^2} \right)$$
$$= \left(\frac{x_1 x + y_1 y}{x^2 + y^2}, \frac{-x_1 y + y_1 x}{x^2 + y^2} \right) \in \mathbb{C}.$$

Examples. 1) If $z = (1, 2)$, then

$$z^{-1} = \left(\frac{1}{1^2 + 2^2}, \frac{-2}{1^2 + 2^2} \right) = \left(\frac{1}{5}, \frac{-2}{5} \right).$$

2) If $z_1 = (1, 2)$ and $z_2 = (3, 4)$, then

$$\frac{z_1}{z_2} = \left(\frac{3 + 8}{9 + 16}, \frac{-4 + 6}{9 + 16} \right) = \left(\frac{11}{25}, \frac{2}{25} \right).$$

An integer power of a complex number $z \in \mathbb{C}^*$ is defined by

$$z^0 = 1; \quad z^1 = z; \quad z^2 = z \cdot z;$$

$$z^n = \underbrace{z \cdot z \cdots z}_{n \text{ times}} \text{ for all integers } n > 0$$

and $z^n = (z^{-1})^{-n}$ for all integers $n < 0$.

The following properties hold for all complex numbers $z, z_1, z_2 \in \mathbb{C}^*$ and for all integers m, n:

1) $z^m \cdot z^n = z^{m+n}$;

2) $\frac{z^m}{z^n} = z^{m-n}$;

3) $(z^m)^n = z^{mn}$;

4) $(z_1 \cdot z_2)^n = z_1^n \cdot z_2^n$;

5) $\left(\frac{z_1}{z_2} \right)^n = \frac{z_1^n}{z_2^n}$.

When $z = 0$, we define $0^n = 0$ for all integers $n > 0$.

e) **Distributive law**

$$z_1 \cdot (z_2 + z_3) = z_1 \cdot z_2 + z_1 \cdot z_3 \text{ for all } z_1, z_2, z_3 \in \mathbb{C}.$$

The above properties of addition and multiplication show that the set $\mathbb{C}$ of all complex numbers, together with these operations, forms a field.

1.1.4 Complex numbers in algebraic form

For algebraic manipulation it is not convenient to represent a complex number as an ordered pair. For this reason another form of writing is preferred.

To introduce this new algebraic representation, consider the set $\mathbb{R} \times \{0\}$, together with the addition and multiplication operations defined on $\mathbb{R}^2$. The function

$$f : \mathbb{R} \to \mathbb{R} \times \{0\}, \quad f(x) = (x, 0)$$

is bijective and moreover,

$$(x, 0) + (y, 0) = (x + y, 0) \text{ and } (x, 0) \cdot (y, 0) = (xy, 0).$$

The reader will not fail to notice that the algebraic operations on $\mathbb{R} \times \{0\}$ are similar to the operations on $\mathbb{R}$; therefore we can identify the ordered pair $(x, 0)$ with the number x for all $x \in \mathbb{R}$. Hence we can use, by the above bijection f, the notation $(x, 0) = x$.

Setting $i = (0, 1)$ we obtain

$$z = (x, y) = (x, 0) + (0, y) = (x, 0) + (y, 0) \cdot (0, 1)$$

$$= x + yi = (x, 0) + (0, 1) \cdot (y, 0) = x + iy.$$

In this way we obtain

Proposition. *Any complex number $z = (x, y)$ can be uniquely represented in the form*

$$z = x + yi,$$

where x, y are real numbers. The relation $i^2 = -1$ holds.

The formula $i^2 = -1$ follows directly from the definition of multiplication: $i^2 = i \cdot i = (0, 1) \cdot (0, 1) = (-1, 0) = -1$.

The expression $x + yi$ is called the *algebraic representation* (*form*) of the complex number $z = (x, y)$, so we can write $\mathbb{C} = \{x + yi \mid x \in \mathbb{R},\ y \in \mathbb{R},\ i^2 = -1\}$. From now on we will denote the complex number $z = (x, y)$ by $x + iy$. The real number $x = \operatorname{Re}(z)$ is called the *real part* of the complex number z and similarly, $y = \operatorname{Im}(z)$ is called the *imaginary part* of z. Complex numbers of the form iy, $y \in \mathbb{R}$ — in other words, complex numbers whose real part is 0 — are called *imaginary*. On the other hand, complex numbers of the form iy, $y \in \mathbb{R}^*$ are called *purely imaginary* and the complex number i is called the *imaginary unit*.

The following relations are easy to verify:

a) $z_1 = z_2$ if and only if $\text{Re}(z)_1 = \text{Re}(z)_2$ and $\text{Im}(z)_1 = \text{Im}(z)_2$.

b) $z \in \mathbb{R}$ if and only if $\text{Im}(z) = 0$.

c) $z \in \mathbb{C} \setminus \mathbb{R}$ if and only if $\text{Im}(z) \neq 0$.

Using the algebraic representation, the usual operations with complex numbers can be performed as follows:

1. Addition

$$z_1 + z_2 = (x_1 + y_1 i) + (x_2 + y_2 i) = (x_1 + x_2) + (y_1 + y_2)i \in \mathbb{C}.$$

It is easy to observe that the sum of two complex numbers is a complex number whose real (imaginary) part is the sum of the real (imaginary) parts of the given numbers:

$$\text{Re}(z_1 + z_2) = \text{Re}(z)_1 + \text{Re}(z)_2;$$
$$\text{Im}(z_1 + z_2) = \text{Im}(z)_1 + \text{Im}(z)_2.$$

2. Multiplication

$$z_1 \cdot z_2 = (x_1 + y_1 i)(x_2 + y_2 i) = (x_1 x_2 - y_1 y_2) + (x_1 y_2 + x_2 y_1)i \in \mathbb{C}.$$

In other words,

$$\text{Re}(z_1 z_2) = \text{Re}(z)_1 \cdot \text{Re}(z)_2 - \text{Im}(z)_1 \cdot \text{Im}(z)_2$$

and

$$\text{Im}(z_1 z_2) = \text{Im}(z)_1 \cdot \text{Re}(z)_2 + \text{Im}(z)_2 \cdot \text{Re}(z)_1.$$

For a real number λ and a complex number $z = x + yi$,

$$\lambda \cdot z = \lambda(x + yi) = \lambda x + \lambda y i \in \mathbb{C}$$

is the product of a real number with a complex number. The following properties are obvious:

1) $\lambda(z_1 + z_2) = \lambda z_1 + \lambda z_2$;

2) $\lambda_1(\lambda_2 z) = (\lambda_1 \lambda_2)z$;

3) $(\lambda_1 + \lambda_2)z = \lambda_1 z + \lambda_2 z$ for all $z, z_1, z_2 \in \mathbb{C}$ and $\lambda, \lambda_1, \lambda_2 \in \mathbb{R}$.

Actually, relations 1) and 3) are special cases of the distributive law and relation 2) comes from the associative law of multiplication for complex numbers.

3. Subtraction

$$z_1 - z_2 = (x_1 + y_1 i) - (x_2 + y_2 i) = (x_1 - x_2) + (y_1 - y_2)i \in \mathbb{C}.$$

That is,

$$\operatorname{Re}(z_1 - z_2) = \operatorname{Re}(z)_1 - \operatorname{Re}(z)_2;$$
$$\operatorname{Im}(z_1 - z_2) = \operatorname{Im}(z)_1 - \operatorname{Im}(z)_2.$$

1.1.5 Powers of the number i

The formulas for the powers of a complex number with integer exponents are preserved for the algebraic form $z = x + iy$. Setting $z = i$, we obtain

$$i^0 = 1; \quad i^1 = i; \quad i^2 = -1; \quad i^3 = i^2 \cdot i = -i;$$

$$i^4 = i^3 \cdot i = 1; \quad i^5 = i^4 \cdot i = i; \quad i_6 = i^5 \cdot i = -1; \quad i^7 = i^6 \cdot i = -i.$$

One can prove by induction that for any positive integer n,

$$i^{4n} = 1; \quad i^{4n+1} = i; \quad i^{4n+2} = -1; \quad i^{4n+3} = -i.$$

Hence $i^n \in \{-1, 1, -i, i\}$ for all integers $n \geq 0$. If n is a negative integer, we have

$$i^n = (i^{-1})^{-n} = \left(\frac{1}{i}\right)^{-n} = (-i)^{-n}.$$

Examples. 1) We have

$$i^{105} + i^{23} + i^{20} - i^{34} = i^{4\cdot 26+1} + i^{4\cdot 5+3} + i^{4\cdot 5} - i^{4\cdot 8+2} = i - i + 1 + 1 = 2.$$

2) Let us solve the equation $z^3 = 18 + 26i$, where $z = x + yi$ and x, y are integers. We can write

$$(x + yi)^3 = (x + yi)^2(x + yi) = (x^2 - y^2 + 2xyi)(x + yi)$$
$$= (x^3 - 3xy^2) + (3x^2y - y^3)i = 18 + 26i.$$

Using the definition of equality of complex numbers, we obtain

$$\begin{cases} x^3 - 3xy^2 = 18 \\ 3x^2y - y^3 = 26. \end{cases}$$

Setting $y = tx$ in the equality $18(3x^2y - y^3) = 26(x^3 - 3xy^2)$, let us observe that $x \neq 0$ and $y \neq 0$ implies $18(3t - t^3) = 26(1 - 3t^2)$. The last relation is equivalent to $(3t - 1)(3t^2 - 12t - 13) = 0$.

The only rational solution of this equation is $t = \dfrac{1}{3}$; hence,

$$x = 3, \ \ y = 1 \text{ and } z = 3 + i.$$

1.1.6 Conjugate of a complex number

For a complex number $z = x + yi$ the number $\overline{z} = x - yi$ is called the *complex conjugate* or the *conjugate complex* of z.

Proposition. *1) The relation $z = \overline{z}$ holds if and only if $z \in \mathbb{R}$.*

2) For any complex number z the relation $z = \overline{\overline{z}}$ holds.

3) For any complex number z the number $z \cdot \overline{z} \in \mathbb{R}$ is a nonnegative real number.

4) $\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2$ (the conjugate of a sum is the sum of the conjugates).

5) $\overline{z_1 \cdot z_2} = \overline{z}_1 \cdot \overline{z}_2$ (the conjugate of a product is the product of the conjugates).

6) For any nonzero complex number z the relation $\overline{z^{-1}} = (\overline{z})^{-1}$ holds.

7) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z}_1}{\overline{z}_2}$, $z_2 \neq 0$ (the conjugate of a quotient is the quotient of the conjugates).

8) The formulas

$$\operatorname{Re}(z) = \frac{z + \overline{z}}{2} \quad \textit{and} \quad \operatorname{Im}(z) = \frac{z - \overline{z}}{2i}$$

are valid for all $z \in \mathbb{C}$.

Proof. 1) If $z = x + yi$, then the relation $z = \overline{z}$ is equivalent to $x + yi = x - yi$. Hence $2yi = 0$, so $y = 0$ and finally $z = x \in \mathbb{R}$.

2) We have $\overline{z} = x - yi$ and $\overline{\overline{z}} = x - (-y)i = x + yi = z$.

3) Observe that $z \cdot \overline{z} = (x + yi)(x - yi) = x^2 + y^2 \geq 0$.

4) Note that

$$\overline{z_1 + z_2} = \overline{(x_1 + x_2) + (y_1 + y_2)i} = (x_1 + x_2) - (y_1 + y_2)i$$

$$= (x_1 - y_1 i) + (x_2 - y_2 i) = \overline{z}_1 + \overline{z}_2.$$

5) We can write

$$\overline{z_1 \cdot z_2} = \overline{(x_1x_2 - y_1y_2) + i(x_1y_2 + x_2y_1)}$$

$$= (x_1x_2 - y_1y_2) - i(x_1y_2 + x_2y_1) = (x_1 - iy_1)(x_2 - iy_2) = \overline{z}_1 \cdot \overline{z}_2.$$

6) Because $z \cdot \frac{1}{z} = 1$, we have $\overline{\left(z \cdot \frac{1}{z}\right)} = \overline{1}$, and consequently $\overline{z} \cdot \overline{\left(\frac{1}{z}\right)} = 1$, yielding $\overline{(z^{-1})} = (\overline{z})^{-1}$.

7) Observe that $\overline{\left(\frac{z_1}{z_2}\right)} = \overline{\left(z_1 \cdot \frac{1}{z_2}\right)} = \overline{z_1} \cdot \overline{\left(\frac{1}{z_2}\right)} = \overline{z_1} \cdot \frac{1}{\overline{z_2}} = \frac{\overline{z_1}}{\overline{z_2}}$.

8) From the relations

$$z + \overline{z} = (x + yi) + (x - yi) = 2x,$$

$$z - \overline{z} = (x + yi) - (x - yi) = 2yi$$

it follows that

$$\operatorname{Re}(z) = \frac{z + \overline{z}}{2} \quad \text{and} \quad \operatorname{Im}(z) = \frac{z - \overline{z}}{2i}$$

as desired. □

The properties 4) and 5) can be easily extended to give

4′) $\overline{\left(\sum_{k=1}^{n} z_k\right)} = \sum_{k=1}^{n} \overline{z}_k$;

5′) $\overline{\left(\prod_{k=1}^{n} z_k\right)} = \prod_{k=1}^{n} \overline{z}_k$ for all $z_k \in \mathbb{C}$, $k = 1, 2, \ldots, n$.

As a consequence of 5′) and 6) we have

5″) $(\overline{z^n}) = (\overline{z})^n$ for any integers n and for any $z \in \mathbb{C}$.

Comments. a) To obtain the multiplication inverse of a complex number $z \in \mathbb{C}^*$ one can use the following approach:

$$\frac{1}{z} = \frac{\overline{z}}{z \cdot \overline{z}} = \frac{x - yi}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2} i.$$

b) The complex conjugate allows us to obtain the quotient of two complex numbers as follows:

$$\frac{z_1}{z_2} = \frac{z_1 \cdot \overline{z_2}}{z_2 \cdot \overline{z_2}} = \frac{(x_1 + y_1 i)(x_2 - y_2 i)}{x_2^2 + y_2^2} = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + \frac{-x_1 y_2 + x_2 y_1}{x_2^2 + y_2^2} i.$$

Examples. (1) Compute $z = \dfrac{5 + 5i}{3 - 4i} + \dfrac{20}{4 + 3i}$.

Solution. We can write

$$z = \frac{(5 + 5i)(3 + 4i)}{9 - 16i^2} + \frac{20(4 - 3i)}{16 - 9i^2} = \frac{-5 + 35i}{25} + \frac{80 - 60i}{25}$$

$$= \frac{75 - 25i}{25} = 3 - i.$$

(2) Let $z_1, z_2 \in \mathbb{C}$. Prove that the number $E = z_1 \cdot \overline{z}_2 + \overline{z}_1 \cdot z_2$ is a real number.

Solution. We have

$$\overline{E} = \overline{z_1 \cdot \overline{z}_2 + \overline{z}_1 \cdot z_2} = \overline{z}_1 \cdot z_2 + z_1 \cdot \overline{z}_2 = E, \text{ so } E \in \mathbb{R}.$$

1.1.7 Modulus of a complex number

The number $|z| = \sqrt{x^2 + y^2}$ is called the *modulus* or the *absolute value* of the complex number $z = x + yi$. For example, the complex numbers

$$z_1 = 4 + 3i, \quad z_2 = -3i, \quad z_3 = 2$$

have the moduli

$$|z_1| = \sqrt{4^2 + 3^2} = 5, \quad |z_2| = \sqrt{0^2 + (-3)^2} = 3, \quad |z_3| = \sqrt{2^2} = 2.$$

Proposition. *The following properties are satisfied:*

(1) $-|z| \leq \operatorname{Re}(z) \leq |z|$ *and* $-|z| \leq \operatorname{Im}(z) \leq |z|$.

(2) $|z| \geq 0$ *for all* $z \in \mathbb{C}$. *Moreover, we have* $|z| = 0$ *if and only if* $z = 0$.

(3) $|z| = |-z| = |\overline{z}|$.

(4) $z \cdot \overline{z} = |z|^2$.

(5) $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$ *(the modulus of a product is the product of the moduli).*

(6) $|z_1| - |z_2| \leq |z_1 + z_2| \leq |z_1| + |z_2|$.

(7) $|z^{-1}| = |z|^{-1}$, $z \neq 0$.

(8) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \neq 0$ *(the modulus of a quotient is the quotient of the moduli).*

9) $|z_1| - |z_2| \leq |z_1 - z_2| \leq |z_1| + |z_2|$.

Proof. One can easily check that (1)–(4) hold.

(5) We have $|z_1 \cdot z_2|^2 = (z_1 \cdot z_2)(\overline{z_1 \cdot z_2}) = (z_1 \cdot \overline{z}_1)(z_2 \cdot \overline{z}_2) = |z_1|^2 \cdot |z_2|^2$ and consequently $|z_1 \cdot z_2| = |z_1| \cdot |z_2|$, since $|z| \geq 0$ for all $z \in \mathbb{C}$.

(6) Observe that

$$|z_1 + z_2|^2 = (z_1 + z_2)(\overline{z_1 + z_2}) = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) = |z_1|^2 + z_1 \cdot \overline{z}_2 + \overline{z}_1 \cdot z_2 + |z_2|^2.$$

Because $\overline{z_1 \cdot \overline{z}_2} = \overline{z}_1 \cdot \overline{\overline{z}}_2 = \overline{z}_1 \cdot z_2$ it follows that

$$z_1\overline{z}_2 + \overline{z}_1 \cdot z_2 = 2 \operatorname{Re}(z_1 \cdot \overline{z}_2) \leq 2|z_1 \cdot \overline{z}_2| = 2|z_1| \cdot |z_2|,$$

hence

$$|z_1 + z_2|^2 \leq (|z_1| + |z_2|)^2,$$

and consequently, $|z_1 + z_2| \leq |z_1| + |z_2|$, as desired.

In order to obtain inequality on the left-hand side note that

$$|z_1| = |z_1 + z_2 + (-z_2)| \leq |z_1 + z_2| + |-z_2| = |z_1 + z_2| + |z_2|,$$

hence

$$|z_1| - |z_2| \leq |z_1 + z_2|.$$

(7) Note that the relation $z \cdot \dfrac{1}{z} = 1$ implies $|z| \cdot \left|\dfrac{1}{z}\right| = 1$, or $\left|\dfrac{1}{z}\right| = \dfrac{1}{|z|}$. Hence $|z^{-1}| = |z|^{-1}$.

(8) We have

$$\left|\frac{z_1}{z_2}\right| = \left|z_1 \cdot \frac{1}{z_2}\right| = |z_1 \cdot z_2^{-1}| = |z_1| \cdot |z_2^{-1}| = |z_1| \cdot |z_2|^{-1} = \frac{|z_1|}{|z_2|}.$$

(9) We can write $|z_1| = |z_1 - z_2 + z_2| \leq |z_1 - z_2| + |z_2|$, so $|z_1 - z_2| \geq |z_1| - |z_2|$. On the other hand,

$$|z_1 - z_2| = |z_1 + (-z_2)| \leq |z_1| + |-z_2| = |z_1| + |z_2|. \qquad \square$$

Remarks. (1) The inequality $|z_1 + z_2| \leq |z_1| + |z_2|$ becomes an equality if and only if $\mathrm{Re}(z_1\overline{z}_2) = |z_1||z_2|$. This is equivalent to $z_1 = tz_2$, where t is a nonnegative real number.

(2) The properties 5) and 6) can be easily extended to give

(5′) $\left|\prod_{k=1}^{n} z_k\right| = \prod_{k=1}^{n} |z_k|$;

(6′) $\left|\sum_{k=1}^{n} z_k\right| \leq \sum_{k=1}^{n} |z_k|$ for all $z_k \in \mathbb{C}$, $k = \overline{1, n}$.

As a consequence of(5′) and (7) we have

(5″) $|z^n| = |z|^n$ for any integer n and any complex number z.

Problem 1. *Prove the identity*

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2(|z_1|^2 + |z_2|^2)$$

for all complex numbers z_1, z_2.

Solution. Using property 4 in the proposition above, we obtain

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = (z_1 + z_2)(\overline{z}_1 + \overline{z}_2) + (z_1 - z_2)(\overline{z}_1 - \overline{z}_2)$$

$$= |z_1|^2 + z_1 \cdot \overline{z}_2 + z_2 \cdot \overline{z}_1 + |z_2|^2 + |z_1|^2 - z_1 \cdot \overline{z}_2 - z_2 \cdot \overline{z}_1 + |z_2|^2$$
$$= 2(|z_1|^2 + |z_2|^2).$$

Problem 2. *Prove that if* $|z_1| = |z_2| = 1$ *and* $z_1z_2 \neq -1$, *then* $\dfrac{z_1 + z_2}{1 + z_1z_2}$ *is a real number.*

Solution. Using again property 4 in the above proposition, we have

$$z_1 \cdot \overline{z}_1 = |z_1|^2 = 1 \text{ and } \overline{z}_1 = \frac{1}{z_1}.$$

Likewise, $\overline{z}_2 = \dfrac{1}{z_2}$. Hence denoting by A the number in the problem we have

$$\overline{A} = \frac{\overline{z}_1 + \overline{z}_2}{1 + \overline{z}_1 \cdot \overline{z}_2} = \frac{\dfrac{1}{z_1} + \dfrac{1}{z_2}}{1 + \dfrac{1}{z_1} \cdot \dfrac{1}{z_2}} = \frac{z_1 + z_2}{1 + z_1z_2} = A,$$

so A is a real number.

Problem 3. *Let a be a positive real number and let*

$$M_a = \left\{ z \in \mathbb{C}^* : \left| z + \frac{1}{z} \right| = a \right\}.$$

Find the minimum and maximum value of $|z|$ when $z \in M_a$.

Solution. Squaring both sides of the equality $a = \left| z + \frac{1}{z} \right|$, we get

$$a^2 = \left| z + \frac{1}{z} \right|^2 = \left(z + \frac{1}{z} \right) \left(\overline{z} + \frac{1}{\overline{z}} \right) = |z|^2 + \frac{z^2 + (\overline{z})^2}{|z|^2} + \frac{1}{|z|^2}$$

$$= \frac{|z|^4 + (z + \overline{z})^2 - 2|z|^2 + 1}{|z|^2}.$$

Hence

$$|z|^4 - |z|^2 \cdot (a^2 + 2) + 1 = -(z + \overline{z})^2 \le 0$$

and consequently

$$|z|^2 \in \left[\frac{a^2 + 2 - \sqrt{a^4 + 4a^2}}{2}, \frac{a^2 + 2 + \sqrt{a^4 + 4a^2}}{2} \right].$$

It follows that $|z| \in \left[\frac{-a + \sqrt{a^2 + 4}}{2}, \frac{a + \sqrt{a^2 + 4}}{2} \right]$, so

$$\max |z| = \frac{a + \sqrt{a^2 + 4}}{2}, \quad \min |z| = \frac{-a + \sqrt{a^2 + 4}}{2}$$

and the extreme values are obtained for the complex numbers in M satisfying $z = -\overline{z}$.

Problem 4. *Prove that for any complex number z,*

$$|z + 1| \ge \frac{1}{\sqrt{2}} \text{ or } |z^2 + 1| \ge 1.$$

Solution. Suppose by way of contradiction that

$$|1 + z| < \frac{1}{\sqrt{2}} \text{ and } |1 + z^2| < 1.$$

Setting $z = a + bi$, with $a, b \in \mathbb{R}$ yields $z^2 = a^2 - b^2 + 2abi$. We obtain

$$(1 + a^2 - b^2)^2 + 4a^2b^2 < 1 \text{ and } (1 + a)^2 + b^2 < \frac{1}{2},$$

and consequently

$$(a^2 + b^2)^2 + 2(a^2 - b^2) < 0 \text{ and } 2(a^2 + b^2) + 4a + 1 < 0.$$

Summing these inequalities implies

$$(a^2 + b^2)^2 + (2a + 1)^2 < 0,$$

which is a contradiction.

Problem 5. *Prove that*

$$\sqrt{\frac{7}{2}} \le |1 + z| + |1 - z + z^2| \le 3\sqrt{\frac{7}{6}}$$

for all complex numbers with $|z| = 1$.

Solution. Let $t = |1 + z| \in [0, 2]$. We have

$$t^2 = (1 + z) \cdot (1 + \overline{z}) = 2 + 2\,\mathrm{Re}(z), \text{ so } \mathrm{Re}(z) = \frac{t^2 - 2}{2}.$$

Then $|1 - z + z^2| = \sqrt{|7 - 2t^2|}$. It suffices to find the extreme values of the function

$$f : [0, 2] \to \mathbb{R}, \quad f(t) = t + \sqrt{|7 - 2t^2|}.$$

We obtain

$$f\left(\sqrt{\frac{7}{2}}\right) = \sqrt{\frac{7}{2}} \le t + \sqrt{|7 - 2t^2|} \le f\left(\sqrt{\frac{7}{6}}\right) = 3\sqrt{\frac{7}{6}}$$

as we can see from the figure below.

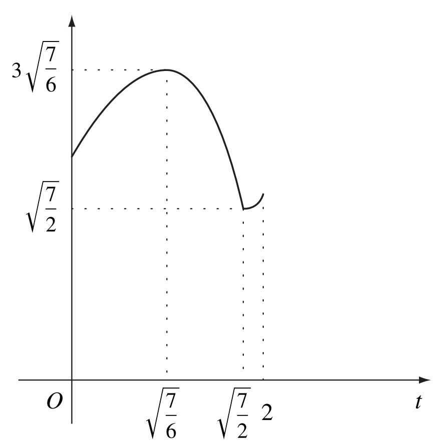

Figure 1.1.

Problem 6. *Consider the set*

$$H = \{z \in \mathbb{C} : \; z = x - 1 + xi, \quad x \in \mathbb{R}\}.$$

Prove that there is a unique number $z \in H$ *such that* $|z| \le |w|$ *for all* $w \in H$.

Solution. Let $\omega = y - 1 + yi$, with $y \in \mathbb{R}$.

It suffices to prove that there is a unique number $x \in \mathbb{R}$ such that

$$(x-1)^2 + x^2 \le (y-1)^2 + y^2$$

for all $y \in \mathbb{R}$.

In other words, x is the minimum point of the function

$$f : \mathbb{R} \to \mathbb{R},\ f(y) = (y-1)^2 + y^2 = 2y^2 - 2y + 1 = 2\left(y - \frac{1}{2}\right)^2 + \frac{1}{2},$$

hence $x = \frac{1}{2}$ and $z = -\frac{1}{2} + \frac{1}{2}i$.

Problem 7. *Let x, y, z be distinct complex numbers such that*

$$y = tx + (1-t)z, \quad t \in (0,1).$$

Prove that

$$\frac{|z| - |y|}{|z - y|} \ge \frac{|z| - |x|}{|z - x|} \ge \frac{|y| - |x|}{|y - x|}.$$

Solution. The relation $y = tx + (1-t)z$ is equivalent to

$$z - y = t(z - x).$$

The inequality

$$\frac{|z| - |y|}{|z - y|} \ge \frac{|z| - |x|}{|z - x|}$$

becomes

$$|z| - |y| \ge t(|z| - |x|),$$

and consequently

$$|y| \le (1-t)|z| + t|x|.$$

This is the triangle inequality for

$$y = (1-t)z + tx.$$

The second inequality can be proved similarly, writing the equality

$$y = tx + (1-t)z$$

as

$$y - x = (1-t)(z - x).$$

1.1.8 Solving quadratic equations

We are now able to solve the quadratic equation with real coefficients

$$ax^2 + bx + c = 0, \quad a \neq 0$$

in the case when its discriminant $\Delta = b^2 - 4ac$ is negative.

By completing the square, we easily get the equivalent form

$$a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{-\Delta}{4a^2}\right] = 0.$$

Therefore

$$\left(x + \frac{b}{2a}\right)^2 - i^2\left(\frac{\sqrt{-\Delta}}{2a}\right)^2 = 0,$$

and so $x_1 = \dfrac{-b + i\sqrt{-\Delta}}{2a}$, $x_2 = \dfrac{-b - i\sqrt{-\Delta}}{2a}$.

Observe that the roots are conjugate complex numbers and the factorization formula

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

holds even in the case $\Delta < 0$.

Let us consider now the general quadratic equation with complex coefficients

$$az^2 + bz + c = 0, \quad a \neq 0.$$

Using the same algebraic manipulation as in the case of real coefficients, we get

$$a\left[\left(z + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] = 0.$$

This is equivalent to

$$\left(z + \frac{b}{2a}\right)^2 = \frac{\Delta}{4a^2}$$

or

$$(2az + b)^2 = \Delta,$$

where $\Delta = b^2 - 4ac$ is also called the discriminant of the quadratic equation. Setting $y = 2az + b$, the equation is reduced to

$$y^2 = \Delta = u + vi,$$

where u and v are real numbers.

This equation has the solutions

$$y_{1,2} = \pm\left(\sqrt{\frac{r+u}{2}} + (\operatorname{sgn} v)\sqrt{\frac{r-u}{2}}i\right),$$

where $r = |\Delta|$ and sign$v$ is the sign of the real number v.

The roots of the initial equation are

$$z_{1,2} = \frac{1}{2a}(-b + y_{1,2}).$$

Observe that the relations between roots and coefficients

$$z_1 + z_2 = -\frac{b}{a}, \quad z_1 z_2 = \frac{c}{a},$$

as well as the factorization formula

$$az^2 + bz + c = a(z - z_1)(z - z_2)$$

are also preserved when the coefficients of the equation are elements of the field of complex numbers $\mathbb{C}$.

Problem 1. *Solve, in complex numbers, the quadratic equation*

$$z^2 - 8(1-i)z + 63 - 16i = 0.$$

Solution. We have

$$\Delta' = (4-4i)^2 - (63 - 16i) = -63 - 16i$$

and

$$r = |\Delta'| = \sqrt{63^2 + 16^2} = 65,$$

where $\Delta' = \left(\frac{b}{2}\right)^2 - ac$.

The equation

$$y^2 = -63 - 16i$$

has the solution $y_{1,2} = \pm\left(\sqrt{\frac{65-63}{2}} + i\sqrt{\frac{65+63}{2}}\right) = \pm(1-8i)$. It follows that $z_{1,2} = 4 - 4i \pm (1-8i)$. Hence

$$z_1 = 5 - 12i \text{ and } z_2 = 3 + 4i.$$

Problem 2. *Let p and q be complex numbers with $q \neq 0$. Prove that if the roots of the quadratic equation $x^2 + px + q^2 = 0$ have the same absolute value, then $\frac{p}{q}$ is a real number.*

(1999 Romanian Mathematical Olympiad – Final Round)

Solution. Let x_1 and x_2 be the roots of the equation and let $r = |x_1| = |x_2|$. Then

$$\frac{p^2}{q^2} = \frac{(x_1+x_2)^2}{x_1x_2} = \frac{x_1}{x_2} + \frac{x_2}{x_1} + 2 = \frac{x_1\overline{x_2}}{r^2} + \frac{x_2\overline{x_1}}{r^2} + 2 = 2 + \frac{2}{r^2}\text{Re}(x_1\overline{x_2})$$

is a real number. Moreover,

$$\text{Re}(x_1\overline{x_2}) \geq -|x_1\overline{x_2}| = -r^2, \text{ so } \frac{p^2}{q^2} \geq 0.$$

Therefore $\frac{p}{q}$ is a real number, as claimed.

Problem 3. *Let a, b, c be distinct nonzero complex numbers with $|a| = |b| = |c|$.*

a) Prove that if a root of the equation $az^2 + bz + c = 0$ has modulus equal to 1, then $b^2 = ac$.

b) If each of the equations

$$az^2 + bz + c = 0 \quad \textit{and} \quad bz^2 + cz + a = 0$$

has a root having modulus 1, then $|a - b| = |b - c| = |c - a|$.

Solution. a) Let z_1, z_2 be the roots of the equation with $|z_1| = 1$. From $z_2 = \frac{c}{a} \cdot \frac{1}{z_1}$ it follows that $|z_2| = \left|\frac{c}{a}\right| \cdot \frac{1}{|z_1|} = 1$. Because $z_1 + z_2 = -\frac{b}{a}$ and $|a| = |b|$, we have $|z_1 + z_2|^2 = 1$. This is equivalent to

$$(z_1 + z_2)(\overline{z_1} + \overline{z_2}) = 1, \text{ i.e., } (z_1 + z_2)\left(\frac{1}{z_1} + \frac{1}{z_2}\right) = 1.$$

We find that

$$(z_1 + z_2)^2 = z_1z_2, \text{ i.e., } \left(-\frac{b}{a}\right)^2 = \frac{c}{a},$$

which reduces to $b^2 = ac$, as desired.

b) As we have already seen, we have $b^2 = ac$ and $c^2 = ab$. Multiplying these relations yields $b^2c^2 = a^2bc$, hence $a^2 = bc$. Therefore

$$a^2 + b^2 + c^2 = ab + bc + ca. \tag{1}$$

Relation (1) is equivalent to

$$(a - b)^2 + (b - c)^2 + (c - a)^2 = 0,$$

i.e.,

$$(a - b)^2 + (b - c)^2 + 2(a - b)(b - c) + (c - a)^2 = 2(a - b)(b - c).$$

It follows that $(a-c)^2 = (a-b)(b-c)$. Taking absolute values we find $\beta^2 = \gamma\alpha$, where $\alpha = |b-c|$, $\beta = |c-a|$, $\gamma = |a-b|$. In an analogous way we obtain $\alpha^2 = \beta\gamma$ and $\gamma^2 = \alpha\beta$. Adding these relations yields $\alpha^2 + \beta^2 + \gamma^2 = \alpha\beta + \beta\gamma + \gamma\alpha$, i.e., $(\alpha-\beta)^2 + (\beta-\gamma)^2 + (\gamma-\alpha)^2 = 0$. Hence $\alpha = \beta = \gamma$.

1.1.9 Problems

1. Consider the complex numbers $z_1 = (1, 2)$, $z_2 = (-2, 3)$ and $z_3 = (1, -1)$. Compute the following complex numbers:

a) $z_1 + z_2 + z_3$; b) $z_1z_2 + z_2z_3 + z_3z_1$; c) $z_1z_2z_3$;

d) $z_1^2 + z_2^2 + z_3^2$; e) $\dfrac{z_1}{z_2} + \dfrac{z_2}{z_3} + \dfrac{z_3}{z_1}$; f) $\dfrac{z_1^2 + z_2^2}{z_2^2 + z_3^2}$.

2. Solve the equations:

a) $z + (-5, 7) = (2, -1)$; b) $(2, 3) + z = (-5, -1)$;

c) $z \cdot (2, 3) = (4, 5)$; d) $\dfrac{z}{(-1, 3)} = (3, 2)$.

3. Solve in $\mathbb{C}$ the equations:

a) $z^2 + z + 1 = 0$; b) $z^3 + 1 = 0$.

4. Let $z = (0, 1) \in \mathbb{C}$. Express $\displaystyle\sum_{k=0}^{n} z^k$ in terms of the positive integer n.

5. Solve the equations:

a) $z \cdot (1, 2) = (-1, 3)$; b) $(1, 1) \cdot z^2 = (-1, 7)$.

6. Let $z = (a, b) \in \mathbb{C}$. Compute z^2, z^3 and z^4.

7. Let $z_0 = (a, b) \in \mathbb{C}$. Find $z \in \mathbb{C}$ such that $z^2 = z_0$.

8. Let $z = (1, -1)$. Compute z^n, where n is a positive integer.

9. Find real numbers x and y in each of the following cases:

a) $(1-2i)x + (1+2i)y = 1+i$; b) $\dfrac{x-3}{3+i} + \dfrac{y-3}{3-i} = i$;

c) $(4-3i)x^2 + (3+2i)xy = 4y^2 - \dfrac{1}{2}x^2 + (3xy - 2y^2)i$.

10. Compute:

a) $(2-i)(-3+2i)(5-4i)$; b) $(2-4i)(5+2i) + (3+4i)(-6-i)$;

c) $\left(\dfrac{1+i}{1-i}\right)^{16} + \left(\dfrac{1-i}{1+i}\right)^{8}$; d) $\left(\dfrac{-1+i\sqrt{3}}{2}\right)^{6} + \left(\dfrac{1-i\sqrt{7}}{2}\right)^{6}$;

e) $\dfrac{3+7i}{2+3i} + \dfrac{5-8i}{2-3i}$.

11. Compute:
a) $i^{2000} + i^{1999} + i^{201} + i^{82} + i^{47}$;
b) $E_n = 1 + i + i^2 + i^3 + \cdots + i^n$ for $n \geq 1$;
c) $i^1 \cdot i^2 \cdot i^3 \cdots i^{2000}$;
d) $i^{-5} + (-i)^{-7} + (-i)^{13} + i^{-100} + (-i)^{94}$.

12. Solve in $\mathbb{C}$ the equations:
a) $z^2 = i$; b) $z^2 = -i$; c) $z^2 = \frac{1}{2} - i\frac{\sqrt{2}}{2}$.

13. Find all complex numbers $z \neq 0$ such that $z + \frac{1}{z} \in \mathbb{R}$.

14. Prove that:
a) $E_1 = (2 + i\sqrt{5})^7 + (2 - i\sqrt{5})^7 \in \mathbb{R}$;
b) $E_2 = \left(\frac{19 + 7i}{9 - i}\right)^n + \left(\frac{20 + 5i}{7 + 6i}\right)^n \in \mathbb{R}$.

15. Prove the following identities:
a) $|z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_1 + z_2 + z_3|^2$;
b) $|1 + z_1\overline{z}_2|^2 + |z_1 - z_2|^2 = (1 + |z_1|^2)(1 + |z_2|^2)$;
c) $|1 - z_1\overline{z}_2|^2 - |z_1 - z_2|^2 = (1 - |z_1|^2)(1 - |z_2|^2)$;
d) $|z_1 + z_2 + z_3|^2 + |-z_1 + z_2 + z_3|^2 + |z_1 - z_2 + z_3|^2 + |z_1 + z_2 - z_3|^2$
$= 4(|z_1|^2 + |z_2|^2 + |z_3|^2)$.

16. Let $z \in \mathbb{C}^*$ such that $\left|z^3 + \frac{1}{z^3}\right| \leq 2$. Prove that $\left|z + \frac{1}{z}\right| \leq 2$.

17. Find all complex numbers z such that

$$|z| = 1 \text{ and } |z^2 + \overline{z}^2| = 1.$$

18. Find all complex numbers z such that

$$4z^2 + 8|z|^2 = 8.$$

19. Find all complex numbers z such that $z^3 = \overline{z}$.

20. Consider $z \in \mathbb{C}$ with $\text{Re}(z) > 1$. Prove that

$$\left|\frac{1}{z} - \frac{1}{2}\right| < \frac{1}{2}.$$

21. Let a, b, c be real numbers and $\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Compute

$$(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega).$$

22. Solve the equations:
a) $|z| - 2z = 3 - 4i$;
b) $|z| + z = 3 + 4i$;
c) $z^3 = 2 + 11i$, where $z = x + yi$ and $x, y \in \mathbb{Z}$;
d) $iz^2 + (1 + 2i)z + 1 = 0$;
e) $z^4 + 6(1 + i)z^2 + 5 + 6i = 0$;
f) $(1 + i)z^2 + 2 + 11i = 0$.

23. Find all real numbers m for which the equation

$$z^3 + (3 + i)z^2 - 3z - (m + i) = 0$$

has at least a real root.

24. Find all complex numbers z such that

$$z' = (z - 2)(\overline{z} + i)$$

is a real number.

25. Find all complex numbers z such that $|z| = \left|\frac{1}{z}\right|$.

26. Let $z_1, z_2 \in \mathbb{C}$ be complex numbers such that $|z_1 + z_2| = \sqrt{3}$ and $|z_1| = |z_2| = 1$. Compute $|z_1 - z_2|$.

27. Find all positive integers n such that

$$\left(\frac{-1 + i\sqrt{3}}{2}\right)^n + \left(\frac{-1 - i\sqrt{3}}{2}\right)^n = 2.$$

28. Let $n > 2$ be an integer. Find the number of solutions to the equation

$$z^{n-1} = i\overline{z}.$$

29. Let z_1, z_2, z_3 be complex numbers with

$$|z_1| = |z_2| = |z_3| = R > 0.$$

Prove that

$$|z_1 - z_2| \cdot |z_2 - z_3| + |z_3 - z_1| \cdot |z_1 - z_2| + |z_2 - z_3| \cdot |z_3 - z_1| \leq 9R^2.$$

30. Let u, v, w, z be complex numbers such that $|u| < 1$, $|v| = 1$ and $w = \dfrac{v(u - z)}{\overline{u} \cdot z - 1}$. Prove that $|w| \leq 1$ if and only if $|z| \leq 1$.

31. Let z_1, z_2, z_3 be complex numbers such that

$$z_1 + z_2 + z_3 = 0 \quad \text{and} \quad |z_1| = |z_2| = |z_3| = 1.$$

Prove that

$$z_1^2 + z_2^2 + z_3^2 = 0.$$

32. Consider the complex numbers $z_1, z_2, \ldots, z_n$ with

$$|z_1| = |z_2| = \cdots = |z_n| = r > 0.$$

Prove that the number

$$E = \frac{(z_1 + z_2)(z_2 + z_3)\cdots(z_{n-1} + z_n)(z_n + z_1)}{z_1 \cdot z_2 \cdots z_n}$$

is real.

33. Let z_1, z_2, z_3 be distinct complex numbers such that

$$|z_1| = |z_2| = |z_3| > 0.$$

If $z_1 + z_2 z_3$, $z_2 + z_1 z_3$ and $z_3 + z_1 z_2$ are real numbers, prove that $z_1 z_2 z_3 = 1$.

34. Let x_1 and x_2 be the roots of the equation $x^2 - x + 1 = 0$. Compute:

a) $x_1^{2000} + x_2^{2000}$; b) $x_1^{1999} + x_2^{1999}$; c) $x_1^n + x_2^n$, for $n \in \mathbb{N}$.

35. Factorize (in linear polynomials) the following polynomials:

a) $x^4 + 16$; b) $x^3 - 27$; c) $x^3 + 8$; d) $x^4 + x^2 + 1$.

36. Find all quadratic equations with real coefficients that have one of the following roots:

a) $(2 + i)(3 - i)$; b) $\dfrac{5 + i}{2 - i}$; c) $i^{51} + 2i^{80} + 3i^{45} + 4i^{38}$.

37. (Hlawka's inequality) Prove that the following inequality

$$|z_1 + z_2| + |z_2 + z_3| + |z_3 + z_1| \leq |z_1| + |z_2| + |z_3| + |z_1 + z_2 + z_3|$$

holds for all complex numbers z_1, z_2, z_3.

1.2 Geometric Interpretation of the Algebraic Operations

1.2.1 Geometric interpretation of a complex number

We have defined a complex number $z = (x, y) = x + yi$ to be an ordered pair of real numbers $(x, y) \in \mathbb{R} \times \mathbb{R}$, so it is natural to let a complex number $z = x + yi$ correspond to a point $M(x, y)$ in the plane $\mathbb{R} \times \mathbb{R}$.

For a formal introduction, let us consider P to be the set of points of a given plane Π equipped with a coordinate system xOy. Consider the bijective function $\varphi : \mathbb{C} \to P$, $\varphi(z) = M(x, y)$.

Definition. The point $M(x, y)$ is called the *geometric image* of the complex number $z = x + yi$.

The complex number $z = x + yi$ is called the *complex coordinate* of the point $M(x, y)$. We will use the notation $M(z)$ to indicate that the complex coordinate of M is the complex number z.

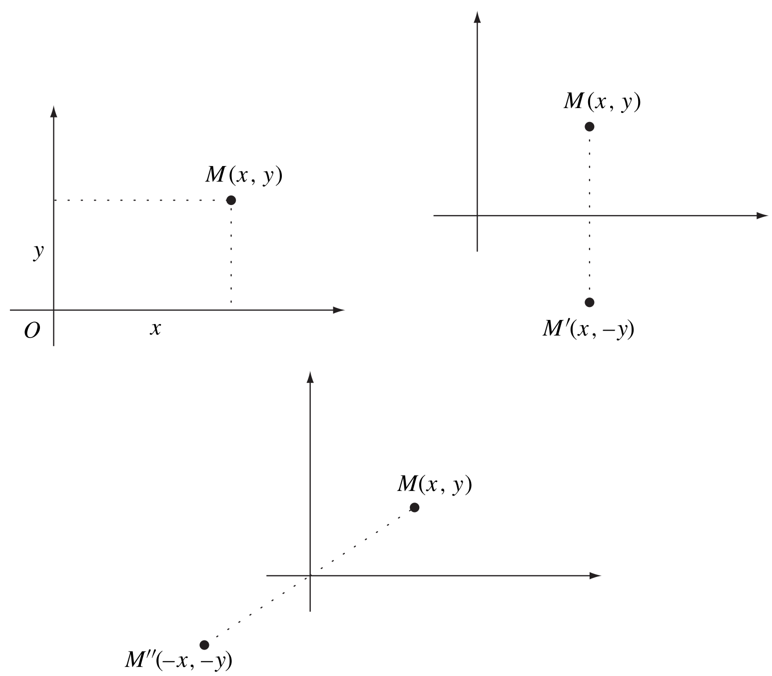

Figure 1.2.

The geometric image of the complex conjugate $\bar{z}$ of a complex number $z = x + yi$ is the reflection point $M'(x, -y)$ across the x-axis of the point $M(x, y)$ (see Fig. 1.2).

The geometric image of the additive inverse $-z$ of a complex number $z = x + yi$ is the reflection $M''(-x, -y)$ across the origin of the point $M(x, y)$ (see Fig. 1.2).

The bijective function φ maps the set $\mathbb{R}$ onto the x-axis, which is called the *real axis*. On the other hand, the imaginary complex numbers correspond to the y-axis, which is called the *imaginary axis*. The plane Π, whose points are identified with complex numbers, is called the *complex plane*.

On the other hand, we can also identify a complex number $z = x + yi$ with the vector $\vec{v} = \overrightarrow{OM}$, where $M(x, y)$ is the geometric image of the complex number z.

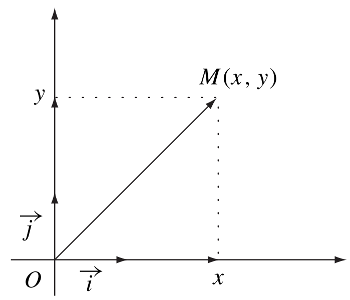

Figure 1.3.

Let V_0 be the set of vectors whose initial points are the origin O. Then we can define the bijective function

$$\varphi' : \mathbb{C} \to V_0, \quad \varphi'(z) = \overrightarrow{OM} = \overrightarrow{v} = x\overrightarrow{i} + y\overrightarrow{j},$$

where $\overrightarrow{i}$, $\overrightarrow{j}$ are the vectors of the x-axis and y-axis, respectively.

1.2.2 Geometric interpretation of the modulus

Let us consider a complex number $z = x + yi$ and the geometric image $M(x, y)$ in the complex plane. The Euclidean distance OM is given by the formula

$$OM = \sqrt{(x_M - x_O)^2 + (y_M - y_O)^2},$$

hence $OM = \sqrt{x^2 + y^2} = |z| = |\overrightarrow{v}|$. In other words, the absolute value $|z|$ of a complex number $z = x + yi$ is the length of the segment OM or the magnitude of the vector $\overrightarrow{v} = x\overrightarrow{i} + y\overrightarrow{j}$.

Remarks. a) For a positive real number r, the set of complex numbers with moduli r corresponds in the complex plane to $\mathcal{C}(O; r)$, our notation for the circle $\mathcal{C}$ with center O and radius r.

b) The complex numbers z with $|z| < r$ correspond to the interior points of circle $\mathcal{C}$; on the other hand, the complex numbers z with $|z| > r$ correspond to the points in the exterior of circle $\mathcal{C}$.

Example. The numbers $z_k = \pm\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$, $k = 1, 2, 3, 4$, are represented in the complex plane by four points on the unit circle centered on the origin, since

$$|z_1| = |z_2| = |z_3| = |z_4| = 1.$$

1.2.3 Geometric interpretation of the algebraic operations

a) **Addition and subtraction.** Consider the complex numbers $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ and the corresponding vectors $\overrightarrow{v}_1 = x_1\overrightarrow{i} + y_1\overrightarrow{j}$ and $\overrightarrow{v}_2 = x_2\overrightarrow{i} + y_2\overrightarrow{j}$. Observe that the sum of the complex numbers is

$$z_1 + z_2 = (x_1 + x_2) + (y_1 + y_2)i,$$

and the sum of the vectors is

$$\overrightarrow{v}_1 + \overrightarrow{v}_2 = (x_1 + x_2)\overrightarrow{i} + (y_1 + y_2)\overrightarrow{j}.$$

Therefore, the sum $z_1 + z_2$ corresponds to the sum $\overrightarrow{v}_1 + \overrightarrow{v}_2$.

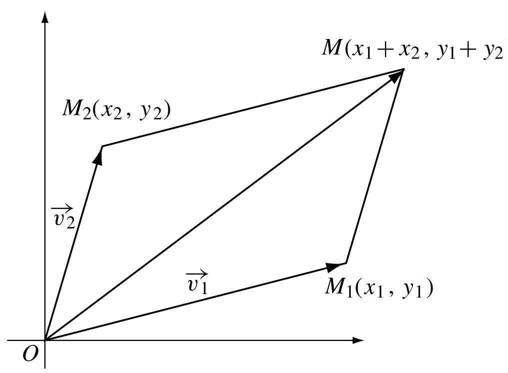

Figure 1.4.

Examples. 1) We have $(3 + 5i) + (6 + i) = 9 + 6i$; hence the geometric image of the sum is given in Fig. 1.5.

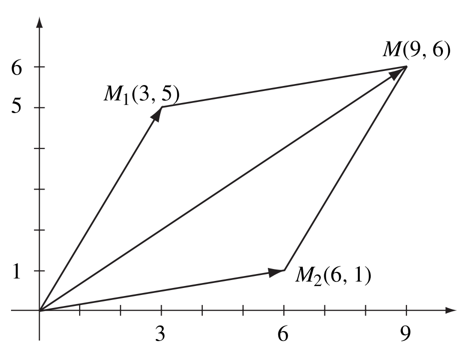

Figure 1.5.

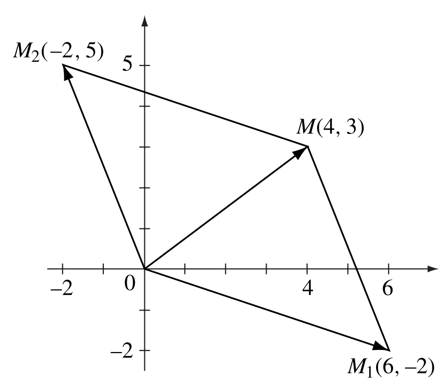

Figure 1.6.

2) Observe that $(6 - 2i) + (-2 + 5i) = 4 + 3i$. Therefore the geometric image of the sum of these two complex numbers is the point $M(4, 3)$ (see Fig. 1.6).

On the other hand, the difference of the complex numbers z_1 and z_2 is

$$z_1 - z_2 = (x_1 - x_2) + (y_1 - y_2)i,$$

and the difference of the vectors v_1 and v_2 is

$$\overrightarrow{v}_1 - \overrightarrow{v}_2 = (x_1 - x_2)\overrightarrow{i} + (y_1 - y_2)\overrightarrow{j}.$$

Hence, the difference $z_1 - z_2$ corresponds to the difference $\overrightarrow{v}_1 - \overrightarrow{v}_2$.

3) We have $(-3 + i) - (2 + 3i) = (-3 + i) + (-2 - 3i) = -5 - 2i$; hence the geometric image of difference of these two complex numbers is the point $M(-5, -2)$ given in Fig. 1.7.

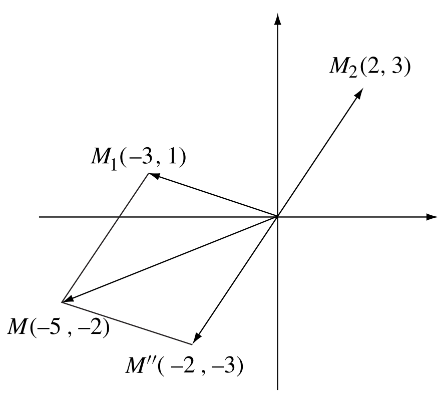

Figure 1.7.

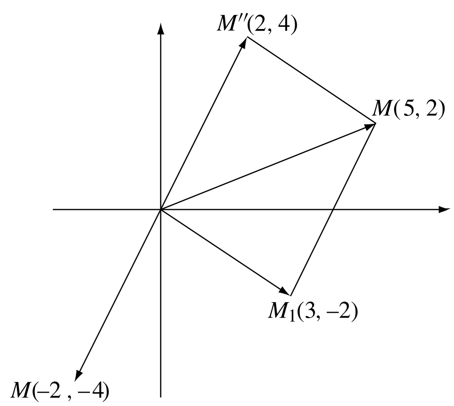

Figure 1.8.

4) Note that $(3 - 2i) - (-2 - 4i) = (3 - 2i) + (2 + 4i) = 5 + 2i$, and obtain the point $M(-2, -4)$ as the geometric image of the difference of these two complex numbers (see Fig. 1.8).

Remark. The distance $M_1(x_1, y_1)$ and $M_2(x_2, y_2)$ is equal to the modulus of the complex number $z_1 - z_2$ or to the length of the vector $\overrightarrow{v}_1 - \overrightarrow{v}_2$. Indeed,

$$|M_1M_2| = |z_1 - z_2| = |\overrightarrow{v}_1 - \overrightarrow{v}_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

b) **Real multiples of a complex number.** Consider a complex number $z = x + iy$ and the corresponding vector $\overrightarrow{v} = x\overrightarrow{i} + y\overrightarrow{j}$. If λ is a real number, then the real multiple $\lambda z = \lambda x + i\lambda y$ corresponds to the vector

$$\lambda\overrightarrow{v} = \lambda x\overrightarrow{i} + \lambda y\overrightarrow{j}.$$

Note that if $\lambda > 0$ then the vectors $\lambda\overrightarrow{v}$ and $\overrightarrow{v}$ have the same orientation and

$$|\lambda\overrightarrow{v}| = \lambda|\overrightarrow{v}|.$$

When $\lambda < 0$, the vector $\lambda \overrightarrow{v}$ changes to the opposite orientation and $|\lambda \overrightarrow{v}| = -\lambda |\overrightarrow{v}|$. Of course, if $\lambda = 0$, then $\lambda \overrightarrow{v} = \overrightarrow{0}$.

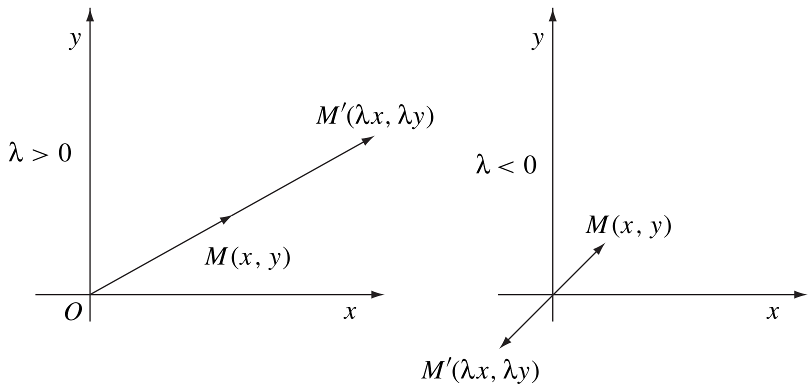

Figure 1.9.

Examples. 1) We have $3(1 + 2i) = 3 + 6i$; therefore $M'(3, 6)$ is the geometric image of the product of 3 and $z = 1 + 2i$.

2) Observe that $-2(-3 + 2i) = 6 - 4i$, and obtain the point $M'(6, -4)$ as the geometric image of the product of -2 and $z = -3 + 2i$.

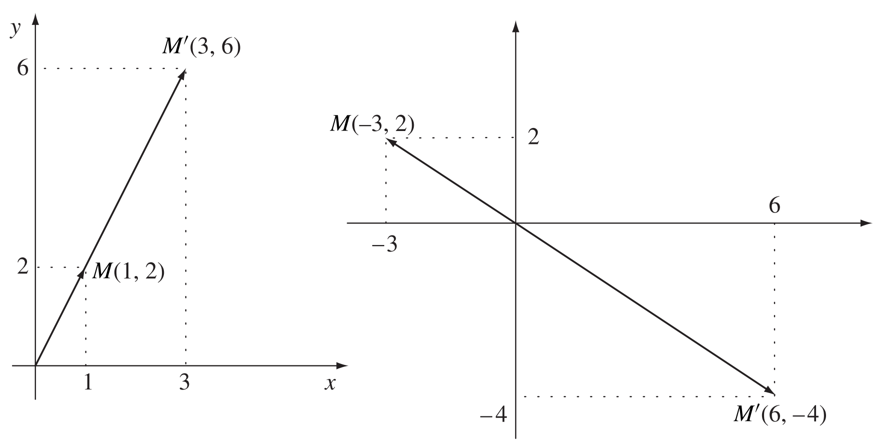

Figure 1.10.

1.2.4 Problems

1. Represent the geometric images of the following complex numbers:
$z_1 = 3 + i$; $z_2 = -4 + 2i$; $z_3 = -5 - 4i$; $z_4 = 5 - i$;
$z_5 = 1$; $z_6 = -3i$; $z_7 = 2i$; $z_8 = -4$.

2. Find the geometric interpretation for the following equalities:
a) $(-5 + 4i) + (2 - 3i) = -3 + i$;
b) $(4 - i) + (-6 + 4i) = -2 + 3i$;
c) $(-3 - 2i) - (-5 + i) = 2 - 3i$;
d) $(8 - i) - (5 + 3i) = 3 - 4i$;
e) $2(-4 + 2i) = -8 + 4i$;
f) $-3(-1 + 2i) = 3 - 6i$.

3. Find the geometric image of the complex number z in each of the following cases:
a) $|z - 2| = 3$; b) $|z + i| < 1$; c) $|z - 1 + 2i| > 3$;
d) $|z - 2| - |z + 2| < 2$; e) $0 < \mathrm{Re}(iz) < 1$; f) $-1 < \mathrm{Im}(z) < 1$;
g) $\mathrm{Re}\left(\dfrac{z-2}{z-1}\right) = 0$; h) $\dfrac{1+\overline{z}}{z} \in \mathbb{R}$.

4. Find the set of points $P(x, y)$ in the complex plane such that

$$|\sqrt{x^2+4} + i\sqrt{y-4}| = \sqrt{10}.$$

5. Let $z_1 = 1 + i$ and $z_2 = -1 - i$. Find $z_3 \in \mathbb{C}$ such that triangle z_1, z_2, z_3 is equilateral.

6. Find the geometric images of the complex numbers z such that the triangle with vertices at z, z^2 and z^3 is right-angled.

7. Find the geometric images of the complex numbers z such that

$$\left|z + \frac{1}{z}\right| = 2.$$

2

Complex Numbers in Trigonometric Form

2.1 Polar Representation of Complex Numbers

2.1.1 Polar coordinates in the plane

Let us consider a coordinate plane and a point $M(x, y)$ that is not the origin.

The real number $r = \sqrt{x^2 + y^2}$ is called the *polar radius* of the point M. The direct angle $t^* \in [0, 2\pi)$ between the vector $\overrightarrow{OM}$ and the positive x-axis is called the *polar argument* of the point M. The pair (r, t^*) is called the *polar coordinates* of the point M. We will write $M(r, t^*)$. Note that the function $h : \mathbb{R} \times \mathbb{R} \setminus \{(0, 0)\} \to (0, \infty) \times [0, 2\pi)$, $h((x, y)) = (r, t^*)$ is bijective.

The origin O is the unique point such that $r = 0$; the argument t^* of the origin is not defined.

For any point M in the plane there is a unique intersection point P of the ray $(OM$ with the unit circle centered at the origin. The point P has the same polar argument t^*. Using the definition of the sine and cosine functions we find that

$$x = r \cos t^* \text{ and } y = r \sin t^*.$$

Therefore, it is easy to obtain the cartesian coordinates of a point from its polar coordinates.

Conversely, let us consider a point $M(x, y)$. The polar radius is $r = \sqrt{x^2 + y^2}$. To determine the polar argument we study the following cases:

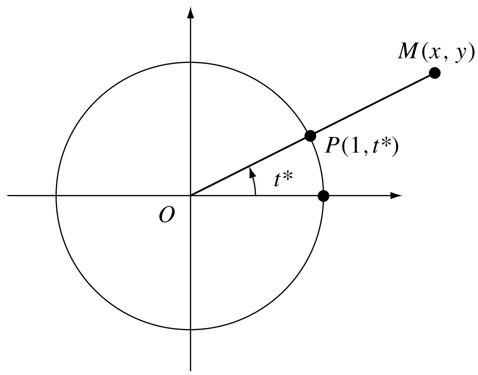

Figure 2.1.

a) If $x \neq 0$, from $\tan t^* = \dfrac{y}{x}$ we deduce that

$$t^* = \arctan \frac{y}{x} + k\pi,$$

where

$$k = \begin{cases} 0, & \text{for } \; x > 0 \text{ and } y \geq 0 \\ 1, & \text{for } \; x < 0 \text{ and any } y \\ 2, & \text{for } \; x > 0 \text{ and } y < 0. \end{cases}$$

b) If $x = 0$ and $y \neq 0$, then

$$t^* = \begin{cases} \pi/2, & \text{for } \; y > 0 \\ 3\pi/2, & \text{for } \; y < 0. \end{cases}$$

Examples. 1. Let us find the polar coordinates of the points $M_1(2, -2)$, $M_2(-1, 0)$, $M_3(-2\sqrt{3}, -2)$, $M_4(\sqrt{3}, 1)$, $M_5(3, 0)$, $M_6(-2, 2)$, $M_7(0, 1)$ and $M_8(0, -4)$.

In this case we have $r_1 = \sqrt{2^2 + (-2)^2} = 2\sqrt{2}$; $t_1^* = \arctan(-1) + 2\pi = -\dfrac{\pi}{4} + 2\pi = \dfrac{7\pi}{4}$, so $M_1\left(2\sqrt{2}, \dfrac{7\pi}{4}\right)$.

Observe that $r_2 = 1$, $t_2^* = \arctan 0 + \pi = \pi$, so $M_2(1, \pi)$.

We have $r_3 = 4$, $t_3^* = \arctan \dfrac{\sqrt{3}}{3} + \pi = \dfrac{\pi}{6} + \pi = \dfrac{7\pi}{6}$, so $M_3\left(4, \dfrac{7\pi}{6}\right)$.

Note that $r_4 = 2$, $t_4^* = \arctan \dfrac{\sqrt{3}}{3} = \dfrac{\pi}{6}$, so $M_4\left(2, \dfrac{\pi}{6}\right)$.

We have $r_5 = 3$, $t_5^* = \arctan 0 + 0 = 0$, so $M_5(3, 0)$.

We have $r_6 = 2\sqrt{2}$, $t_6^* = \arctan(-1) + \pi = -\dfrac{\pi}{4} + \pi = \dfrac{3\pi}{4}$, so $M_6\left(2\sqrt{2}, \dfrac{3\pi}{4}\right)$.

Note that $r_7 = 1$, $t_7^* = \dfrac{\pi}{2}$, so $M_7\left(1, \dfrac{\pi}{2}\right)$.

Observe that $r_8 = 4$, $t_8^* = \dfrac{3\pi}{2}$, so $M_8\left(1, \dfrac{3\pi}{2}\right)$.

2. Let us find the cartesian coordinates of the points $M_1\left(2, \dfrac{2\pi}{3}\right)$, $M_2\left(3, \dfrac{7\pi}{4}\right)$ and $M_3(1, 1)$.

We have $x_1 = 2\cos\dfrac{2\pi}{3} = 2\left(-\dfrac{1}{2}\right) = -1$, $y_1 = 2\sin\dfrac{2\pi}{3} = 2\dfrac{\sqrt{3}}{2} = \sqrt{3}$, so $M_1(-1, \sqrt{3})$.

Note that $x_2 = 3\cos\dfrac{7\pi}{4} = \dfrac{3\sqrt{2}}{2}$, $y_2 = 3\sin\dfrac{7\pi}{4} = -\dfrac{3\sqrt{2}}{2}$, so $M_2\left(\dfrac{3\sqrt{2}}{2}, -\dfrac{3\sqrt{2}}{2}\right)$.

Observe that $x_3 = \cos 1$, $y_2 = \sin 1$, so $M_3(\cos 1, \sin 1)$.

2.1.2 Polar representation of a complex number

For a complex number $z = x + yi$ we can write the polar representation

$$z = r(\cos t^* + i\sin t^*),$$

where $r \in [0, \infty)$ and $t^* \in [0, 2\pi)$ are the polar coordinates of the geometric image of z.

The polar argument t^* of the geometric image of z is called the argument of z, denoted by $\arg z$. The polar radius r of the geometric image of z is equal to the modulus of z. For $z \neq 0$, the modulus and argument of z are uniquely determined.

Consider $z = r(\cos t^* + i\sin t^*)$ and let $t = t^* + 2k\pi$ for an integer k. Then

$$z = r[\cos(t - 2k\pi) + i\sin(t - 2k\pi)] = r(\cos t + i\sin t),$$

i.e., any complex number z can be represented as $z = r(\cos t + i\sin t)$, where $r \geq 0$ and $t \in \mathbb{R}$. The set $\operatorname{Arg} z = \{t :\ t^* + 2k\pi,\ k \in \mathbb{Z}\}$ is called the *extended argument* of the complex number z.

Therefore, two complex numbers $z_1, z_2 \neq 0$ represented as

$$z_1 = r_1(\cos t_1 + i\sin t_1) \text{ and } z_2 = r_2(\cos t_2 + i\sin t_2)$$

are equal if and only if $r_1 = r_2$ and $t_1 - t_2 = 2k\pi$, for an integer k.

Example 1. Let us find the polar representation of the numbers:

a) $z_1 = -1 - i$,

b) $z_2 = 2 + 2i$,

c) $z_3 = -1 + i\sqrt{3}$,

d) $z_4 = 1 - i\sqrt{3}$

and determine their extended argument.

a) As in the figure below the geometric image $P_1(-1, -1)$ lies in the third quadrant. Then $r_1 = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$ and

$$t_1^* = \arctan \frac{y}{x} + \pi = \arctan 1 + \pi = \frac{\pi}{4} + \pi = \frac{5\pi}{4}.$$

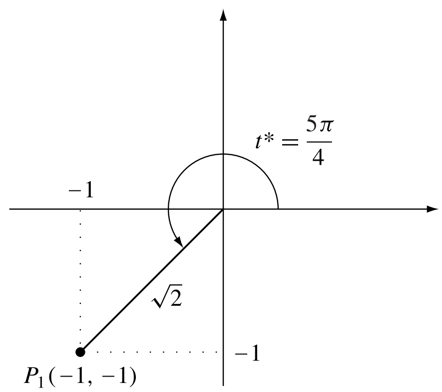

Figure 2.2.

Hence

$$z_1 = \sqrt{2}\left(\cos \frac{5\pi}{4} + i \sin \frac{5\pi}{4}\right)$$

and

$$\operatorname{Arg} z_1 = \left\{\frac{5\pi}{4} + 2k\pi \mid k \in \mathbb{Z}\right\}.$$

b) The point $P_2(2, 2)$ lies in the first quadrant, so we can write

$$r_2 = \sqrt{2^2 + 2^2} = 2\sqrt{2} \text{ and } t_2^* = \arctan 1 = \frac{\pi}{4}.$$

Hence

$$z_2 = 2\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$$

and

$$\operatorname{Arg} z = \left\{\frac{\pi}{4} + 2k\pi \mid k \in \mathbb{Z}\right\}.$$

c) The point $P_3(-1, \sqrt{3})$ lies in the second quadrant, so

$$r_3 = 2 \text{ and } t_3^* = \arctan(-\sqrt{3}) + \pi = -\frac{\pi}{3} + \pi = \frac{2\pi}{3}.$$

Therefore,

$$z_3 = 2\left(\cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3}\right)$$

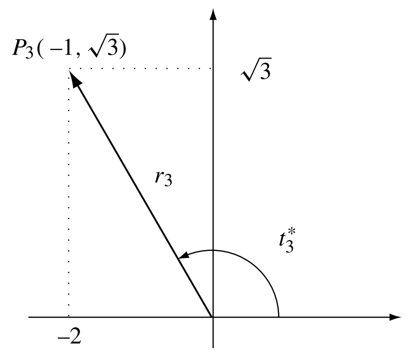

Figure 2.3.

and

$$\operatorname{Arg} z_3 = \left\{ \frac{2\pi}{3} + 2k\pi \mid k \in \mathbb{Z} \right\}.$$

d) The point $P_4(1, -\sqrt{3})$ lies in the fourth quadrant (Fig. 2.4), so

$$r_4 = 2 \text{ and } t_4^* = \arctan(-\sqrt{3}) + 2\pi = -\frac{\pi}{3} + 2\pi = \frac{5\pi}{3}.$$

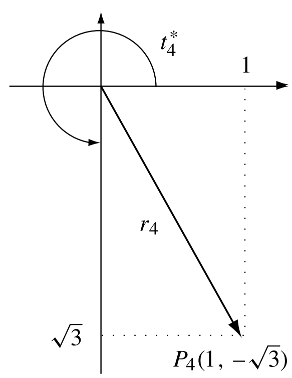

Figure 2.4.

Hence

$$z_4 = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right),$$

and

$$\operatorname{Arg} z_4 = \left\{ \frac{5\pi}{3} + 2k\pi \mid k \in \mathbb{Z} \right\}.$$

Example 2. Let us find the polar representation of the numbers

a) $z_1 = 2i$,

b) $z_2 = -1$,

c) $z_3 = 2$,

d) $z_4 = -3i$

and determine their extended argument.

a) The point $P_1(0, 2)$ lies on the positive y-axis, so

$$r_1 = 2, \quad t_1^* = \frac{\pi}{2}, \quad z_1 = 2\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

and

$$\operatorname{Arg} z_1 = \left\{\frac{\pi}{2} + 2k\pi \mid k \in \mathbb{Z}\right\}.$$

b) The point $P_2(-1, 0)$ lies on the negative x-axis, so

$$r_2 = 1, \quad t_2^* = \pi, \quad z_2 = \cos\pi + i\sin\pi$$

and

$$\operatorname{Arg} z_2 = \{\pi + 2k\pi \mid k \in \mathbb{Z}\}.$$

c) The point $P_3(2, 0)$ lies on the positive x-axis, so

$$r_3 = 2, \quad t_3^* = 0, \quad z_3 = 2(\cos 0 + i\sin 0)$$

and

$$\operatorname{Arg} z_3 = \{2k\pi \mid k \in \mathbb{Z}\}.$$

d) The point $P_4(0, -3)$ lies on the negative y-axis, so

$$r_4 = 3, \quad t_4^* = \frac{3\pi}{2}, \quad z_3 = 2\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

and

$$\operatorname{Arg} z_4 = \left\{\frac{3\pi}{2} + 2k\pi \mid k \in \mathbb{Z}\right\}.$$

Remark. The following formulas should be memorized:

$$1 = \cos 0 + i\sin 0; \quad i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2};$$

$$-1 = \cos\pi + i\sin\pi; \quad -i = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}.$$

Problem 1. *Find the polar representation of the complex number*

$$z = 1 + \cos a + i\sin a, \quad a \in (0, 2\pi).$$

Solution. The modulus is

$$|z| = \sqrt{(1+\cos a)^2 + \sin^2 a} = \sqrt{2(1+\cos a)} = \sqrt{4\cos^2\frac{a}{2}} = 2\left|\cos\frac{a}{2}\right|.$$

The argument of z is determined as follows:

a) If $a \in (0, \pi)$, then $\frac{a}{2} \in \left(0, \frac{\pi}{2}\right)$ and the point $P(1+\cos a, \sin a)$ lies on the first quadrant. Hence

$$t^* = \arctan\frac{\sin a}{1+\cos a} = \arctan\left(\tan\frac{a}{2}\right) = \frac{a}{2},$$

and in this case

$$z = 2\cos\frac{a}{2}\left(\cos\frac{a}{2} + i\sin\frac{a}{2}\right).$$

b) If $a \in (\pi, 2\pi)$, then $\frac{a}{2} \in \left(\frac{\pi}{2}, \pi\right)$ and the point $P(1+\cos a, \sin a)$ lies on the fourth quadrant. Hence

$$t^* = \arctan\left(\tan\frac{a}{2}\right) + 2\pi = \frac{a}{2} - \pi + 2\pi = \frac{a}{2} + \pi$$

and

$$z = -2\cos\frac{a}{2}\left(\cos\left(\frac{a}{2}+\pi\right) + i\sin\left(\frac{a}{2}+\pi\right)\right).$$

c) If $a = \pi$, then $z = 0$.

Problem 2. *Find all complex numbers z such that $|z| = 1$ and*

$$\left|\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right| = 1.$$

Solution. Let $z = \cos x + i\sin x$, $x \in [0, 2\pi)$. Then

$$\begin{aligned} 1 &= \left|\frac{z}{\overline{z}} + \frac{\overline{z}}{z}\right| = \frac{|z^2 + \overline{z}^2|}{|z|^2} \\ &= |\cos 2x + i\sin 2x + \cos 2x - i\sin 2x| \\ &= 2|\cos 2x| \end{aligned}$$

hence

$$\cos 2x = \frac{1}{2} \text{ or } \cos 2x = -\frac{1}{2}.$$

If $\cos 2x = \frac{1}{2}$, then

$$x_1 = \frac{\pi}{6}, \quad x_2 = \frac{5\pi}{6}, \quad x_3 = \frac{7\pi}{6}, \quad x_4 = \frac{11\pi}{6}.$$

If $\cos 2x = -\frac{1}{2}$, then

$$x_5 = \frac{\pi}{3}, \quad x_6 = \frac{2\pi}{3}, \quad x_7 = \frac{4\pi}{3}, \quad x_8 = \frac{5\pi}{3}.$$

Hence there are eight solutions

$$z_k = \cos x_k + i \sin x_k, \quad k = 1, 2, \dots, 8.$$

2.1.3 Operations with complex numbers in polar representation

1. Multiplication

Proposition. *Suppose that*

$$z_1 = r_1(\cos t_1 + i \sin t_1) \textit{ and } z_2 = r_2(\cos t_2 + i \sin t_2).$$

Then

$$z_1 z_2 = r_1 r_2(\cos(t_1 + t_2) + i \sin(t_1 + t_2)). \tag{1}$$

Proof. Indeed,

$$z_1 z_2 = r_1 r_2(\cos t_1 + i \sin t_1)(\cos t_2 + i \sin t_2)$$

$$= r_1 r_2((\cos t_1 \cos t_2 - \sin t_1 \sin t_2) + i(\sin t_1 \cos t_2 + \sin t_2 \cos t_1))$$

$$= r_1 r_2(\cos(t_1 + t_2) + i \sin(t_1 + t_2)).$$ □

Remarks. a) We find again that $|z_1 z_2| = |z_1| \cdot |z_2|$.

b) We have $\arg(z_1 z_2) = \arg z_1 + \arg z_2 - 2k\pi$, where

$$k = \begin{cases} 0, & \text{for} \quad \arg z_1 + \arg z_2 < 2\pi, \\ 1, & \text{for} \quad \arg z_1 + \arg z_2 \geq 2\pi. \end{cases}$$

c) Also we can write $\operatorname{Arg}(z_1 z_2) = \{\arg z_1 + \arg z_2 + 2k\pi : k \in \mathbb{Z}\}$.

d) Formula (1) can be extended to $n \geq 2$ complex numbers. If $z_k = r_k(\cos t_k + i \sin t_k)$, $k = 1, \dots, n$, then

$$z_1 z_2 \cdots z_n = r_1 r_2 \cdots r_n(\cos(t_1 + t_2 + \cdots + t_n) + i \sin(t_1 + t_2 + \cdots + t_n)).$$

The proof by induction is immediate. This formula can be written as

$$\prod_{k=1}^{n} z_k = \prod_{k=1}^{n} r_k \left(\cos \sum_{k=1}^{n} t_k + i \sin \sum_{k=1}^{n} t_k \right). \tag{2}$$

Example. Let $z_1 = 1 - i$ and $z_2 = \sqrt{3} + i$. Then

$$z_1 = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right), \quad z_2 = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

and

$$\begin{aligned} z_1 z_2 &= 2\sqrt{2}\left[\cos\left(\frac{7\pi}{4} + \frac{\pi}{6}\right) + i\sin\left(\frac{7\pi}{4} + \frac{\pi}{6}\right)\right] \\ &= 2\sqrt{2}\left(\cos\frac{23\pi}{12} + i\sin\frac{23\pi}{12}\right). \end{aligned}$$

2. The power of a complex number

Proposition. (De Moivre[1]) *For $z = r(\cos t + i\sin t)$ and $n \in \mathbb{N}$, we have*

$$z^n = r^n(\cos nt + i\sin nt). \tag{3}$$

Proof. Apply formula (2) for $z = z_1 = z_2 = \cdots = z_n$ to obtain

$$\begin{aligned} z^n &= \underbrace{r \cdot r \cdots r}_{n \text{ times}}(\cos(\underbrace{t + t + \cdots + t}_{n \text{ times}}) + i\sin(\underbrace{t + t + \cdots + t}_{n \text{ times}})) \\ &= r^n(\cos nt + i\sin nt). \end{aligned}$$

□

Remarks. a) We find again that $|z^n| = |z|^n$.
b) If $r = 1$, then $(\cos t + i\sin t)^n = \cos nt + i\sin nt$.
c) We can write $\operatorname{Arg} z^n = \{n \arg z + 2k\pi \;:\; k \in \mathbb{Z}\}$.

Example. Let us compute $(1 + i)^{1000}$.
The polar representation of $1 + i$ is $\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. Applying de Moivre's formula we obtain

$$\begin{aligned} (1 + i)^{1000} &= (\sqrt{2})^{1000}\left(\cos 1000\frac{\pi}{4} + i\sin 1000\frac{\pi}{4}\right) \\ &= 2^{500}(\cos 250\pi + i\sin 250\pi) = 2^{500}. \end{aligned}$$

Problem. *Prove that*

$$\sin 5t = 16\sin^5 t - 20\sin^3 t + 5\sin t;$$

$$\cos 5t = 16\cos^5 t - 20\cos^3 t + 5\cos t.$$

---

[1] Abraham de Moivre (1667–1754), French mathematician, a pioneer in probability theory and trigonometry.

Solution. Using de Moivre's theorem to expand $(\cos t + i \sin t)^5$, then using the binomial theorem, we have

$$\cos 5t + i \sin 5t = \cos^5 t + 5i \cos^4 t \sin t + 10i^2 \cos^3 t \sin^2 t$$
$$+ 10i^3 \cos^2 t \sin^3 t + 5i^4 \cos t \sin^4 t + i^5 \sin^5 t.$$

Hence

$$\cos 5t + i \sin 5t = \cos^5 t - 10 \cos^3 t(1 - \cos^2 t) + 5 \cos t(1 - \cos^2 t)^2$$
$$+ i(sint(1 - \sin^2 t)^2 \sin t - 10(1 - \sin^2 t) \sin^3 t + \sin^5 t).$$

Simple algebraic manipulation leads to the desired result.

3. Division

Proposition. *Suppose that*

$$z_1 = r_1(\cos t_1 + i \sin t_2), \quad z_2 = r_2(\cos t_2 + i \sin t_2) \neq 0.$$

Then

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}[\cos(t_1 - t_2) + i \sin(t_1 - t_2)].$$

Proof. We have

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{r_1(\cos t_1 + i \sin t_1)}{r_2(\cos t_2 + i \sin t_2)} = \\ &= \frac{r_1(\cos t_1 + i \sin t_1)(\cos t_2 - i \sin t_2)}{r_2(\cos^2 t_2 + \sin^2 t_2)} \\ &= \frac{r_1}{r_2}[(\cos t_1 \cos t_2 + \sin t_1 \sin t_2) + i(\sin t_1 \cos t_2 - \sin t_2 \cos t_1)] \\ &= \frac{r_1}{r_2}(\cos(t_1 - t_2) + i \sin(t_1 - t_2)).\end{aligned}$$ □

Remarks. a) We have again $\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$;

b) We can write $\operatorname{Arg}\left(\frac{z_1}{z_2}\right) = \{\arg z_1 - \arg z_2 + 2k\pi : k \in \mathbb{Z}\}$;

c) For $z_1 = 1$ and $z_2 = z$,

$$\frac{1}{z} = z^{-1} = \frac{1}{r}(\cos(-t) + i \sin(-t));$$

d) De Moivre's formula also holds for negative integer exponents n, i.e., we have

$$z^n = r^n(\cos nt + i \sin nt).$$

Problem. *Compute*

$$z = \frac{(1-i)^{10}(\sqrt{3}+i)^5}{(-1-i\sqrt{3})^{10}}.$$

Solution. We can write

$$\begin{aligned}
z &= \frac{(\sqrt{2})^{10}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)^{10} \cdot 2^5\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)^5}{2^{10}\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)^{10}} \\
&= \frac{2^{10}\left(\cos\frac{35\pi}{2} + i\sin\frac{35\pi}{2}\right)\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)}{2^{10}\left(\cos\frac{40\pi}{3} + i\sin\frac{40\pi}{3}\right)} \\
&= \frac{\cos\frac{55\pi}{3} + i\sin\frac{55\pi}{3}}{\cos\frac{40\pi}{3} + i\sin\frac{40\pi}{3}} = \cos 5\pi + i\sin 5\pi = -1.
\end{aligned}$$

2.1.4 Geometric interpretation of multiplication

Consider the complex numbers

$$z_1 = r_1(\cos t_1^* + i\sin t_1^*), \quad z_2 = r_2(\cos t_2^* + i\sin t_2^*)$$

and their geometric images $M_1(r_1, t_1^*)$, $M_2(r_2, t_2^*)$. Let P_1, P_2 be the intersection points of the circle $\mathcal{C}(O; 1)$ with the rays $(OM_1$ and $(OM_2$. Construct the point $P_3 \in \mathcal{C}(O; 1)$ with the polar argument $t_1^* + t_2^*$ and choose the point $M_3 \in (OP_3$ such that $OM_3 = OM_1 \cdot OM_2$. Let z_3 be the complex coordinate of M_3. The point $M_3(r_1 r_2, t_1^* + t_2^*)$ is the geometric image of the product $z_1 \cdot z_2$.

Let A be the geometric image of the complex number 1. Because

$$\frac{OM_3}{OM_1} = \frac{OM_2}{1}, \quad \text{i.e.,} \quad \frac{OM_3}{OM_2} = \frac{OM_2}{OA}$$

and $\widehat{M_2OM_3} = \widehat{AOM_1}$, it follows that triangles OAM_1 and OM_2M_3 are similar.

In order to construct the geometric image of the quotient, note that the image of $\frac{z_3}{z_2}$ is M_1.

2.1.5 Problems

1. Find the polar coordinates for the following points, given their cartesian coordinates:
a) $M_1(-3, 3)$; b) $M_2(-4\sqrt{3}, -4)$; c) $M_3(0, -5)$;
d) $M_4(-2, -1)$; e) $M_5(4, -2)$.

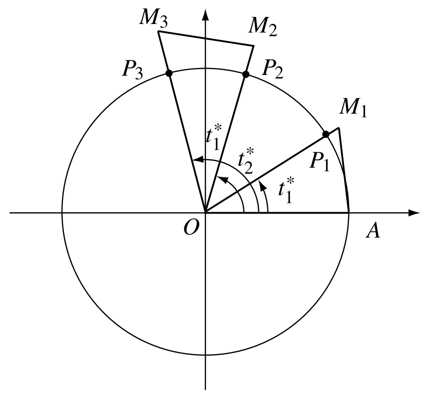

Figure 2.5.

2. Find the cartesian coordinates for the following points, given their polar coordinates:
a) $P_1\left(2, \dfrac{\pi}{3}\right)$; b) $P_2\left(4, 2\pi - \arcsin\dfrac{3}{5}\right)$; c) $P_3(2, \pi)$;
d) $P_4(3, -\pi)$; e) $P_5\left(1, \dfrac{\pi}{2}\right)$; f) $P_6\left(4, \dfrac{3\pi}{2}\right)$.

3. Express $\arg(\overline{z})$ and $\arg(-z)$ in terms of $\arg(z)$.

4. Find the geometric images for the complex numbers z in each of the following cases:
a) $|z| = 2$; b) $|z+i| \geq 2$; c) $|z-i| \leq 3$;
d) $\pi < \arg z < \dfrac{5\pi}{4}$; e) $\arg z \geq \dfrac{3\pi}{2}$; f) $\arg z < \dfrac{\pi}{2}$;
g) $\arg(-z) \in \left(\dfrac{\pi}{6}, \dfrac{\pi}{3}\right)$; h) $|z+1+i| < 3$ and $0 < \arg z < \dfrac{\pi}{6}$.

5. Find polar representations for the following complex numbers:
a) $z_1 = 6 + 6i\sqrt{3}$; b) $z_2 = -\dfrac{1}{4} + i\dfrac{\sqrt{3}}{4}$; c) $z_3 = -\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$;
d) $z_4 = 9 - 9i\sqrt{3}$; e) $z_5 = 3 - 2i$; f) $z_6 = -4i$.

6. Find polar representations for the following complex numbers:
a) $z_1 = \cos a - i\sin a, \quad a \in [0, 2\pi)$;
b) $z_2 = \sin a + i(1 + \cos a), \quad a \in [0, 2\pi)$;
c) $z_3 = \cos a + \sin a + i(\sin a - \cos a), \quad a \in [0, 2\pi)$;
d) $z_4 = 1 - \cos a + i\sin a, \quad a \in [0, 2\pi)$.

7. Compute the following products using the polar representation of a complex number:
a) $\left(\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right)(-3+3i)(2\sqrt{3}+2i)$; b) $(1+i)(-2-2i)\cdot i$;
c) $-2i \cdot (-4 + 4\sqrt{3}i) \cdot (3+3i)$; d) $3 \cdot (1-i)(-5+5i)$.

Verify your results using the algebraic form.

8. Find $|z|$, $\arg z$, $\operatorname{Arg} z$, $\arg \overline{z}$, $\arg(-z)$ for

a) $z = (1-i)(6+6i)$; b) $z = (7 - 7\sqrt{3}i)(-1-i)$.

9. Find $|z|$ and $\arg z$ for

a) $z = \dfrac{(2\sqrt{3}+2i)^8}{(1-i)^6} + \dfrac{(1+i)^6}{(2\sqrt{3}-2i)^8}$;

b) $z = \dfrac{(-1+i)^4}{(\sqrt{3}-i)^{10}} + \dfrac{1}{(2\sqrt{3}+2i)^4}$;

c) $z = (1+i\sqrt{3})^n + (1-i\sqrt{3})^n$.

10. Prove that de Moivre's formula holds for negative integer exponents.

11. Compute:

a) $(1 - \cos a + i \sin a)^n$ for $a \in [0, 2\pi)$ and $n \in \mathbb{N}$;

b) $z^n + \dfrac{1}{z^n}$, if $z + \dfrac{1}{z} = \sqrt{3}$.

2.2 The n^{th} Roots of Unity

2.2.1 Defining the n^{th} roots of a complex number

Consider a positive integer $n \geq 2$ and a complex number $z_0 \neq 0$. As in the field of real numbers, the equation

$$Z^n - z_0 = 0 \tag{1}$$

is used for defining the n^{th} roots of number z_0. Hence we call any solution Z of the equation (1) an *n^{th} root of the complex number z_0*.

Theorem. *Let $z_0 = r(\cos t^* + i \sin t^*)$ be a complex number with $r > 0$ and $t^* \in [0, 2\pi)$.*

The number z_0 has n distinct n^{th} roots, given by the formulas

$$Z_k = \sqrt[n]{r}\left(\cos \frac{t^* + 2k\pi}{n} + i \sin \frac{t^* + 2k\pi}{n}\right),$$

$k = 0, 1, \ldots, n-1$.

Proof. We use the polar representation of the complex number Z with the extended argument

$$Z = \rho(\cos\varphi + i \sin\varphi).$$

By definition, we have $Z^n = z_0$ or equivalently

$$\rho^n(\cos n\varphi + i \sin n\varphi) = r(\cos t^* + i \sin t^*).$$

We obtain $\rho^n = r$ and $n\varphi = t^* + 2k\pi$ for $k \in \mathbb{Z}$; hence $\rho = \sqrt[n]{r}$ and $\varphi_k = \dfrac{t^*}{n} + k \cdot \dfrac{2\pi}{n}$ for $k \in \mathbb{Z}$.

So far the roots of equation (1) are

$$Z_k = \sqrt[n]{r}(\cos\varphi_k + i\sin\varphi_k) \text{ for } k \in \mathbb{Z}.$$

Now observe that $0 \le \varphi_0 < \varphi_1 < \cdots < \varphi_{n-1} < 2\pi$, so the numbers φ_k, $k \in \{0, 1, \ldots, n-1\}$, are reduced arguments, i.e., $\varphi_k^* = \varphi_k$. Until now we had n distinct roots of z_0:

$$Z_0, Z_1, \ldots, Z_{n-1}.$$

Consider some integer k and let $r \in \{0, 1, \ldots, n-1\}$ be the residue of k modulo n. Then $k = nq + r$ for $q \in \mathbb{Z}$, and

$$\varphi_k = \frac{t^*}{n} + (nq + r)\frac{2\pi}{n} = \frac{t^*}{n} + r\frac{2\pi}{n} + 2q\pi = \varphi_r + 2q\pi.$$

It is clear that $Z_k = Z_r$. Hence

$$\{Z_k : k \in \mathbb{Z}\} = \{Z_0, Z_1, \ldots, Z_{n-1}\}.$$

In other words, there are exactly n distinct n^{th} roots of z_0, as claimed. □

The geometric images of the n^{th} roots of a complex number $z_0 \neq 0$ are the vertices of a regular n-gon inscribed in a circle with center at the origin and radius $\sqrt[n]{r}$.

To prove this, denote $M_0, M_1, \ldots, M_{n-1}$ the points with complex coordinates Z_0, $Z_1, \ldots, Z_{n-1}$. Because $OM_k = |Z_k| = \sqrt[n]{r}$ for $k \in \{0, 1, \ldots, n-1\}$, it follows that the points M_k lie on the circle $\mathcal{C}(O; \sqrt[n]{r})$. On the other hand, the measure of the arc $\overset{\frown}{M_k M_{k+1}}$ is equal to

$$\arg Z_{k+1} - \arg Z_k = \frac{t^* + 2(k+1)\pi - (t^* + 2k\pi)}{n} = \frac{2\pi}{n},$$

for all $k \in \{0, 1, \ldots, n-2\}$ and the remaining arc $\overset{\frown}{M_{n-1} M_0}$ is

$$\frac{2\pi}{n} = 2\pi - (n-1)\frac{2\pi}{n}.$$

Because all of the arcs $\overset{\frown}{M_0 M_1}$, $\overset{\frown}{M_1 M_2}, \ldots, \overset{\frown}{M_{n-1} M_0}$ are equal, the polygon $M_0 M_1 \cdots M_{n-1}$ is regular.

Example. Let us find the third roots of the number $z = 1 + i$ and represent them in the complex plane.

The polar representation of $z = 1 + i$ is

$$z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$

The cube roots of the number z are

$$Z_k = \sqrt[6]{2}\left(\cos\left(\frac{\pi}{12}+k\frac{2\pi}{3}\right)+i\sin\left(\frac{\pi}{12}+k\frac{2\pi}{3}\right)\right),\ k=0,1,2,$$

or, in explicit form,

$$Z_0 = \sqrt[6]{2}\left(\cos\frac{\pi}{12}+i\sin\frac{\pi}{12}\right),$$

$$Z_1 = \sqrt[6]{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$$

and

$$Z_2 = \sqrt[6]{2}\left(\cos\frac{17\pi}{12}+i\sin\frac{17\pi}{12}\right).$$

Using polar coordinates, the geometric images of the numbers Z_0, Z_1, Z_2 are

$$M_0\left(\sqrt[6]{2},\frac{\pi}{12}\right),\ M_1\left(\sqrt[6]{2},\frac{3\pi}{4}\right),\ M_2\left(\sqrt[6]{2},\frac{17\pi}{12}\right).$$

The resulting equilateral triangle $M_0M_1M_2$ is shown in the following figure:

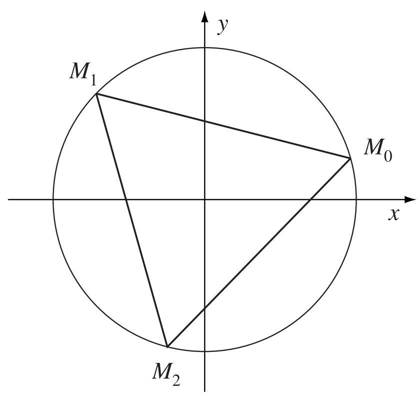

Figure 2.6.

2.2.2 The n^{th} roots of unity

The roots of the equation $Z^n - 1 = 0$ are called the n^{th} *roots of unity*. Since $1 = \cos 0 + i\sin 0$, from the formulas for the n^{th} roots of a complex number we derive that the n^{th} roots of unity are

$$\varepsilon_k = \cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n},\quad k\in\{0,1,2,\ldots,n-1\}.$$

Explicitly, we have

$$\varepsilon_0 = \cos 0 + i\sin 0 = 1;$$

$$\varepsilon_1 = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n} = \varepsilon;$$

$$\varepsilon_2 = \cos\frac{4\pi}{n} + i\sin\frac{4\pi}{n} = \varepsilon^2;$$

$$\dots$$

$$\varepsilon_{n-1} = \cos\frac{2(n-1)\pi}{n} + i\sin\frac{2(n-1)\pi}{n} = \varepsilon^{n-1}.$$

The set $\{1, \varepsilon, \varepsilon^2, \dots, \varepsilon^{n-1}\}$ is denoted by U_n. Observe that the set U_n is generated by the element ε, i.e., the elements of U_n are the powers of ε.

As stated before, the geometric images of the n^{th} roots of unity are the vertices of a regular polygon with n sides inscribed in the unit circle with one of the vertices at 1.

We take a brief look at some particular values of n.

i) For $n = 2$, the equation $Z^2 - 1 = 0$ has the roots -1 and 1, which are the square roots of unity.

ii) For $n = 3$, the cube roots of unity, i.e., the roots of equation $Z^3 - 1 = 0$ are given by

$$\varepsilon_k = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3} \text{ for } k \in \{0, 1, 2\}.$$

Hence

$$\varepsilon_0 = 1, \quad \varepsilon_1 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2} = \varepsilon$$

and

$$\varepsilon_2 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = -\frac{1}{2} - i\frac{\sqrt{3}}{2} = \varepsilon^2.$$

They form an equilateral triangle inscribed in the circle $\mathcal{C}(O; 1)$ as in the figure below.

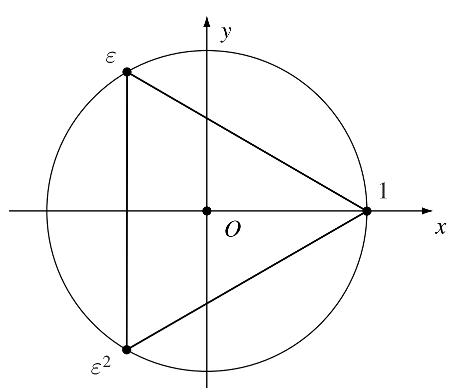

Figure 2.7.

iii) For $n = 4$, the fourth roots of unity are

$$\varepsilon_k = \cos\frac{2k\pi}{4} + i\sin\frac{2k\pi}{4} \text{ for } k = 0, 1, 2, 3.$$

In explicit form, we have

$$\varepsilon_0 = \cos 0 + i\sin 0 = 1; \quad \varepsilon_1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i;$$

$$\varepsilon_2 = \cos\pi + i\sin\pi = -1 \text{ and } \varepsilon_3 = \cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2} = -i.$$

Observe that $U_4 = \{1, i, i^2, i^3\} = \{1, i, -1, -i\}$. The geometric images of the fourth roots of unity are the vertices of a square inscribed in the circle $\mathcal{C}(O; 1)$.

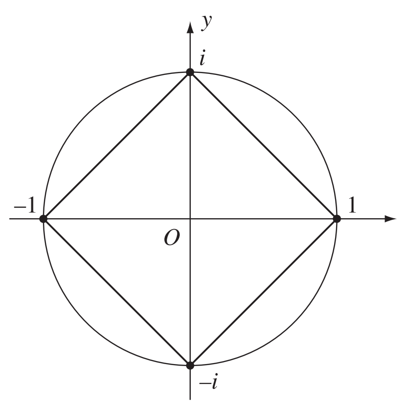

Figure 2.8.

The root $\varepsilon_k \in U_n$ is called *primitive* if for all positive integer $m < n$ we have $\varepsilon_k^m \neq 1$.

Proposition 1. *a) If $n|q$, then any root of $Z^n - 1 = 0$ is a root of $Z^q - 1 = 0$.*

b) The common roots of $Z^m - 1 = 0$ and $Z^n - 1 = 0$ are the roots of $Z^d - 1 = 0$, where $d = \gcd(m, n)$, i.e., $U_m \cap U_n = U_d$.

c) The primitive roots of $Z^m - 1 = 0$ are $\varepsilon_k = \cos\frac{2k\pi}{m} + i\sin\frac{2k\pi}{m}$, where $0 \leq k \leq m$ and $\gcd(k, m) = 1$.

Proof. a) If $q = pn$, then $Z^q - 1 = (Z^n)^p - 1 = (Z^n - 1)(Z^{(p-1)n} + \cdots + Z^n + 1)$ and the conclusion follows.

b) Consider $\varepsilon_p = \cos\frac{2p\pi}{m} + i\sin\frac{2p\pi}{m}$ a root of $Z^m - 1 = 0$ and $\varepsilon'_q = \cos\frac{2q\pi}{n} + i\sin\frac{2q\pi}{n}$ a root of $Z^n - 1 = 0$. Since $|\varepsilon_p| = |\varepsilon'_q| = 1$, we have $\varepsilon_p = \varepsilon'_q$ if and only if $\arg \varepsilon_p = \arg \varepsilon'_q$, i.e., $\dfrac{2p\pi}{m} = \dfrac{2q\pi}{n} + 2r\pi$ for some integer r. The last relation is equivalent to $\dfrac{p}{m} - \dfrac{q}{n} = r$, that is, $pn - qm = rmn$.

On the other hand we have $m = m'd$ and $n = n'd$, where $\gcd(m', n') = 1$. From the relation $pn - qm = rmn$ we find $n'p - m'q = rm'n'd$. Hence $m'|n'p$, so $m'|p$. That is, $p = p'm'$ for some positive integer p' and

$$\arg \varepsilon_p = \frac{2p\pi}{m} = \frac{2p'm'\pi}{m'd} = \frac{2p'\pi}{d} \text{ and } \varepsilon_p^d = 1.$$

Conversely, since $d|m$ and $d|n$ (from property a), any root of $Z^d - 1 = 0$ is a root of $Z^m - 1 = 0$ and $Z^n - 1 = 0$.

c) First we will find the smallest positive integer p such that $\varepsilon_k^p = 1$. From the relation $\varepsilon_k^p = 1$ it follows that $\dfrac{2kp\pi}{m} = 2k'\pi$ for some positive integer k'. That is, $\dfrac{kp}{m} = k' \in \mathbb{Z}$. Consider $d = \gcd(k, m)$ and $k = k'd$, $m = m'd$, where $\gcd(k', m') = 1$. We obtain $\dfrac{k'pd}{m'd} = \dfrac{k'p}{m'} \in \mathbb{Z}$. Since k' and m' are relatively primes, we get $m'|p$. Therefore, the smallest positive integer p with $\varepsilon_k^p = 1$ is $p = m'$. Substituting in the relation $m = m'd$, it follows that $p = \dfrac{m}{d}$, where $d = \gcd(k, m)$.

If ε_k is a primitive root of unity, then from relation $\varepsilon_k^p = 1$, $p = \dfrac{m}{\gcd(k, m)}$, it follows that $p = m$, i.e., $\gcd(k, m) = 1$. □

Remark. From Proposition 1.b) one obtains that the equations $Z^m - 1 = 0$ and $Z^n - 1 = 0$ have the unique common root 1 if and only if $\gcd(m, n) = 1$.

Proposition 2. *If $\varepsilon \in U_n$ is a primitive root of unity, then the roots of the equation $z^n - 1 = 0$ are $\varepsilon^r, \varepsilon^{r+1}, \dots, \varepsilon^{r+n-1}$, where r is an arbitrary positive integer.*

Proof. Let r be a positive integer and consider $h \in \{0, 1, \dots, n-1\}$. Then $(\varepsilon^{r+h})^n = (\varepsilon^n)^{r+h} = 1$, i.e., ε^{r+h} is a root of $Z^n - 1 = 0$.

We need only prove that $\varepsilon^r, \varepsilon^{r+1}, \dots, \varepsilon^{r+n-1}$ are distinct. Assume by way of contradiction that for $r + h_1 \neq r + h_2$ and $h_1 > h_2$, we have $\varepsilon^{r+h_1} = \varepsilon^{r+h_2}$. Then $\varepsilon^{r+h_2}(\varepsilon^{h_1-h_2} - 1) = 0$. But $\varepsilon^{r+h_2} \neq 0$ implies $\varepsilon^{h_1-h_2} = 1$. Taking into account that $h_1 - h_2 < n$ and ε is a primitive root of $Z^n - 1 = 0$, we get a contradiction. □

Proposition 3. *Let $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}$ be the n^{th} roots of unity. For any positive integer k the following relation holds:*

$$\sum_{j=0}^{n-1} \varepsilon_j^k = \begin{cases} n, & \text{if } n|k, \\ 0, & \text{otherwise.} \end{cases}$$

Proof. Consider $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Then $\varepsilon \in U_n$ is a primitive root of unity, hence $\varepsilon^m = 1$ if and only if $n|m$. Assume that n does not divides k. We have

$$\sum_{j=0}^{n-1}\varepsilon_j^k = \sum_{j=0}^{n-1}(\varepsilon^j)^k = \sum_{j=0}^{n-1}(\varepsilon^k)^j = \frac{1-(\varepsilon^k)^n}{1-\varepsilon^k} = \frac{1-(\varepsilon^n)^k}{1-\varepsilon^k} = 0.$$

If $n|k$, then $k = qn$ for some positive integer q, and we obtain

$$\sum_{j=0}^{n-1}\varepsilon_j^k = \sum_{j=0}^{n-1}\varepsilon_j^{qn} = \sum_{j=0}^{n-1}(\varepsilon_j^n)^q = \sum_{j=0}^{n-1}1 = n. \qquad \square$$

Proposition 4. *Let p be a prime number and let $\varepsilon = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$. If $a_0, a_1, \dots, a_{p-1}$ are nonzero integers, the relation*

$$a_0 + a_1\varepsilon + \cdots + a_{p-1}\varepsilon^{p-1} = 0$$

holds if and only if $a_0 = a_1 = \cdots = a_{p-1}$.

Proof. If $a_0 = a_1 = \cdots = a_{p-1}$, then the above relation is clearly true.

Conversely, define the polynomials $f, g \in \mathbb{Z}[X]$ by $f = a_1 + a_1X + \cdots + a_{p-1}X^{p-1}$ and $g = 1 + X + \cdots + X^{p-1}$. If the polynomials f, g have common zeros, then $\gcd(f, g)$ divides g. But it is well known (for example by Eisenstein's irreducibility criterion) that g is irreducible over $\mathbb{Z}$. Hence $\gcd(f, g) = g$, so $g|f$ and we obtain $g = kf$ for some nonzero integer k, i.e., $a_0 = a_1 = \cdots = a_{n-1}$. $\square$

Problem 1. *Find the number of ordered pairs (a, b) of real numbers such that $(a + bi)^{2002} = a - bi$.*

(American Mathematics Contest 12A, 2002, Problem 24)

Solution. Let $z = a + bi$, $\overline{z} = a - bi$, and $|z| = \sqrt{a^2 + b^2}$. The given relation becomes $z^{2002} = \overline{z}$. Note that

$$|z|^{2002} = |z^{2002}| = |\overline{z}| = |z|,$$

from which it follows that

$$|z|(|z|^{2001} - 1) = 0.$$

Hence $|z| = 0$, and $(a, b) = (0, 0)$, or $|z| = 1$. In the case $|z| = 1$, we have $z^{2002} = \overline{z}$, which is equivalent to $z^{2003} = \overline{z}\cdot z = |z|^2 = 1$. Since the equation $z^{2003} = 1$ has 2003 distinct solutions, there are altogether $1 + 2003 = 2004$ ordered pairs that meet the required conditions.

Problem 2. *Two regular polygons are inscribed in the same circle. The first polygon has 1982 sides and the second has 2973 sides. If the polygons have any common vertices, how many such vertices will there be?*

Solution. The number of common vertices is given by the number of common roots of $z^{1982} - 1 = 0$ and $z^{2973} - 1 = 0$. Applying Proposition 1.b), the desired number is $d = \gcd(1982, 2973) = 991$.

Problem 3. *Let $\varepsilon \in U_n$ be a primitive root of unity and let z be a complex number such that $|z - \varepsilon^k| \le 1$ for all $k = 0, 1, \dots, n-1$. Prove that $z = 0$.*

Solution. From the given condition it follows that $(z - \varepsilon^k)\overline{(z - \varepsilon^k)} \le 1$, yielding $|z|^2 \le z\overline{(\varepsilon^k)} + \overline{z} \cdot \varepsilon^k$, $k = 0, 1, \dots, n-1$. By summing these relations we obtain

$$n|z|^2 \le z\overline{\left(\sum_{k=0}^{n-1} \varepsilon^k\right)} + \overline{z} \cdot \sum_{k=0}^{n-1} \varepsilon^k = 0.$$

Thus $z = 0$.

Problem 4. *Let $P_0P_1 \cdots P_{n-1}$ be a regular polygon inscribed in a circle of radius 1. Prove that:*

a) $P_0P_1 \cdot P_0P_2 \cdots P_0P_{n-1} = n$;
b) $\sin\frac{\pi}{n} \sin\frac{2\pi}{n} \cdots \sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}$;
c) $\sin\frac{\pi}{2n} \sin\frac{3\pi}{2n} \cdots \sin\frac{(2n-1)\pi}{2n} = \frac{1}{2^{n-1}}$.

Solution. a) Without loss of generality we may assume that the vertices of the polygon are the geometric images of the n^{th} roots of unity, and $P_0 = 1$. Consider the polynomial $f = z^n - 1 = (z-1)(z-\varepsilon)\cdots(z-\varepsilon^{n-1})$, where $\varepsilon = \cos\frac{2\pi}{n} + i \sin\frac{2\pi}{n}$. Then it is clear that

$$n = f'(1) = (1-\varepsilon)(1-\varepsilon^2)\cdots(1-\varepsilon^{n-1}).$$

Taking the modulus of each side, the desired result follows.

b) We have

$$1 - \varepsilon^k = 1 - \cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n} = 2\sin^2\frac{k\pi}{n} - 2i\sin\frac{k\pi}{n}\cos\frac{k\pi}{n}$$
$$= 2\sin\frac{k\pi}{n}\left(\sin\frac{k\pi}{n} - i\cos\frac{k\pi}{n}\right),$$

hence $|1 - \varepsilon^k| = 2\sin\frac{k\pi}{n}$, $k = 1, 2, \dots, n-1$, and the desired trigonometric identity follows from a).

c) Consider the regular polygon $Q_0Q_1 \cdots Q_{2n-1}$ inscribed in the same circle whose vertices are the geometric images of the $(2n)^{\text{th}}$ roots of unity. According to a),

$$Q_0Q_1 \cdot Q_0Q_2 \cdots Q_0Q_{2n-1} = 2n.$$

Now taking into account that $Q_0Q_2 \cdots Q_{n-2}$ is also a regular polygon, we deduce from a) that

$$Q_0Q_2 \cdot Q_0Q_4 \cdots Q_0Q_{2n-2} = n.$$

Combining the last two relations yields

$$Q_0Q_1 \cdot Q_0Q_3 \cdots Q_0Q_{2n-1} = 2.$$

A similar computation to the one in b) leads to

$$Q_0Q_{2k-1} = 2\sin\frac{(2k-1)\pi}{2n}, \quad k = 1, 2, \ldots, n,$$

and the desired result follows.

Let n be a positive integer and let $\varepsilon_n = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. The n^{th}*-cyclotomic polynomial* is defined by

$$\phi_n(x) = \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} (x - \varepsilon_n^k).$$

Clearly the degree of ϕ_n is $\varphi(n)$, where φ is the Euler "totient" function. ϕ_n is a monic polynomial with integer coefficients and is irreducible over $\mathbb{Q}$. The first sixteen cyclotomic polynomials are given below:

$\phi(x) = x - 1$
$\phi_2(x) = x + 1$
$\phi_3(x) = x^2 + x + 1$
$\phi_4(x) = x^2 + 1$
$\phi_5(x) = x^4 + x^3 + x^2 + x + 1$
$\phi_6(x) = x^2 - x + 1$
$\phi_7(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
$\phi_8(x) = x^4 + 1$
$\phi_9(x) = x^6 + x^3 + 1$
$\phi_{10}(x) = x^4 - x^3 + x^2 - x + 1$
$\phi_{11}(x) = x^{10} + x^9 + x^8 + \cdots + x + 1$
$\phi_{12}(x) = x^4 - x^2 + 1$
$\phi_{13}(x) = x^{12} + x^{11} + x^{10} + \cdots + x + 1$
$\phi_{14}(x) = x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$
$\phi_{15}(x) = x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$
$\phi_{16}(x) = x^8 + 1$

The following properties of cyclotomic polynomials are well known:

1) If $q > 1$ is an odd integer, then $\phi_{2q}(x) = \phi_q(-x)$.

2) If $n > 1$, then

$$\phi_n(1) = \begin{cases} p, & \text{when } n \text{ is a power of a prime } p, \\ 1, & \text{otherwise.} \end{cases}$$

The next problem extends the trigonometric identity in Problem 4.b).

Problem 5. *The following identities hold:*

a) $\displaystyle\prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \sin\frac{k\pi}{n} = \frac{1}{2^{\varphi(n)}}$*, whenever n is not a power of a prime;*

b) $\displaystyle\prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \cos\frac{k\pi}{n} = \frac{(-1)^{\frac{\varphi(n)}{2}}}{2^{\varphi(n)}}$*, for all odd positive integers n.*

Solution. a) As we have seen in Problem 4.b),

$$1-\varepsilon_n^k = 2\sin\frac{k\pi}{n}\left(\sin\frac{k\pi}{n} - i\cos\frac{k\pi}{n}\right) = \frac{2}{i}\sin\frac{k\pi}{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right).$$

We have

$$1 = \phi_n(1) = \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} (1-\varepsilon_n^k) = \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \frac{2}{i}\sin\frac{k\pi}{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right)$$

$$= \frac{2^{\varphi(n)}}{i^{\varphi(n)}}\left(\prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \sin\frac{k\pi}{n}\right)\left(\cos\frac{\varphi(n)}{2}\pi + i\sin\frac{\varphi(n)}{2}\pi\right)$$

$$= \frac{2^{\varphi(n)}}{(-1)^{\frac{\varphi(n)}{2}}}\left(\prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \sin\frac{k\pi}{n}\right)(-1)^{\frac{\varphi(n)}{2}},$$

where we have used the fact that $\varphi(n)$ is even, and also the well-known relation

$$\sum_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} k = \frac{1}{2}n\varphi(n).$$

The conclusion follows.

b) We have

$$1+\varepsilon_n^k = 1+\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = 2\cos^2\frac{k\pi}{n} + 2i\sin\frac{k\pi}{n}\cos\frac{k\pi}{n}$$

$$= 2\cos\frac{k\pi}{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right), \quad k = 0, 1, \ldots, n-1.$$

Because n is odd, from the relation $\phi_{2n}(x) = \phi_n(-1)$ it follows that $\phi_n(-1) = \phi_{2n}(1) = 1$. Then

$$1 = \phi_n(-1) = \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} (1-\varepsilon_n^k) = (-1)^{\varphi(n)} \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} (1+\varepsilon_n^k)$$

$$= (-1)^{\varphi(n)} \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} 2\cos\frac{k\pi}{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right)$$

$$= (-1)^{\varphi(n)} 2^{\varphi(n)} \left(\prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \cos\frac{k\pi}{n}\right)\left(\cos\frac{\varphi(n)}{2}\pi + i\sin\frac{\varphi(n)}{2}\pi\right)$$

$$= (-1)^{\frac{\varphi(n)}{2}} 2^{\varphi(n)} \prod_{\substack{1\le k\le n-1\\ \gcd(k,n)=1}} \cos\frac{k\pi}{n},$$

yielding the desired identity.

2.2.3 Binomial equations

A binomial equation is an equation of the form $Z^n + a = 0$, where $a \in \mathbb{C}^*$ and $n \ge 2$ is an integer.

Solving for Z means finding the n^{th} roots of the complex number $-a$. This is in fact a simple polynomial equation of degree n with complex coefficients. From the well-known fundamental theorem of algebra it follows that it has exactly n complex roots, and it is obvious that the roots are distinct.

Example. 1) Let us find the roots of $Z^3 + 8 = 0$.

We have $-8 = 8(\cos\pi + i\sin\pi)$, so the roots are

$$Z_k = 2\left(\cos\frac{\pi + 2k\pi}{3} + i\sin\frac{\pi+2k\pi}{3}\right), \quad k \in \{0,1,2\}.$$

2) Let us solve the equation $Z^6 - Z^3(1+i) + i = 0$.

Observe that the equation is equivalent to

$$(Z^3 - 1)(Z^3 - i) = 0.$$

Solving for Z the binomial equations $Z^3 - 1 = 0$ and $Z^3 - i = 0$, we obtain the solutions

$$\varepsilon_k = \cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3} \text{ for } k \in \{0,1,2\}$$

and

$$Z_k = \cos\frac{\frac{\pi}{2} + 2k\pi}{3} + i\sin\frac{\frac{\pi}{2}+2k\pi}{3} \text{ for } k \in \{0,1,2\}.$$

2.2.4 Problems

1. Find the square roots of the following complex numbers:
a) $z = 1 + i$; b) $z = i$; c) $z = \dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}$;
d) $z = -2(1 + i\sqrt{3})$; e) $z = 7 - 24i$.

2. Find the cube roots of the following complex numbers:
a) $z = -i$; b) $z = -27$; c) $z = 2 + 2i$;
d) $z = \dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$; e) $z = 18 + 26i$.

3. Find the fourth roots of the following complex numbers:
a) $z = 2 - i\sqrt{12}$; b) $z = \sqrt{3} + i$; c) $z = i$;
d) $z = -2i$; e) $z = -7 + 24i$.

4. Find the fifth, sixth, seventh, eighth, and twefth roots of the complex numbers given above.

5. Let $U_n = \{\varepsilon_0, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}\}$. Prove that:
a) $\varepsilon_j \cdot \varepsilon_k \in U_n$, for all $j, k \in \{0, 1, \dots, n-1\}$;
b) $\varepsilon_j^{-1} \in U_n$, for all $j \in \{0, 1, \dots, n-1\}$.

6. Solve the equations:
a) $z^3 - 125 = 0$; b) $z^4 + 16 = 0$;
c) $z^3 + 64i = 0$; d) $z^3 - 27i = 0$.

7. Solve the equations:
a) $z^7 - 2iz^4 - iz^3 - 2 = 0$; b) $z^6 + iz^3 + i - 1 = 0$;
c) $(2 - 3i)z^6 + 1 + 5i = 0$; d) $z^{10} + (-2 + i)z^5 - 2i = 0$.

8. Solve the equation

$$z^4 = 5(z - 1)(z^2 - z + 1).$$

3

Complex Numbers and Geometry

3.1 Some Simple Geometric Notions and Properties

3.1.1 The distance between two points

Suppose that the complex numbers z_1 and z_2 have the geometric images M_1 and M_2. Then the distance between the points M_1 and M_2 is given by

$$M_1M_2 = |z_1 - z_2|.$$

The distance function $d : \mathbb{C} \times \mathbb{C} \to [0, \infty)$ is defined by

$$d(z_1, z_2) = |z_1 - z_2|,$$

and it satisfies the following properties:

a) (positiveness and nondegeneration):

$$d(z_1, z_2) \geq 0 \text{ for all } z_1, z_2 \in \mathbb{C};$$

$$d(z_1, z_2) = 0 \text{ if and only if } z_1 = z_2.$$

b) (symmetry):

$$d(z_1, z_2) = d(z_2, z_1) \text{ for all } z_1, z_2 \in \mathbb{C}.$$

c) (triangle inequality):

$$d(z_1, z_2) \leq d(z_1, z_3) + d(z_3, z_2) \text{ for all } z_1, z_2, z_3 \in \mathbb{C}.$$

To justify c) let us observe that

$$|z_1 - z_2| = |(z_1 - z_3) + (z_3 - z_2)| \leq |z_1 - z_3| + |z_3 - z_2|,$$

from the modulus property. Equality holds if and only if there is a positive real number k such that

$$z_3 - z_1 = k(z_2 - z_3).$$

3.1.2 Segments, rays and lines

Let A and B be two distinct points with complex coordinates a and b. We say that the point M with complex coordinate z is between the points A and B if $z \neq a$, $z \neq b$ and the following relation holds:

$$|a - z| + |z - b| = |a - b|.$$

We use the notation $A - M - B$.

The set $(AB) = \{M : A - M - B\}$ is called the *open segment* determined by the points A and B. The set $[AB] = (AB) \cup \{A, B\}$ represents the *closed segment* defined by the points A and B.

Theorem 1. *Suppose $A(a)$ and $B(b)$ are two distinct points. The following statements are equivalent:*

1) $M \in (AB)$;

2) there is a positive real number k such that $z - a = k(b - z)$;

3) there is a real number $t \in (0, 1)$ such that $z = (1 - t)a + tb$, where z is the complex coordinate of M.

Proof. We first prove that 1) and 2) are equivalent. Indeed, we have $M \in (AB)$ if and only if $|a - z| + |z - b| = |a - b|$. That is, $d(a, z) + d(z, b) = d(a, b)$, or equivalently there is a real $k > 0$ such that $z - a = k(b - z)$.

To prove that 2) $\Leftrightarrow$ 3), set $t = \dfrac{k}{k+1} \in (0, 1)$ or $k = \dfrac{t}{1-t} > 0$. Then we have $z - a = k(b - z)$ if and only if $z = \dfrac{1}{k+1}a + \dfrac{k}{k+1}b$. That is, $z = (1 - t)a + tb$ and we are done. □

The set $(AB = \{M | A - M - B \text{ or } A - B - M\}$ is called the *open ray* with endpoint A that contains B.

Theorem 2. *Suppose $A(a)$ and $B(b)$ are two distinct points. The following statements are equivalent:*

1) $M \in (AB$;

2) there is a positive real number t such that $z = (1 - t)a + tb$, where z is the complex coordinate of M;

3) $\arg(z-a)=\arg(b-a)$*;*
4) $\dfrac{z-a}{b-a}\in\mathbb{R}^+$.

Proof. It suffices to prove that 1) $\Rightarrow$ 2) $\Rightarrow$ 3) $\Rightarrow$ 4) $\Rightarrow$ 1).

1) $\Rightarrow$ 2). Since $M\in(AB$ we have $A-M-B$ or $A-B-M$. There are numbers $t,l\in(0,1)$ such that

$$z=(1-t)a+tb \text{ or } b=(1-l)a+lz.$$

In the first case we are done; for the second case set $t=\dfrac{1}{l}$, hence

$$z=tb-(t-1)a=(1-t)a+tb,$$

as claimed.

2) $\Rightarrow$ 3). From $z=(1-t)a+tb$, $t>0$ we obtain

$$z-a=t(b-a),\ \ t>0.$$

Hence

$$\arg(z-a)=\arg(b-a).$$

3) $\Rightarrow$ 4). The relation

$$\arg\frac{z-a}{b-a}=\arg(z-a)-\arg(b-a)+2k\pi \text{ for some } k\in\mathbb{Z}$$

implies $\arg\dfrac{z-a}{b-a}=2k\pi$, $k\in\mathbb{Z}$. Since $\arg\dfrac{z-a}{b-a}\in[0,2\pi)$, it follows that $k=0$ and $\arg\dfrac{z-a}{b-a}=0$. Thus $\dfrac{z-a}{b-a}\in\mathbb{R}^+$, as desired.

4) $\Rightarrow$ 1). Let $t=\dfrac{z-a}{b-a}\in\mathbb{R}^*$. Hence

$$z=a+t(b-a)=(1-t)a+tb,\ \ t>0.$$

If $t\in(0,1)$, then $M\in(AB)\subset(AB$.

If $t=1$, then $z=b$ and $M=B\in(AB$. Finally, if $t>1$ then, setting $l=\dfrac{1}{t}\in(0,1)$, we have

$$b=lz+(1-l)a.$$

It follows that $A-B-M$ and $M\in(AB$.

The proof is now complete. □

Theorem 3. *Suppose $A(a)$ and $B(b)$ are two distinct points. The following statements are equivalent:*

1) $M(z)$ lies on the line AB.

2) $\dfrac{z-a}{b-a} \in \mathbb{R}$.

3) There is a real number t such that $z = (1-t)a + tb$.

4) $\begin{vmatrix} z-a & \overline{z}-\overline{a} \\ b-a & \overline{b}-\overline{a} \end{vmatrix} = 0;$

5) $\begin{vmatrix} z & \overline{z} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{vmatrix} = 0.$

Proof. To obtain the equivalences 1) $\Leftrightarrow$ 2) $\Leftrightarrow$ 3) observe that for a point C such that $C - A - B$ the line AB is the union $(AB \cup \{A\} \cup (AC$. Then apply Theorem 2.

Next we prove the equivalences 2) $\Leftrightarrow$ 4) $\Leftrightarrow$ 5).

Indeed, we have $\dfrac{z-a}{b-a} \in \mathbb{R}$ if and only if $\dfrac{z-a}{b-a} = \overline{\left(\dfrac{z-a}{b-a}\right)}$.

That is, $\dfrac{z-a}{b-a} = \dfrac{\overline{z}-\overline{a}}{\overline{b}-\overline{a}}$, or, equivalently, $\begin{vmatrix} z-a & \overline{z}-\overline{a} \\ b-a & \overline{b}-\overline{a} \end{vmatrix} = 0$, so we obtain that 2) is equivalent to 4).

Moreover, we have

$$\begin{vmatrix} z & \overline{z} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{vmatrix} = 0 \text{ if and only if } \begin{vmatrix} z-a & \overline{z}-\overline{a} & 0 \\ a & \overline{a} & 1 \\ b-a & \overline{b}-\overline{a} & 0 \end{vmatrix} = 0$$

The last relation is equivalent to

$$\begin{vmatrix} z-a & \overline{z}-\overline{a} \\ b-a & \overline{b}-\overline{a} \end{vmatrix} = 0,$$

so we obtain that 4) is equivalent to 5), and we are done. □

Problem 1. *Let z_1, z_2, z_3 be complex numbers such that $|z_1| = |z_2| = |z_3| = R$ and $z_2 \neq z_3$. Prove that*

$$\min_{a \in \mathbb{R}} |az_2 + (1-a)z_3 - z_1| = \frac{1}{2R}|z_1 - z_2| \cdot |z_1 - z_3|.$$

(Romanian Mathematical Olympiad – Final Round, 1984)

Solution. Let $z = az_2 + (1-a)z_3$, $a \in \mathbb{R}$ and consider the points A_1, A_2, A_3, A of complex coordinates z_1, z_2, z_3, z, respectively. From the hypothesis it follows that the circumcenter of triangle $A_1A_2A_3$ is the origin of the complex plane. Notice that point A lies on the line A_2A_3, so $A_1A = |z - z_1|$ is greater than or equal to the altitude A_1B of the triangle $A_1A_2A_3$.

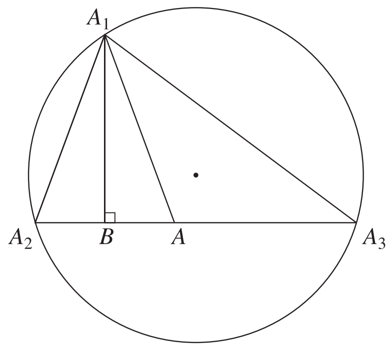

Figure 3.1.

It suffices to prove that

$$A_1B = \frac{1}{2R}|z_1 - z_2||z_1 - z_3| = \frac{1}{2R}A_1A_2 \cdot A_1A_3.$$

Indeed, since R is the circumradius of the triangle $A_1A_2A_3$, we have

$$A_1B = \frac{2\text{area}[A_1A_2A_3]}{A_2A_3} = \frac{2\dfrac{A_1A_2 \cdot A_2A_3 \cdot A_3A_1}{4R}}{A_2A_3} = \frac{A_1A_2 \cdot A_3A_1}{2R},$$

as claimed.

3.1.3 Dividing a segment into a given ratio

Consider two distinct points $A(a)$ and $B(b)$. A point $M(z)$ on the line AB divides the segments AB into the ratio $k \in \mathbb{R} \setminus \{1\}$ if the following vectorial relation holds:

$$\overrightarrow{MA} = k \cdot \overrightarrow{MB}.$$

In terms of complex numbers this relation can be written as

$$a - z = k(b - z) \text{ or } (1 - k)z = a - kb.$$

Hence, we obtain

$$z = \frac{a - kb}{1 - k}.$$

Observe that for $k < 0$ the point M lies on the line segment joining the points A and B. If $k \in (0, 1)$, then $M \in (AB \setminus [AB]$. Finally, if $k > 1$, then $M \in (BA \setminus [AB]$.

As a consequence, note that for $k = -1$ we obtain that the coordinate of the midpoint of segment $[AB]$ is given by $z_M = \dfrac{a+b}{2}$.

Example. Let $A(a)$, $B(b)$, $C(c)$ be noncollinear points in the complex plane. Then the midpoint M of segment $[AB]$ has the complex coordinate $z_M = \dfrac{a+b}{2}$. The centroid G of triangle ABC divides the median $[CM]$ into $2 : 1$ internally, hence its complex coordinate is given by $k = -2$, i.e.,

$$z_G = \frac{c + 2z_M}{1+2} = \frac{a+b+c}{3}.$$

3.1.4 Measure of an angle

Recall that a triangle is oriented if an ordering of its vertices is specified. It is positively or directly oriented if the vertices are oriented counterclockwise. Otherwise, we say that the triangle is negatively oriented. Consider two distinct points $M_1(z_1)$ and $M_2(z_2)$, other than the origin of a complex plane. The angle $\widehat{M_1OM_2}$ is oriented if the points M_1 and M_2 are ordered counterclockwise (Fig. 3.2 below).

Proposition. *The measure of the directly oriented angle* $\widehat{M_1OM_2}$ *equals* $\arg \dfrac{z_2}{z_1}$.

Proof. We consider the following two cases.

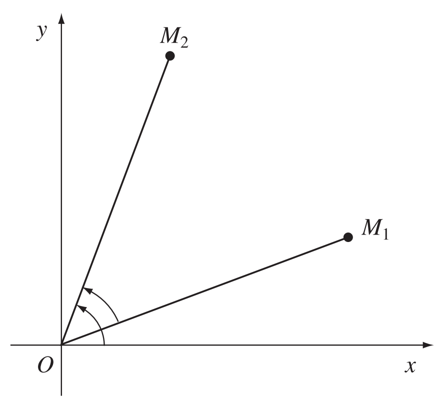

Figure 3.2.

a) If the triangle M_1OM_2 is negatively oriented (Fig. 3.2), then

$$\widehat{M_1OM_2} = \widehat{xOM_2} - \widehat{xOM_1} = \arg z_2 - \arg z_1 = \arg \frac{z_2}{z_1}.$$

b) If the triangle M_1OM_2 is positively oriented (Fig. 3.3), then

$$\widehat{M_1OM_2} = 2\pi - \widehat{M_2OM_1} = 2\pi - \arg \frac{z_2}{z_1},$$

since the triangle M_2OM_1 is negatively oriented. Thus

$$\widehat{M_1OM_2} = 2\pi - \arg\frac{z_1}{z_2} = 2\pi - \left(2\pi - \arg\frac{z_2}{z_1}\right) = \arg\frac{z_2}{z_1},$$

as claimed. □

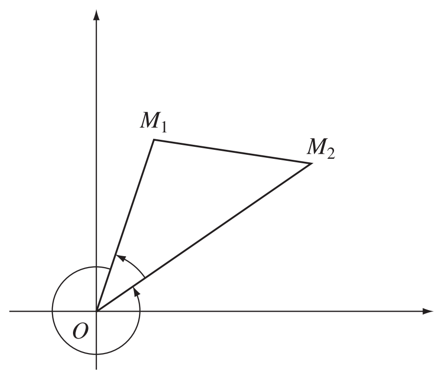

Figure 3.3.

Remark. The result also holds if the points O, M_1, M_2 are collinear.

Examples. a) Suppose that $z_1 = 1 + i$ and $z_2 = -1 + i$. Then (see Fig. 3.4)

$$\frac{z_2}{z_1} = \frac{-1+i}{1+i} = \frac{(-1+i)(1-i)}{2} = i,$$

so

$$\widehat{M_1OM_2} = \arg i = \frac{\pi}{2} \text{ and } \widehat{M_2OM_1} = \arg(-i) = \frac{3\pi}{2}.$$

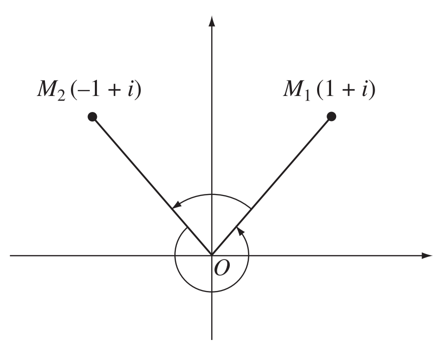

Figure 3.4.

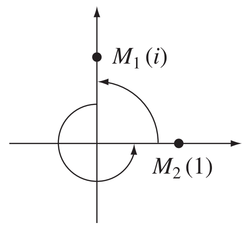

Figure 3.5.

b) Suppose that $z_1 = i$ and $z_2 = 1$. Then $\dfrac{z_2}{z_1} = \dfrac{1}{i} = -i$, so (see Fig. 3.5)

$$\widehat{M_1 O M_2} = \arg(-i) = \frac{3\pi}{2} \text{ and } \widehat{M_2 O M_1} = \arg(i) = \frac{\pi}{2}.$$

Theorem. *Consider three distinct points $M_1(z_1)$, $M_2(z_2)$ and $M_3(z_3)$. The measure of the oriented angle $\widehat{M_2 M_1 M_3}$ is* $\arg \dfrac{z_3 - z_1}{z_2 - z_1}$.

Proof. The translation with the vector $-z_1$ maps the points M_1, M_2, M_3 into the points O, M_2', M_3', with complex coordinates $O, z_2 - z_1, z_3 - z_1$. Moreover, we have $\widehat{M_2 M_1 M_3} = \widehat{M_2' O M_3'}$. By the previous result, we obtain

$$\widehat{M_2' O M_3'} = \arg \frac{z_3 - z_1}{z_2 - z_1},$$

as claimed. □

Example. Suppose that $z_1 = 4 + 3i$, $z_2 = 4 + 7i$, $z_3 = 8 + 7i$. Then

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{4i}{4 + 4i} = \frac{i(1 - i)}{2} = \frac{1 + i}{2},$$

so

$$\widehat{M_3 M_1 M_2} = \arg \frac{1 + i}{2} = \frac{\pi}{4}$$

and

$$\widehat{M_2 M_1 M_3} = \arg \frac{2}{1 + i} = \arg(1 - i) = \frac{7\pi}{4}.$$

Remark. Using polar representation, from the above result we have

$$\frac{z_3 - z_1}{z_2 - z_1} = \left|\frac{z_3 - z_1}{z_2 - z_1}\right| \left(\cos\left(\arg \frac{z_3 - z_1}{z_2 - z_1}\right) + i \sin\left(\arg \frac{z_3 - z_1}{z_2 - z_1}\right)\right)$$

$$= \left|\frac{z_3 - z_1}{z_2 - z_1}\right| (\cos \widehat{M_2 M_1 M_3} + i \sin \widehat{M_2 M_1 M_3}).$$

3.1.5 Angle between two lines

Consider four distinct points $M_i(z_i)$, $i \in \{1, 2, 3, 4\}$. The measure of the angle determined by the lines M_1M_3 and M_2M_4 equals $\arg \dfrac{z_3 - z_1}{z_4 - z_2}$ or $\arg \dfrac{z_4 - z_2}{z_3 - z_1}$. The proof is obtained following the same ideas as in the previous subsection.

3.1.6 Rotation of a point

Consider an angle α and the complex number given by

$$\varepsilon = \cos\alpha + i \sin\alpha.$$

Let $z = r(\cos t + i \sin t)$ be a complex number and M its geometric image.

Form the product $z\varepsilon = r(\cos(t + \alpha) + i \sin(t + \alpha))$ and let us observe that $|z\varepsilon| = r$ and

$$\arg(z\varepsilon) = \arg z + \alpha.$$

It follows that the geometric image M' of $z\varepsilon$ is the rotation of M with respect to the origin by the angle α.

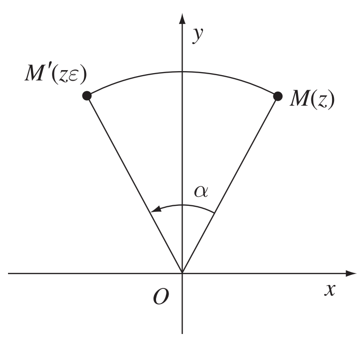

Figure 3.6.

Now we have all the ingredients to establish the following result:

Proposition. *Suppose that the point C is the rotation of B with respect to A by the angle α.*

If a, b, c are the coordinates of the points A, B, C, respectively, then

$$c = a + (b - a)\varepsilon, \ \textit{where } \varepsilon = \cos\alpha + i \sin\alpha.$$

Proof. The translation with vector $-a$ maps the points A, B, C into the points O, B', C', with complex coordinates $O, b - a, c - a$, respectively (see Fig. 3.7). The point C' is the image of B' under rotation about the origin through the angle α, so $c - a = (b - a)\varepsilon$, or $c = a + (b - a)\varepsilon$, as desired. □

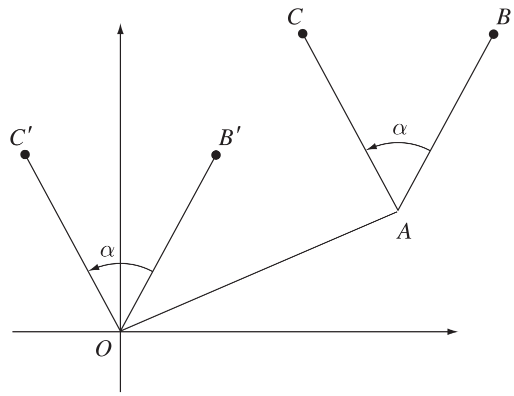

Figure 3.7.

We will call the formula in the above proposition the *rotation formula*.

Problem 1. *Let $ABCD$ and $BNMK$ be two nonoverlapping squares and let E be the midpoint of AN. If point F is the foot of the perpendicular from B to the line CK, prove that points E, F, B are collinear.*

Solution. Consider the complex plane with origin at F and the axis CK and FB, where FB is the imaginary axis.

Let c, k, bi be the complex coordinates of points C, K, B with $c, k, b \in \mathbb{R}$. The rotation with center B through the angle $\theta = \frac{\pi}{2}$ maps point C to A, so A has the complex coordinate $a = b(1-i)+ci$. Similarly, point N is obtained by rotating point K around B through the angle $\theta = -\frac{\pi}{2}$ and its complex coordinate is

$$n = b(1+i) - ki.$$

The midpoint E of segment AN has the complex coordinate

$$e = \frac{a+n}{2} = b + \frac{c-k}{2}i,$$

so E lies on the line FB, as desired.

Problem 2. *On the sides AB, BC, CD, DA of quadrilateral $ABCD$, and exterior to the quadrilateral, we construct squares of centers O_1, O_2, O_3, O_4, respectively. Prove that*

$$O_1O_3 \perp O_2O_4 \quad \textit{and} \quad O_1O_3 = O_2O_4.$$

Solution. Let $ABMM'$, $BCNN'$, $CDPP'$ and $DAQQ'$ be the constructed squares with centers O_1, O_2, O_3, O_4, respectively.

Denote by a lowercase letter the coordinate of each of the points denoted by an uppercase letter, i.e., o_1 is the coordinate of O_1, etc.

Point M is obtained from point A by a rotation about B through the angle $\theta = \frac{\pi}{2}$; hence $m = b + (a - b)i$. Likewise,

$$n = c + (b - c)i, \quad p = d + (c - d)i \text{ and } q = a + (d - a)i.$$

It follows that

$$o_1 = \frac{a+m}{2} = \frac{a+b+(a-b)i}{2}, \quad o_2 = \frac{b+c+(b-c)i}{2},$$

$$o_3 = \frac{c+d+(c-d)i}{2} \text{ and } o_4 = \frac{d+a+(d-a)i}{2}.$$

Then

$$\frac{o_3 - o_1}{o_4 - o_2} = \frac{c+d-a-b+i(c-d-a+b)}{a+d-b-c+i(d-a-b+c)} = -i \in i\mathbb{R}^*,$$

so $O_1O_3 \perp O_2O_4$. Moreover,

$$\left|\frac{o_3 - o_1}{o_4 - o_2}\right| = |-i| = 1;$$

hence $O_1O_3 = O_2O_4$, as desired.

Problem 3. *In the exterior of the triangle ABC we construct triangles ABR, BCP, and CAQ such that*

$$m(\widehat{PBC}) = m(\widehat{CAQ}) = 45^\circ,$$

$$m(\widehat{BCP}) = m(\widehat{QCA}) = 30^\circ,$$

and

$$m(\widehat{ABR}) = m(\widehat{RAB}) = 15^\circ.$$

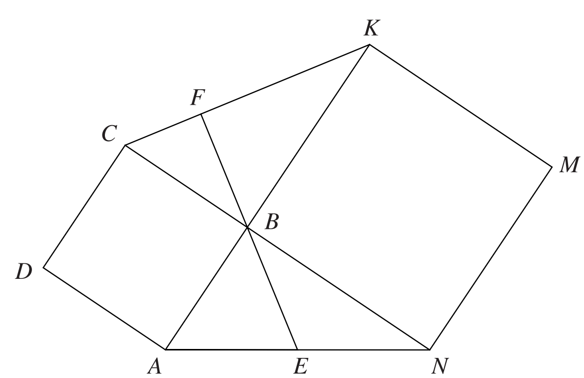

Figure 3.8.

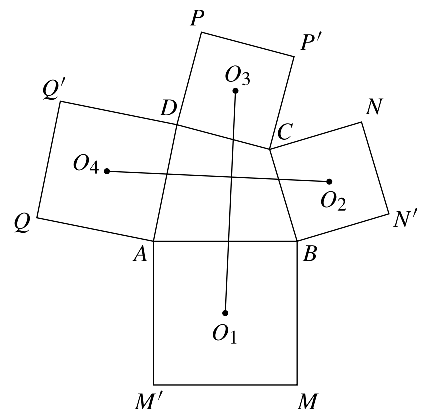

Figure 3.9.

Prove that

$$m(\widehat{QRP}) = 90^\circ \textit{ and } RQ = RP.$$

Solution. Consider the complex plane with origin at point R and let M be the foot of the perpendicular from P to the line BC.

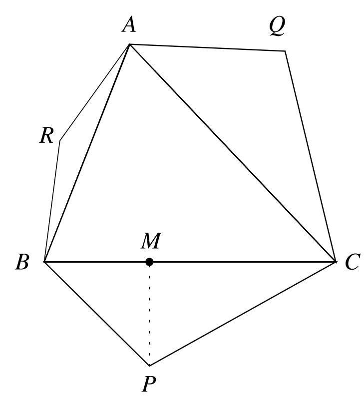

Figure 3.10.

Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter. From $MP = MB$ and $\dfrac{MC}{MP} = \sqrt{3}$ it follows that

$$\frac{p-m}{b-m} = i \text{ and } \frac{c-m}{p-m} = i\sqrt{3},$$

hence

$$p = \frac{c + \sqrt{3}b}{1 + \sqrt{3}} + \frac{b - c}{1 + \sqrt{3}}i.$$

Likewise,

$$q = \frac{c + \sqrt{3}a}{1 + \sqrt{3}} + \frac{a - c}{1 + \sqrt{3}}i.$$

Point B is obtained from point A by a rotation about R through an angle $\theta = 150°$, so

$$b = a\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right).$$

Simple algebraic manipulations show that $\frac{p}{q} = i \in i\mathbb{R}^*$, hence $QR \perp PR$. Moreover, $|p| = |iq| = |q|$, so $RP = RQ$ and we are done.

3.2 Conditions for Collinearity, Orthogonality and Concyclicity

In this section we consider four distinct points $M_i(z_i)$, $i \in \{1, 2, 3, 4\}$.

Proposition 1. *The points M_1, M_2, M_3 are collinear if and only if*

$$\frac{z_3 - z_1}{z_2 - z_1} \in \mathbb{R}^*.$$

Proof. The collinearity of the points M_1, M_2, M_3 is equivalent to $\widehat{M_2M_1M_3} \in \{0, \pi\}$. It follows that $\arg \frac{z_3 - z_1}{z_2 - z_1} \in \{0, \pi\}$ or equivalently $\frac{z_3 - z_1}{z_2 - z_1} \in \mathbb{R}^*$, as claimed. □

Proposition 2. *The lines M_1M_2 and M_3M_4 are orthogonal if and only if*

$$\frac{z_1 - z_2}{z_3 - z_4} \in i\mathbb{R}^*.$$

Proof. We have $M_1M_2 \perp M_3M_4$ if and only if $(M_1M_2, M_3M_4) \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$. This is equivalent to $\arg \frac{z_1 - z_2}{z_3 - z_4} \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$. We obtain $\frac{z_1 - z_2}{z_3 - z_4} \in i\mathbb{R}^*$. □

Remark. Suppose that $M_2 = M_4$. Then $M_1M_2 \perp M_3M_2$ if and only if $\frac{z_1 - z_2}{z_3 - z_2} \in i\mathbb{R}^*$.

Examples. 1) Consider the points $M_1(2-i)$, $M_2(-1+2i)$, $M_3(-2-i)$, $M_4(1+2i)$. Simple algebraic manipulation shows that

$$\frac{z_1 - z_2}{z_3 - z_4} = i, \text{ hence } M_1M_2 \perp M_3M_4.$$

2) Consider the points $M_1(2-i)$, $M_2(-1+2i)$, $M_3(1+2i)$, $M_4(-2-i)$. Then we have $\dfrac{z_1 - z_2}{z_3 - z_4} = -i$ hence $M_1M_2 \perp M_3M_4$.

Problem 1. *Let z_1, z_2, z_3 be the coordinates of vertices A, B, C of a triangle. If $w_1 = z_1 - z_2$ and $w_2 = z_3 - z_1$, prove that $\widehat{A} = 90^\circ$ if and only if* $\operatorname{Re}(w_1 \cdot \overline{w_2}) = 0$.

Solution. We have $\widehat{A} = 90^\circ$ if and only if $\dfrac{z_2 - z_1}{z_3 - z_1} \in i\mathbb{R}$, which is equivalent to $\dfrac{w_1}{-w_2} \in i\mathbb{R}$, i.e., $\operatorname{Re}\left(\dfrac{w_1}{-w_2}\right) = 0$. The last relation is equivalent to $\operatorname{Re}\left(\dfrac{w_1 \cdot \overline{w_2}}{-|w_2|^2}\right) = 0$, i.e., $\operatorname{Re}(w_1 \cdot \overline{w_2}) = 0$, as desired.

Proposition 3. *The distinct points $M_1(z_1)$, $M_2(z_2)$, $M_3(z_3)$, $M_4(z_4)$ are concyclic or collinear if and only if*

$$k = \frac{z_3 - z_2}{z_1 - z_2} : \frac{z_3 - z_4}{z_1 - z_4} \in \mathbb{R}^*.$$

Proof. Assume that the points are collinear. We can arrange four points on a circle in $(4-1)! = 3! = 6$ different ways. Consider the case when M_1, M_2, M_3, M_4 are given in this order. Then M_1, M_2, M_3, M_4 are concyclic if and only if

$$\widehat{M_1M_2M_3} + \widehat{M_1M_4M_3} \in \{3\pi, \pi\}.$$

That is,

$$\arg \frac{z_3 - z_2}{z_1 - z_2} + \arg \frac{z_1 - z_4}{z_3 - z_4} \in \{3\pi, \pi\}.$$

We obtain

$$\arg \frac{z_3 - z_2}{z_1 - z_2} - \arg \frac{z_3 - z_4}{z_1 - z_4} \in \{3\pi, \pi\},$$

i.e., $k < 0$.

For any other arrangements of the four points the proof is similar. Note that $k > 0$ in three cases and $k < 0$ in the other three. □

The number k is called the *cross ratio* of the four points $M_1(z_1)$, $M_2(z_2)$, $M_3(z_3)$ and $M_4(z_4)$.

Remarks. 1) The points M_1, M_2, M_3, M_4 are collinear if and only if

$$\frac{z_3 - z_2}{z_1 - z_2} \in \mathbb{R}^* \text{ and } \frac{z_3 - z_4}{z_1 - z_4} \in \mathbb{R}^*.$$

2) The points M_1, M_2, M_3, M_4 are concyclic if and only if

$$k = \frac{z_3 - z_2}{z_1 - z_2} : \frac{z_3 - z_4}{z_1 - z_4} \in \mathbb{R}^*, \text{ but } \frac{z_3 - z_2}{z_1 - z_2} \notin \mathbb{R} \text{ and } \frac{z_3 - z_4}{z_1 - z_4} \notin \mathbb{R}.$$

Examples. 1) The geometric images of the complex numbers $1, i, -1, -i$ are concyclic. Indeed, we have the cross ratio $k = \dfrac{-1-i}{1-i} : \dfrac{-1+i}{1+i} = -1 \in \mathbb{R}^*$ and clearly $\dfrac{-1-i}{1-i} \notin \mathbb{R}$ and $\dfrac{-1+i}{1+i} \notin \mathbb{R}$.

2) The points $M_1(2-i)$, $M_2(3-2i)$, $M_3(-1+2i)$ and $M_4(-2+3i)$ are collinear. Indeed, $k = \dfrac{-4+4i}{-1+i} : \dfrac{1-i}{4-4i} = 1 \in \mathbb{R}^*$ and $\dfrac{-4+4i}{-1+i} = 4 \in \mathbb{R}^*$.

Problem 2. *Find all complex numbers z such that the points of complex coordinates z, z^2, z^3, z^4 – in this order – are the vertices of a cyclic quadrilateral.*

Solution. If the points of complex coordinates z, z^2, z^3, z^4 – in this order – are the vertices of a cyclic quadrilateral, then

$$\frac{z^3-z^2}{z-z^2} : \frac{z^3-z^4}{z-z^4} \in \mathbb{R}^*.$$

It follows that

$$-\frac{1+z+z^2}{z} \in \mathbb{R}^*, \text{ i.e., } -1-\left(z+\frac{1}{z}\right) \in \mathbb{R}^*.$$

We obtain $z+\frac{1}{z} \in \mathbb{R}$, i.e., $z+\frac{1}{z} = \overline{z}+\frac{1}{\overline{z}}$. Hence $(z-\overline{z})(|z|^2-1)=0$, hence $z \in \mathbb{R}$ or $|z|=1$.

If $z \in \mathbb{R}$, then the points of complex coordinates z, z^2, z^3, z^4 are collinear, hence it is left to consider the case $|z|=1$.

Let $t = \arg z \in [0, 2\pi)$. We prove that the points of complex coordinates z, z^2, z^3, z^4 lie in this order on the unit circle if and only if $t \in \left(0, \frac{2\pi}{3}\right) \cup \left(\frac{4\pi}{3}, 2\pi\right)$. Indeed,

a) If $t \in \left(0, \frac{\pi}{2}\right)$, then $0 < t < 2t < 3t < 4t < 2\pi$ or

$$0 < \arg z < \arg z^2 < \arg z^3 < \arg z^4 < 2\pi.$$

b) If $t \in \left[\frac{\pi}{2}, \frac{2\pi}{3}\right)$, then $0 \le 4t-2\pi < t < 2t < 3t < 2\pi$ or

$$0 \le \arg z^4 < \arg z < \arg z^2 < \arg z^3 < 2\pi.$$

c) If $t \in \left[\frac{2\pi}{3}, \pi\right)$, then $0 \le 3t-2\pi < t \le 4t-2\pi < 2t < 2\pi$ or

$$0 \le \arg z^3 < \arg z \le \arg z^4 < \arg z^2.$$

In the same manner we can analyze the case $t \in [\pi, 2\pi)$.

To conclude, the complex numbers satisfying the desired property are

$$z = \cos t + i \sin t, \text{ with } t \in \left(0, \frac{2\pi}{3}\right) \cup \left(\frac{4\pi}{3}, \pi\right).$$

3.3 Similar Triangles

Consider six points $A_1(a_1)$, $A_2(a_2)$, $A_3(a_3)$, $B_1(b_1)$, $B_2(b_2)$, $B_3(b_3)$ in the complex plane. We say that the triangles $A_1A_2A_3$ and $B_1B_2B_3$ are similar if the angle at A_k is equal to the angle at B_k, $k \in \{1, 2, 3\}$.

Proposition 1. *The triangles $A_1A_2A_3$ and $B_1B_2B_3$ are similar, having the same orientation, if and only if*

$$\frac{a_2 - a_1}{a_3 - a_1} = \frac{b_2 - b_1}{b_3 - b_1}. \tag{1}$$

Proof. We have $\triangle A_1A_2A_3 \sim \triangle B_1B_2B_3$ if and only if $\dfrac{A_1A_2}{A_1A_3} = \dfrac{B_1B_2}{B_1B_3}$ and $\widehat{A_3A_1A_2} \equiv \widehat{B_3B_1B_2}$. This is equivalent to $\dfrac{|a_2 - a_1|}{|a_3 - a_1|} = \dfrac{|b_2 - b_1|}{|b_3 - b_1|}$ and $\arg \dfrac{a_2 - a_1}{a_3 - a_1} = \arg \dfrac{b_2 - b_1}{b_3 - b_1}$. We obtain $\dfrac{a_2 - a_1}{a_3 - a_1} = \dfrac{b_2 - b_1}{b_3 - b_1}$. □

Remarks. 1) The condition (1) is equivalent to

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = 0.$$

2) The triangles $A_1(0)$, $A_2(1)$, $A_3(2i)$ and $B_1(0)$, $B_2(-i)$, $B_3(-2)$ are similar, but opposite oriented. In this case the condition (1) is not satisfied. Indeed,

$$\frac{a_2 - a_1}{a_3 - a_1} = \frac{1 - 0}{2i - 0} = \frac{1}{2i} \neq \frac{b_2 - b_1}{b_3 - b_1} = \frac{-i - 0}{-2 - 0} = \frac{i}{2}.$$

Proposition 2. *The triangles $A_1A_2A_3$ and $B_1B_2B_3$ are similar, having opposite orientation, if and only if*

$$\frac{a_2 - a_1}{a_3 - a_1} = \frac{\overline{b}_2 - \overline{b}_1}{\overline{b}_3 - \overline{b}_1}.$$

Proof. Reflection across the x-axis maps the points B_1, B_2, B_3 into the points $M_1(\overline{b}_1)$, $M_2(\overline{b}_2)$, $M_3(\overline{b}_3)$. The triangles $B_1B_2B_3$ and $M_1M_2M_3$ are similar and have opposite orientation, hence triangles $A_1A_2A_3$ and $M_1M_2M_3$ are similar with the same orientation. The conclusion follows from the previous proposition. □

Problem 1. *On sides AB, BC, CA of a triangle ABC we draw similar triangles ADB, BEC, CFA, having the same orientation. Prove that triangles ABC and DEF have the same centroid.*

Solution. Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter.

Triangles ADB, BEC, CFA are similar with the same orientation, hence

$$\frac{d-a}{b-a} = \frac{e-b}{c-b} = \frac{f-c}{a-c} = z,$$

and consequently

$$d = a + (b-a)z, \quad e = b + (c-b)z, \quad f = c + (a-c)z.$$

Then

$$\frac{d+c+f}{3} = \frac{a+b+c}{3},$$

so triangles ABC and DEF have the same centroid.

Problem 2. *Let M, N, P be the midpoints of sides AB, BC, CA of triangle ABC. On the perpendicular bisectors of segments $[AB]$, $[BC]$, $[CA]$ points C', A', B' are chosen inside the triangle such that*

$$\frac{MC'}{AB} = \frac{NA'}{BC} = \frac{PB'}{CA}.$$

Prove that ABC and $A'B'C'$ have the same centroid.

Solution. Note that from

$$\frac{MC'}{AB} = \frac{NA'}{BC} = \frac{PB'}{CA}$$

it follows that $\tan(\widehat{C'AB}) = \tan(\widehat{A'BC}) = \tan(\widehat{B'CA})$. Hence triangles $AC'B$, $BA'C$, $CB'A$ are similar and we can proceed as in the previous problem.

Problem 3. *Let ABO be an equilateral triangle with center S and let $A'B'O$ be another equilateral triangle with the same orientation and $S \neq A'$, $S \neq B'$. Consider M and N the midpoints of the segments $A'B$ and AB'.*

Prove that triangles $SB'M$ and $SA'N$ are similar.

(30th IMO – Shortlist)

Solution. Let R be the circumradius of the triangle ABO and let

$$\varepsilon = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}.$$

Consider the complex plane with origin at point S such that point O lies on the positive real axis. Then the coordinates of points O, A, B are R, $R\varepsilon$, $R\varepsilon^2$, respectively.

Let $R+z$ be the coordinate of point B', so $R - z\varepsilon$ is the coordinate of point A'. It follows that the midpoints M, N have the coordinates

$$z_M = \frac{z_B + z_{A'}}{2} = \frac{R\varepsilon^2 + R - z\varepsilon}{2} = \frac{R(\varepsilon^2+1) - z\varepsilon}{2}$$

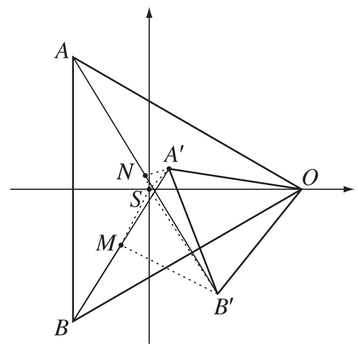

Figure 3.11.

$$= \frac{-R\varepsilon - z\varepsilon}{2} = \frac{-\varepsilon(R+z)}{2}$$

and

$$z_N = \frac{z_A + z_{B'}}{2} = \frac{R\varepsilon + R + z}{2} = \frac{R(\varepsilon+1)+z}{2} = \frac{-E\varepsilon^2 + z}{2}$$
$$= \frac{z - \frac{R}{\varepsilon}}{2} = \frac{R - z\varepsilon}{-2\varepsilon}.$$

Now we have

$$\frac{z_{B'} - z_S}{z_M - z_S} = \frac{\overline{z_{A'}} - \overline{z_S}}{\overline{z_N} - \overline{z_S}}$$

if and only if

$$\frac{R+z}{\frac{-\varepsilon(R+z)}{2}} = \frac{\overline{R - z\varepsilon}}{\overline{\left(\frac{R - z\varepsilon}{-2\varepsilon}\right)}}.$$

The last relation is equivalent to $\varepsilon \cdot \overline{\varepsilon} = 1$, i.e., $|\varepsilon|^2 = 1$. Hence the triangles $SB'M$ and $SA'N$ are similar, with opposite orientation.

3.4 Equilateral Triangles

Proposition 1. *Suppose z_1, z_2, z_3 are the coordinates of the vertices of the triangle $A_1A_2A_3$. The following statements are equivalent:*

a) $A_1A_2A_3$ is an equilateral triangle;

b) $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$;

c) $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$;

d) $\dfrac{z_2 - z_1}{z_3 - z_2} = \dfrac{z_3 - z_2}{z_1 - z_2}$;

e) $\dfrac{1}{z-z_1}+\dfrac{1}{z-z_2}+\dfrac{1}{z-z_3}=0$, *where* $z=\dfrac{z_1+z_2+z_3}{3}$;

f) $(z_1+\varepsilon z_2+\varepsilon^2 z_3)(z_1+\varepsilon^2 z_2+\varepsilon z_3)=0$, *where* $\varepsilon=\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}$;

g) $\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{vmatrix}=0.$

Proof. The triangle $A_1A_2A_3$ is equilateral if and only if $A_1A_2A_3$ is similar with same orientation with $A_2A_3A_1$, or

$$\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{vmatrix}=0,$$

thus a) $\Leftrightarrow$ g).

Computing the determinant we obtain

$$\begin{aligned} 0 &= \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ z_2 & z_3 & z_1 \end{vmatrix} \\ &= z_1z_2+z_2z_3+z_3z_1-(z_1^2+z_2^2+z_3^2) \\ &= -(z_1+\varepsilon z_2+\varepsilon^2 z_3)(z_1+\varepsilon^2 z_2+\varepsilon z_3), \end{aligned}$$

hence g) $\Leftrightarrow$ c) $\Leftrightarrow$ f).

Simple algebraic manipulation shows that d) $\Leftrightarrow$ c). Since a) $\Leftrightarrow$ b) is obvious, we leave for the reader to prove that a) $\Leftrightarrow$ e). □

The next results bring some refinements to this issue.

Proposition 2. *Let z_1, z_2, z_3 be the coordinates of the vertices A_1, A_2, A_3 of a positively oriented triangle. The following statements are equivalent.*

a) $A_1A_2A_3$ is an equilateral triangle;

b) $z_3-z_1=\varepsilon(z_2-z_1)$, *where* $\varepsilon=\cos\dfrac{\pi}{3}+i\sin\dfrac{\pi}{3}$;

c) $z_2-z_1=\varepsilon(z_3-z_1)$, *where* $\varepsilon=\cos\dfrac{5\pi}{3}+i\sin\dfrac{5\pi}{3}$;

d) $z_1+\varepsilon z_2+\varepsilon^2 z_3=0$, *where* $\varepsilon=\cos\dfrac{2\pi}{3}+i\sin\dfrac{2\pi}{3}$.

Proof. $A_1A_2A_3$ is equilateral and positively oriented if and only if A_3 is obtained from A_2 by rotation about A_1 through an angle of $\dfrac{\pi}{3}$. That is,

$$z_3=z_1+\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)(z_2-z_1),$$

hence a) $\Leftrightarrow$ b).

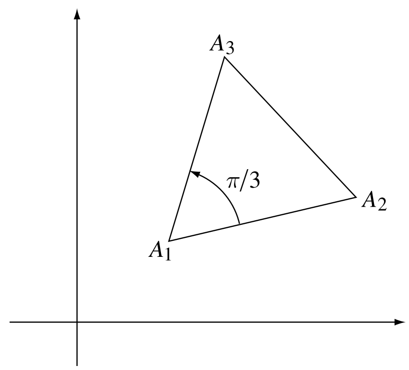

Figure 3.12.

The rotation about A_1 through an angle of $\dfrac{5\pi}{3}$ maps A_3 into A_2. Similar considerations show that a) $\Leftrightarrow$ c).

To prove that b) $\Leftrightarrow$ d), observe that b) is equivalent to

b′) $z_3 = z_1 + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)(z_2 - z_1) = \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) z_1 + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2$. Hence

$$\begin{aligned}
z_1 + \varepsilon z_2 + \varepsilon^2 z_3 &= z_1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2 + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) z_3 \\
&= z_1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2 \\
&\quad + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)\left[\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) z_1 + \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2\right] \\
&= z_1 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) z_2 - z_1 + \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) z_2 = 0,
\end{aligned}$$

or b) $\Leftrightarrow$ d). □

Proposition 3. *Let z_1, z_2, z_3 be the coordinates of the vertices A_1, A_2, A_3 of a negatively oriented triangle.*

The following statements are equivalent:

a) $A_1A_2A_3$ is an equilateral triangle;

b) $z_3 - z_1 = \varepsilon(z_2 - z_1)$, where $\varepsilon = \cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}$;

c) $z_2 - z_1 = \varepsilon(z_3 - z_1)$, where $\varepsilon = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$;

d) $z_1 + \varepsilon^2 z_2 + \varepsilon z_3 = 0$, where $\varepsilon = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$.

Proof. Equilateral triangle $A_1A_2A_3$ is negatively oriented if and only if $A_1A_3A_2$ is a positively oriented equilateral triangle. The rest follows from the previous proposition. □

Proposition 4. *Let z_1, z_2, z_3 be the coordinates of the vertices of equilateral triangle $A_1A_2A_3$. Consider the statements:*

1) $A_1A_2A_3$ is an equilateral triangle;

2) $z_1 \cdot \overline{z}_2 = z_2 \cdot \overline{z}_3 = z_3 \cdot \overline{z}_1$;

3) $z_1^2 = z_2 \cdot z_3$ and $z_2^2 = z_1 \cdot z_3$.

Then 2) $\Rightarrow$ 1), 3) $\Rightarrow$ 1) and 2) $\Leftrightarrow$ 3).

Proof. 2) $\Rightarrow$ 1). Taking the modulus of the terms in the given relation we obtain

$$|z_1| \cdot |\overline{z}_2| = |z_2| \cdot |\overline{z}_3| = |z_3| \cdot |\overline{z}_1|,$$

or equivalently

$$|z_1| \cdot |z_2| = |z_2| \cdot |z_3| = |z_3| \cdot |z_1|.$$

This implies

$$r = |z_1| = |z_2| = |z_3|$$

and

$$\overline{z}_1 = \frac{r^2}{z_1}, \quad \overline{z}_2 = \frac{r^2}{z_2}, \quad \overline{z}_3 = \frac{r^2}{z_3}.$$

Returning to the given relation we have

$$\frac{z_1}{z_2} = \frac{z_2}{z_3} = \frac{z_3}{z_1},$$

or

$$z_1^2 = z_2z_3, \quad z_2^2 = z_3z_1, \quad z_3^2 = z_1z_2.$$

Summing up these relations yields

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1,$$

so triangle $A_1A_2A_3$ is equilateral.

Observe that we have also proved that 2) $\Rightarrow$ 3) and that the arguments are reversible; hence 2) $\Leftrightarrow$ 3). As a consequence, 3) $\Rightarrow$ 1) and we are done. □

Problem 1. *Let z_1, z_2, z_3 be nonzero complex coordinates of the vertices of the triangle $A_1A_2A_3$. If $z_1^2 = z_2z_3$ and $z_2^2 = z_1z_3$, show that triangle $A_1A_2A_3$ is equilateral.*

Solution. Multiplying the relations $z_1^2 = z_2z_3$ and $z_2^2 = z_1z_3$ yields $z_1^2z_2^2 = z_1z_2z_3^2$, and consequently $z_1z_2 = z_3^2$. Thus

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1,$$

so triangle $A_1A_2A_3$ is equilateral, by Proposition 1 in this section.

Problem 2. *Let z_1, z_2, z_3 be the coordinates of the vertices of triangle $A_1A_2A_3$. If $|z_1| = |z_2| = |z_3|$ and $z_1 + z_2 + z_3 = 0$, prove that triangle $A_1A_2A_3$ is equilateral.*

Solution. The following identity holds for any complex numbers z_1 and z_2 (see Problem 1 in Subsection 1.1.7):

$$|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2(|z_1|^2 + |z_2|^2). \qquad (1)$$

From $z_1 + z_2 + z_3 = 0$ it follows that $z_1 + z_2 = -z_3$, so $|z_1 + z_2| = |z_3|$. Using the relations $|z_1| = |z_2| = |z_3|$ and (1) we get $|z_1 - z_2|^2 = 3|z_1|^2$. Analogously, we find the relations $|z_2 - z_3|^2 = 3|z_1|^2$ and $|z_3 - z_1|^2 = 3|z_1|^2$. Therefore $|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$, i.e., triangle $A_1A_2A_3$ is equilateral.

Alternative solution 1. If we pass to conjugates, then we obtain $\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 0$. Combining this with the hypothesis yields $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1 = 0$, from which the desired conclusion follows by Proposition 1.

Alternative solution 2. Taking into account the hypotheses $|z_1| = |z_2| = |z_3|$ it follows that we can consider the complex plane with its origin at the circumcenter of triangle $A_1A_2A_3$. Then, the coordinate of orthocenter H is $z_H = z_1 + z_2 + z_3 = 0 = z_O$. Hence $H = O$, and triangle $A_1A_2A_3$ is equilateral.

Problem 3. *In the exterior of triangle ABC three positively oriented equilateral triangles $AC'B$, $BA'C$ and $CB'A$ are constructed. Prove that the centroids of these triangles are the vertices of an equilateral triangle.*

(Napoleon's problem)

Solution.

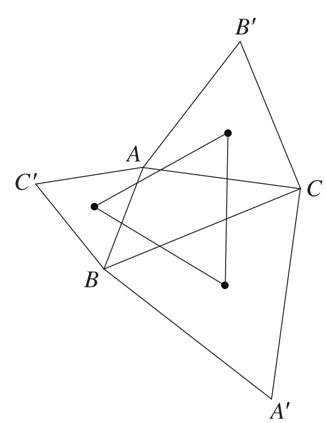

Figure 3.13.

Let a, b, c be the coordinates of vertices A, B, C, respectively.

Using Proposition 2, we have

$$a + c'\varepsilon + b\varepsilon^2 = 0, \quad b + a'\varepsilon + c\varepsilon^2 = 0, \quad c + b'\varepsilon + a\varepsilon^2 = 0, \tag{1}$$

where a', b', c' are the coordinates of points A', B', C'.

The centroids of triangles $A'BC$, $AB'C$, ABC' have the coordinates

$$a'' = \frac{1}{3}(a' + b + c), \quad b'' = \frac{1}{3}(a + b' + c), \quad c'' = \frac{1}{3}(a + b + c'),$$

respectively. We have to check that $c'' + a''\varepsilon + b''\varepsilon^2 = 0$. Indeed,

$$\begin{aligned} 3(c'' + a''\varepsilon + b''\varepsilon^2) &= (a + b + c') + (a' + b + c)\varepsilon + (a + b' + c)\varepsilon^2 \\ &= (b + a'\varepsilon + c\varepsilon^2) + (c + b'\varepsilon + a\varepsilon^2)\varepsilon + (a + c'\varepsilon + b\varepsilon^2)\varepsilon^2 = 0. \end{aligned}$$

Problem 4. *On the sides of the triangle ABC we draw three regular n-gons, external to the triangle. Find all values of n for which the centers of the n-gons are the vertices of an equilateral triangle.*

(Balkan Mathematical Olympiad 1990 – Shortlist)

Solution. Let A_0, B_0, C_0 be the centers of the regular n-gons constructed externally on the sides BC, CA, AB, respectively.

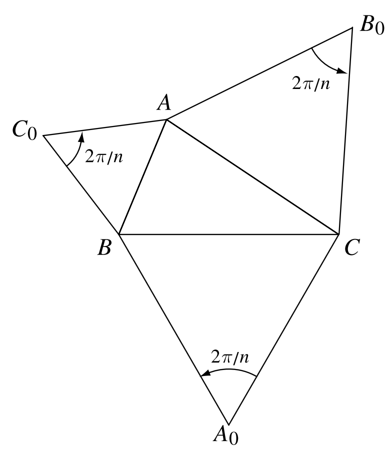

Figure 3.14.

The angles $\widehat{AC_0B}$, $\widehat{BA_0C}$, $\widehat{AB_0C}$ have the measures of $\dfrac{2\pi}{n}$. Let

$$\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

and denote by a, b, c, a_0, b_0, c_0 the coordinates of the points A, B, C, A_0, B_0, C_0, respectively.

Using the rotation formula, we obtain

$$a = c_0 + (b - c_0)\varepsilon;$$

$$b = a_0 + (c - a_0)\varepsilon;$$

$$c = b_0 + (a - b_0)\varepsilon.$$

Thus

$$a_0 = \frac{b - c\varepsilon}{1 - \varepsilon}, \quad b_0 = \frac{c - a\varepsilon}{1 - \varepsilon}, \quad c_0 = \frac{a - b\varepsilon}{1 - \varepsilon}.$$

Triangle $A_0B_0C_0$ is equilateral if and only if

$$a_0^2 + b_0^2 + c_0^2 = a_0b_0 + b_0c_0 + c_0a_0.$$

Substituting the above values of a_0, b_0, c_0 we obtain

$$(b - c\varepsilon)^2 + (c - a\varepsilon)^2 + (a - b\varepsilon)^2$$

$$= (b - c\varepsilon)(c - a\varepsilon) + (c - a\varepsilon)(a - b\varepsilon) + (a - b\varepsilon)(c - a\varepsilon).$$

This is equivalent to

$$(1 + \varepsilon + \varepsilon^2)[(a - b)^2 + (b - c)^2 + (c - a)^2] = 0.$$

It follows that $1 + \varepsilon + \varepsilon^2 = 0$, i.e., $\frac{2\pi}{n} = \frac{2\pi}{3}$ and we get $n = 3$. Therefore $n = 3$ is the only value with the desired property.

3.5 Some Analytic Geometry in the Complex Plane

3.5.1 Equation of a line

Proposition 1. *The equation of a line in the complex plane is*

$$\overline{\alpha} \cdot \overline{z} + \alpha z + \beta = 0,$$

where $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{R}$ *and* $z = x + iy \in \mathbb{C}$.

Proof. The equation of a line in the cartesian plane is

$$Ax + By + C = 0,$$

where $A, B, C \in \mathbb{R}$ and $A^2 + B^2 \neq 0$. If we set $z = x + iy$, then $x = \frac{z + \overline{z}}{2}$ and $y = \frac{z - \overline{z}}{2i}$. Thus,

$$A\frac{z + \overline{z}}{2} - Bi\frac{z - \overline{z}}{2} + C = 0,$$

or equivalently

$$\overline{z}\left(\frac{A+Bi}{2}\right)+z\frac{A-Bi}{2}+C=0.$$

Let $\alpha=\dfrac{A-Bi}{2}\in\mathbb{C}^*$ and $\beta=C\in\mathbb{R}$. Then

$$\overline{\alpha}\cdot\overline{z}+\alpha z+\beta=0,$$

as claimed. □

If $\alpha=\overline{\alpha}$, then $B=0$ and we have a vertical line. If $\alpha\neq\overline{\alpha}$, then we define the *angular coefficient* of the line as

$$m=-\frac{A}{B}=\frac{\alpha+\overline{\alpha}}{\dfrac{\alpha-\overline{\alpha}}{i}}=\frac{\alpha+\overline{\alpha}}{\alpha-\overline{\alpha}}i.$$

Proposition 2. *Consider the lines d_1 and d_2 with equations*

$$\overline{\alpha}_1\cdot\overline{z}+\alpha_1\cdot z+\beta_1=0$$

and

$$\overline{\alpha}_2\cdot\overline{z}+\alpha_2\cdot z+\beta_2=0,$$

respectively.

Then the lines d_1 and d_2 are:

1) parallel if and only if $\dfrac{\overline{\alpha}_1}{\alpha_1}=\dfrac{\overline{\alpha}_2}{\alpha_2}$;

2) perpendicular if and only if $\dfrac{\overline{\alpha}_1}{\alpha_2}+\dfrac{\overline{\alpha}_2}{\alpha_2}=0$;

3) concurrent if and only if $\dfrac{\overline{\alpha}_1}{\alpha_1}\neq\dfrac{\overline{\alpha}_2}{\alpha_2}$.

Proof. 1) We have $d_1\|d_2$ if and only if $m_1=m_2$. Therefore $\dfrac{\alpha_1+\overline{\alpha}_1}{\alpha_1-\overline{\alpha}_1}i=\dfrac{\alpha_2+\overline{\alpha}_2}{\alpha_2-\overline{\alpha}_2}i$, so $\alpha_2\overline{\alpha}_1=\alpha_1\overline{\alpha}_2$ and we get $\dfrac{\overline{\alpha}_1}{\alpha_1}=\dfrac{\overline{\alpha}_2}{\alpha_2}$.

2) We have $d_1\perp d_2$ if and only if $m_1m_2=-1$. That is, $\alpha_2\overline{\alpha}_1+\alpha_2\overline{\alpha}_2=0$, or $\dfrac{\overline{\alpha}_1}{\alpha}+\dfrac{\overline{\alpha}_2}{\alpha_2}=0$.

3) The lines d_1 and d_2 are concurrent if and only if $m_1\neq m_2$. This condition yields $\dfrac{\overline{\alpha}_1}{\alpha_1}\neq\dfrac{\overline{\alpha}_2}{\alpha_2}$.

The results for *angular coefficient* correspond to the properties of *slope*. □

The ratio $m_d=-\dfrac{\overline{\alpha}}{\alpha}$ is called the complex angular coefficient of the line d of equation

$$\overline{\alpha}\cdot\overline{z}+\alpha\cdot z+\beta=0.$$

3.5.2 Equation of a line determined by two points

Proposition. *The equation of a line determined by the points* $P_1(z_1)$ *and* $P_2(z_2)$ *is*

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z & \overline{z} & 1 \end{vmatrix} = 0.$$

Proof. The equation of a line determined by the points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ in the cartesian plane is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x & y & 1 \end{vmatrix} = 0.$$

Using complex numbers we have

$$\begin{vmatrix} \dfrac{z_1 + \overline{z_1}}{2} & \dfrac{z_1 - \overline{z_1}}{2i} & 1 \\ \dfrac{z_2 + \overline{z_2}}{2} & \dfrac{z_2 - \overline{z_2}}{2i} & 1 \\ \dfrac{z_1 + \overline{z}}{2} & \dfrac{z - \overline{z}}{2i} & 1 \end{vmatrix} = 0$$

if and only if

$$\frac{1}{4i}\begin{vmatrix} z_1 + \overline{z_1} & z_1 - \overline{z_1} & 1 \\ z_2 + \overline{z_2} & z_2 - \overline{z_2} & 1 \\ z + \overline{z} & z - \overline{z} & 1 \end{vmatrix} = 0.$$

That is,

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z & \overline{z} & 1 \end{vmatrix} = 0,$$

as desired. □

Remarks. 1) The points $M_1(z_1)$, $M_2(z_2)$, $M_3(z_3)$ are collinear if and only if

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0.$$

2) The complex angular coefficient of a line determined by the points with coordinates z_1 and z_2 is

$$m = \frac{z_2 - z_1}{\overline{z_2} - \overline{z_1}}.$$

Indeed, the equation is

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = 0 \Leftrightarrow z_1\overline{z_2} + z_2 z + z\overline{z_1} - z\overline{z_2} - z_1\overline{z} - z_2\overline{z_1} = 0$$

$$\Leftrightarrow \overline{z}(z_2 - z_1) - z(\overline{z_2} - \overline{z_1}) + z_1\overline{z_2} - z_2\overline{z_1} = 0.$$

Using the definition of the complex angular coefficient we obtain

$$m = \frac{z_2 - z_1}{\overline{z_2} - \overline{z_1}}.$$

3.5.3 The area of a triangle

Theorem. *The area of triangle $A_1A_2A_3$ whose vertices have coordinates z_1, z_2, z_3 is equal to the absolute value of the number*

$$\frac{i}{4}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix}. \tag{1}$$

Proof. Using cartesian coordinates, the area of a triangle with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3) is equal to the absolute value of the determinant

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

Since

$$x_k = \frac{z_k + \overline{z_k}}{2}, \quad y_k = \frac{z_k - \overline{z_k}}{2i}, \quad k = 1, 2, 3$$

we obtain

$$\Delta = \frac{1}{8i}\begin{vmatrix} z_1 + \overline{z_1} & z_1 - \overline{z_1} & 1 \\ z_2 + \overline{z_2} & z_2 - \overline{z_2} & 1 \\ z_3 + \overline{z_3} & z_3 - \overline{z_3} & 1 \end{vmatrix} = -\frac{1}{4i}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix}$$

$$= \frac{i}{4}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix},$$

as claimed. □

It is easy to see that for positively oriented triangle $A_1A_2A_3$ with vertices with coordinates z_1, z_2, z_3 the following inequality holds:

$$\frac{i}{4}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} > 0.$$

Corollary. *The area of a directly oriented triangle $A_1A_2A_3$ whose vertices have coordinates z_1, z_2, z_3 is*

$$\text{area}[A_1A_2A_3] = \frac{1}{2}\operatorname{Im}(\overline{z_1}z_2 + \overline{z_2}z_3 + \overline{z_3}z_1). \tag{2}$$

Proof. The determinant in the above theorem is

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix} = (z_1\overline{z_2} + z_2\overline{z_3} + z_3\overline{z_1} - \overline{z_2}z_3 - z_1\overline{z_3} - z_2\overline{z_1})$$

$$= [(z_1\overline{z_2} + z_2\overline{z_3} + z_3\overline{z_1}) - \overline{(z_1\overline{z_2} + z_2\overline{z_3} + z_3\overline{z_1})}]$$

$$= 2i\operatorname{Im}(z_1\overline{z_2} + z_2\overline{z_3} + z_3\overline{z_1}) = -2i\operatorname{Im}(\overline{z_1}z_2 + \overline{z_2}z_3 + \overline{z_3}z_1).$$

Replacing this value in (1), the desired formula follows. □

We will see that formula (2) can be extended to a convex directly oriented polygon $A_1A_2\cdots A_n$ (see Section 4.3).

Problem 1. *Consider the triangle $A_1A_2A_3$ and the points M_1, M_2, M_3 situated on lines A_2A_3, A_1A_3, A_1A_2, respectively. Assume that M_1, M_2, M_3 divide segments $[A_2A_3]$, $[A_3A_1]$, $[A_1A_2]$ into ratios $\lambda_1, \lambda_2, \lambda_3$, respectively. Then*

$$\frac{\text{area}[M_1M_2M_3]}{\text{area}[A_1A_2A_3]} = \frac{1 - \lambda_1\lambda_2\lambda_3}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)}. \tag{3}$$

Solution. The coordinates of the points M_1, M_2, M_3 are

$$m_1 = \frac{a_2 - \lambda_1 a_3}{1-\lambda_1}, \quad m_2 = \frac{a_3 - \lambda_2 a_1}{1-\lambda_2}, \quad m_3 = \frac{a_1 - \lambda_3 a_2}{1-\lambda_3}.$$

Applying formula (2) we find that

$$\begin{aligned}
\text{area}[M_1M_2M_3] &= \frac{1}{2}\,\text{Im}(\overline{m_1}m_2 + \overline{m_2}m_3 + \overline{m_3}m_1)\\
&= \frac{1}{2}\,\text{Im}\Bigg[\frac{(\overline{a_2} - \lambda_1\overline{a_3})(a_3 - \lambda_2 a_1)}{(1-\lambda_1)(1-\lambda_2)} + \frac{(\overline{a_3} - \lambda_2\overline{a_1})(a_1 - \lambda_3 a_2)}{(1-\lambda_2)(1-\lambda_3)}\\
&\quad + \frac{(\overline{a_1} - \lambda_3\overline{a_2})(a_2 - \lambda_1 a_3)}{(1-\lambda_3)(1-\lambda_1)}\Bigg]\\
&= \frac{1}{2}\,\text{Im}\left[\frac{1-\lambda_1\lambda_2\lambda_3}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)}(\overline{a_1}a_2 + \overline{a_2}a_3 + \overline{a_3}a_1)\right]\\
&= \frac{1-\lambda_1\lambda_2\lambda_3}{(1-\lambda_1)(1-\lambda_2)(1-\lambda_3)}\,\text{area}[A_1A_2A_3].
\end{aligned}$$

Remark. From formula (3) we derive the well-known theorem of Menelaus: *The points M_1, M_2, M_3 are collinear if and only if $\lambda_1\lambda_2\lambda_3 = 1$, i.e.,*

$$\frac{M_1A_2}{M_1A_3}\cdot\frac{M_2A_3}{M_2A_1}\cdot\frac{M_3A_1}{M_3A_2} = 1$$

Problem 2. *Let a, b, c be the coordinates of the vertices A, B, C of a triangle. It is known that $|a| = |b| = |c| = 1$ and that there exists $\alpha \in \left(0, \frac{\pi}{2}\right)$ such that $a + b\cos\alpha + c\sin\alpha = 0$. Prove that*

$$1 < \text{area}[ABC] \le \frac{1+\sqrt{2}}{2}.$$

(Romanian Mathematical Olympiad – Final Round, 2003)

Solution. Observe that

$$\begin{aligned}
1 = |a|^2 &= |b\cos\alpha + c\sin\alpha|^2\\
&= (b\cos\alpha + c\sin\alpha)(\overline{b}\cos\alpha + \overline{c}\sin\alpha)\\
&= |b|^2\cos^2\alpha + |c|^2\sin^2\alpha + (b\overline{c} + \overline{b}c)\sin\alpha\cos\alpha\\
&= 1 + \frac{b^2+c^2}{bc}\cos\alpha\sin\alpha.
\end{aligned}$$

It follows that $b^2 + c^2 = 0$, hence $b = \pm ic$. Applying formula (2) we obtain

$$\begin{aligned}
\text{area}[ABC] &= \frac{1}{2}|\operatorname{Im}(\overline{a}b + \overline{b}c + \overline{c}a)| \\
&= \frac{1}{2}|\operatorname{Im}[(-\overline{b}\cos\alpha - \overline{c}\sin\alpha)b + \overline{b}c - \overline{c}(b\cos\alpha + c\sin\alpha)]| \\
&= \frac{1}{2}|\operatorname{Im}(-\cos\alpha - \sin\alpha - b\overline{c}\sin\alpha - b\overline{c}\cos\alpha + \overline{b}c)| \\
&= \frac{1}{2}|\operatorname{Im}[\overline{b}c - (\sin\alpha + \cos\alpha)b\overline{c}]| = \frac{1}{2}|\operatorname{Im}[(1 + \sin\alpha + \cos\alpha)\overline{b}c]| \\
&= \frac{1}{2}(1 + \sin\alpha + \cos\alpha)|\operatorname{Im}(\overline{b}c)| = \frac{1}{2}(1 + \sin\alpha + \cos\alpha)|\operatorname{Im}(\pm ic\overline{c})| \\
&= \frac{1}{2}(1 + \sin\alpha + \cos\alpha)|\operatorname{Im}(\pm i)| = \frac{1}{2}(1 + \sin\alpha + \cos\alpha) \\
&= \frac{1}{2}\left[1 + \sqrt{2}\left(\frac{\sqrt{2}}{2}\sin\alpha + \frac{\sqrt{2}}{2}\cos\alpha\right)\right] = \frac{1}{2}\left(1 + \sqrt{2}\sin\left(\alpha + \frac{\pi}{4}\right)\right).
\end{aligned}$$

Taking into account that $\frac{\pi}{4} < \alpha + \frac{\pi}{4} < \frac{3\pi}{4}$ we get that $\frac{\sqrt{2}}{2} < \sin\left(\alpha + \frac{\pi}{4}\right) \le 1$ and the conclusion follows.

3.5.4 Equation of a line determined by a point and a direction

Proposition 1. *Let $d : \overline{\alpha z} + \alpha \cdot z + \beta = 0$ be a line and let $P_0(z_0)$ be a point. The equation of a line parallel to d and passing through point P_0 is*

$$z - z_0 = -\frac{\overline{\alpha}}{\alpha}(\overline{z} - \overline{z_0}).$$

Proof. Using cartesian coordinates, the line parallel to d and passing through point $P_0(x_0, y_0)$ has the equation

$$y - y_0 = i\frac{\alpha + \overline{\alpha}}{\alpha - \overline{\alpha}}(x - x_0).$$

Using complex numbers the equation takes the form

$$\frac{z - \overline{z}}{2i} - \frac{z_0 - \overline{z_0}}{2i} = i\frac{\alpha + \overline{\alpha}}{\alpha - \overline{\alpha}}\left(\frac{z + \overline{z}}{2} - \frac{z_0 + \overline{z_0}}{2}\right).$$

This is equivalent to $(\alpha - \overline{\alpha})(z - z_0 - \overline{z} + \overline{z_0}) = (\alpha + \overline{\alpha})(z + \overline{z} - z_0 - \overline{z_0})$, or $\alpha(z - z_0) = -\overline{\alpha}(\overline{z} - \overline{z_0})$. We obtain $z - z_0 = -\frac{\overline{\alpha}}{\alpha}(\overline{z} - \overline{z_0})$. □

Proposition 2. *Let $d : \overline{\alpha z} + \alpha \cdot z + \beta = 0$ be a line and let $P_0(z_0)$ be a point. The line passing through point P_0 and perpendicular to d has the equation $z - z_0 = \frac{\overline{\alpha}}{\alpha}(\overline{z} - \overline{z_0})$.*

Proof. Using cartesian coordinates, the line passing through point P_0 and perpendicular to d has the equation

$$y - y_0 = -\frac{1}{i} \cdot \frac{\alpha - \overline{\alpha}}{\alpha + \overline{\alpha}}(x - x_0).$$

Then we obtain

$$\frac{z - \overline{z}}{2i} - \frac{z_0 - \overline{z_0}}{2i} = i \cdot \frac{\alpha - \overline{\alpha}}{\alpha + \overline{\alpha}} \left(\frac{z + \overline{z}}{2} - \frac{z_0 + \overline{z_0}}{2} \right).$$

That is, $(\alpha + \overline{\alpha})(z - z_0 - \overline{z} + \overline{z_0}) = -(\alpha - \overline{\alpha})(z - z_0 + \overline{z} - \overline{z_0})$ or

$$(z - z_0)(\alpha + \overline{\alpha} + \alpha - \overline{\alpha}) = (\overline{z} - \overline{z_0})(-\alpha + \overline{\alpha} + \alpha + \overline{\alpha}).$$

We obtain $\alpha(z - z_0) = \overline{\alpha}(\overline{z} - \overline{z_0})$ and $z - z_0 = \frac{\overline{\alpha}}{\alpha}(\overline{z} - \overline{z_0})$. □

3.5.5 The foot of a perpendicular from a point to a line

Proposition. *Let $P_0(z_0)$ be a point and let $d : \overline{\alpha z} + \alpha z + \beta = 0$ be a line. The foot of the perpendicular from P_0 to d has the coordinate*

$$z = \frac{\alpha z_0 - \overline{\alpha}\,\overline{z_0} - \beta}{2\alpha}.$$

Proof. The point z is the solution of the system

$$\begin{cases} \overline{\alpha} \cdot \overline{z} + \alpha \cdot z + \beta = 0, \\ \alpha(z - z_0) = \overline{\alpha}(\overline{z} - \overline{z_0}). \end{cases}$$

The first equation gives

$$\overline{z} = \frac{-\alpha z - \beta}{\overline{\alpha}}.$$

Substituting in the second equation yields

$$\alpha z - \alpha z_0 = -\alpha z - \beta - \overline{\alpha} \cdot \overline{z_0}.$$

Hence

$$z = \frac{\alpha z_0 - \overline{\alpha}\,\overline{z_0} - \beta}{2\alpha},$$

as claimed. □

3.5.6 Distance from a point to a line

Proposition. *The distance from a point $P_0(z_0)$ to a line $d : \overline{\alpha} \cdot \overline{z} + \alpha \cdot z + \beta = 0$, $\alpha \in \mathbb{C}^*$ is equal to*

$$D = \frac{|\alpha z_0 + \overline{\alpha} \cdot \overline{z_0} + \beta|}{2\sqrt{\alpha \cdot \overline{\alpha}}}.$$

Proof. Using the previous result, we can write

$$D = \left| \frac{\alpha z_0 - \overline{\alpha} \cdot \overline{z_0} - \beta}{2\alpha} - z_0 \right| = \left| \frac{-\alpha z_0 - \overline{\alpha z_0} - \beta}{2\alpha} \right|$$
$$= \frac{|\alpha \cdot z_0 + \overline{\alpha z_0} + \beta|}{2|\alpha|} = \frac{|\alpha z_0 + \overline{\alpha z_0} + \beta|}{2\sqrt{\alpha\overline{\alpha}}}. \qquad \square$$

3.6 The Circle

3.6.1 Equation of a circle

Proposition. *The equation of a circle in the complex plane is*

$$z \cdot \overline{z} + \alpha \cdot z + \overline{\alpha} \cdot \overline{z} + \beta = 0,$$

where $\alpha \in \mathbb{C}$ *and* $\beta \in \mathbb{R}$.

Proof. The equation of a circle in the cartesian plane is

$$x^2 + y^2 + mx + ny + p = 0,$$

$m, n, p \in \mathbb{R}$, $p < \dfrac{m^2 + n^2}{4}$.

Setting $x = \dfrac{z + \overline{z}}{2}$ and $y = \dfrac{z - \overline{z}}{2i}$ we obtain

$$|z|^2 + m\frac{z + \overline{z}}{2} + n\frac{z - \overline{z}}{2i} + p = 0$$

or

$$z \cdot \overline{z} + z\frac{m - ni}{2} + \overline{z}\frac{m + ni}{2} + p = 0.$$

Take $\alpha = \dfrac{m - ni}{2} \in \mathbb{C}$ and $\beta = p \in \mathbb{R}$ in the above equation and the claim is proved. $\square$

Note that the radius of the circle is equal to

$$r = \sqrt{\frac{m^2}{4} + \frac{n^2}{4} - p} = \sqrt{\alpha\overline{\alpha} - \beta}.$$

Then the equation is equivalent to

$$(\overline{z} + \alpha)(z + \overline{\alpha}) = r^2.$$

Setting

$$\gamma = -\overline{\alpha} = -\frac{m}{2} - \frac{n}{2}i$$

the equation of the circle with center at γ and radius r is

$$(\overline{z}-\overline{\gamma})(z-\gamma)=r^2.$$

Problem. *Let z_1, z_2, z_3 be the coordinates of the vertices of triangle $A_1A_2A_3$. The coordinate z_O of the circumcenter of triangle $A_1A_2A_3$ is*

$$z_O=\frac{\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ |z_1|^2 & |z_2|^2 & |z_3|^2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \overline{z_1} & \overline{z_2} & \overline{z_3} \end{vmatrix}}. \tag{1}$$

Solution. The equation of the line passing through $P(z_0)$ which is perpendicular to the line A_1A_2 can be written in the form

$$z(\overline{z_1}-\overline{z_2})+\overline{z}(z_1-z_2)=z_0(\overline{z_1}-\overline{z_2})+\overline{z_0}(z_1-z_2). \tag{2}$$

Applying this formula for the midpoints of the sides $[A_2A_3]$, $[A_1A_3]$ and for the lines A_2A_3, A_1A_3, we find the equations

$$z(\overline{z_2}-\overline{z_3})+\overline{z}(z_2-z_3)=|z_2|^2-|z_3|^2$$
$$z(\overline{z_3}-\overline{z_1})+\overline{z}(z_3-z_1)=|z_3|^2-|z_1|^2.$$

By eliminating $\overline{z}$ from these two equations, it follows that

$$\begin{aligned} &z[(\overline{z_2}-\overline{z_3})+(\overline{z_3}-\overline{z_1})(z_2-z_3)] \\ &\qquad =(z_1-z_3)(|z_2|^2-|z_3|^2)+(z_2-z_3)(|z_3|^2-|z_1|^2), \end{aligned}$$

hence

$$z\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \overline{z_1} & \overline{z_2} & \overline{z_2} \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ |z_1|^2 & |z_2|^2 & |z_3|^2 \end{vmatrix}$$

and the desired formula follows.

Remark. We can write this formula in the following equivalent form:

$$z_O=\frac{z_1\overline{z_1}(z_2-z_3)+z_2\overline{z_2}(z_3-z_1)+z_3\overline{z_3}(z_1-z_2)}{\begin{vmatrix} 1 & 1 & 1 \\ \overline{z_1} & \overline{z_2} & \overline{z_3} \\ z_1 & z_2 & z_3 \end{vmatrix}}. \tag{3}$$

3.6.2 The power of a point with respect to a circle

Proposition. *Consider a point $P_0(z_0)$ and a circle with equation*

$$z \cdot \overline{z} + \alpha \cdot z + \overline{\alpha} \cdot \overline{z} + \beta = 0,$$

for $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{R}$.

The power of P_0 with respect to the circle is

$$\rho(z_0) = z_0 \cdot \overline{z_0} + \alpha z_0 + \overline{\alpha} \cdot \overline{z_0} + \beta.$$

Proof. Let $O(-\overline{\alpha})$ be the center of the circle. The power of P_0 with respect to the circle of radius r is defined by $\rho(z_0) = OP_0^2 - r^2$. In this case we obtain

$$\rho(z_0) = OP_0^2 - r^2 = |z_0 + \overline{\alpha}|^2 - r^2 = z_0 \cdot \overline{z_0} + \alpha z_0 + \overline{\alpha z_0} + \alpha\overline{\alpha} - \alpha\overline{\alpha} + \beta$$

$$= z_0 \cdot \overline{z_0} + \alpha z_0 + \overline{\alpha} \cdot \overline{z_0} + \beta,$$

as claimed. □

Given two circles of equations

$$z \cdot \overline{z} + \alpha_1 \cdot z + \overline{\alpha_1} \cdot \overline{z} + \beta_1 = 0 \quad \text{and} \quad z \cdot \overline{z} + \alpha_2 \cdot z + \overline{\alpha_2} \cdot \overline{z} + \beta_2 = 0,$$

where $\alpha_1, \alpha_2 \in \mathbb{C}$, $\beta_1, \beta_2 \in \mathbb{R}$, their *radical axis* is the locus of points having equal powers with respect to the circles. If $P(z)$ is a point of this locus, then

$$z \cdot \overline{z} + \alpha_1 z + \overline{\alpha_1} \cdot \overline{z} + \beta_1 = z \cdot \overline{z} + \alpha_2 z + \overline{\alpha_2} \cdot z + \beta_2,$$

or equivalently $(\alpha_1 - \alpha_2)z + (\overline{\alpha_1} - \overline{\alpha_2})\overline{z} + \beta_1 - \beta_2 = 0$, which is the equation of a line.

3.6.3 Angle between two circles

The angle between two circles with equations

$$z \cdot \overline{z} + \alpha_1 \cdot z + \overline{\alpha_1} \cdot \overline{z} + \beta_1 = 0$$

and

$$z \cdot \overline{z} + \alpha_2 \cdot z + \overline{\alpha_2} \cdot \overline{z} + \beta_2 = 0, \quad \alpha_1, \alpha_2 \in \mathbb{C}, \quad \beta_1, \beta_2 \in \mathbb{R},$$

is the angle θ determined by the tangents to the circles at a common point.

Proposition. *The following formula*

$$\cos\theta = \left|\frac{\beta_1 + \beta_2 - (\alpha_1\overline{\alpha_2} + \overline{\alpha_1}\alpha_2)}{2r_1r_2}\right|$$

holds.

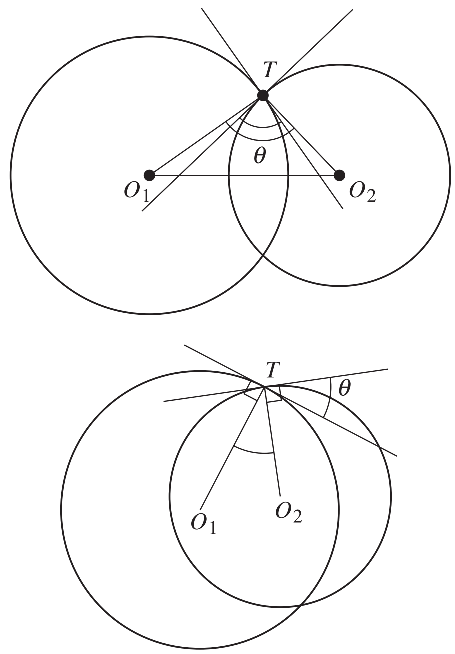

Figure 3.15.

Proof. Let T be a common point and let $O_1(-\overline{\alpha_1})$, $O_2(-\overline{\alpha_2})$ be the centers of the circles.

The angle θ is equal to $\widehat{O_1 T O_2}$ or $\pi - \widehat{O_1 T O_2}$, hence

$$\cos\theta = |\cos\widehat{O_1 T O_2}| = \frac{|r_1^2 + r_2^2 - O_1 O_2^2|}{2r_1 r_2}$$

$$= \frac{|\alpha_1\overline{\alpha_1} - \beta_1 + \alpha_2\overline{\alpha_2} - \beta_2 - |\overline{\alpha_1} - \overline{\alpha_2}|^2|}{2r_1 r_2}$$

$$= \frac{|\alpha_1\overline{\alpha_1} + \alpha_2\overline{\alpha_2} - \beta_1 - \beta_2 - \overline{\alpha_1}\alpha_1 - \alpha_2\overline{\alpha_2} + \overline{\alpha_1}\alpha_2 + \alpha_1\overline{\alpha_2}|}{2r_1 r_2}$$

$$= \frac{|\beta_1 + \beta_2 - (\alpha_1\overline{\alpha_2} + \overline{\alpha_1}\alpha_2)|}{2r_1 r_2},$$

as claimed. □

Note that the circles are orthogonal if and only if

$$\beta_1 + \beta_2 = \alpha_1\overline{\alpha_2} + \overline{\alpha_1}\alpha_2.$$

Problem 1. *Let a, b be real numbers such that $|b| \leq 2a^2$. Prove that the set of points with coordinates z such that*

$$|z^2 - a^2| = |2az + b|$$

is the union of two orthogonal circles.

Solution. The relation

$$|z^2 - a^2| = |2az + b|$$

is equivalent to

$$|z^2 - a^2|^2 = |2az + b|^2, \text{ i.e.,}$$
$$(z^2 - a^2)(\overline{z}^2 - \overline{a}^2) = (2az + b)(2a\overline{z} + b).$$

We can rewrite the last relation as

$$|z|^4 - a^2(z^2 + \overline{z}^2) + a^4 = 4a^2|z|^2 + 2ab(z + \overline{z}) + b^2, \text{ i.e.,}$$
$$|z|^4 - a^2[(z + \overline{z})^2 - 2|z|^2] + a^4 = 4a^2|z|^2 + 2ab(z + \overline{z}) + b^2.$$

Hence

$$|z|^4 - 2a^2|z|^2 + a^4 = a^2(z + \overline{z})^2 + 2ab(z + \overline{z}) + b^2, \text{ i.e.,}$$
$$(|z|^2 - a^2)^2 = (a(z + \overline{z}) + b)^2.$$

It follows that

$$z \cdot \overline{z} - a^2 = a(z + \overline{z}) + b \text{ or } z \cdot \overline{z} - a^2 = -a(z + \overline{z}) - b.$$

This is equivalent to

$$(z - a)(\overline{z} - a) = 2a^2 + b \text{ or } (z + a)(\overline{z} + a) = 2a^2 - b.$$

Finally

$$|z - a|^2 = 2a^2 + b \text{ or } |z + a|^2 = 2a^2 - b. \tag{1}$$

Since $|b| \leq 2a^2$, it follows that $2a^2 + b \geq 0$ and $2a^2 - b \geq 0$. Hence the relations (1) are equivalent to

$$|z - a| = \sqrt{2a^2 + b} \text{ or } |z + a| = \sqrt{2a^2 - b}.$$

Therefore, the points with coordinates z that satisfy $|z^2 - a^2| = |2az + b|$ lie on two circles of centers C_1 and C_2, whose coordinates a and $-a$, and with radii $R_1 = \sqrt{2a^2 + b}$ and $R_2 = \sqrt{2a^2 - b}$. Furthermore,

$$C_1C_2^2 = 4a^2 = (\sqrt{2a^2 + b})^2 + (\sqrt{2a^2 - b})^2 = R_1^2 + R_2^2,$$

hence the circles are orthogonal, as claimed.

4

More on Complex Numbers and Geometry

4.1 The Real Product of Two Complex Numbers

The concept of the scalar product of two vectors is well known. In what follows we will introduce this concept for complex numbers. We will see that in many situations use of this product simplifies the solution to the problem considerably.

Let a and b be two complex numbers.

Definition. We call the *real product* of complex numbers a and b the number given by

$$a \cdot b = \frac{1}{2}(\overline{a}b + a\overline{b}).$$

It is easy to see that

$$\overline{a \cdot b} = \frac{1}{2}(a\overline{b} + \overline{a}b) = a \cdot b;$$

hence $a \cdot b$ is a real number, which justifies the name of this product.

The following properties are easy to verify.

Proposition 1. *For all complex numbers a, b, c, z the following relations hold:*
1) $a \cdot a = |a|^2$.
2) $a \cdot b = b \cdot a$; *(the real product is commutative).*
3) $a \cdot (b+c) = a \cdot b + a \cdot c$; *(the real product is distributive with respect to addition).*
4) $(\alpha a) \cdot b = \alpha(a \cdot b) = a \cdot (\alpha b)$ *for all* $\alpha \in \mathbb{R}$.

5) $a \cdot b = 0$ if and only if $OA \perp OB$, where A has coordinate a and B has coordinate b.

6) $(az) \cdot (bz) = |z|^2(a \cdot b)$.

Remark. Suppose that A and B are points with coordinates a and b. Then the real product $a \cdot b$ is equal to the power of the origin with respect to the circle of diameter AB.

Indeed, let $M\left(\dfrac{a+b}{2}\right)$ be the midpoint of $[AB]$, hence the center of this circle, and let $r = \dfrac{1}{2}AB = \dfrac{1}{2}|a-b|$ be the radius of this circle. The power of the origin with respect to the circle is

$$OM^2 - r^2 = \left|\frac{a+b}{2}\right|^2 - \left|\frac{a-b}{2}\right|^2$$
$$= \frac{(a+b)(\overline{a}+\overline{b})}{4} - \frac{(a-b)(\overline{a}-\overline{b})}{4} = \frac{a\overline{b}+b\overline{a}}{2} = a \cdot b,$$

as claimed.

Proposition 2. *Suppose that $A(a), B(b), C(c)$ and $D(d)$ are four distinct points. The following statements are equivalent:*

1) $AB \perp CD$;

2) $(b-a) \cdot (c-d) = 0$;

3) $\dfrac{b-a}{d-c} \in i\mathbb{R}^*$ (or, equivalently, $\operatorname{Re}\left(\dfrac{b-a}{d-c}\right) = 0$).

Proof. Take points $M(b-a)$ and $N(d-c)$ such that $OABM$ and $OCDN$ are parallelograms. Then we have $AB \perp CD$ if and only if $OM \perp ON$. That is, $m \cdot n = (b-a) \cdot (d-c) = 0$, using property 5) of the real product.

The equivalence 2) $\Leftrightarrow$ 3) follows immediately from the definition of the real product. □

Proposition 3. *The circumcenter of triangle ABC is at the origin of the complex plane. If a, b, c are the coordinates of vertices A, B, C, then the orthocenter H has the coordinate $h = a + b + c$.*

Proof. Using the real product of the complex numbers, the equations of the altitudes AA', BB', CC' of the triangle are

$$AA' : (z-a) \cdot (b-c) = 0,\; BB' : (z-b) \cdot (c-a) = 0,\; CC' : (z-c) \cdot (a-b) = 0.$$

We will show that the point with coordinate $h = a+b+c$ lies on all three altitudes. Indeed, we have $(h-a) \cdot (b-c) = 0$ if and only if $(b+c) \cdot (b-c) = 0$. The last relation is equivalent to $b \cdot b - c \cdot c = 0$, or $|b|^2 = |c|^2$. Similarly, $H \in BB'$ and $H \in CC'$, and we are done. □

Remark. If the numbers a, b, c, o, h are the coordinates of the vertices of triangle ABC, the circumcenter O and the orthocenter H of the triangle, then $h = a+b+c-2o$.

Indeed, taking A' diametrically opposite A in the circumcircle of triangle ABC, the quadrilateral $HBA'C$ is a parallelogram. If $\{M\} = HA' \cap BC$, then

$$z_M = \frac{b+c}{2} = \frac{z_H + z_{A'}}{2} = \frac{z_H + 2o - a}{2}, \text{ i.e., } z_H = a + b + c - 2o.$$

Problem 1. *Let $ABCD$ be a convex quadrilateral. Prove that*

$$AB^2 + CD^2 = AD^2 + BC^2$$

if and only if $AC \perp BD$.

Solution. Using the properties of the real product of complex numbers, we have

$$AB^2 + CD^2 = BC^2 + DA^2$$

if and only if

$$(b-a)\cdot(b-a) + (d-c)\cdot(d-c) = (c-b)\cdot(c-b) + (a-d)\cdot(a-d).$$

That is,

$$a \cdot b + c \cdot d = b \cdot c + d \cdot a$$

and finally

$$(c-a)\cdot(d-b) = 0,$$

or, equivalently, $AC \perp BD$, as required.

Problem 2. *Let M, N, P, Q, R, S be the midpoints of the sides AB, BC, CD, DE, EF, FA of a hexagon. Prove that*

$$RN^2 = MQ^2 + PS^2$$

if and only if $MQ \perp PS$.

(Romanian Mathematical Olympiad – Final Round, 1994)

Solution. Let a, b, c, d, e, f be the coordinates of the vertices of the hexagon. The points M, N, P, Q, R, S have the coordinates

$$m = \frac{a+b}{2}, \quad n = \frac{b+c}{2}, \quad p = \frac{c+d}{2},$$
$$q = \frac{d+e}{2}, \quad r = \frac{e+f}{2}, \quad s = \frac{f+a}{2},$$

respectively.

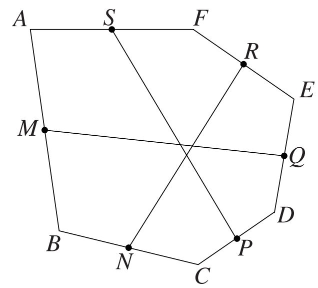

Figure 4.1.

Using the properties of the real product of complex numbers, we have

$$RN^2 = MQ^2 + PS^2$$

if and only if

$$(e + f - b - c) \cdot (e + f - b - c)$$
$$= (d + e - a - b) \cdot (d + e - a - b) + (f + a - c - d) \cdot (f + a - c - d).$$

That is,

$$(d + e - a - b) \cdot (f + a - c - d) = 0;$$

hence $MQ \perp PS$, as claimed.

Problem 3. *Let $A_1 A_2 \cdots A_n$ be a regular polygon inscribed in a circle of center O and radius R. Prove that for all points M in the plane the following relation holds:*

$$\sum_{k=1}^{n} MA_k^2 = n(OM^2 + R^2).$$

Solution. Consider the complex plane with origin at point O and let $R\varepsilon_k$ be the coordinate of vertex A_k, where ε_k are the n^{th}-roots of unity, $k = 1, \ldots, n$. Let m be the coordinate of M.

Using the properties of the real product of the complex numbers, we have

$$\begin{aligned}
\sum_{k=1}^{n} MA_k^2 &= \sum_{k=1}^{n} (m - R\varepsilon_k) \cdot (m - R\varepsilon_k) \\
&= \sum_{k=1}^{n} (m \cdot m - 2R\varepsilon_k \cdot m + R^2 \varepsilon_k \cdot \varepsilon_k) \\
&= n|m|^2 - 2R\Big(\sum_{k=1}^{n} \varepsilon_k\Big) \cdot m + R^2 \sum_{k=1}^{n} |\varepsilon_k|^2 \\
&= n \cdot OM^2 + nR^2 = n(OM^2 + R^2),
\end{aligned}$$

since $\sum_{k=1}^{n} \varepsilon_k = 0$.

Remark. If M lies on the circumcircle of the polygon, then

$$\sum_{k=1}^{n} MA_k^2 = 2nR^2.$$

Problem 4. *Let O be the circumcenter of the triangle ABC, let D be the midpoint of the segment AB, and let E is the centroid of triangle ACD. Prove that lines CD and OE are perpendicular if and only if $AB = AC$.*

(Balkan Mathematical Olympiad, 1985)

Solution. Let O be the origin of the complex plane and let a, b, c, d, e be the coordinates of points A, B, C, D, E, respectively. Then

$$d = \frac{a+b}{2} \text{ and } e = \frac{a+c+d}{3} = \frac{3a+b+2c}{6}.$$

Using the real product of complex numbers, if R is the circumradius of triangle ABC, then

$$a \cdot a = b \cdot b = c \cdot c = R^2.$$

Lines CD and DE are perpendicular if and only if $(d - c) \cdot e = 0$ That is,

$$(a + b - 2c) \cdot (3a + b + 2c) = 0.$$

The last relation is equivalent to

$$3a \cdot a + a \cdot b + 2a \cdot c + 3a \cdot b + b \cdot b + 2b \cdot c - 6a \cdot c - 2b \cdot c - 4c \cdot c = 0,$$

that is,

$$a \cdot b = a \cdot c. \quad (1)$$

On the other hand, $AB = AC$ is equivalent to

$$|b - a|^2 = |c - a|^2.$$

That is,

$$(b - a) \cdot (b - a) = (c - a) \cdot (c - a)$$

or

$$b \cdot b - 2a \cdot b + a \cdot a = c \cdot c - 2a \cdot c + a \cdot a,$$

hence

$$a \cdot b = a \cdot c. \quad (2)$$

The relations (1) and (2) show that $CD \perp OE$ if and only if $AB = AC$.

Problem 5. *Let a, b, c be distinct complex numbers such that $|a| = |b| = |c|$ and $|b+c-a| = |a|$.*

Prove that $b + c = 0$.

Solution. Let A, B, C be the geometric images of the complex numbers a, b, c, respectively. Choose the circumcenter of triangle ABC as the origin of the complex plane and denote by R the circumradius of triangle ABC. Then

$$a\overline{a} = b\overline{b} = c\overline{c} = R^2,$$

and using the real product of the complex numbers, we have

$$|b+c-a| = |a| \text{ if and only if } |b+c-a|^2 = |a|^2.$$

That is,

$$(b+c-a)\cdot(b+c-a) = |a|^2, \text{ i.e.,}$$

$$|a|^2 + |b|^2 + |c|^2 + 2b\cdot c - 2a\cdot c - 2a\cdot b = |a|^2.$$

We obtain

$$2(R^2 + b\cdot c - a\cdot c - a\cdot b) = 0, \text{ i.e.,}$$

$$a\cdot a + b\cdot c - a\cdot c - a\cdot b = 0.$$

It follows that $(a-b)\cdot(a-c) = 0$, hence $AB \perp AC$, i.e., $\widehat{BAC} = 90°$. Therefore, $[BC]$ is the diameter of the circumcircle of triangle ABC, so $b + c = 0$.

Problem 6. *Let E, F, G, H be the midpoints of sides AB, BC, CD, DA of the convex quadrilateral $ABCD$. Prove that lines AB and CD are perpendicular if and only if*

$$BC^2 + AD^2 = 2(EG^2 + FH^2).$$

Solution. Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter. Then

$$e = \frac{a+b}{2}, \quad f = \frac{b+c}{2}, \quad g = \frac{c+d}{2}, \quad h = \frac{d+a}{2}.$$

Using the real product of the complex numbers, the relation

$$BC^2 + AD^2 = 2(EG^2 + FH^2)$$

becomes

$$(c-b)\cdot(c-b) + (d-a)\cdot(d-a)$$

$$= \frac{1}{2}(c+d-a-b)\cdot(c+d-a-b) + \frac{1}{2}(a+d-b-c)\cdot(a+d-b-c).$$

This is equivalent to

$$c\cdot c + b\cdot b + d\cdot d + a\cdot a - 2b\cdot c - 2a\cdot d$$
$$= a\cdot a + b\cdot b + c\cdot c + d\cdot d - 2a\cdot c - 2b\cdot d,$$

or

$$a\cdot d + b\cdot c = a\cdot c + b\cdot d.$$

The last relation shows that $(a-b)\cdot(d-c) = 0$ if and only if $AB \perp CD$, as desired.

Problem 7. *Let G be the centroid of triangle ABC and let A_1, B_1, C_1 be the midpoints of sides BC, CA, AB, respectively. Prove that*

$$MA^2 + MB^2 + MC^2 + 9MG^2 = 4(MA_1^2 + MB_1^2 + MC_1^2)$$

for all points M in the plane.

Solution. Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter. Then

$$g = \frac{a+b+c}{3}, \quad a_1 = \frac{b+c}{2}, \quad b_1 = \frac{c+a}{2}, \quad c_1 = \frac{a+b}{2}.$$

Using the real product of the complex numbers, we have

$$MA^2 + MB^2 + MC^2 + 9MG^2$$
$$= (m-a)\cdot(m-a) + (m-b)\cdot(m-b) + (m-c)\cdot(m-c)$$
$$+ 9\left(m - \frac{a+b+c}{3}\right)\cdot\left(m - \frac{a+b+c}{3}\right)$$
$$= 12|m|^2 - 8(a+b+c)\cdot m + 2(|a|^2+|b|^2+|c|^2) + 2a\cdot b + 2b\cdot c + 2c\cdot a.$$

On the other hand,

$$4(MA_1^2 + MB_1^2 + MC_1^2)$$
$$= 4\left[\left(m - \frac{b+c}{2}\right)\cdot\left(m - \frac{b+c}{2}\right) + \left(m - \frac{c+a}{2}\right)\right.$$
$$\left.\cdot\left(m - \frac{c+a}{2}\right) + \left(m - \frac{a+b}{2}\right)\cdot\left(m - \frac{a+b}{2}\right)\right]$$
$$= 12|m|^2 - 8(a+b+c)\cdot m + 2(|a|^2+|b|^2+|c|^2) + 2a\cdot b + 2b\cdot c + 2c\cdot a,$$

so we are done.

Remark. The following generalization can be proved similarly.

Let $A_1A_2\cdots A_n$ be a polygon with the centroid G and let A_{ij} be the midpoint of the segment $[A_iA_j]$, $i<j$, $i,j\in\{1,2,\dots,n\}$.

Then

$$(n-2)\sum_{k=1}^{n} MA_k^2 + n^2MG^2 = 4\sum_{i<j} MA_{ij}^2,$$

for all points M in the plane. A nice generalization is given in Theorem 5, Section 4.11.

4.2 The Complex Product of Two Complex Numbers

The cross product of two vectors is a central concept in vector algebra, with numerous applications in various branches of mathematics and science. In what follows we adapt this product to complex numbers. The reader will see that this new interpretation has multiple advantages in solving problems involving area or collinearity.

Let a and b be two complex numbers.

Definition. The complex number

$$a\times b = \frac{1}{2}(\overline{a}b - a\overline{b})$$

is called the *complex product* of the numbers a and b.

Note that

$$a\times b + \overline{a\times b} = \frac{1}{2}(\overline{a}b - a\overline{b}) + \frac{1}{2}(a\overline{b} - \overline{a}b) = 0,$$

so $\operatorname{Re}(a\times b)=0$, which justifies the definition of this product.

The following properties are easy to verify:

Proposition 1. *Suppose that a, b, c are complex numbers. Then:*

1) $a\times b = 0$ if and only if $a=0$ or $b=0$ or $a=\lambda b$, where λ is a real number.

2) $a\times b = -b\times a$; (the complex product is anticommutative*).*

3) $a\times(b+c) = a\times b + a\times c$ (the complex product is distributive *with respect to addition).*

4) $\alpha(a\times b) = (\alpha a)\times b = a\times(\alpha b)$, for all real numbers α.

5) If $A(a)$ and $B(b)$ are distinct points other than the origin, then $a\times b=0$ if and only if O, A, B are collinear.

Remarks. a) Suppose $A(a)$ and $B(b)$ are distinct points in the complex plane, different from the origin.

The complex product of the numbers a and b has the following useful geometric interpretation:

$$a\times b = \begin{cases} 2i\cdot \text{area}[AOB], & \text{if triangle } OAB \text{ is positively oriented;}\\ -2i\cdot \text{area}[AOB], & \text{if triangle } OAB \text{ is negatively oriented.}\end{cases}$$

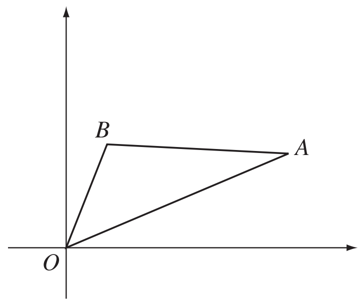

Figure 4.2.

Indeed, if triangle OAB is positively (directly) oriented, then

$$\begin{aligned} 2i \cdot \text{area}[OAB] &= i \cdot OA \cdot OB \cdot \sin(\widehat{AOB}) \\ &= i|a| \cdot |b| \cdot \sin\left(\arg\frac{b}{a}\right) = i \cdot |a| \cdot |b| \cdot \operatorname{Im}\left(\frac{b}{a}\right) \cdot \frac{|a|}{|b|} \\ &= \frac{1}{2}|a|^2\left(\frac{b}{a} - \frac{\overline{b}}{\overline{a}}\right) = \frac{1}{2}(\overline{a}b - a\overline{b}) = a \times b. \end{aligned}$$

In the other case, note that triangle OBA is positively oriented, hence

$$2i \cdot \text{area}[OBA] = b \times a = -a \times b.$$

b) Suppose $A(a)$, $B(b)$, $C(c)$ are three points in the complex plane.

The complex product allows us to obtain the following useful formula for the area of the triangle ABC:

$$\text{area}[ABC] = \begin{cases} \dfrac{1}{2i}(a \times b + b \times c + c \times a), \\ \text{if triangle } ABC \text{ is positively oriented;} \\ \\ -\dfrac{1}{2i}(a \times b + b \times c + c \times a), \\ \text{if triangle } ABC \text{ is negatively oriented.} \end{cases}$$

Moreover, simple algebraic manipulation shows that

$$\text{area}[ABC] = \frac{1}{2}\operatorname{Im}(\overline{a}b + \overline{b}c + \overline{c}a)$$

if triangle ABC is directly (positively) oriented.

To prove the above formula, translate points A, B, C with vector $-c$. The images of A, B, C are points A', B', O with coordinates $a - c$, $b - c$, 0, respectively. Triangles ABC and $A'B'O$ are congruent with the same orientation. If ABC is positively oriented, then

$$\begin{aligned}\text{area}[ABC] &= \text{area}[OA'B'] = \frac{1}{2i}((a-c)\times(b-c))\\ &= \frac{1}{2i}((a-c)\times b-(a-c)\times c) = \frac{1}{2i}(c\times(a-c)-b\times(a-c))\\ &= \frac{1}{2i}(c\times a-c\times c-b\times a+b\times c) = \frac{1}{2i}(a\times b+b\times c+c\times a),\end{aligned}$$

as claimed.

The other situation can be similarly solved.

Proposition 2. *Suppose $A(a)$, $B(b)$ and $C(c)$ are distinct points. The following statements are equivalent:*

1) Points A, B, C are collinear.

2) $(b-a)\times(c-a)=0$.

3) $a\times b+b\times c+c\times a=0$.

Proof. Points A, B, C are collinear if and only if area$[ABC]=0$, i.e., $a\times b+b\times c+c\times a=0$. The last equation can be written in the form $(b-a)\times(c-a)=0$. □

Proposition 3. *Let $A(a)$, $B(b)$, $C(c)$, $D(d)$ be four points, no three of which are collinear. Then $AB\|CD$ if and only if $(b-a)\times(d-c)=0$.*

Proof. Choose the points $M(m)$ and $N(n)$ such that $OABM$ and $OCDN$ are parallelograms; then $m=b-a$ and $n=d-c$.

Lines AB and CD are parallel if and only if points O, M, N are collinear. Using property 5, this is equivalent to $0=m\times n=(b-a)\times(d-c)$. □

Problem 1. *Points D and E lie on sides AB and AC of the triangle ABC such that*

$$\frac{AD}{AB}=\frac{AE}{AC}=\frac{3}{4}.$$

Consider points E' and D' on the rays $(BE$ and $(CD$ such that $EE'=3BE$ and $DD'=3CD$. Prove that:

1) points D', A, E' are collinear;

2) $AD'=AE'$.

Solution. The points D, E, D', E' have the coordinates: $d=\dfrac{a+3b}{4}$, $e=\dfrac{a+3c}{4}$,

$$e'=4e-3b=a+3c-3b \text{ and } d'=4d-3c=a+3b-3c,$$

respectively.

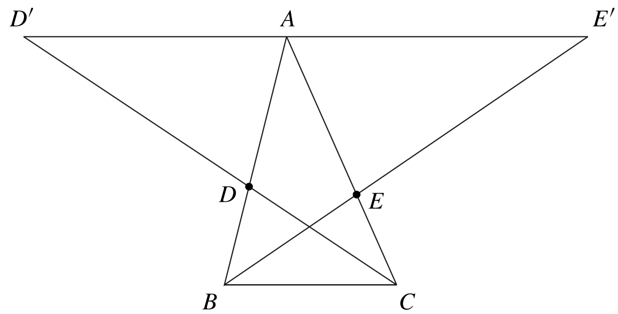

Figure 4.3.

1) Since

$$(a-d')\times(e'-d')=(3c-3b)\times(6c-6b)=18(c-b)\times(c-b)=0,$$

using Proposition 2 it follows that the points D', A, E' are collinear.

2) Note that

$$\frac{AD'}{D'E'}=\left|\frac{a-d'}{e'-d'}\right|=\frac{1}{2},$$

so A is the midpoint of segment $D'E'$.

Problem 2. *Let $ABCDE$ be a convex pentagon and let M, N, P, Q, X, Y be the midpoints of the segments BC, CD, DE, EA, MP, NQ, respectively.*

Prove that $XY \parallel AB$.

Solution. Let a, b, c, d, e be the coordinates of vertices A, B, C, D, E, respectively.

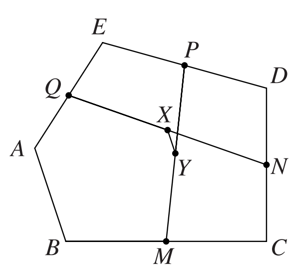

Figure 4.4.

Points M, N, P, Q, X, Y have the coordinates

$$m=\frac{b+c}{2},\quad n=\frac{c+d}{2},\quad p=\frac{d+e}{2},$$
$$q=\frac{e+a}{2},\quad x=\frac{b+c+d+e}{4},\quad y=\frac{c+d+e+a}{4},$$

respectively. Then

$$\frac{y-x}{b-a} = \frac{\frac{a-b}{4}}{b-a} = -\frac{1}{4} \in \mathbb{R},$$

hence

$$(y-x) \times (b-a) = -\frac{1}{4}(b-a) \times (b-a) = 0.$$

From Proposition 3 it follows that $XY \| AB$.

4.3 The Area of a Convex Polygon

We say that the convex polygon $A_1A_2 \cdots A_n$ is *directly* (or *positively*) *oriented* if for any point M situated in the interior of the polygon the triangles MA_kA_{k+1}, $k = 1, 2, \ldots, n$, are directly oriented, where $A_{n+1} = A_1$.

Theorem. *Consider a directly oriented convex polygon* $A_1A_2 \cdots A_n$ *with vertices with coordinates* $a_1, a_2, \ldots, a_n$*. Then*

$$\text{area}[A_1A_2 \cdots A_n] = \frac{1}{2}\,\text{Im}(\overline{a_1}a_2 + \overline{a_2}a_3 + \cdots + \overline{a_{n-1}}a_n + \overline{a_n}a_1).$$

Proof. We use induction on n. The base case $n = 3$ was proved above using the complex product. Suppose that the claim holds for $n = k$ and note that

$$\text{area}[A_1A_2 \cdots A_kA_{k+1}] = \text{area}[A_1A_2 \cdots A_k] + \text{area}[A_kA_{k+1}A_1]$$

$$= \frac{1}{2}\,\text{Im}(\overline{a_1}a_2 + \overline{a_2}a_3 + \cdots + \overline{a_{k-1}}a_k + \overline{a_k}a_1) + \frac{1}{2}\,\text{Im}(\overline{a_k}a_{k+1} + \overline{a_{k+1}}a_1 + \overline{a_1}a_k)$$

$$= \frac{1}{2}\,\text{Im}(\overline{a_1}a_2 + \overline{a_2}a_3 + \cdots + \overline{a_{k-1}}a_k + \overline{a_k}a_{k+1} + \overline{a_{k+1}}a_1)$$

$$+ \frac{1}{2}\,\text{Im}(\overline{a_k}a_1 + \overline{a_1}a_k) = \frac{1}{2}\,\text{Im}(\overline{a_1}a_2 + \overline{a_2}a_3 + \cdots + \overline{a_k}a_{k+1} + \overline{a_{k+1}}a_1),$$

$$\text{since} \qquad \text{Im}(\overline{a_k}a_1 + \overline{a_1}a_k) = 0.$$

Alternative proof. Choose a point M in the interior of the polygon. Applying the formula (2) in Subsection 3.5.3 we have

$$\begin{aligned}\text{area}[A_1A_2\cdots A_n] &= \sum_{k=1}^{n}\text{area}[MA_kA_{k+1}]\\ &= \frac{1}{2}\sum_{k=1}^{n}\operatorname{Im}(\overline{z}a_k+\overline{a_k}a_{k+1}+\overline{a_{k+1}}z)\\ &= \frac{1}{2}\sum_{k=1}^{n}\operatorname{Im}(\overline{a_k}a_{k+1})+\frac{1}{2}\sum_{k=1}^{n}\operatorname{Im}(\overline{z}a_k+\overline{a_{k+1}}z)\\ &= \frac{1}{2}\operatorname{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+1}\right)+\frac{1}{2}\operatorname{Im}\left(\overline{z}\sum_{k=1}^{n}a_k+z\sum_{j=1}^{n}\overline{a_j}\right)=\frac{1}{2}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+1}\right),\end{aligned}$$

since for any complex numbers z, w the relation $\operatorname{Im}(\overline{z}w+z\overline{w})=0$ holds. □

Remark. From the above formula it follows that the points $A_1(a_1)$, $A_2(a_2),\ldots,$ $A_n(a_n)$ are collinear if and only if

$$\operatorname{Im}(\overline{a}_1a_2+\overline{a}_2a_3+\cdots+\overline{a}_{n-1}a_n+\overline{a}_na_1)=0.$$

Problem 1. *Let $P_0P_1\cdots P_{n-1}$ be the polygon whose vertices have coordinates $1,\varepsilon,\ldots,\varepsilon^{n-1}$ and let $Q_0Q_1\cdots Q_{n-1}$ be the polygon whose vertices have coordinates $1, 1+\varepsilon,\ldots,1+\varepsilon+\cdots+\varepsilon^{n-1}$, where $\varepsilon=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$. Find the ratio of the areas of these polygons.*

Solution. Consider $a_k=1+\varepsilon+\cdots+\varepsilon^k$, $k=0,1,\ldots,n-1$, and observe that

$$\begin{aligned}\text{area}[Q_0Q_1\cdots Q_{n-1}] &= \frac{1}{2}\operatorname{Im}\left(\sum_{k=0}^{n-1}\overline{a_k}a_{k+1}\right)\frac{1}{2}\operatorname{Im}\left(\sum_{k=0}^{n-1}\frac{(\overline{\varepsilon})^{k+1}-1}{\overline{\varepsilon}-1}\cdot\frac{\varepsilon^{k+2}-1}{\varepsilon-1}\right)\\ &= \frac{1}{2|\varepsilon-1|^2}\operatorname{Im}\left[\sum_{k=0}^{n-1}(\varepsilon-(\overline{\varepsilon})^{k+1}-\varepsilon^{k+2}+1)\right]\\ &= \frac{1}{2|\varepsilon-1|^2}\operatorname{Im}(n\varepsilon+n)=\frac{1}{2|\varepsilon-1|^2}n\sin\frac{2\pi}{n}\\ &= \frac{n}{8\sin^2\frac{\pi}{n}}2\sin\frac{\pi}{n}\cos\frac{\pi}{n}=\frac{n}{4}\operatorname{cotan}\frac{\pi}{n},\end{aligned}$$

$$\text{since}\quad \sum_{k=0}^{n-1}\overline{\varepsilon}^{k+1}=0\quad\text{and}\quad\sum_{k=0}^{n-1}\varepsilon^{k+2}=0.$$

On the other hand, it is clear that

$$\text{area}[P_0P_1\cdots P_{n-1}]=n\,\text{area}[P_0OP_1]=\frac{n}{2}\sin\frac{2\pi}{n}=n\sin\frac{\pi}{n}\cos\frac{\pi}{n}.$$

We obtain

$$\frac{\text{area}[P_0P_1\cdots P_{n-1}]}{\text{area}[Q_0Q_1\cdots Q_{n-1}]} = \frac{n\sin\dfrac{\pi}{n}\cos\dfrac{\pi}{n}}{\dfrac{n}{4}\text{cotan}\dfrac{\pi}{n}} = 4\sin^2\frac{\pi}{n}. \tag{1}$$

Remark. We have $Q_kQ_{k+1} = |a_{k+1} - a_k| = |\varepsilon^{k+1}| = 1$, and $P_kP_{k+1} = |\varepsilon^{k+1} - \varepsilon^k| = |\varepsilon^k(\varepsilon - 1)| = |\varepsilon^k||1-\varepsilon| = |1-\varepsilon| = 2\sin\dfrac{\pi}{n}$, $k = 0, 1, \ldots, n-1$. It follows that

$$\frac{P_kP_{k+1}}{Q_kQ_{k+1}} = 2\sin\frac{\pi}{n}, \quad k = 0, 1, \ldots, n-1.$$

That is, the polygons $P_0P_1\cdots P_{n-1}$ and $Q_0Q_1\cdots Q_{n-1}$ are similar and the result in (1) follows.

Problem 2. *Let $A_1A_2\cdots A_n$ $(n \geq 5)$ be a convex polygon and let B_k be the midpoint of the segment $[A_kA_{k+1}]$, $k = 1, 2, \ldots, n$, where $A_{n+1} = A_1$. Then the following inequality holds:*

$$\text{area}[B_1B_2\cdots B_n] \geq \frac{1}{2}\text{area}[A_1A_2\cdots A_n].$$

Solution. Let a_k and b_k be the coordinates of points A_k and B_k, $k = 1, 2, \ldots, n$. It is clear that the polygon $B_1B_2\cdots B_n$ is convex and if we assume that $A_1A_2\cdots A_n$ is positively oriented, then $B_1B_2\cdots B_n$ also has this property. Choose as the origin O of the complex plane a point situated in the interior of polygon $A_1A_2\cdots A_n$.

We have $b_k = \dfrac{1}{2}(a_k + a_{k+1})$, $k = 1, 2, \ldots, n$, and

$$\begin{aligned}
\text{area}[B_1B_2\cdots B_n] &= \frac{1}{2}\,\text{Im}\left(\sum_{k=1}^{n}\overline{b_k}b_{k+1}\right) = \frac{1}{8}\,\text{Im}\sum_{k=1}^{n}(\overline{a_k} + \overline{a_{k+1}})(a_{k+1} + a_{k+2}) \\
&= \frac{1}{8}\,\text{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+1}\right) + \frac{1}{8}\,\text{Im}\left(\sum_{k=1}^{n}\overline{a_{k+1}}a_{k+2}\right) + \frac{1}{8}\,\text{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+2}\right) \\
&= \frac{1}{2}\text{area}[A_1A_2\cdots A_n] + \frac{1}{8}\,\text{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+2}\right) \\
&= \frac{1}{2}\text{area}[A_1A_2\cdots A_n] + \frac{1}{8}\sum_{k=1}^{n}\text{Im}(\overline{a_k}a_{k+2}) \\
&= \frac{1}{2}\text{area}[A_1A_2\cdots A_n] + \frac{1}{8}\sum_{k=1}^{n}OA_k\cdot OA_{k+2}\sin\widehat{A_kOA_{k+2}} \\
&\geq \frac{1}{2}\text{area}[A_1A_2\cdots A_n].
\end{aligned}$$

We have used the relations

$$\operatorname{Im}\left(\sum_{k=1}^{n}\overline{a_k}a_{k+1}\right)=\operatorname{Im}\left(\sum_{k=1}^{n}\overline{a_{k+1}}a_{k+2}\right)=2\,\text{area}[A_1A_2\cdots A_n],$$

and $\sin \widehat{A_kOA_{k+2}} \geq 0$, $k=1,2,\dots,n$, where $A_{n+2}=A_2$.

4.4 Intersecting Cevians and Some Important Points in a Triangle

Proposition 1. *Consider the points A', B', C' on the sides BC, CA, AB of the triangle ABC such that AA', BB', CC' intersect at point Q and let*

$$\frac{BA'}{A'C}=\frac{p}{n},\quad \frac{CB'}{B'A}=\frac{m}{p},\quad \frac{AC'}{C'B}=\frac{n}{m}.$$

If a,b,c are the coordinates of points A,B,C, respectively, then the coordinate of point Q is

$$q=\frac{ma+nb+pc}{m+n+p}.$$

Proof. The coordinates of A',B',C' are $a'=\dfrac{nb+pc}{n+p}$, $b'=\dfrac{ma+pc}{m+p}$ and $c'=\dfrac{ma+nb}{m+n}$, respectively. Let Q be the point with coordinate $q=\dfrac{ma+nb+pc}{m+n+p}$. We prove that AA', BB', CC' meet at Q.

The points A, Q, A' are collinear if and only if $(q-a)\times(a'-a)=0$. This is equivalent to

$$\left(\frac{ma+nb+pc}{m+n+p}-a\right)\times\left(\frac{nb+pc}{n+p}-a\right)=0$$

or $(nb+pc-(n+p)a)\times(nb+pc-(n+p)a)=0$, which is clear by definition of the complex product.

Likewise, Q lies on lines BB' and CC', so the proof is complete. □

Some important points in a triangle. 1) If $Q=G$, the centroid of the triangle ABC, we have $m=n=p=1$. Then we obtain again that the coordinate of G is

$$z_G=\frac{a+b+c}{3}.$$

2) Suppose that the lengths of the sides of triangle ABC are $BC=\alpha$, $CA=\beta$, $AB=\gamma$. If $Q=I$, the incenter of triangle ABC, then, using the known result concerning the angle bisector, it follows that $m=\alpha$, $n=\beta$, $p=\gamma$. Therefore the coordinate of I is

$$z_I=\frac{\alpha a+\beta b+\gamma c}{\alpha+\beta+\gamma}=\frac{1}{2s}[\alpha a+\beta b+\gamma c],$$

where $s = \frac{1}{2}(\alpha + \beta + \gamma)$.

3) If $Q = H$, the orthocenter of the triangle ABC, we easily obtain the relations

$$\frac{BA'}{A'C} = \frac{\tan C}{\tan B}, \quad \frac{CB'}{B'A} = \frac{\tan A}{\tan C}, \quad \frac{AC'}{C'B} = \frac{\tan B}{\tan A}.$$

It follows that $m = \tan A$, $n = \tan B$, $p = \tan C$, and the coordinate of H is given by

$$z_H = \frac{(\tan A)a + (\tan B)b + (\tan C)c}{\tan A + \tan B + \tan C}.$$

Remark. The above formula can also be extended to the limiting case when the triangle ABC is a right triangle. Indeed, assume that $A \to \frac{\pi}{2}$. Then $\tan A \to \pm\infty$ and $\frac{(\tan B)b + (\tan C)c}{\tan A} \to 0$, $\frac{\tan B + \tan C}{\tan A} \to 0$. In this case $z_H = a$, i.e., the orthocenter of triangle ABC is the vertex A.

4) The Gergonne[1] point J is the intersection of the cevians AA', BB', CC', where A', B', C' are the points of tangency of the incircle to the sides BC, CA, AB, respectively. Then

$$\frac{BA'}{A'C} = \frac{\frac{1}{s-\gamma}}{\frac{1}{s-\beta}}, \quad \frac{CB'}{B'A} = \frac{\frac{1}{s-\alpha}}{\frac{1}{s-\gamma}}, \quad \frac{AC'}{C'B} = \frac{\frac{1}{s-\beta}}{\frac{1}{s-\alpha}},$$

and the coordinate z_J is obtained from the same proposition, where

$$z_J = \frac{r_\alpha a + r_\beta b + r_\gamma c}{r_\alpha + r_\beta + r_\gamma}.$$

Here $r_\alpha, r_\beta, r_\gamma$ denote the radii of the three excircles of triangle. It is not difficult to show that the following formulas hold:

$$r_\alpha = \frac{K}{s-\alpha}, \quad r_\beta = \frac{K}{s-\beta}, \quad r_\gamma = \frac{K}{s-\gamma},$$

where $K = \text{area}[ABC]$ and $s = \frac{1}{2}(\alpha + \beta + \gamma)$.

5) The Lemoine[2] point K is the intersection of the symmedians of the triangle (the symmedian is the reflection of the bisector across the median). Using the notation from

[1] Joseph-Diaz Gergonne (1771–1859), French mathematician, founded the journal *Annales de Mathématiques Pures et Appliquées* in 1810.

[2] Emile Michel Hyacinthe Lemoine (1840–1912), French mathematician, made important contributions to geometry.

the proposition we obtain

$$\frac{BA'}{A'C} = \frac{\gamma^2}{\beta^2}, \quad \frac{CB'}{B'A} = \frac{\alpha^2}{\gamma^2}, \quad \frac{AC'}{C'B} = \frac{\beta^2}{\alpha^2}.$$

It follows that

$$z_K = \frac{\alpha^2 a + \beta^2 b + \gamma^2 c}{\alpha^2 + \beta^2 + \gamma^2}.$$

6) The Nagel[3] point N is the intersection of the cevians AA', BB', CC', where A', B', C' are the points of tangency of the excircles with the sides BC, CA, AB, respectively. Then

$$\frac{BA'}{A'C} = \frac{s-\gamma}{s-\beta}, \quad \frac{CB'}{B'A} = \frac{s-\alpha}{s-\gamma}, \quad \frac{AC'}{C'B} = \frac{s-\beta}{s-\alpha},$$

and the proposition mentioned before gives the coordinate z_N of the Nagel point N,

$$z_N = \frac{(s-\alpha)a + (s-\beta)b + (s-\gamma)c}{(s-\alpha)+(s-\beta)+(s-\gamma)} = \frac{1}{s}[(s-\alpha)a + (s-\beta)b + (s-\gamma)c]$$
$$= \left(1 - \frac{\alpha}{s}\right)a + \left(1 - \frac{\beta}{s}\right)b + \left(1 - \frac{\gamma}{s}\right)c.$$

Problem. *Let α, β, γ be the lengths of sides BC, CA, AB of triangle ABC and suppose $\alpha < \beta < \gamma$. If points O, I, H are the circumcenter, the incenter and the orthocenter of the triangle ABC, respectively. Prove that*

$$\text{area}[OIH] = \frac{1}{8r}(\alpha-\beta)(\beta-\gamma)(\gamma-\alpha),$$

where r is the inradius of ABC.

Solution. Consider triangle ABC, directly oriented in the complex plane centered at point O.

Using the complex product and the coordinates of I and H, we have

$$\text{area}[OIH] = \frac{1}{2i}(I \times h) = \frac{1}{2i}\left[\frac{\alpha a + \beta b + \gamma c}{\alpha+\beta+\gamma} \times (a+b+c)\right]$$
$$= \frac{1}{4si}[(\alpha-\beta)a \times b + (\beta-\gamma)b \times c + (\gamma-\alpha)c \times a]$$
$$= \frac{1}{2s}[(\alpha-\beta)\cdot \text{area}[OAB] + (\beta-\gamma)\cdot \text{area}[OBC] + (\gamma-\alpha)\cdot \text{area}[OCA]]$$
$$= \frac{1}{2s}\left[(\alpha-\beta)\frac{R^2 \sin 2C}{2} + (\beta-\gamma)\frac{R^2 \sin 2A}{2} + (\gamma-\alpha)\frac{R^2 \sin 2B}{2}\right]$$

[3]Christian Heinrich von Nagel (1803–1882), German mathematician. His contributions to triangle geometry were included in the book *The Development of Modern Triangle Geometry* [13].

$$= \frac{R^2}{4s}[(\alpha - \beta)\sin 2C + (\beta - \gamma)\sin 2A + (\gamma - \alpha)\sin 2B]$$
$$= \frac{1}{8r}(\alpha - \beta)(\beta - \gamma)(\gamma - \alpha),$$

as desired.

4.5 The Nine-Point Circle of Euler

Given a triangle ABC, choose its circumcenter O to be the origin of the complex plane and let a, b, c be the coordinates of the vertices A, B, C. We have seen in Section 2.22, Proposition 3, that the coordinate of the orthocenter H is $z_H = a + b + c$.

Let us denote by A_1, B_1, C_1 the midpoints of sides BC, CA, AB, by A', B', C' the feet of the altitudes and by A'', B'', C'' the midpoints of segments AH, BH, CH, respectively.

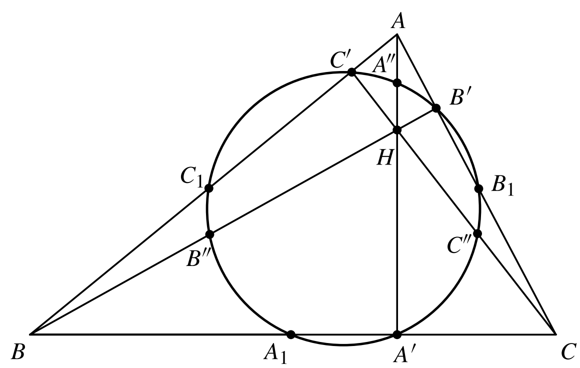

Figure 4.5.

It is clear that for the points $A_1, B_1, C_1, A'', B'', C''$ we have the following coordinates:

$$z_{A_1} = \frac{1}{2}(b + c), \quad z_{B_1} = \frac{1}{2}(c + a), \quad z_{C_1} = \frac{1}{2}(a + b),$$
$$z_{A''} = a + \frac{1}{2}(b + c), \quad z_{B''} = b + \frac{1}{2}(c + a), \quad z_{C''} = c + \frac{1}{2}(a + b).$$

It is not so easy to find the coordinates of A', B', C'.

Proposition 1. *Consider the point $X(x)$ in the plane of triangle ABC. Let P be the projection of X onto line BC. Then the coordinate of P is given by*

$$p = \frac{1}{2}\left(x - \frac{bc}{R^2}\overline{x} + b + c\right)$$

where R is the circumradius of triangle ABC.

Proof. Using the complex product and the real product we can write the equations of lines BC and XP as follows:

$$BC : (z-b)\times(c-b)=0,$$

$$XP : (z-x)\cdot(c-b)=0.$$

The coordinate p of P satisfies both equations; hence we have

$$(p-b)\times(c-b)=0 \quad \text{and} \quad (p-x)\cdot(c-b)=0.$$

These equations are equivalent to

$$(p-b)(\overline{c}-\overline{b})-(\overline{p}-\overline{b})(c-b)=0$$

and

$$(p-x)(\overline{c}-\overline{b})+(\overline{p}-\overline{x})(c-b)=0.$$

Adding the above relations we find

$$(2p-b-x)(\overline{c}-\overline{b})+(\overline{b}-\overline{x})(c-b)=0.$$

It follows that

$$p=\frac{1}{2}\left[b+x+\frac{c-b}{\overline{c}-\overline{b}}(\overline{x}-\overline{b})\right]=\frac{1}{2}\left[b+x+\frac{c-b}{\dfrac{R^2}{c}-\dfrac{R^2}{b}}(\overline{x}-\overline{b})\right]$$

$$=\frac{1}{2}\left[b+x-\frac{bc}{R^2}(\overline{x}-\overline{b})\right]=\frac{1}{2}\left(x-\frac{bc}{R^2}\overline{x}+b+c\right).$$ □

From the above Proposition 1, the coordinates of A', B', C' are

$$z_{A'}=\frac{1}{2}\left(a+b+c-\frac{bc\overline{a}}{R^2}\right),$$

$$z_{B'}=\frac{1}{2}\left(a+b+c-\frac{ca\overline{b}}{R^2}\right),$$

$$z_{C'}=\frac{1}{2}\left(a+b+c-\frac{ab\overline{c}}{R^2}\right).$$

Theorem 2. (The nine-point circle.) *In any triangle ABC the points A_1, B_1, C_1, A', B', C', A'', B'', C'' are all on the same circle, whose center is at the midpoint of the segment OH, and the radius is one-half of the circumcircle.*

Proof. Denote by O_9 the midpoint of the segment OH. Using our initial assumption, it follows that $z_{O_9} = \frac{1}{2}(a+b+c)$. Also we have $|a| = |b| = |c| = R$, where R is the circumradius of triangle ABC.

Observe that $O_9A_1 = |z_{A_1} - z_{O_9}| = \frac{1}{2}|a| = \frac{1}{2}R$, and also $O_9B_1 = O_9C_1 = \frac{1}{2}R$.

We can write $O_9A'' = |z_{A''} - z_{O_0}| = \frac{1}{2}|a| = \frac{1}{2}R$, and also $O_0B'' = O_9C'' = \frac{1}{2}R$.

The distance O_9A' is also not difficult to compute:

$$O_9A' = |z_{A'} - z_{O_9}| = \left|\frac{1}{2}\left(a+b+c-\frac{bc\overline{a}}{R^2}\right) - \frac{1}{2}(a+b+c)\right|$$
$$= \frac{1}{2R^2}|bc\overline{a}| = \frac{1}{2R^2}|\overline{a}||b||c| = \frac{R^3}{2R^2} = \frac{1}{2}R.$$

Similarly, we get $O_9B' = O_9C' = \frac{1}{2}R$. Therefore $O_9A_1 = O_0B_1 = O_9C_1 = O_9A' = O_9B' = O_9C' = O_9A'' = O_9B'' = O_9C'' = \frac{1}{2}R$ and the desired property follows. □

Theorem 3. 1) (Euler[4] line of a triangle.) *In any triangle ABC the points O, G, H are collinear.*

2) (Nagel line of a triangle.) *In any triangle ABC the points I, G, N are collinear.*

Proof. 1) If the circumcenter O is the origin of the complex plane, we have $z_O = 0$, $z_G = \frac{1}{3}(a+b+c)$, $z_H = a+b+c$. Hence these points are collinear by Proposition 2 in Section 2.22.

2) We have $z_I = \frac{\alpha}{2s}a + \frac{\beta}{2s}b + \frac{\gamma}{2s}c$, $z_G = \frac{1}{3}(a+b+c)$, and $z_N = \left(1-\frac{\alpha}{s}\right)a + \left(1-\frac{\beta}{s}\right)b + \left(1-\frac{\gamma}{s}\right)c$ and we can write $z_N = 3z_G - 2z_I$.

Applying the result mentioned above and properties of the complex product we obtain $(z_G - z_I) \times (z_N - z_I) = (z_G - z_I) \times [3(z_G - z_I)] = 0$; hence the points I, H, N are collinear. □

Remark. Note that $NG = 2GI$, hence the triangles OGI and HGN are similar. It follows that the lines OI and NH are parallel and we have the following basic configuration of triangle ABC (in Figure 4.6):

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[4]Leonhard Euler (1707–1783), one of the most important mathematicians, created a good deal of analysis, and revised almost all the branches of pure mathematics which were then known, adding proofs, and arranging the whole in a consistent form. Euler wrote an immense number of memoirs on all kinds of mathematical subjects. We recommend William Dunham's book *Euler. The Master of Us All* (The Mathematical Association of America, 1999) for more details concerning Euler's contributions to mathematics.

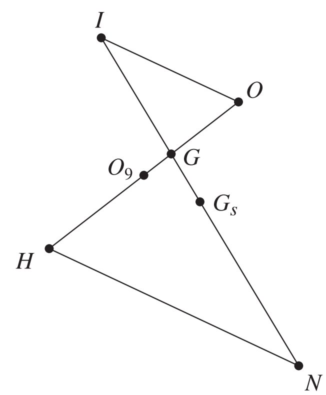

Figure 4.6.

If G_s is the midpoint of segment $[IN]$, then its coordinate is

$$z_{G_s} = \frac{1}{2}(z_I + z_N) = \frac{(\beta+\gamma)}{4s}a + \frac{(\gamma+\alpha)}{4s}b + \frac{(\alpha+\beta)}{4s}c.$$

The point G_s is called the *Spiecker point* of triangle ABC and it is easy to verify that it is the incenter of the medial triangle $A_1B_1C_1$.

Problem 1. *Consider a point M on the circumcircle of the triangle ABC. Prove that the nine-point centers of the triangles MBC, MCA, MAB are the vertices of a triangle similar to triangle ABC.*

Solution. Let A', B', C' be the nine-point centers of the triangles MBC, MCD, MAB, respectively. Take the origin of the complex plane to be at the circumcenter of triangle ABC. Denote by a lowercase letter the coordinate of the point denoted by an uppercase letter. Then

$$a' = \frac{m+b+c}{2}, \quad b' = \frac{m+c+a}{2}, \quad c' = \frac{m+a+b}{2},$$

since M lies on the circumcircle of triangle ABC. Then

$$\frac{b'-a'}{c'-a'} = \frac{a-b}{a-c} = \frac{b-a}{c-a},$$

and hence triangles $A'B'C'$ and ABC are similar.

Problem 2. *Show that triangle ABC is a right triangle if and only if its circumcircle and its nine-point circle are tangent.*

Solution. Take the origin of the complex plane to be at circumcenter O of triangle ABC and denote by a, b, c the coordinates of vertices A, B, C, respectively. Then the circumcircle of triangle ABC is tangent to the nine-point circle of triangle ABC if and only if $OO_9 = \frac{R}{2}$. This is equivalent to $OO_9^2 = \frac{R^2}{4}$, that is, $|a+b+c|^2 = R^2$.

Using properties of the real product, we have

$$\begin{aligned} |a+b+c|^2 &= (a+b+c)\cdot(a+b+c) = a^2+b^2+c^2+2(a\cdot b+b\cdot c+c\cdot a) \\ &= 3R^2+2(a\cdot b+b\cdot c+c\cdot a) = 3R^2+(2R^2-\alpha^2+2R^2-\beta^2+2R^2-\gamma^2) \\ &= 9R^2-(\alpha^2+\beta^2+\gamma^2), \end{aligned}$$

where α, β, γ are the lengths of the sides of triangle ABC. We have used the formulas $a\cdot b = R^2 - \frac{\gamma^2}{2}$, $b\cdot c = R^2 - \frac{\alpha^2}{2}$, $c\cdot a = R^2 - \frac{\beta^2}{2}$, which can be easily derived from the definition of the real product of complex numbers (see also the lemma in Subsection 4.6.2).

Therefore, $\alpha^2+\beta^2+\gamma^2 = 8R^2$, which is the same as $\sin^2 A+\sin^2 B+\sin^2 C = 2$. We can write the last relation as $1-\cos 2A+1-\cos 2B+1-\cos 2C = 4$. This is equivalent to $2\cos(A+B)\cos(A-B)+2\cos^2 C = 0$, i.e., $4\cos A\cos B\cos C = 0$, and the desired conclusion follows.

Problem 3. *Let $ABCD$ be a cyclic quadrilateral and let E_a, E_b, E_c, E_d be the nine-point centers of triangles BCD, CDA, DAB, ABC, respectively. Prove that the lines AE_a, BE_b, CE_c, DE_d are concurrent.*

Solution. Take the origin of the complex plane to be the center O of the circumcircle of $ABCD$. Then the coordinates of the nine-point centers are

$$e_a = \frac{1}{2}(b+c+d), \quad e_b = \frac{1}{2}(c+d+a), \quad e_c = \frac{1}{2}(d+a+b), \quad e_d = \frac{1}{2}(a+b+c).$$

We have $AE_a : z = ka+(1-k)e_a$, $k \in \mathbb{R}$, and the analogous equations for the lines BE_b, CE_c, DE_d. Observe that the point with coordinate $\frac{1}{3}(a+b+c+d)$ lies on all of the four lines $\left(k = \frac{1}{3}\right)$, and we are done.

4.6 Some Important Distances in a Triangle

4.6.1 Fundamental invariants of a triangle

Consider the triangle ABC with sides α, β, γ, the semiperimeter $s = \frac{1}{2}(\alpha+\beta+\gamma)$, the inradius r and the circumradius R. The numbers s, r, R are called the *fundamental invariants* of triangle ABC.

Theorem 1. *The sides α, β, γ are the roots of the cubic equation*

$$t^3 - 2st^2 + (s^2+r^2+4Rr)t - 4sRr = 0.$$

Proof. Let us prove that α satisfies the equation. We have

$$\alpha = 2R\sin A = 4R\sin\frac{A}{2}\cos\frac{A}{2} \text{ and } s-\alpha = r\operatorname{cotan}\frac{A}{2} = r\frac{\cos\frac{A}{2}}{\sin\frac{A}{2}},$$

hence

$$\cos^2\frac{A}{2} = \frac{\alpha(s-\alpha)}{4Rr} \text{ and } \sin^2\frac{A}{2} = \frac{\alpha r}{4R(s-\alpha)}.$$

From the formula $\cos^2\frac{A}{2} + \sin^2\frac{A}{2} = 1$, it follows that

$$\frac{\alpha(s-\alpha)}{4Rr} + \frac{\alpha r}{4R(s-\alpha)} = 1.$$

That is, $\alpha^3 - 2s\alpha^2 + (s^2 + r^2 + 4Rr)\alpha - 4sRr = 0$. We can show analogously that β and γ are roots of the above equation. □

From the above theorem, by using the relations between the roots and the coefficients, it follows that

$$\alpha + \beta + \gamma = 2s,$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = s^2 + r^2 + 4Rr,$$

$$\alpha\beta\gamma = 4sRr.$$

Corollary 2. *In any triangle ABC, the following formulas hold:*

$$\alpha^2 + \beta^2 + \gamma^2 = 2(s^2 - r^2 - 4Rr),$$

$$\alpha^3 + \beta^3 + \gamma^3 = 2s(s^2 - 3r^2 - 6Rr).$$

Proof. We have

$$\alpha^2 + \beta^2 + \gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = 4s^2 - 2(s^2 + r^2 + 4Rr)$$

$$= 2s^2 - 2r^2 - 8Rr = 2(s^2 - r^2 - 4Rr).$$

In order to prove the second identity, we can write

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha+\beta+\gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) + 3\alpha\beta\gamma$$

$$= 2s(2s^2 - 2r^2 - 8Rr - s^2 - r^2 - 4Rr) + 12sRr = 2s(s^2 - 3r^2 - 6Rr). \quad \square$$

4.6.2 The distance OI

Assume that the circumcenter O of the triangle ABC is the origin of the complex plane and let a, b, c be the coordinates of the vertices A, B, C, respectively.

Lemma. *The real products $a \cdot b, b \cdot c, c \cdot a$ are given by*

$$a \cdot b = R^2 - \frac{\gamma^2}{2}, \quad b \cdot c = R^2 - \frac{\alpha^2}{2}, \quad c \cdot a = R^2 - \frac{\beta^2}{2}.$$

Proof. Using the properties of the real product we have

$$\gamma^2 = |a-b|^2 = (a-b)\cdot(a-b) = a^2 - 2a \cdot b - b^2 = 2R^2 - 2a \cdot b,$$

and the first formula follows. □

Theorem 4. (Euler) *The following formula holds:*

$$OI^2 = R^2 - 2Rr.$$

Proof. The coordinate of the incenter is given by

$$z_I = \frac{\alpha}{2s}a + \frac{\beta}{2s}b + \frac{\gamma}{2s}c$$

so we can write

$$OI^2 = |z_I|^2 = \left(\frac{\alpha}{2s}a + \frac{\beta}{2s}b + \frac{\gamma}{2s}c\right) \cdot \left(\frac{\alpha}{2s}a + \frac{\beta}{2s}b + \frac{\gamma}{2s}c\right)$$

$$= \frac{1}{4s^2}(\alpha^2 + \beta^2 + \gamma^2)R^2 + 2\frac{1}{4s^2}\sum_{\text{cyc}}(\alpha\beta)a \cdot b.$$

Using the lemma above we find that

$$OI^2 = \frac{1}{4s^2}(\alpha^2 + \beta^2 + \gamma^2)R^2 + \frac{2}{4s^2}\sum_{\text{cyc}}\alpha\beta\left(R^2 - \frac{\gamma^2}{2}\right)$$

$$= \frac{1}{4s^2}(\alpha+\beta+\gamma)^2R^2 - \frac{1}{4s^2}\sum_{\text{cyc}}\alpha\beta\gamma^2 = R^2 - \frac{1}{4s^2}\alpha\beta\gamma(\alpha+\beta+\gamma)$$

$$= R^2 - \frac{1}{2s}\alpha\beta\gamma = R^2 - 2\frac{\alpha\beta\gamma}{4K} \cdot \frac{K}{s} = R^2 - 2Rr,$$

where the well-known formulas

$$R = \frac{\alpha\beta\gamma}{4K}, \quad r = \frac{K}{s},$$

are used. Here K is the area of triangle ABC. □

Corollary 5. (Euler's inequality.) *In any triangle ABC the following inequality holds:*

$$R \geq 2r.$$

We have equality if and only if the triangle ABC is equilateral.

Proof. From Theorem 4 we have $OI^2 = R(R-2r) \geq 0$, hence $R \geq 2r$. The equality $R - 2r = 0$ holds if and only if $OI^2 = 0$, i.e., $O = I$. Therefore triangle ABC is equilateral. □

4.6.3 The distance ON

Theorem 6. *If N is the Nagel point of triangle ABC, then*

$$ON = R - 2r.$$

Proof. The coordinate of the Nagel point of the triangle is given by

$$z_N = \left(1 - \frac{\alpha}{s}\right)a + \left(1 - \frac{\beta}{s}\right)b + \left(1 - \frac{\gamma}{s}\right)c.$$

Therefore

$$\begin{aligned} ON^2 = |z_N|^2 = z_N \cdot z_N &= R^2 \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)^2 + 2\sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)\left(1 - \frac{\beta}{s}\right) a \cdot b \\ &= R^2 \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)^2 + 2\sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)\left(1 - \frac{\beta}{s}\right)\left(R^2 - \frac{\gamma^2}{2}\right) \\ &= R^2 \left(3 - \frac{\alpha+\beta+\gamma}{s}\right)^2 - \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)\left(1 - \frac{\beta}{s}\right)\gamma^2 \\ &= R^2 - \sum_{\text{cyc}} \left(1 - \frac{\alpha}{s}\right)\left(1 - \frac{\beta}{s}\right)\gamma^2 = R^2 - E. \end{aligned}$$

To calculate E we note that

$$\begin{aligned} E &= \sum_{\text{cyc}} \left(1 - \frac{\alpha+\beta}{s} + \frac{\alpha\beta}{s^2}\right)\gamma^2 = \sum_{\text{cyc}} \gamma^2 - \frac{1}{s}\sum_{\text{cyc}} (\alpha+\beta)\gamma^2 + \frac{1}{s^2}\sum_{\text{cyc}} \alpha\beta\gamma^2 \\ &= \sum_{\text{cyc}} \gamma^2 - \frac{1}{s}\sum_{\text{cyc}} (2s-\gamma)\gamma^2 + \frac{2\alpha\beta\gamma}{s} = -\sum_{\text{cyc}} \alpha^2 + \frac{1}{s}\sum_{\text{cyc}} \alpha^3 + 8\frac{\alpha\beta\gamma}{4K} \cdot \frac{K}{s} \\ &= -\sum_{\text{cyc}} \alpha^2 + \frac{1}{s}\sum_{\text{cyc}} \alpha^3 + 8Rr. \end{aligned}$$

Applying the formula in Corollary 2, we conclude that

$$E = -2(s^2 - r^2 - 4Rr) + 2(s^2 - 3r^2 - 6Rr) + 8Rr = -4r^2 + 4Rr.$$

Hence $ON^2 = R^2 - E = R^2 - 4Rr + 4r^2 = (R - 2r)^2$ and the desired formula is proved by Euler's inequality. □

Theorem 7. (Feuerbach[5]) *In any triangle the incircle and the nine-point circle of Euler are tangent.*

Proof. Using the configuration in Section 4.5 we observe that

$$\frac{1}{2} = \frac{GI}{GN} = \frac{GO_9}{GO}.$$

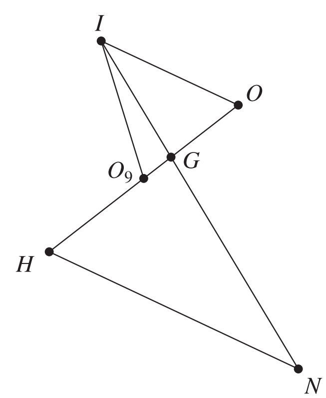

Figure 4.7.

Therefore triangles GIO_9 and GNO are similar. It follows that the lines IO_9 and ON are parallel and $IO_9 = \frac{1}{2}ON$. Applying Theorem 6 we get $IO_9 = \frac{1}{2}(R - 2r) = \frac{R}{2} - r = R_9 - r$, hence the incircle is tangent to the nine-point circle. □

The point of tangency of these two circles is denoted by φ and is called the *Feuerbach point* of triangle.

4.6.4 The distance *OH*

Theorem 8. *If H is the orthocenter of triangle ABC, then*

$$OH^2 = 9R^2 + 2r^2 + 8Rr - 2s^2.$$

Proof. Assuming that the circumcenter O is the origin of the complex plane, the coordinate of H is

$$z_H = a + b + c.$$

[5]Karl Wilhelm Feuerbach (1800–1834), German geometer, published the result of Theorem 7 in 1822.

Using the real product we can write

$$OH^2 = |z_H|^2 = z_H \cdot z_H = (a+b+c)\cdot(a+b+c)$$
$$= \sum_{\text{cyc}} |a|^2 + 2\sum ab = 3R^2 + 2\sum_{\text{cyc}} a\cdot b.$$

Applying the formulas in the lemma (p. 112) and then the first formula in Corollary 2, we obtain

$$OH^2 = 3R^2 + 2\sum_{\text{cyc}}\left(R^2 - \frac{\gamma^2}{2}\right) = 9R^2 - (\alpha^2+\beta^2+\gamma^2)$$
$$= 9R^2 - 2(s^2 - r^2 - 4Rr) = 9R^2 + 2r^2 + 8Rr - 2s^2. \qquad \square$$

Corollary 9. *The following formulas hold:*

1) $OG^2 = R^2 + \frac{2}{9}r^2 + \frac{8}{9}Rr - \frac{2}{9}s^2$*;*

2) $OO_9^2 = \frac{9}{4}R^2 + \frac{1}{2}r^2 + 2Rr - \frac{1}{2}s^2$.

Corollary 10. *In any triangle ABC the inequality*

$$\alpha^2 + \beta^2 + \gamma^2 \le 9R^2$$

is true. Equality holds if and only if the triangle is equilateral.

4.7 Distance between Two Points in the Plane of a Triangle

4.7.1 Barycentric coordinates

Consider a triangle ABC and let α, β, γ be the lengths of sides BC, CA, AB, respectively.

Proposition 1. *Let a, b, c be the coordinates of vertices A, B, C and let P be a point in the plane of triangle. If z_P is the coordinate of P, then there exist unique real numbers μ_a, μ_b, μ_c such that*

$$z_P = \mu_a a + \mu_b b + \mu_c c \text{ and } \mu_a + \mu_b + \mu_c = 1.$$

Proof. Assume that P is in the interior of triangle ABC and consider the point A' such that $AP \cap BC = \{A'\}$. Let $k_1 = \frac{PA}{PA'}$, $k_2 = \frac{A'B}{A'C}$ and observe that

$$z_P = \frac{a + k_1 z_{A'}}{1+k_1}, \quad z_{A'} = \frac{b + k_2 c}{1+k_2}.$$

Hence in this case we can write

$$z_P = \frac{1}{1+k_1}a + \frac{k_1}{(1+k_1)(1+k_2)}b + \frac{k_1k_2}{(1+k_1)(1+k_2)}c.$$

Moreover, if we consider

$$\mu_a = \frac{1}{1+k_1}, \quad \mu_b = \frac{k_1}{(1+k_1)(1+k_2)}, \quad \mu_c = \frac{k_1k_2}{(1+k_1)(1+k_2)}$$

we have

$$\mu_a + \mu_b + \mu_c = \frac{1}{1+k_1} + \frac{k_1}{(1+k_1)(1+k_2)} + \frac{k_1k_2}{(1+k_1)(1+k_2)}$$
$$= \frac{1+k_1+k_2+k_1k_2}{(1+k_1)(1+k_2)} = 1.$$

We proceed in an analogous way in the case when the point P is situated in the exterior of triangle ABC.

If the point P is situated on the support line of a side of triangle ABC (i.e., the line determined by two vertices)

$$z_P = \frac{1}{1+k}b + \frac{k}{1+k}c = 0 \cdot a + \frac{1}{1+k}b + \frac{k}{1+k}c,$$

where $k = \dfrac{PB}{PC}$. □

The real numbers μ_a, μ_b, μ_c are called the *absolute barycentric coordinates* of P with respect to the triangle ABC.

The signs of numbers μ_a, μ_b, μ_c depend on the regions of the plane where the point P is situated. Triangle ABC determines seven such regions.

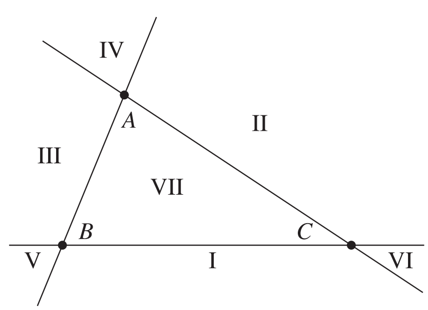

Figure 4.8.

In the next table we give the signs of μ_a, μ_b, μ_c:

	I	II	III	IV	V	VI	VII
μ_a	$-$	$+$	$+$	$+$	$-$	$-$	$+$
μ_b	$+$	$-$	$+$	$-$	$+$	$-$	$+$
μ_c	$+$	$+$	$-$	$-$	$-$	$+$	$+$

4.7.2 Distance between two points in barycentric coordinates

In what follows, in order to simplify the formulas, we will use the symbol called "cyclic sum." That is, $\sum_{\text{cyc}} f(x_1, x_2, \dots, x_n)$, the sum of terms considered in the cyclic order. The most important example for our purposes is

$$\sum_{\text{cyc}} f(x_1, x_2, x_3) = f(x_1, x_2, x_3) + f(x_2, x_3, x_1) + f(x_3, x_1, x_2).$$

Theorem 2. *In the plane of triangle ABC consider the points P_1 and P_2 with coordinates z_{P_1} and z_{P_2}, respectively. If $z_{P_k} = \alpha_k a + \beta_k b + \gamma_k c$, where $\alpha_k, \beta_k, \gamma_k$ are real numbers such that $\alpha_k + \beta_k + \gamma_k = 1$, $k = 1, 2$, then*

$$P_1P_2^2 = -\sum_{\text{cyc}} (\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\gamma^2.$$

Proof. Choose the origin of the complex plane at the circumcenter O of the triangle ABC. Using properties of the real product, we have

$$\begin{aligned} P_1P_2^2 &= |z_{P_2} - z_{P_1}|^2 = |(\alpha_2 - \alpha_1)a + (\beta_2 - \beta_1)b + (\gamma_2 - \gamma_1)c|^2 \\ &= \sum_{\text{cyc}} (\alpha_2 - \alpha_1)^2 a \cdot a + 2\sum_{\text{cyc}} (\alpha_2 - \alpha_1)(\beta_2 - \beta_1) a \cdot b \\ &= \sum_{\text{cyc}} (\alpha_2 - \alpha_1)^2 R^2 + 2\sum_{\text{cyc}} (\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\left(R^2 - \frac{\gamma^2}{2}\right) \\ &= R^2(\alpha_2 + \beta_2 + \gamma_2 - \alpha_1 - \beta_1 - \gamma_1)^2 - \sum_{\text{cyc}} (\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\gamma^2 \\ &= -\sum_{\text{cyc}} (\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\gamma^2, \end{aligned}$$

since $\alpha_1 + \beta_1 + \gamma_1 = \alpha_2 + \beta_2 + \gamma_2 = 1$. □

Theorem 3. *The points $A_1, A_2, B_1, B_2, C_1, C_2$ are situated on the sides BC, CA, AB of triangle ABC such that lines AA_1, BB_1, CC_1 meet at point P_1 and lines AA_2, BB_2, CC_2 meet at point P_2. If*

$$\frac{BA_k}{A_kC} = \frac{p_k}{n_k}, \quad \frac{CB_k}{B_kA} = \frac{m_k}{p_k}, \quad \frac{AC_k}{C_kB} = \frac{n_k}{m_k}, \quad k = 1, 2$$

where m_k, n_k, p_k are nonzero real numbers, $k = 1, 2$, and $S_k = m_k + n_k + p_k$, $k = 1, 2$, then

$$P_1P_2^2 = \frac{1}{S_1^2S_2^2}\left[S_1S_2\sum_{\text{cyc}}(n_1p_2 + p_1n_2)\alpha^2 - S_1^2\sum_{\text{cyc}} n_2p_2\alpha^2 - S_2^2\sum_{\text{cyc}} n_1p_1\alpha^2\right].$$

Proof. The coordinates of points P_1 and P_2 are

$$z_{P_k} = \frac{m_ka + n_kb + p_kc}{m_k + n_k + p_k}, \quad k = 1, 2.$$

It follows that in this case the absolute barycentric coordinates of points P_1 and P_2 are given by

$$\alpha_k = \frac{m_k}{m_k + n_k + p_k} = \frac{m_k}{S_k}, \quad \beta_k = \frac{n_k}{m_k + n_k + p_k} = \frac{n_k}{S_k},$$

$$\gamma_k = \frac{p_k}{m_k + n_k + p_k} = \frac{p_k}{S_k}, \quad k = 1, 2.$$

Substituting in the formula in Theorem 2 we find

$$\begin{aligned}
P_1P_2^2 &= -\sum_{\text{cyc}}\left(\frac{n_2}{S_2} - \frac{n_1}{S_1}\right)\left(\frac{p_2}{S_2} - \frac{p_1}{S_1}\right)\alpha^2 \\
&= -\frac{1}{S_1^2S_2^2}\sum_{\text{cyc}}(S_1n_2 - S_2n_1)(S_1p_2 - S_2p_1)\alpha^2 \\
&= -\frac{1}{S_1^2S_2^2}\sum_{\text{cyc}}[S_1^2n_2p_2 + S_2^2n_1p_1 - S_1S_2(n_1p_2 + n_2p_1)]\alpha^2 \\
&= \frac{1}{S_1^2S_2^2}\left[S_1S_2\sum_{\text{cyc}}(n_1p_2 + p_1n_2)\alpha^2 - S_1^2\sum_{\text{cyc}} n_2p_2\alpha^2 - S_2^2\sum_{\text{cyc}} n_1p_1\alpha^2\right]
\end{aligned}$$

and the desired formula follows. □

Corollary 4. *For any real numbers $\alpha_k, \beta_k, \gamma_k$ with $\alpha_k + \beta_k + \gamma_k = 1$, $k = 1, 2$, the following inequality holds:*

$$\sum_{\text{cyc}}(\alpha_2 - \alpha_1)(\beta_2 - \beta_1)\gamma^2 \leq 0,$$

with equality if and only if $\alpha_1 = \alpha_2$, $\beta_1 = \beta_2$, $\gamma_1 = \gamma_2$.

Corollary 5. *For any nonzero real numbers m_k, n_k, p_k, $k = 1, 2$, with $S_k = m_k + n_k + p_k$, $k = 1, 2$, the lengths of sides α, β, γ of triangle ABC satisfy the inequality*

$$\sum_{\text{cyc}}(n_1p_2 + p_1n_2)^2 \geq \frac{S_1}{S_2}\sum_{\text{cyc}} n_2p_2\alpha^2 + \frac{S_2}{S_1}\sum_{\text{cyc}} n_1p_1\alpha^2$$

with equality if and only if $\frac{p_1}{n_1} = \frac{p_2}{n_2}$, $\frac{m_1}{p_1} = \frac{m_2}{p_2}$, $\frac{n_1}{m_1} = \frac{n_2}{m_2}$.

Applications. 1) Let us use the formula in Theorem 3 to compute the distance GI, where G is the centroid and I is the incenter of the triangle.

We have $m_1 = n_1 = p_1 = 1$ and $m_2 = \alpha$, $n_2 = \beta$, $p_2 = \gamma$; hence

$$S_1 = \sum_{\text{cyc}} m_1 = 3; \quad S_2 = \sum_{\text{cyc}} m_2 = \alpha + \beta + \gamma = 2s;$$

$$\sum_{\text{cyc}} (n_1 p_2 + n_2 p_1)\alpha^2 = (\beta + \gamma)\alpha^2 + (\gamma + \alpha)\beta^2 + (\alpha + \beta)\gamma^2$$
$$= (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha) - 3\alpha\beta\gamma = 2s(s^2 + r^2 + 4rR) - 12sRr$$
$$= 2s^3 + 2sr^2 - 4sRr.$$

On the other hand,

$$\sum_{\text{cyc}} n_2 p_2 \alpha^2 = \alpha^2\beta\gamma + \beta^2\gamma\alpha + \gamma^2\alpha\beta = \alpha\beta\gamma(\alpha + \beta + \gamma) = 8s^2 Rr$$

and

$$\sum_{\text{cyc}} n_1 p_1 \alpha^2 = \alpha^2 + \beta^2 + \gamma^2 = 2s^2 - 2r^2 - 8Rr.$$

Then

$$GI^2 = \frac{1}{9}(s^2 + 5r^2 - 16Rr).$$

2) Let us prove that in any triangle ABC with sides α, β, γ, the following inequality holds:

$$\sum_{\text{cyc}} (2\alpha - \beta - \gamma)(2\beta - \alpha - \gamma)\gamma^2 \le 0.$$

In the inequality in Corollary 4 we consider the points $P_1 = G$ and $P_2 = I$. Then $\alpha_1 = \beta_1 = \gamma_1 = \frac{1}{3}$ and $\alpha_2 = \frac{\alpha}{2s}$, $\beta_2 = \frac{\beta}{2s}$, $\gamma_2 = \frac{\gamma}{2s}$, and the above inequality follows. We have equality if and only if $P_1 = P_2$; that is, $G = I$, so the triangle is equilateral.

4.8 The Area of a Triangle in Barycentric Coordinates

Consider the triangle ABC with a, b, c the coordinates of its vertices, respectively. Let α, β, γ be the lengths of sides BC, CA and AB.

Theorem 1. *Let $P_j(z_{P_j})$, $j = 1, 2, 3$, be three points in the plane of triangle ABC with $z_{P_j} = \alpha_j a + \beta_j b + \gamma_j c$, where α_j, β_j, γ_j are the barycentric coordinates of P_j. If the triangles ABC and $P_1P_2P_3$ have the same orientation, then*

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]} = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

Proof. Suppose that the triangles ABC and $P_1P_2P_3$ are positively oriented. If O denotes the origin of the complex plane, then using the complex product we can write

$$2i\ \text{area}[P_1OP_2] = z_{P_1} \times z_{P_2} = (\alpha_1 a + \beta_1 b + \gamma_1 c) \times (\alpha_2 a + \beta_2 b + \gamma_2 c)$$

$$= (\alpha_1\beta_2 - \alpha_2\beta_1)a \times b + (\beta_1\gamma_2 - \beta_2\gamma_1)b \times c + (\gamma_1\alpha_2 - \gamma_2\alpha_1)c \times a$$

$$= \begin{vmatrix} a \times b & b \times c & c \times a \\ \gamma_1 & \alpha_1 & \beta_1 \\ \gamma_2 & \alpha_2 & \beta_2 \end{vmatrix} = \begin{vmatrix} a \times b & b \times c & 2i\ \text{area}[ABC] \\ \gamma_1 & \alpha_1 & 1 \\ \gamma_2 & \alpha_2 & 1 \end{vmatrix}.$$

Analogously, we find

$$2i\ \text{area}[P_2OP_3] = \begin{vmatrix} a \times b & b \times c & 2i\ \text{area}[ABC] \\ \gamma_2 & \alpha_2 & 1 \\ \gamma_3 & \alpha_3 & 1 \end{vmatrix},$$

$$2i\ \text{area}[P_3OP_1] = \begin{vmatrix} a \times b & b \times c & 2i\ \text{area}[ABC] \\ \gamma_3 & \alpha_3 & 1 \\ \gamma_1 & \alpha_1 & 1 \end{vmatrix}.$$

Assuming that the origin O is situated in the interior of triangle $P_1P_2P_3$, it follows that

$$\text{area}[P_1P_2P_3] = \text{area}[P_1OP_2] + \text{area}[P_2OP_3] + \text{area}[P_3OP_1]$$

$$= \frac{1}{2i}(\alpha_1 - \alpha_2 + \alpha_2 - \alpha_3 + \alpha_3 - \alpha_1)a \times b - \frac{1}{2i}(\gamma_1 - \gamma_2 + \gamma_2 - \gamma_3 + \gamma_3 - \gamma_1)b \times c$$

$$+ (\gamma_1\alpha_2 - \gamma_2\alpha_1 + \gamma_2\alpha_3 - \gamma_3\alpha_2 + \gamma_3\alpha_1 - \gamma_1\alpha_3)\text{area}[ABC]$$

$$= (\gamma_1\alpha_2 - \gamma_2\alpha_1 + \gamma_2\alpha_3 - \gamma_3\alpha_2 + \gamma_3\alpha_1 - \gamma_1\alpha_3)\text{area}[ABC]$$

$$= \text{area}[ABC] \begin{vmatrix} 1 & \gamma_1 & \alpha_1 \\ 1 & \gamma_2 & \alpha_2 \\ 1 & \gamma_3 & \alpha_3 \end{vmatrix} = \text{area}[ABC] \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

and the desired formula is obtained. □

Corollary 2. *Consider the triangle ABC and the points A_1, B_1, C_1 situated on the lines BC, CA, AB, respectively, such that*

$$\frac{A_1B}{A_1C} = k_1, \quad \frac{B_1C}{B_1A} = k_2, \quad \frac{C_1A}{C_1B} = k_3.$$

If $AA_1 \cap BB_1 = \{P_1\}$, $BB_1 \cap CC_1 = \{P_2\}$ and $CC_1 \cap AA_1 = \{P_3\}$, then

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]} = \frac{(1 - k_1k_2k_3)^2}{(1 + k_1 + k_1k_2)(1 + k_2 + k_2k_3)(1 + k_3 + k_3k_1)}.$$

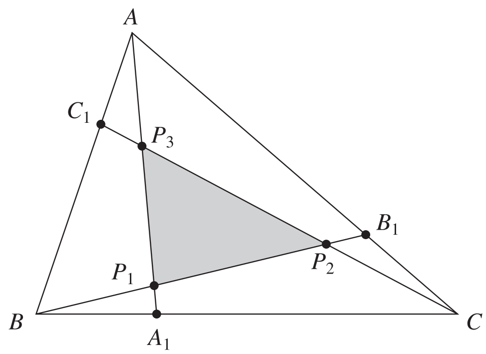

Figure 4.9.

Proof. Applying Menelaus's well-known theorem in triangle AA_1B we find that

$$\frac{C_1A}{C_1B} \cdot \frac{CB}{CA_1} \cdot \frac{P_3A_1}{P_3A} = 1.$$

Hence

$$\frac{P_3A}{P_3A_1} = \frac{C_1A}{C_1B} \cdot \frac{CB}{CA_1} = k_3(1+k_1).$$

The coordinate of P_3 is given by

$$z_{P_3} = \frac{a + k_3(1+k_1)z_{A_1}}{1 + k_3(1+k_1)} = \frac{a + k_3(1+k_1)\dfrac{b+k_1c}{1+k_1}}{1+k_3+k_3k_1} = \frac{a + k_3b + k_3k_1c}{1+k_3+k_3k_1}.$$

In an analogous way we find that

$$z_{P_1} = \frac{k_1k_2a + b + k_1c}{1+k_1+k_1k_2} \quad \text{and} \quad z_{P_2} = \frac{k_2a + k_2k_3b + c}{1+k_2+k_2k_3}.$$

The triangles ABC and $P_1P_2P_3$ have the same orientation; hence by applying the formula in Theorem 1 we find that

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]}$$

$$= \frac{1}{(1+k_1+k_1k_2)(1+k_2+k_2k_3)(1+k_3+k_3k_1)} \begin{vmatrix} k_1k_2 & 1 & k_1 \\ k_2 & k_2k_3 & 1 \\ 1 & k_3 & k_3k_1 \end{vmatrix}$$

$$= \frac{(1-k_1k_2k_3)^2}{(1+k_1+k_1k_2)(1+k_2+k_2k_3)(1+k_3+k_3k_1)}. \qquad \square$$

Remark. When $k_1 = k_2 = k_3 = k$, from Corollary 2 we obtain Problem 3 from the 23^{rd} Putnam Mathematical Competition.

Let A_j, B_j, C_j be points on the lines BC, CA, AB, respectively, such that

$$\frac{BA_j}{A_jC} = \frac{p_j}{n_j}, \quad \frac{CB_j}{B_jA} = \frac{m_j}{p_j}, \quad \frac{AC_j}{C_jB} = \frac{n_j}{m_j}, \quad j = 1, 2, 3.$$

Corollary 3. *If P_j is the intersection point of lines AA_j, BB_j, CC_j, $j = 1, 2, 3$, and the triangles ABC, $P_1P_2P_3$ have the same orientation, then*

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]} = \frac{1}{S_1S_2S_3}\begin{vmatrix} m_1 & n_1 & p_1 \\ m_2 & n_2 & p_2 \\ m_3 & n_3 & p_3 \end{vmatrix}$$

where $S_j = m_j + n_j + p_j$, $j = 1, 2, 3$.

Proof. In terms of the coordinates of the triangle, the coordinates of the points P_j are

$$z_{P_j} = \frac{m_ja + n_jb + p_jc}{m_j + n_j + p_j} = \frac{1}{S_j}(m_ja + n_jb + p_jc), \quad j = 1, 2, 3.$$

The formula above follows directly from Theorem 1. □

Corollary 4. *In triangle ABC let us consider the cevians AA', BB' and CC' such that*

$$\frac{A'B}{A'C} = m, \quad \frac{B'C}{B'A} = n, \quad \frac{C'A}{C'B} = p.$$

Then the following formula holds:

$$\frac{\text{area}[A'B'C']}{\text{area}[ABC]} = \frac{1 + mnp}{(1+m)(1+n)(1+p)}.$$

Proof. Observe that the coordinates of A', B', C' are given by

$$z_{A'} = \frac{1}{1+m}b + \frac{m}{1+m}c, \quad z_{B'} = \frac{1}{1+n}c + \frac{n}{1+n}a, \quad z_{C'} = \frac{1}{1+p}a + \frac{p}{1+p}b.$$

Applying the formula in Corollary 3 we obtain

$$\frac{\text{area}[A'B'C']}{\text{area}[ABC]} = \frac{1}{(1+m)(1+n)(1+p)}\begin{vmatrix} 0 & 1 & m \\ n & 0 & 1 \\ 1 & p & 0 \end{vmatrix}$$

$$= \frac{1 + mnp}{(1+m)(1+n)(1+p)}.$$ □

Applications. 1) (Steinhaus[6]) Let A_j, B_j, C_j be points on lines BC, CA, AB, respectively, $j = 1, 2, 3$. Assume that

$$\frac{BA_1}{A_1C} = \frac{2}{4}, \quad \frac{CB_1}{B_1A} = \frac{1}{2}, \quad \frac{AC_1}{C_1B} = \frac{4}{1};$$

[6]Hugo Dyonizy Steinhaus (1887–1972), Polish mathematician, made important contributions in functional analysis and other branches of modern mathematics.

$$\frac{BA_2}{A_2C} = \frac{4}{1}, \quad \frac{CB_2}{B_2A} = \frac{2}{4}, \quad \frac{AC_2}{C_2B} = \frac{1}{2};$$
$$\frac{BA_3}{A_3C} = \frac{1}{2}, \quad \frac{CB_3}{B_3A} = \frac{4}{1}, \quad \frac{AC_3}{C_3B} = \frac{2}{4}.$$

If P_j is the intersection point of lines AA_j, BB_j, CC_j, $j = 1, 2, 3$, and triangles ABC, $P_1P_2P_3$ are of the same orientation, then from Corollary 3 we obtain

$$\frac{\text{area}[P_1P_2P_3]}{\text{area}[ABC]} = \frac{1}{7 \cdot 7 \cdot 7} \begin{vmatrix} 1 & 4 & 2 \\ 2 & 1 & 4 \\ 4 & 2 & 1 \end{vmatrix} = \frac{49}{7^3} = \frac{1}{7}.$$

2) If the cevians AA', BB', CC' are concurrent at point P, let us denote by K_P the area of triangle $A'B'C'$. We can use the formula in Corollary 4 to compute the areas of some triangles determined by the feet of the cevians of some remarkable points in a triangle.

(i) If I is the incenter of triangle ABC we have

$$K_I = \frac{1 + \frac{\gamma}{\beta} \cdot \frac{\beta}{\alpha} \cdot \frac{\alpha}{\gamma}}{\left(1 + \frac{\gamma}{\beta}\right)\left(1 + \frac{\beta}{\alpha}\right)\left(1 + \frac{\alpha}{\gamma}\right)} \text{area}[ABC]$$

$$= \frac{2\alpha\beta\gamma}{(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)} \text{area}[ABC] = \frac{2\alpha\beta\gamma sr}{(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)}.$$

(ii) For the orthocenter H of the acute triangle ABC we obtain

$$K_H = \frac{1 + \frac{\tan C}{\tan B} \cdot \frac{\tan B}{\tan A} \cdot \frac{\tan A}{\tan C}}{\left(1 + \frac{\tan C}{\tan B}\right)\left(1 + \frac{\tan B}{\tan A}\right)\left(1 + \frac{\tan A}{\tan C}\right)} \text{area}[ABC]$$

$$= (2 \cos A \cos B \cos C)\text{area}[ABC] = (2 \cos A \cos B \cos C)sr.$$

(iii) For the Nagel point of triangle ABC we can write

$$K_N = \frac{1 + \frac{s-\gamma}{s-\beta} \cdot \frac{s-\alpha}{s-\gamma} \cdot \frac{s-\beta}{s-\alpha}}{\left(1 + \frac{s-\gamma}{s-\beta}\right)\left(1 + \frac{s-\alpha}{s-\gamma}\right)\left(1 + \frac{s-\beta}{s-\alpha}\right)} \text{area}[ABC]$$

$$= \frac{2(s-\alpha)(s-\beta)(s-\gamma)}{\alpha\beta\gamma} \text{area}[ABC] = \frac{4\text{area}^2[ABC]}{2s\alpha\beta\gamma} \text{area}[ABC]$$

$$= \frac{r}{2R} \text{area}[ABC] = \frac{sr^2}{2R}.$$

If we proceed in the same way for the Gergonne point J we find the relation

$$K_J = \frac{r}{2R}\text{area}[ABC] = \frac{sr^2}{2R}.$$

Remark. Two cevians AA' and AA'' are *isotomic* if the points A' and A'' are symmetric with respect to the midpoint of the segment BC. Assuming that

$$\frac{A'B}{A'C} = m, \quad \frac{B'C}{B'A} = n, \quad \frac{C'A}{C'B} = p,$$

then for the corresponding isotomic cevians we have

$$\frac{A''B}{A''C} = \frac{1}{m}, \quad \frac{B''C}{B''A} = \frac{1}{n}, \quad \frac{C''A}{C''B} = \frac{1}{p}.$$

Applying the formula in Corollary 4, it follows that

$$\frac{\text{area}[A'B'C']}{\text{area}[ABC]} = \frac{1+mnp}{(1+m)(1+n)(1+p)}$$

$$= \frac{1+\dfrac{1}{mnp}}{\left(1+\dfrac{1}{m}\right)\left(1+\dfrac{1}{n}\right)\left(1+\dfrac{1}{p}\right)} = \frac{\text{area}[A''B''C'']}{\text{area}[ABC]}.$$

Therefore $\text{area}[A'B'C'] = \text{area}[A''B''C'']$. A special case of this relation is $K_N = K_J$, since the points N and J are isotomic (i.e., these points are intersections of isotomic cevians).

3) Consider the excenters $I_\alpha, I_\beta, I_\gamma$ of triangle ABC. It is not difficult to see that the coordinates of these points are

$$z_{I_\alpha} = -\frac{\alpha}{2(s-\alpha)}a + \frac{\beta}{2(s-\beta)}b + \frac{\gamma}{2(s-\gamma)}c,$$

$$z_{I_\beta} = \frac{\alpha}{2(s-\alpha)}a - \frac{\beta}{2(s-\beta)}b + \frac{\gamma}{2(s-\gamma)}c,$$

$$z_{I_\gamma} = \frac{\alpha}{2(s-\alpha)}a + \frac{\beta}{2(s-\beta)}b - \frac{\gamma}{2(s-\gamma)}c.$$

From the formula in Theorem 1, it follows that

$$\text{area}[I_\alpha I_\beta I_\gamma] = \begin{vmatrix} -\dfrac{\alpha}{2(s-\alpha)} & \dfrac{\beta}{2(s-\beta)} & \dfrac{\gamma}{2(s-\gamma)} \\ \dfrac{\alpha}{2(s-\alpha)} & -\dfrac{\beta}{2(s-\beta)} & \dfrac{\gamma}{2(s-\gamma)} \\ \dfrac{\alpha}{2(s-\alpha)} & \dfrac{\beta}{2(s-\beta)} & -\dfrac{\gamma}{2(s-\gamma)} \end{vmatrix} \text{area}[ABC]$$

$$= \frac{\alpha\beta\gamma}{8(s-\alpha)(s-\beta)(s-\gamma)} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \text{area}[ABC]$$

$$= \frac{s\alpha\beta\gamma \text{ area}[ABC]}{2s(s-\alpha)(s-\beta)(s-\gamma)} = \frac{s\alpha\beta\gamma \text{ area}[ABC]}{2 \text{ area}^2[ABC]} = \frac{2s\alpha\beta\gamma}{4 \text{ area}[ABC]} = 2sR.$$

4) (Nagel line.) Using the formula in Theorem 1, we give a different proof for the so-called Nagel line: the points I, G, N are collinear. We have seen that the coordinates of these points are

$$z_I = \frac{\alpha}{2s}a + \frac{\beta}{2s}b + \frac{\gamma}{2s}c,$$

$$z_G = \frac{1}{3}a + \frac{1}{3}b + \frac{1}{3}c,$$

$$z_N = \left(1 - \frac{\alpha}{s}\right)a + \left(1 - \frac{\beta}{s}\right)b + \left(1 - \frac{\gamma}{s}\right)c.$$

Then

$$\text{area}[IGN] = \begin{vmatrix} \frac{\alpha}{2s} & \frac{\beta}{2s} & \frac{\gamma}{2s} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 - \frac{\alpha}{s} & 1 - \frac{\beta}{s} & 1 - \frac{\gamma}{s} \end{vmatrix} \cdot \text{area}[ABC] = 0,$$

hence the points I, G, N are collinear.

4.9 Orthopolar Triangles

4.9.1 The Simson–Wallance line and the pedal triangle

Consider the triangle ABC, and let M be a point situated in the triangle plane. Let P, Q, R be the projections of M onto lines BC, CA, AB, respectively.

Theorem 1. (The Simson[7] line[8]) *The points P, Q, R are collinear if and only if M is on the circumcircle of triangle ABC.*

---

[7] Robert Simson (1687–1768), Scottish mathematician.

[8] This line was attributed to Simson by Poncelet, but is now frequently known as the Simson–Wallance line since it does not actually appear in any work of Simson. William Wallance (1768–1843) was also a Scottish mathematician, who possibly published the theorem above concerning the Simson line in 1799.

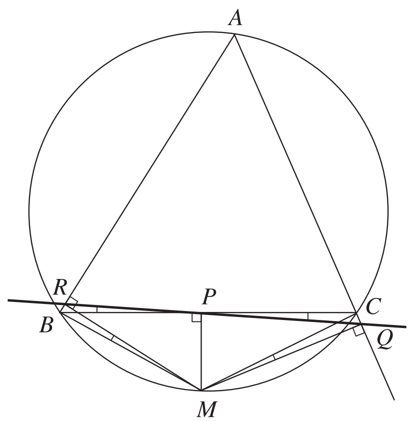

Figure 4.10.

Proof. We will give a standard geometric argument.

Suppose that M lies on the circumcircle of triangle ABC. Without loss of generality we may assume that M is on the arc $\overset{\frown}{BC}$. In order to prove the collinearity of R, P, Q, it suffices to show that the angles $\widehat{BPR}$ and $\widehat{CPQ}$ are congruent. The quadrilaterals $PRBM$ and $PCQM$ are cyclic (since $\widehat{BRM} \equiv \widehat{BPM}$ and $\widehat{MPC} + \widehat{MQC} = 180^\circ$), hence we have $\widehat{BPR} \equiv \widehat{BMR}$ and $\widehat{CPQ} \equiv \widehat{CMQ}$. But $\widehat{BMR} = 90^\circ - \widehat{ABM} = 90^\circ - \widehat{MCQ}$, since the quadrilateral $ABMC$ is cyclic too. Finally, we obtain $\widehat{BMR} = 90^\circ - \widehat{MCQ} = \widehat{CMQ}$, so the angles $\widehat{BPR}$ and $\widehat{CPQ}$ are congruent.

To prove the converse, we note that if the points P, Q, R are collinear, then the angles $\widehat{BPR}$ and $\widehat{CPQ}$ are congruent, hence $\widehat{ABM} + \widehat{ACM} = 180^\circ$, i.e., the quadrilateral $ABMC$ is cyclic. Therefore the point M is situated on the circumcircle of triangles ABC. □

When M lies on the circumcircle of triangle ABC, the line in the above theorem is called the *Simson–Wallance line of M* with respect to triangle ABC.

We continue with a nice generalization of the property contained in Theorem 1. For an arbitrary point X in the plane of triangle ABC consider its projections P, Q and R on the lines BC, CA and AB, respectively.

The triangle *PQR* is called the *pedal triangle* of point X with respect to the triangle ABC. Let us choose the circumcenter O of triangle ABC as the origin of the complex plane.

Theorem 2. *The area of the pedal triangle of X with respect to the triangle ABC is given by*

$$\text{area}[PQR] = \frac{\text{area}[ABC]}{4R^2}|x\overline{x} - R^2| \tag{1}$$

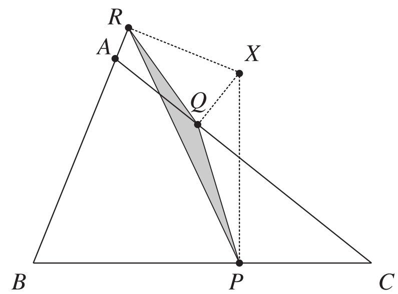

Figure 4.11.

where R is the circumradius of triangle ABC.

Proof. Applying the formula in Proposition 1, Section 4.5, we obtain the coordinates p, q, r of the points P, Q, R, respectively:

$$p = \frac{1}{2}\left(x - \frac{bc}{R^2}\overline{x} + b + c\right),$$

$$q = \frac{1}{2}\left(x - \frac{ca}{R^2}\overline{x} + c + a\right),$$

$$r = \frac{1}{2}\left(x - \frac{ab}{R^2}\overline{x} + a + b\right).$$

Taking into account the formula in Section 2.5.3 we have

$$\text{area}[PQR] = \frac{i}{4}\begin{vmatrix} p & \overline{p} & 1 \\ q & \overline{q} & 1 \\ r & \overline{r} & 1 \end{vmatrix} = \frac{i}{4}\begin{vmatrix} q-p & \overline{q}-\overline{p} \\ r-p & \overline{r}-\overline{p} \end{vmatrix}.$$

For the coordinates p, q, r we obtain

$$\overline{p} = \frac{1}{2}\left(\overline{x} - \frac{\overline{b}\,\overline{c}}{R^2}x + \overline{b} + \overline{c}\right),$$

$$\overline{q} = \frac{1}{2}\left(\overline{x} - \frac{\overline{c}\,\overline{a}}{R^2}x + \overline{c} + \overline{a}\right),$$

$$\overline{r} = \frac{1}{2}\left(\overline{x} - \frac{\overline{a}\,\overline{b}}{R^2}x + \overline{a} + \overline{b}\right).$$

It follows that

$$q - p = \frac{1}{2}(a-b)\left(1 - \frac{c\overline{x}}{R^2}\right) \text{ and } r - p = \frac{1}{2}(a-c)\left(1 - \frac{b\overline{x}}{R^2}\right), \tag{2}$$

$$\overline{q} - \overline{p} = \frac{1}{2abc}(a-b)(x-c)R^2 \quad \text{and} \quad \overline{r} - \overline{p} = \frac{1}{2abc}(a-c)(x-b)R^2.$$

Therefore

$$\begin{aligned}
\text{area}[PQR] &= \frac{i}{4}\begin{vmatrix} q-p & \overline{q}-\overline{p} \\ r-p & \overline{r}-\overline{p} \end{vmatrix} \\
&= \frac{i(a-b)(a-c)}{16abc}\begin{vmatrix} 1-\dfrac{c\overline{x}}{R^2} & (x-c)R^2 \\ & \\ 1-\dfrac{b\overline{x}}{R^2} & (x-b)R^2 \end{vmatrix} \\
&= \frac{i(a-b)(a-c)}{16abc}\begin{vmatrix} R^2-c\overline{x} & x-c \\ R^2-b\overline{x} & x-b \end{vmatrix} \\
&= \frac{i(a-b)(a-c)}{16abc}\begin{vmatrix} (b-c)\overline{x} & b-c \\ R^2-b\overline{x} & x-b \end{vmatrix} \\
&= \frac{i(a-b)(b-c)(a-c)}{16abc}\begin{vmatrix} \overline{x} & 1 \\ R^2-b\overline{x} & x-b \end{vmatrix} \\
&= \frac{i(a-b)(b-c)(a-c)}{16abc}(x\overline{x}-R^2).
\end{aligned}$$

Proceeding to moduli we find that

$$\begin{aligned}
\text{area}[PQR] &= \frac{|a-b||b-c||c-a|}{16|a||b||c|}|x\overline{x}-R^2| = \frac{\alpha\beta\gamma}{16R^3}|x\overline{x}-R^2| \\
&= \frac{\text{area}[ABC]}{4R^2}|x\overline{x}-R^2|,
\end{aligned}$$

where α, β, γ are the length of sides of triangle ABC. □

Remarks. 1) The formula in Theorem 2 contains the Simson–Wallance line property. Indeed, points P, Q, R are collinear if and only if area$[PQR] = 0$. That is, $|x\overline{x} - R^2| = 0$, i.e., $x\overline{x} = R^2$. It follows that $|x| = R$, so X lies on the circumcircle of triangle ABC.

2) If X lies on a circle of radius R_1 and center O (the circumcenter of triangle ABC), then $x\overline{x} = R_1^2$, and from Theorem 2 we obtain

$$\text{area}[PQR] = \frac{\text{area}[ABC]}{4R^2}|R_1^2 - R^2|.$$

It follows that the area of triangle PQR does not depend on the point X.

The converse is also true. The locus of all points X in the plane of triangle ABC such that area$[PQR] = k$ (constant) is defined by

$$|x\overline{x} - R^2| = \frac{4R^2k}{\text{area}[ABC]}.$$

This is equivalent to

$$|x|^2 = R^2 \pm \frac{4R^2k}{\text{area}[ABC]} = R^2\left(1 \pm \frac{4k}{\text{area}[ABC]}\right).$$

If $k > \frac{1}{4}\text{area}[ABC]$, then the locus is a circle of center O and radius $R_1 = R\sqrt{1 + \frac{4k}{\text{area}[ABC]}}$.

If $k \le \frac{1}{4}\text{area}[ABC]$, then the locus consists of two circles of center O and radii $R\sqrt{1 \pm \frac{4k}{\text{area}[ABC]}}$, one of which degenerated to O when $k = \frac{1}{4}\text{area}[ABC]$.

Theorem 3. *For any point X in the plane of triangle ABC, we can construct a triangle with sides $AX \cdot BC$, $BX \cdot CA$, $CX \cdot AB$. This triangle is then similar to the pedal triangle of point X with respect to the triangle ABC.*

Proof. Let PQR be the pedal triangle of X with respect to triangle ABC. From formula (2) we obtain

$$q - p = \frac{1}{2}(a-b)(x-c)\frac{R^2 - c\overline{x}}{R^2(x-c)}. \tag{3}$$

Proceeding to moduli in (3), it follows that

$$|q - p| = \frac{1}{2R^2}|a-b||x-c|\left|\frac{R^2 - c\overline{x}}{x-c}\right|. \tag{4}$$

On the other hand,

$$\left|\frac{R^2 - c\overline{x}}{x-c}\right|^2 = \frac{R^2 - c\overline{x}}{x-c} \cdot \frac{R^2 - \overline{c}x}{\overline{x} - \overline{c}} = \frac{R^2 - c\overline{x}}{x-c} \cdot \frac{R^2 - \overline{c}x}{\overline{x} - \frac{R^2}{c}}$$

$$= \frac{R^2 - c\overline{x}}{x-c} \cdot \frac{R^2(c-x)}{c\overline{x} - R^2} = R^2,$$

hence from (4) we derive the relation

$$|q - p| = \frac{1}{2R}|a-b||x-c|. \tag{5}$$

Therefore

$$\frac{PQ}{CX \cdot AB} = \frac{QR}{AX \cdot BC} = \frac{RP}{BX \cdot CA} = \frac{1}{2R}, \tag{6}$$

and the conclusion follows. □

Corollary 4. *In the plane of triangle ABC consider the point X and denote by $A'B'C'$ the triangle with sides $AX \cdot BC$, $BX \cdot CA$, $CX \cdot AB$. Then*

$$\text{area}[A'B'C'] = \text{area}[ABC]|x\overline{x} - R^2|. \tag{7}$$

Proof. From formula (6) it follows that $\text{area}[A'B'C'] = 4R^2\text{area}[PQR]$, where PQR is the pedal triangle of X with respect to triangle ABC. Replacing this result in (1), we find the desired formula. □

Corollary 5. (Ptolemy's inequality) *For any quadrilateral $ABCD$ the following inequality holds:*

$$AC \cdot BD \leq AB \cdot CD + BC \cdot AD. \tag{8}$$

Corollary 6. (Ptolemy's theorem) *The convex quadrilateral $ABCD$ is cyclic if and only if*

$$AC \cdot BD = AB \cdot CD + BC \cdot AD. \tag{9}$$

Proof. If the relation (9) holds, then triangle $A'B'C'$ in Corollary 4 is degenerate; i.e., $\text{area}[A'B'C'] = 0$. From formula (7) it follows that $d \cdot \overline{d} = R^2$, where d is the coordinate of D and R is the circumradius of triangle ABC. Hence the point D lies on the circumcircle of triangle ABC.

If quadrilateral $ABCD$ is cyclic, then the pedal triangle of point D with respect to triangle ABC is degenerate. From (6) we obtain the relation (9). □

Corollary 7. (Pompeiu's Theorem[9]) *For any point X in the plane of the equilateral triangle ABC, three segments with lengths XA, XB, XC can be taken as the sides of a triangle.*

Proof. In Theorem 3 we have $BC = CA = AB$ and the desired conclusion follows. □

The triangle in Corollary 7 is called the *Pompeiu triangle* of X with respect to the equilateral triangle ABC. This triangle is degenerate if and only if X lies on the circumcircle of ABC. Using the second part of Theorem 3 we find that Pompeiu's triangle of point X is similar to the pedal triangle of X with respect to triangle ABC and

$$\frac{CX}{PQ} = \frac{AX}{QR} = \frac{BX}{RP} = \frac{2R}{\alpha} = \frac{2\sqrt{3}}{3}. \tag{10}$$

Problem 1. *Let A, B and C be equidistant points on the circumference of a circle of unit radius centered at O, and let X be any point in the circle's interior. Let d_A, d_B, d_C be the distances from X to A, B, C, respectively. Show that there is a triangle with*

[9]Dimitrie Pompeiu (1873–1954), Romanian mathematician, made important contributions in the fields of mathematical analysis, functions of a complex variable, and rational mechanics.

sides d_A, d_B, d_C, and the area of this triangle depends only on the distance from X to O.

(2003 Putnam Mathematical Competition)

Solution. The first assertion is just the property contained in Corollary 7. Taking into account the relations (10), it follows that the area of Pompeiu's triangle of point X is $\frac{2}{3}\text{area}[PQR]$. From Theorem 2 we get that area$[PQR]$ depends only on the distance from P to O, as desired.

Problem 2. *Let X be a point in the plane of the equilateral triangle ABC such that X does not lie on the circumcircle of triangle ABC, and let $XA = u$, $XB = v$, $XC = w$. Express the length side α of triangle ABC in terms of real numbers u, v, w.*

(1978 GDR Mathematical Olympiad)

Solution. The segments $[XA]$, $[XB]$, $[XC]$ are the sides of Pompeiu's triangle of point X with respect to equilateral triangle ABC. Denote this triangle by $A'B'C'$. From relations (10) and from Theorem 2 it follows that

$$\text{area}[A'B'C'] = \left(\frac{2\sqrt{3}}{3}\right)^2 \text{area}[PQR] = \frac{1}{3R^2}\text{area}[ABC]|x \cdot \overline{x} - R^2|$$

$$= \frac{1}{3R^2} \cdot \frac{\alpha^2\sqrt{3}}{4}|x \cdot \overline{x} - R^2| = \frac{\sqrt{3}}{4}|XO^2 - R^2|. \tag{11}$$

On the other hand, using the well-known formula of Hero we obtain, after a few simple computations:

$$\text{area}[A'B'C'] = \frac{1}{4}\sqrt{(u^2+v^2+w^2)^2 - 2(u^4+v^4+w^4)}.$$

Substituting in (11) we find

$$|XO^2 - R^2| = \frac{1}{\sqrt{3}}\sqrt{(u^2+v^2+w^2)^2 - 2(u^4+v^4+w^4)}. \tag{12}$$

Now we consider the following two cases:

Case 1. If X lies in the interior of the circumcircle of triangle ABC, then $XO^2 < R^2$. Using the relation (see also formula (4) in Section 4.11)

$$XO^2 = \frac{1}{3}(u^2+v^2+w^2-3R^2),$$

from (12) we find that

$$2R^2 = \frac{1}{3}(u^2+v^2+w^2) + \frac{1}{\sqrt{3}}\sqrt{(u^2+v^2+w^2)^2 - 2(u^4+v^4+w^4)},$$

hence

$$\alpha^2 = \frac{1}{2}(u^2 + v^2 + w^2) + \frac{\sqrt{3}}{2}\sqrt{(u^2 + v^2 + w^2)^2 - 2(u^4 + v^4 + w^4)}.$$

Case 2. If X lies in the exterior of circumcircle of triangle ABC, then $XO^2 > R^2$ and after some similar computations we find

$$\alpha^2 = \frac{1}{2}(u^2 + v^2 + w^2) - \frac{\sqrt{3}}{2}\sqrt{(u^2 + v^2 + w^2)^2 - 2(u^4 + v^4 + w^4)}.$$

4.9.2 Necessary and sufficient conditions for orthopolarity

Consider a triangle ABC and points X, Y, Z situated on its circumcircle. Triangles ABC and XYZ are called *orthopolar triangles* (or *S-triangles*)[10] if the Simson–Wallance line of point X with respect to triangle ABC is perpendicular (orthogonal) to line YZ.

Let us choose the circumcenter O of triangle ABC at the origin of the complex plane. Points A, B, C, X, Y, Z have the coordinates a, b, c, x, y, z with

$$|a| = |b| = |c| = |x| = |y| = |z| = R,$$

where R is the circumradius of the triangle ABC.

Theorem 3. *Triangles ABC and XYZ are orthopolar triangles if and only if $abc = xyz$.*

Proof. Let P, Q, R be the feet of the orthogonal lines from the point X to the lines BC, CA, AB, respectively.

Points P, Q, R are on the same line; that is, the Simson–Wallance line of point X with respect to triangle ABC.

The coordinates of P, Q, R are denoted by p, q, r, respectively. Using the formula in Proposition 1, Section 4.5, we have

$$p = \frac{1}{2}\left(x - \frac{bc}{R^2}\overline{x} + b + c\right)$$
$$q = \frac{1}{2}\left(x - \frac{ca}{R^2}\overline{x} + c + a\right),$$
$$r = \frac{1}{2}\left(x - \frac{ab}{R^2}\overline{x} + a + b\right).$$

We study two cases.

[10]This definition was given in 1915 by Romanian mathematician Traian Lalescu (1882–1929). He is famous for his book *La géometrie du triangle* published by Librairie Vuibert, Paris, 1937.

Case 1. Point X is not a vertex of triangle ABC.

Then PQ is orthogonal to YZ if and only if $(p-q)\cdot(y-z)=0$. That is,

$$\left[(b-a)\left(1-\frac{c\overline{x}}{R^2}\right)\right]\cdot(y-z)=0$$

or

$$(\overline{b}-\overline{a})(R^2-\overline{c}x)(y-z)+(b-a)(R^2-c\overline{x})(\overline{y}-\overline{z})=0.$$

We obtain

$$\left(\frac{R^2}{b}-\frac{R^2}{a}\right)\left(R^2-\frac{R^2}{c}x\right)(y-z)+(b-a)\left(R^2-c\frac{R^2}{x}\right)\left(\frac{R^2}{y}-\frac{R^2}{z}\right)=0,$$

hence

$$\frac{1}{abc}(a-b)(c-x)(y-z)-\frac{1}{xyz}(a-b)(c-x)(y-z)=0.$$

The last relation is equivalent to

$$(abc-xyz)(a-b)(c-x)(y-z)=0$$

and finally we get $abc=xyz$, as desired.

Case 2. Point X is a vertex of triangle ABC. Without loss of generality, assume that $X=B$.

Then the Simson–Wallance line of point $X=B$ is the orthogonal line from B to AC. It follows that BQ is orthogonal to YZ if and only if lines AC and YZ are parallel. This is equivalent to $ac=yz$. Because $b=x$, we obtain $abc=xyz$, as desired. □

Remark. Due to the symmetry of the relation $abc=xyz$, we observe that the Simson–Wallance line of any vertex of triangle XYZ with respect to ABC is orthogonal to the opposite side of the triangle XYZ. Moreover, the same property holds for the vertices of triangle ABC.

Hence ABC and XYZ are orthopolar triangles if and only if XYZ and ABC are orthopolar triangles. Therefore the orthopolarity relation is symmetric.

Problem 1. *The median and the orthic triangles of a triangle ABC are orthopolar in the nine-point circle.*

Solution. Consider the origin of the complex plane at the circumcenter O of triangle ABC. Let M, N, P be the midpoints of AB, BC, CA and let A', B', C' be the feet of the altitudes of triangles ABC from A, B, C, respectively.

If m, n, p, a', b', c' are coordinates of M, N, P, A', B', C' then we have

$$m=\frac{1}{2}(a+b),\quad n=\frac{1}{2}(b+c),\quad p=\frac{1}{2}(c+a)$$

and

$$a' = \frac{1}{2}\left(a+b+c-\frac{bc}{R^2}\overline{a}\right) = \frac{1}{2}\left(a+b+c-\frac{bc}{a}\right),$$

$$b' = \frac{1}{2}\left(a+b+c-\frac{ca}{b}\right), \quad c' = \frac{1}{2}\left(a+b+c-\frac{ab}{2}\right).$$

The nine-point center O_9 is the midpoint of the segment OH, where $H(a+b+c)$ is the orthocenter of triangle ABC. The coordinate of O_9 is $\omega = \frac{1}{2}(a+b+c)$.

Now observe that

$$(a-\omega)(b-\omega)(c-\omega) = (m-\omega)(n-\omega)(p-\omega) = \frac{1}{8}abc,$$

and the claim is proved.

Problem 2. *The altitudes of triangle ABC meet its circumcircle at points A_1, B_1, C_1, respectively. If A_1', B_1', C_1' are the antipodal points of A_1, B_1, C_1 on the circumcircle ABC, then ABC and $A_1'B_1'C_1'$ are orthopolar triangles.*

Solution. The coordinates of A_1, B_1, C_1 are $-\frac{bc}{a}, -\frac{ca}{b}, -\frac{ab}{c}$, respectively. Indeed, the equation of line AH in terms of the real product is $AH : (z-a)\cdot(b-c) = 0$. It suffices to show that the point with coordinate $-\frac{bc}{a}$ lies both on AH and on the circumcircle of triangle ABC. First, let us note that $\left|-\frac{bc}{a}\right| = \frac{|b||c|}{|a|} = \frac{R\cdot R}{R} = R$, hence this point is situated on the circumcircle of triangle ABC. Now, we show that the complex number $-\frac{bc}{a}$ satisfies the equation of the line AH. This is equivalent to

$$\left(\frac{bc}{a}+a\right)\cdot(b-c) = 0.$$

Using the definition of the real product, this reduces to

$$\left(\frac{\overline{b}\,\overline{c}}{\overline{a}}+\overline{a}\right)(b-c) + \left(\frac{bc}{a}+a\right)(\overline{b}-\overline{c}) = 0$$

or

$$\left(\frac{a\overline{b}\,\overline{c}}{R^2}+\overline{a}\right)(b-c) + \left(\frac{bc}{a}+a\right)\left(\frac{R^2}{b}-\frac{R^2}{c}\right) = 0.$$

Finally, this comes down to

$$(b-c)\left(\frac{a\overline{b}\overline{c}}{R^2}+\overline{a}-\frac{R^2}{a}-\frac{aR^2}{bc}\right) = 0,$$

a relation that is clearly true.

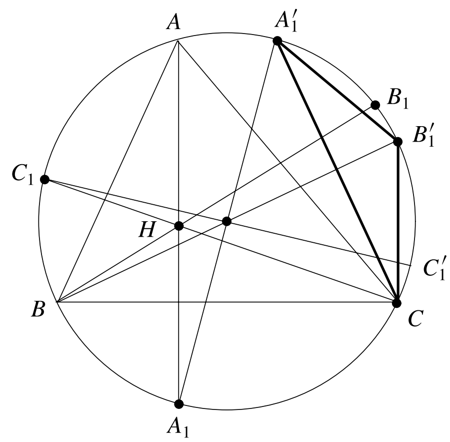

Figure 4.12.

It follows that A_1', B_1', C_1' have coordinates $\frac{bc}{a}$, $\frac{ca}{b}$, $\frac{ab}{c}$, respectively. Because

$$\frac{bc}{a} \cdot \frac{ca}{c} \cdot \frac{ab}{c} = abc,$$

we obtain that the triangles ABC and $A_1'B_1'C_1'$ are orthopolar.

Problem 3. *Let P and P' be distinct points on the circumcircle of triangle ABC such that lines AP and AP' are symmetric with respect to the bisector of angle $\widehat{BAC}$. Then triangles ABC and APP' are orthopolar.*

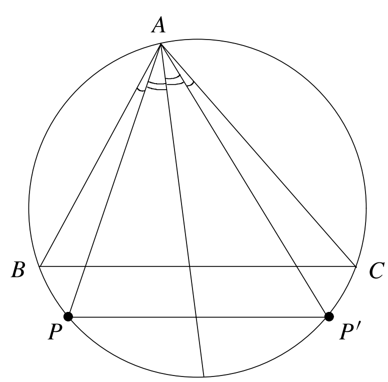

Figure 4.13.

Solution. Let us consider p and p' the coordinates of points P and P', respectively. It is clear that the lines PP' and BC are parallel. Using the complex product, it follows that $(p - p') \times (b - c) = 0$. This relation is equivalent to

$$(p - p')(\overline{b} - \overline{c}) - (\overline{p} - \overline{p'})(b - c) = 0.$$

Considering the origin of the complex plane at the circumcenter O of triangle ABC, we have

$$(p - p')\left(\frac{R^2}{b} - \frac{R^2}{c}\right) - \left(\frac{R^2}{p} - \frac{R^2}{p'}\right)(b - c) = 0,$$

so

$$R^2(p - p')(b - c)\left(\frac{1}{bc} - \frac{1}{pp'}\right) = 0.$$

Therefore $bc = pp'$, i.e., $abc = app'$. From Theorem 3 it follows that ABC and APP' are orthopolar triangles.

4.10 Area of the Antipedal Triangle

Consider a triangle ABC and a point M. The perpendicular lines from A, B, C to MA, MB, MC, respectively, determine a triangle; we call this triangle the *antipedal* triangle of M with respect to ABC.

Recall that M' is the *isogonal point* of M if the pairs of lines AM, AM'; BM, BM'; CM, CM' are isogonal, i.e., the following relations hold: $\widehat{MAC} \equiv \widehat{M'AB}$, $\widehat{MBC} \equiv \widehat{M'BA}$, $\widehat{MCA} \equiv \widehat{M'CB}$.

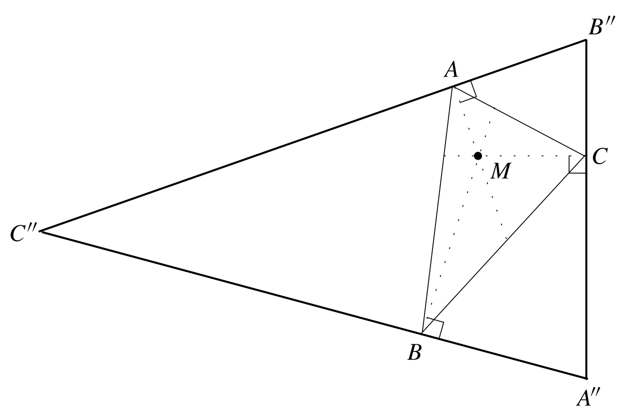

Figure 4.14.

Theorem. *Consider M a point in the plane of triangle ABC, M' the isogonal point of M and $A''B''C''$ the antipedal triangle of M with respect to ABC. Then*

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|R^2 - OM'^2|}{4R^2} = \frac{|\rho(M')|}{4R^2},$$

where $\rho(M')$ is the power of M' with respect to the circumcircle of triangle ABC.

Proof. Consider point O the origin of the complex plane and let m, a, b, c be the coordinates of M, A, B, C. Then

$$R^2 = a\overline{a} = b\overline{b} = c\overline{c} \text{ and } \rho(M) = R^2 - m\overline{m}. \tag{1}$$

Let O_1, O_2, O_3 be the circumcenters of triangles BMC, CMA, AMB, respectively. It is easy to verify that O_1, O_2, O_3 are the midpoints of segments MA'', MB'', MC'', respectively, and so

$$\frac{\text{area}[O_1O_2O_3]}{\text{area}[A''B''C'']} = \frac{1}{4}. \tag{2}$$

The coordinate of the circumcenter of the triangle with vertices with coordinates z_1, z_2, z_3 is given by the following formula (see formula (1) in Subsection 3.6.1):

$$z_O = \frac{z_1\overline{z_1}(z_2 - z_3) + z_2\overline{z_2}(z_3 - z_1) + z_3\overline{z_3}(z_1 - z_2)}{\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix}}.$$

The bisector line of the segment $[z_1, z_2]$ has the following equation in terms of real product: $\left[z - \frac{1}{2}(z_1 + z_2)\right] \cdot (z_1 - z_2) = 0$. It is sufficient to check that z_O satisfies this equation as this implies, by symmetry, that z_O belongs to the perpendicular bisectors of segments $[z_2, z_3]$ and $[z_3, z_1]$.

The coordinate of O_1 is

$$z_{O_1} = \frac{m\overline{m}(b - c) + b\overline{b}(c - m) + c\overline{c}(m - b)}{\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}}$$

$$= \frac{(R^2 - m\overline{m})(c - b)}{\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}} = \frac{\rho(M)(c - b)}{\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}}.$$

Let

$$\Delta = \begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}$$

and consider

$$\alpha = \frac{1}{\Delta}\begin{vmatrix} m & \overline{m} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}, \quad \beta = \frac{1}{\Delta}\begin{vmatrix} m & \overline{m} & 1 \\ c & \overline{c} & 1 \\ a & \overline{a} & 1 \end{vmatrix},$$

and

$$\gamma = \frac{1}{\Delta}\begin{vmatrix} m & \overline{m} & 1 \\ a & \overline{a} & 1 \\ b & \overline{b} & 1 \end{vmatrix}.$$

With this notation we obtain

$$\begin{aligned}&(\alpha a + \beta b + \gamma c)\cdot\Delta\\ &= \sum_{\text{cyc}} m(a\overline{b} - a\overline{c}) - \sum_{\text{cyc}} \overline{m}(ab - ac) + \sum_{\text{cyc}} a(b\overline{c} - \overline{b}c)\\ &= m\Delta - \overline{m}\cdot 0 + \sum_{\text{cyc}} a\left(b\frac{R^2}{c} - \frac{R^2}{c}a\right)\\ &= m\Delta + R^2\sum_{\text{cyc}}\left(\frac{ab}{c} - \frac{ac}{b}\right) = m\Delta,\end{aligned}$$

and consequently

$$\alpha a + \beta b + \gamma c = m,$$

since it is clear that $\Delta \neq 0$.

We note that α, β, γ are real numbers and $\alpha + \beta + \gamma = 1$, so α, β, γ are the barycentric coordinates of point M.

Since

$$z_{O_1} = \frac{(c-b)\cdot\rho(M)}{\alpha\cdot\Delta}, \quad z_{O_2} = \frac{(c-a)\cdot\rho(M)}{\beta\Delta}, \quad z_{O_3} = \frac{(a-b)\cdot\rho(M)}{\gamma\cdot\Delta},$$

we have

$$\begin{aligned}\frac{\text{area}[O_1O_2O_3]}{\text{area}[ABC]} &= \left|\frac{\dfrac{i}{4}\begin{vmatrix} z_{O_1} & \overline{z_{O_1}} & 1 \\ z_{O_2} & \overline{z_{O_2}} & 1 \\ z_{O_3} & \overline{z_{O_3}} & 1 \end{vmatrix}}{\dfrac{i}{4}\cdot\Delta}\right|\\ &= \left|\frac{1}{\Delta}\cdot\frac{\rho^2(M)}{\Delta^2}\cdot\frac{1}{\alpha\beta\gamma}\begin{vmatrix} b-c & \overline{b}-\overline{c} & \alpha \\ c-a & \overline{c}-\overline{a} & \beta \\ a-b & \overline{a}-\overline{b} & \gamma \end{vmatrix}\right|\\ &= \left|\frac{\rho^2(M)}{\Delta^3}\cdot\frac{1}{\alpha\beta\gamma}\cdot\begin{vmatrix} c-a & \overline{c}-\overline{a} \\ a-b & \overline{a}-\overline{b} \end{vmatrix}\right|\\ &= \left|\frac{\rho^2(M)}{\Delta^3}\cdot\frac{1}{\alpha\beta\gamma}\cdot\Delta\right| = \left|\frac{\rho^2(M)}{\Delta^2}\cdot\frac{1}{\alpha\beta\gamma}\right|.\end{aligned} \tag{3}$$

Relations (2) and (3) imply that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|\Delta^2\alpha\beta\gamma|}{4\rho^2(M)}. \tag{4}$$

Because α, β, γ are the barycentric coordinates of M, it follows that

$$z_M = \alpha z_A + \beta z_B + \gamma z_C.$$

Using the real product we find that

$$\begin{aligned} OM^2 = z_M \cdot z_M &= (\alpha z_A + \beta z_B + \gamma z_C) \cdot (\alpha z_A + \beta z_B + \gamma z_C) \\ &= (\alpha^2 + \beta^2 + \gamma^2)R^2 + 2\sum_{\text{cyc}} \alpha\beta z_A \cdot z_B \\ &= (\alpha^2 + \beta^2 + \gamma^2)R^2 + 2\sum_{\text{cyc}} \alpha\beta \left(R^2 - \frac{AB^2}{2}\right) \\ &= (\alpha + \beta + \gamma)^2 R^2 - \sum_{\text{cyc}} \alpha\beta AB^2 = R^2 - \sum_{\text{cyc}} \alpha\beta AB^2. \end{aligned}$$

Therefore the power of M' with respect to the circumcircle of triangle ABC can be expressed in the form

$$\rho(M) = R^2 - OM^2 = \sum_{\text{cyc}} \alpha\beta AB^2.$$

On the other hand, if α, β, γ are the barycentric coordinates of the point M, then its isogonal point M' has the barycentric coordinates given by

$$\alpha' = \frac{\beta\gamma BC^2}{\beta\gamma BC^2 + \alpha\gamma CA^2 + \alpha\beta AB^2}, \quad \beta' = \frac{\gamma\alpha CA^2}{\beta\gamma BC^2 + \alpha\gamma CA^2 + \alpha\beta AB^2},$$

$$\gamma' = \frac{\alpha\beta AB^2}{\beta\gamma BC^2 + \alpha\gamma CA^2 + \alpha\beta AB^2}.$$

Therefore

$$\begin{aligned} \rho(M') &= \sum_{\text{cyc}} \alpha'\beta' AB^2 \\ &= \frac{\alpha\beta\gamma AB^2 \cdot BC^2 \cdot CA^2}{(\beta\gamma BC^2 + \alpha\gamma CA^2 + \alpha\beta AB^2)^2} = \frac{\alpha\beta\gamma AB^2 BC^2 CA^2}{\rho^2(M)}. \end{aligned} \tag{5}$$

On the other hand, we have

$$\Delta^2 = \left|\left(\frac{4}{i} \cdot \frac{i}{4}\Delta\right)^2\right| = \left|\frac{4}{i} \cdot \text{area}[ABC]\right|^2 = \frac{AB^2 \cdot BC^2 \cdot CA^2}{R^2}. \tag{6}$$

The desired conclusion follows from the relations (4), (5), and (6). □

Applications. 1) If M is the orthocenter H, then M' is the circumcenter O and

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{R^2}{4R^2} = \frac{1}{4}.$$

2) If M is the circumcenter O, then M' is the orthocenter H and we obtain

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|R^2 - OH^2|}{4R^2}.$$

Using the formula in Theorem 8, Subsection 4.6.4, it follows that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|(2R + r)^2 - s^2|}{2R^2}.$$

3) If M is the Lemoine point K, then M' is the centroid G and

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|R^2 - OG^2|}{4R^2}.$$

Applying the formula in Corollary 9, Subsection 4.6.4, then the first formula in Corollary 2, Subsection 4.6.1, it follows that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{2(s^2 - r^2 - 4Rr)}{36R^2} = \frac{\alpha^2 + \beta^2 + \gamma^2}{36R^2}$$

where α, β, γ are the sides of triangle ABC.

From the inequality $\alpha^2 + \beta^2 + \gamma^2 \leq 9R^2$ (Corollary 10, Subsection 4.6.4) we obtain

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} \leq \frac{1}{4}.$$

4) If M is the incenter I of triangle ABC, then $M' = I$ and using Euler's formula $OI^2 = R^2 - 2Rr$ (see Theorem 4 in Subsection 4.6.2) we find that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} = \frac{|R^2 - OI^2|}{4R^2} = \frac{2Rr}{4R^2} = \frac{r}{4R}.$$

Applying Euler's inequality $R \geq 2r$ (Corollary 5 in Subsection 4.6.2) it follows that

$$\frac{\text{area}[ABC]}{\text{area}[A''B''C'']} \leq \frac{1}{4}.$$

4.11 Lagrange's Theorem and Applications

Consider the distinct points $A_1(z_1), \ldots, A_n(z_n)$ in the complex plane. Let $m_1, \ldots, m_n$ be nonzero real numbers such that $m_1 + \cdots + m_n \neq 0$. Let $m = m_1 + \cdots + m_n$.

The point G with coordinate

$$z_G = \frac{1}{m}(m_1 z_1 + \cdots + m_n z_n)$$

is called the *barycenter of set* $\{A_1, \ldots, A_n\}$ with respect to the weights $m_1, \ldots, m_n$.

In the case $m_1 = \cdots = m_n = 1$, the point G is the *centroid* of the set $\{A_1, \ldots, A_n\}$.

When $n = 3$ and the points A_1, A_2, A_3 are not collinear, we obtain the absolute barycentric coordinates of G with respect to the triangle $A_1A_2A_3$ (see Subsection 4.7.1):

$$\mu_{z_1} = \frac{m_1}{m}, \quad \mu_{z_2} = \frac{m_2}{m}, \quad \mu_{z_3} = \frac{m_3}{m}.$$

Theorem 1. (Lagrange[11]) *Consider the points $A_1, \ldots, A_n$ and the nonzero real numbers $m_1, \ldots, m_n$ such that $m = m_1 + \cdots + m_n \neq 0$. If G denotes the barycenter of set $\{A_1, \ldots, A_n\}$ with respect to the weights $m_1, \ldots, m_n$, then for any point M in the plane the following relation holds:*

$$\sum_{j=1}^{n} m_j MA_j^2 = mMG^2 + \sum_{j=1}^{n} m_j GA_j^2 \tag{1}$$

Proof. Without loss of generality we can assume that the barycenter G is the origin of the complex plane; that is, $z_G = 0$.

Using properties of the real product we obtain for all $j = 1, \ldots, n$, the relations

$$\begin{aligned} MA_j^2 &= |z_M - z_j|^2 = (z_M - z_j) \cdot (z_M - z_j) \\ &= |z_M|^2 - 2z_M \cdot z_j + |z_j|^2, \end{aligned}$$

i.e.,

$$MA_j^2 = |z_M|^2 - 2z_M \cdot z_j + |z_j|^2.$$

Multiplying by m_j and adding the relations obtained for $j = 1, \ldots, n$, it follows that

$$\begin{aligned} \sum_{j=1}^{n} m_j MA_j^2 &= \sum_{j=1}^{n} m_j (|z_M|^2 - 2z_M \cdot z_j + |z_j|^2) \\ &= m|z_M|^2 - 2z_M \cdot \left(\sum_{j=1}^{n} m_j z_j \right) + \sum_{j=1}^{n} m_j |z_j|^2 \end{aligned}$$

[11] Joseph Louis Lagrange (1736–1813), French mathematician, one of the greatest mathematicians of the eighteenth century. He made important contributions in all branches of mathematics and his results have greatly influenced modern science.

$$= m|z_M|^2 - 2z_M \cdot (mz_G) + \sum_{j=1}^{n} m_j|z_j|^2$$

$$= m|z_M|^2 + \sum_{j=1}^{n} m_j|z_j|^2 = m|z_M - z_G|^2 + \sum_{j=1}^{n} m_j|z_j - z_G|^2$$

$$= mMG^2 + \sum_{j=1}^{n} m_j GA_j^2. \qquad \square$$

Corollary 2. *Consider the distinct points $A_1, \dots, A_n$ and the nonzero real numbers $m_1, \dots, m_n$ such that $m_1 + \cdots + m_n \neq 0$. For any point M in the plane the following inequality holds:*

$$\sum_{j=1}^{n} m_j MA_j^2 \geq \sum_{j=1}^{n} m_j GA_j^2, \tag{2}$$

with equality if and only if $M = G$, the barycenter of set $\{A_1, \dots, A_n\}$ with respect to the weights $m_1, \dots, m_n$.

Proof. The inequality (2) follows directly from Lagrange's relation (1). $\square$

If $m_1 = \cdots = m_n = 1$, from Theorem 1 one obtains:

Corollary 3. (Leibniz[12]) *Consider the distinct points $A_1, \dots, A_n$ and the centroid G of the set $\{A_1, \dots, A_n\}$. For any point M in the plane the following relation holds:*

$$\sum_{j=1}^{n} MA_j^2 = nMG^2 + \sum_{j=1}^{n} GA_j^2. \tag{3}$$

Remark. The relation (3) is equivalent to the following identity: For any complex numbers $z, z_1, \dots, z_n$ we have

$$\frac{1}{n}\sum_{j=1}^{n} |z - z_j|^2 = n\left|z - \frac{z_1 + \cdots + z_n}{n}\right|^2 + \sum_{j=1}^{n}\left|z_j - \frac{z_1 + \cdots + z_n}{n}\right|^2.$$

Applications. We will use formula (3) in determining some important distances in a triangle. Let us consider the triangle ABC and let us take $n = 3$ in the formula (3). We find that for any point M in the plane of triangle ABC the following formula holds:

$$MA^2 + MB^2 + MC^2 = 3MG^2 + GA^2 + GB^2 + GC^2 \tag{4}$$

where G is the centroid of triangle ABC. Assume that the circumcenter O of the triangle ABC is the origin of complex plane.

[12]Gottfried Wilhelm Leibniz (1646–1716) was a German philosopher, mathematician, and logician who is probably most well known for having invented the differential and integral calculus independently of Sir Isaac Newton.

1) In the relation (4) we choose $M = 0$ and we get

$$3R^2 = 3OG^2 + GA^2 + GB^2 + GC^2.$$

Applying the well-known median formula it follows that

$$\begin{aligned} GA^2 + GB^2 + GC^2 &= \frac{4}{9}(m_\alpha^2 + m_\beta^2 + m_\gamma^2) \\ &= \frac{4}{9}\sum_{\text{cyc}} \frac{1}{4}[2(\beta^2+\gamma^2) - \alpha^2] = \frac{1}{3}(\alpha^2+\beta^2+\gamma^2), \end{aligned}$$

where α, β, γ are the sides of triangle ABC. We find

$$OG^2 = R^2 - \frac{1}{9}(\alpha^2+\beta^2+\gamma^2). \qquad (5)$$

An equivalent form of the distance OG is given in terms of the basic invariants of triangle in Corollary 9, Subsection 4.6.4.

2) Using the collinearity of points O, G, H and the relation $OH = 3OG$ (see Theorem 3.1 in Section 4.5) it follows that

$$OH^2 = 9OG^2 = 9R^2 - (\alpha^2+\beta^2+\gamma^2) \qquad (6)$$

An equivalent form for the distance OH was obtained in terms of the fundamental invariants of the triangle in Theorem 8, Subsection 4.6.4.

3) Consider in (4) $M = I$, the incenter of triangle ABC. We obtain

$$IA^2 + IB^2 + IC^2 = 3IG^2 + \frac{1}{3}(\alpha^2+\beta^2+\gamma^2).$$

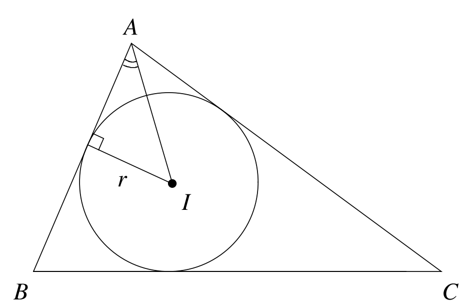

Figure 4.15.

On the other hand, we have the following relations:

$$IA = \frac{r}{\sin\dfrac{A}{2}}, \quad IB = \frac{r}{\sin\dfrac{B}{2}}, \quad IC = \frac{r}{\sin\dfrac{C}{2}},$$

where r is the inradius of triangle ABC. It follows that

$$IG^2 = \frac{1}{3}\left[r^2\left(\frac{1}{\sin^2\frac{A}{2}} + \frac{1}{\sin^2\frac{B}{2}} + \frac{1}{\sin^2\frac{C}{2}}\right) - \frac{1}{3}(\alpha^2+\beta^2+\gamma^2)\right].$$

Taking into account the well-known formula

$$\sin^2\frac{A}{2} = \frac{(s-\beta)(s-\gamma)}{\beta\gamma}$$

we obtain

$$\sum_{\text{cyc}}\frac{1}{\sin^2\frac{A}{2}} = \sum_{\text{cyc}}\frac{\beta\gamma}{(s-\beta)(s-\gamma)} = \sum_{\text{cyc}}\frac{\beta\gamma(s-\alpha)}{(s-\alpha)(s-\beta)(s-\gamma)}$$

$$= \frac{s}{K^2}\sum_{\text{cyc}}\beta\gamma(s-\alpha) = \frac{s}{K^2}\left[s\sum\beta\gamma - 3\alpha\beta\gamma\right]$$

$$= \frac{s}{K^2}[s(s^2+r^2+4Rr) - 12sRr] = \frac{1}{r^2}(s^2+r^2-8Rr),$$

where we have used the formulas in Subsection 4.6.1. Therefore

$$IG^2 = \frac{1}{3}\left[s^2+r^2-8Rr-\frac{1}{3}(\alpha^2+\beta^2+\gamma^2)\right]$$

$$= \frac{1}{3}\left[s^2+r^2-8Rr-\frac{2}{3}(s^2-r^2-4Rr)\right] = \frac{1}{9}(s^2+5r^2-16Rr),$$

where the first formula in Corollary 2 was used. That is,

$$IG^2 = \frac{1}{9}(s^2+5r^2-16Rr), \tag{7}$$

hence we obtain again the formula in Application 1), Subsection 4.7.2.

Problem 1. *Let z_1, z_2, z_3 be distinct complex numbers having modulus R. Prove that*

$$\frac{9R^2 - |z_1+z_2+z_3|^2}{|z_1-z_2|\cdot|z_2-z_3|\cdot|z_3-z_1|} \geq \frac{\sqrt{3}}{R}.$$

Solution. Let A, B, C be the geometric images of the complex numbers z_1, z_2, z_3 and let G be the centroid of the triangle ABC.

The coordinate of G is equal to $\frac{z_1+z_2+z_3}{3}$, and $|z_1-z_2| = \gamma$, $|z_2-z_3| = \alpha$, $|z_3-z_1| = \beta$.

The inequality becomes

$$\frac{9R^2-9OG^2}{\alpha\beta\gamma} \geq \frac{\sqrt{3}}{R}. \tag{1}$$

Using the formula

$$OG^2 = R^2 - \frac{1}{9}(\alpha^2 + \beta^2 + \gamma^2),$$

(1) is equivalent to

$$\alpha^2 + \beta^2 + \gamma^2 \geq \frac{\alpha\beta\gamma\sqrt{3}}{R} = \frac{4rK}{R}\sqrt{3} = 4K\sqrt{3}.$$

Here is a proof of this famous inequality, by using Hero's formula and the AM-GM inequality:

$$K = \sqrt{s(s-\alpha)(s-\beta)(s-\gamma)} \leq \sqrt{s\frac{(s-\alpha+s-\beta+s-\gamma)^3}{27}} = \sqrt{s\frac{s^3}{27}}$$

$$= \frac{s^2}{3\sqrt{3}} = \frac{(\alpha+\beta+\gamma)^2}{12\sqrt{3}} \leq \frac{3(\alpha^2+\beta^2+\gamma^2)}{12\sqrt{3}} = \frac{\alpha^2+\beta^2+\gamma^2}{4\sqrt{3}}.$$

We now extend Leibniz's relation in Corollary 3. First, we need the following result.

Theorem 4. *Let $n \geq 2$ be a positive integer. Consider the distinct points $A_1, \ldots, A_n$ and let G be the centroid of the set $\{A_1, \ldots, A_n\}$. Then for any point in the plane the following formula holds:*

$$n^2 MG^2 = n\sum_{j=1}^{n} MA_j^2 - \sum_{1\leq i<k\leq n} A_iA_k^2. \tag{8}$$

Proof. We assume that the barycenter G is the origin of the complex plane. Using properties of the real product we have

$$MA_j^2 = |z_M - z_j|^2 = (z_M - z_j)\cdot(z_M - z_j) = |z_M|^2 - 2z_M \cdot z_j + |z_j|^2$$

and

$$A_iA_k^2 = |z_i - z_k|^2 = |z_i|^2 - 2z_i \cdot z_k + |z_k|^2,$$

where the complex number z_j is the coordinate of the point A_j, $j = 1, 2, \ldots, n$.

The relation (8) is equivalent to

$$n^2|z_M|^2 = n\sum_{j=1}^{n}(|z_M|^2 - 2z_M\cdot z_j + |z_j|^2) - \sum_{1\leq i<k\leq n}|(|z_i|^2 - 2z_i\cdot z_k + |z_k|^2).$$

That is,

$$n\sum_{j=1}^{n}|z_j|^2 = 2n\sum_{j=1}^{n} z_M\cdot z_j + \sum_{1\leq i<k\leq n}(|z_i|^2 - 2z_i\cdot z_k + |z_k|^2).$$

Taking into account the hypothesis that G is the origin of the complex plane, we have

$$\sum_{j=1}^{n} z_M \cdot z_j = z_M \cdot \left(\sum_{j=1}^{n} z_j\right) = n(z_M \cdot z_G) = n(z_M \cdot 0) = 0.$$

Hence, the relation (8) is equivalent to

$$\sum_{j=1}^{n} |z_j|^2 = -2 \sum_{1\le i<k\le n} z_i \cdot z_k.$$

The last relation can be obtained as follows:

$$0 = |z_G|^2 = z_G \cdot z_G = \frac{1}{n^2}\left(\sum_{i=1}^{n} z_i\right)\cdot\left(\sum_{k=1}^{n} z_k\right)$$

$$= \frac{1}{n^2}\left(\sum_{j=1}^{n} |z_j|^2 + 2\sum_{1\le i<k\le n} z_i \cdot z_k\right).$$

Therefore the relation (8) is proved. □

Remark. The formula (8) is equivalent to the following identity: For any complex numbers $z, z_1, \dots, z_n$, we have

$$\frac{1}{n}\sum_{j=1}^{n} |z - z_j|^2 - \left|z - \frac{z_1 + \cdots + z_n}{n}\right|^2 = \frac{1}{n}\sum_{1\le i<k\le n} |z_i - z_k|^2.$$

Applications. 1) If $A_1, \dots, A_n$ are points on the circle of center O and radius R, then taking in (8) $M = O$, it follows that

$$\sum_{1\le i<k\le n} A_i A_k^2 = n^2(R^2 - OG^2).$$

If $n = 3$ we obtain the formula (5).

2) For any point M in the plane the following inequality holds:

$$\sum_{j=1}^{n} MA_j^2 \ge \frac{1}{n}\sum_{1\le i<k\le n} A_i A_k^2,$$

with equality if and only if $M = G$, the centroid of the set $\{A_1, \dots, A_n\}$.

Let $n \ge 2$ be a positive integer, and let k be an integer such that $2 \le k \le n$. Consider the distinct points $A_1, \dots, A_n$ and let G be the centroid of the set $\{A_1, \dots, A_n\}$. For indices $i_1 < \cdots < i_k$ let us denote by $G_{i_1,\dots,i_k}$ the centroid of the set $\{A_{i_1}, \dots, A_{i_k}\}$. We have the following result:

Theorem 5. *For any point M in the plane,*

$$(n-k)\binom{n}{k}\sum_{j=1}^{n} MA_j^2 + n^2(k-1)\binom{n}{k}MG^2$$
$$= kn(n-1)\sum_{1\le i_1<\cdots,i_k\le n} MG_{i_1\cdots i_k}^2. \tag{9}$$

Proof. It is not difficult to see that the barycenter of the set $\{G_{i_1\cdots i_k} : 1 \le i_1 < \cdots < i_k \le n\}$ is G. Applying Leibniz's relation one obtains

$$\sum_{j=1}^{n} MA_j^2 = nMG^2 + \sum_{j=1}^{n} GA_j^2, \tag{10}$$

$$\sum_{1\le i_1<\cdots<i_k\le n} MG_{i_1\cdots i_k}^2 = \binom{n}{k}MG^2 + \sum_{1\le i_1<\cdots<i_k\le n} GG_{i_1\cdots i_k}^2 \tag{11}$$

$$\sum_{s=1}^{k} MA_{i_s}^2 = kMG_{i_1\cdots i_k}^2 + \sum_{s=1}^{k} G_{i_1\cdots i_k}A_{i_s}^2. \tag{12}$$

Considering in (12) $M = G$ and adding all these relations, it follows that

$$\sum_{1\le i_1<\cdots<i_k\le n}\sum_{s=1}^{k} GA_{i_s}^2 = k\sum_{1\le i_1<\cdots<i_k\le n} GG_{i_1\cdots i_k}^2$$
$$+ \sum_{1\le i_1<\cdots<i_k\le n}\sum_{s=1}^{k} G_{i_1\cdots i_k}A_{i_s}^2. \tag{13}$$

Applying formula (8) in Theorem 5 for the sets $\{A_1, \ldots, A_n\}$ and $\{A_{i_1}, \ldots, A_{i_k}\}$, respectively, we get

$$n^2MG^2 = n\sum_{j=1}^{n} MA_j^2 - \sum_{1\le i<k\le n} A_iA_k^2, \tag{14}$$

$$k^2MG_{i_1\cdots i_k}^2 = k\sum_{s=1}^{k} MA_{i_s}^2 - \sum_{1\le p<q\le k} A_{i_p}A_{i_q}^2. \tag{15}$$

Taking $M = G_{i_1\cdots i_k}$ in (15), it follows that

$$\sum_{s=1}^{k} G_{i_1\cdots i_k}A_{i_s}^2 = \frac{1}{k}\sum_{1\le p<q\le k} A_{i_p}A_{i_q}^2. \tag{16}$$

From (16) and (13) we obtain

$$\sum_{1\le i_1<\cdots<i_k\le n}\sum_{s=1}^{k} GA_{i_s}^2 = k\sum_{1\le i_1<\cdots<i_k\le n} GG_{i_1\cdots i_k}^2$$

$$+\frac{1}{k}\sum_{1\le i_1<\cdots<i_k\le n}\ \sum_{1\le p<q\le n} A_{i_p}A_{i_q}^2 \tag{17}$$

If we rearrange the terms in formula (17), we get

$$\frac{\binom{k}{1}\binom{n}{k}}{\binom{n}{1}}\sum_{j=1}^{n} GA_j^2 = k\sum_{1\le i_1<\cdots<i_k\le n} GG_{i_1\cdots i_k}^2 + \frac{1}{k}\frac{\binom{k}{2}\binom{n}{k}}{\binom{n}{2}}\sum_{1\le i<k\le n} A_iA_j^2. \tag{18}$$

From relations (10), (11), (14) and (18) we readily derive formula (9). □

Remark. The relation (9) is equivalent to the following identity: For any complex numbers $z, z_1, \ldots, z_n$ we have

$$(n-k)\binom{n}{k}\sum_{j=1}^{n}|z-z_j|^2 + n^2(k-1)\binom{n}{k}\left|z-\frac{z_1+\cdots+z_n}{n}\right|^2$$

$$= kn(n-1)\sum_{1\le i_1<\cdots<i_k\le n}\left|z-\frac{z_{i_1}+\cdots+z_{i_k}}{k}\right|^2.$$

Applications. 1) In the case $k=2$, from (9) we obtain that for any point M in the plane, the following relation holds:

$$(n-2)\sum_{j=1}^{n} MA_j^2 + n^2MG^2 = 4\sum_{1\le i_1<i_2\le n} MG_{i_1i_2}^2.$$

In this case $G_{i_1i_2}$ is the midpoint of the segment $[A_{i_1}A_{i_2}]$.

2) If $k=3$, from (9) we get that for any point M in the plane, the relation

$$(n-3)(n-2)\sum_{j=1}^{n} MA_j^2 + 2n^2(n-2)MG^2 = 18\sum_{1\le i_1<i_2<i_3\le n} MG_{i_1i_2i_3}^2$$

holds. Here the point $G_{i_1i_2i_3}$ is the centroid of triangle $A_{i_1}A_{i_2}A_{i_3}$.

4.12 Euler's Center of an Inscribed Polygon

Consider a polygon $A_1A_2\cdots A_n$ inscribed in a circle centered at the origin of a complex plane and let $a_1, a_2, \ldots, a_n$ be the coordinates of its vertices.

By definition, the point E with coordinate

$$z_E = \frac{a_1+a_2+\cdots+a_n}{2}$$

is called *Euler's center* of the polygon $A_1A_2\cdots A_n$. In the case $n=3$ it is clear that $E=O_9$, the center of Euler's nine-point circle.

Remarks. a) Let $G(z_G)$ and $H(z_H)$ be the centroid and orthocenter of the inscribed polygon $A_1A_2\cdots A_n$. Then

$$z_E = \frac{nz_G}{2} = \frac{z_H}{2} \text{ and } OE = \frac{nOG}{2} = \frac{OH}{2}.$$

Recall that the orthocenter of the polygon $A_1A_2\cdots A_n$ is the point H with coordinate $z_H = a_1 + a_2 + \cdots + a_n$.

b) For $n = 4$, point E is also called *Mathot's point* of the inscribed quadrilateral $A_1A_2A_3A_4$.

Proposition. *In the above notation, the following relation holds:*

$$\sum_{i=1}^{n} EA_i^2 = nR^2 + (n-4)EO^2. \tag{1}$$

Proof. Using the identity (8) in Theorem 4, Section 2.17,

$$n^2 \cdot MG^2 = n\sum_{i=1}^{n} MA_i^2 - \sum_{1\le i<j\le n} A_iA_j^2$$

for $M = E$ and $M = O$, we obtain

$$n^2 \cdot EG^2 = n\sum_{i=1}^{n} EA_i^2 - \sum_{1\le i<j\le n} A_iA_j^2, \tag{2}$$

and

$$n^2 \cdot OG^2 = nR^2 - \sum_{1\le i<j\le n} A_iA_j^2. \tag{3}$$

Setting $s = \sum_{i=1}^{n} a_i$, we have

$$EG = |z_E - z_G| = \left|\frac{s}{2} - \frac{s}{n}\right| = \left|\frac{s}{2}\right| \cdot \frac{n-2}{n} = \frac{n-2}{n} \cdot OE. \tag{4}$$

From the relations (2), (3) and (4) we derive that

$$n\sum_{i=1}^{n} EA_i^2 = n^2 \cdot EG^2 - n^2 \cdot OG^2 + n^2R^2$$

$$= (n-2)^2OE^2 - 4OE^2 + n^2R^2 = n(n-4)\cdot EO^2 + n^2R^2$$

or, equivalently,

$$\sum_{i=1}^{n} EA_i^2 = nR^2 + (n-4)EO^2,$$

as desired. □

Applications. 1) For $n = 3$, from relation (1) we obtain

$$O_9A_1^2 + O_9A_2^2 + O_9A_3^2 = 3R^2 - OO_9^2. \tag{5}$$

Using the formula in Corollary 9.2, Subsection 4.6.4, we can express the right-hand side in (5) in terms of the fundamental invariants of triangle $A_1A_2A_3$:

$$O_9A_1^2 + O_9A_2^2 + O_9A_3^2 = \frac{3}{4}R^2 - \frac{1}{2}r^2 - 2Rr + \frac{1}{2}s^2. \tag{6}$$

From formula (5) it follows that for any triangle $A_1A_2A_3$ the following inequality holds:

$$O_9A_1^2 + O_9A_2^2 + O_9A_3^2 \le 3R^2, \tag{7}$$

with equality if and only if the triangle is equilateral.

2) For $n = 4$ we obtain the interesting relation

$$\sum_{i=1}^{4} EA_i^2 = 4R^2. \tag{8}$$

The point E is the unique point in the plane of the quadrilateral $A_1A_2A_3A_4$ satisfying relation (8).

3) For $n > 4$, from relation (1) the inequality

$$\sum_{i=1}^{n} EA_i^2 \ge nR^2 \tag{9}$$

follows. Equality holds only in the polygon $A_1A_2\cdots A_n$ with the property $E = O$.

4) The Cauchy–Schwarz inequality and inequality (7) give

$$\Big(\sum_{i=1}^{3} R \cdot O_9A_i\Big)^2 \le (3R^2)\sum_{i=1}^{3} O_9A_i^2 \le 9R^2.$$

This is equivalent to

$$O_9A_1 + O_9A_2 + O_9A_3 \le 3R. \tag{10}$$

5) Using the same inequality and the relation (8) we have

$$\Big(R\sum_{i=1}^{4} EA_i\Big)^2 \le 4R^2 \cdot \sum_{i=1}^{4} EA_i = 16R^4$$

or, equivalently,

$$\sum_{i=1}^{4} EA_i \le 4R. \tag{11}$$

6) Using the relation

$$2EA_i = 2|e - a_i| = 2\left|\frac{s}{2} - a_i\right| = |s - 2a_i|,$$

the inequalities (4), (5) become

$$\sum_{\text{cyc}} |-a_1 + a_2 + a_3| \le 6R$$

and, respectively,

$$\sum_{\text{cyc}} |-a_1 + a_2 + a_3 + a_4| \le 8R.$$

The above inequalities hold for all complex numbers of the same modulus R.

4.13 Some Geometric Transformations of the Complex Plane

4.13.1 Translation

Let z_0 be a fixed complex number and let t_{z_0} be the mapping defined by

$$t_{z_0} : \mathbb{C} \to \mathbb{C}, \quad t_{z_0}(z) = z + z_0.$$

The mapping t_{z_0} is called the *translation* of the complex plane with complex number z_0.

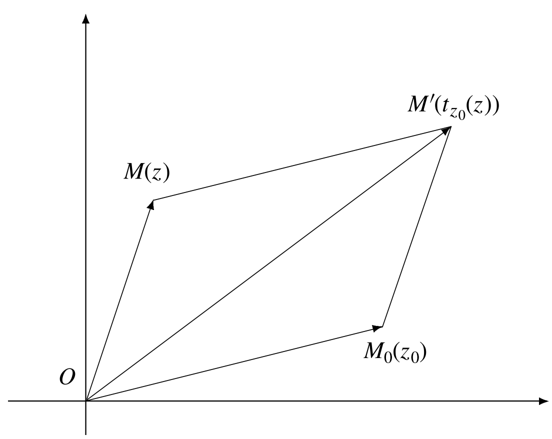

Figure 4.16.

Taking into account the geometric interpretation of the addition of two complex numbers (see Subsection 1.2.3), we have Fig. 4.16, giving the geometric image of $t_{z_0}(z)$.

In Fig. 4.16 $OM_0M'M$ is a parallelogram and OM' is one of its diagonals. Therefore, the mapping t_{z_0} corresponds in the complex plane $\mathbb{C}$ to the translation $t_{\overrightarrow{OM_0}}$ of vector $\overrightarrow{OM_0}$ in the case of a Euclidean plane.

It is clear that the composition of two translations t_{z_1} and t_{z_2} satisfies the relation

$$t_{z_1} \circ t_{z_2} = t_{z_1+z_2}.$$

It is also clear that the set $\mathcal{T}$ of all translations of a complex plane is a group with respect to the composition of mappings. The group $(\mathcal{T}, \circ)$ is Abelian and its unity is $t_O = 1_{\mathbb{C}}$, the translation of the complex number 0.

4.13.2 Reflection in the real axis

Consider the mappings $s : \mathbb{C} \to \mathbb{C}$, $s(z) = \overline{z}$. If M is the point with coordinate z, then the point $M'(s(z))$ is obtained by reflecting M across the real axis (see Fig. 4.17). The mapping s is called the *reflection in the real axis*. It is clear that $s \circ s = 1_{\mathbb{C}}$.

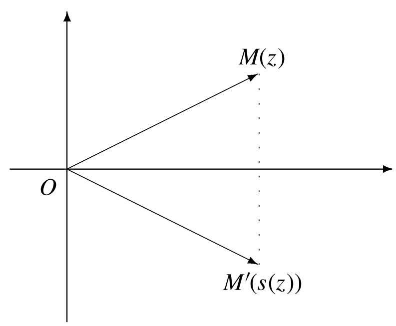

Figure 4.17.

4.13.3 Reflection in a point

Consider the mapping $s_0 : \mathbb{C} \to \mathbb{C}$, $s_0(z) = -z$. Since $s_0(z) + z = 0$, the origin O is the midpoint of the segment $[M(z)M'(z)]$. Hence M' is the reflection of point M across O (Fig. 4.18).

The mapping s_0 is called the *reflection in the origin*.

Consider a fixed complex number z_0 and the mapping

$$s_{z_0} : \mathbb{C} \to \mathbb{C}, \quad s_{z_0}(z) = 2z_0 - z.$$

If z_0, z, $s_{z_0}(z)$ are the coordinates of points M_0, M, M', then M_0 is the midpoint of the segment $[MM']$, hence M' is the reflection of M in M_0 (Fig. 4.19).

The mapping s_{z_0} is called the *reflection in the point* $M_0(z_0)$. It is clear that the following relation holds: $s_{z_0} \circ s_{z_0} = 1_{\mathbb{C}}$.

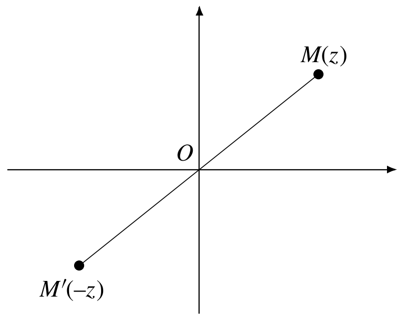

Figure 4.18.

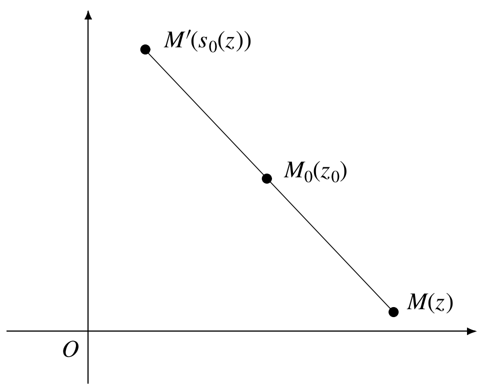

Figure 4.19.

4.13.4 Rotation

Let $a = \cos t_0 + i \sin t_0$ be a complex number having modulus 1 and let r_a be the mapping given by $r_a : \mathbb{C} \to \mathbb{C}$, $r_a(z) = az$. If $z = \rho(\cos t + i \sin t)$, then

$$r_a(z) = az = \rho[\cos(t + t_0) + i \sin(t + t_0)],$$

hence $M'(r_a(z))$ is obtained by rotating point $M(z)$ about the origin through the angle t_0 (Fig. 4.20).

The mapping r_a is called the *rotation* with center O and angle $t_0 = \arg a$.

4.13.5 Isometric transformation of the complex plane

A mapping $f : \mathbb{C} \to \mathbb{C}$ is called an *isometry* if it preserves distance, i.e., for all $z_1, z_2 \in \mathbb{C}$, $|f(z_1) - f(z_2)| = |z_1 - z_2|$.

Theorem 1. *Translations, reflections (in the real axis or in a point) and rotations about center O are isometries of the complex plane.*

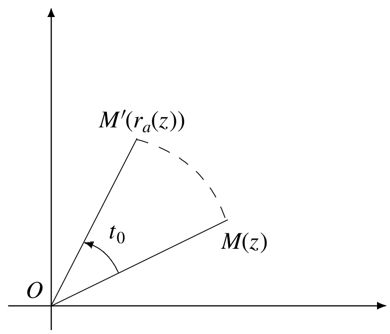

Figure 4.20.

Proof. For the translation t_{z_0} we have

$$|t_{z_0}(z_1) - t_{z_0}(z_2)| = |(z_1 + z_0) - (z_2 + z_0)| = |z_1 - z_2|.$$

For the reflection s across the real axis we obtain

$$|s(z_1) - s(z_2)| = |\overline{z_1} - \overline{z_2}| = |\overline{z_1 - z_2}| = |z_1 - z_2|,$$

and the same goes for the reflection in a point. Finally, if r_a is a rotation, then

$$|r_a(z_1) - r_a(z_2)| = |az_1 - az_2| = |a||z_1 - z_2| = |z_1 - z_2|, \text{ since } |a| = 1. \qquad \square$$

We can easily check that the composition of two isometries is also an isometry. The set $Izo(\mathbb{C})$ of all isometries of the complex plane is a group with respect to the composition of mappings and $(\mathcal{T}, \circ)$ is a subgroup of it.

Problem. *Let $A_1A_2A_3A_4$ be a cyclic quadrilateral inscribed in a circle of center O and let H_1, H_2, H_3, H_4 be the orthocenters of triangles $A_2A_3A_4$, $A_1A_3A_4$, $A_1A_2A_4$, $A_1A_2A_3$, respectively.*

Prove that quadrilaterals $A_1A_2A_3A_4$ and $H_1H_2H_3H_4$ are congruent.

(Balkan Mathematical Olympiad, 1984)

Solution. Consider the complex plane with origin at the circumcenter, and denote by lowercase letters the coordinates of the points denoted by uppercase letters.

If $s = a_1 + a_2 + a_3 + a_4$, then $h_1 = a_2 + a_3 + a_4 = s - a_1$, $h_2 = s - a_2$, $h_3 = s - a_3$, $h_4 = s - a_4$. Hence the quadrilateral $H_1H_2H_3H_4$ is the reflection of quadrilateral $A_1A_2A_3A_4$ across the point with coordinate $\frac{s}{2}$.

The following result describes all isometries of the complex plane.

Theorem 2. *Any isometry is a mapping $f : \mathbb{C} \to \mathbb{C}$ with $f(z) = az + b$ or $f(z) = a\overline{z} + b$, where $a, b \in \mathbb{C}$ and $|a| = 1$.*

Proof. Let $b = f(0)$, $c = f(1)$ and $a = c - b$. Then

$$|a| = |c - b| = |f(1) - f(0)| = |1 - 0| = 1.$$

Consider the mapping $g : \mathbb{C} \to \mathbb{C}$, given by $g(z) = az + b$. It is not difficult to prove that g is an isometry, with $g(0) = b = f(0)$ and $g(1) = a + b = c = f(1)$. Hence $h = g^{-1} \circ f$ is an isometry, with 0 and 1 as fixed points. By definition, it follows that any real number is a fixed point of h, hence $h = 1_{\mathbb{C}}$ or $h = s$, the reflection in the real axis. Hence $g = f$ or $g = f \circ s$, and the proof is complete. □

The above result shows that any isometry of the complex plane is the composition of a rotation and a translation, or the composition of a rotation with the reflection in the origin O and a translation.

4.13.6 Morley's theorem

In 1899, Frank Morley, then professor of mathematics at Haverford College, came across a result so surprising that it entered mathematical folklore under the name of "Morley's Miracle." Morley's marvelous theorem states that: *The three points of intersection of the adjacent trisectors of the angles of any triangle form an equilateral triangle*.

The theorem was mistakenly attributed to Napoleon Bonaparte, who made some contributions to geometry.

There are various proofs of this nice result: J. Conway's proof, D.J. Newman's proof, L. Bankoff's proof, and N. Dergiades's proof.

Here we present the new proof published in 1998, by Alain Connes. His proof is derived from the following result:

Theorem 3. (Alain Connes) *Consider the transformations of a complex plane $f_i : \mathbb{C} \to \mathbb{C}$, $f_i(z) = a_i z + b_i$, $i = 1, 2, 3$, where all coefficients a_i are different from zero. Assume that the mappings $f_1 \circ f_2$, $f_2 \circ f_3$, $f_3 \circ f_1$ and $f_1 \circ f_2 \circ f_3$ are not translations, i.e., equivalently, $a_1a_2, a_2a_3, a_3a_1, a_1a_2a_3 \in \mathbb{C} \setminus \{1\}$. Then the following statements are equivalent:*

(1) $f_1^3 \circ f_2^3 \circ f_3^3 = 1_{\mathbb{C}}$;

(2) $j^3 = 1$ and $\alpha + j\beta + j^2\gamma = 0$, where $j = a_1a_2a_3 \neq 1$ and α, β, γ are the unique fixed points of mappings $f_1 \circ f_2$, $f_2 \circ f_3$, $f_3 \circ f_1$, respectively.

Proof. Note that $(f_1 \circ f_2)(z) = a_1a_2z + a_1b_2 + b_1, \quad a_1a_2 \neq 1,$

$$(f_2 \circ f_3)(z) = a_2a_3z + a_2b_3 + b_2, \quad a_2a_3 \neq 1,$$

$$(f_3 \circ f_1)(z) = a_3a_1z + a_3b_1 + b_3, \quad a_3a_1 \neq 1.$$

$$\mathrm{Fix}(f_1 \circ f_2) = \left\{\frac{a_1 b_2 + b_1}{1 - a_1 a_2}\right\} = \left\{\frac{a_1 a_3 b_2 + a_3 b_1}{a_3 - j} =: \alpha\right\},$$
$$\mathrm{Fix}(f_2 \circ f_3) = \left\{\frac{a_2 b_3 + b_2}{1 - a_2 a_3}\right\} = \left\{\frac{a_1 a_2 b_3 + a_1 b_2}{a_1 - j} =: \beta\right\},$$
$$\mathrm{Fix}(f_3 \circ f_1) = \left\{\frac{a_3 b_1 + b_3}{1 - a_3 a_1}\right\} = \left\{\frac{a_2 a_3 b_1 + a_2 b_3}{a_2 - j} =: \gamma\right\},$$

where $\mathrm{Fix}(f)$ denotes the set of fixed points of the mapping f.

For the cubes of f_1, f_2, f_3 we have the formulas

$$f_1^3(z) = a_1^3 z + b_1(a_1^2 + a_1 + 1),$$
$$f_2^3(z) = a_2^3 z + b_2(a_2^2 + a_2 + 1),$$
$$f_3^3(z) = a_3^3 + b_3(a_3^2 + a_3 + 1),$$

hence

$$(f_1^3 \circ f_2^3 \circ f_3^3)(z) = a_1^3 a_2^3 a_3^3 z + a_1^3 a_2^3 b_3(a_3^2 + a_3 + 1)$$
$$+ a_1^3 b_2(a_2^2 + a_2 + 1) + b_1(a_1^2 + a_1 + 1).$$

Therefore $f_1^3 \circ f_2^3 \circ f_3^3 = id_C$ if and only if $a_1^3 a_2^3 a_3^3 = 1$ and

$$a_1^3 a_2^3 b_3(a_3^2 + a_3 + 1) + a_1^3 b_2(a_2^2 + a_2 + 1) + b_1(a_1^2 + a_1 + 1) = 0.$$

To prove the equivalence of statements (1) and (2) we have to show that $\alpha + j\beta + j^2\gamma$ is different from the free term of $f_1^3 \circ f_2^3 \circ f_3^3$ by a multiplicative constant. Indeed, using the relation $j^3 = 1$ and implicitly $j^2 + j + 1 = 0$, we have successively:

$$\alpha + j\beta + j^2\gamma = \alpha + j\beta + (-1 - j)\gamma = \alpha - \gamma + j(\beta - \gamma)$$
$$= \frac{a_1 a_3 b_2 + a_3 b_1}{a_3 - j} - \frac{a_2 a_3 b_1 + a_2 b_3}{a_2 - j} + j\Big(\frac{a_1 a_2 b_3 + a_1 b_2}{a_1 - j} - \frac{a_2 a_3 b_1 + a_2 b_3}{a_2 - j}\Big)$$
$$= \frac{a_1 a_2 a_3 b_2 + a_2 a_3 b_1 - a_1 a_3 b_2 j - a_3 b_1 j - a_2 a_3^2 b_1 - a_2 a_3 b_3 + a_2 a_3 b_1 j + a_2 b_3 j}{(a_2 - j)(a_3 - j)}$$
$$+ j\frac{a_1 a_2^2 b_3 + a_1 a_2 b_2 - a_1 a_2 b_3 j - a_1 b_2 j - a_1 a_2 a_3 b_1 - a_1 a_2 b_3 + a_2 a_3 b_1 j + a_2 b_3 j}{(a_1 - j)(a_2 - j)}$$
$$= \frac{1}{a_2 - j}\left(\frac{b_2 j - a_2 a_3 b_1 j^2 - a_1 a_3 b_2 j - a_3 b_1 j - a_2 a_3^2 b_1 - a_2 a_3 b_3 + a_2 b_3 j}{a_3 - j}\right.$$
$$\left. + \frac{a_1 a_2^2 b_3 j + a_1 a_2 b_2 j + a_1 a_2 b_3 - a_1 b_2 j^2 - b_1 j^2 + a_2 a_3 b_1 j^2 + a_2 b_3 j^2}{a_1 - j}\right)$$

$$= \frac{1}{(a_1 - j)(a_2 - j)(a_3 - j)}(a_1 b_2 j - b_1 - a_1^2 a_3 b_2 j - a_1 a_3 b_1 j - a_1 a_2 a_3^2 b_1 - b_3 j$$

$$+ a_1 a_2 b_3 j - b_2 j^2 + a_2 a_3 b_1 + a_1 a_3 b_2 j^2 + a_3 b_1 j^2 + a_2 a_3^2 b_1 j + a_2 a_3 b_3 j - a_2 b_3 j^2$$

$$+ a_2 b_3 j^2 + b_2 j^2 + b_3 j - a_1 a_3 b_2 j^2 - a_3 b_1 j^2 + a_2 a_3 b_1 j^2 + a_2 a_3 b_3 j^2$$

$$- a_1 a_2^2 b_3 j^2 - a_1 a_2 b_2 j^2 - a_1 a_2 b_3 j + a_1 b_2 + b_1 - a_2 a_3 b_1 - a_2 b_3)$$

$$= \frac{1}{(a_1 - j)(a_2 - j)(a_3 - j)}(-a_1 b_2 j^2 - a_1^2 a_3 b_2 j - a_1 a_3 b_1 j - a_3 b_1 j$$

$$- a_2 a_3^2 b_1 - a_2 a_3 b_3 - a_1 a_2^2 b_3 j^2 - a_1 a_2 b_2 j^2 - a_2 b_3)$$

$$= -\frac{1}{(a_1 - j)(a_2 - j)(a_3 - j)}(a_1^2 a_2^2 a_3^2 b_2 + a_1^3 a_2 a_3^2 b_2$$

$$+ a_1^2 a_2 a_3^2 b_1 + a_1 a_2 a_3^2 b_1 + a_2 a_3^2 b_1 + a_2 a_3 b_3 + a_1^3 a_2^4 a_3^2 b_3 + a_1^3 a_2^3 a_3^2 b_2 + a_2 b_3)$$

$$= -\frac{1}{(a_1 - j)(a_2 - j)(a_3 - j)}[a_2 a_3^2 b_1 (1 + a_1 + a_1^2) + a_1^3 a_2 a_3^2 b_2 (1 + a_2 + a_2^2)$$

$$+ a_2 b_3 (1 + a_3 + a_1^3 + a_1^3 a_2^3 a_3^2)]$$

$$= -\frac{a_2 a_3^2}{(a_1 - j)(a_2 - j)(a_3 - j)}[a_1^3 a_2^3 b_3 (1 + a_3 + a_3^2)$$

$$+ a_1^3 b_2 (1 + a_2 + a_2^2) + b_1 (1 + a_1 + a_1^2)]. \qquad \square$$

Theorem 4. (Morley) *The three points $A'(\alpha)$, $B'(\beta)$, $C'(\gamma)$ of the adjacent trisectors of the angles of any triangle ABC form an equilateral triangle.*

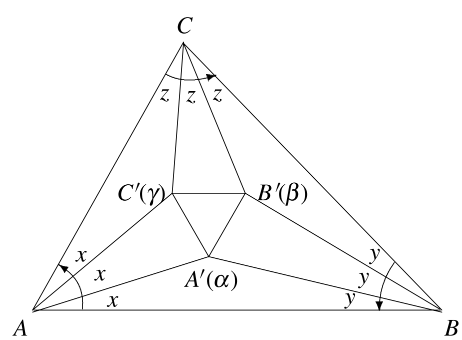

Figure 4.21.

Proof. (Alain Connes) Let us consider the rotations $f_1 = r_{A,2x}$, $f_2 = r_{B,2y}$, $f_3 = r_{C,2z}$ of centers A, B, C and of angles $x = \frac{1}{3}\widehat{A}$, $y = \frac{1}{3}\widehat{B}$, $z = \frac{1}{3}\widehat{C}$ (Figure 4.21).

Note that $\mathrm{Fix}(f_1 \circ f_2) = \{A'\}$, $\mathrm{Fix}(f_2 \circ f_3) = \{B'\}$, $\mathrm{Fix}(f_3 \circ f_1) = \{C'\}$ (see Figure 4.22).

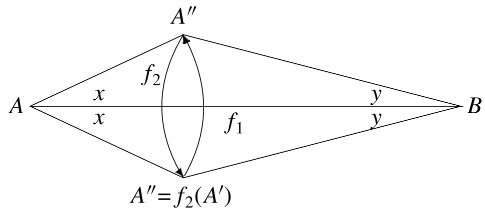

Figure 4.22.

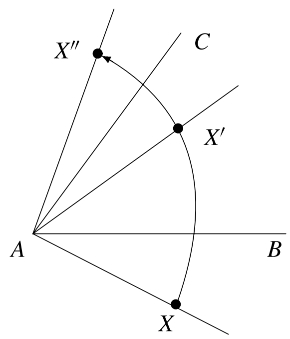

Figure 4.23.

To prove that triangle $A'B'C'$ is equilateral it is sufficient to show, by Proposition 2 in Section 2.4 and above Theorem 3, that $f_1^3 \circ f_2^3 \circ f_3^3 = 1_{\mathbb{C}}$. The composition $s_{AC} \circ s_{AB}$ of reflections s_{AC} and s_{AB} across the lines AC and AB is a rotation about center A through angle $6x$.

Therefore $f_1^3 = s_{AC} \circ s_{AB}$ and analogously $f_2^3 = s_{BA} \circ s_{BC}$ and $f_3^3 = s_{CB} \circ s_{CA}$. It follows that

$$f_1^3 \circ f_2^3 \circ f_3^3 = s_{AC} \circ s_{AB} \circ s_{BA} \circ s_{BC} \circ s_{CB} \circ s_{CA} = 1_{\mathbb{C}}. \qquad \square$$

4.13.7 Homothecy

Given a fixed nonzero real number k, the mapping $h_k : \mathbb{C} \to \mathbb{C}$, $h_k(z) = kz$, is called the *homothecy* of the complex plane with center O and magnitude k.

Figures 4.24 and 4.25 show the position of point $M'(h_k(z))$ in the cases $k > 0$ and $k < 0$.

Points $M(z)$ and $M'(h_k(z))$ are collinear with the center O, which lies on the line segment MM' if and only if $k < 0$.

Moreover, the following relation holds:

$$|OM'| = |k||OM|.$$

Point M' is called the *homothetic* point of M with center O and magnitude k.

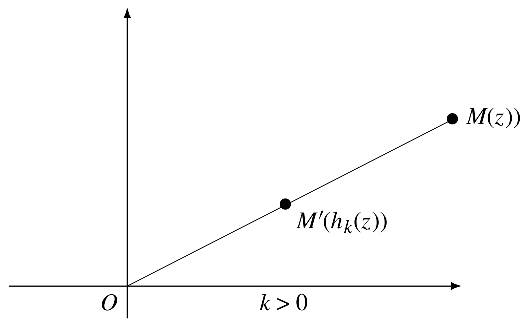

Figure 4.24.

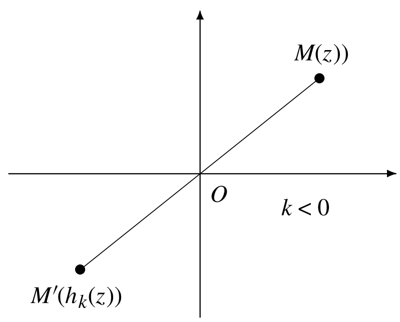

Figure 4.25.

It is clear that the composition of two homothecies h_{k_1} and h_{k_2} is also a homothecy, that is,

$$h_{k_1} \circ h_{k_2} = h_{k_1 k_2}.$$

The set $\mathcal{H}$ of all homothecies of the complex plane is an Abelian group with respect to the composition of mappings. The identity of the group $(\mathcal{H}, \circ)$ is $h_1 = 1_{\mathbb{C}}$, the homothecy of magnitude 1.

Problem. *Let M be a point inside an equilateral triangle ABC and let M_1, M_2, M_3 be the feet of the perpendiculars from M to the sides BC, CA, AB, respectively. Find the locus of the centroid of the triangle $M_1M_2M_3$.*

Solution. Let $1, \varepsilon, \varepsilon^2$ be the coordinates of points A, B, C, where $\varepsilon = \cos 120° + i \sin 120°$. Recall that

$$\varepsilon^2 + \varepsilon + 1 = 0 \text{ and } \varepsilon^3 = 1.$$

If m, m_1, m_2, m_3 are the coordinates of points M, M_1, M_2, M_3, we have

$$m_1 = \frac{1}{2}(1 + \varepsilon + m - \varepsilon\overline{m}),$$

$$m_2 = \frac{1}{2}(\varepsilon + \varepsilon^2 + m - \overline{m}),$$

$$m_3 = \frac{1}{2}(\varepsilon^2 + 1 + m - \varepsilon^2\overline{m}).$$

Let g be the coordinate of the centroid of the triangle $M_1M_2M_3$. Then

$$g = \frac{1}{3}(m_1 + m_2 + m_3) = \frac{1}{6}(2(1 + \varepsilon + \varepsilon^2) + 3m - \overline{m}(1 + \varepsilon + \varepsilon^2)) = \frac{m}{2},$$

hence $OG = \frac{1}{2}OM$.

The locus of G is the interior of the triangle obtained from ABC under a homothecy of center O and magnitude $\frac{1}{2}$. In other words, the vertices of this triangle have coordinates $\frac{1}{2}$, $\frac{1}{2}\varepsilon$, $\frac{1}{2}\varepsilon^2$.

4.13.8 Problems

1. Prove that the composition of two isometries of the complex plane is an isometry.

2. An isometry of the complex plane has two fixed points A and B. Prove that any point M of line AB is a fixed point of the transformation.

3. Prove that any isometry of the complex plane is a composition of a rotation with a translation and possibly also with the reflection in the real axis.

4. Prove that the mapping $f : \mathbb{C} \to \mathbb{C}$, $f(z) = i \cdot \overline{z} + 4 - i$ is an isometry. Analyze f as in problem 3.

5. Prove that the mapping $g : \mathbb{C} \to \mathbb{C}$, $g(z) = -iz + 1 + 2i$ is an isometry. Analyze g as in problem 4.

5

Olympiad-Caliber Problems

The use of complex numbers is helpful in solving Olympiad problems. In many instances, a rather complicated problem can be solved unexpectedly by employing complex numbers. Even though the methods of Euclidean geometry, coordinate geometry, vector algebra and complex numbers look similar, in many situations the use of the latter has multiple advantages. This chapter will illustrate some classes of Olympiad-caliber problems where the method of complex numbers works efficiently.

5.1 Problems Involving Moduli and Conjugates

Problem 1. *Let z_1, z_2, z_3 be complex numbers such that*

$$|z_1| = |z_2| = |z_3| = r > 0$$

and $z_1 + z_2 + z_3 \neq 0$. Prove that

$$\left| \frac{z_1 z_2 + z_2 z_3 + z_3 z_1}{z_1 + z_2 + z_3} \right| = r.$$

Solution. Observe that

$$z_1 \cdot \overline{z_1} = z_2 \cdot \overline{z_2} = z_3 \cdot \overline{z_3} = r^2.$$

Then

$$\left| \frac{z_1 z_2 + z_2 z_3 + z_3 z_1}{z_1 + z_2 + z_3} \right|^2 = \frac{z_1 z_2 + z_2 z_3 + z_3 z_1}{z_1 + z_2 + z_3} \cdot \frac{\overline{z_1 z_2} + \overline{z_2 z_3} + \overline{z_3 z_1}}{\overline{z_1} + \overline{z_2} + \overline{z_3}}$$

$$= \frac{z_1z_2 + z_2z_3 + z_3z_1}{z_1 + z_2 + z_3} \cdot \frac{\dfrac{r^2}{z_1} \cdot \dfrac{r^2}{z_2} + \dfrac{r^2}{z_2} \cdot \dfrac{r^2}{z_3} + \dfrac{r^2}{z_3} \cdot \dfrac{r^2}{z_1}}{\dfrac{r^2}{z_1} + \dfrac{r^2}{z_2} + \dfrac{r^2}{z_3}} = r^2,$$

as desired.

Problem 2. *Let z_1, z_2 be complex numbers such that*

$$|z_1| = |z_2| = r > 0.$$

Prove that

$$\left(\frac{z_1 + z_2}{r^2 + z_1z_2}\right)^2 + \left(\frac{z_1 - z_2}{r^2 - z_1z_2}\right)^2 \geq \frac{1}{r^2}.$$

Solution. The desired inequality is equivalent to

$$\left(\frac{r(z_1 + z_2)}{r^2 + z_1z_2}\right)^2 + \left(\frac{r(z_1 - z_2)}{r^2 - z_1z_2}\right)^2 \geq 1.$$

Setting

$$z_1 = r(\cos 2x + i \sin 2x) \text{ and } z_2 = r(\cos 2y + i \sin 2y)$$

yields

$$\frac{r(z_1 + z_2)}{r^2 + z_1z_2} = \frac{r^2(\cos 2x + i \sin 2x + \cos 2y + i \sin 2y)}{r^2(1 + \cos(2x + 2y) + i \sin(2x + 2y))} = \frac{\cos(x - y)}{\cos(x + y)}.$$

Similarly,

$$\frac{r(z_1 - z_2)}{r^2 - z_1z_2} = \frac{\sin(y - x)}{\sin(y + x)}.$$

Thus

$$\left(\frac{r(z_1 + z_2)}{r^2 + z_1z_2}\right)^2 + \left(\frac{r(z_1 - z_2)}{r^2 - z_1z_2}\right)^2 = \frac{\cos^2(x - y)}{\cos^2(x + y)} + \frac{\sin^2(x - y)}{\sin^2(x + y)}$$
$$\geq \cos^2(x - y) + \sin^2(x - y) = 1,$$

as claimed.

Problem 3. *Let z_1, z_2, z_3 be complex numbers such that*

$$|z_1| = |z_2| = |z_3| = 1$$

and

$$\frac{z_1^2}{z_2z_3} + \frac{z_2^2}{z_1z_3} + \frac{z_3^2}{z_1z_2} + 1 = 0.$$

Prove that

$$|z_1 + z_2 + z_3| \in \{1, 2\}.$$

Solution. The given equality can be written as

$$z_1^3 + z_2^3 + z_3^3 + z_1 z_2 z_3 = 0$$

or

$$\begin{aligned} -4z_1 z_2 z_3 &= z_1^3 + z_2^3 + z_3^3 - 3z_1 z_2 z_3 \\ &= (z_1 + z_2 + z_3)(z_1^2 + z_2^2 + z_3^2 - z_1 z_2 - z_2 z_3 - z_3 z_1). \end{aligned}$$

Setting $z = z_1 + z_2 + z_3$, yields

$$z^3 - 3z(z_1 z_2 + z_2 z_3 + z_3 z_1) = -4z_1 z_2 z_3.$$

This is equivalent to

$$z^3 = z_1 z_2 z_3 \left[3z \left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} \right) - 4 \right].$$

The last relation can be written as

$$z^3 = z_1 z_2 z_3 [3z(\overline{z_1} + \overline{z_2} + \overline{z_3}) - 4], \quad \text{i.e.,} \quad z^3 = z_1 z_2 z_3 (3|z|^2 - 4).$$

Taking the absolute values of both sides yields $|z|^3 = |3|z|^2 - 4|$. If $|z| \geq \dfrac{2}{\sqrt{3}}$, then $|z|^3 - 3|z|^2 + 4 = 0$, implying $|z| = 2$. If $|z| < \dfrac{2}{\sqrt{3}}$, then $|z|^3 + 3|z|^2 - 4 = 0$, giving $|z| = 1$, as needed.

Alternate solution. It is not difficult to see that $|z_1^3 + z_2^3 + z_3^3| = 1$. By using the algebraic identity

$$(u + v)(v + w)(w + u) = (u + v + w)(uv + vw + wu) - uvw$$

for $u = z_1^3$, $v = z_2^3$, $w = z_3^3$, it follows that

$$\begin{aligned} (z_1^3 + z_2^3)(z_2^3 + z_3^3)(z_3^3 + z_1^3) &= (z_1^3 + z_2^3 + z_3^3)(z_1^3 z_2^3 + z_2^3 z_3^3 + z_3^3 z_1^3) - z_1^3 z_2^3 z_3^3 \\ &= z_1^3 z_2^3 z_3^3 (z_1^3 + z_2^3 + z_3^3) \left(\frac{1}{z_1^3} + \frac{1}{z_2^3} + \frac{1}{z_3^3} \right) - z_1^3 z_2^3 z_3^3 \\ &= z_1^3 z_2^3 z_3^3 (z_1^3 + z_2^3 + z_3^3) \overline{(z_1^3 + z_2^3 + z_3^3)} - z_1^3 z_2^3 z_3^3 \\ &= z_1^3 z_2^3 z_3^3 - z_1^3 z_2^3 z_3^3 = 0. \end{aligned}$$

Suppose that $z_1^3 + z_2^3 = 0$. Then $z_1 + z_2 = 0$ or $z_1^2 - z_1 z_2 + z_3^2 = 0$ implying $z_1^2 + z_2^2 = -2z_1 z_2$ or $z_1^2 + z_2^2 = z_1 z_2$.

On the other hand, from the given relation it follows that $z_3^3 = -z_1z_2z_3$, yielding $z_3^2 = -z_1z_2$.

We have

$$|z_1 + z_2 + z_3|^2 = (z_1 + z_2 + z_3)\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right)$$
$$= 3 + \left(\frac{z_1}{z_2} + \frac{z_2}{z_1}\right) + \left(\frac{z_1}{z_3} + \frac{z_3}{z_2}\right) + \left(\frac{z_2}{z_3} + \frac{z_3}{z_1}\right)$$
$$= 3 + \frac{z_1^2 + z_2^2}{z_1z_2} + \frac{z_3^2 + z_1z_2}{z_2z_3} + \frac{z_3^2 + z_1z_2}{z_3z_1} = 3 + \frac{z_1^2 + z_2^2}{z_1z_2}.$$

This leads to $|z_1 + z_2 + z_3|^2 = 1$ if $z_1^2 + z_2^2 = -2z_1z_2$ and $|z_1 + z_2 + z_3|^2 = 4$ if $z_1^2 + z_2^2 = z_1z_2$. The conclusion follows.

Problem 4. *Let z_1, z_2 be complex numbers with $|z_1| = |z_2| = 1$. Prove that*

$$|z_1 + 1| + |z_2 + 1| + |z_1z_2 + 1| \geq 2.$$

Solution. We have

$$|z_1 + 1| + |z_2 + 1| + |z_1z_2 + 1|$$
$$\geq |z_1 + 1| + |z_1z_2 + 1 - (z_2 + 1)| = |z_1 + 1| + |z_1z_2 - z_2|$$
$$\geq |z_1 + 1| + |z_2||z_1 - 1| = |z_1 + 1| + |z_1 - 1|$$
$$\geq |z_1 + 1 + z_1 - 1| = 2|z_1| = 2,$$

as claimed.

Problem 5. *Let $n > 0$ be an integer and let z be a complex number such that $|z| = 1$. Prove that*

$$n|1 + z| + |1 + z^2| + |1 + z^3| + \cdots + |1 + z^{2n}| + |1 + z^{2n+1}| \geq 2n.$$

Solution. We have

$$n|1 + z| + |1 + z^2| + |1 + z^3| + \cdots + |1 + z^{2n}| + |1 + z^{2n+1}|$$
$$= \sum_{k=1}^{n}(|1 + z| + |1 + z^{2k+1}|) + \sum_{k=1}^{n}|1 + z^{2k}|$$
$$\geq \sum_{k=1}^{n}|z - z^{2k+1}| + \sum_{k=1}^{n}|1 + z^{2k}| = \sum_{k=1}^{n}(|z||1 - z^{2k}| + |1 + z^{2k}|)$$
$$= \sum_{k=1}^{n}(|1 - z^{2k}| + |z + z^{2k}|) \geq \sum_{k=1}^{n}|1 - z^{2k} + 1 + z^{2k}| = 2n,$$

as claimed.

Alternate solution. We use induction on n.

For $n = 1$, we prove that $|1+z| + |1+z^2| + |1+z^3| \geq 2$. Indeed,

$$2 = |1 + z + z^3 + 1 - z(1+z^2)| \leq |1+z| + |z^3+1| + |z||1+z^2|$$
$$= |1+z| + |1+z^2| + |1+z^3|.$$

Assume that the inequality is valid for some n, so

$$n|1+z| + |1+z^2| + \cdots + |1+z^{2n+1}| \geq 2n.$$

We prove that

$$(n+1)|1+z| + |1+z^2| + \cdots + |1+z^{2n+1}| + |1+z^{2n+2}| + |1+z^{2n+3}| \geq 2n+2.$$

Using the inductive hypothesis yields

$$(n+1)|1+z| + |1+z^2| + \cdots + |1+z^{2n+2}| + |1+z^{2n+3}|$$
$$\geq 2n + |1+z| + |1+z^{2n+2}| + |1+z^{2n+3}|$$
$$= 2n + |1+z| + |z||1+z^{2n+2}| + |1+z^{2n+3}|$$
$$\geq 2n + |1 + z - z(1+z^{2n+2}) + 1 + z^{2n+3}| = 2n+2,$$

as needed.

Problem 6. *Let z_1, z_2, z_3 be complex numbers such that*

1) $|z_1| = |z_2| = |z_3| = 1$*;*

2) $z_1 + z_2 + z_3 \neq 0$*;*

3) $z_1^2 + z_2^2 + z_3^2 = 0$.

Prove that for all integers $n \geq 2$,

$$|z_1^n + z_2^n + z_3^n| \in \{0, 1, 2, 3\}.$$

Solution. Let

$$s_1 = z_1 + z_2 + z_3, \quad s_2 = z_1z_2 + z_2z_3 + z_3z_1, \quad s_3 = z_1z_2z_3$$

and consider the cubic equation

$$z^3 - s_1z^2 + s_2z - s_3 = 0$$

with roots z_1, z_2, z_3.

Because $z_1^2 + z_2^2 + z_3^2 = 0$, we have

$$s_1^2 = 2s_2. \tag{1}$$

On the other hand,

$$s_2 = s_3\left(\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}\right) = s_3(\overline{z_1}+\overline{z_2}+\overline{z_3}) = s_3 \cdot \overline{s_1}. \tag{2}$$

The relations (1) and (2) imply $s_1^2 = 2s_3 \cdot \overline{s_1}$ and, consequently, $|s_1|^2 = 2|s_3| \cdot |\overline{s_1}| = 2|s_1|$. Because $s_1 \neq 0$, we have $|s_1| = 2$, so $s_1 = 2\lambda$ with $|\lambda| = 1$.

From relations (1) and (2) it follows that $s_2 = \frac{1}{2}s_1^2 = 2\lambda^2$ and $s_3 = \frac{s_2}{\overline{s_1}} = \frac{2\lambda^2}{2\overline{\lambda}} = \lambda^3$.

The equation with roots z_1, z_2, z_3 becomes

$$z^3 - 2\lambda z^2 + 2\lambda^2 z - \lambda^3 = 0.$$

This is equivalent to

$$(z-\lambda)(z^2 - \lambda z + \lambda^2) = 0.$$

The roots are $\lambda, \lambda\varepsilon, -\lambda\varepsilon^2$,where $\varepsilon = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.

Without loss of generality we may assume that $z_1 = \lambda$, $z_2 = \lambda\varepsilon$, $z_3 = -\lambda\varepsilon^2$. Using the relations $\varepsilon^2 - \varepsilon + 1 = 0$ and $\varepsilon^3 = -1$, it follows that

$$E_n = |z_1^n + z_2^n + z_3^n| = |\lambda^n + \lambda^n\varepsilon^n + (-1)^n\lambda^n\varepsilon^{2n}|$$

$$= |1 + \varepsilon^n + (-1)^n\varepsilon^{2n}|.$$

It is not difficult to see that $E_{k+6} = E_k$ for all integers k and that the equalities

$$E_0 = 3, \quad E_1 = 2, \quad E_2 = 0, \quad E_3 = 1, \quad E_4 = 0, \quad E_5 = 2,$$

settle the claim.

Alternate solution. It is clear that z_1^2, z_2^2, z_3^2 are distinct. Otherwise, if, for example, $z_1^2 = z_2^2$, then $1 = |z_3^2| = |-(z_1^2 + z_2^2)| = 2|z_1^2| = 2$, a contradiction.

From $z_1^2 + z_2^2 + z_3^2 = 0$ it follows that z_1^2, z_2^2, z_3^2 are the coordinates of the vertices of an equilateral triangle. Hence we may assume that $z_2^2 = \varepsilon z_1^2$ and $z_3^2 = \varepsilon^2 z_1^2$, where $\varepsilon^2 + \varepsilon + 1 = 0$. Because $z_2^2 = \varepsilon^4 z_1^2$ and $z_3^2 = \varepsilon^2 z_1^2$ it follows that $z_2 = \pm\varepsilon^2 z_1$ and $z_3 = \pm\varepsilon z_1$. Then

$$|z_1^n + z_2^n + z_3^n| = |(1 + (\pm\varepsilon)^n + (\pm\varepsilon^2)^n)z_1^n| = |1 + (\pm\varepsilon)^n + (\pm\varepsilon^2)^n| \in \{0, 1, 2, 3\}$$

by the same argument used at the end of the previous solution.

Problem 7. *Find all complex numbers z such that*

$$|z - |z+1|| = |z + |z-1||.$$

Solution. We have

$$|z - |z+1|| = |z + |z-1||$$

if and only if

$$|z - |z+1||^2 = |z + |z-1||^2,$$

i.e.,

$$(z - |z+1|) \cdot (\overline{z} - |z+1|) = (z + |z-1|) \cdot (\overline{z} + |z-1|).$$

The last equation is equivalent to

$$z \cdot \overline{z} - (z+\overline{z})|z+1| + |z+1|^2 = z \cdot \overline{z} + (z+\overline{z}) \cdot |z-1| + |z-1|^2.$$

This can be written as

$$|z+1|^2 - |z-1|^2 = (z+\overline{z}) \cdot (|z+1| + |z-1|),$$

i.e.,

$$(z+1)(\overline{z}+1) - (z-1)(\overline{z}-1) = (z+\overline{z}) \cdot (|z+1| + |z-1|).$$

The last equation is equivalent to

$$2(z+\overline{z}) = (z+\overline{z}) \cdot (|z+1| + |z-1|), \text{ i.e., } z + \overline{z} = 0,$$

or $|z+1| + |z-1| = 2$.

The triangle inequality

$$2 = |(z+1) - (z-1)| \le |z+1| + |z-1|$$

shows that the solutions to the equation $|z+1| + |z-1| = 2$ satisfy $z + 1 = t(1-z)$, where t is a real number and $t \ge 0$.

It follows that $z = \dfrac{t-1}{t+1}$, so z is any real number with $-1 \le z \le 1$.

The equation $z + \overline{z} = 0$ has the solutions $z = bi$, $b \in \mathbb{R}$. Hence, the solutions to the equation are

$$\{bi : b \in \mathbb{R}\} \cup \{a \in \mathbb{R} : a \in [-1, 1]\}.$$

Problem 8. *Let $z_1, z_2, \ldots, z_n$ be complex numbers such that $|z_1| = |z_2| = \cdots = |z_n| > 0$. Prove that*

$$\operatorname{Re}\left(\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{z_j}{z_k}\right) = 0$$

if and only if

$$\sum_{k=1}^{n} z_k = 0.$$

(Romanian Mathematical Olympiad – Second Round, 1987)

Solution. Let

$$S = \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{z_j}{z_k}.$$

Then

$$S = \left(\sum_{k=1}^{n} z_k\right) \cdot \left(\sum_{k=1}^{n} \frac{1}{z_k}\right),$$

and since $z_k \cdot \overline{z_k} = r^2$ for all k, we have

$$S = \left(\sum_{k=1}^{n} z_k\right) \cdot \left(\sum_{k=1}^{n} \frac{\overline{z_k}}{r^2}\right)$$

$$= \frac{1}{r^2}\left(\sum_{k=1}^{n} z_k\right)\left(\overline{\sum_{k=1}^{n} z_k}\right) = \frac{1}{r^2}\left|\sum_{k=1}^{n} z_k\right|^2.$$

Hence S is a real number, so $\operatorname{Re} S = S = 0$ if and only if $\sum_{k=1}^{n} z_k = 0$.

Problem 9. *Let λ be a real number and let $n \geq 2$ be an integer. Solve the equation*

$$\lambda(\overline{z} + z^n) = i(\overline{z} - z^n).$$

Solution. The equation is equivalent to

$$z^n(\lambda + i) = \overline{z}(-\lambda + i).$$

Taking the absolute values of both sides of the equation, we obtain $|z|^n = |\overline{z}| = |z|$, hence $|z| = 0$ or $|z| = 1$.

If $|z| = 0$, then $z = 0$ which satisfies the equation. If $|z| = 1$, then $\overline{z} = \dfrac{1}{z}$ and the equation may be rewritten as

$$z^{n+1} = \frac{-\lambda + i}{\lambda + i}.$$

Because $\left|\dfrac{-\lambda + i}{\lambda + i}\right| = 1$, there exists $t \in [0, 2\pi)$ such that

$$\frac{-\lambda + i}{\lambda + i} = \cos t + i \sin t.$$

Then

$$z_k = \cos\frac{t + 2k\pi}{n+1} + i \sin\frac{t + 2k\pi}{n+1}$$

for $k = 0, 1, \ldots, n$ are the other solutions to the equation (besides $z = 0$).

Problem 10. *Prove that*

$$\left|\frac{6z-i}{2+3iz}\right| \le 1 \text{ if and only if } |z| \le \frac{1}{3}.$$

Solution. We have

$$\left|\frac{6z-i}{2+3iz}\right| \le 1 \text{ if and only if } |6z-i| \le |2+3iz|.$$

The last inequality is equivalent to

$$|6z-i|^2 \le |2+3iz|^2, \text{ i.e., } (6z-i)(6\overline{z}+i) \le (2+3iz)(2-3i\overline{z}).$$

We find

$$36z\cdot\overline{z} + 6iz - 6i\overline{z} + 1 \le 4 - 6i\overline{z} + 6iz + 9z\overline{z},$$

i.e., $27z\cdot\overline{z} \le 3$. Finally, $z\overline{z} \le \frac{1}{9}$ or, equivalently, $|z| \le \frac{1}{3}$, as desired.

Problem 11. *Let z be a complex number such that $z \in \mathbb{C}\setminus\mathbb{R}$ and*

$$\frac{1+z+z^2}{1-z+z^2} \in \mathbb{R}.$$

Prove that $|z| = 1$.

Solution. We have

$$\frac{1+z+z^2}{1-z+z^2} = 1 + 2\frac{z}{1-z+z^2} \in \mathbb{R} \text{ if and only if } \frac{z}{1-z+z^2} \in \mathbb{R}.$$

That is,

$$\frac{1-z+z^2}{z} = \frac{1}{z} - 1 + z \in \mathbb{R}, \text{ i.e., } z + \frac{1}{z} \in \mathbb{R}.$$

The last relation is equivalent to

$$z + \frac{1}{z} = \overline{z} + \frac{1}{\overline{z}}, \text{ i.e., } (z-\overline{z})(1-|z|^2) = 0.$$

We find $z = \overline{z}$ or $|z| = 1$.

Because z is not a real number, it follows that $|z| = 1$, as desired.

Problem 12. *Let $z_1, z_2, \ldots, z_n$ be complex numbers such that $|z_1| = \cdots = |z_n| = 1$*

$$z = \left(\sum_{k=1}^{n} z_k\right)\cdot\left(\sum_{k=1}^{n}\frac{1}{z_k}\right).$$

Prove that z is a real number and $0 \le z \le n^2$.

Solution. Note that $\overline{z_k} = \dfrac{1}{z_k}$ for all $k = 1, \ldots, n$. Because

$$\overline{z} = \left(\sum_{k=1}^{n} \overline{z_k}\right)\left(\sum_{k=1}^{n} \frac{1}{\overline{z_k}}\right) = \left(\sum_{k=1}^{n} \frac{1}{z_k}\right)\left(\sum_{k=1}^{n} z_k\right) = z,$$

it follows that z is a real number.

Let $z_k = \cos\alpha_k + i\sin\alpha_k$, where α_k are real numbers. for $k = \overline{1, n}$. Then

$$z = \left(\sum_{k=1}^{n} \cos\alpha_k + i\sum_{k=1}^{n} \sin\alpha_k\right)\left(\sum_{k=1}^{n} \cos\alpha_k - i\sum_{k=1}^{n} \sin\alpha_k\right)$$

$$= \left(\sum_{k=1}^{n} \cos\alpha_k\right)^2 + \left(\sum_{k=1}^{n} \sin\alpha_k\right)^2 \geq 0.$$

On the other hand, we have

$$z = \sum_{k=1}^{n} (\cos^2\alpha_k + \sin^2\alpha_k) + 2\sum_{1 \leq i < j \leq n} (\cos\alpha_i \cos\alpha_j + \sin\alpha_i \sin\alpha_j)$$

$$= n + 2\sum_{1 \leq i < j \leq n} \cos(\alpha_i - \alpha_j) \leq n + 2\binom{n}{2} = n + 2\frac{n(n-1)}{2} = n^2,$$

as desired.

Remark. An alternative solution to the inequalities $0 \leq z \leq n^2$ is as follows:

$$z = \left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \frac{1}{z_k}\right) = \left(\sum_{k=1}^{n} z_k\right)\left(\sum_{k=1}^{n} \overline{z}_k\right)$$

$$= \left(\sum_{k=1}^{n} z_k\right)\overline{\left(\sum_{k=1}^{n} z_k\right)} = \left|\sum_{k=1}^{n} z_k\right|^2 \leq \left(\sum_{k=1}^{n} |z_k|\right)^2 = n^2,$$

so $0 \leq z \leq n^2$.

Problem 13. *Let z_1, z_2, z_3 be complex numbers such that*

$$z_1 + z_2 + z_3 \neq 0 \quad \textit{and} \quad |z_1| = |z_2| = |z_3|.$$

Prove that

$$\operatorname{Re}\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) \cdot \operatorname{Re}\left(\frac{1}{z_1 + z_2 + z_3}\right) \geq 0.$$

Solution. Let $r = |z_1| = |z_2| = |z_3| > 0$. Then

$$z_1\overline{z_1} = z_2\overline{z_2} = z_3\overline{z_3} = r^2$$

and

$$\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = \frac{\overline{z_1} + \overline{z_2} + \overline{z_3}}{r^2} = \frac{\overline{z_1 + z_2 + z_3}}{r^2}.$$

On the other hand, we have

$$\frac{1}{z_1 + z_2 + z_3} = \frac{\overline{z_1 + z_2 + z_3}}{|z_1 + z_2 + z_3|^2}$$

and, consequently,

$$\operatorname{Re}\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) \cdot \operatorname{Re}\left(\frac{1}{z_1 + z_2 + z_3}\right)$$

$$= \operatorname{Re}\left(\frac{\overline{z_1 + z_2 + z_3}}{r^2}\right) \cdot \operatorname{Re}\left(\frac{\overline{z_1 + z_2 + z_3}}{|z_1 + z_2 + z_3|^2}\right) = \frac{(\operatorname{Re}(\overline{z_1 + z_2 + z_3}))^2}{r^2|z_1 + z_2 + z_3|^2} \geq 0,$$

as desired.

Problem 14. *Let x, y, z be complex numbers.*

a) Prove that

$$|x| + |y| + |z| \leq |x + y - z| + |x - y + z| + |-x + y + z|.$$

b) If x, y, z are distinct and the numbers $x + y - z$, $x - y + z$, $-x + y + z$ have equal absolute values, prove that

$$2(|x| + |y| + |z|) \leq |x + y - z| + |x - y + z| + |-x + y + z|.$$

Solution. Let

$$m = -x + y + z, \quad n = x - y + z, \quad p = x + y - z.$$

We have

$$x = \frac{n + p}{2}, \quad y = \frac{m + p}{2}, \quad z = \frac{m + n}{2}.$$

a) Adding the inequalities

$$|x| \leq \frac{1}{2}(|n| + |p|), \quad |y| \leq \frac{1}{2}(|m| + |p|), \quad |z| \leq \frac{1}{2}(|m| + |n|)$$

yields

$$|x| + |y| + |z| \leq |m| + |n| + |p|,$$

as desired.

b) Let A, B, C be the points with coordinates m, n, p and observe that numbers m, n, p are distinct and that $|m| = |n| = |p| = R$, the circumradius of triangle ABC.

Let the origin of the complex plane be the circumcenter of triangle ABC.

The orthocenter H of triangle ABC has the coordinate $h = m + n + p$. The desired inequality becomes

$$|h - m| + |h - n| + |h - p| \le |m| + |n| + |p|$$

or

$$AH + BH + CH \le 3R.$$

This is equivalent to

$$\cos A + \cos B + \cos C \le \frac{3}{2}. \tag{1}$$

Inequality (1) can be written as

$$2\cos\frac{A+B}{2}\cos\frac{A-B}{2} + 1 - 2\sin^2\frac{C}{2} \le \frac{3}{2}$$

or

$$0 \le \left(2\sin\frac{C}{2} - \cos\frac{A-B}{2}\right)^2 + \sin^2\frac{A-B}{2},$$

which is clear. We have equality in (1) if and only if triangle ABC is equilateral, i.e., $m = a$, $n = a\varepsilon$, $p = a\varepsilon^2$, where a is a complex parameter and $\varepsilon = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. In this case $x = -\frac{a}{2}$, $y = -\frac{a}{2}\varepsilon$, $z = -\frac{a}{2}\varepsilon^2$.

Problem 15. *Let $z_0, z_1, z_2, \ldots, z_n$ be complex numbers such that*

$$(k+1)z_{k+1} - i(n-k)z_k = 0$$

for all $k \in \{0, 1, 2, \ldots, n-1\}$.

1) Find z_0 such that

$$z_0 + z_1 + \cdots + z_n = 2^n.$$

2) For the value of z_0 determined above, prove that

$$|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 < \frac{(3n+1)^n}{n!}.$$

Solution. a) Use induction to prove that

$$z_k = i^k \binom{n}{k} z_0, \text{ for all } k \in \{0, 1, \ldots, n\}.$$

Then

$$z_0 + z_1 + \cdots + z_n = 2^n \text{ if and only if } z_0(1+i)^n = 2^n,$$

i.e., $z_0 = (1-i)^n$.

b) Applying the AM-GM inequality, we have

$$|z_0|^2 + |z_1|^2 + \cdots + |z_n|^2 = |z_0|^2 \left(\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 \right)$$

$$= |z_0|^2 \cdot \binom{2n}{n} = 2^n \cdot \binom{2n}{n} = \frac{2^n}{n!} 2n(2n-1)\cdots(n+1)$$

$$< \frac{2^n}{n!} \left(\frac{2n + (2n-1) + \cdots + (n+1)}{n} \right)^n = \frac{(3n+1)^n}{n!},$$

as desired.

Problem 16. *Let z_1, z_2, z_3 be complex numbers such that*

$$z_1 + z_2 + z_3 = z_1 z_2 + z_2 z_3 + z_3 z_1 = 0.$$

Prove that $|z_1| = |z_2| = |z_3|$.

Solution. Substituting $z_1 + z_2 = -z_3$ in $z_1 z_2 + z_3(z_1 + z_3) = 0$ gives $z_1 z_2 = z_3^2$, so $|z_1| \cdot |z_2| = |z_3|^2$. Likewise, $|z_2| \cdot |z_3| = |z_1|^2$ and $|z_3||z_1| = |z_2|^2$. Then

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = |z_1||z_2| + |z_2||z_3| + |z_3||z_1|,$$

i.e.,

$$(|z_1| - |z_2|)^2 + (|z_2| - |z_3|)^2 + (|z_3| - |z_1|)^2 = 0,$$

yielding $|z_1| = |z_2| = |z_3|$.

Alternate solution. Using the relations between the roots and the coefficients, it follows that z_1, z_2, z_3 are the roots of polynomial $z^3 - p$, where $p = z_1 z_2 z_3$. Hence $z_1^3 - p = z_2^3 - p = z_3^3 - p = 0$, implying $z_1^3 = z_2^3 = z_3^3$, and the conclusion follows.

Problem 17. *Prove that for all complex numbers z with $|z| = 1$ the following inequalities hold:*

$$\sqrt{2} \leq |1 - z| + |1 + z^2| \leq 4.$$

Solution. Setting $z = \cos t + i \sin t$ yields

$$|1 - z| = \sqrt{(1 - \cos t)^2 + \sin^2 t} = \sqrt{2 - 2\cos t} = 2\left|\sin \frac{t}{2}\right|$$

and

$$|1 + z^2| = \sqrt{(1 + \cos 2t)^2 + \sin^2 2t} = \sqrt{2 + 2\cos 2t}$$

$$= 2|\cos t| = 2\left|1 - 2\sin^2 \frac{t}{2}\right|.$$

It suffices to prove that $\frac{\sqrt{2}}{2} \leq |a| + |1 - 2a^2| \leq 2$, for $a = \sin \frac{t}{2} \in [-1, 1]$. We leave this to the reader.

Problem 18. *Let z_1, z_2, z_3, z_4 be distinct complex numbers such that*

$$\operatorname{Re}\frac{z_2 - z_1}{z_4 - z_1} = \operatorname{Re}\frac{z_2 - z_3}{z_4 - z_3} = 0.$$

a) Find all real numbers x such that

$$|z_1 - z_2|^x + |z_1 - z_4|^x \le |z_2 - z_4|^x \le |z_2 - z_3|^x + |z_4 - z_3|^x.$$

b) Prove that $|z_3 - z_1| \le |z_4 - z_2|$.

Solution. Consider the points A, B, C, D with coordinates z_1, z_2, z_3, z_4, respectively. The conditions

$$\operatorname{Re}\frac{z_2 - z_1}{z_4 - z_1} = \operatorname{Re}\frac{z_2 - z_3}{z_4 - z_3} = 0$$

imply $\widehat{BAD} = \widehat{BCD} = 90°$. Then $|z_1 - z_2| = AB$ and $|z_1 - z_4| = AD$ are the lengths of the sides of the right triangle ABD with hypotenuses $BD = |z_2 - z_4|$.

The inequality $AB^x + AD^x \le BD^x$ holds for $x \ge 2$.

Similarly, $|z_2 - z_3| = BC$ and $|z_4 - z_3| = CD$ are the sides of the right triangle BCD, so the inequality $BD^x \le BC^x + CD^x$ holds for $x \le 2$. Consequently, $x = 2$.

Finally, $AC = |z_3 - z_1| \le BD = |z_4 - z_2|$, since AC is a chord in the circle of diameter BD.

Problem 19. *Let x and y be distinct complex numbers such that $|x| = |y|$. Prove that*

$$\frac{1}{2}|x + y| < |x|.$$

Solution. Let $x = a + ib$ and $y = c + id$, with $a, b, c, d \in \mathbb{R}$ and $a^2 + b^2 = c^2 + d^2$. The inequality is equivalent to

$$(a + c)^2 + (b + d)^2 < 4(a^2 + b^2)$$

or

$$(a - c)^2 + (b - d)^2 > 0,$$

which is clear, since $x \ne y$.

Alternate solution. Consider points $X(x)$ and $Y(y)$. In triangle XOY we have $OX = OY$. Hence $OM < OX$, where M is the midpoint of segment $[XY]$. The coordinate of point M is $\dfrac{x + y}{2}$, and the desired inequality follows.

Problem 20. *Consider the set*

$$A = \{z \in \mathbb{C} \colon z = a + bi,\ a > 0, |z| < 1\}.$$

Prove that for any $z \in A$ there is a number $x \in A$ such that

$$z = \frac{1-x}{1+x}.$$

Solution. Let $z \in A$. The equation $z = \dfrac{1-x}{1+x}$ has the root

$$x = \frac{1-z}{1+z} = \frac{1-a-ib}{1+a+ib},$$

where $a > 0$ and $a^2 + b^2 < 1$.

To prove that $x \in A$, it suffices to show that $|x| < 1$ and $\mathrm{Re}(x) > 0$. Indeed, we have

$$|x|^2 = \frac{(1-a)^2 + b^2}{(1+a)^2 + b^2} < 1 \text{ if and only if } (1-a)^2 < (1+a)^2,$$

i.e., $0 < 4a$, as needed.

Moreover, $\mathrm{Re}(x) = \dfrac{1-|z|^2}{|1+z|^2} > 0$, since $|z| < 1$.

Here are more problems involving moduli and conjugates of complex numbers.

Problem 21. Consider the set

$$A = \{z \in \mathbb{C} \colon |z| < 1\},$$

a real number a with $|a| > 1$, and the function

$$f \colon A \to A, \quad f(z) = \frac{1+az}{z+a}.$$

Prove that f is bijective.

Problem 22. Let z be a complex number such that $|z| = 1$ and both $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$ are rational numbers. Prove that $|z^{2n} - 1|$ is rational for all integers $n \geq 1$.

Problem 23. Consider the function

$$f \colon \mathbb{R} \to \mathbb{C}, \quad f(t) = \frac{1+ti}{1-ti}.$$

Prove that f is injective and determine its range.

Problem 24. Let $z_1, z_2 \in \mathbb{C}^*$ such that $|z_1 + z_2| = |z_1| = |z_2|$. Compute $\dfrac{z_1}{z_2}$.

Problem 25. Prove that for any complex numbers $z_1, z_2, \ldots, z_n$ the following inequality holds:

$$\begin{aligned}&\big(|z_1| + |z_2| + \cdots + |z_n| + |z_1 + z_2 + \cdots + z_n|\big)^2\\ &\geq 2\big(|z_n|^2 + \cdots + |z_n|^2 + |z_1 + z_2 + \cdots + z_n|^2\big).\end{aligned}$$

Problem 26. Let $z_1, z_2, \ldots, z_{2n}$ be complex numbers such that $|z_1| = |z_2| = \cdots = |z_{2n}|$ and $\arg z_1 \le \arg z_2 \le \cdots \le \arg z_{2n} \le \pi$. Prove that

$$|z_1 + z_{2n}| \le |z_2 + z_{2n-1}| \le \cdots \le |z_n + z_{n+1}|.$$

Problem 27. Find all positive real numbers x and y satisfying the system of equations

$$\sqrt{3x}\left(1 + \frac{1}{x+y}\right) = 2,$$
$$\sqrt{7y}\left(1 - \frac{1}{x+y}\right) = 4\sqrt{2}.$$

(1996 Vietnamese Mathematical Olympiad)

Problem 28. Let z_1, z_2, z_3 be complex numbers. Prove that $z_1 + z_2 + z_3 = 0$ if and only if $|z_1| = |z_2 + z_3|$, $|z_2| = |z_3 + z_1|$ and $|z_3| = |z_1 + z_2|$.

Problem 29. Let $z_1, z_2, \ldots, z_n$ be distinct complex numbers with the same modulus such that

$$z_3 z_4 \cdots z_{n-1} z_n + z_1 z_4 \cdots z_{n-1} z_n + \cdots + z_1 z_2 \cdots z_{n-2} = 0.$$

Prove that

$$z_1 z_2 + z_2 z_3 + \cdots + z_{n-1} z_n = 0.$$

Problem 30. *Let a and z be complex numbers such that $|z + a| = 1$. Prove that*

$$|z^2 + a^2| \ge \frac{|1 - 2|a||}{\sqrt{2}}.$$

Problem 31. Find the geometric images of the complex numbers z for which

$$z^n \cdot \operatorname{Re}(z) = \overline{z}^n \cdot \operatorname{Im}(z),$$

where n is an integer.

Problem 32. Let a, b be real numbers with $a + b = 1$ and let z_1, z_2 be complex numbers with $|z_1| = |z_2| = 1$.

Prove that

$$|az_1 + bz_2| \ge \frac{|z_1 + z_2|}{2}.$$

Problem 33. Let k, n be positive integers and let $z_1, z_2, \ldots, z_n$ be nonzero complex numbers with the same modulus such that

$$z_1^k + z_2^k + \cdots + z_n^k = 0.$$

Prove that

$$\frac{1}{z_1^k} + \frac{1}{z_2^k} + \cdots + \frac{1}{z_n^k} = 0.$$

5.2 Algebraic Equations and Polynomials

Problem 1. *Consider the quadratic equation*

$$a^2z^2 + abz + c^2 = 0$$

where $a, b, c \in \mathbb{C}^*$ and denote by z_1, z_2 its roots. Prove that if $\dfrac{b}{c}$ is a real number then $|z_1| = |z_2|$ or $\dfrac{z_1}{z_2} \in \mathbb{R}$.

Solution. Let $t = \dfrac{b}{c} \in \mathbb{R}$. Then $b = tc$ and

$$\Delta = (ab)^2 - 4a^2 \cdot c^2 = a^2c^2(t^2 - 4).$$

If $|t| \geq 2$, the roots of the equation are

$$z_{1,2} = \frac{-tac \pm ac\sqrt{t^2 - 4}}{2a^2} = \frac{c}{2a}(-t \pm \sqrt{t^2 - 4}),$$

and it is obvious that $\dfrac{z_1}{z_2}$ is a real number.

If $|t| < 2$, the roots of the equation are

$$z_{1,2} = \frac{c}{2a}(-t \pm i\sqrt{4 - t^2}),$$

hence $|z_1| = |z_2| = \dfrac{|c|}{|a|}$, as claimed.

Problem 2. *Let a, b, c, z be complex numbers such that $|a| = |b| = |c| > 0$ and $az^2 + bz + c = 0$. Prove that*

$$\frac{\sqrt{5} - 1}{2} \leq |z| \leq \frac{\sqrt{5} + 1}{2}.$$

Solution. Let $r = |a| = |b| = |c| > 0$. We have

$$|az^2| = |-bz - c| \leq |b||z| + |c|,$$

hence $r|z^2| \leq r|z| + r$. It follows that $|z|^2 - |z| - 1 \leq 0$, so $|z| \leq \dfrac{1 + \sqrt{5}}{2}$.

On the other hand, $|c| = |-az^2 - bz| \leq |a||z|^2 + b|z|$, such that $|z|^2 + |z| - 1 \geq 0$. Thus $|z| \geq \dfrac{\sqrt{5} - 1}{2}$, and we are done.

Problem 3. *Let p, q be complex numbers such that $|p| + |q| < 1$. Prove that the moduli of the roots of the equation $z^2 + pz + q = 0$ are less than 1.*

Solution. Because $z_1+z_2=-p$ and $z_1z_2=q$, the inequality $|p|+|q|<1$ implies $|z_1+z_2|+|z_1z_2|<1$. But $||z_1|-|z_2||\le|z_1+z_2|$, hence

$$|z_1|-|z_2|+|z_1||z_2|-1<0 \text{ if and only if } (1+|z_2|)(|z_1|-1)<0$$

and

$$|z_2|-|z_1|+|z_2||z_1|-1<0 \text{ if and only if } (1+|z_1|)(|z_2|-1)<0.$$

Consequently, $|z_1|<1$ and $|z_2|<1$, as desired.

Problem 4. *Let $f=x^2+ax+b$ be a quadratic polynomial with complex coefficients with both roots having modulus 1. Prove that $f=x^2+|a|x+|b|$ has the same property.*

Solution. Let x_1 and x_2 be the complex roots of the polynomial $f=x^2+ax+b$ and let y_1 and y_2 be the complex roots of the polynomial $g=x^2+|a|x+|b|$.

We have to prove that if $|x_1|=|x_2|=1$, then $|y_1|=|y_2|=1$.

Since $x_1\cdot x_2=b$ and $x_1+x_2=-a$, then $|b|=|x_1||x_2|=1$ and $|a|\le|x_1|+|x_2|=2$.

The quadratic polynomial $g=x^2+|a|x+1$ has the discriminant $\Delta=|a|^2-4\le 0$, hence

$$y_{1,2}=\frac{-|a|\pm i\sqrt{4-|a|^2}}{2}.$$

It is easy to see that $|y_1|=|y_2|=1$, as desired.

Problem 5. *Let a, b be nonzero complex numbers. Prove that the equation*

$$az^3+bz^2+\overline{b}z+\overline{a}=0$$

has at least one root with absolute value equal to 1.

Solution. Observe that if z is a root of the equation, then $\dfrac{1}{\overline{z}}$ is also a root of the equation. Consequently, if z_1, z_2, z_3 are the roots of the equation, then $\dfrac{1}{\overline{z_1}}, \dfrac{1}{\overline{z_2}}, \dfrac{1}{\overline{z_3}}$ are the same roots, not necessarily in the same order.

If $z_k=\dfrac{1}{\overline{z_k}}$ for some $k=1,2,3$, then $|z_k|^2=z_k\overline{z_k}=1$ and we are done. If $z_k\ne\dfrac{1}{\overline{z_k}}$ for all $k=1,2,3$, we may consider without loss of generality that

$$z_1=\frac{1}{\overline{z_2}},\quad z_2=\frac{1}{\overline{z_3}},\quad z_3=\frac{1}{\overline{z_1}}.$$

The first two equalities yield $z_1\cdot\overline{z_2}\cdot z_2\cdot\overline{z_3}=1$, hence $|z_1|\cdot|z_2|^2\cdot|z_3|=1$. On the other hand, $z_1z_2z_3=-\dfrac{\overline{a}}{a}$, so $|z_1||z_2||z_3|=1$. It follows that $|z_2|=1$, as claimed.

Problem 6. *Let $f=x^4+ax^3+bx^2+cx+d$ be a polynomial with real coefficients and real roots. Prove that if $|f(i)|=1$ then $a=b=c=d=0$.*

Solution. Let x_1, x_2, x_3, x_4 be the real roots of the polynomial f. Then

$$f = (x - x_1)(x - x_2)(x - x_3)(x - x_4)$$

and

$$|f(i)| = \sqrt{1 + x_1^2} \cdot \sqrt{1 + x_2^2} \cdot \sqrt{1 + x_3^2} \cdot \sqrt{1 + x_4^2}.$$

Because $|f(i)| = 1$, we deduce that $x_1 = x_2 = x_3 = x_4 = 0$ and consequently $a = b = c = d = 0$, as desired.

Problem 7. *Prove that if* $11z^{10} + 10iz^9 + 10iz - 11 = 0$, *then* $|z| = 1$.

(1989 Putnam Mathematical Competition)

Solution. The equation can be rewritten as $z^9 = \dfrac{11 - 10iz}{11z + 10i}$. If $z = a + bi$, then

$$|z|^9 = \left|\frac{11 - 10iz}{11z + 10i}\right| = \frac{\sqrt{11^2 + 220b + 10^2(a^2 + b^2)}}{\sqrt{11^2(a^2 + b^2) + 220b + 10^2}}.$$

Let $f(a, b)$ and $g(a, b)$ denote the numerator and denominator of the right-hand side. If $|z| > 1$, then $a^2 + b^2 > 1$, so $g(a, b) > f(a, b)$, leading to $|z^9| < 1$, a contradiction. If $|z| < 1$, then $a^2 + b^2 < 1$, so $g(a, b) < f(a, b)$, yielding $|z^9| > 1$, again a contradiction. Hence $|z| = 1$.

Problem 8. *Let* $n \geq 3$ *be an integer and let* a *be a nonzero real number. Show that any nonreal root* z *of the equation* $x^n + ax + 1 = 0$ *satisfies the inequality*

$$|z| \geq \sqrt[n]{\frac{1}{n - 1}}.$$

(Romanian Mathematical Olympiad – Final Round, 1995)

Solution. Let $z = r(\cos\alpha + i\sin\alpha)$ be a nonreal root of the equation, where $\alpha \in (0, 2\pi)$ and $\alpha \neq \pi$. Substituting back into the equation we find $r^n \cos n\alpha + ra\cos\alpha + 1 + i(r^n \sin n\alpha + ra\sin\alpha) = 0$. Hence

$$r^n \cos n\alpha + ra\cos\alpha + 1 = 0 \text{ and } r^n \sin n\alpha + ra\sin\alpha = 0.$$

Multiplying the first relation by $\sin\alpha$, the second by $\cos\alpha$, and then subtracting them we find that $r^n \sin(n - 1)\alpha = \sin\alpha$. It follows that

$$r^n|\sin(n - 1)\alpha| = |\sin\alpha|.$$

The inequality $|\sin k\alpha| \leq k|\sin\alpha|$ is valid for any positive integer k. The proof is based on a simple inductive argument on k.

Applying this inequality, from $r^n|\sin(n - 1)\alpha| = |\sin\alpha|$, we obtain $|\sin\alpha| \leq r^n(n - 1)|\sin\alpha|$. Because $\sin\alpha \neq 0$, it follows that $r^n \geq \dfrac{1}{n - 1}$, i.e., $|z| \geq \sqrt[n]{\dfrac{1}{n - 1}}$.

Problem 9. *Suppose P is a polynomial with complex coefficients and an even degree. If all the roots of P are complex nonreal numbers with modulus 1, prove that*

$$P(1) \in \mathbb{R} \textit{ if and only if } P(-1) \in \mathbb{R}.$$

Solution. It suffices to prove that $\dfrac{P(1)}{P(-1)} \in \mathbb{R}$.

Let $x_1, x_2, \ldots, x_{2n}$ be the roots of P. Then

$$P(x) = \lambda(x - x_1)(x - x_2)\cdots(x - x_{2n})$$

for some $\lambda \in \mathbb{C}^*$, and

$$\frac{P(1)}{P(-1)} = \frac{\lambda(1 - x_1)(1 - x_2)\cdots(1 - x_{2n})}{\lambda(-1 - x_1)(-1 - x_2)\cdots(-1 - x_{2n})} = \prod_{k=1}^{2n} \frac{1 - x_k}{1 + x_k}.$$

From the hypothesis we have $|x_k| = 1$ for all $k = 1, 2, \ldots, 2n$. Then

$$\overline{\left(\frac{1 - x_k}{1 + x_k}\right)} = \frac{1 - \overline{x_k}}{1 + \overline{x_k}} = \frac{1 - \dfrac{1}{x_k}}{1 + \dfrac{1}{x_k}} = \frac{x_k - 1}{x_k + 1} = -\frac{1 - x_k}{1 + x_k},$$

hence

$$\overline{\left(\frac{P(1)}{P(-1)}\right)} = \prod_{k=1}^{2n} \overline{\left(\frac{1 - x_k}{1 + x_k}\right)} = \prod_{k=1}^{2n} \left(-\frac{1 - x_k}{1 + x_k}\right)$$

$$= (-1)^{2n} \prod_{k=1}^{2n} \frac{1 - x_k}{1 + x_k} = \frac{P(1)}{P(-1)}.$$

This proves that $\dfrac{P(1)}{P(-1)}$ is a real number, as desired.

Problem 10. *Consider the sequence of polynomials defined by $P_1(x) = x^2 - 2$ and $P_j(x) = P_1(P_{j-1}(x))$ for $j = 2, 3, \ldots$. Show that for any positive integer n the roots of equation $P_n(x) = x$ are all real and distinct.*

(18th IMO – Shortlist)

Solution. Put $x = z + z^{-1}$, where z is a nonzero complex number. Then $P_1(x) = x^2 - 2 = (z + z^{-1})^2 - 2 = z^2 + z^{-2}$. A simple inductive argument shows that for all positive integers n we have $P_n(x) = z^{2^n} + z^{-2^n}$.

The equation $P_n(x) = x$ is equivalent to $z^{2^n} + z^{-2^n} = z + z^{-1}$. We obtain $z^{2^n} - z = z^{-1} - z^{-2^n}$, i.e., $z(z^{2^n-1} - 1) = z^{-2^n}(z^{2^n-1} - 1)$. It follows that $(z^{2^n-1} - 1)(z^{2^n+1} - 1) = 0$. Because $\gcd(2^n - 1, 2^n + 1) = 1$ the unique common root of equations $z^{2^n-1} - 1 = 0$ and $z^{2^n+1} - 1 = 0$ is $z = 1$ (see Proposition 1 in Section 2.2). Moreover, for any root of equation $(z^{2^n-1}-1)(z^{2^n+1}-1)=0$ we have $|z|=1$, i.e., $z^{-1}=\overline{z}$. Also, observe that for two roots z and w of $(z^{2^n-1}-1)(z^{2^n+1}-1)=0$ which are different from 1, we have $z+z^{-1}=w+w^{-1}$ if and only if $(z-w)(1-(zw)^{-1})=0$. This is equivalent to $zw=1$, i.e., $w=z^{-1}=\overline{z}$, a contradiction to the fact that the unique common root of $z^{2^n-1}-1=0$ and $z^{2^n+1}-1=0$ is 1.

It is clear that the degree of polynomial P_n is 2^n. As we have seen before, all the roots of $P_n(x)=x$ are given by $x=z+z^{-1}$, where $z=1$, $z=\cos\dfrac{2k\pi}{n}+i\sin\dfrac{2k\pi}{2^n-1}$, $k=1,\dots,2^n-2$ and $z=\cos\dfrac{2s\pi}{2^n+1}+i\sin\dfrac{2s\pi}{2^n+1}$, $s=1,\dots,2^n$.

Taking into account the symmetry of expression $z+z^{-1}$, the total number of these roots is $1+\dfrac{1}{2}(2^n-2)+\dfrac{1}{2}2^n=2^n$ and all of them are real and distinct.

Here are other problems involving algebraic equations and polynomials.

Problem 11. *Let a, b, c be complex numbers with $a\neq 0$. Prove that if the roots of the equation $az^2+bz+c=0$ have equal moduli then $\overline{a}b|c|=|a|\overline{b}c$.*

Problem 12. *Let z_1, z_2 be the roots of the equation $z^2+z+1=0$ and let z_3, z_4 be the roots of the equation $z^2-z+1=0$. Find all integers n such that $z_1^n+z_2^n=z_3^n+z_4^n$.*

Problem 13. *Consider the equation with real coefficients*

$$x^6+ax^5+bx^4+cx^3+bx^2+ax+1=0,$$

and denote by $x_1, x_2, \dots, x_6$ the roots of the equation.

Prove that

$$\prod_{k=1}^{6}(x_k^2+1)=(2a-c)^2.$$

Problem 14. *Let a and b be complex numbers and let $P(z)=az^2+bz+i$. Prove that there is a $z_0\in\mathbb{C}$ with $|z_0|=1$ such that $|P(z_0)|\geq 1+|a|$.*

Problem 15. *Find all polynomials f with real coefficients satisfying, for any real number x, the relation $f(x)f(2x^2)=f(2x^3+x)$.*

(21^{st} IMO – Shortlist)

5.3 From Algebraic Identities to Geometric Properties

Problem 1. *Consider equilateral triangles ABC and $A'B'C'$, both in the same plane and having the same orientation. Show that the segments $[AA']$, $[BB']$, $[CC']$ can be the sides of a triangle.*

Solution. Let a, b, c be the coordinates of vertices A, B, C and let a', b', c' be the coordinates of vertices A', B', C'. Because triangles ABC and $A'B'C'$ are similar, we have the relation (see Remark 1 in Section 3.3):

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a' & b' & c' \end{vmatrix} = 0. \tag{1}$$

That is,

$$a'(b-c) + b'(c-a) + c'(a-b) = 0. \tag{2}$$

On the other hand the following relation is clear:

$$a(b-c) + b(c-a) + c(a-b) = 0. \tag{3}$$

By subtracting relation (3) from relation (2), we find

$$(a'-a)(b-c) + (b'-b)(c-a) + (c'-c)(a-b) = 0. \tag{4}$$

Passing to moduli, it follows that

$$|a'-a||b-c| \le |b'-b||c-a| + |c'-c||a-b|. \tag{5}$$

Taking into account that $|b-c| = |c-a| = |a-b|$, we obtain $AA' \le BB' + CC'$. Similarly we prove the inequalities $BB' \le CC' + AA'$ and $CC' \le AA' + BB'$, hence the desired conclusion follows.

Remarks. 1) If ABC and $A'B'C'$ are two similar triangles situated in the same plane and having the same orientation, then from (5) the inequality

$$AA' \cdot BC \le BB' \cdot CA + CC' \cdot AB \tag{6}$$

follows. This is the *generalized Ptolemy inequality*. Ptolemy's inequality is obtained when the triangle $A'B'C'$ degenerates to a point.

2) Taking into account the inequality (6), we have also $BB' \cdot CA \le CC' \cdot AB + AA' \cdot BC$ and $CC' \cdot AB \le AA' \cdot BC + BB' \cdot CA$. It follows that for any two similar triangles ABC and $A'B'C'$ with the same orientation and situated in the same plane, we can construct a triangle of sides lengths $AA' \cdot BC$, $BB' \cdot CA$, $CC' \cdot AB$.

3) In the case when the triangle $A'B'C'$ degenerates to the point M, from the property in our problem it follows that the segments MA, MB, MC are the sides of a triangle, i.e., Pompeiu's theorem (see also Subsection 4.9.1).

Problem 2. *Let P be an arbitrary point in the plane of a triangle ABC. Then*

$$\alpha \cdot PB \cdot PC + \beta \cdot PC \cdot PA + \gamma \cdot PA \cdot PB \ge \alpha\beta\gamma,$$

where α, β, γ are the sides of ABC.

Solution. Let us consider the origin of the complex plane at P and let a, b, c be the coordinates of vertices of triangle ABC. From the algebraic identity

$$\frac{bc}{(a-b)(a-c)} + \frac{ca}{(b-c)(b-a)} + \frac{ab}{(c-a)(c-b)} = 1. \tag{1}$$

Passing to the absolute value, it follows that

$$\frac{|b||c|}{|a-b||a-c|} + \frac{|c||a|}{|b-c||b-a|} + \frac{|a||b|}{|c-a||c-b|} \geq 1. \tag{2}$$

Taking into account that $|a| = PA$, $|b| = PB$, $|c| = PC$, and $|b-c| = \alpha$, $|c-a| = \beta$, $|a-b| = \gamma$, the inequality (2) is equivalent to

$$\frac{PB \cdot PC}{\beta\gamma} + \frac{PC \cdot PA}{\gamma\alpha} + \frac{PA \cdot PB}{\alpha\beta} \geq 1,$$

i.e., the desired inequality.

Remarks. 1) If P is the circumcenter O of triangle ABC, we can derive Euler's inequality $R \geq 2r$. Indeed, in this case the inequality is equivalent to $R^2(\alpha+\beta+\gamma) \geq \alpha\beta\gamma$. Therefore

$$R^2 \geq \frac{\alpha\beta\gamma}{\alpha+\beta+\gamma} = \frac{\alpha\beta\gamma}{2s} = \frac{4R}{2s} \cdot \frac{\alpha\beta\gamma}{4R} = 2R \cdot \frac{\text{area}[ABC]}{s} = 2Rr,$$

hence $R \geq 2r$.

2) If P is the centroid G of triangle ABC we obtain the following inequality involving the medians $m_\alpha, m_\beta, m_\gamma$:

$$\frac{m_\alpha m_\beta}{\alpha\beta} + \frac{m_\beta m_\gamma}{\beta\gamma} + \frac{m_\gamma m_\alpha}{\gamma\alpha} \geq \frac{9}{4}$$

with equality if and only if triangle ABC is equilateral. A good argument for the case of acute-angled triangles is given in the next problem.

Problem 3. *Let ABC be an acute-angled triangle and let P be a point in its interior. Prove that*

$$\alpha \cdot PB \cdot PC + \beta \cdot PC \cdot PA + \gamma \cdot PA \cdot PB = \alpha\beta\gamma,$$

if and only if P is the orthocenter of triangle ABC.

(1998 Chinese Mathematical Olympiad)

Solution. Let P be the origin of the complex plane and let a, b, c be the coordinates of A, B, C, respectively. The relation in the problem is equivalent to

$$|ab(a-b)| + |bc(b-c)| + |ca(c-a)| = |(a-b)(b-c)(c-a)|.$$

Let

$$z_1 = \frac{ab}{(a-c)(b-c)}, \quad z_2 = \frac{bc}{(b-a)(c-a)}, \quad z_3 = \frac{ca}{(c-b)(a-b)}.$$

It follows that

$$|z_1| + |z_2| + |z_3| = 1 \quad \text{and} \quad z_1 + z_2 + z_3 = 1,$$

the latter from identity (1) in the previous problem.

We will prove that P is the orthocenter of triangle ABC if and only if z_1, z_2, z_3 are positive real numbers. Indeed, if P is the orthocenter, then since the triangle ABC is acute-angled, it follows that P is in the interior of ABC. Hence there are positive real numbers r_1, r_2, r_3 such that

$$\frac{a}{b-c} = -r_1 i, \quad \frac{b}{c-a} = -r_2 i, \quad \frac{c}{a-b} = -r_3 i,$$

implying $z_1 = r_1 r_2 > 0$, $z_2 = r_2 r_3 > 0$, $z_3 = r_3 r_1 > 0$, and we are done. Conversely, suppose that z_1, z_2, z_3 are all positive real numbers. Because

$$-\frac{z_1 z_2}{z_3} = \left(\frac{b}{c-a}\right)^2, \quad -\frac{z_2 z_3}{z_1} = \left(\frac{c}{a-b}\right)^2, \quad -\frac{z_3 z_1}{z_2} = \left(\frac{a}{b-c}\right)^2$$

it follows that $\frac{a}{b-c}$, $\frac{b}{c-a}$, $\frac{c}{a-b}$ are pure imaginary numbers, thus $AP \perp BC$ and $BP \perp CA$, showing that P is the orthocenter of triangle ABC.

Problem 4. *Let G be the centroid of triangle ABC and let R_1, R_2, R_3 be the circumradii of triangles GBC, GCA, GAB, respectively. Then*

$$R_1 + R_2 + R_3 \geq 3R,$$

where R is the circumradius of triangle ABC.

Solution. In Problem 2, consider P the centroid G of triangle ABC. Then

$$\alpha \cdot GB \cdot GC + \beta \cdot GC \cdot GA + \gamma \cdot GA \cdot GB \geq \alpha\beta\gamma, \tag{1}$$

where α, β, γ are the lengths of the sides of triangle ABC.

But

$$\alpha \cdot GB \cdot GC = 4R_1 \cdot \text{area}[GBC] = 4R_1 \cdot \frac{1}{3}\text{area}[ABC].$$

Likewise,

$$\beta \cdot GC \cdot GA = 4R_2 \cdot \frac{1}{3}\text{area}[ABC], \quad \gamma \cdot GA \cdot GB = 4R_3 \cdot \frac{1}{3}\text{area}[ABC].$$

Hence, the inequality (1) is equivalent to

$$\frac{4}{3}(R_1 + R_2 + R_3) \cdot \text{area}[ABC] \geq 4R \cdot \text{area}[ABC],$$

i.e., $R_1 + R_2 + R_3 \geq 3R$.

Problem 5. *Let ABC be a triangle and let P be a point in its interior. Let R_1, R_2, R_3 be the radii of the circumcircles of triangles PBC, PCA, PAB, respectively. Lines PA, PB, PC intersect sides BC, CA, AB at A_1, B_1, C_1, respectively. Let*

$$k_1 = \frac{PA_1}{AA_1}, \quad k_2 = \frac{PB_1}{BB_1}, \quad k_3 = \frac{PC_1}{CC_1}.$$

Prove that $k_1R_1 + k_2R_2 + k_3R_3 \geq R$, where R is the circumradius of triangle ABC.

(2004 Romanian IMO Team Selection Test)

Solution. Note that

$$k_1 = \frac{\text{area}[PBC]}{\text{area}[ABC]}, \quad k_2 = \frac{\text{area}[PCA]}{\text{area}[ABC]}, \quad k_3 = \frac{\text{area}[PAB]}{\text{area}[ABC]}.$$

But $\text{area}[ABC] = \dfrac{\alpha\beta\gamma}{4R}$ and $\text{area}[PBC] = \dfrac{\alpha \cdot PB \cdot PC}{4R_1}$. Two similar relations for area$[PCA]$ and area$[PAB]$ hold.

The desired inequality is equivalent to

$$R\frac{\alpha \cdot PB \cdot PC}{\alpha\beta\gamma} + R\frac{\beta \cdot PC \cdot PA}{\alpha\beta\gamma} + R\frac{\gamma \cdot PA \cdot PB}{\alpha\beta\gamma} \geq R$$

which reduces to the inequality in Problem 2.

In the case when triangle ABC is acute-angled, from Problem 3 it follows that equality holds if and only if P is the orthocenter of ABC.

Problem 6. *For any point M in the plane of triangle ABC the following inequality holds:*

$$AM^3 \sin A + BM^3 \sin B + CM^3 \sin C \geq 6 \cdot MG \cdot \text{area}[ABC],$$

where G is the centroid of triangle ABC.

Solution. The identity

$$x^3(y-z) + y^3(z-x) + z^3(x-y) = (x-y)(y-z)(z-x)(x+y+z) \qquad (1)$$

holds for any complex numbers x, y, z. Passing to the absolute value we obtain the inequality

$$|x^3(y-z)| + |y^3(z-x)| + |z^3(x-y)| \geq |x-y||y-z||z-x||x+y+z|. \qquad (2)$$

Let a, b, c, m be the coordinates of points A, B, C, M, respectively. In (2) consider $x = m - a$, $y = m - b$, $z = m - c$ and obtain

$$AM^3 \cdot \alpha + BM^3 \cdot \beta + CM^3 \cdot \gamma \geq 3\alpha\beta\gamma MG. \tag{3}$$

Using the formula $\text{area}[ABC] = \dfrac{\alpha\beta\gamma}{4R}$ and the law of sines the desired inequality follows from (3).

Problem 7. *Let $ABCD$ be a cyclic quadrilateral inscribed in circle $\mathcal{C}(O; R)$ having the sides lengths $\alpha, \beta, \gamma, \delta$ and the diagonals lengths d_1 and d_2. Then*

$$\text{area}[ABCD] \geq \frac{\alpha\beta\gamma\delta d_1 d_2}{8R^4}.$$

Solution. Take the center O to be the origin of the complex plane and consider a, b, c, d the coordinates of vertices A, B, C, D. From the well-known Euler identity

$$\sum_{\text{cyc}} \frac{a^3}{(a-b)(a-c)(a-d)} = 1 \tag{1}$$

by passing to the absolute value, it follows that

$$\sum_{\text{cyc}} \frac{|a|^3}{|a-b||a-c||a-d|} \geq 1. \tag{2}$$

The inequality (2) is equivalent to

$$\sum_{\text{cyc}} \frac{R^3}{AB \cdot AC \cdot AD} \geq 1 \tag{3}$$

or

$$\sum_{\text{cyc}} R^3 \cdot BD \cdot CD \cdot BC \geq \alpha\beta\gamma\delta d_1 d_2. \tag{4}$$

But we have the known relation $BD \cdot CD \cdot BC = 4R \cdot \text{area}[BCD]$ and three other such relations. The inequality (4) can be written in the form

$$4R^4(\text{area}[ABC] + \text{area}[BCD] + \text{area}[CDA] + \text{area}[DAB]) \geq \alpha\beta\gamma\delta d_1 d_2$$

or equivalently $8R^4\text{area}[ABCD] \geq \alpha\beta\gamma\delta d_1 d_2$.

Problem 8. *Let a, b, c be distinct complex numbers such that*

$$(a-b)^7 + (b-c)^7 + (c-a)^7 = 0.$$

Prove that a, b, c are the coordinates of the vertices of an equilateral triangle.

Solution. Setting $x = a - b$, $y = b - c$, $z = c - a$ implies $x + y + z = 0$ and $x^7 + y^7 + z^7 = 0$. Since $z \neq 0$, we may set $\alpha = \frac{x}{z}$ and $\beta = \frac{y}{z}$. Hence $\alpha + \beta = -1$ and $\alpha^7 + \beta^7 = -1$. Then

$$\alpha^6 - \alpha^5\beta + \alpha^4\beta^2 - \alpha^3\beta^3 + \alpha^2\beta^4 - \alpha\beta^5 + \beta^6 = 1. \tag{1}$$

Let $s = \alpha + \beta = -1$ and $p = ab$. The relation (1) becomes

$$(\alpha^6 + \beta^6) - p(\alpha^4 + \beta^4) + p^2(\alpha^2 + \beta^2) - p^3 = 1. \tag{2}$$

Because $\alpha^2 + \beta^2 = s^2 - 2p = 1 - 2p$,

$$\alpha^4 + \beta^4 = (\alpha^2 + \beta^2)^2 - 2\alpha^2\beta^2 = (1 - 2p)^2 - 2p^2 = 1 - 4p + 2p^2,$$

$$\alpha^6 + \beta^6 = (\alpha^2 + \beta^2)((\alpha^4 + \beta^4) - \alpha^2\beta^2) = (1 - 2p)(1 - 4p + p^2),$$

the equality (2) is equivalent to

$$(1 - 2p)(1 - 4p + p^2) - p(1 - 4p + 2p^2) + p^2(1 - 2p) - p^3 = 1.$$

That is, $1 - 4p + p^2 - 2p + 8p^2 - 2p^3 - p + 4p^2 - 2p^3 + p^2 - 2p^3 - p^3 = 1$; i.e., $-7p^3 + 14p^2 - 7p + 1 = 1$. We obtain $-7p(p - 1)^2 = 0$, hence $p = 0$ or $p = 1$.

If $p = 0$, then $\alpha = 0$ or $\beta = 0$, and consequently $x = 0$ or $y = 0$. It follows that $a = b$ or $b = c$, which is false; hence $p = 1$.

From $\alpha\beta = 1$ and $\alpha + \beta = -1$ we deduce that α and β are the roots of the quadratic equation $x^2 + x + 1 = 0$. Thus $\alpha^3 = \beta^3 = 1$ and $|\alpha| = |\beta| = 1$. Therefore $|x| = |y| = |z|$ or $|a - b| = |b - c| = |c - a|$, as claimed.

Alternate solution. Let $x = a - b$, $y = b - c$, $z = c - a$. Because $x + y + z = 0$ and $x^7 + y^7 + z^7 = 0$, we find that $(x + y)^7 - x^7 - y^7 = 0$. This is equivalent to $7xy(x + y)(x^2 + xy + y^2)^2 = 0$.

But $xyz \neq 0$, so $x^2 + xy + y^2 = 0$, i.e., $x^3 = y^3$. From symmetry, $x^3 = y^3 = z^3$, hence $|x| = |y| = |z|$.

Problem 9. *Let M be a point in the plane of the square $ABCD$ and let $MA = x$, $MB = y$, $MC = z$, $MD = t$. Prove that the numbers xy, yz, zt, tx are the sides of a quadrilateral.*

Solution. Consider the complex plane such that $1, i, -1, -i$ are the coordinates of vertices A, B, C, D of the square. If z is the coordinate of point M, then we have the identity

$$1(z - i)(z + 1) + i(z + 1)(z + i) - 1(z + i)(z - 1) - i(z - 1)(z - i) = 0. \tag{1}$$

Subtracting the first term of the sum from both sides yields

$$i(z+1)(z+i) - 1(z+i)(z-i) - i(z-1)(z-i) = -1(z-i)(z+1),$$

and using the triangle inequality we obtain

$$|z-i||z+i| + |z+1||z+i| + |z+i||z-1| \geq |z-1||z-i|$$

or $yz + zt + tx \geq xy$.

In the same manner we prove that

$$xy + zt + tx \geq yz, \quad xy + yz + tx \geq yz$$

and $xy + yz + zt \geq tx$, as needed.

Problem 10. *Let z_1, z_2, z_3 be distinct complex numbers such that $|z_1| = |z_2| = |z_3| = R$. Prove that*

$$\frac{1}{|z_1 - z_2||z_1 - z_3|} + \frac{1}{|z_2 - z_1||z_2 - z_3|} + \frac{1}{|z_3 - z_1||z_3 - z_2|} \geq \frac{1}{R^2}.$$

Solution. The following identity is easy to verify

$$\frac{z_1^2}{(z_1 - z_2)(z_1 - z_3)} + \frac{z_2^2}{(z_2 - z_1)(z_2 - z_3)} + \frac{z_3^2}{(z_3 - z_1)(z_3 - z_2)} = 1.$$

Passing to the absolute value we find that

$$1 = \left| \sum_{\text{cyc}} \frac{z_1^2}{(z_1 - z_2)(z_1 - z_3)} \right| \leq \sum_{\text{cyc}} \frac{|z_1|^2}{|z_1 - z_2||z_1 - z_3|}$$

$$= R^2 \sum_{\text{cyc}} \frac{1}{|z_1 - z_2||z_1 - z_3|},$$

i.e., the desired inequality.

Alternate solution. Let

$$\alpha = |z_2 - z_3|, \quad \beta = |z_3 - z_1|, \quad \gamma = |z_1 - z_2|.$$

From Problem 29 in Section 1.1 we have

$$\alpha\beta + \beta\gamma + \gamma\alpha \leq 9R^2.$$

Using the inequality

$$(\alpha\beta + \beta\gamma + \gamma\alpha)\left(\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha}\right) \geq 9$$

it follows that

$$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} \geq \frac{9}{\alpha\beta + \beta\gamma + \gamma\alpha} \geq \frac{1}{R^2},$$

as desired.

Remark. Consider the triangle with vertices at z_1, z_2, z_3 and whose circumcenter is the origin of the complex plane. Then the circumradius R equals $|z_1| = |z_2| = |z_3|$ and the sides are

$$\alpha = |z_2 - z_3|, \quad \beta = |z_1 - z_3|, \quad \gamma = |z_1 - z_2|.$$

The above inequality is equivalent to

$$\frac{1}{\alpha\beta} + \frac{1}{\beta\gamma} + \frac{1}{\gamma\alpha} \geq \frac{1}{R^2},$$

i.e.,

$$\alpha + \beta + \gamma \geq \frac{\alpha\beta\gamma}{R^2} = \frac{4K}{R} = \frac{4sr}{R}.$$

We obtain $R \geq 2r$, i.e., Euler's inequality for a triangle.

Problem 11. *Let ABC be a triangle and let P be a point in its plane. Prove that*

$$\alpha \cdot PA^3 + \beta \cdot PB^3 + \gamma \cdot PC^3 \geq 3\alpha\beta\gamma \cdot PG,$$

where G is the centroid of ABC.

2) Prove that

$$R^2(R^2 - 4r^2) \geq 4r^2[8R^2 - (\alpha^2 + \beta^2 + \gamma^2)].$$

Solution. 1) The identity

$$x^3(y-z) + y^3(z-x) + z^3(x-y) = (x-y)(y-z)(z-x)(x+y+z) \qquad (1)$$

holds for any complex numbers x, y, z. Passing to absolute values we obtain

$$|x|^3|y-z| + |y|^3|z-x| + |z|^3|x-y| \geq |x-y||y-z||z-x||x+y+z|.$$

Let a, b, c, z_P be the coordinates of A, B, C, P, respectively. In (2) take $x = z_P - a$, $y = z_P - b$, $z = z_P - c$ and obtain the desired inequality.

2) If P is the circumcenter O of triangle ABC, after some elementary transformations the previous inequality becomes $R^2 \geq 6r \cdot OG$. Squaring both sides yields $R^4 \geq 36r^2 \cdot OG^2$. Using the well-known relation $OG^2 = R^2 - \frac{1}{9}(\alpha^2 + \beta^2 + \gamma^2)$ we obtain $R^4 \geq 36R^2r^2 - 4r^2(\alpha^2 + \beta^2 + \gamma^2)$ and the conclusion follows.

Remark. The inequality 2) improves Euler's inequality for the class of obtuse triangles. This is equivalent to proving that $\alpha^2 + \beta^2 + \gamma^2 < 8R^2$ in any such triangle. The last relation can be written as $\sin^2 A + \sin^2 B + \sin^2 C < 2$, or $\cos^2 A + \cos^2 B - \sin^2 C > 0$. That is,

$$\frac{1 + \cos 2A}{2} + \frac{1 + \cos 2B}{2} - 1 + \cos^2 C > 0,$$

which reduces to $\cos(A + B)\cos(A - B) + \cos^2 C > 0$. This is equivalent to $\cos C[\cos(A - B) - \cos(A + B)] > 0$, i.e., $\cos A \cos B \cos C < 0$.

Here are some other problems involving this topic.

Problem 12. Let a, b, c, d be distinct complex numbers with $|a| = |b| = |c| = |d|$ and $a + b + c + d = 0$.

Then the geometric images of a, b, c, d are the vertices of a rectangle.

Problem 13. The complex numbers z_i, $i = 1, 2, 3, 4, 5$, have the same nonzero modulus and

$$\sum_{i=1}^{5} z_i = \sum_{i=1}^{5} z_i^2 = 0.$$

Prove that $z_1, z_2, \ldots, z_5$ are the coordinates of the vertices of a regular pentagon.

(Romanian Mathematical Olympiad – Final Round, 2003)

Problem 14. Let ABC be a triangle.

a) Prove that if M is any point in its plane, then

$$AM \sin A \le BM \sin B + CM \sin C.$$

b) Let A_1, B_1, C_1 be points on the sides BC, AC and AB, respectively, such that the angles of the triangle $A_1B_1C_1$ are in this order α, β, γ. Prove that

$$\sum_{\text{cyc}} AA_1 \sin\alpha \le \sum_{\text{cyc}} BC \sin\alpha.$$

(Romanian Mathematical Olympiad – Second Round, 2003)

Problem 15. Let M and N be points inside triangle ABC such that

$$\widehat{MAB} = \widehat{NAC} \quad \text{and} \quad \widehat{MBA} = \widehat{NBC}.$$

Prove that

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB} = 1.$$

(39^{th} IMO – Shortlist)

5.4 Solving Geometric Problems

Problem 1. *On each side of a parallelogram a square is drawn external to the figure. Prove that the centers of the squares are the vertices of another square.*

Solution. Consider the complex plane with origin at the intersection point of the diagonals and let $a, b, -a, -b$ be the coordinates of the vertices A, B, C, D, respectively.

Using the rotation formulas, we obtain

$$b = z_{O_1} + (a - z_{O_1})(-i) \text{ or } z_{O_1} = \frac{b + ai}{1 + i}.$$

Likewise,

$$z_{O_2} = \frac{a - bi}{1 + i}, \quad z_{O_3} = \frac{-b - ai}{1 + i}, \quad z_{O_4} = \frac{-a + bi}{1 + i}.$$

It follows that

$$\widehat{O_4 O_1 O_2} = \arg \frac{z_{O_2} - z_{O_1}}{z_{O_4} - z_{O_1}} = \arg \frac{a - bi - b - ai}{-a + bi - b - ai} = \arg i = \frac{\pi}{2},$$

so $O_1 O_2 = O_1 O_4$, and

$$\widehat{O_2 O_3 O_4} = \arg \frac{z_{O_4} - z_{O_4}}{z_{O_2} - z_{O_3}} = \arg \frac{-a + bi + b + ai}{a - bi + b + ai} = \arg i = \frac{\pi}{2},$$

so $O_3 O_4 = O_3 O_2$. Therefore $O_1 O_2 O_3 O_4$ is a square.

Problem 2. *Given a point on the circumcircle of a cyclic quadrilateral, prove that the products of the distances from the point to any pair of opposite sides or to the diagonals are equal.*

(Pappus's theorem)

Solution. Let a, b, c, d be the coordinates of the vertices A, B, C, D of the quadrilateral and consider the complex plane with origin at the circumcenter of $ABCD$. Without loss of generality assume that the circumradius equals 1.

The equation of line AB is

$$\begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ z & \overline{z} & 1 \end{vmatrix} = 0.$$

This is equivalent to

$$z(\overline{a} - \overline{b}) - \overline{z}(a - b) = \overline{a}b - a\overline{b}, \text{ i.e., } z + ab\overline{z} = a + b.$$

Let point M_1 be the foot of the perpendicular from a point M on the circumcircle to the line AB. If m is the coordinate point M, then (see Proposition 1 in Section 4.5)

$$z_{M_1} = \frac{m - ab\overline{m} + a + b}{2}$$

and

$$d(M, AB) = |m - m_1| = \left| m - \frac{m - ab\overline{m} + a + b}{2} \right| = \left| \frac{(m - a)(m - b)}{2m} \right|,$$

since $m\overline{m} = 1$.

Likewise,

$$d(M, BC) = \left|\frac{(m-b)(m-c)}{2m}\right|, \quad d(M, CD) = \left|\frac{(m-c)(m-d)}{2m}\right|,$$

$$d(M, DA) = \left|\frac{(m-d)(m-a)}{2m}\right|, \quad d(M, AC) = \left|\frac{(m-a)(m-c)}{2m}\right|$$

and

$$d(M, BD) = \left|\frac{(m-b)(m-d)}{2m}\right|.$$

Thus,

$$d(M, AB) \cdot d(M, CD) = d(M, BC) \cdot d(M, DA) = d(M, AC) \cdot d(M, BD),$$

as claimed.

Problem 3. *Three equal circles $\mathcal{C}_1(O_1; r)$, $\mathcal{C}_2(O_2; r)$ and $\mathcal{C}_3(O_3; r)$ have a common point O. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$, $\mathcal{C}_2$ and $\mathcal{C}_3$, $\mathcal{C}_3$ and $\mathcal{C}_1$, meet again at points A, B, C respectively. Prove that the circumradius of triangle ABC is equal to r.*

(Tzitzeica's[1] "five-coin problem")

Solution. Consider the complex plane with origin at point O and let z_1, z_2, z_3 be the coordinates of the centers O_1, O_2, O_3, respectively. It follows that points A, B, C have the coordinates $z_1 + z_2, z_2 + z_3, z_3 + z_1$, hence

$$AB = |(z_1 + z_2) - (z_2 + z_3)| = |z_1 - z_3| = O_1O_3.$$

Likewise, $BC = O_1O_2$ and $AC = O_2O_3$, hence triangles ABC and $O_1O_2O_3$ are congruent. Consequently, their circumradii are equal. Since $OO_1 = OO_2 = OO_3 = r$, the circumradius of triangles $O_1O_2O_3$ and ABC is equal to r, as desired.

Problem 4. *On the sides AB and BC of triangle ABC draw squares with centers D and E such that points C and D lie on the same side of line AB and points A and E lie opposite sides of line BC. Prove that the angle between lines AC and DE is equal to* $45°$.

Solution. The rotation about E through angle $90°$ mappings point C to point B, hence

$$z_B = z_E + (z_C - z_E)i \text{ and } z_E = \frac{z_B - z_C i}{1 - i}.$$

Similarly, $z_D = \dfrac{z_B - z_A i}{1 - i}$.

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[1] Gheorghe Tzitzeica (1873–1939), Romanian mathematician, made important contributions in geometry.

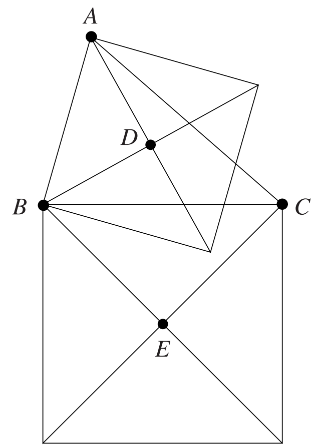

Figure 5.1.

The angle between the lines AC and DE is equal to

$$\arg \frac{z_C - z_A}{z_E - z_D} = \arg \frac{(z_C - z_A)(1-i)}{z_B - z_C i - z_B + z_A i} = \arg \frac{1-i}{-i} = \arg(1+i) = \frac{\pi}{4},$$

as desired.

Remark. If the squares are replaced in the same conditions by rectangles with centers D and E, then the angle between lines AC and DE is equal to $90^\circ - \widehat{BAD}$.

Problem 5. *On the sides AB and BC of triangles ABC equilateral triangles ABN and ACM are drawn external to the figure. If P, Q, R are the midpoints of segments BC, AM, AN, respectively, prove that triangle PQR is equilateral.*

Solution. Consider the complex plane with origin at A and denote by a lowercase letter the coordinate of the point denoted by an uppercase letter.

The rotation about center A through angle 60° maps points N and C to B and M, respectively. Setting $\varepsilon = \cos 60^\circ + i \sin 60^\circ$, we have $b = n \cdot \varepsilon$ and $m = c \cdot \varepsilon$. Thus

$$p = \frac{b+c}{2}, \quad q = \frac{m}{2} = \frac{c \cdot \varepsilon}{2}, \quad r = \frac{n}{2} = \frac{b}{2\varepsilon} = \frac{b\varepsilon^5}{2} = -\frac{b\varepsilon^2}{2}.$$

To prove that triangle PQR is equilateral, using Proposition 1 in Section 3.4, it suffices to observe that

$$p^2 + q^2 + r^2 = pq + qr + rp.$$

Problem 6. *Let $AA'BB'CC'$ be a hexagon inscribed in the circle $\mathcal{C}(O; R)$ such that*

$$AA' = BB' = CC' = R.$$

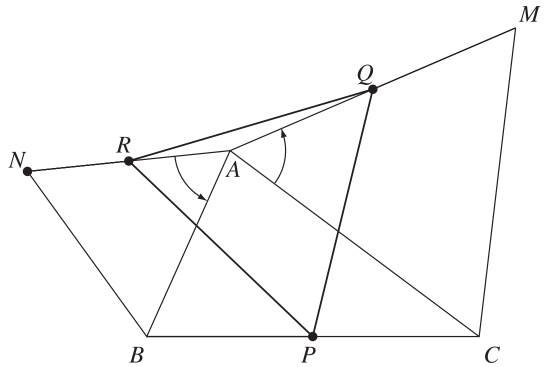

Figure 5.2.

If M, N, P are midpoints of sides AA', BB', CC' respectively, prove that triangle MNP is equilateral.

Solution. Consider the complex plane with origin at the circumcenter O and let a, b, c, a', b', c' be the coordinates of the vertices A, B, C, A', B', C', respectively. If $\varepsilon = \cos 60° + i \sin 60°$, then

$$a' = a \cdot \varepsilon, \quad b' = b \cdot \varepsilon, \quad c' = c \cdot \varepsilon.$$

The points M, N, P have the coordinates

$$m = \frac{a\varepsilon + b}{2}, \quad n = \frac{b\varepsilon + c}{2}, \quad p = \frac{c\varepsilon + a}{2}.$$

It is easy to observe that

$$m^2 + n^2 + p^2 = mn + np + pm;$$

therefore MNP is an equilateral triangle (see Proposition 1 in Section 3.4).

Problem 7. *On the sides AB and AC of triangle ABC squares $ABDE$ and $ACFG$ are drawn external to the figure. If M is the midpoint of side BC, prove that $AM \perp EG$ and $EG = 2AM$.*

Solution. Consider the complex plane with origin at A and let b, c, g, e, m be the coordinates of points B, C, G, E, M.

Observe that $g = ci$, $e = -bi$, $m = \dfrac{b+c}{2}$, hence

$$\frac{m-a}{g-e} = \frac{-(b+c)}{2i(b+c)} = \frac{i}{2} \in i\mathbb{R}^*$$

and

$$|m-a| = \frac{1}{2}|e-g|.$$

Thus, $AM \perp EG$ and $2AM = EG$.

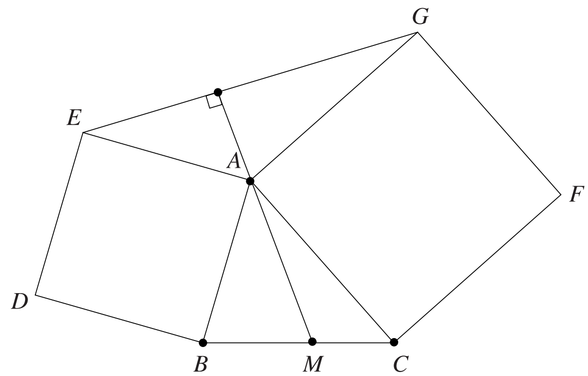

Figure 5.3.

Problem 8. *The sides AB, BC and CA of the triangle ABC are divided into three equals parts by points M, N; P, Q and R, S, respectively. Equilateral triangles MND, PQE, RSF are constructed exterior to triangle ABC. Prove that triangle DEF is equilateral.*

Solution. Denote by lowercase letters the coordinates of the points denoted by uppercase letters. Then

$$m = \frac{2a+b}{3}, \quad n = \frac{a+2b}{3}, \quad p = \frac{2b+c}{3},$$

$$q = \frac{b+2c}{3}, \quad r = \frac{2c+a}{3}, \quad s = \frac{c+2a}{3}.$$

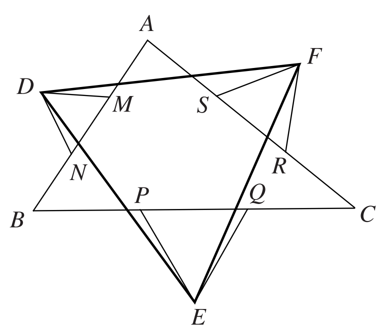

Figure 5.4.

The point D is obtained from point M by a rotation of center N and angle $60°$. Hence

$$d = n + (m-n)\varepsilon = \frac{a + 2b + (a-b)\varepsilon}{3},$$

where $\varepsilon = \cos 60° + i \sin 60°$. Likewise

$$e = q + (p - q)\varepsilon = \frac{b + 2c + (b - c)\varepsilon}{3}$$

and

$$f = s + (r - s)\varepsilon = \frac{c + 2a + (c - a)\varepsilon}{3}.$$

Since

$$\frac{f - d}{e - d} = \frac{c + a - 2b + (b + c - 2a)\varepsilon}{2c - a - b + (2b - a - c)\varepsilon}$$

$$= \frac{\varepsilon(b + c - 2a + (c + a - 2b)(-\varepsilon^2))}{2c - a - b + (2b - a - c)\varepsilon}$$

$$= \frac{\varepsilon(b + c - 2a) + (c + a - 2b)(\varepsilon - 1))}{2c - a - b + (2b - a - c)\varepsilon} = \varepsilon,$$

we have $\widehat{FDE} = 60°$ and $FD = FE$, so triangle DEF is equilateral.

Problem 9. *Let $ABCD$ be a square of length side a and consider a point P on the incircle of the square. Find the value of*

$$PA^2 + PB^2 + PC^2 + PD^2.$$

Solution. Consider the complex plane such that point A, B, C, D have coordinates

$$z_A = \frac{a\sqrt{2}}{2}, \quad z_B = \frac{a\sqrt{2}}{2}i, \quad z_C = -\frac{a\sqrt{2}}{2}, \quad z_D = -\frac{a\sqrt{2}}{2}i.$$

Let $z_P = \frac{a}{2}(\cos x + i \sin x)$ be the coordinate of point P.

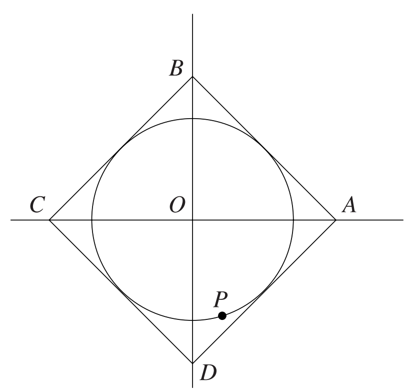

Figure 5.5.

Then

$$PA^2 + PB^2 + PC^2 + PD^2 = |z_A - z_P|^2 + |z_B - z_P|^2 + |z_C - z_P|^2 + |z_D - z_P|^2$$
$$= \sum_{\text{cyc}} (z_A - z_P)(\overline{z_A} - \overline{z_P}) = 4\frac{a^2}{2} + 2\frac{a\sqrt{2}}{2} \cdot \frac{a}{2}\left(2\cos x + 2\cos\left(x + \frac{\pi}{2}\right) + \right.$$
$$\left. +2\cos(x+\pi) + 2\cos\left(x + \frac{3\pi}{2}\right)\right) + 4\frac{a^2}{4} = 2a^2 + 0 + a^2 = 3a^2.$$

Problem 10. *On the sides AB and AD of the triangle ABD draw externally squares $ABEF$ and $ADGH$ with centers O and Q, respectively. If M is the midpoint of the side BD, prove that OMQ is an isosceles triangle with a right angle at M.*

Solution. Let a, b, d be the coordinates of the points A, B, D, respectively.

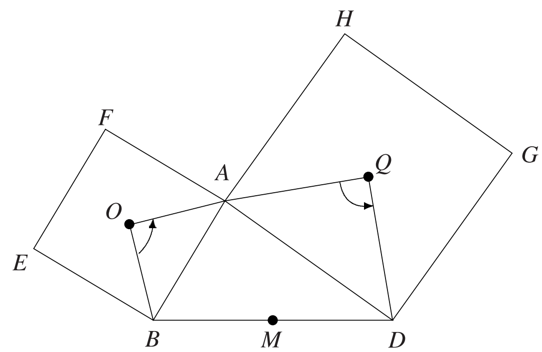

Figure 5.6.

The rotation formula gives

$$\frac{a - z_O}{b - z_O} = \frac{d - z_Q}{a - z_Q} = i,$$

so

$$z_O = \frac{b + a + (a-b)i}{2} \text{ and } z_Q = \frac{a + d + (d-a)i}{2}.$$

The coordinate of the midpoint M of segment $[BD]$ is $z_M = \dfrac{b+d}{2}$, hence

$$\frac{z_O - z_M}{z_Q - z_M} = \frac{a - d + (a-b)i}{a - b + (d-a)i} = i.$$

Therefore $QM \perp OM$ and $OM = QM$, as desired.

Problem 11. *On the sides of a convex quadrilateral $ABCD$, equilateral triangles ABM and CDP are drawn external to the figure, and equilateral triangles BCN and ADQ are drawn internal to the figure. Describe the shape of the quadrilateral $MNPQ$.*

(23rd IMO – Shortlist)

Solution. Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter.

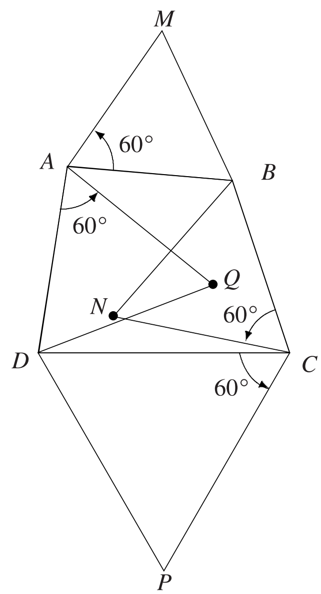

Figure 5.7.

Using the rotation formula, we obtain

$$m = a + (b - a)\varepsilon, \quad n = c + (b - c)\varepsilon,$$
$$p = c + (d - c)\varepsilon, \quad q = a + (d - a)\varepsilon,$$

where

$$\varepsilon = \cos 60^\circ + i \sin 60^\circ.$$

It is easy to notice that

$$m + p = a + c + (b + d - a - c)\varepsilon = n + q,$$

hence $MNPQ$ is a parallelogram or points M, N, P, Q are collinear.

Problem 12. *On the sides of a triangle ABC draw externally the squares $ABMM'$, $ACNN'$ and $BCPP'$. Let A', B', C' be the midpoints of the segments $M'N'$, $P'M$, PN, respectively.*

Prove that triangles ABC and $A'B'C'$ have the same centroid.

Solution. Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter.

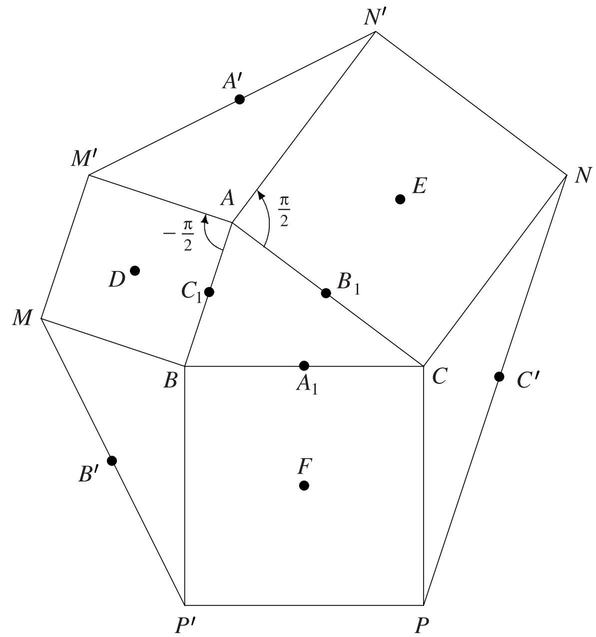

Figure 5.8.

Using the rotation formula we obtain

$$n' = a + (c - a)i \text{ and } m' = a + (b - a)(-i),$$

hence

$$a' = \frac{m' + n'}{2} = \frac{2a + (c - b)i}{2}.$$

Likewise,

$$b' = \frac{2b + (a - c)i}{2} \text{ and } c' = \frac{2c + (b - a)i}{2}.$$

Triangles $A'B'C'$ and ABC have the same centroid if and only if

$$\frac{a' + b' + c'}{3} = \frac{a + b + c}{3}.$$

Since

$$a' + b' + c' = \frac{2a + 2b + 2c + (c - b + a - c + b - a)i}{2} = a + b + c,$$

the conclusion follows.

Problem 13. *Let ABC be an acute-angled triangle. On the same side of line AC as point B draw isosceles triangles DAB, BCE, AFC with right angles at A, C, F, respectively.*

Prove that the points D, E, F are collinear.

Solution. Denote by lowercase letters the coordinates of the points denoted by uppercase letters. The rotation formula gives

$$d = a + (b - a)(-i), \quad e = c + (b - c)i, \quad a = f + (c - f)i.$$

Then

$$f = \frac{a - ci}{1 - i} = \frac{a + c + (a - c)i}{2} = \frac{d + e}{2},$$

so points F, D, E are collinear.

Problem 14. *On sides AB and CD of the parallelogram $ABCD$ draw externally equilateral triangles ABE and CDF. On the sides AD and BC draw externally squares of centers G and H.*

Prove that $EHFG$ is a parallelogram.

Solution. Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter.

Since $ABCD$ is a parallelogram, we have $a + c = b + d$.

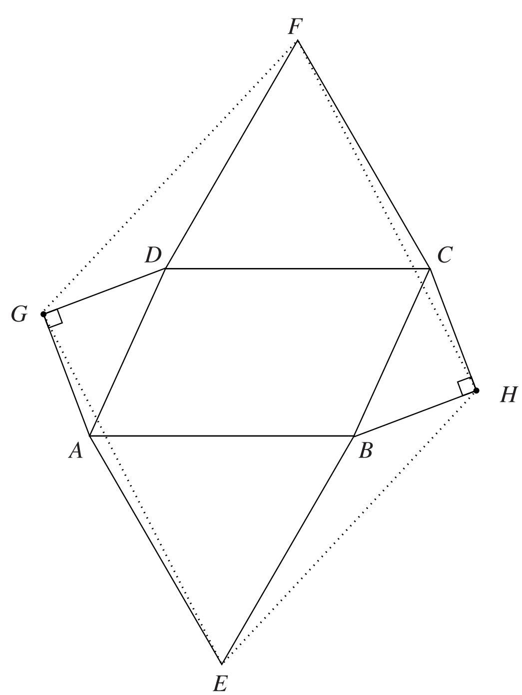

Figure 5.9.

The rotations with 90° and centers G and H mapping the points A and C into D and B, respectively. Then $d-g=(a-g)i$ and $b-h=(c-h)i$, hence $g=\dfrac{d-ai}{1-i}$ and $h=\dfrac{b-ci}{1-i}$.

The rotations with 60° and centers E and F mapping the point B and D into A and C, respectively. Then $a-e=(b-e)\varepsilon$ and $c-f=(d-f)\varepsilon$, where $\varepsilon=\cos 60^\circ+i\sin 60^\circ$. Hence $e=\dfrac{a-b\varepsilon}{1-\varepsilon}$ and $f=\dfrac{c-d\varepsilon}{1-\varepsilon}$.

Observe that

$$g+h=\frac{d+b-(a+c)i}{1-i}=\frac{(a+c)-(a+c)i}{1-i}=a+c$$

and

$$e+f=\frac{a+c-(b+d)\varepsilon}{1-\varepsilon}=\frac{a+c-(a_c)\varepsilon}{1-\varepsilon}=a+c,$$

hence $EHFG$ is a parallelogram.

Problem 15. *Let ABC be a right-angled triangle with $\widehat{C}=90^\circ$ and let D be the foot of the altitude from C. If M and N are the midpoints of the segments $[DC]$ and $[BD]$, prove that lines AM and CN are perpendicular.*

Solution. Consider the complex plane with origin at point C, and let a, b, d, m, n be the coordinates of points A, B, D, M, N, respectively.

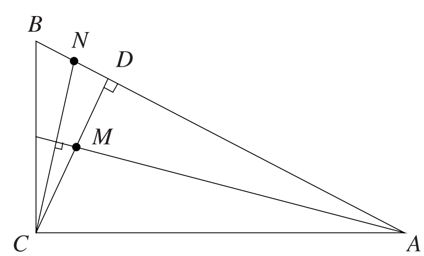

Figure 5.10.

Triangles ABC and CDB are similar with the same orientation, hence

$$\frac{a-d}{d-0}=\frac{0-d}{d-b} \text{ or } d=\frac{ab}{a+b}.$$

Then

$$m=\frac{d}{2}=\frac{ab}{2(a+b)} \text{ and } n=\frac{b+d}{2}=\frac{2ab+b^2}{2(a+b)}.$$

Thus

$$\arg\frac{m-a}{n-0}=\arg\frac{\dfrac{ab}{2(a+b)}-a}{\dfrac{2ab+b^2}{2(a+b)}}=\arg\left(-\frac{a}{b}\right)=\frac{\pi}{2},$$

so $AM\perp CN$.

Alternate solution. Using the properties of real product in Proposition 1, Section 4.1, and taking into account that $CA \perp CB$, we have

$$(m-a)\cdot(n-c) = \left(\frac{ab}{2(a+b)} - a\right)\cdot\left(\frac{2ab+b^2}{2(a+b)}\right)$$

$$= \left(a\frac{2a+b}{2(a+b)}\right)\cdot\left(b\frac{2a+b}{2(a+b)}\right) = \left|\frac{2a+b}{2(a+b)}\right|^2 (a\cdot b) = 0.$$

The conclusion follows from Proposition 2 in Section 4.1.

Problem 16. *Let ABC be an equilateral triangle with the circumradius equal to 1. Prove that for any point P on the circumcircle we have*

$$PA^2 + PB^2 + PC^2 = 6.$$

Solution. Consider the complex plane such that the coordinates of points A, B, C are the cube roots of unity $1, \varepsilon, \varepsilon^2$, respectively, and let z be the coordinate of point P. Then $|z| = 1$ and we have

$$\begin{aligned} PA^2 + PB^2 + PC^2 &= |z-1|^2 + |z-\varepsilon|^2 + |z-\varepsilon^2|^2 \\ &= (z-1)(\overline{z}-1) + (z-\varepsilon)(\overline{z}-\overline{\varepsilon}) + (z-\varepsilon^2)(\overline{z}-\overline{\varepsilon}^2) \\ &= 3|z|^2 - (1+\varepsilon+\varepsilon^2)\overline{z} - (1+\overline{\varepsilon}+\overline{\varepsilon}^2)z + 1 + |\varepsilon|^2 + |\varepsilon^2|^2 \\ &= 3 - 0\cdot\overline{z} - 0\cdot z + 1 + 1 + 1 = 6, \end{aligned}$$

as desired.

Problem 17. *Point B lies inside the segment $[AC]$. Equilateral triangles ABE and BCF are constructed on the same side of line AC. If M and N are the midpoints of segments AF and CE, prove that triangle BMN is equilateral.*

Solution. Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter. The point E is obtained from point B by a rotation with center A and angle of $60°$, hence

$$e = a + (b-a)\varepsilon, \text{ where } \varepsilon = \cos 60° + i\sin 60°.$$

Likewise, $f = b + (c-b)\varepsilon$.

The coordinates of points M and N are

$$m = \frac{a+b+(c-b)\varepsilon}{2} \text{ and } n = \frac{c+a+(b-a)\varepsilon}{2}.$$

It suffices to prove that $\dfrac{m-b}{n-b} = \varepsilon$. Indeed, we have

$$m - b = (n-b)\varepsilon$$

if and only if

$$a - b + (c - b)\varepsilon = (c + a - 2b)\varepsilon + (b - a)\varepsilon^2.$$

That is,

$$a - b = (a - b)\varepsilon + (b - a)(\varepsilon - 1),$$

as needed.

Problem 18. *Let $ABCD$ be a square with center O and let M, N be the midpoints of segments BO, CD respectively.*

Prove that triangle AMN is isosceles and right-angled.

Solution. Consider the complex plane with center at O such that $1, i, -1, -i$ are the coordinates of points A, B, C, D respectively.

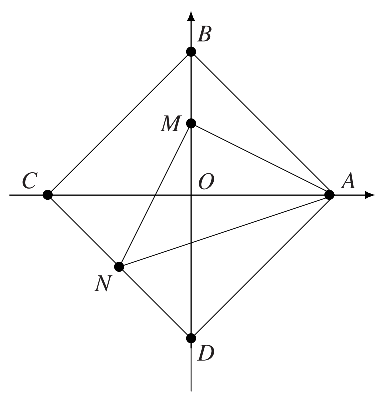

Figure 5.11.

The points M and N have the coordinates $m = \dfrac{i}{2}$ and $n = \dfrac{-1-i}{2}$, so

$$\frac{a-m}{n-m} = \frac{1 - \dfrac{i}{2}}{\dfrac{-1-i}{2} - \dfrac{i}{2}} = \frac{2-i}{-1-2i} = i.$$

Then $AM \perp MN$ and $AM = NM$, as needed.

Problem 19. *In the plane of the nonequilateral triangle $A_1A_2A_3$ consider points B_1, B_2, B_3 such that triangles $A_1A_2B_3$, $A_2A_3B_1$ and $A_3A_1B_2$ are similar with the same orientation.*

Prove that triangle $B_1B_2B_3$ is equilateral if and only if triangles $A_1A_2B_3$, $A_2A_3B_1$, $A_3A_1B_2$ are isosceles with the bases A_1A_2, A_2A_3, A_3A_1 and the base angles equal to $30°$.

Solution. Triangles $A_1A_2B_3$, $A_2A_3B_1$, $A_3A_1B_2$ are similar with the same orientation, hence $\frac{b_3 - a_2}{a_1 - a_2} = \frac{b_1 - a_3}{a_2 - a_3} = \frac{b_2 - a_1}{a_3 - a_1} = z$. Then

$$b_3 = a_2 + z(a_1 - a_2), \quad b_1 = a_3 + z(a_2 - a_3), \quad b_2 = a_1 + z(a_3 - a_1).$$

Triangle $B_1B_2B_3$ is equilateral if and only if

$$b_1 + \varepsilon b_2 + \varepsilon^2 b_3 = 0 \quad \text{or} \quad b_1 + \varepsilon b_3 + \varepsilon^2 b_2 = 0.$$

Assume the first is valid.

Then, we have

$$b_1 + \varepsilon b_2 + \varepsilon^2 b_3 = 0 \text{ if and only if}$$

$$a_3 + z(a_2 - a_3) + \varepsilon a_1 + \varepsilon z(a_3 - a_1) + \varepsilon^2 a_2 + \varepsilon^2 z(a_1 - a_2) = 0, \text{ i.e.,}$$

$$a_3 + \varepsilon a_1 + \varepsilon^2 a_2 + z(a_2 - a_3 + \varepsilon a_3 - \varepsilon a_1 + \varepsilon^2 a_1 - \varepsilon^2 a_2) = 0.$$

The last relation is equivalent to

$$\Leftrightarrow \; z[a_2(1-\varepsilon)(1+\varepsilon) - a_1\varepsilon(1-\varepsilon) - a_3(1-\varepsilon)] = -(a_3 + \varepsilon a_1 + \varepsilon^2 a_2), \text{ i.e.,}$$

$$z = +\frac{a_3 + \varepsilon a_1 + \varepsilon^2 a_2}{(1-\varepsilon)(a_3 + \varepsilon a_1 + \varepsilon^2 a_2)} = \frac{1}{1-\varepsilon} = \frac{1}{\sqrt{3}}(\cos 30^\circ + i \sin 30^\circ),$$

which shows that triangles $A_1A_2B_3$, $A_2A_3B_1$ and $A_3A_1B_2$ are isosceles with angles of 30°.

Notice that $a_3 + \varepsilon a_1 + \varepsilon^2 a_2 \neq 0$, since triangle $A_1A_2A_3$ is not equilateral.

Problem 20. *The diagonals AC and CE of a regular hexagon $ABCDEF$ are divided by interior points M and N, respectively, such that*

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r knowing that points B, M and N are collinear.

(23^{rd} IMO)

Solution. Consider the complex plane with origin at the center of the regular hexagon such that $1, \varepsilon, \varepsilon^2, \varepsilon^3, \varepsilon^4, \varepsilon^5$ are the coordinates of the vertices B, C, D, E, F, A, where

$$\varepsilon = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1 + i\sqrt{3}}{2}.$$

Since

$$\frac{MC}{MA} = \frac{NE}{NC} = \frac{1-r}{r},$$

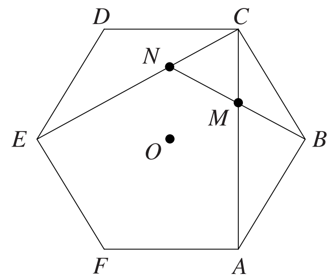

Figure 5.12.

the coordinates of points M and N are

$$m = \varepsilon r + \varepsilon^5(1 - r)$$

and

$$n = \varepsilon^2 r + \varepsilon(1 - r),$$

respectively.

The points B, M, N are collinear if and only if $\frac{m-1}{n-1} \in \mathbb{R}^*$. We have

$$m - 1 = \varepsilon r + \varepsilon^5(1 - r) - 1 = \varepsilon r - \varepsilon^2(1 - r) - 1$$

$$= \frac{1 + i\sqrt{3}}{2}r - \frac{-1 + i\sqrt{3}}{2}(1 - r) = -\frac{1}{2} + \frac{i\sqrt{3}}{2}(2r - 1)$$

and

$$n - 1 = \varepsilon^3 r + \varepsilon(1 - r) - 1 = -r + \frac{1 + i\sqrt{3}}{2}(1 - r) - 1$$

$$= -\frac{1}{2} - \frac{3r}{2} + \frac{i\sqrt{3}}{2}(1 - r),$$

hence

$$\frac{m-1}{n-1} = \frac{-1 + i\sqrt{3}(2r - 1)}{-(1 + 3r) + i\sqrt{3}(1 - r)} \in \mathbb{R}^*$$

if and only if

$$\sqrt{3}(1 - r) - (1 + 3r) \cdot \sqrt{3}(2r - 1) = 0.$$

This is equivalent to $1 - r = 6r^2 - r - 1$, i.e., $r^2 = \frac{1}{3}$. It follows $r = \frac{1}{\sqrt{3}}$.

Problem 21. *Let G be the centroid of quadrilateral $ABCD$. Prove that if lines GA and GD are perpendicular, then AD is congruent to the line segment joining the midpoints of sides AD and BC.*

Solution. Consider a, b, c, d, g the coordinates of points A, B, C, D, G, respectively. Using properties of the real product of complex numbers we have

$$GA \perp GD \text{ if and only if } (a-g)\cdot(d-g)=0, \text{ i.e.,}$$

$$\left(a-\frac{a+b+c+d}{4}\right)\cdot\left(d-\frac{a+b+c+d}{4}\right)=0.$$

That is,

$$(3a-b-c-d)\cdot(3d-a-b-c)=0$$

and we obtain

$$[a-b-c+d+2(a-d)]\cdot[a-b-c+d-2(a-d)]=0.$$

The last relation is equivalent to

$$(a+d-b-c)\cdot(a+d-b-c)=4(a-d)\cdot(a-d), \text{ i.e.,}$$

$$\left|\frac{a+d}{2}-\frac{b+c}{2}\right|^2=|a-d|^2. \tag{1}$$

Let M and N be the midpoints of the sides AD and BC. The coordinates of points M and N are $\frac{a+d}{2}$ and $\frac{b+c}{2}$, hence relation (1) shows that $MN = AD$ and we are done.

Problem 22. *Consider a convex quadrilateral $ABCD$ with the nonparallel opposite sides AD and BC. Let G_1, G_2, G_3, G_4 be the centroids of the triangles BCD, ACD, ABD, ABC, respectively. Prove that if $AG_1 = BG_2$ and $CG_3 = DG_4$ then $ABCD$ is an isosceles trapezoid.*

Solution. Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter. Setting $s = a+b+c+d$ yields

$$g_1=\frac{b+c+d}{3}=\frac{s-a}{3},\quad g_2=\frac{s-b}{3},\quad g_3=\frac{s-c}{3},\quad g_4=\frac{s-d}{3}.$$

The relation $AG_1 = BG_2$ can be written as

$$|a-g_1|=|b-g_2|,\quad \text{that is,}\quad |4a-s|=|4b-s|.$$

Using the real product of complex numbers, the last relation is equivalent to

$$(4a-s)\cdot(4a-s)=(4b-s)\cdot(4b-s), \text{ i.e.,}$$

$$16|a|^2-8a\cdot s=16|b|^2-8b\cdot s.$$

We find

$$2(|a|^2 - |b|^2) = (a - b) \cdot s. \tag{1}$$

Likewise, we have

$$CG_3 = DG_4 \text{ if and only if } 2(|c|^2 - |d|^2) = (c - d) \cdot s. \tag{2}$$

Subtracting the relations (1) and (2) gives

$$2(|a|^2 - |b|^2 - |c|^2 + |d|^2) = (a - b - c + d) \cdot (a + b + c + d).$$

That is,

$$2(|a|^2 - |b|^2 - |c|^2 + |d|^2) = |a + d|^2 - |b + c|^2, \text{ i.e.,}$$

$$2(a\overline{a} - b\overline{b} - c\overline{c} + d\overline{d}) = a\overline{c} + a\overline{d} + \overline{a}d + d\overline{d} - b\overline{b} - b\overline{c} - \overline{b}c - c\overline{c}.$$

We obtain

$$a\overline{a} - a\overline{d} - \overline{a}d + d\overline{d} = b\overline{b} - b\overline{c} - \overline{b}c + c\overline{c}, \text{ i.e.,}$$

$$|a - d|^2 = |b - c|^2.$$

Hence

$$AD = BC. \tag{3}$$

Adding relations (1) and (2) gives

$$2(|a|^2 - |b|^2 - |d|^2 + |c|^2) = (a - b - d + c) \cdot (a + b + c + d),$$

and similarly we obtain

$$AC = BD. \tag{4}$$

From relations (3) and (4) we deduce that $AB \| CD$ and consequently $ABCD$ is an isosceles trapezoid.

Problem 23. *Prove that in any quadrilateral $ABCD$,*

$$AC^2 \cdot BD^2 = AB^2 \cdot CD^2 + AD^2 \cdot BC^2 - 2AB \cdot BC \cdot CD \cdot DA \cdot \cos(A + C).$$

(Bretschneider relation or a first generalization of Ptolemy's theorem)

Solution. Let z_A, z_B, z_C, z_D be the coordinates of the points A, B, C, D in the complex plane with origin at A and point B on the positive real axis (see Fig. 5.13).

Using the identities

$$(z_A - z_C)(z_B - z_D) = -(z_A - z_B)(z_D - z_C) - (z_A - z_D)(z_C - z_B)$$

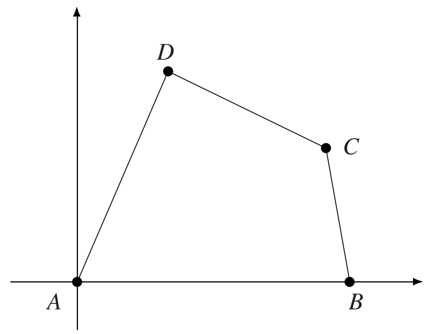

Figure 5.13.

and

$$(\overline{z_A - z_C})(\overline{z_B - z_D}) = -(\overline{z_A - z_B})(\overline{z_D - z_C}) - (\overline{z_A - z_D})(\overline{z_C - z_B}),$$

by multiplication we obtain

$$AC^2 \cdot BD^2 = AB^2 \cdot DC^2 + AD \cdot BC^2 + z + \overline{z},$$

where

$$z = (z_A - z_B)(z_D - z_C)(\overline{z_A - z_D})(\overline{z_C - z_B}).$$

It suffices to prove that

$$z + \overline{z} = -2AB \cdot BC \cdot CD \cdot DA \cdot \cos(A + C).$$

We have

$$z_A - z_B = AB(\cos\pi + i\sin\pi),$$

$$z_D - z_C = DC[\cos(2\pi - B - C) + i\sin(2\pi - B - C)],$$

$$\overline{z_A - z_D} = DA[\cos(\pi - A) + i\sin(\pi - A)]$$

and

$$\overline{z_C - z_D} = BC[\cos(\pi + B) + i\sin(\pi + B)].$$

Then

$$z + \overline{z} = 2\text{Re}z = 2AB \cdot BC \cdot CD \cdot DA\cos(5\pi - A - C)$$

$$= -2AB \cdot BC \cdot CD \cdot DA \cdot \cos(A + C)$$

and we are done.

Remark. Since $\cos(A + C) \geq -1$, this relation gives Ptolemy's inequality

$$AC \cdot BD \leq AB \cdot DC + AD \cdot BC,$$

with equality only for cyclic quadrilaterals.

Problem 24. *Let $ABCD$ be a quadrilateral and $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = d_1$ and $BC = d_2$.*

Prove that

$$d_2^2[a^2d^2 + b^2c^2 - 2abcd\cos(B - D)] = d_1^2[a^2b^2 + c^2d^2 - 2abcd\cos(A - C)]$$

(A second generalization of Ptolemy's theorem)

Solution. Let z_A, z_B, z_C, z_D be the coordinates of the points A, B, C, D in the complex plane with origin at D and point C on the positive real axis (see the figure in the previous problem but with different notation).

Multiplying the identities

$$\begin{aligned}&(z_B - z_D)[(z_A - z_B)(z_A - z_B) - (z_C - z_D)(z_C - z_D)]\\ &= (z_C - z_A) \cdot [(z_B - z_A)(z_B - z_C) - (z_D - z_A)(z_D - z_C)]\end{aligned}$$

and

$$\begin{aligned}&(\overline{z_B - z_D})[(\overline{z_A - z_B})(\overline{z_A - z_D}) - (\overline{z_C - z_B})(\overline{z_C - z_D})]\\ &= (\overline{z_C - z_A}) \cdot [(\overline{z_B - z_A})(\overline{z_B - z_C}) - (\overline{z_D - z_A})(\overline{z_D - z_C})]\end{aligned}$$

yields

$$\begin{aligned}&d_2^2[a^2 \cdot d^2 + b^2 \cdot c^2 - (z_A - z_B)(z_A - z_D)(\overline{z_C - z_B})(\overline{z_C - z_D})\\ &\quad - (z_C - z_B)(z_C - z_D)(\overline{z_A - z_B})(\overline{z_A - z_D})]\\ &= d_1^2[a^2 \cdot b^2 + c^2 \cdot d^2 - (z_B - z_A)(z_B - z_C)(\overline{z_D - z_A})(\overline{z_D - z_C})\\ &\quad - (z_D - z_A)(z_D - z_C)(\overline{z_B - z_A})(\overline{z_B - z_C})].\end{aligned}$$

It suffices to prove that

$$2\mathrm{Re}(z_A - z_B)(z_A - z_D)(\overline{z_C - z_B})(\overline{z_C - z_D}) = 2abcd\cos(B - D)$$

and

$$2\mathrm{Re}(z_B - z_A)(z_B - z_C)(\overline{z_B - z_A})(\overline{z_D - z_C}) = 2abcd\cos(A - C).$$

We have

$$z_B - z_A = a[\cos(\pi + A + D) + i\sin(\pi + A + D)],$$

$$z_B - z_C = b[\cos(\pi - C) + i\sin(\pi - C)],$$
$$\overline{z_D - z_A} = d[\cos(\pi - D) + i\sin(\pi - D)],$$
$$\overline{z_D - z_C} = c[\cos\pi + i\sin\pi],$$
$$z_A - z_B = a[\cos(A + D) + i\sin(A + D)],$$
$$z_A - z_D = d[\cos D + i\sin D],$$
$$\overline{z_C - z_B} = b[\cos B + i\sin B],$$
$$\overline{z_C - z_D} = c[\cos 0 + i\sin 0];$$

hence

$$2\mathrm{Re}(z_A - z_B)(z_A - z_D)(\overline{z_C - z_B})(\overline{z_C - z_D})$$
$$= 2abcd\cos(A + D + D + C) = 2abcd\cos(2\pi - B + D) = 2abcd\cos(B - D)$$

and

$$2\mathrm{Re}(z_B - z_A)(z_B - z_C)(\overline{z_D - z_A})(\overline{z_D - z_C})$$
$$= 2abcd\cos(\pi + A + D + \pi - C + \pi - D + \pi)$$
$$= 2abcd\cos(4\pi + A - C) = 2abcd\cos(A - C),$$

as desired.

Remark. If $ABCD$ is a cyclic quadrilateral, then $B + D = A + C = \pi$. It follows that

$$\cos(B - A) = \cos(2B - \pi) = -\cos 2B$$

and

$$\cos(A - C) = \cos(2A - \pi) = -\cos 2A.$$

The relation becomes

$$d_2^2[(ad + bc)^2 - 2abcd(1 - \cos 2B)] = d_1^2[(ab + cd)^2 - 2abcd(1 - \cos 2A)].$$

This is equivalent to

$$d_2^2(ad + bc)^2 - 4abcdd_2^2\sin^2 B = d_1^2(ab + cd)^2 - 2abcdd_1^2\sin^2 A. \tag{1}$$

The law of sines applied to the triangles ABC and ABD with circumradii R gives $d_1 = 2R\sin B$ and $d_2 = 2R\sin A$, hence $d_1\sin A = d_2\sin B$. The relation (1) is equivalent to

$$d_2^2(ad + bc)^2 = d_1^2(ab + cd)^2,$$

and consequently

$$\frac{d_2}{d_1} = \frac{ab + cd}{ad + bc}. \tag{2}$$

Relation (2) is known as *Ptolemy's second theorem.*

Problem 25. *In a plane three equilateral triangles OAB, OCD and OEF are given. Prove that the midpoints of the segments BC, DE and FA are the vertices of an equilateral triangle.*

Solution. Consider the complex plane with origin at O and assume that triangles OAB, OCD, OEF are positively orientated. Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter.

Let $\varepsilon = \cos 60° + i \sin 60°$. Then

$$b = a\varepsilon, \quad d = c\varepsilon, \quad f = e\varepsilon$$

and

$$m = \frac{b+c}{2} = \frac{a\varepsilon + c}{2}, \quad n = \frac{d+e}{2} = \frac{c\varepsilon + e}{2}, \quad p = \frac{f+a}{2} = \frac{e\varepsilon + a}{2}.$$

Triangle MNP is equilateral if and only if

$$m + \omega n + \omega^2 p = 0,$$

where

$$\omega = \cos 120° + i \sin 120° = \varepsilon^2.$$

Because

$$m + \varepsilon^2 n + \varepsilon^4 p = m + \varepsilon^2 n - \varepsilon p = \frac{1}{2}(a\varepsilon + c - c + e\varepsilon^2 - e\varepsilon^2 - \varepsilon a) = 0,$$

we are done.

We invite the reader to solve the following problems by using complex numbers.

Problem 26. Let ABC be a triangle such that $AC^2 + AB^2 = 5BC^2$. Prove that the medians from the vertices B and C are perpendicular.

Problem 27. On the sides BC, CA, AB of a triangle ABC the points A', B', C' are chosen such that

$$\frac{A'B}{A'C} = \frac{B'C}{B'A} = \frac{C'A}{C'B} = k.$$

Consider the points A'', B'', C'' on the segments $B'C'$, $C'A'$, $A'B'$ such that

$$\frac{A''C'}{A''B'} = \frac{C''B'}{C''A'} = \frac{B''A'}{B''C'} = k.$$

Prove that triangles ABC and $A''B''C''$ are similar.

Problem 28. Prove that in any triangle the following inequality is true

$$\frac{R}{2r} \geq \frac{m_\alpha}{h_\alpha}.$$

Equality holds only for equilateral triangles.

Problem 29. Let $ABCD$ be a quadrilateral inscribed in the circle $\mathcal{C}(O;\,R)$. Prove that

$$AB^2 + BC^2 + CD^2 + DA^2 = 8R^2$$

if and only if $AC \perp BD$ or one of the diagonals is a diameter of $\mathcal{C}$.

Problem 30. On the sides of convex quadrilateral $ABCD$ equilateral triangles ABM, BCN, CDP and DAQ are drawn external to the figure. Prove that quadrilaterals $ABCD$ and $MNPQ$ have the same centroid.

Problem 31. Let $ABCD$ be a quadrilateral and consider the rotations $\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4$ with centers A, B, C, D through angle α and of the same orientation.

Points M, N, P, Q are the images of points A, B, C, D under the rotations $\mathcal{R}_2$, $\mathcal{R}_3$, $\mathcal{R}_4$, $\mathcal{R}_1$, respectively.

Prove that the midpoints of the diagonals of the quadrilaterals $ABCD$ and $MNPQ$ are the vertices of a parallelogram.

Problem 32. Prove that in any cyclic quadrilateral $ABCD$ the following holds:

a) $AD + BC\cos(A+B) = AB\cos A + CD\cos D$;

b) $BC\sin(A+B) = AB\sin A - CD\sin D$.

Problem 33. Let O_9, I, G be the 9-point center, the incenter and the centroid, respectively, of a triangle ABC. Prove that lines O_9G and AI are perpendicular if and only if $\widehat{A} = \dfrac{\pi}{3}$.

Problem 34. Two circles ω_1 and ω_2 are given in the plane, with centers O_1 and O_2, respectively. Let M_1' and M_2' be two points on ω_1 and ω_2, respectively, such that the lines O_1M_1' and O_2M_2' intersect. Let M_1 and M_2 be points on ω_1 and ω_2, respectively, such that when measured clockwise the angles $\widehat{M_1'O_1M_1}$ and $\widehat{M_2'O_2M_2}$ are equal.

(a) Determine the locus of the midpoint of $[M_1M_2]$.

(b) Let P be the point of intersection of lines O_1M_1 and O_2M_2. The circumcircle of triangle M_1PM_2 intersects the circumcircle of triangle O_1PO_2 at P and another point Q. Prove that Q is fixed, independent of the locations of M_1 and M_2.

(2000 Vietnamese Mathematical Olympiad)

Problem 35. Isosceles triangles $A_3A_1O_2$ and $A_1A_2O_3$ are constructed externally along the sides of a triangle $A_1A_2A_3$ with $O_2A_3 = O_2A_1$ and $O_3A_1 = O_3A_2$. Let O_1 be a point on the opposite side of line A_2A_3 from A_1, with $\widehat{O_1A_3A_2} = \frac{1}{2}\widehat{A_1O_3A_2}$ and $\widehat{O_1A_2A_3} = \frac{1}{2}\widehat{A_1O_2A_3}$, and let T be the foot of the perpendicular from O_1 to A_2A_3. Prove that $A_1O_1 \perp O_2O_3$ and that

$$\frac{A_1O_1}{O_2O_3} = 2\frac{O_1T}{A_2A_3}.$$

(2000 Iranian Mathematical Olympiad)

Problem 36. A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. We define $A_s = A_{s-3}$ for all $s \geq 4$. We construct a sequence of points $P_0, P_1, P_2, \dots$ such that P_{k+1} is the image of P_k under rotation with center A_{k+1} through angle 120° clockwise ($k = 0, 1, 2, \dots$). Prove that if $P_{1986} = P_0$ then the triangle $A_1A_2A_3$ is equilateral.

(27th IMO)

Problem 37. Two circles in a plane intersect. Let A be one of the points of intersection. Starting simultaneously from A two points move with constant speeds, each point travelling along its own circle in the same direction. After one revolution the two points return simultaneously to A. Prove that there exists a fixed point P in the plane such that, at any time, the distances from P to the moving points are equal.

(21st IMO)

Problem 38. Inside the square $ABCD$, the equilateral triangles ABK, BCL, CDM, DAN are inscribed. Prove that the midpoints of the segments KL, LM, MN, NK and the midpoints of the segments AK, BK, BL, CL, CM, DM, DN, AN are the vertices of a regular dodecagon.

(19th IMO)

Problem 39. Let ABC be an equilateral triangle and let M be a point in the interior of angle $\widehat{BAC}$. Points D and E are the images of points B and C under the rotations with center M and angle 120°, counterclockwise and clockwise, respectively.

Prove that the fourth vertex of the parallelogram with sides MD and ME is the reflection of point A across point M.

Problem 40. Prove that for any point M inside parallelogram $ABCD$ the following inequality holds:

$$MA \cdot MC + MB \cdot MD \geq AB \cdot BC.$$

Problem 41. Let ABC be a triangle, H its orthocenter, O its circumcenter, and R its circumradius. Let D be the reflection of A across BC, let E be that of B across CA, and F that of C across AB. Prove that D, E and F are collinear if and only if $OH = 2R$.

(39th IMO – Shortlist)

Problem 42. Let ABC be a triangle such that $\widehat{ACB} = 2\widehat{ABC}$. Let D be the point on the side BC such that $CD = 2BD$. The segment AD is extended to E so that $AD = DE$. Prove that

$$\widehat{ECB} + 180° = 2\widehat{EBC}.$$

(39th IMO – Shortlist)

5.5 Solving Trigonometric Problems

Problem 1. *Prove that*

$$\cos\frac{\pi}{11}+\cos\frac{3\pi}{11}+\cos\frac{5\pi}{11}+\cos\frac{7\pi}{11}+\cos\frac{9\pi}{11}=\frac{1}{2}.$$

Solution. Setting $z=\cos\frac{\pi}{11}+i\sin\frac{\pi}{11}$ implies that

$$z+z^3+z^5+z^7+z^9=\frac{z^{11}-z}{z^2-1}=\frac{-1-z}{z^2-1}=\frac{1}{1-z}.$$

Taking the real parts of both sides of the equality gives the desired result.

Problem 2. *Compute the product* $P=\cos 20^\circ\cdot\cos 40^\circ\cdot\cos 80^\circ$.

Solution. Setting $z=\cos 20^\circ+i\sin 20^\circ$ implies $z^9=-1$, $\overline{z}=\cos 20^\circ-i\sin 20^\circ$ and $\cos 20^\circ=\frac{z^2+1}{2z}$, $\cos 40^\circ=\frac{z^4+1}{2z^2}$, $\cos 80^\circ=\frac{z^8+1}{2z^4}$. Then

$$P=\frac{(z^2+1)(z^4+1)(z^8+1)}{8z^7}=\frac{(z^2-1)(z^2+1)(z^4+1)(z^8+1)}{8z^7(z^2-1)}$$
$$=\frac{z^{16}-1}{8(z^9-z^7)}=\frac{-z^7-1}{8(-1-z^7)}=\frac{1}{8}.$$

Solution II. This is a classic problem with a classic solution. Let $S=\cos 20\cos 40\cos 80$. Then

$$\begin{aligned}S\sin 20&=\sin 20\cos 20\cos 40\cos 80\\&=\frac{1}{2}\sin 40\cos 40\cos 80\\&=\frac{1}{4}\sin 80\cos 80\\&=\frac{1}{8}\cos 160=\frac{1}{8}\sin 20.\end{aligned}$$

So $S=\frac{1}{8}$.

Note that this classic solution is contrived, with no motivation. The solution using complex numbers, however, is a straightforward computation.

Problem 3. *Let x, y, z be real numbers such that*

$$\sin x+\sin y+\sin z=0 \quad \textit{and} \quad \cos x+\cos y+\cos z=0.$$

Prove that

$$\sin 2x+\sin 2y+\sin 2z=0 \quad \textit{and} \quad \cos 2x+\cos 2y+\cos 2z=0.$$

Solution. Setting $z_1 = \cos x + i\sin x$, $z_2 = \cos y + i\sin y$, $z_3 = \cos z + i\sin z$, we have $z_1 + z_2 + z_3 = 0$ and $|z_1| = |z_2| = |z_3| = 1$.

We have

$$z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1)$$

$$= -2z_1z_2z_3\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) = -2z_1z_2z_3(\overline{z}_1 + \overline{z}_2 + \overline{z}_3)$$

$$= -2z_1z_2z_3(\overline{z_1 + z_2 + z_3}) = 0.$$

Thus $(\cos 2x + \cos 2y + \cos 2z) + i(\sin 2x + \sin 2y + \sin 2z) = 0$, and the conclusion is obvious.

Problem 4. *Prove that*

$$\cos^2 10^\circ + \cos^2 50^\circ + \cos^2 70^\circ = \frac{3}{2}.$$

Solution. Setting $z = \cos 10^\circ + i\sin 10^\circ$, we have $z^9 = i$ and

$$\cos 10^\circ = \frac{z^2+1}{2z}, \quad \cos 50^\circ = \frac{z^{10}+1}{2z^5}, \quad \cos 70^\circ = \frac{z^{14}+1}{2z^7}.$$

The identity is equivalent to

$$\left(\frac{z^2+1}{2z}\right)^2 + \left(\frac{z^{10}+1}{2z^5}\right)^2 + \left(\frac{z^{14}+1}{2z^7}\right)^2 = \frac{3}{2}.$$

That is,

$$z^{16} + 2z^{14} + z^{12} + z^{24} + 2z^{14} + z^4 + z^{28} + 2z^{14} + 1 = 6z^{14}, \text{ i.e.,}$$

$$z^{28} + z^{24} + z^{16} + z^{12} + z^4 + 1 = 0.$$

Using relation $z^{18} = -1$, we obtain

$$z^{16} + z^{12} - z^{10} - z^6 + z^4 + 1 = 0$$

or equivalently

$$(z^4+1)(z^{12} - z^6 + 1) = 0.$$

That is,

$$\frac{(z^4+1)(z^{18}+1)}{z^6+1} = 0,$$

which is obvious.

Problem 5. *Solve the equation*

$$\cos x + \cos 2x - \cos 3x = 1.$$

Solution. Setting $z = \cos x + i \sin x$ yields

$$\cos x = \frac{z^2+1}{2z}, \quad \cos 2x = \frac{z^4+1}{2z^2}, \quad \cos 3x = \frac{z^6+1}{2z^3}.$$

The equation may be rewritten as

$$\frac{z^2+1}{2z} + \frac{z^4+1}{2z^2} - \frac{z^6+1}{2z^3} = 1, \text{ i.e., } z^4 + z^2 + z^5 + z - z^6 - 1 - 2z^3 = 0$$

This is equivalent to

$$(z^6 - z^5 - z^4 - z^3) + (z^3 - z^2 - z + 1) = 0$$

or

$$(z^3+1)(z^3 - z^2 - z + 1) = 0.$$

Finally we obtain

$$(z^3+1)(z-1)^2(z+1) = 0.$$

Thus, $z = 1$ or $z = -1$ or $z^3 = -1$ and consequently $x \in \{2k\pi | k \in \mathbb{Z}\}$ or $x \in \{\pi + 2k\pi | k \in \mathbb{Z}\}$ or $x \in \left\{\frac{\pi + 2k\pi}{3} | k \in \mathbb{Z}\right\}$. Therefore $x \in \{k\pi | k \in \mathbb{Z}\} \cup \left\{\frac{2k+1}{2}\pi | k \in \mathbb{Z}\right\}$.

Problem 6. *Compute the sums*

$$S = \sum_{k=1}^{n} q^k \cdot \cos kx \quad \textit{and} \quad T = \sum_{k=1}^{n} q^k \cdot \sin kx.$$

Solution. We have

$$1 + S + iT = \sum_{k=0}^{n} q^k (\cos kx + i \sin kx) = \sum_{k=0}^{n} q^k (\cos x + i \sin x)^k$$

$$= \frac{1 - q^{n+1}(\cos x + i \sin x)^{n+1}}{1 - q \cos x - iq \sin x}$$

$$= \frac{1 - q^{n+1}[\cos(n+1)x + i \sin(n+1)x]}{1 - q \cos x - iq \sin x}$$

$$= \frac{[1 - q^{n+1} \cos(n+1)x - iq^{n+1} \sin(n+1)x][1 - q \cos x + iq \sin x]}{q^2 - 2q \cos x + 1},$$

hence

$$1 + S = \frac{q^{n+2}\cos nx - q^{n+1}\cos(n+1)x - q\cos x + 1}{q^2 - 2q\cos x + 1}$$

and

$$T = \frac{q^{n+2}\sin nx - q^{n+1}\sin(n+1)x + q\sin x}{q^2 - 2q\cos x + 1}.$$

Remark. If $q = 1$ then we find the well-known formulas

$$\sum_{k=1}^{n}\cos kx = \frac{\sin\dfrac{nx}{2}\cos\dfrac{(n+1)x}{2}}{\sin\dfrac{x}{2}} \quad \text{and} \quad \sum_{k=1}^{n}\sin kx = \frac{\sin\dfrac{nx}{2}\sin\dfrac{(n+1)x}{2}}{\sin\dfrac{x}{2}}.$$

Indeed, we have

$$\sum_{k=1}^{n}\cos kx = \frac{\cos nx - \cos(n+1)x - (1-\cos x)}{2(1-\cos x)}$$

$$= \frac{2\sin\dfrac{x}{2}\sin\dfrac{(2n+1)x}{2} - 2\sin^2\dfrac{x}{2}}{4\sin^2\dfrac{x}{2}}$$

$$= \frac{\sin\dfrac{(2n+1)x}{2} - \sin\dfrac{x}{2}}{2\sin\dfrac{x}{2}} = \frac{\sin\dfrac{nx}{2}\cos\dfrac{(n+1)x}{2}}{\sin\dfrac{x}{2}}$$

and

$$\sum_{k=1}^{n}\sin kx = \frac{\sin nx - \sin(n+1)x + \sin x}{2(1-\cos x)}$$

$$= \frac{2\sin\dfrac{x}{2}\cos\dfrac{x}{2} - 2\sin\dfrac{x}{2}\cos\dfrac{(2n+1)x}{2}}{4\sin^2\dfrac{x}{2}}$$

$$= \frac{\cos\dfrac{x}{2} - \cos\dfrac{(2n+1)x}{2}}{2\sin\dfrac{x}{2}} = \frac{\sin\dfrac{nx}{2}\sin\dfrac{(n+1)x}{2}}{\sin\dfrac{x}{2}}.$$

Problem 7. *The points $A_1, A_2, \ldots, A_{10}$ are equally distributed on a circle of radius R (in that order). Prove that $A_1A_4 - A_1A_2 = R$.*

Solution. Let $z = \cos\frac{\pi}{10} + i\sin\frac{\pi}{10}$. Without loss of generality we may assume that $R = 1$. We need to show that $2\sin\frac{3\pi}{10} - 2\sin\frac{\pi}{10} = 1$.

In general, if $z = \cos a + i \sin a$, then $\sin a = \dfrac{z^2-1}{2iz}$ and we have to prove that $\dfrac{z^6-1}{iz^3} - \dfrac{z^2-1}{iz} = 1$. This reduces to $z^6 - z^4 + z^2 - 1 = iz^3$. Because $z^5 = i$, the previous relation is equivalent to $z^8 - z^6 + z^4 - z^2 + 1 = 0$. But this is true because $(z^8 - z^6 + z^4 - z^2 + 1)(z^2+1) = z^{10} + 1 = 0$ and $z^2 + 1 \neq 0$.

Problem 8. *Show that*

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \frac{1}{2}.$$

(5^{th} IMO)

Solution. Let $z = \cos\frac{\pi}{7} + i\sin\frac{\pi}{7}$. Then $z^7 + 1 = 0$. Because $z \neq -1$ and $z^7 + 1 = (z+1)(z^6 - z^5 + z^4 - z^3 + z^2 - z + 1) = 0$ it follows that the second factor from the above product is zero. The condition is equivalent to $z(z^2 - z + 1) = \dfrac{1}{1-z^3}$.

The given sum is

$$\cos\frac{\pi}{7} - \cos\frac{2\pi}{7} + \cos\frac{3\pi}{7} = \operatorname{Re}(z^3 - z^2 + z).$$

Therefore, we have to prove that $\operatorname{Re}\left(\dfrac{1}{1-z^3}\right) = \dfrac{1}{2}$. This follows from the well-known:

Lemma. *If* $z = \cos t + i \sin t$ *and* $z \neq 1$, *then* $\operatorname{Re}\dfrac{1}{1-z} = \dfrac{1}{2}$.

Proof. $\dfrac{1}{1-z} = \dfrac{1}{1-(\cos t + i\sin t)} = \dfrac{1}{(1-\cos t) - i\sin t} =$

$$= \frac{1}{2\sin^2\frac{t}{2} - 2i\sin\frac{t}{2}\cos\frac{t}{2}} = \frac{1}{2\sin\frac{t}{2}\left(\sin\frac{t}{2} - i\cos\frac{t}{2}\right)}$$

$$= \frac{\sin\frac{t}{2} + i\cos\frac{t}{2}}{2\sin\frac{t}{2}} = \frac{1}{2} + i\frac{\cos\frac{t}{2}}{2\sin\frac{t}{2}}.$$

Problem 9. *Prove that the average of the numbers* $k\sin k^\circ$ $(k = 2, 4, 6, \ldots, 180)$ *is* $\cot 1^\circ$.

(1996 USA Mathematical Olympiad)

Solution. Denote $z = \cos t + i\sin t$. From the identity

$$z + 2z^2 + \cdots + nz^n = (z + \cdots + z^n) + (z^2 + \cdots + z^n) + \cdots + z^n$$

$$= \frac{1}{z-1}[(z^{n+1}-z)+(z^{n+1}-z^2)+\cdots+(z^{n+1}-z^n)]$$

$$= \frac{nz^{n+1}}{z-1} - \frac{z^{n+1}-z}{(z-1)^2}$$

we derive the formulas:

$$\sum_{k=1}^{n} k\cos kt = \frac{(n+1)\sin\dfrac{(2n+1)t}{2}}{2\sin\dfrac{t}{2}} - \frac{1-\cos(n+1)t}{4\sin^2\dfrac{t}{2}}, \tag{1}$$

$$\sum_{k=1}^{n} k\sin kt = \frac{\sin(n+1)t}{4\sin^2\dfrac{t}{2}} - \frac{n\cos\dfrac{(2n+1)t}{2}}{2\sin\dfrac{t}{2}}. \tag{2}$$

Using relation (2) one obtains:

$$2\sin 2^\circ + 4\sin 4^\circ + \cdots + 178\sin 178^\circ = 2(\sin 2^\circ + 2\sin 2\cdot 2^\circ + \cdots + 89\sin 89\cdot 2^\circ)$$

$$= 2\left(\frac{\sin 90\cdot 2^\circ}{4\sin^2 1^\circ} - \frac{90\cos 179^\circ}{2\sin 1^\circ}\right) = -\frac{90\cos 179^\circ}{\sin 1^\circ} = 90\cot 1^\circ.$$

Finally,

$$\frac{1}{90}(2\sin 2^\circ + 4\sin 4^\circ + \cdots + 178\sin 178^\circ + 180\sin 180^\circ) = \cot 1^\circ.$$

Problem 10. *Let n be a positive integer. Find real numbers a_0 and a_{kl}, $k, l = \overline{1, n}$, $k > l$, such that*

$$\frac{\sin^2 nx}{\sin^2 x} = a_0 + \sum_{1\le l<k\le n} a_{kl}\cos 2(k-l)x$$

for all real numbers $x \neq m\pi$, $m \in \mathbb{Z}$.

(Romanian Mathematical Regional Contest "Grigore Moisil", 1995)

Solution. Using the identities

$$S_1 = \sum_{j=1}^{n} \cos 2jx = \frac{\sin nx\cos(n+1)x}{\sin x}$$

and

$$S_2 = \sum_{j=1}^{n} \sin 2jx = \frac{\sin nx\sin(n+1)x}{\sin x}$$

we obtain

$$S_1^2 + S_2^2 = \left(\frac{\sin nx}{\sin x}\right)^2.$$

On the other hand,

$$S_1^2 + S_2^2 = (\cos 2x + \cos 4x + \cdots + \cos 2nx)^2$$

$$+ (\sin 2x + \sin 4x + \cdots + \sin 2nx)^2$$

$$= n + 2 \sum_{1 \le l < k \le n} (\cos 2kx \cos 2lx + \sin 2kx \sin 2lx)$$

$$= n + 2 \sum_{1 \le l < k \le n} \cos 2(k-l)x,$$

hence

$$\left(\frac{\sin nx}{\sin x}\right)^2 = n + 2 \sum_{1 \le l < k \le n} \cos 2(k-l)x.$$

Set $a_0 = n$ and $a_{kl} = 2$, $1 \le l < k \le n$, and the problem is solved.

Here are some more problems.

Problem 11. Sum the following two n-term series for $\theta = 30°$:

i) $1 + \dfrac{\cos\theta}{\cos\theta} + \dfrac{\cos(2\theta)}{\cos^2\theta} + \dfrac{\cos(3\theta)}{\cos^3\theta} + \cdots + \dfrac{\cos((n-1)\theta)}{\cos^{n-1}\theta}$, and

ii) $\cos\theta\cos\theta + \cos^2\theta\cos(2\theta) + \cos^3\theta\cos(3\theta) + \cdots + \cos^n\theta\cos(n\theta)$.

(Crux Mathematicorum, 2003)

Problem 12. Prove that

$$1 + \cos^{2n}\left(\frac{\pi}{n}\right) + \cos^{2n}\left(\frac{2\pi}{n}\right) + \cdots + \cos^{2n}\left(\frac{(n-1)\pi}{n}\right)$$

$$= n \cdot 4^{-n}\left(2 + \binom{2n}{n}\right),$$

for all integers $n \ge 2$.

Problem 13. For any integer $p \ge 0$ there are real numbers $a_0, a_1, \ldots, a_p$ with $a_p \ne 0$ such that

$$\cos 2p\alpha = a_0 + a_1 \sin^2\alpha + \cdots + a_p \cdot (\sin^2\alpha)^p, \text{ for all } \alpha \in \mathbb{R}.$$

5.6 More on the n^{th} Roots of Unity

Problem 1. *Let $n \ge 3$ and $k \ge 2$ be positive integers and consider the complex numbers*

$$z = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

and

$$\theta = 1 - z + z^2 - z^3 + \cdots + (-1)^{k-1}z^{k-1}.$$

a) If k is even, prove that $\theta^n = 1$ if and only if n is even and $\frac{n}{2}$ divides $k-1$ or $k+1$.

b) If k is odd, prove that $\theta^n = 1$ if and only if n divide $k-1$ or $k+1$.

Solution. Since $z \neq -1$, we have

$$\theta = \frac{1+(-1)^{k+1}z^k}{1+z}.$$

a) If k is even, then

$$\theta = \frac{1-z^k}{1+z} = \frac{1-\cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n}}{1+\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}} = \frac{\sin\frac{k\pi}{n}\left(\sin\frac{k\pi}{n} - i\cos\frac{k\pi}{n}\right)}{\cos\frac{\pi}{n}\left(\cos\frac{\pi}{n} + i\sin\frac{\pi}{n}\right)}$$

$$= -i\frac{\sin\frac{k\pi}{n}}{\cos\frac{\pi}{n}}\left(\cos\frac{(k-1)\pi}{n} + i\sin\frac{(k-1)\pi}{n}\right),$$

and

$$|\theta| = \left|\frac{\sin\frac{k\pi}{n}}{\cos\frac{\pi}{n}}\right|.$$

We have

$$|\theta| = 1 \text{ if and only if } \left|\sin\frac{k\pi}{n}\right| = \left|\cos\frac{\pi}{n}\right|.$$

That is,

$$\sin^2\frac{k\pi}{n} = \cos^2\frac{\pi}{n} \quad \text{or} \quad \cos\frac{2k\pi}{n} + \cos\frac{2\pi}{n} = 0.$$

The last relation is equivalent to

$$\cos\frac{(k+1)\pi}{n}\cos\frac{(k-1)\pi}{n} = 0, \text{ i.e., } \frac{2(k+1)}{n} \in 2\mathbb{Z}+1$$

or $\frac{2(k-1)}{n} \in 2\mathbb{Z}+1$. This is equivalent to the statement that n is even and $\frac{n}{2}$ divides $k+1$ or $k-1$. Hence, it suffices to prove that $\theta^n = 1$ is equivalent to $|\theta| = 1$.

The direct implication is obvious. Conversely, if $|\theta| = 1$, then $n = 2t$, $t \in \mathbb{Z}_+$ and t divides $k+1$ or $k-1$. Since k is even, numbers $k+1$, $k-1$ are odd, hence $t = 2l+1$ and $n = 4l+2$, $l \in \mathbb{Z}$.

Then

$$\theta = \pm i\left(\cos\frac{(k-1)\pi}{n} + i\sin\frac{(k-1)\pi}{n}\right)$$

and

$$\theta^n = -\cos(k-1)\pi = 1,$$

as desired.

b) If k is odd, then

$$\theta = \frac{1+z^k}{1+z} = \frac{1+\cos\dfrac{2k\pi}{n} + i\sin\dfrac{2k\pi}{n}}{1+\cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}} = \frac{\cos\dfrac{k\pi}{n}\left(\cos\dfrac{k\pi}{n} + i\sin\dfrac{k\pi}{n}\right)}{\cos\dfrac{\pi}{n}\left(\cos\dfrac{\pi}{n} + i\sin\dfrac{\pi}{n}\right)}$$

$$= \frac{\cos\dfrac{k\pi}{n}}{\cos\dfrac{\pi}{n}}\left(\cos\frac{k-1}{n}\pi + i\sin\frac{k-1}{n}\pi\right).$$

We have

$$|\theta| = 1 \text{ if and only if } \left|\cos\frac{k\pi}{n}\right| = \left|\cos\frac{\pi}{n}\right|.$$

That is,

$$\cos^2\frac{k\pi}{n} = \cos^2\frac{\pi}{n} \quad \text{so} \quad \cos\frac{2k\pi}{n} = \cos\frac{2\pi}{n}.$$

It follows that

$$\sin\frac{(k+1)\pi}{n}\sin\frac{(k-1)\pi}{n} = 0,$$

i.e., n divides $k+1$ or $k-1$.

It suffices to prove that $\theta^n = 1$ is equivalent to $|\theta| = 1$. Since the direct implication is obvious, let us prove the converse. If $|\theta| = 1$, then $k \pm 1 = nt$, $t \in \mathbb{Z}$. Then $k = nt \pm 1$ and

$$\theta = (-1)^t\left(\cos\frac{(k-1)\pi}{n} + i\sin\frac{(k-1)\pi}{n}\right).$$

It follows that

$$\theta^n = (-1)^{k\pm1}(\cos(k-1)\pi + i\sin(k-1)\pi) = (-1)^{k\pm1}(-1)^{k-1} = 1,$$

as desired.

Problem 2. *Consider the cube root of unity*

$$\varepsilon = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}.$$

Compute

$$(1+\varepsilon)(1+\varepsilon^2)\cdots(1+\varepsilon^{1987}).$$

Solution. Notice that $\varepsilon^3 = 1$, $\varepsilon^2 + \varepsilon + 1 = 0$ and $1987 = 662 \cdot 3 + 1$. Then

$$(1+\varepsilon)(1+\varepsilon^2)\cdots(1+\varepsilon^{1987})$$

$$= \prod_{k=0}^{661} [(1+\varepsilon^{3k+1})(1+\varepsilon^{3k+2})(1+\varepsilon^{3k+3})](1+\varepsilon^{1987})$$

$$= \prod_{k=0}^{661} [(1+\varepsilon)(1+\varepsilon^2)(1+1)](1+\varepsilon) = (1+\varepsilon)[2(1+\varepsilon+\varepsilon^2+\varepsilon^3)]^{662}$$

$$= (1+\varepsilon)[2(0+1)]^{662} = 2^{662}(1+\varepsilon)$$

$$= 2^{662}(-\varepsilon^2) = 2^{662}\frac{1+i\sqrt{3}}{2} = 2^{661}(1+i\sqrt{3}).$$

Problem 3. *Let $\varepsilon \neq 1$ be a cube root of unity. Compute*

$$(1-\varepsilon+\varepsilon^2)(1-\varepsilon^2+\varepsilon^4)\cdots(1-\varepsilon^n+\varepsilon^{2n}).$$

Solution. Notice that $1+\varepsilon+\varepsilon^2 = 0$ and $\varepsilon^3 = 1$. Hence $1-\varepsilon+\varepsilon^2 = -2\varepsilon$ and $1+\varepsilon-\varepsilon^2 = -2\varepsilon^2$.

Then

$$1-\varepsilon^n+\varepsilon^{2n} = \begin{cases} 1, & \text{if} \quad n \equiv 0 (\text{mod} 3), \\ -2\varepsilon, & \text{if} \quad n \equiv 1 (\text{mod} 3), \\ -2\varepsilon^2, & \text{if} \quad n \equiv 2 (\text{mod} 3), \end{cases}$$

the product of any three consecutive factors of the given product equals

$$1 \cdot (-2\varepsilon) \cdot (-2\varepsilon^2) = 2^2.$$

Therefore

$$(1-\varepsilon+\varepsilon^2)(1-\varepsilon^2+\varepsilon^4)\cdots(1-\varepsilon^n+\varepsilon^{2n})$$

$$= \begin{cases} 2^{\frac{2n}{3}}, & \text{if} \quad n \equiv 0 (\text{mod} 3), \\ -2^{2[\frac{n}{3}]+1}\varepsilon, & \text{if} \quad n \equiv 1 (\text{mod} 3), \\ 2^{2[\frac{n}{3}]+2}, & \text{if} \quad n \equiv 2 (\text{mod} 3). \end{cases}$$

Problem 4. *Prove that the complex number*

$$z = \frac{2+i}{2-i}$$

has modulus equal to 1, but z is not an n^{th}-root of unity for any positive integer n.

Solution. Obviously $|z| = 1$. Assume by contradiction that there is an integer $n \geq 1$ such that $z^n = 1$.

Then $(2+i)^n = (2-i)^n$, and writing $2+i = (2-i)+2i$ it follows that

$$(2-i)^n = (2+i)^n$$

$$= (2-i)^n + \binom{n}{1}(2-i)^{n-1}2i + \cdots + \binom{n}{n-1}(2-i)(2i)^{n-1} + (2i)^n.$$

This is equivalent to

$$(2i)^n = (-2+i)\left[\binom{n}{1}(2i-1)^{n-2}2i + \cdots + \binom{n}{n-1}(2i)^{n-1}\right]$$
$$= (-2+i)(a+bi),$$

with $a, b \in \mathbb{Z}$.

Taking the modulus of both members of the equality gives $2^n = 5(a^2+b^2)$, a contradiction.

Problem 5. *Let U_n be the set of n^{th}-roots of unity. Prove that the following statements are equivalent:*

a) there is $\alpha \in U_n$ such that $1+\alpha \in U_n$;

b) there is $\beta \in U_n$ such that $1-\beta \in U_n$.

(Romanian Mathematical Olympiad – Second Round, 1990)

Solution. Assume that there exists $\alpha \in U_n$ such that $1+\alpha \in U_n$. Setting $\beta = \dfrac{1}{1+\alpha}$ we have $\beta^n = \left(\dfrac{1}{1+\alpha}\right)^n = \dfrac{1}{(1+\alpha)^n} = 1$, hence $\beta \in U_n$. On the other hand, $1-\beta = \dfrac{\alpha}{\alpha+1}$ and $(1-\beta)^n = \dfrac{\alpha^n}{(1+\alpha)^n} = 1$, hence $1-\beta \in U_n$, as desired.

Conversely, if $\beta, 1-\beta \in U_n$, set $\alpha = \dfrac{1-\beta}{\beta}$. Since $\alpha^n = \dfrac{(1-\beta)^n}{\beta^n} = 1$ and $(1+\alpha)^n = \dfrac{1}{\beta^n} = 1$, we have $\alpha \in U_n$ and $1+\alpha \in U_n$, as desired.

Remark. The statements a) and b) are equivalent with $6|n$. Indeed, if $\alpha, 1+\alpha \in U_n$, then $|\alpha| = |1+\alpha| = 1$. It follows that $1 = |1+\alpha|^2 = (1+\alpha)(1+\overline{\alpha}) = 1+\alpha+\overline{\alpha}+|\alpha|^2 = 1+\alpha+\overline{\alpha}+1 = 2+\alpha+\dfrac{1}{\alpha}$, i.e., $\alpha = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$, hence

$$1+\alpha = \frac{1}{2} \pm i\frac{\sqrt{3}}{2} = \cos\frac{2\pi}{6} \pm i\sin\frac{2\pi}{6}.$$

Since $(1+\alpha)^n = 1$ it follows that 6 divides n.

Conversely, if n is a multiple of 6, then both $\alpha = -\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$ and $1+\alpha = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$ belong to U_n.

Problem 6. *Let $n \geq 3$ be a positive integer and let $\varepsilon \neq 1$ be an n^{th} root of unity.*

1) Show that $|1-\varepsilon| > \dfrac{2}{n-1}$.

2) If k is a positive integer such that n does not divides k, then

$$\left|\sin\frac{k\pi}{n}\right| > \frac{1}{n-1}.$$

(Romanian Mathematical Olympiad – Final Round, 1988)

Solution. 1) We have $\varepsilon^n - 1 = (\varepsilon - 1)(\varepsilon^{n-1} + \cdots + \varepsilon + 1)$ hence, taking into account that $\varepsilon \neq 1$, we find $\varepsilon^{n-1} + \cdots + \varepsilon + 1 = 0$. The last relation is equivalent to $(\varepsilon^{n-1}-1)+\cdots+(\varepsilon-1) = -n$, i.e., $(\varepsilon-1)[\varepsilon^{n-2}+2\varepsilon^{n-3}+\cdots+(n-2)\varepsilon+(n-1)] = -n$. Passing to the absolute value we find that

$$n = |\varepsilon-1||\varepsilon^{n-2}+2\varepsilon^{n-3}+\cdots+(n-1)| \le |\varepsilon-1|(|\varepsilon^{n-2}|+2|\varepsilon|^{n-3}+\cdots+(n-1)).$$

Therefore

$$n \le |1-\varepsilon|(1+2+\cdots+(n-1)) = |1-\varepsilon|\frac{n(n-1)}{2},$$

i.e., we find the inequality $|1-\varepsilon| \ge \dfrac{2}{n-1}$. Moreover, equality is not possible since the geometric images of $1, \varepsilon, \ldots, \varepsilon^{n-1}$ are not collinear.

2) Consider $\varepsilon = \cos\dfrac{2k\pi}{n} + i\sin\dfrac{2k\pi}{n}$ and obtain

$$1-\varepsilon = 1-\cos\frac{2k\pi}{n} - i\sin\frac{2k\pi}{n}.$$

Hence

$$|1-\varepsilon|^2 = \left(1-\cos\frac{2k\pi}{n}\right)^2 + \sin^2\frac{2k\pi}{n} = 2-2\cos\frac{2k\pi}{n} = 4\sin^2\frac{k\pi}{n}.$$

Applying the inequality in 1), the desired inequality follows.

Problem 7. *Let U_n be the set of the n^{th}-roots of unity. Prove that*

$$\prod_{\varepsilon\in U_n}\left(\varepsilon+\frac{1}{\varepsilon}\right) = \begin{cases} 0, & \textit{if} \quad n\equiv 0 \pmod 4,\\ 2, & \textit{if} \quad n\equiv 1 \pmod 2,\\ -4, & \textit{if} \quad n\equiv 2 \pmod 4,\\ 2, & \textit{if} \quad n\equiv 3 \pmod 4.\end{cases}$$

Solution. Consider the polynomial

$$f(x) = X^n - 1 = \prod_{\varepsilon\in U_n}(X-\varepsilon).$$

Denoting by P_n the product in our problem, we have

$$P_n = \prod_{\varepsilon\in U_n}\left(\varepsilon+\frac{1}{\varepsilon}\right) = \prod_{\varepsilon\in U_n}\frac{\varepsilon^2+1}{\varepsilon} = \frac{\displaystyle\prod_{\varepsilon\in U_n}(\varepsilon+i)(\varepsilon-i)}{\displaystyle\prod_{\varepsilon\in U_n}\varepsilon}$$

$$= \frac{\prod_{\varepsilon \in U_n} (i+\varepsilon) \prod_{\varepsilon \in U_n} (-i+\varepsilon)}{(-1)^n f(0)} = \frac{f(-1) \cdot f(i)}{(-1)^{n-1}} = \frac{[(-i)^n - 1](i^n - 1)}{(-1)^{n-1}}.$$

If $n \equiv 0 \pmod 4$, then $i^n = 1$ and $P_n = 0$.

If $n \equiv 1 \pmod 2$, then $(-1)^{n-1} = 1$ and

$$P_n = (-i^n - 1)(i^n - 1) = -(i^{2n} - 1) = -((-1)^n - 1) = -(-1-1) = 2.$$

If $n \equiv 2 \pmod 4$, then $(-1)^{n-1} = -1$, $(-i)^n = i^n = i^2 = -1$, $i^n = -1$, hence $P_n = \dfrac{(-1-1)(-1-1)}{-1} = -4$.

If $n \equiv 3 \pmod 4$, then $(-1)^{n-1} = 1$ and

$$P_n = (-i^n - 1)(i^n - 1) = (i^3 - 1)(-i^3 - 1) = -(i^6 - 1) = -((-1)^3 - 1) = 2,$$

and we are done.

Problem 8. *Let*

$$\omega = \cos \frac{2\pi}{2n+1} + i \sin \frac{2\pi}{2n+1}, \quad n \geq 0,$$

and let

$$z = \frac{1}{2} + \omega + \omega^2 + \cdots + \omega^n.$$

Prove that:

a) $\mathrm{Im}(z^{2k}) = \mathrm{Re}(z^{2k+1}) = 0$ *for all* $k \in \mathbb{N}$*;*

b) $(2z+1)^{2n+1} + (2z-1)^{2n+1} = 0$.

Solution. We have $\omega^{2n+1} = 1$ and

$$1 + \omega + \omega^2 + \cdots + \omega^{2n} = 0.$$

Then

$$\frac{1}{2} + \omega + \omega^2 + \cdots + \omega^n + \omega^n(\omega + \omega^2 + \cdots + \omega^n) + \frac{1}{2} = 0$$

or

$$z + \omega^n \left(z - \frac{1}{2}\right) + \frac{1}{2} = 0,$$

hence

$$z = \frac{1}{2} \cdot \frac{\omega^n - 1}{\omega^n + 1}.$$

a) We have $\overline{z} = \dfrac{1}{2} \dfrac{\dfrac{1}{\omega^n} - 1}{\dfrac{1}{\omega^n} + 1} = -z$. Thus $z^{2k} = \overline{z^{2k}}$ and $z^{2k+1} = -\overline{z^{2k+1}}$. The conclusion follows from these two equalities.

b) From the relation

$$z + \omega^n \left(z - \frac{1}{2} \right) + \frac{1}{2} = 0$$

we obtain $2z + 1 = -\omega^n(2z - 1)$. Taking into account that $\omega^{2n+1} = 1$, we obtain $(2z + 1)^{2n+1} = -(2z - 1)^{2n+1}$, and we are done.

Problem 9. *Let n be an odd positive integer and $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}$ the complex roots of unity of order n. Prove that*

$$\prod_{k=0}^{n-1} (a + b\varepsilon_k^2) = a^n + b^n$$

for all complex numbers a and b.

(Romanian Mathematical Olympiad – Second Round, 2000)

Solution. If $ab = 0$, then the claim is obvious, so consider the case when $a \neq 0$ and $b \neq 0$.

We start with a useful lemma.

Lemma. *If $\varepsilon_0, \varepsilon_1, \dots, \varepsilon_{n-1}$ are the complex roots of unity of order n, where n is an odd integer, then*

$$\prod_{k=0}^{n-1} (A + B\varepsilon_k) = A^n + B^n,$$

for all complex numbers A and B.

Proof. Using the identity

$$x^n - 1 = \prod_{k=0}^{n-1} (x - \varepsilon_k)$$

for $x = -\dfrac{A}{B}$ yields

$$-\left(\frac{A^n}{B^n} + 1 \right) = -\prod_{k=0}^{n-1} \left(\frac{A}{B} + \varepsilon_k \right),$$

and the conclusion follows. □

Because n is odd, the function $f : U_n \to U_n$ is bijective. To prove this, it suffices to show that it is injective. Indeed, assume that $f(x) = f(y)$. It follows that $(x - y)(x + y) = 0$. If $x + y = 0$, then $x^n = (-y)^n$, i.e., $1 = -1$, a contradiction. Hence $x = y$.

From the lemma we have

$$\prod_{k=0}^{n-1} (a + b\varepsilon_k^2) = \prod_{j=0}^{n-1} (a + b\varepsilon_j) = a^n + b^n.$$

Problem 10. *Let n be an even positive integer such that $\frac{n}{2}$ is odd and let $\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_{n-1}$ be the complex roots of unity of order n. Prove that*

$$\prod_{k=0}^{n-1}(a + b\varepsilon_k^2) = (a^{\frac{n}{2}} + b^{\frac{n}{2}})^2$$

for any complex numbers a and b.

(Romanian Mathematical Olympiad – Second Round, 2000)

Solution. If $b = 0$ the claim is obvious. If not, let $n = 2(2s+1)$. Consider a complex number α such that $\alpha^2 = \frac{a}{b}$ and the polynomial

$$f = X^n - 1 = (X - \varepsilon_0)(X - \varepsilon_1)\cdots(X - \varepsilon_{n-1}).$$

We have

$$f\left(\frac{\alpha}{i}\right) = \left(\frac{1}{i}\right)^a (\alpha - i\varepsilon_0)\cdots(\alpha - i\varepsilon_{n-1})$$

and

$$f\left(-\frac{\alpha}{i}\right) = \left(\frac{-1}{i}\right)^a (\alpha + i\varepsilon_0)\cdots(\alpha + i\varepsilon_{n-1}),$$

hence

$$f\left(\frac{\alpha}{i}\right) f\left(-\frac{\alpha}{i}\right) = (\alpha^2 + \varepsilon_0^2)\cdots(\alpha^2 + \varepsilon_{n-1}^2).$$

Therefore

$$\prod_{k=0}^{n-1}(a + b\varepsilon_k^2) = b^n \prod_{k=0}^{n-1}\left(\frac{a}{b} + \varepsilon_k^2\right) = b^n \prod_{k=0}^{n-1}(\alpha^2 + \varepsilon_k^2)$$

$$= b^n f\left(\frac{\alpha}{i}\right) f\left(-\frac{\alpha}{i}\right) = b^n[(\alpha^2)^{2s+1} + 1]^2 = b^n \left[\left(\frac{a}{b}\right)^{2s+1} + 1\right]^2$$

$$= b^{2(2s+1)}\left(\frac{a^{2s+1} + b^{2s+1}}{b^{2s+1}}\right)^2 = (a^{\frac{n}{2}} + b^{\frac{n}{2}})^2.$$

The following problems also involve n^{th} roots of unity.

Problem 11. For all positive integers k define

$$U_k = \{z \in \mathbb{C} \mid z^k = 1\}.$$

Prove that for any integers m and n with $0 < m < n$ we have

$$U_1 \cup U_2 \cup \cdots \cup U_m \subset U_{n-m+1} \cup U_{n-m+2} \cup \cdots \cup U_n.$$

(Romanian Mathematical Regional Contest “Grigore Moisil”, 1997)

Problem 12. Let a, b, c, d, α be complex numbers such that $|a| = |b| \neq 0$ and $|c| = |d| \neq 0$. Prove that all roots of the equation

$$c(bx + a\alpha)^n - d(ax + b\overline{\alpha})^n = 0, \quad n \geq 1,$$

are real numbers.

Problem 13. Suppose that $z \neq 1$ is a complex number such that $z^n = 1$, $n \geq 1$. Prove that

$$|nz - (n+2)| \leq \frac{(n+1)(2n+1)}{6}|z-1|^2.$$

(Crux Mathematicorum, 2003)

Problem 14. Let M be a set of complex numbers such that if $x, y \in M$, then $\frac{x}{y} \in M$. Prove that if the set M has n elements, then M is the set of the n^{th}-roots of 1.

Problem 15. A finite set A of complex numbers has the property: $z \in A$ implies $z^n \in A$ for every positive integer n.

a) Prove that $\sum_{z \in A} z$ is an integer.

b) Prove that for every integer k one can choose a set A which fulfills the above condition and $\sum_{z \in A} z = k$.

(Romanian Mathematical Olympiad – Final Round, 2003)

5.7 Problems Involving Polygons

Problem 1. *Let $z_1, z_2, \ldots, z_n$ be distinct complex numbers such that $|z_1| = |z_2| = \cdots = |z_n|$. Prove that*

$$\sum_{1 \leq i < j \leq n} \left|\frac{z_i + z_j}{z_i - z_j}\right|^2 \geq \frac{(n-1)(n-2)}{2}.$$

Solution. Consider the points $A_1, A_2, \ldots, A_n$ with coordinates $z_1, z_2, \ldots, z_n$. The polygon $A_1A_2 \cdots A_n$ is inscribed in the circle with center at origin and radius $R = |z_1|$.

The coordinate of the midpoint A_{ij} of the segment $[A_iA_j]$ is equal to $\frac{z_i + z_j}{2}$, for $1 \leq i < j \leq n$. Hence

$$|z_i + z_j|^2 = 4OA_{ij}^2 \quad \text{and} \quad |z_i - z_j|^2 = A_iA_j^2.$$

Moreover, $4OA_{ij}^2 = 4R^2 - A_iA_j^2$.

The sum

$$\sum_{1 \leq i < j \leq n} \left|\frac{z_i + z_j}{z_i - z_j}\right|^2$$

equals

$$\sum_{1\le i<j\le n}\frac{4OA_{ij}^2}{A_iA_j^2}=\sum_{1\le i<j\le n}\frac{4R^2-A_iA_j^2}{A_iA_j^2}=4R^2\sum_{1\le i<j\le n}\frac{1}{A_iA_j^2}-\binom{n}{2}.$$

The $AM-HM$ inequality gives

$$\sum_{1\le i<j\le n}\frac{1}{A_iA_j^2}\ge\frac{\left(\binom{n}{2}\right)^2}{\sum\limits_{1\le i<j\le n}A_iA_j^2}.$$

Since $\sum\limits_{1\le i<j\le n}A_iA_j^2\le n^2\cdot R^2$, it follows that

$$\sum_{1\le i<j\le n}\left|\frac{z_i+z_j}{z_i-z_j}\right|^2\ge 4R^2\frac{\left(\binom{n}{2}\right)^2}{\sum\limits_{1\le i<j\le n}A_iA_j^2}-\binom{n}{2}$$

$$\ge\frac{4\left(\binom{n}{2}\right)^2}{n^2}-\binom{n}{2}=\frac{\left(4\binom{n}{2}-n^2\right)\cdot\binom{n}{2}}{n^2}=\frac{(n-1)(n-2)}{2},$$

as claimed.

Problem 2. *Let $A_1A_2\cdots A_n$ be a polygon and let $a_1,a_2,\ldots,a_n$ be the coordinates of the vertices $A_1,A_2,\ldots,A_n$. If $|a_1|=|a_2|=\cdots=|a_n|=R$, prove that*

$$\sum_{1\le i<j\le n}|a_i+a_j|^2\ge n(n-2)R^2.$$

Solution. We have

$$\sum_{1\le i<j\le n}|a_i+a_j|^2=\sum_{1\le i<j\le n}(a_i+a_j)(\overline{a_i}+\overline{a_j})$$

$$=\sum_{1\le i<j\le n}(|a_i|^2+|a_j|^2+a_i\overline{a_j}+\overline{a_i}a_j)$$

$$=2R^2\binom{n}{2}+\sum_{i\ne j}a_i\overline{a_j}=n(n-1)R^2+\sum_{i=1}^n\sum_{j=1}^n a_i\overline{a_j}-\sum_{i=1}^n a_i\overline{a_i}$$

$$=n(n-1)R^2+\left(\sum_{i=1}^n a_i\right)\left(\sum_{i=1}^n\overline{a_i}\right)-nR^2$$

$$=n(n-2)R^2+\left|\sum_{i=1}^n a_i\right|^2\ge n(n-2)R^2,$$

as desired.

Problem 3. *Let $z_1, z_2, \dots, z_n$ be the coordinates of the vertices of a regular polygon with the circumcenter at the origin of the complex plane. Prove that there are $i, j, k \in \{1, 2, \dots, n\}$ such that $z_i + z_j = z_k$ if and only if 6 divides n.*

Solution. Let $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Then $z_p = z_1 \cdot \varepsilon^{p-1}$, for all $p = \overline{1, n}$.

We have $z_i + z_j = z_k$ if and only if $1 + \varepsilon^{j-i} = \varepsilon^{k-i}$, i.e.,

$$2\cos\frac{(j-i)\pi}{n}\left[\cos\frac{(j-i)\pi}{n} + i\sin\frac{(j-i)\pi}{n}\right] = \cos\frac{2(k-i)\pi}{n} + i\sin\frac{2(k-i)\pi}{n}.$$

The last relation is equivalent to

$$\frac{(j-i)\pi}{n} = \frac{\pi}{3} = \frac{2(k-i)\pi}{n}, \text{ i.e., } n = 6(k-i) = 3(j-i),$$

hence 6 divides n.

Conversely, if 6 divides n, let

$$i = 1, \quad j = \frac{n}{3} + 1, \quad k = \frac{n}{6} + 1$$

and we have $z_i + z_l = z_k$, as desired.

Problem 4. *Let $z_1, z_2, \dots, z_n$ be the coordinates of the vertices of a regular polygon. Prove that*

$$z_1^2 + z_2^2 + \cdots + z_n^2 = z_1z_2 + z_2z_3 + \cdots + z_nz_1.$$

Solution. Without loss of generality we may assume that the center of the polygon is the origin of the complex plane.

Let $z_k = z_1\varepsilon^{k-1}$, where

$$\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}, \quad k = 1, \dots, n.$$

The right-hand side is equal to

$$z_1z_2 + z_2z_3 + \cdots + z_nz_1 = \sum_{k=1}^{n} z_i z_{k+1}$$

$$= \sum_{k=1}^{n} z_1^2\varepsilon^{2k-1} = z_1^2 \cdot \varepsilon \cdot \frac{1-\varepsilon^{2n}}{1-\varepsilon^2} = 0.$$

On the other hand,

$$z_1^2 + z_2^2 + \cdots + z_n^2 = \sum_{k=1}^{n} z_i^2 = \sum_{k=1}^{n} z_1^2\varepsilon^{2k-2} = z_1^2\frac{1-\varepsilon^{2n}}{1-\varepsilon^2} = 0$$

and we are done.

Problem 5. *Let $n \geq 4$ and let $a_1, a_2, \dots, a_n$ be the coordinates of the vertices of a regular polygon. Prove that*

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 = a_1a_3 + a_2a_4 + \cdots + a_na_2.$$

Solution. Assume that the center of the polygon is the origin of the complex plane and $a_k = a_1\varepsilon^{k-1}$, $k = 1, \dots, n$, where

$$\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}.$$

The left-hand side of the equality is

$$a_1a_2 + a_2a_3 + \cdots + a_na_1 = a_1^2\sum_{k=1}^{n}\varepsilon^{2k-1} = a_1^2\varepsilon\frac{1-\varepsilon^{2n}}{1-\varepsilon^2} = 0.$$

The right-hand side of the equality is

$$a_1^2\sum_{k=1}^{n}\varepsilon^{2k} = a_1^2\varepsilon^2\frac{1-\varepsilon^{2n}}{1-\varepsilon^2} = 0,$$

and we are done.

Problem 6. *Let $z_1, z_2, \dots, z_n$ be distinct complex numbers such that*

$$|z_1| = |z_2| = \cdots = |z_n| = 1.$$

Consider the statements:

a) $z_1, z_2, \dots, z_n$ are the coordinates of the vertices of a regular polygon.

b) $z_1^n + z_2^n + \cdots + z_n^n = n(-1)^{n+1}z_1z_2\cdots z_n$.

Decide with proof if the implications a) $\Rightarrow$ b) and b) $\Rightarrow$ a) are true.

Solution. We study at first the implication a) $\Rightarrow$ b).

Let $\varepsilon = \frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Since $z_1, z_2, \dots, z_n$ are coordinates of the vertices of a regular polygon, without loss of generality we may assume that

$$z_k = z_1\varepsilon^{k-1} \quad \text{for} \quad k = \overline{1, n}.$$

The relation b) becomes

$$z_1^n(1 + \varepsilon^n + \varepsilon^{2n} + \cdots + \varepsilon^{n(n-1)}) = n(-1)^{n+1}z_1^n\varepsilon^{1+2+\cdots+(n-1)}.$$

This is equivalent to

$$n = n(-1)^{n+1}\varepsilon^{\frac{n(n-1)}{2}}, \text{ i.e.,}$$

$$1 = (-1)^{n+1}\left(\cos\frac{n(n-1)}{2}\cdot\frac{2\pi}{n} + i\sin\frac{n(n-1)}{2}\cdot\frac{2\pi}{n}\right).$$

We obtain

$$1 = (-1)^{n+1}(\cos(n-1)\pi + i\sin(n-1)\pi), \text{ i.e., } 1 = (-1)^{n+1}(-1)^{n-1},$$

which is valid. Therefore the implication a) $\Rightarrow$ b) holds.

We prove now that the implication b) $\Rightarrow$ a) is also valid.

Observe that

$$|n \cdot (-1)^{n+1} z_1 z_2 \cdots z_n| = n|z_1| \cdot |z_2| \cdots |z_n| = n,$$

hence

$$|z_1^n + z_2^n + \cdots + z_n^n| = n.$$

Using the triangle inequality we obtain

$$n = |z_1^n + z_2^n + \cdots + z_n^n| \le |z_1^n| + |z_2^n| + \cdots + |z_n^n| = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n,$$

hence the numbers $z_1^n, z_2^n, \ldots, z_n^n$ have the same argument. Since $|z_1^n| = |z_2^n| = \cdots = |z_n^n| = 1$, it follows that $z_1^n = z_2^n = \cdots = z_n^n = a$, where a is a complex number with $|a| = 1$. Numbers $z_1, z_2, \ldots, z_n$ are distinct, therefore there are the n^{th}-roots of a, and consequently the coordinates of the vertices of a regular polygon.

Problem 7. *Let A, B, C be 3 consecutive vertices of a regular n-gon and consider the point M on the circumcircle such that points B and M lie on opposite sides of line AC.*

Prove that $MA + MC = 2MB\cos\frac{\pi}{n}$.

(A generalization of the Van Schouten theorem; see the first remark below)

Solution. Consider the complex plane with origin at the center of the polygon and let 1 be the coordinate of A_1.

If $\varepsilon = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$, then ε^{k-1} is the coordinate of A_k, $k = \overline{1, n}$.

Without loss of generality, assume that $A = A_1$, $B = A_2$ and $C = A_3$. Let $z_M = \cos t + i\sin t$, $t \in [0, 2\pi)$ be the coordinate of point M. Since point B and M are separated by the line AC, it follows that $\dfrac{4\pi}{n} < t$.

Then

$$MA = |z_M - 1| = \sqrt{(\cos t - 1)^2 + \sin^2 t} = \sqrt{2 - 2\cos t} = 2\sin\frac{t}{2};$$

$$MB = |z_M - \varepsilon| = 2\sin\left(\frac{t}{2} - \frac{\pi}{n}\right)$$

and

$$MC = |z_M - \varepsilon^2| = 2\sin\left(\frac{t}{2} - \frac{2\pi}{n}\right).$$

The equality

$$MA + MB = 2MC \cos\frac{\pi}{n}$$

is equivalent to

$$2\sin\frac{t}{2} + 2\sin\left(\frac{t}{2} - \frac{2\pi}{n}\right) = 4\sin\left(\frac{t}{2} - \frac{\pi}{n}\right)\cos\frac{\pi}{n},$$

which follows using the sum-to-product formula in the left-hand side.

Remarks. 1) If $n = 3$ then we obtain the Van Schouten theorem: *For any point M on the circumcircle of equilateral triangle ABC such that M belongs on the arc $\widehat{AC}$, the following relation holds:*

$$MA + MC = MB.$$

Note that this result also follows from Ptolemy's theorem.

2) If $n = 4$, then for any point M on the circumcircle of square $ABCD$ such that B and M lie on opposite sides of line AC, we have the relation

$$MA + MC = \sqrt{2}MB.$$

Problem 8. *Let P be a point on the circumcircle of square $ABCD$. Find all integers $n > 0$ such that the sum*

$$S_n(P) = PA^n + PB^n + PC^n + PD^n$$

is constant with respect to point P.

Solution. Consider the complex plane with origin at the center of the square such that A, B, C, D have coordinates $1, i, -1, -i$, respectively.

Let $z = a + bi$ be the coordinate of point P, where $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$.

The sum $S_n(P)$ is equal to

$$S_n(P) = [(a-1)^2 + b^2]^{\frac{n}{2}} + [a^2 + (b-1)^2]^{\frac{n}{2}} + [(a+1)^2 + b^2]^{\frac{n}{2}} + [a^2 + (b+1)^2]^{\frac{n}{2}}$$
$$= 2^{\frac{n}{2}}\left[(1+a)^{\frac{n}{2}} + (1-a)^{\frac{n}{2}} + (1+b)^{\frac{n}{2}} + (1-b)^{\frac{n}{2}}\right].$$

Set $P = A(1, 0)$. Then $S_n(A) = 2^{\frac{n+2}{2}} + 2^n$. For $P = E\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, we get

$$S_n(E) = 2(2-\sqrt{2})^{\frac{n}{2}} + 2(2+\sqrt{2})^{\frac{n}{2}}.$$

Since $S_n(P)$ is constant with respect to P, it follows that $S_n(A) = S_n(E)$ or $2^{\frac{n+2}{2}} + 2^n = 2(2-\sqrt{2})^{\frac{n}{2}} + 2(2+\sqrt{2})^{\frac{n}{2}}$.

It is obvious that $2^{\frac{n+2}{2}} > 2(2-\sqrt{2})^{\frac{n}{2}}$ for all $n \geq 1$. We also have $2^n > 2(2+\sqrt{2})^{\frac{n}{2}}$ for all $n \geq 9$. The last inequality is equivalent to

$$\frac{1}{4} > \left(\frac{2+\sqrt{2}}{4}\right)^n \quad \text{for} \quad n \geq 9.$$

The left-hand side member of the inequality decreases with n, so it suffices to notice that

$$\frac{1}{4} > \left(\frac{2+\sqrt{2}}{4}\right)^9.$$

Therefore the inequality $S_n(A) = S_n(E)$ can hold only for $n \leq 8$. Now it is not difficult to verify that $S_n(P)$ is constant only for $n \in \{2, 4, 6\}$.

Problem 9. *A function $f : \mathbb{R}^2 \to \mathbb{R}$ is called Olympic if it has the following property: given $n \geq 3$ distinct points $A_1, A_2, \ldots, A_n \in \mathbb{R}^2$, if $f(A_1) = f(A_2) = \cdots = f(A_n)$ then the points $A_1, A_2, \ldots, A_n$ are the vertices of a convex polygon. Let $P \in \mathbb{C}[X]$ be a nonconstant polynomial. Prove that the function $f : \mathbb{R}^2 \to \mathbb{R}$, defined by $f(x, y) = |P(x + iy)|$, is Olympic if and only if all the roots of P are equal.*

(Romanian Mathematical Olympiad – Final Round, 2000)

Solution. First suppose that all the roots of P are equal, and write $P(x) = a(z-z_0)^n$ for some $a, z_0 \in \mathbb{C}$ and $n \in \mathbb{N}$. If $A_1, A_2, \ldots, A_n$ are distinct point in $\mathbb{R}^2$ such that $f(A_1) = f(A_2) = \cdots = f(A_n)$, then $A_1, \ldots, A_n$ are situated on a circle with center $(\text{Re}(z_0), \text{Im}(z_0))$ and radius $\sqrt[n]{|f(A_1)/a|}$, implying that the points are the vertices of a convex polygon.

Conversely, suppose that not all the roots of P are equal, and write $P(x) = (z-z_1)(z-z_2)Q(z)$ where z_1 and z_2 are distinct roots of $P(x)$ such that $|z_1-z_2|$ is minimal. Let l be the line containing $Z_1 = (\text{Re}(z_1), \text{Im}(z_1))$ and $Z_2 = (\text{Re}(z_2), \text{Im}(z_2))$, and let $z_3 = \frac{1}{2}(z_1 + z_2)$ so that $Z_3 = (\text{Re}(z_3), \text{Im}(z_3))$ is the midpoint of $[Z_1Z_2]$. Also, let s_1, s_2 denote the rays Z_3Z_1 and Z_3Z_2, and let $d = f(Z_3) \geq 0$. We must have $r > 0$, because otherwise z_3 would be a root of P such that $|z_1 - z_3| < |z_1 - z_2|$, which is impossible. Because $f(Z_3) = 0$,

$$\lim_{\substack{Z_3 \to \infty \\ Z \in s_1}} f(Z) = +\infty,$$

and f is continuous, there exists a point $Z_4 \in s_1$, on the side of Z_1 opposite Z_3, such that $f(Z_4) = r$. Similarly, there exists $Z_5 \in s_2$, on the side of Z_2 opposite Z_3, such that $f(Z_5) = r$. Thus, $f(Z_3) = f(Z_4) = f(Z_5)$ and Z_3, Z_4, Z_5 are not vertices of a convex polygon. Hence, f is not Olympic.

Problem 10. *In a convex hexagon $ABCDEF$, $\widehat{A}+\widehat{C}+\widehat{E}=360^\circ$ and*

$$AB \cdot CD \cdot EF = BC \cdot DE \cdot FA.$$

Prove that $AB \cdot FC \cdot EC = BF \cdot DE \cdot CA$.

(1999 Polish Mathematical Olympiad)

Solution. Position the hexagon in the complex plane and let $a = B - A$, $b = C - B, \dots, f = A - F$. The product identity implies that $|ace| = |bdf|$, and the angle equality implies $\frac{-b}{a} \cdot \frac{-d}{c} \cdot \frac{-f}{e}$ is real and positive. Hence, $ace = -bdf$. Also, $a+b+c+d+e+f = 0$. Multiplying this by ad and adding $ace + bdf = 0$ gives $a^2d + abd + acd + ad^2 + ade + adf + ace + bdf = 0$ which factors to $a(d+e)(c+d) + d(a+b)(f+a) = 0$. Thus

$$|a(d+e)(c+d)| = |d(a+b)(f+a)|,$$

which is what we wanted.

Problem 11. *Let $n > 2$ be an integer and $f : \mathbb{R}^2 \to \mathbb{R}$ be a function such that for any regular n-gon $A_1A_2\cdots A_n$,*

$$f(A_1) + f(A_2) + \cdots + f(A_n) = 0.$$

Prove that f is identically zero.

(Romanian Mathematical Olympiad – Final Round, 1996)

Solution. We identify $\mathbb{R}^2$ with the complex plane and let $\zeta = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$. Then the condition is that for any $z \in \mathbb{C}$ and any positive real t,

$$\sum_{j=1}^{n} f(z + t\zeta^j) = 0.$$

In particular, for each of $k = 1, \dots, n$, we have

$$\sum_{j=1}^{n} f(z - \zeta^k + \zeta^j) = 0.$$

Summing over k, we have

$$\sum_{m=1}^{n}\sum_{k=1}^{n} f(z - (1-\zeta^m)\zeta^k) = 0.$$

For $m = n$ the inner sum is $nf(z)$; for other m, the inner sum again runs over a regular polygon, hence is 0. Thus $f(z) = 0$ for all $z \in \mathbb{C}$.

Here are some proposed problems.

Problem 12. Prove that there exists a convex 1990-gon with the following two properties:

a) all angles are equal;

b) the lengths of the sides are the numbers $1^2, 2^2, 3^2, \ldots, 1989^2, 1990^2$ in some order.

(31^{st} IMO)

Problem 13. Let A and E be opposite vertices of a regular octagon. Let a_n be the number of paths of length n of the form $(P_0, P_1, \ldots, P_n)$ where P_i are vertices of the octagon and the paths are constructed using the rule: $P_0 = A$, $P_n = E$, P_i and P_{i+1} are adjacent vertices for $i = 0, \ldots, n-1$ and $P_i \neq E$ for $i = 0, \ldots, n-1$.

Prove that $a_{2n-1} = 0$ and $a_{2n} = \dfrac{1}{\sqrt{2}}(x^{n-1} - y^{n-1})$, for all $n = 1, 2, 3, \ldots$, where $x = 2 + \sqrt{2}$ and $y = 2 - \sqrt{2}$.

(21^{st} IMO)

Problem 14. Let A, B, C be three consecutive vertices of a regular polygon and let us consider a point M on the major arc AC of the circumcircle.

Prove that

$$MA \cdot MC = MB^2 - AB^2.$$

Problem 15. Let $A_1A_2 \cdots A_n$ be a regular polygon with the circumradius equal to 1. Find the maximum value of $\max \prod_{j=1}^{n} PA_j$ when P describes the circumcircle.

(Romanian Mathematical Regional Contest "Grigore Moisil", 1992)

Problem 16. Let $A_1A_2 \cdots A_{2n}$ be a regular polygon with circumradius equal to 1 and consider a point P on the circumcircle. Prove that

$$\sum_{k=0}^{n-1} PA_{k+1}^2 \cdot PA_{n+k+1}^2 = 2n.$$

5.8 Complex Numbers and Combinatorics

Problem 1. *Compute the sum*

$$\sum_{k=0}^{3n-1} (-1)^k \binom{6n}{2k+1} 3^k.$$

Solution. We have

$$\sum_{k=0}^{3n-1} (-1)^k \binom{6n}{2k+1} 3^k = \sum_{k=0}^{3n-1} \binom{6n}{2k+1} (-3)^k$$

$$= \sum_{k=0}^{3n-1} \binom{6n}{2k+1} (i\sqrt{3})^{2k} = \frac{1}{i\sqrt{3}} \sum_{k=0}^{3n-1} \binom{6n}{2k+1} (i\sqrt{3})^{2k+1}$$

$$= \frac{1}{i\sqrt{3}} \text{Im}(1+i\sqrt{3})^{6n} = \frac{1}{i\sqrt{3}} \text{Im}\left[2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)\right]^{6n}$$

$$= \frac{1}{i\sqrt{3}} \text{Im}[2^{6n}(\cos 2\pi n + i \sin 2\pi n)] = 0.$$

Problem 2. *Calculate the sum* $S_n = \sum_{k=0}^{n} \binom{n}{k} \cos k\alpha$, *where* $\alpha \in [0, \pi]$.

Solution. Consider the complex number $z = \cos\alpha + i\sin\alpha$ and the sum $T_n = \sum_{k=0}^{n} \binom{n}{k} \sin k\alpha$. We have

$$S_n + iT_n = \sum_{k=0}^{n} \binom{n}{k} (\cos k\alpha + i \sin k\alpha) = \sum_{k=0}^{n} \binom{n}{k} (\cos\alpha + i\sin\alpha)^k$$

$$= \sum_{k=0}^{n} \binom{n}{k} z^k = (1+z)^n. \tag{1}$$

The polar form of complex number $1+z$ is

$$1 + \cos\alpha + i\sin\alpha = 2\cos^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$= 2\cos\frac{\alpha}{2}\left(\cos\frac{\alpha}{2} + i\sin\frac{\alpha}{2}\right)$$

since $\alpha \in [0, \pi]$. From (1) it follows that

$$S_n + iT_n = \left(2\cos\frac{\alpha}{2}\right)^n \left(\cos\frac{n\alpha}{2} + i\sin\frac{n\alpha}{2}\right),$$

i.e.,

$$S_n = \left(2\cos\frac{\alpha}{2}\right)^n \cos\frac{n\alpha}{2} \quad \text{and} \quad T_n = \left(2\cos\frac{\alpha}{2}\right)^n \sin\frac{n\alpha}{2}.$$

Problem 3. *Prove the identity*

$$\left(\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots\right)^2 + \left(\binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \cdots\right)^2 = 2^n.$$

Solution. Denote

$$x_n = \binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \cdots \quad \text{and} \quad y_n = \binom{n}{1} - \binom{n}{3} + \binom{n}{5} - \cdots$$

and observe that

$$(1+i)^n = x_n + y_n i. \tag{1}$$

Passing to the absolute value it follows that

$$|x_n + y_n i| = |(1+i)^n| = |1+i|^n = 2^{\frac{n}{2}}.$$

This is equivalent to $x_n^2 + y_n^2 = 2^n$.

Remark. We can write the explicit formulas for x_n and y_n as follows. Observe that

$$(1+i)^n = \left(\sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\right)^n = 2^{\frac{n}{2}}\left(\cos\frac{n\pi}{4} + i\sin\frac{n\pi}{4}\right).$$

From relation (1) we get

$$x_n = 2^{\frac{n}{2}}\cos\frac{n\pi}{4} \quad \text{and} \quad y_n = 2^{\frac{n}{2}}\sin\frac{n\pi}{4}.$$

Problem 4. *If m and p are positive integers and $m > p$, then*

$$\binom{m}{0} + \binom{m}{p} + \binom{m}{2p} + \binom{m}{3p} + \cdots$$

$$= \frac{2^m}{p}\left(1 + \sum_{k=1}^{\left[\frac{p-1}{2}\right]}\left(\cos\frac{k\pi}{p}\right)^m \cos\frac{mk\pi}{p}\right).$$

Solution. We begin with the following simple but useful remark: If $f \in \mathbb{R}[X]$ is a polynomial, $f = a_0 + a_1X + \cdots + a_mX^m$, and $\varepsilon = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p}$ is the p^{th} primitive root of unity, then for all real numbers n the following relation holds:

$$a_0 + a_px^p + a_{2p}x^{2p} + \cdots = \frac{1}{p}(f(x) + f(\varepsilon x) + \cdots + f(\varepsilon^{p-1}x)). \tag{1}$$

To prove (1) we use the relation

$$1 + \varepsilon^k + \varepsilon^{2k} + \cdots + \varepsilon^{(p-1)k} = \begin{cases} p, & \text{if } p|k, \\ 0, & \text{otherwise,} \end{cases}$$

on the right-hand side.

Consider the case when p is odd. Using relation (1) for polynomial $f = (1+X)^m = \binom{m}{0} + \binom{m}{1}X + \cdots + \binom{m}{m}X^m$ we obtain

$$\binom{m}{0} + \binom{m}{p}x^p + \binom{m}{2p}x^{2p} + \cdots = \frac{1}{p}((1+x)^m + (1+\varepsilon x)^m + \cdots + (1+\varepsilon^{p-1}x)^m) \tag{2}$$

Substituting $x = 1$ in relation (2) we find

$$S_p = \binom{m}{0} + \binom{m}{p} + \binom{m}{2p} + \cdots = \frac{1}{p}(2^m + (1+\varepsilon)^m + \cdots + (1+\varepsilon^{p-1})^m). \tag{3}$$

From $\varepsilon^k = \cos\dfrac{2k\pi}{p} + i\sin\dfrac{2k\pi}{p}$ it follows that for all $k = 0, 1, \dots, p-1$

$$(1+\varepsilon^k)^m = 2^m \left(\cos\frac{k\pi}{p}\right)^m \left(\cos\frac{mk\pi}{p} + i\sin\frac{mk\pi}{p}\right).$$

Using the relation $\varepsilon^{p-k} = \overline{\varepsilon^k}$ we find

$$(1+\varepsilon^{p-k})^m = (1+\overline{\varepsilon^k})^m = \overline{(1+\varepsilon^k)^m}$$
$$= 2^m \left(\cos\frac{k\pi}{p}\right)^m \left(\cos\frac{mk\pi}{p} - i\sin\frac{mk\pi}{p}\right).$$

Replacing in (3) we obtain

$$S_p = \frac{1}{p}\sum_{k=0}^{p-1}(1+\varepsilon^k)^m = \frac{1}{p}\left[\sum_{k=0}^{\frac{p-1}{2}}(1+\varepsilon^k)^m + \sum_{k=1}^{\frac{p-1}{2}}(1+\varepsilon^{p-k})^m\right]$$
$$= \frac{1}{p}\left[2^m + 2^m\sum_{k=1}^{\frac{p-1}{2}}\left(\cos\frac{k\pi}{p}\right)^m\left(\cos\frac{mk\pi}{p} + i\sin\frac{mk\pi}{p}\right)\right.$$
$$\left. + 2^m\sum_{k=1}^{\frac{p-1}{2}}\left(\cos\frac{k\pi}{p}\right)^m\left(\cos\frac{mk\pi}{p} - i\sin\frac{mk\pi}{p}\right)\right]$$
$$= \frac{2^m}{p}\left(1 + 2\sum_{k=1}^{\frac{p-1}{2}}\left(\cos\frac{k\pi}{p}\right)^m\cos\frac{mk\pi}{p}\right).$$

Consider now the case when p is an even positive integer. Because $\varepsilon^{\frac{p}{2}} = -1$ we have

$$S_p = \frac{1}{p}\sum_{k=0}^{p-1}(1+\varepsilon^k)^m = \frac{1}{p}\left[2^m + \sum_{k=1}^{\frac{p}{2}-1}(1+\varepsilon^k)^m + \sum_{k=\frac{p}{2}+1}^{p-1}(1+\varepsilon^k)^m\right]$$
$$= \frac{1}{p}\left[2^m + \sum_{k=1}^{\frac{p}{2}-1}2^m\left(\cos\frac{k\pi}{p}\right)^m\left(\cos\frac{mk\pi}{p} + i\sin\frac{mk\pi}{p}\right) + \right.$$
$$\left. + \sum_{k=1}^{\frac{p}{2}-1}2^m\left(\cos\frac{k\pi}{p}\right)^m\left(\cos\frac{mk\pi}{p} - i\sin\frac{mk\pi}{p}\right)\right]$$
$$= \frac{2^m}{p}\left(1 + 2\sum_{k=1}^{\frac{p}{2}-1}\left(\cos\frac{k\pi}{p}\right)^m\cos\frac{mk\pi}{p}\right).$$

Problem 5. *The following identity holds:*

$$\binom{n}{m}+\binom{n}{m+p}+\binom{n}{m+2p}+\cdots=\frac{2^n}{p}\sum_{k=0}^{p-1}\left(\cos\frac{k\pi}{p}\right)^n\cos\frac{(n-2m)k\pi}{p}.$$

Solution. Let $\varepsilon_0,\varepsilon_1,\ldots,\varepsilon_{p-1}$ be the p^{th} roots of unity. Then

$$\sum_{k=0}^{p-1}\varepsilon_k^{-m}(1+\varepsilon_k)^n=\sum_{k=0}^{n}\binom{n}{k}(\varepsilon_0^{k-m}+\cdots+\varepsilon_{p-1}^{k-m}). \tag{1}$$

Using the result in Proposition 3, Subsection 2.2.2, it follows that

$$\varepsilon_0^{k-m}+\cdots+\varepsilon_{p-1}^{k-m}=\begin{cases} p, & \text{if } p|(k-m),\\ 0, & \text{otherwise.}\end{cases} \tag{2}$$

Taking into account that

$$\begin{aligned}&\varepsilon_k^{-m}(1+\varepsilon_k)^m\\ &=\left(\cos\frac{2mk\pi}{p}-i\sin\frac{2mk\pi}{p}\right)\left(2\cos\frac{k\pi}{p}\right)^n\left(\cos\frac{nk\pi}{p}+i\sin\frac{nk\pi}{p}\right)\\ &=2^n\left(\cos\frac{k\pi}{p}\right)^n\left(\cos\frac{(n-2m)k\pi}{p}+i\sin\frac{(n-2m)k\pi}{p}\right)\end{aligned}$$

and using (1) and (2) the desired identity follows.

Remark. The following interesting trigonometric relation holds:

$$\sum_{k=0}^{p-1}\left(\cos\frac{k\pi}{p}\right)^n\sin\frac{(n-2m)k\pi}{p}=0. \tag{3}$$

Problem 6. *Consider the integers a_n, b_n, c_n, where*

$$a_n=\binom{n}{0}+\binom{n}{3}+\binom{n}{6}+\cdots,$$

$$b_n=\binom{n}{1}+\binom{n}{4}+\binom{n}{7}+\cdots,$$

$$c_n=\binom{n}{2}+\binom{n}{5}+\binom{n}{8}+\cdots.$$

Show that:
1) $a_n^3+b_n^3+c_n^3-3a_nb_nc_n=2^n$.
2) $a_n^2+b_n^2+c_n^2-a_nb_n-b_nc_n-c_na_n=1$.
3) Two of integers a_n, b_n, c_n are equal and the third differs by one.

Solution. 1) Let ε be a cube root of unity different from 1. We have

$$(1+1)^n = a_n + b_n + c_n,\ (1+\varepsilon)^n = a_n + b_n\varepsilon + c_n\varepsilon^2,\ (1+\varepsilon^2)^n = a_n + b_n\varepsilon^2 + c_n\varepsilon.$$

Therefore

$$\begin{aligned} a_n^3 + b_n^3 + c_n^3 - 3a_nb_nc_n &= (a_n + b_n + c_n)(a_n + b_n\varepsilon + c_n\varepsilon^2)(a_n + b_n\varepsilon^2 + c_n\varepsilon) \\ &= 2^n(1+\varepsilon)^n(1+\varepsilon^2)^n = 2^n(-\varepsilon^2)^n(-\varepsilon)^n = 2^n. \end{aligned}$$

2) Using the identity

$$x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-xy-yz-zx)$$

and the above relation it follows that

$$a_n^2 + b_n^2 + c_n^2 - a_nb_n - b_nc_n - c_na_n = 1.$$

3) Multiplying the above relation by 2 we find

$$(a_n - b_n)^2 + (b_n - c_n)^2 + (c_n - a_n)^2 = 2. \tag{1}$$

From (1) it follows that two of a_n, b_n, c_n are equal and the third differs by one.

Remark. From Problem 5 it follows that

$$\begin{aligned} a_n &= \frac{1}{3}\left[2^n + \cos\frac{n\pi}{3} + (-1)^n\cos\frac{2n\pi}{3}\right] = \frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right), \\ b_n &= \frac{1}{3}\left[2^n + \cos\frac{(n-2)\pi}{3} + (-1)^n\cos\frac{(2n-4)\pi}{3}\right] \\ &= \frac{1}{3}\left(2^n + 2\cos\frac{(n-2)\pi}{3}\right), \\ c_n &= \frac{1}{3}\left[2^n + \cos\frac{(n-4)\pi}{3} + (-1)^n\cos\frac{(2n-8)\pi}{3}\right] \\ &= \frac{1}{3}\left(2^n + 2\cos\frac{(n-4)\pi}{3}\right). \end{aligned}$$

It is not difficult to see that

$$a_n = b_n \text{ if and only if } n \equiv 1 \pmod 3,$$

$$a_n = c_n \text{ if and only if } n \equiv 2 \pmod 3,$$

$$b_n = c_n \text{ if and only if } n \equiv 0 \pmod 3.$$

Problem 7. *How many positive integers of n digits chosen from the set* $\{2, 3, 7, 9\}$ *are divisible by 3?*

(Romanian Mathematical Regional Contest "Traian Lalescu", 2003)

Solution. Let x_n, y_n, z_n be the number of all positive integers of n digits 2, 3, 7 or 9 which are congruent to 0, 1 and 2 modulo 3. We have to find x_n.

Consider $\varepsilon = \cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}$. It is clear that $x_n + y_n + z_n = 4^n$ and

$$x_n + \varepsilon y_n + \varepsilon^2 z_n = \sum_{j_1+j_2+j_3+j_4=n} \varepsilon^{2j_1+3j_2+7j_3+9j_4} = (\varepsilon^2 + \varepsilon^3 + \varepsilon^7 + \varepsilon^9)^n = 1.$$

It follows that $x_n - 1 + \varepsilon y_n + \varepsilon^2 z_n = 0$. Applying Proposition 4 in Subsection 2.2.2 we obtain $x_n - 1 = y_n = z_n = k$. Then $3k = x_n + y_n + z_n - 1 = 4^n - 1$ and we find $k = \dfrac{1}{3}(4^n - 1)$. Finally $x_n = k + 1 = \dfrac{1}{3}(4^n + 2)$.

Problem 8. *Let n be a prime number and let $a_1, a_2, \ldots, a_m$ be positive integers. Consider $f(k)$ the number of all m-tuples $(c_1, \ldots, c_m)$ satisfying $1 \le c_i \le a_i$ and $\sum_{i=1}^{m} c_i \equiv k \pmod{n}$. Show that $f(0) = f(1) = \cdots = f(n-1)$ if and only if $n | a_j$ for some $j \in \{1, \ldots, m\}$.*

(Rookie Contest, 1999)

Solution. Let $\varepsilon = \cos\dfrac{2\pi}{n} + i\sin\dfrac{2\pi}{n}$. Note that the following relations hold:

$$\prod_{i=1}^{m}(X + X^2 + \cdots + X^{a_i}) = \sum_{1 \le c_i \le a_i} X^{c_1+\cdots+c_m}$$

and

$$f(0) + f(1)\varepsilon + \cdots + f(n-1)\varepsilon^{n-1} = \sum_{1\le c_i \le a_i} \varepsilon^{c_1+\cdots+c_m} = \prod_{i=1}^{m}(\varepsilon + \varepsilon^2 + \cdots + \varepsilon^{a_i}).$$

Applying the result in Proposition 4, Subsection 2.2.2, we have $f(0) = f(1) = \cdots = f(n-1)$ if and only if $f(0) + f(1)\varepsilon + \cdots + f(n-1)\varepsilon^{n-1} = 0$. This is equivalent to $\prod_{i=1}^{m}(\varepsilon + \varepsilon^2 + \cdots + \varepsilon^{a_i}) = 0$, i.e., $\varepsilon + \varepsilon^2 + \cdots + \varepsilon^{a_j} = 0$ for some $j \in \{1, \ldots, m\}$. It follows that $\varepsilon^{a_j} - 1 = 0$, i.e., $n | a_j$.

Problem 9. *For a finite set of real numbers A denote by $|A|$ the cardinal number of A and by $m(A)$ the sum of elements of A.*

Let p be a prime and $A = \{1, 2, \ldots, 2p\}$. Find the number of all subsets $B \subset A$ such that $|B| = p$ and $p | m(B)$.

(36^{th} IMO)

Solution. The case $p = 2$ is trivial. Consider $p \ge 3$ and $\varepsilon = \cos\dfrac{2\pi}{p} + i\sin\dfrac{2\pi}{p}$. Denote by x_j the number of all subsets $B \subset A$ with properties $|B| = p$ and $m(B) \equiv j \pmod{p}$.

Then

$$\sum_{j=0}^{p-1} x_j \varepsilon^j = \sum_{B\subset A, |B|=p} \varepsilon^{m(B)} = \sum_{1\le c_1<\cdots<\le c_p\le 2p} \varepsilon^{c_1+\cdots+c_p}$$

The last sum is the coefficient of X^p in $(X+\varepsilon)(X+\varepsilon^2)\cdots(X+\varepsilon^{2p})$. Taking into account the relation $X^p - 1 = (X-1)(X-\varepsilon)\cdots(X-\varepsilon^{p-1})$ we obtain $(X+\varepsilon)(X+\varepsilon^2)\cdots(X+\varepsilon^{2p}) = (X^p+1)^2$, hence the coefficient of X^p is 2. Therefore

$$\sum_{j=0}^{p-1} x_j \varepsilon^j = 2,$$

i.e., $x_0 - 2 + x_1\varepsilon + \cdots + x_{p-1}\varepsilon^{p-1} = 0$. From Proposition 4, Subsection 2.2.2, it follows that $x_0 - 2 = x_1 = \cdots = x_{p-1} = k$. We find $pk = x_0 + \cdots + x_{p-1} - 2 = \binom{2p}{p} - 2$ hence $k = \frac{1}{p}\left(\binom{2p}{p} - 2\right)$. Therefore, the desired number is

$$x_0 = 2 + k = 2 + \frac{1}{p}\left(\binom{2p}{p} - 2\right).$$

Problem 10. *Prove that the number* $\sum_{k=0}^{n} \binom{2n+1}{2k+1} 2^{3k}$ *is not divisible by 5 for any integer* $n \ge 0$.

(16th IMO)

Solution. Since $2^3 \equiv -2 \pmod 5$, an equivalent problem is to prove that $S_n = \sum_{k=0}^{n} \binom{2n+1}{2k+1}(-2)^k$ is not divisible by 5. Expanding $(1+i\sqrt{2})^{2n+1}$ and then separating the even and odd terms we get

$$(1+i\sqrt{2})^{2n+1} = R_n + i\sqrt{2}S_n, \tag{1}$$

where $R_n = \sum_{k=0}^{n} \binom{2n+1}{2k}(-2)^k$.

Passing to the absolute value from (1) it follows that

$$3^{2n+1} = R_n^2 + 2S_n^2 \tag{2}$$

Since $3^2 \equiv -1 \pmod 5$, the relation (2) leads to

$$R_n^2 + 2S_n^2 \equiv \pm 3 \pmod 5. \tag{3}$$

Assume by contradiction that $S_n \equiv 0 \pmod 5$ for some positive integer n. Then from (3) we obtain $R_n^2 \equiv \pm 3 \pmod 5$, a contradiction since any square is congruent to 0, 1, or 4 modulo 5.

Here are other problems concerning complex numbers and combinatorics.

Problem 11. Calculate the sum $s_n = \sum_{k=0}^{n} \binom{n}{k}^2 \cos kt$, where $t \in [0, \pi]$.

Problem 12. Prove that following identities:

1) $\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots = \frac{1}{4}\left(2^n + 2^{\frac{n}{2}+1} \cos \frac{n\pi}{4}\right).$

(Romanian Mathematical Olympiad – Second Round, 1981)

2) $\binom{n}{0} + \binom{n}{5} + \binom{n}{10} + \cdots =$

$$= \frac{1}{5}\left[2^n + \frac{(\sqrt{5}+1)^n}{2^{n-1}} \cos \frac{n\pi}{5} + \frac{(\sqrt{5}-1)^n}{2^{n-1}} \cos \frac{2n\pi}{5}\right].$$

Problem 13. Consider the integers A_n, B_n, C_n defined by

$$A_n = \binom{n}{0} - \binom{n}{3} + \binom{n}{6} - \cdots ,$$

$$B_n = -\binom{n}{1} + \binom{n}{4} - \binom{n}{7} + \cdots ,$$

$$C_n = \binom{n}{2} - \binom{n}{5} + \binom{n}{8} - \cdots .$$

The following identities hold:

1) $A_n^2 + B_n^2 + C_n^2 - A_n B_n - B_n C_n - C_n A_n = 3^n$;

2) $A_n^2 + A_n B_n + B_n^2 = 3^{n-1}$.

Problem 14. Let $p \geq 3$ be a prime and let m, n be positive integers divisible by p such that n is odd. For each m-tuple $(c_1, \ldots, c_m)$, $c_i \in \{1, 2, \ldots, n\}$, with the property that $p | \sum_{i=1}^{m} c_i$, let us consider the product $c_1 \cdots c_m$. Prove that the sum of all these products are divisible by $\left(\frac{n}{p}\right)^m$.

Problem 15. Let k be a positive integer and $a = 4k - 1$. Prove that for any positive integer n, the integer

$$s_n = \binom{n}{0} - \binom{n}{2}a + \binom{n}{4}a^2 - \binom{n}{6}a^3 + \cdots \text{ is divisible by } 2^{n-1}.$$

(Romanian Mathematical Olympiad – Second Round, 1984)

5.9 Miscellaneous Problems

Problem 1. *Two unit squares K_1, K_2 with centers M, N are situated in the plane so that $MN = 4$. Two sides of K_1 are parallel to the line MN, and one of the diagonals of K_2 lies on MN. Find the locus of the midpoint of XY as X, Y vary over the interior of K_1, K_2, respectively.*

(1997 Bulgarian Mathematical Olympiad)

Solution. Introduce complex numbers with $M = -2$, $N = 2$. Then the locus is the set of points of the form $-(w + xi) + (y + zi)$, where $|w|, |x| < 1/2$ and $|x + y|, |x - y| < \sqrt{2}/2$. The result is an octagon with vertices $(1+\sqrt{2})/2+i/2$, $1/2+(1+\sqrt{2})i/2$, and so on.

Problem 2. *Curves A, B, C and D are defined in the plane as follows:*

$$A = \left\{(x, y) : x^2 - y^2 = \frac{x}{x^2 + y^2}\right\},$$
$$B = \left\{(x, y) : 2xy + \frac{y}{x^2 + y^2} = 3\right\},$$
$$C = \{(x, y) : x^3 - 3xy^2 + 3y = 1\},$$
$$D = \{(x, y) : 3x^2y - 3x - y^3 = 0\}.$$

Prove that $A \cap B = C \cap D$.

(1987 Putnam Mathematical Competition)

Solution. Let $z = x + yi$. The equations defining A and B are the real and imaginary parts of the equation $z^2 = z^{-1} + 3i$, and similarly the equations defining C and D are the real and imaginary parts of $z^3 - 3iz = 1$. Hence for all real x and y, we have $(x, y) \in A \cap B$ if and only if $z^2 = z^{-1} + 3i$. This is equivalent to $z^3 - 3iz = 1$, i.e., $(x, y) \in C \cap D$.

Thus $A \cap B = C \cap D$.

Problem 3. *Determine with proof whether or not it is possible to consider 1975 points on the unit circle such that the distances between any two points are rational numbers (the distances being taken along the chord).*

(17^{th} IMO)

Solution. There are infinitely many points with rational coordinates on the unit circle. This is a well-known result arising from Pythagorean triangles and the corresponding equation:

$$m^2 + n^2 = p^2.$$

Any such point $A(x_A, y_A)$ can be represented by a complex number

$$z_A = x_A + iy_A = \cos\alpha_A + i\sin\alpha_A$$

where α_A is the argument of the complex number z_A and $\cos\alpha_A$, $\sin\alpha_A$ are rational numbers.

Taking on the unit circle complex numbers of the form

$$z_A^2 = \cos 2\alpha_A + i\sin 2\alpha_A$$

we have for two such points:

$$|z_A^2 - z_B^2| = \sqrt{(\cos 2\alpha_A - \cos 2\alpha_B)^2 + (\sin 2\alpha_A - \sin 2\alpha_B)^2}$$

$$= \sqrt{2[1-\cos 2(\alpha_B - \alpha_A)]} = \sqrt{2\cdot 2\sin^2(\alpha_B-\alpha_A)} = 2|\sin(\alpha_B - \alpha_A)|$$

$$= 2|\sin\alpha_B\cos\alpha_A - \sin\alpha_A\cos\alpha_B| \in \mathbb{Q}.$$

Answer: Yes, it is possible.

Problem 4. *A tourist takes a trip through a city in stages. Each stage consists of three segments of length 100 meters separated by right turns of* $60°$. *Between the last segment of one stage and the first segment of the next stage, the tourist makes a left turn of* $60°$. *At what distance will the tourist be from his initial position after 1997 stages?*

(1997 Rio Plata Mathematical Olympiad)

Solution. In one stage, the tourist traverses the complex number

$$x = 100 + 100\overline{\varepsilon} + 100\overline{\varepsilon}^2 = 100 - 100\sqrt{3}i,$$

where $\varepsilon = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}$.

Thus in 1997 stages, the tourist traverses the complex number

$$z = x + x\varepsilon + x\varepsilon^2 + \cdots + x\varepsilon^{1996} = x\frac{1-\varepsilon^{1997}}{1-\varepsilon} = x\varepsilon^2.$$

Hence, the tourist ends up $|z| = |x\varepsilon^2| = |x| = 200$ meters away from his initial position.

Problem 5. *Let A, B, C, be fixed points in the plane. A man starts from a certain point P_0 and walks directly to A. At A he turns by* $60°$ *to the left and walks to P_1 such that $P_0A = AP_1$. After he performs the same action* 1986 *times successively around points A, B, C, A, B, C, ..., he returns to the starting point. Prove that ABC is an equilateral triangle, and that the vertices A, B, C, are arranged counterclockwise.*

(27th IMO)

Solution. For convenience, let $A_1, A_2, A_3, A_4, A_5, \ldots$ be $A, B, C, A, B, \ldots$, respectively, and let P_0 be the origin. After the k^{th} step, the position P_k will be $P_k = A_k + (P_{k-1} - A_k)\varepsilon$ for $k = 1, 2, \ldots$, where $\varepsilon = \cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}$. We easily obtain

$$P_k = (1-\varepsilon)(A_k + \varepsilon A_{k-1} + \varepsilon^2 A_{k-2} + \cdots + \varepsilon^{k-1}A_1).$$

The condition $P = P_{1986}$ is equivalent to $A_{1986}+\varepsilon A_{1985}+\cdots+\varepsilon^{1984}A_2+\varepsilon^{1985}A_1 = 0$, which having in mind that $A_1 = A_4 = A_7 = \cdots$, $A_2 = A_5 = A_8 = \cdots$, $A_3 = A_6 = A_9 = \cdots$, reduces to

$$662(A_3 + \varepsilon A_2 + \varepsilon^2 A_1) = (1 + \varepsilon^3 + \cdots + \varepsilon^{1983})(A_3 + \varepsilon A_2 + \varepsilon^2 A_1) = 0,$$

and the assertion follows from Proposition 2 in Section 3.4.

Problem 6. *Let a, n be integers and let p be prime such that $p > |a| + 1$. Prove that the polynomial $f(x) = x^n + ax + p$ cannot be represented as a product of two nonconstant polynomials with integer coefficients.*

(1999 Romanian Mathematical Olympiad)

Solution. Let z be a complex root of the polynomial. We shall prove that $|z| > 1$. Suppose $|z| \leq 1$. Then, $z^n + az = -p$, we deduce that

$$p = |z^n + az| = |z||z^{n-1} + a| \leq |z^{n-1}| + |a| \leq 1 + |a|,$$

which contradicts the hypothesis.

Now, suppose $f = gh$ is a decomposition of f into nonconstant polynomials with integer coefficients. Then $p = f(0) = g(0)h(0)$, and either $|g(0)| = 1$ or $|h(0)| = 1$. Assume without loss generality that $|g(0)| = 1$. If $z_1, z_2, \ldots, z_k$ are the roots of g, then they are also roots of f. Therefore

$$1 = |g(0)| = |z_1 z_2 \cdots z_k| = |z_1||z_2|\cdots|z_k| > 1,$$

a contradiction.

Problem 7. *Prove that if a, b, c are complex numbers such that*

$$\begin{cases} (a+b)(a+c) = b, \\ (b+c)(b+a) = c, \\ (c+a)(c+b) = a, \end{cases}$$

then a, b, c are real numbers.

(2001 Romanian IMO Team Selection Test)

Solution. Let $P(x) = x^3 - sx^2 + qx - p$ be the polynomial with roots a, b, c. We have $s = a + b + c$, $q = ab + bc + ca$, $p = abc$. The given equalities are equivalent to

$$\begin{cases} sa + bc = b, \\ sb + ca = c, \\ sc + ab = a. \end{cases} \tag{1}$$

Adding these equalities, we obtain $q = s - s^2$. Multiplying the equalities in (1) by a, b, c, respectively, and adding them we obtain $s(a^2 + b^2 + c^2) + 3p = q$ or, after a short computation,

$$3p = -3s^3 + s^2 + s. \tag{2}$$

If we write the given equations in the form

$$(s-c)(s-b) = b, \quad (s-a)(s-c) = c, \quad (s-b)(s-a) = a,$$

we obtain $((s-a)(s-b)(s-c))^2 = abc$, and, by performing standard computations and using (2), we finally get

$$s(4s-3)(s+1)^2 = 0.$$

If $s = 0$, then $P(x) = x^3$, so $a = b = c = 0$. If $s = -1$, then $P(x) = x^3 + x^2 - 2x - 1$, which has the roots $2\cos\frac{2\pi}{7}$, $2\cos\frac{4\pi}{7}$, $2\cos\frac{6\pi}{7}$ (this is not obvious, but we can see that P changes its sign on the intervals $(-2,-1)$, $(-1,0)$, $(1,2)$ of the real line, hence its roots are real). Finally, if $s = 3/4$, then $P(x) = x^3 - \frac{3}{4}x^2 + \frac{3}{16}x - \frac{1}{64}$, which has roots $a = b = c = 1/4$.

Alternate solution. Subtract the second equation from the first. We obtain $(a+b)(a-b) = b-c$. Analogously, $(b+c)(b-c) = c-a$ and $(c+a)(c-a) = a-b$. We can see that if two of the numbers are equal, then all three are equal and the conclusion is obvious. Suppose that the numbers are distinct. Then, after multiplying the equalities above, we obtain $(a+b)(b+c)(c+a) = 1$, and next: $b(b+c) = c(c+a) = a(a+b) = 1$. Now, if one of the numbers is real, it follows immediately that all three are real. Suppose all numbers are not real. Then $\arg a, \arg b, \arg c \in (0, 2\pi)$. Two of the numbers $\arg a, \arg b, \arg c$ are contained in either $(0,\pi)$ or in $[\pi, 2\pi)$. Suppose these are $\arg a, \arg b$ and that $\arg a \le \arg b$. Then $\arg a \le \arg(a+b) \le \arg b$ and $\arg a \le \arg a(a+b) \le \arg(a+b) \le \arg b$. This is a contradiction, since $a(a+b) = 1$.

Problem 8. *Find the smallest integer n such that an $n \times n$ square can be partitioned into 40×40 and 49×49 squares, with both types of squares present in the partition.*

(2000 Russian Mathematical Olympiad)

Solution. We can partition a 2000×2000 square into 40×40 and 49×49 squares: partition one 1960×1960 corner of the square into 49×49 squares and then partition the remaining portion into 40×40 squares.

We now show that n must be at least 2000. Suppose that an $n \times n$ square has been partitioned into 40×40 and 49×49 squares, using at least one of each type. Let $\zeta = \cos\frac{2\pi}{40} + i\sin\frac{2\pi}{40}$ and $\xi = \cos\frac{2\pi}{49} + i\sin\frac{2\pi}{49}$. Orient the $n \times n$ square so that two sides are horizontal, and number the rows and columns of unit squares from the top left: $0, 1, 2, \ldots, n-1$. For $0 \le j, k \le n-1$, and write $\zeta^j \xi^k$ in square (j, k). If an $m \times m$ square has its top-left corner at (x, y), then the sum of the numbers written in it is

$$\sum_{j=x}^{x+m-1} \sum_{k=y}^{y+m-1} \zeta^j \xi^k = \zeta^x \xi^y \left(\frac{\zeta^m - 1}{\zeta - 1} \right) \left(\frac{\xi^m - 1}{\xi - 1} \right).$$

The first fraction in parentheses is 0 if $m = 40$, and the second fraction is 0 if $m = 49$. Thus, the sum of the numbers written inside each square in the partition is 0, so the sum of all the numbers must be 0. However, applying the above formula with $(m, x, y) = (n, 0, 0)$, we find that the sum of all the numbers equals 0 only if either $\zeta^n - 1$ or $\xi^n - 1$ equals 0. Thus, n must be either a multiple of 40 or a multiple of 49.

Let a and b be the number of 40×40 and 49×49 squares, respectively. The area of the square equals $40^2 \cdot a + 49^2 \cdot b = n^2$. If $40|n$, then $40^2|b$ and hence $b \ge 40^2$. Thus, $n^2 > 49^2 \cdot 40^2 = 1960^2$; because n is a multiple of 40, $n \ge 50 \cdot 40 = 2000$. If instead $49|n$, then $49^2|a$, $a \ge 49^2$, and again $n^2 > 1960^2$. Because n is a multiple of 49, $n \ge 41 \cdot 49 = 2009 > 2000$. In either case, $n \ge 2000$, and 2000 is the minimum possible value of n.

Problem 9. *The pair (z_1, z_2) of nonzero complex numbers has the following property: there is a real number $a \in [-2, 2]$ such that $z_1^2 - az_1z_2 + z_2^2 = 0$. Prove that all pairs (z_1^n, z_2^n), $n = 2, 3, \ldots$, have the same property.*

(Romanian Mathematical Olympiad – Second Round, 2001)

Solution. Denote $t = \frac{z_1}{z_2}$, $t \in \mathbb{C}^*$. The relation $z_1^2 - az_1z_2 + z_2^2 = 0$ is equivalent to $t^2 - at + 1 = 0$. We have $\Delta = a^2 - 4 \le 0$, hence $t = \frac{a \pm i\sqrt{4 - a^2}}{2}$ and $|t| = \sqrt{\frac{a^2}{4} + \frac{4 - a^2}{4}} = 1$. If $t = \cos\alpha + i \sin\alpha$, then $\frac{z_1^n}{z_2^n} = t^n = \cos n\alpha + i \sin n\alpha$ and we can write $z_1^{2n} - a_n z_1^n z_2^n + z_2^{2n} = 0$, where $a_n = 2\cos n\alpha \in [-2, 2]$.

Alternate solution. Because $a \in [-2, 2]$, we can write $a = 2\cos\alpha$. The relation $z_1^2 - az_1z_2 + z_2^2 = 0$ is equivalent to

$$\frac{z_1}{z_2} + \frac{z_2}{z_1} = 2\cos\alpha \tag{1}$$

and, by a simple inductive argument, from (1) it follows that

$$\frac{z_1^n}{z_2^n} + \frac{z_2^n}{z_1^n} = 2\cos n\alpha, \quad n = 1, 2, \ldots.$$

Problem 10. *Find*

$$\min_{z \in \mathbb{C} \setminus \mathbb{R}} \frac{\mathrm{Im} z^5}{\mathrm{Im}^5 z}$$

and the values of z for which the minimum is reached.

Solution. Let a, b be real numbers such that $z = a + bi$, $b \neq 0$. Then $\mathrm{Im}(z)^5 = 5a^4b - 10a^2b^3 + b^5$ and

$$\frac{\mathrm{Im}z^5}{\mathrm{Im}^5 z} = 5\left(\frac{a}{b}\right)^4 - 10\left(\frac{a}{b}\right)^2 + 1.$$

Setting $x = \left(\frac{a}{b}\right)^2$ yields

$$\frac{\mathrm{Im}(z)^5}{\mathrm{Im}^5 z} = 5x^2 - 10x + 1 = 5(x-1)^2 - 4.$$

The minimum value is -4 and is obtained for $x = 1$ i.e., for $z = a(1 \pm i)$, $a \neq 0$.

Problem 11. *Let z_1, z_2, z_3 be complex numbers, not all real, such that $|z_1| = |z_2| = |z_3| = 1$ and $2(z_1 + z_2 + z_3) - 3z_1z_2z_3 \in \mathbb{R}$.*

Prove that

$$\max(\arg z_1, \arg z_2, \arg z_3) \geq \frac{\pi}{6}.$$

Solution. Let $z_k = \cos t_k + i \sin t_k$, $k \in \{1, 2, 3\}$.

The condition $2(z_1 + z_2 + z_3) - 3z_1z_2z_3 \in \mathbb{R}$ implies

$$2(\sin t_1 + \sin t_2 + \sin t_3) = 3\sin(t_1 + t_2 + t_3). \tag{1}$$

Assume by way of contradiction that $\max(t_1, t_2, t_3) < \frac{\pi}{6}$, hence $t_1, t_2, t_3 < \frac{\pi}{6}$. Let $t = \frac{t_1 + t_2 + t_3}{3} \in \left(0, \frac{\pi}{6}\right)$. The sine function is concave on $\left[0, \frac{\pi}{6}\right)$, so

$$\frac{1}{3}(\sin t_1 + \sin t_2 + \sin t_3) \leq \sin\frac{t_1 + t_2 + t_3}{3}. \tag{2}$$

From the relations (1) and (2) we obtain

$$\frac{\sin(t_1 + t_2 + t_3)}{2} \leq \sin\frac{t_1 + t_2 + t_3}{3}.$$

Then

$$\sin 3t \leq 2\sin t.$$

It follows that

$$4\sin^3 t - \sin t \geq 0,$$

i.e., $\sin^2 t \geq \frac{1}{4}$. Hence $\sin t \geq \frac{1}{2}$, then $t \geq \frac{\pi}{6}$, which contradicts that $t \in \left(0, \frac{\pi}{6}\right)$.

Therefore $\max(t_1, t_2, t_3) \geq \frac{\pi}{6}$, as desired.

Here are some more problems.

Problem 12. Solve in complex numbers the system of equations

$$\begin{cases} x|y| + y|x| = 2z^2, \\ y|z| + z|y| = 2x^2, \\ z|x| + x|z| = 2y^2. \end{cases}$$

Problem 13. Solve in complex numbers the following:

$$\begin{cases} x(x-y)(x-z) = 3, \\ y(y-x)(y-z) = 3, \\ z(z-x)(z-y) = 3. \end{cases}$$

(Romanian Mathematical Olympiad – Second Round, 2002)

Problem 14. Let X, Y, Z, T be four points in the plane. The segments $[XY]$ and $[ZT]$ are said to be connected if there is some point O in the plane such that the triangles OXY and OZT are right isosceles triangles in O.

Let $ABCDEF$ be a convex hexagon such that the pairs of segments $[AB]$, $[CE]$, and $[BD]$, $[EF]$ are connected. Show that the points A, C, D and F are the vertices of a parallelogram and that the segments $[BC]$ and $[AE]$ are connected.

(Romanian Mathematical Olympiad – Final Round, 2002)

Problem 15. Let $ABCDE$ be a cyclic pentagon inscribed in a circle of center O which has angles $\widehat{B} = 120^\circ$, $\widehat{C} = 120^\circ$, $\widehat{D} = 130^\circ$, $\widehat{E} = 100^\circ$. Show that the diagonals BD and CE meet at a point belonging to the diameter AO.

(Romanian IMO, Team Selection Test, 2002)

6

Answers, Hints and Solutions to Proposed Problems

In what follows answers and solutions are presented to problems posed in previous chapters. We have preserved the title of the subsection containing the problem and the number of the proposed problem.

6.1 Answers, Hints and Solutions to Routine Problems

6.1.1 Complex numbers in algebraic representation (pp. 18–21)

1. a) $z_1 + z_2 + z_3 = (0, 4)$; b) $z_1 z_2 + z_2 z_3 + z_3 z_1 = (-4, 5)$;
c) $z_1 z_2 z_3 = (-9, 7)$; d) $z_1^2 + z_2^2 + z_3^2 = (-8, -10)$;
e) $\dfrac{z_1}{z_2} + \dfrac{z_2}{z_3} + \dfrac{z_3}{z_1} = \left(-\dfrac{311}{130}, \dfrac{65}{83}\right)$; f) $\dfrac{z_1^2 + z_2^2}{z_2^2 + z_3^2} = \left(\dfrac{152}{221}, -\dfrac{72}{221}\right)$.

2. a) $z = (7, -8)$; b) $z = (-7, -4)$;
c) $z = \left(\dfrac{23}{13}, -\dfrac{2}{13}\right)$; d) $z = (-9, 7)$.

3. a) $z_1 = \left(-\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$, $z_2 = \left(-\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$;
b) $z_1 = (-1, 0)$, $z_2 = \left(\dfrac{1}{2}, \dfrac{\sqrt{3}}{2}\right)$, $z_3 = \left(\dfrac{1}{2}, -\dfrac{\sqrt{3}}{2}\right)$.

4. $\sum_{k=0}^{n} z^k = \begin{cases} (1,0), & \text{for} \quad n = 4k; \\ (1,1), & \text{for} \quad n = 4k+1; \\ (0,1), & \text{for} \quad n = 4k+2; \\ (0,0), & \text{for} \quad n = 4k+3. \end{cases}$

5. a) $z = (1, 1)$; b) $z_1 = (2, 1)$, $z_2 = (-2, -1)$.

6. $z^2 = (a^2 - b^2, 2ab)$; $z^3 = (a^3 - 3ab^2, 3a^2b - b^3)$;
$z^4 = (a^4 - 6a^2b^2 + b^4, 4a^3b - 4ab^3)$.

7. $z_1 = \left(\sqrt{\dfrac{a + \sqrt{a^2+b^2}}{2}},\ \operatorname{sgn} b \sqrt{\dfrac{-a + \sqrt{a^2+b^2}}{2}}\right)$,
$z_2 = \left(-\sqrt{\dfrac{a + \sqrt{a^2+b^2}}{2}},\ -\operatorname{sgn} b \sqrt{\dfrac{-a + \sqrt{a^2+b^2}}{2}}\right)$.

8. For all nonnegative integers k we have
$z^{4k} = ((-4)^k, 0)$; $z^{4k+1} = ((-4)^k, -(-4)^k)$; $z^{4k+2} = (0, -2(-4)^k)$;
$z^{4k+3} = (-2(-4)^k, -2(-4)^k)$; for $k \geq 0$.

9. a) $x = \dfrac{1}{4}$, $y = \dfrac{3}{4}$; b) $x = -2$, $y = 8$; c) $x = 0$, $y = 0$.

10. a) $8 + 51i$; b) $4 - 43i$; c) 2; d) $\dfrac{11}{4} - \dfrac{5\sqrt{7}}{2}i$; e) $\dfrac{61}{13} + \dfrac{4}{13}i$.

11. a) $-i$; b) $E_{4k} = 1$, $E_{4k+1} = 1 + i$, $E_{4k+2} = i$, $E_{4k+3} = 0$; c) 1; d) $-3i$.

12. a) $z_1 = \dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}$, $z_2 = -\dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}$;
b) $z_1 = \dfrac{\sqrt{2}}{2} - i\dfrac{\sqrt{2}}{2}$, $z_2 = -\dfrac{\sqrt{2}}{2} + i\dfrac{\sqrt{2}}{2}$;
c) $z_{1,2} = \pm\left(\dfrac{\sqrt{1+\sqrt{3}}}{2} - \dfrac{\sqrt{\sqrt{3}-1}}{2}i\right)$.

13. $z \in \mathbb{R}$ or $z = x + iy$ with $x^2 + y^2 = 1$.

14. a) $\overline{E}_1 = E_1$;
b) $\overline{E}_2 = E_2$.

15. We substitute a formula for the definition of modulus.

16. From the identity

$$\left(z + \frac{1}{z}\right)^3 = z^3 + \frac{1}{z^3} + 3\left(z + \frac{1}{z}\right)$$

we obtain

$$\left|z + \frac{1}{z}\right|^3 \leq 2 + 3\left|z + \frac{1}{z}\right| \quad \text{or} \quad a^3 - 3a - 2 \leq 0,$$

where

$$a = \left|z + \frac{1}{z}\right|, \ a \geq 0.$$

Since

$$a^3 - 3a - 2 = (a-2)(a^2 + 2a + 1) = (a-2)(a+1)^2,$$

we have $a \leq 2$, as desired.

17. The equation $|z^2 + \overline{z}^2| = 1$ is equivalent to $|z^2 + \overline{z}^2|^2 = 1$. That is, $(z^2 + \overline{z}^2)(\overline{z}^2 + z^2) = 1$. We find $(z^2 + \overline{z}^2)^2 = 1$ or $\left(z^2 + \frac{1}{z^2}\right)^2 = 1$. The last equation is equivalent to $(z^4 + 1)^2 = z^4$ or $(z^4 - z^2 + 1)(z^4 + z^2 + 1) = 0$. The solutions are $\pm\frac{1}{2}i \pm \frac{\sqrt{3}}{2}$ and $\pm\frac{\sqrt{3}}{2} \pm \frac{1}{2}i$.

18. $z \in \left\{\pm\sqrt{\frac{2}{3}}, \pm i\sqrt{2}\right\}$.

19. $z \in \{0, 1, -1, i, -i\}$.

20. Observe that $\left|\frac{1}{z} - \frac{1}{2}\right| < \frac{1}{2}$ is equivalent to $|2 - z| < |z|$, and consequently $(2 - z)(2 - \overline{z}) < z \cdot \overline{z}$. It follows that $4 < 2(z + \overline{z}) = 4\mathrm{Re}(z)$, as needed.

21. $a^2 + b^2 + c^2 - ab - bc - ca$.

22. a) $z_{1,2} = \frac{-6 + \sqrt{21}}{3} + 2i$; b) $z = -\frac{7}{6} + 4i$; c) $z = 2 + i$;
d) $z_{1,2} = \frac{-2 \pm \sqrt{3}}{2} + \frac{1}{2}i$; e) $z^2 = -1$, $z^2 = -5 - 6i$; f) $z^2 = -\frac{13}{2} - \frac{9}{2}i$.

23. $m \in \{1, 5\}$.

24. $z = -2y + 2 + iy$, $y \in \mathbb{R}$.

25. $z = x + iy$ with $x^2 + y^2 = 1$.

26. From $|z_1 + z_2| = \sqrt{3}$ it follows that $|z_1 + z_2|^2 = 3$, i.e., $(z_1 + z_2)(\overline{z_1 + z_2}) = 3$. We obtain $|z_1|^2 + (z_1\overline{z}_2 + \overline{z}_1 z_2) + |z_2|^2 = 3$. That is, $z_1\overline{z}_2 + \overline{z}_1 z_2 = 1$. On the other hand we have $|z_1 - z_2|^2 = |z_1|^2 - (z_1\overline{z}_2 + \overline{z}_1 z_2) + |z_2|^2 = 2 - 1 = 1$, hence $|z_1 - z_2| = 1$.

27. Letting $\varepsilon = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and noticing that $\varepsilon^3 = 1$, we obtain $n = 3k$, $k \in \mathbb{Z}$.

28. Note that $z = 0$ is a solution. For $z \neq 0$ passing to absolute value we obtain $|z|^{n-1} = |z|$, i.e., $|z| = 1$. The equation is equivalent to $z^n = i\overline{z} \cdot z$, which reduces to $z^n = i$. The total number of solutions is $n + 1$.

29. Let

$$\alpha = |z_2 - z_3|, \quad \beta = |z_3 - z_1|, \quad \gamma = |z_1 - z_2|.$$

Since the following inequality,

$$\alpha\beta + \beta\gamma + \gamma\alpha \le \alpha^2 + \beta^2 + \gamma^2$$

holds, and

$$\alpha^2 + \beta^2 + \gamma^2 = 3(|z_1|^2 + |z_2|^2 + |z_2|^2 - |z_1 + z_2 + z_3|^2)$$
$$\le 3(|z_1|^2 + |z_2|^2 + |z_2|^2 = 9R^2,$$

it follows that

$$\alpha\beta + \beta\gamma + \gamma\alpha \le 9r^2.$$

30. Observe that

$$|w| = |v| \cdot \frac{|u - z|}{\overline{u}z - 1} = \frac{|u - z|}{|\overline{u}z - 1|} \le 1$$

if and only if

$$|u - z| \le |\overline{u}z - 1|.$$

This is equivalent to

$$|u - z|^2 \le |\overline{u}z - 1|^2.$$

We obtain

$$(u - z)(\overline{u} - \overline{z}) \le (\overline{u}z - 1)(u\overline{z} - 1),$$

i.e.,

$$|u|^2 + |z|^2 - |u|^2|z|^2 - 1 \le 0.$$

Finally

$$(|u^2| - 1)(|z|^2 - 1) \ge 0.$$

Since $|u| \le 1$, it follows that $|w| \le 1$ if and only if $|z| \le 1$, as desired.

31. $z_1^2 + z_2^2 + z_3^2 = (z_1 + z_2 + z_3)^2 - 2(z_1z_2 + z_2z_3 + z_3z_1)$

$$= -2z_1z_2z_3\left(\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3}\right) = -2z_1z_2z_3(\overline{z_1} + \overline{z_2} + \overline{z_3}) = 0.$$

32. The relation $|z_k| = r$ implies $\overline{z_k} = \dfrac{r^2}{z_k}$ for $k \in \{1, 2, \ldots, n\}$. Then

$$\overline{E} = \frac{\left(\dfrac{r^2}{z_1} + \dfrac{r^2}{z_2}\right)\left(\dfrac{r^2}{z_2} + \dfrac{r^2}{z_3}\right)\cdots\left(\dfrac{r^2}{z_n} + \dfrac{r^2}{z_1}\right)}{\dfrac{r^2}{z_1} \cdot \dfrac{r^2}{z_2} \cdots \dfrac{r^2}{z_n}}$$

$$= \frac{r^{2n} \cdot \dfrac{z_1 + z_2}{z_1z_2} \cdot \dfrac{z_2 + z_3}{z_2z_3} \cdots \dfrac{z_n + z_1}{z_nz_1}}{r^{2n} \cdot \dfrac{1}{z_1z_2 \cdots z_n}} = E,$$

hence $E \in \mathbb{R}$.

33. Notice that

$$z_1 \cdot \overline{z_1} = z_2 \cdot \overline{z_2} = z_3 \cdot \overline{z_3} = r^2$$

and

$$z_1 z_2 + z_3 \in \mathbb{R} \text{ if and only if } z_1 z_2 + z_3 = \overline{z_1} \cdot \overline{z_2} + \overline{z_3}.$$

Then

$$\frac{r^2}{z_1 z_2 z_3} = \frac{z_1 z_2 + z_3}{z_1 z_2 + r^2 z_3} = \frac{z_1 z_3 + z_2}{z_1 z_3 + r^2 z_2} = \frac{z_2 z_3 + z_1}{z_2 z_3 + r^2 z_1}$$

$$\frac{(z_1 - 1)(z_2 - z_3)}{(z_2 - z_3)(z_1 - r^2)} = \frac{z_1 - 1}{z_1 - r^2} = \frac{z_2 - 1}{z_2 - r^2} = \frac{z_3 - 1}{z_3 - r^2} = \frac{z_1 - z_2}{z_1 - z_2} = 1.$$

Hence $z_1 z_2 z_2 = r^2$ and consequently $r^3 = r^2$. Therefore $r = 1$ and $z_1 z_2 z_3 = 1$, as desired.

34. Note that $x_1^3 = x_2^3 = -1$.

a) -1; b) 1; c) Consider $n \in \{6k, 6k \pm 1, 6k \pm 2, 6k \pm 3\}$.

35. a) $x^4 + 16 = x^4 + 2^4 = (x^2 + 4i)(x^2 - 4i)$

$$= [x^2 + (\sqrt{2}(1+i))^2][x^2 - (\sqrt{2}(1+i))^2]$$

$$= (x + \sqrt{2}(-1+i))(x + \sqrt{2}(1-i))(x - \sqrt{2}(1+i))(x + \sqrt{2}(1+i)).$$

b) $x^3 - 27 = x^3 - 3^3 = (x-3)(x-3\varepsilon)(x-3\varepsilon^2)$, where $\varepsilon = -\dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$.

c) $x^3 + 8 = x^3 + 2^3 = (x+2)(x+1+i\sqrt{3})(x+1-i\sqrt{3})$.

d) $x^4 + x^2 + 1 = (x^2 - \varepsilon)(x^2 - \varepsilon^2) = (x^2 - \varepsilon^{-2})(x^2 - \varepsilon^2)$

$$= (x-\varepsilon)(x+\varepsilon)(x-\overline{\varepsilon})(x+\overline{\varepsilon}), \text{ where } \varepsilon = -\frac{1}{2} + \frac{\sqrt{3}}{2}i.$$

36. a) $x^2 - 14x + 50 = 0$; b) $x^2 - \dfrac{18}{5}x + \dfrac{26}{5} = 0$; c) $x^2 + 4x + 8 = 0$.

37. We have

$$2|z_1 + z_2| \cdot |z_2 + z_3| = 2|z_2(z_1 + z_2 + z_3) + z_1 z_3| \le 2|z_2| \cdot |z_1 + z_2 + z_3| + 2|z_1||z_3|,$$

and likewise,

$$2|z_2 + z_3| \cdot |z_3 + z_1| \le 2|z_3||z_1 + z_2 + z_3| + 2|z_2||z_1|,$$

$$2|z_3 + z_1| \cdot |z_1 + z_2| \le 2|z_1||z_1 + z_2 + z_3| + 2|z_2||z_3|.$$

Summing up these inequalities with

$$|z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2 = |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_1 + z_2 + z_3|^2$$

yields

$$(|z_1 + z_2|^2 + |z_2 + z_3|^2 + |z_3 + z_1|^2) \le (|z_1| + |z_2| + |z_3| + |z_1 + z_2 + z_3|^2).$$

The conclusion is now obvious.

6.1.2 Geometric interpretation of the algebraic operations (p. 27)

3. a) The circle of center $(2, 0)$ and radius 3.
b) The disk of center $(0, -1)$ and radius 1.
c) The exterior of the circle of center $(1, -2)$ and radius 3.
d) $M = \left\{(x, y) \in \mathbb{R}^2 | x \ge -\dfrac{1}{2}\right\} \cup \left\{(x, y) \in \mathbb{R}^2 | x < -\dfrac{1}{2}, 3x^2 - y^2 - 3 < 0\right\}$.
e) $M = \{(x, y) \in \mathbb{R}^2 | -1 < y < 0\}$.
f) $M = \{(x, y) \in \mathbb{R}^2 | -1 < y < 1\}$.
g) $M = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 - 3x + 2 = 0\}4$.
h) The union of the lines with equations $x = -\dfrac{1}{2}$ and $y = 0$.

4. $M = \{(x, y) \in \mathbb{R}^2 | y = 10 - x^2,\ y \ge 4\}$.

5. $z_3 = \sqrt{3}(1 - i)$ and $z_3' = \sqrt{3}(1 + i)$.

6. $M = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 + x = 0,\ x \ne 0,\ x \ne -1\}$
$\cup\{(0, y) \in \mathbb{R}^2 | y \ne 0\} \cup \{(-1, y) \in \mathbb{R}^2 | y \ne 0\}$.

7. The union of the circles with equations

$$x^2 + y^2 - 2y - 1 = 0 \quad \text{and} \quad x^2 + y^2 + 2y - 1 = 0.$$

6.1.3 Polar representation of complex numbers (pp. 39–41)

1. a) $r = 3\sqrt{2}, t^* = \dfrac{3\pi}{4}$; b) $r = 8, t^* = \dfrac{7\pi}{6}$; c) $r = 5, t^* = \pi$;
d) $r = \sqrt{5}, t^* = \arctan\dfrac{1}{2} + \pi$; e) $r = 2\sqrt{5}, t^* = \arctan\left(-\dfrac{1}{2}\right) + 2\pi$.

2. a) $x = 1, y = \sqrt{3}$; b) $x = \dfrac{16}{5}, y = -\dfrac{12}{5}$; c) $x = -2, y = 0$;
d) $x = -3, y = 0$ e) $x = 0, y = 1$ f) $x = 0, y = -4$.

3. $\arg(\overline{z}) = \begin{cases} 2\pi - \arg z, & \text{if } \arg z \ne 0, \\ 0, & \text{if } \arg z = 0; \end{cases}$;

$\arg(-z) = \begin{cases} \pi + \arg z, & \text{if } \arg z \in [0, \pi), \\ -\pi + \arg z, & \text{if } \arg z \in [\pi, 2\pi). \end{cases}$

4. a) The circle of radius 2 with center at origin.
b) The circle of center $(0, -1)$ and radius 2 and its exterior.
c) The disk of center $(0, 1)$ and radius 3.

d) The interior of the angle determined by the rays $y = 0$, $x \leq 0$ and $y = x$, $x \leq 0$.

e) The fourth quadrant and the ray $(OY'$.

f) The first quadrant and the ray $(OX$.

g) The interior of the angle determined by the rays $y = \dfrac{\sqrt{3}}{3}x$, $x \leq 0$ and $y = \sqrt{3}x$, $x < 0$.

h) The intersection of the disk of center $(-1, -1)$ and radius 3 with the interior of the angle determined by the rays $y = 0$, $x \geq 0$ and $y = \dfrac{\sqrt{3}}{3}x$, $x > 0$.

5. a) $z_1 = 12\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right)$; b) $z_2 = \dfrac{1}{2}\left(\cos\dfrac{2\pi}{3} + i\sin\dfrac{2\pi}{3}\right)$;

c) $z_3 = \cos\dfrac{4\pi}{3} + i\sin\dfrac{4\pi}{3}$; d) $z_4 = 18\left(\cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3}\right)$;

e) $z_5 = \sqrt{13}\left[\cos\left(2\pi - \arctan\dfrac{2}{3}\right) + i\sin\left(2\pi - \arctan\dfrac{2}{3}\right)\right]$;

f) $z_6 = 4\left(\cos\dfrac{3\pi}{2} + i\sin\dfrac{3\pi}{2}\right)$.

6. a) $z_1 = \cos(2\pi - a) + i\sin(2\pi - a)$, $a \in [0, 2\pi)$;

b) $z_2 = 2\left|\cos\dfrac{a}{2}\right| \cdot \left[\cos\left(\dfrac{\pi}{2} - \dfrac{a}{2}\right) + i\sin\left(\dfrac{\pi}{2} - \dfrac{a}{2}\right)\right]$ if $a \in [0, \pi)$;

$z_2 = 2\left|\cos\dfrac{a}{2}\right| \cdot \left[\cos\left(\dfrac{3\pi}{2} - \dfrac{a}{2}\right) + i\sin\left(\dfrac{3\pi}{2} - \dfrac{a}{2}\right)\right]$ if $a \in (\pi, 2\pi)$;

c) $z_3 = \sqrt{2}\left[\cos\left(a + \dfrac{7\pi}{4}\right) + i\sin\left(a + \dfrac{7\pi}{4}\right)\right]$ if $a \in \left[0, \dfrac{\pi}{4}\right]$;

$z_3 = \sqrt{2}\left[\cos\left(a - \dfrac{\pi}{4}\right) + i\sin\left(a - \dfrac{\pi}{4}\right)\right]$ if $a \in \left(\dfrac{\pi}{4}, 2\pi\right)$;

d) $z_4 = 2\sin\dfrac{a}{2}\left[\cos\left(\dfrac{\pi}{2} - \dfrac{a}{2}\right) + i\sin\left(\dfrac{\pi}{2} - \dfrac{a}{2}\right)\right]$ if $a \in [0, \pi)$;

$z_4 = 2\sin\dfrac{a}{2}\left[\cos\left(\dfrac{5\pi}{2} - \dfrac{a}{2}\right) + i\sin\left(\dfrac{5\pi}{2} - \dfrac{a}{2}\right)\right]$ if $a \in [\pi, 2\pi)$.

7. a) $12\sqrt{2}\left(\cos\dfrac{7\pi}{4} + i\sin\dfrac{7\pi}{4}\right)$; b) $4(\cos 0 + i\sin 0)$;

c) $48\sqrt{2}\left(\cos\dfrac{5\pi}{12} + i\sin\dfrac{5\pi}{12}\right)$; d) $30\left(\cos\dfrac{\pi}{2} + i\sin\dfrac{\pi}{2}\right)$.

8. a) $|z| = 12$, $\arg z = 0$, $\operatorname{Arg} z = 2k\pi$, $\arg\overline{z} = 0$, $\arg(-z) = \pi$;

b) $|z| = 14\sqrt{2}$, $\arg z = \dfrac{11\pi}{12}$, $\operatorname{Arg} z = \dfrac{11\pi}{12} + 2k\pi$, $\arg\overline{z} = \dfrac{13\pi}{12}$, $\arg(-z) = \dfrac{\pi}{12}$.

9. a) $|z| = 2^{13} + \dfrac{1}{2^{13}}$, $\arg z = \dfrac{5\pi}{6}$; b) $|z| = \dfrac{1}{2^9}$, $\arg z = \pi$;

c) $|z| = 2^{n+1}\left|\cos\dfrac{5n\pi}{3}\right|$, $\arg z \in \{0, \pi\}$.

10. If $z = r(\cos t + i\sin t)$ and $n = -m$, where m is a positive integer, then

$$z^n = z^{-m} = \frac{1}{z^m} = \frac{1}{r^m(\cos mt + i\sin mt)} = \frac{1}{r^m} \cdot \frac{\cos 0 + i\sin 0}{\cos mt + i\sin mt}$$

$$= \frac{1}{r^m}[\cos(0-m)t + i\sin(0-m)t] = r^{-m}(\cos(-mt) + i\sin(-mt))$$

$$= r^n(\cos nt + i\sin nt).$$

11. a) $2^n \sin^n \frac{a}{2}\left[\cos\frac{n(\pi-a)}{2} + i\sin\frac{n(\pi-a)}{2}\right]$ if $a \in [0,\pi)$;
$2^n \sin^n \frac{a}{2}\left[\cos\frac{n(5\pi-a)}{2} + i\sin\frac{n(5\pi-a)}{2}\right]$ if $a \in [\pi, 2\pi]$;
b) $z^n + \frac{1}{z^n} = 2\cos\frac{n\pi}{6}$.

6.1.4 The n^{th} roots of unity (p. 52)

1. a) $z_k = \sqrt[4]{2}\left(\cos\frac{\frac{\pi}{4}+2k\pi}{2} + i\sin\frac{\frac{\pi}{4}+2k\pi}{2}\right)$, $k \in \{0,1\}$;

b) $z_k = \cos\frac{\frac{\pi}{2}+2k\pi}{2} + i\sin\frac{\frac{\pi}{2}+2k\pi}{2}$, $k \in \{0,1\}$;

c) $z_k = \cos\frac{\frac{\pi}{4}+2k\pi}{2} + i\sin\frac{\frac{\pi}{4}+2k\pi}{2}$, $k \in \{0,1\}$;

d) $z_k = 2\left(\cos\frac{\frac{4\pi}{3}+2k\pi}{2} + i\sin\frac{\frac{4\pi}{3}+2k\pi}{2}\right)$, $k \in \{0,1\}$;

e) $z_0 = 4-3i$, $z_1 = -4+3i$.

2. a) $z_k = \cos\frac{\frac{3\pi}{2}+2k\pi}{3} + i\sin\frac{\frac{3\pi}{2}+2k\pi}{2}$, $k \in \{0,1,2\}$;

b) $z_k = 3\left(\cos\frac{\pi+2k\pi}{3} + i\sin\frac{\pi+2k\pi}{3}\right)$, $k \in \{0,1,2\}$;

c) $z_k = \sqrt{2}\left(\cos\frac{\frac{\pi}{4}+2k\pi}{3} + i\sin\frac{\frac{\pi}{4}+2k\pi}{3}\right)$, $k \in \{0,1,2\}$;

d) $z_k = \cos\frac{\frac{5\pi}{3}+2k\pi}{3} + i\sin\frac{\frac{5\pi}{3}+2k\pi}{3}$, $k \in \{0,1,2\}$;

e) $z_0 = 3+i$, $z_1 = (3+i)\varepsilon$, $z_2 = (3+i)\varepsilon^2$, where $1, \varepsilon, \varepsilon^2$ are the cube roots of 1.

3. a) $z_k = \sqrt{2}\left(\cos\frac{\frac{5\pi}{4}+2k\pi}{4} + i\sin\frac{\frac{5\pi}{4}+2k\pi}{4}\right)$, $k \in \{0,1,2,3\}$;

b) $z_k = \sqrt[4]{2}\left(\cos\frac{\frac{\pi}{6}+2k\pi}{4} + i\sin\frac{\frac{\pi}{6}+2k\pi}{4}\right)$, $k \in \{0,1,2,3\}$;

c) $z_k = \cos \dfrac{\dfrac{\pi}{2} + 2k\pi}{4} + i \sin \dfrac{\dfrac{\pi}{2} + 2k\pi}{4}$, $k \in \{0, 1, 2, 3\}$;

d) $z_k = \sqrt[4]{2}\left(\cos \dfrac{\dfrac{3\pi}{2} + 2k\pi}{4} + i \sin \dfrac{\dfrac{3\pi}{2} + 2k\pi}{4}\right)$, $k \in \{0, 1, 2, 3\}$;

e) $z_0 = 2 + i$, $z_1 = -2 - i$, $z_2 = -1 + 2i$, $z_3 = 1 - 2i$.

4. $z_k = \cos \dfrac{2k\pi}{n} + i \sin \dfrac{2k\pi}{n}$, $k \in \{0, 1, \dots, n-1\}$, $n \in \{5, 6, 7, 8, 12\}$.

5. a) Consider $\varepsilon_j = \varepsilon^j$, $\varepsilon_k = \varepsilon^k$, where $\varepsilon = \cos \dfrac{2\pi}{n} + i \sin \dfrac{2\pi}{n}$. Then $\varepsilon_j \cdot \varepsilon_k = \varepsilon^{j+k}$. Let r be the remainder modulo n of $j + k$. We have $j + k = p \cdot n + r$, $r \in \{0, 1, \dots, n-1\}$ and $\varepsilon_j \cdot \varepsilon_k =^{p \cdot n + r} = (\varepsilon^n)^p \cdot \varepsilon^r = \varepsilon^r = \varepsilon_r \in U_n$.

b) We can write $\varepsilon_j^{-1} = \dfrac{1}{\varepsilon_j} = \dfrac{1}{\varepsilon^j} = \dfrac{\varepsilon^n}{\varepsilon^j} = \varepsilon^{n-j} \in U_n$.

6. a) $z_k = 5\left(\cos \dfrac{2k\pi}{3} + i \sin \dfrac{2k\pi}{3}\right)$, $k \in \{0, 1, 2\}$;

b) $z_k = 2\left(\cos \dfrac{\pi + 2k\pi}{4} + i \sin \dfrac{\pi + 2k\pi}{4}\right)$, $k \in \{0, 1, 2, 3\}$;

c) $z_k = 4\left(\cos \dfrac{\dfrac{3\pi}{2} + 2k\pi}{3} + i \sin \dfrac{\dfrac{3\pi}{2} + 2k\pi}{3}\right)$, $k \in \{0, 1, 2\}$;

d) $z_k = 3\left(\cos \dfrac{\dfrac{\pi}{2} + 2k\pi}{3} + i \sin \dfrac{\dfrac{\pi}{2} + 2k\pi}{3}\right)$, $k \in \{0, 1, 2\}$.

7. a) The equation is equivalent to $(z^4 - i)(z^3 - 2i) = 0$.

b) We can write the equation as $(z^3 + 1)(z^3 + i - 1) = 0$.

c) The equation is equivalent to $z^6 = -1 + i$.

d) We can write the equation equivalently as $(z^5 - 2)(z^5 + i) = 0$.

8. It is clear that any solution is different from zero. Multiplying by z, the equation is equivalent to $z^5 - 5z^4 + 10z^3 - 10z^2 + 5z - 1 = -1$, $z \neq 0$. We obtain the binomial equation $(z - 1)^5 = -1$, $z \neq 0$. The solutions are $z_k = 1 + \cos \dfrac{(2k+1)\pi}{5} + i \sin \dfrac{(2k+1)\pi}{5}$, $k = 0, 1, 3, 4$.

6.1.5 Some geometric transformations of the complex plane (p. 160)

1. Suppose that f, g are isometries. Then for all complex numbers a, b, we have $|f(g(a)) - f(g(b))| = |g(a) - g(b)| = |a - b|$, so $f \circ g$ is also an isometry.

2. Suppose that f is an isometry and let C be any point on the line AB. Let $f(C) = M$. Then $MA = f(C)f(A) = AC$ and, similarly, $MB = BC$. Thus $|MA - MB| = AB$.

Hence A, M, B are collinear. Now, from $MA = AC$ and $MB = BC$, we conclude that $M = C$. Hence $f(M) = M$ and the conclusion follows.

3. This follows immediately from the fact that any isometry f is of the form $f(z) = az + b$ or $f(z) = a\overline{z} + b$, with $|a| = 1$.

4. The function f is the product of the rotation $z \to iz$, the translation $z \to z + 4 - i$, and the reflection in the real axis. It is clear that f is an isometry.

5. The function f is the product of the rotation $z \to -iz$ with the translation $z \to z + 1 + 2i$.

6.2 Solutions to the Olympiad-Caliber Problems

6.2.1 Problems involving moduli and conjugates (pp. 175–176)

Problem 21. At first we prove that function f is well defined, i.e., $|f(z)| < 1$ for all z with $|z| < 1$.

Indeed, we have $|f(z)| < 1$ if and only if $\left|\frac{1+az}{z+a}\right| < 1$, i.e., $|1 + az|^2 < |z + a|^2$. The last relation is equivalent to $(1 + az)(1 + \overline{az}) < (z + a)(\overline{z} + \overline{a})$. That is, $1 + |a|^2|z|^2 < |a|^2 + |z|^2$ or equivalently $(|a|^2 - 1)(|z|^2 - 1) < 0$. The last inequality is obvious since $|z| < 1$, and $|a| > 1$.

To prove that f is bijective, it suffices to observe that for any $y \in A$ there is a unique $z \in A$ such that

$$f(z) = \frac{1 + az}{z + a} = y.$$

We obtain

$$z = \frac{ay - 1}{a - y} = -f(-y),$$

hence $|z| = |f(-y)| < 1$, as desired.

Problem 22. Let $z = \cos\varphi + i \sin\varphi$ with $\cos\varphi, \sin\varphi \in \mathbb{Q}$. Then

$$z^{2n} - 1 = \cos 2n\varphi + i \sin 2n\varphi - 1 = 1 - 2\sin^2 n\varphi + 2i \sin n\varphi \cos n\varphi - 1$$

$$= -2 \sin n\varphi(\sin n\varphi - i \cos n\varphi)$$

and

$$|z^{2n} - 1| = 2|\sin n\varphi|.$$

It suffices to prove that $\sin n\varphi \in \mathbb{Q}$. We prove by induction on n that both $\sin n\varphi$ and $\cos n\varphi$ are rational numbers. The claim is obvious for $n = 1$.

Assume that $\sin n\varphi, \cos n\varphi \in \mathbb{Q}$. Then

$$\sin(n + 1)\varphi = \sin n\varphi \cos\varphi + \cos n\varphi \cos\varphi \in \mathbb{Q}$$

and

$$\cos(n+1)\varphi = \cos n\varphi \cos\varphi - \sin n\varphi \sin\varphi \in \mathbb{Q},$$

as desired.

Problem 23. To prove that the function f is injective, let $f(a) = f(b)$. Then $\dfrac{1+ai}{1-ai} = \dfrac{1+bi}{1-bi}$. This is equivalent to $1 + ab + (a-b)i = 1 + ab + (b-a)i$, i.e., $a = b$, as needed.

The image of the function f is the set of numbers $z \in \mathbb{C}$ such that there is $t \in \mathbb{R}$ with

$$z = f(t) = \frac{1+ti}{1-ti}.$$

From $z = \dfrac{1+ti}{1-ti}$ we obtain $t = \dfrac{z-1}{i(1+z)}$ if $z \neq 1$. Then $t \in \mathbb{R}$ if and only if $t = \bar{t}$. The last relation is equivalent to $\dfrac{z-1}{i(1+z)} = \dfrac{\overline{z}-1}{-i(1+\overline{z}}$, i.e., $-(z-1)(\overline{z}+1) = (z+1)(\overline{z}-1)$. It follows that $2z\overline{z} = 2$, i.e., $|z| = 1$, hence the image of the function f is the set $\{z \in \mathbb{R} | |z| = 1 \text{ and } z \neq -1\}$, the unit circle without the point with coordinate $z = -1$.

Problem 24. Let $\dfrac{z_2}{z_1} = t \in \mathbb{C}$. Then

$$|z_1 + z_1 t| = |z_1| = |z_1 t| \text{ or } |1+t| = |t| = 1.$$

It follows that $t\bar{t} = 1$ and

$$1 = |1+t|^2 = (1+t)(1+\bar{t}) = 1 + t + \bar{t} + 1,$$

hence $t^2 + t + 1 = 0$.

Therefore t is a nonreal cube root of unity.

Alternate solution. Let A, B, C be the geometric images of the complex numbers $z_1, z_2, z_1 + z_2$, respectively. In the parallelogram $OACB$ we have $OA = OB = OC$, hence $\widehat{AOB} = 120°$. Then

$$\frac{z_2}{z_1} = \cos 120° + i \sin 120° \text{ or } \frac{z_1}{z_2} = \cos 120° + i \sin 120°,$$

therefore

$$\frac{z_2}{z_1} = \cos\frac{2\pi}{3} \pm i \sin\frac{2\pi}{3}.$$

Problem 25. We prove first the inequality

$$|z_k| \leq |z_1| + |z_2| + \cdots + |z_{k-1}| + |z_{k+1}| + \cdots + |z_n| + |z_1 + z_2 + \cdots + z_n|$$

for all $k \in \{1, 2, \dots, n\}$. Indeed,

$$\begin{aligned} |z_k| &= |(z_1 + z_2 + \cdots + z_{k-1} + z_k + z_{k+1} + \cdots + z_n) \\ &\qquad - (z_1 + z_2 + \cdots + z_{k-1} + z_{k+1} + \cdots + z_n)| \\ &\le |z_1 + z_2 + \cdots + z_n| + |z_1| + \cdots + |z_{k-1}| + |z_{k+1}| + \cdots + |z_n|, \end{aligned}$$

as claimed.

Denote $S_k = |z_1| + \cdots + |z_{k-1}| + |z_{k+1}| + \cdots + |z_n|$ for all k. Then

$$|z_k| \le S_k + |z_1 + z_2 + \cdots + z_n|, \text{ for all } k. \tag{1}$$

Moreover,

$$|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|. \tag{2}$$

Multiplying by $|z_k|$ the inequalities (1) and by $|z_1 + z_2 + \cdots + z_n|$ the inequalities (2), we obtained by summation:

$$\begin{aligned} &|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 + |z_1 + z_2 + \cdots + z_n|^2 \\ &\le |z_1 + z_2 + \cdots + z_n| \sum_{k=1}^{n} |z_k| + \sum_{k=1}^{n} |z_k| S_k. \end{aligned}$$

Adding on both sides of the inequality the expression

$$|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 + |z_1 + z_2 + \cdots + z_n|^2$$

yields

$$\begin{aligned} &2(|z_1|^2 + |z_2|^2 + \cdots + |z_n|^2 + |z_1 + z_2 + \cdots + z_n|^2) \\ &\le (|z_1| + \cdots + |z_n| + |z_1 + z_2 + \cdots + z_n|)^2, \end{aligned}$$

as desired.

Problem 26. Let $M_1, M_2, \dots, M_{2n}$ be the points with the coordinates $z_1, z_2, \dots, z_{2n}$ and let $A_1, A_2, \dots, A_n$ be the midpoints of segments M_1M_{2n}, $M_2M_{2n-1}, \dots, M_nM_{n+1}$.

The points M_i, $i = \overline{1, 2n}$ lie on the upper semicircle centered in the origin and with radius 1. Moreover, the lengths of the chords $M_1M_{2n}, M_2M_{2n-1}, \dots, M_nM_{n+1}$ are in a decreasing order, hence $OA_1, OA_2, \dots, OA_n$ are increasing. Thus

$$\left|\frac{z_1 + z_{2n}}{2}\right| \le \left|\frac{z_2 + z_{2n-1}}{2}\right| \le \cdots \le \left|\frac{z_n + z_{n+1}}{2}\right|$$

and the conclusion follows.

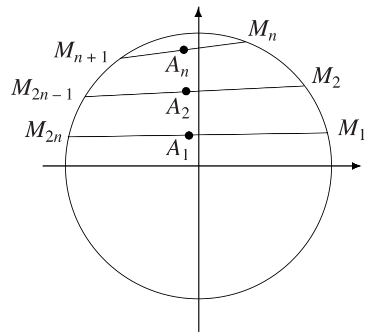

Figure 6.1.

Alternate solution. Consider $z_k = r(\cos t_k + i \sin t_k)$, $k = 1, 2, \ldots, 2n$ and observe that for any $j = 1, 2, \ldots, n$, we have

$$|z_j + z_{2n-j+1}|^2 = |r[(\cos t_j + \cos t_{2n-j+1}) + i(\sin t_j + \sin t_{2n-j+1})]|^2$$

$$= r^2[(\cos t_j + \cos t_{2n-j+1})^2 + (\sin t_j + \sin t_{2n-j+1})^2]$$

$$= r^2[2 + 2(\cos t_j \cos t_{2n-j+1} + \sin t_j \sin t_{2n-j+1})]$$

$$= 2r^2[1 + \cos(t_{2n-j+1} - t_j)] = 4r^2 \cos^2 \frac{t_{2n-j+1} - t_j}{2}.$$

Therefore $|z_j + z_{2n-j+1}| = 2r \cos \dfrac{t_{2n-j+1} - t_j}{2}$ and the inequalities

$$|z_1 + z_{2n}| \leq |z_2 + z_{2n-1}| \leq \cdots \leq |z_n + z_{n+1}|$$

are equivalent to $t_{2n} - t_1 \geq t_{2n-1} - t_2 \geq \cdots \geq t_{n+1} - t_n$. Because $0 \leq t_1 \leq t_2 \leq \cdots \leq t_{2n} \leq \pi$, the last inequalities are obviously satisfied.

Problem 27. It is natural to make the substitution $\sqrt{x} = u$, $\sqrt{y} = v$. The system becomes

$$u\left(1 + \frac{1}{u^2 + v^2}\right) = \frac{2}{\sqrt{3}},$$
$$v\left(1 - \frac{1}{u^2 + v^2}\right) = \frac{4\sqrt{2}}{\sqrt{7}}.$$

But $u^2 + v^2$ is the square of the absolute value of the complex number $z = u + iv$. This suggests that we add the second equation multiplied by i to the first one. We obtain

$$u + iv + \frac{u - iv}{u^2 + v^2} = \left(\frac{2}{\sqrt{3}} + i\frac{4\sqrt{2}}{\sqrt{7}}\right).$$

The quotient $(u - iv)/(u^2 + v^2)$ is equal to $\overline{z}/|z|^2 = \overline{z}/(z\overline{z}) = 1/z$, so the above equation becomes

$$z + \frac{1}{z} = \left(\frac{2}{\sqrt{3}} + i\frac{4\sqrt{2}}{\sqrt{7}}\right).$$

Hence z satisfies the quadratic equation

$$z^2 - \left(\frac{2}{\sqrt{3}} + i\frac{4\sqrt{2}}{\sqrt{7}}\right) z + 1 = 0$$

with solutions

$$\left(\frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{21}}\right) + i\left(\frac{2\sqrt{2}}{\sqrt{7}} \pm \sqrt{2}\right),$$

where the signs + and − correspond.

This shows that the initial system has the solutions

$$x = \left(\frac{1}{\sqrt{3}} \pm \frac{2}{\sqrt{21}}\right)^2, \quad y = \left(\frac{2\sqrt{2}}{\sqrt{7}} \pm \sqrt{2}\right)^2,$$

where the signs + and − correspond.

Problem 28. The direct implication is obvious.

Conversely, let $|z_1| = |z_2 + z_3|$, $|z_2| = |z_1 + z_3|$, $|z_3| = |z_1 + z_2|$. It follows that

$$|z_1|^2 + |z_2|^2 + |z_3|^2 = |z_2 + z_3|^2 + |z_3 + z_1|^2 + |z_1 + z_2|^2.$$

This is equivalent to

$$\begin{aligned} z_1\overline{z_1} + z_2\overline{z_2} + z_3\overline{z_3} &= z_2\overline{z_2} + z_2\overline{z_3} + \overline{z_2}z_3 + z_3\overline{z_3} \\ &\quad + z_3\overline{z_1} + z_1\overline{z_3} + z_1\overline{z_1} + z_1\overline{z_1} + z_1\overline{z_2} + z_2\overline{z_1} + z_2\overline{z_2}, \quad \text{i.e.,} \\ z_1\overline{z_1} + z_2\overline{z_2} + z_3\overline{z_3} &+ z_1\overline{z_2} + z_2\overline{z_1} + z_1\overline{z_3} + \overline{z_1}z_3 + z_2\overline{z_3} + \overline{z_3}z_2 = 0. \end{aligned}$$

We write the last relation as

$$(z_1 + z_2 + z_3)(\overline{z_1} + \overline{z_2} + \overline{z_3}) = 0,$$

and we obtain

$$|z_1 + z_2 + z_3|^2 = 0, \text{ i.e., } z_1 + z_2 + z_3 = 0,$$

as desired.

Problem 29. Let $a = |z_1| = |z_2| = \cdots = |z_n|$. Then

$$\overline{z_k} = \frac{a^2}{z_k}, \quad k = \overline{1, n}$$

and

$$\overline{z_1z_2+z_2z_3+\cdots+z_{n-1}z_n}=\sum_{k=1}^{n-1}\overline{z_kz_{k+1}}=\sum_{k=1}^{n-1}\frac{a^4}{z_kz_{k+1}}$$

$$=\frac{a^4}{z_1z_2\cdots z_n}(z_3z_4\cdots z_n+z_1z_4\cdots z_n+\cdots+z_1z_2\cdots z_{n-2})=0;$$

hence

$$z_1z_2+z_2z_3+\cdots+z_{n-1}z_n=0,$$

as desired.

Problem 30. Let

$$z=r_1(\cos t_1+i\sin t_1)$$

and

$$a=r_2(\cos t_2+i\sin t_2).$$

We have

$$1=|z+a|=\sqrt{(r_1\cos t_1+r_2\cos t_2)^2+(r_1\sin t_1+r_2\sin t_2)^2}$$
$$=\sqrt{r_1^2+r_2^2+2r_1r_2\cos(t_1-t_2)},$$

so

$$\cos(t_1-t_2)=\frac{1-r_1^2-r_2^2}{2r_1r_2}.$$

Then

$$|z^2+a^2|=|r_1^2(\cos 2t_1+i\sin 2t_1)+r_2^2(\cos 2t_2+i\sin 2t_2)|$$
$$=\sqrt{(r_1^2\cos 2t_1+r_2^2\cos 2t_2)^2+(r_1^2\sin 2t_1+r_2^2\sin 2t_2)}$$
$$=\sqrt{r_1^4+r_2^4+2r_1^2r_2^2\cos 2(t_1-t_2)}$$
$$=\sqrt{r_1^4+r_2^4+2r_1r_2(2\cos^2(t_1-t_2)-1)}$$
$$=\sqrt{r_1^4+r_2^4+2r_1^2r_2^2\cdot\left(2\left(\frac{1-r_1^2-r_2^2}{2r_1r_2}\right)^2-1\right)}$$
$$=\sqrt{2r_1^4+2r_2^4+1-2r_1^2-2r_2^2}.$$

The inequality

$$|z^2+a^2|\geq\frac{|1-2|a||}{\sqrt{2}}$$

is equivalent to

$$2r_1^4 + 2r_2^4 + 1 - 2r_1^2 - 2r_2^2 \geq \frac{(1-2r_1^2)^2}{2}, \text{ i.e.,}$$
$$4r_1^4 + 4r_2^4 - 4r_1^2 - 4r_2^2 + 2 \geq 1 - 4r_1^2 + 4r_2^2.$$

We obtain

$$(2r_2^2 - 1)^2 \geq 0,$$

and we are done.

Problem 31. It is easy to see that $z = 0$ is a root of the equation. Consider $z = a+ib \neq 0$, $a, b \in \mathbb{R}$.

Observe that if $a = 0$, then $b = 0$ and if $b = 0$, then $a = 0$. Therefore we may assume that $a, b \neq 0$.

Taking the modulus of both members of the equation

$$az^n = b\overline{z}^n \tag{1}$$

yields $|a| = |b|$ or $a = \pm b$.

Case 1. If $a = b$, the equation (1) becomes

$$(a + ia)^n = (a - ia)^n.$$

This is equivalent to

$$\left(\frac{1+i}{1-i}\right)^n = 1, \text{ i.e., } i^n = 1,$$

which has solutions only for $n = 4k$, $k \in \mathbb{Z}$. In that case the solutions are

$$z = a(1 + i), \quad a \neq 0.$$

Case 2. If $a = -b$, the equation (1) may be rewritten as

$$(a - ia)^n = -(a + ia)^n.$$

That is,

$$\left(\frac{1-i}{1+i}\right)^n = -1, \text{ i.e., } (-i)^n = -1,$$

which has solutions only for $n = 4k + 2$, $k \in \mathbb{Z}$. We obtain

$$z = a(1 - i), \quad a \neq 0.$$

To conclude,

a) if n is odd, then $z = 0$;

b) if $n = 4k$, $k \in \mathbb{Z}$, then $z = \{a(1 + i)|a \in \mathbb{R}\}$, i.e., a line through origin;

c) if $n = 4k + 2$, $k \in \mathbb{Z}$, then $z = \{a(1 - i)|a \in \mathbb{R}\}$, i.e., a line through origin.

Problem 32. Let $z_1 = \cos t_1 + i \sin t_1$ and $z_2 = \cos t_2 + i \sin t_2$. The inequality

$$|az_1 + bz_2| \geq \frac{|z_1 + z_2|}{2}$$

is equivalent to

$$\sqrt{(a\cos t_1 + b\cos t_2)^2 + (a\sin t_1 + b\sin t_2)^2} \geq \frac{1}{2}\sqrt{(\cos t_1 + \cos t_2)^2 + (\sin t_1 + \sin t_2)^2}.$$

That is,

$$2\sqrt{a^2 + b^2 + 2ab\cos(t_1 - t_2)} \geq \sqrt{2 + \cos(t_1 - t_2)}, \text{ i.e.,}$$

$$4a^2 + 4(1-a)^2 + 8a(1-a)\cos(t_1 - t_2) \geq 2 + 2\cos(t_1 - t_2).$$

We obtain

$$8a^2 - 8a + 2 \geq (8a^2 - 8a + 2)\cos(t_1 - t_2), \text{ i.e., } 1 \geq \cos(t_1 - t_2),$$

which is obvious.

The equality holds if and only if $t_1 = t_2$, i.e., $z_1 = z_2$ or $a = b = \dfrac{1}{2}$.

Problem 33. Let $r = |z_1| = |z_2| = \cdots = |z_n| > 0$. Then

$$\frac{1}{z_1^k} + \frac{1}{z_2^k} + \cdots + \frac{1}{z_n^k} = \frac{\overline{z_1}^k}{r^{2k}} + \frac{\overline{z_2}^k}{r^{2k}} + \cdots + \frac{\overline{z_n}^k}{r^{2k}} = \frac{1}{r^{2k}}\overline{(z_1^k + z_2^k + \cdots + z_n^k)} = 0,$$

as desired.

6.2.2 Algebraic equations and polynomials (p. 181)

Problem 11. Let $r = |z_1| = |z_2|$.

The relation $\overline{a}b|c| = |a|\overline{b}c$ is equivalent to

$$\frac{\overline{a}b|c|}{\overline{a}a|a|} = \frac{|a|\overline{b}c}{\overline{a}a|a|}.$$

This relation can be written as

$$\frac{b}{a} \cdot \left|\frac{c}{a}\right| = -\left(\overline{\frac{b}{a}}\right) \cdot \frac{c}{a}.$$

That is,

$$-(x_1 + x_2) \cdot |x_1 x_2| = -(\overline{x_1} + \overline{x_2}) \cdot x_1 x_2, \text{ i.e.,}$$

$$(x_1 + x_2)r^2 = |x_1|^2 x_2 + x_1|x_2|^2.$$

It follows that

$$(x_1 + x_2)r^2 = (x_1 + x_2)r^2,$$

which is certainly true.

Problem 12. Observe that $z_1^3 = z_2^3 = 1$ and $z_3^3 = z_4^3 = -1$. If $n = 6k + r$, with $k \in \mathbb{Z}$ and $r \in \{0, 1, 2, 3, 4, 5\}$, then $z_1^n + z_2^n = z_1^r + z_2^r$ and $z_3^n + z_4^n = z_3^r + z_4^r$.

The equality $z_1^n + z_2^n = z_3^n + z_4^n$ is equivalent to $z_1^r + z_2^r = z_3^r + z_4^r$ and holds only for $r \in \{0, 2, 4\}$. Indeed,

i) if $r = 0$, then $z_1^0 + z_2^0 = 2 = z_3^0 + z_4^0$;

ii) if $r = 2$, then $z_1^2 + z_2^2 = (z_1 + z_2)^2 - 2z_1z_2 = (-1)^2 - 2 \cdot 1 = -1$ and $z_3^2 + z_4^2 = (z_3 + z_4)^2 - 2z_3z_4 = 1^2 - 2 \cdot 1 = -1$;

iii) if $r = 4$, then $z_1^4 + z_2^4 = z_1 + z_2 = 1$ and $z_3^3 + z_4^4 = -(z_3 + z_4) = -(-1) = 1$.

The other cases are:

iv) $r = 1$ then $z_1 + z_2 = -1 \neq z_3 + z_4 = 1$;

v) $r = 3$, then $z_1^3 + z_2^3 = 1 + 1 = 2 \neq z_3^3 + z_4^3 = -1 - 1 = -2$;

vi) $r = 5$, then $z_1^5 + z_2^5 = z_1^2 + z_2^2 = -1 \neq z_3^5 + z_4^5 = -(z_3^2 + z_4^2) = 1$.

Therefore, the desired numbers are the even numbers.

Problem 13. Let

$$f(x) = x^6 + ax^5 + bx^4 + cx^3 + bx^2 + ax + 1$$

$$= \prod_{k=1}^{6}(x - x_k) = \prod_{k=1}^{6}(x_k - x), \text{ for all } x \in \mathbb{C}.$$

We have

$$\prod_{k=1}^{6}(x_k^2 + 1) = \prod_{k=1}^{6}(x_k + i) \cdot \prod_{k=1}^{6}(x_k - i) = f(-i) \cdot f(i)$$
$$= (i^6 + ai^5 + bi^4 + ci^3 + bi^2 + ai + 1) \cdot (i^6 - ai^5 + bi^4 - ci^3 + bi^2 - ai + 1)$$
$$= (2ai - ci)(-2ai + ci) = (2a - c)^2,$$

as desired.

Problem 14. For a complex number z with $|z| = 1$, observe that

$$P(z) + P(-z) = az^2 + bz + i + az^2 - bz + i = 2(az^2 + i).$$

It suffices to choose z_0 such that $az_0^2 = |a|i$. Let

$$a = |a|(\cos t + i \sin t), \quad t \in [0, 2\pi).$$

The equation $az^2 = |a|i$ is equivalent to

$$z_0^2 = \cos\left(\frac{\pi}{2} - t\right) + i \sin\left(\frac{\pi}{2} - t\right).$$

Set

$$z_0 = \cos\left(\frac{\pi}{4} - \frac{t}{2}\right) + i \sin\left(\frac{\pi}{4} - \frac{t}{2}\right),$$

and we are done.

Therefore, we have

$$P(z_0) + P(-z_0) = 2(|z|i + i) = 2i(1 + |a|).$$

Passing to absolute values it follows that

$$|P(z_0)| + |P(-z_0)| \geq 2(1 + |a|).$$

That is, $|P(z_0)| \geq 1 + |a|$ or $|P(-z_0)| \geq 1 + |a|$.

Note that $|z_0| = |-z_0| = 1$, as needed.

Problem 15. Let z be a complex root of polynomial f. From the given relation it follows that $2z^3 + z$ is also a root of f. Observe that if $|z| > 1$, then

$$|2z^3 + z| = |z||2z^2 + 1| \geq |z|(2|z|^2 - 1) > |z|.$$

Hence, if f has a root z_1 with $|z_1| > 1$, then f has a root $z_2 = 2z_1^3 + z_1$ with $|z_2| > |z_1|$. We can continue this procedure and obtain an infinite number of roots of f, $z_1, z_2, \ldots$ with $\cdots > |z_2| > |z_1|$, a contradiction.

Therefore, all roots of f satisfy $|z| \leq 1$.

We will show that f is not divisible by x. Assume, by contradiction, the contrary and choose the greatest $k \geq 1$ with the property that x^k divides f. It follows that $f(x) = x^k(a + xg(x))$ with $a \neq 0$, hence

$$f(2x^2) = x^{2k}(a_1 + 2^{k+1}x^2 g(2x^2)) = x^{2k}(a_1 + xg_1(x))$$

and

$$f(2x^3 + x) = x^k(2x^2 + 1)^k(a + (2x^2 + 1)xg(x)) = x^k(a + xg_2(x)),$$

where g, g_1, g_2 are polynomials and $a_1 \neq 0$ is a real number. The relation $f(x)f(2x^2) = f(2x^3 + x)$ is equivalent to $x^k(a + xg(x))x^{2k}(a_1 + xg_1(x)) = x^k(a + xg_2(x))$ which is not possible for $a \neq 0$ and $k > 0$.

Let m be the degree of polynomial f. The polynomials $f(2x^2)$ and $f(2x^3+x)$ have degrees $2m$ and $3m$, respectively.

If $f(x) = b_m x^m + \cdots + b_0$, then $f(2x^2) = 2^m b_m x^{2m} + \cdots$ and $f(2x^3 + x) = 2^m b_m x^{3m} + \cdots$ From the given relation we find $b_m \cdot 2^m \cdot b_m = 2^m b_m$, hence $b_m = 1$. Again using the given relation it follows that $f^2(0) = f(0)$, i.e., $b_0^2 = b_0$, hence $b_0 = 1$.

The product of the roots of polynomial f is ± 1. Taking into account that for any root z of f we have $|z| \le 1$, it follows that the roots of f have modulus 1.

Consider z a root of f. Then $|z| = 1$ and $1 = |2z^3 + z| = |z||2z^2 + 1| = |2z^2 + 1| \ge |2z^2| - 1 = 2|z|^2 - 1 = 1$. Equality is possible if and only if the complex numbers $2z^2$ and -1 have the same argument; that is, $z = \pm i$.

Because f has real coefficients and its roots are $\pm i$, it follows that f is of the form $(x^2 + 1)^n$ for some positive integer n. Using the identity $(x^2 + 1)(4x^4 + 1) = (2x^3 + x)^2 + 1$ we obtain that the desired polynomials are $f(x) = (x^2 + 1)^n$, where n is an arbitrary positive integer.

6.2.3 From algebraic identities to geometric properties (p. 190)

Problem 12. Let A, B, C, D be the points with coordinates a, b, c, d, respectively.

If $a + b = 0$, then $c + d = 0$. Hence $a + b = c + d$, i.e., $ABCD$ is a parallelogram inscribed in the circle of radius $R = |a|$ and we are done.

If $a + b \ne 0$, then the points M and N with coordinates $a + b$ and $c + d$, respectively, are symmetric with respect to the origin O of the complex plane. Since AB is a diagonal in the rhombus $OAMB$, it follows that AB is the perpendicular bisector of the segment OM. Likewise, CD is the perpendicular bisector of the segment ON. Therefore A, B, C, D are the intersection points of the circle of radius R with the perpendicular bisector s of the segments OM and ON, so A, B, C, D are the vertices of a rectangle.

Alternate solution. First, let us note that from $a + b + c + d = 0$ it follows that $a + d = -(b + c)$, i.e., $|a + d| = |b + c|$. Hence $|a + d|^2 = |b + c|^2$ and using properties of the real product we find that $(a + d) \cdot (a + d) = (b + c) \cdot (b + c)$. That is, $|a|^2 + |d|^2 + 2a \cdot d = |b|^2 + |c|^2 + 2b \cdot c$. Taking into account that $|a| = |b| = |c| = |d|$ one obtains $a \cdot d = b \cdot c$.

On the other hand, $AD^2 = |d - a|^2 = (d - a) \cdot (d - a) = |d|^2 + |a|^2 - 2a \cdot d = 2(R^2 - a \cdot d)$. Analogously, we have $BC^2 = 2(R^2 - b \cdot c)$. Since $a \cdot d = b \cdot c$, it follows that $AD = BC$, so $ABCD$ is a rectangle.

Problem 13. Consider the polynomial

$$P(X) = X^5 + aX^4 + bX^3 + cX^2 + dX + e$$

with roots z_k, $k = \overline{1,5}$. Then

$$a = -\sum z_1 = 0 \quad \text{and} \quad b = \sum z_1 z_2 = \frac{1}{2}\left(\sum z_1\right)^2 - \frac{1}{2}\sum z_1^2 = 0.$$

Denoting by r the common modulus and taking conjugates we also get

$$0 = \sum \overline{z}_1 = \sum \frac{r^2}{z_1} = \frac{r^2}{z_1 z_2 z_3 z_4 z_5} \sum z_1 z_2 z_3 z_4,$$

from which $d = 0$ and

$$0 = \sum \overline{z}_1 \overline{z}_2 = \sum \frac{r^4}{z_1 z_2} = \frac{r^4}{z_1 z_2 z_3 z_4 z_5} \sum z_1 z_2 z_3;$$

therefore $c = 0$. It follows that $P(X) = X^5 + e$, so $z_1, z_2, \ldots, z_5$ are the fifth roots of e and the conclusion is proved.

Problem 14. a) Consider a complex plane with origin at M. Denote by a, b, c the coordinates of A, B, C, respectively. As $a(b-c) = b(a-c) + c(b-a)$ we have $|a||b-a| = |b(a-c) + c(b-a)| \leq |b||a-c| + |c||b-a|$. Thus $AM \cdot BC \leq BM \cdot AC + CM \cdot AB$ or $2R \cdot AM \cdot \sin A \leq 2R \cdot BM \cdot \sin B + 2R \cdot CM \cdot \sin C$ which gives $AM \cdot \sin A \leq BM \cdot \sin B + CM \cdot \sin C$.

b) From a) we have

$$\begin{aligned} AA_1 \cdot \sin\alpha &\leq AB_1 \cdot \sin\beta + AC_1 \cdot \sin\gamma, \\ BB_1 \cdot \sin\beta &\leq BA_1 \cdot \sin\alpha + BC_1 \cdot \sin\gamma, \\ CC_1 \cdot \sin\gamma &\leq CA_1 \cdot \sin\alpha + CB_1 \cdot \sin\beta, \end{aligned}$$

which, summed up, give the desired conclusion.

Problem 15. Let the coordinates of A, B, C, M and N be a, b, c, m and n, respectively. Since the lines AM, BM and CM are concurrent, as well as the lines AN, BN and CN, it follows from Ceva's theorem that

$$\frac{\sin \widehat{BAM}}{\sin \widehat{MAC}} \cdot \frac{\sin \widehat{CBM}}{\sin \widehat{MBA}} \cdot \frac{\sin \widehat{ACM}}{\sin \widehat{MCB}} = 1, \tag{1}$$

$$\frac{\sin \widehat{BAN}}{\sin \widehat{NAC}} \cdot \frac{\sin \widehat{CBN}}{\sin \widehat{NBA}} \cdot \frac{\sin \widehat{ACN}}{\sin \widehat{NCB}} = 1. \tag{2}$$

By hypotheses, $\widehat{BAM} = \widehat{NAC}$ and $\widehat{MBA} = \widehat{CBN}$. Hence $\widehat{BAN} = \widehat{MAC}$ and $\widehat{NBA} = \widehat{CBM}$. Combined with (1) and (2), these equalities imply

$$\sin \widehat{ACM} \cdot \sin \widehat{ACN} = \sin \widehat{MCB} \cdot \sin \widehat{NCB}.$$

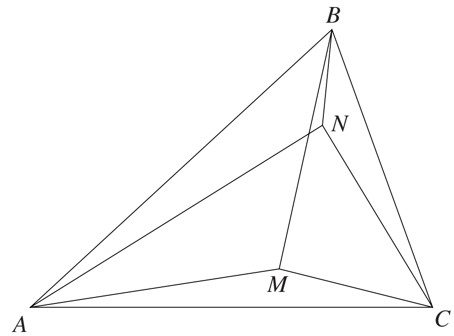

Figure 6.2.

Thus,

$$\cos(\widehat{NCM} + 2\widehat{ACM}) - \cos\widehat{NCM} = \cos(\widehat{NCM} + 2\widehat{NCB}) - \cos\widehat{NCM},$$

and hence $\widehat{ACM} = \widehat{NCB}$.

Since $\widehat{BAM} = \widehat{NAC}$, $\widehat{MBA} = \widehat{CBN}$ and $\widehat{ACN} = \widehat{MCB}$, the following complex ratios are all positive real numbers:

$$\frac{m-a}{b-a} : \frac{c-a}{n-a}, \quad \frac{m-b}{a-b} : \frac{c-b}{n-b} \quad \text{and} \quad \frac{m-c}{b-c} : \frac{a-c}{n-c}.$$

Hence each of these equals its absolute value, and so

$$\frac{AM \cdot AN}{AB \cdot AC} + \frac{BM \cdot BN}{BA \cdot BC} + \frac{CM \cdot CN}{CA \cdot CB}$$
$$= \frac{(m-a)(n-a)}{(b-a)(c-a)} + \frac{(m-b)(n-b)}{(a-b)(c-b)} + \frac{(m-c)(n-c)}{(b-c)(a-c)} = 1.$$

6.2.4 Solving geometric problems (pp. 211–213)

Problem 26. Let a, b, c be the coordinates of the points A, B, C, respectively. Using the real product of the complex numbers, we have

$$AC^2 + AB^2 = 5BC^2 \text{ if and only if } |c-a|^2 + |b-a|^2 = 5|c-b|^2, \text{ i.e.,}$$

$$(c-a)\cdot(c-a) + (b-a)\cdot(b-a) = 5(c-b)\cdot(c-b).$$

The last relation is equivalent to

$$c^2 - 2a\cdot c + a^2 + b^2 - 2a\cdot b + a^2 = 5c^2 - 10b\cdot c + 5b^2, \text{ i.e.,}$$

$$2a^2 - 4b^2 - 4c^2 - 2a\cdot b - 2a\cdot c + 10b\cdot c = 0.$$

It follows that

$$a^2 - 2b^2 - 2c^2 - a \cdot b - a \cdot c + 5b \cdot c = 0, \text{ i.e.,}$$

$$(a + c - 2b) \cdot (a + b - 2c) = 0, \text{ so } \left(\frac{a+c}{2} - b\right) \cdot \left(\frac{a+b}{2} - c\right) = 0.$$

The last relation shows that the medians from B and C are perpendicular, as desired.

Problem 27. Denoting by a lowercase letter the coordinates of a point with an uppercase letter, we obtain

$$a' = \frac{b - kc}{1 - k}, \quad b' = \frac{c - ka}{1 - k}, \quad c' = \frac{a - kb}{1 - k}$$

and

$$a'' = \frac{c' - kb'}{1 - k} = \frac{(1 + k^2)a - k(b + c)}{(1 - k)^2},$$

$$b'' = \frac{a' - kc'}{1 - k} = \frac{(1 + k^2)b - k(a + c)}{(1 - k)^2},$$

$$c'' = \frac{b' - ka'}{1 - k} = \frac{(1 + k^2)c - k(b + a)}{(1 - k)^2}.$$

Then

$$\frac{c'' - a''}{b'' - a''} = \frac{(1 + k^2)(c - a) - k(a - c)}{(1 + k^2)(b - a) - k(a - b)} = \frac{c - a}{b - a},$$

which proves that triangles ABC and $A''B''C''$ are similar.

Problem 28. Consider the complex plane with origin at the circumcircle of triangle ABC and let z_1, z_2, z_3 be the coordinates of points A, B, C.

The inequality $\dfrac{R}{2r} \geq \dfrac{m_\alpha}{h_\alpha}$ is equivalent to

$$2rm_\alpha \leq Rh_\alpha, \text{ i.e., } 2\frac{K}{s}m_\alpha \leq R\frac{2K}{\alpha}.$$

Hence $\alpha m_\alpha \leq Rs$.

Using complex numbers, we have

$$2\alpha m_\alpha = 2|z_2 - z_3| \left| z_1 - \frac{z_2 + z_3}{2} \right| = |(z_2 - z_3)(2z_1 - z_2 - z_3)|$$

$$= |z_2(z_1 - z_2) + z_1(z_2 - z_3) + z_3(z_3 - z_1)|$$

$$\leq |z_2||z_1 - z_2| + |z_1||z_2 - z_3| + |z_3||z_3 - z_1| = R(\alpha + \beta + \gamma) = 2Rs.$$

Hence $\alpha m_\alpha \leq Rs$, as desired.

Problem 29. Consider the complex plane with origin at the circumcenter O and let a, b, c, d be the coordinates of points A, B, C, D.

The midpoints E and F of the diagonals AC and BD have the coordinates $\frac{a+c}{2}$ and $\frac{b+d}{2}$.

Using the real product the complex numbers we have

$$AB^2 + BC^2 + CD^2 + DA^2 = 8R^2 \text{ if and only if}$$

$$(b-a)\cdot(b-a)+(c-b)\cdot(c-b)+(d-c)\cdot(d-c)+(a-d)\cdot(a-d) = 8R^2, \text{ i.e.,}$$

$$2a\cdot b + 2b\cdot c + 2c\cdot d + 2d\cdot a = 0.$$

The last relation is equivalent to

$$b\cdot(a+c) + d\cdot(a+c) = 0, \text{ i.e., } (b+d)\cdot(a+c) = 0.$$

We find

$$\frac{b+d}{2}\cdot\frac{a+c}{2} = 0, \text{ i.e., } OE \perp OF$$

or $E = O$ or $F = O$.

That is, $AC \perp BD$ or one of the diagonals AC and BD is a diameter of the circle $\mathcal{C}$.

Problem 30. Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter and let

$$\varepsilon = \cos 120^\circ + i\sin 120^\circ.$$

Since triangles ABM, BCN, COP and DAQ are equilateral we have

$$m + b\varepsilon + a\varepsilon^2 = 0, \quad n + c\varepsilon + b\varepsilon^2 = 0, \quad p + d\varepsilon + c\varepsilon^2 = 0, \quad q + a\varepsilon + d\varepsilon^2 = 0.$$

Summing these equalities yields

$$(m+n+p+q) + (a+b+c+d)(\varepsilon+\varepsilon^2) = 0,$$

and since $\varepsilon + \varepsilon^2 = -1$ it follows that $m+n+p+q = a+b+c+d$. Therefore the quadrilaterals $ABCD$ and $MNPQ$ have the same centroid.

Problem 31. Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter. Using the rotation formula, we obtain

$$m = b + (a-b)\varepsilon, \quad n = c + (b-c)\varepsilon, \quad p = d + (c-d)\varepsilon, \quad q = a + (d-a)\varepsilon,$$

where $\varepsilon = \cos\alpha + i\sin\alpha$.

Let E, F, G, H be the midpoints of the diagonals BD, AC, MP, NQ respectively; then

$$e = \frac{b+d}{2}, \quad f = \frac{a+c}{2}, \quad g = \frac{b+d+(a+c-b-d)\varepsilon}{2}$$

$$\text{and } h = \frac{a+c+(b+d-a-c)\varepsilon}{2}.$$

Since $e + f = g + h$, then $EGFH$ is a parallelogram, as desired.

Problem 32. Consider the points E, F, G, H such that

$$OE \perp AB, \quad OE = CD, \quad OF \perp BC, \quad OF = AD,$$

$$OG \perp CD, \quad OG = AB, \quad OH \perp AD, \quad OH = BC,$$

where O is the circumcenter of $ABCD$.

We prove that $EFGH$ is a parallelogram. Since $OE = CD$, $OF = AD$ and $\widehat{EOF} = 180^\circ - \widehat{ABC} = \widehat{ADC}$ follows that triangles EOF and ADC are congruent, hence $EF = GH$. Likewise $FG = EH$ and the claim is proved.

Consider the complex plane with origin at O such that F is on the positive real axis. Denote by a lowercase letter the coordinate of a point denoted by an uppercase letter. We have

$$|e| = CD, \quad |f| = AD, \quad |g| = AB, \quad |h| = BC.$$

Furthermore,

$$\widehat{FOG} = 180^\circ - \widehat{C} = \widehat{A}, \quad \widehat{GOH} = \widehat{B}, \quad \widehat{HOE} = \widehat{C},$$

hence

$$f = |f| = AD, \quad g = |g|(\cos A + i \sin A) = AD(\cos A + i \sin A),$$

$$h = |h|[\cos(A+B) + i \sin(A+B)] = BC[\cos(A+B) + i \sin(A+B)],$$

$$e = |e|[\cos(A+B+C) + i \sin(A+B+C)] = CD(\cos D - i \sin D).$$

Since $e + g = f + h$, we obtain

$$AD + BC\cos(A+B) + iBC\sin(A+B)$$

$$= CD(\cos D - i \sin D) + AB(\cos A + i \sin A)$$

and the conclusion follows.

Problem 33. Consider the complex plane with origin at the circumcenter O of the triangle. Let a, b, c, ω, g, z_I be the coordinates of the points A, B, C, O_9, G, I, respectively.

Without loss of generality, we may assume that the circumradius of the triangle ABC is equal to 1, hence $|a| = |b| = |c| = 1$.

We have

$$\omega = \frac{a+b+c}{2}, \quad g = \frac{a+b+c}{3}, \quad z_I = \frac{a|b-c| + b|a-c| + c|a-b|}{|a-b| + |b-c| + |a-c|}.$$

Using the properties of the real product of complex numbers, we have

$$O_9G \perp AI \text{ if and only if } (\omega - g) \cdot (a - z_I) = 0, \text{ i.e.,}$$

$$\frac{a+b+c}{6} \cdot \frac{(a-b)|a-c| + (a-c)|a-b|}{|a-b| + |b-c| + |a-c|} = 0.$$

This is equivalent to

$$(a+b+c) \cdot [(a-b)|a-c| + (a-c)|a-b|] = 0, \text{ i.e.,}$$

$$\operatorname{Re}\{(a+b+c)[(\overline{a} - \overline{b})|a-c| + (\overline{a} - \overline{c})|a-b|]\} = 0.$$

We find that

$$\begin{aligned} &\operatorname{Re}\{|a-c|(a\overline{a} + b\overline{a} + c\overline{a} - a\overline{b} - b\overline{b} - c\overline{b}) \\ &+ |a-b|(a\overline{a} + b\overline{a} + c\overline{a} - a\overline{c} - b\overline{c} - c\overline{c})\} = 0. \end{aligned} \tag{1}$$

Observe that

$$a\overline{a} = b\overline{b} = c\overline{c} = 1 \quad \text{and} \quad \operatorname{Re}(b\overline{a} - \overline{a}b) = \operatorname{Re}(c\overline{a} - a\overline{c}) = 0,$$

hence the relation (1) is equivalent to

$$\operatorname{Re}\{|a-c|(c\overline{a} - c\overline{b}) + |a-b|(b\overline{a} - b\overline{c})\} = 0, \text{ i.e.,}$$

$$|a-c|(c\overline{a} + \overline{c}a - \overline{c}b - c\overline{b}) + |a-b|(\overline{a}b + a\overline{b} - b\overline{c} - \overline{b}c) = 0.$$

It follows that

$$|a-c|[(b\overline{b} - \overline{b}c - \overline{c}b + c\overline{c}) - (a\overline{a} - c\overline{a} - \overline{c}a + c\overline{c})]$$

$$+ |a-b|[(b\overline{b} - \overline{b}c - \overline{c}b + c\overline{c}) - (a\overline{a} - \overline{a}b - a\overline{b} + b\overline{b})] = 0, \text{ i.e.,}$$

$$|a-c|(|b-c|^2 - |a-c|^2) + |a-b|(|b-c|^2 - |a-b|^2) = 0.$$

This is equivalent to

$$AC \cdot BC^2 - AC^3 + AB \cdot BC^2 - AB^3 = 0.$$

The last relation can be written as

$$BC^2(AC + AB) = (AC + AB)(AC^2 - AC \cdot AB + AB^2),$$

$$\text{so } AC \cdot AB = AC^2 + AB^2 - BC^2.$$

We obtain

$$\cos A = \frac{1}{2}, \text{ i.e., } \widehat{A} = \frac{\pi}{3},$$

as desired.

Problem 34. (a) Let a lowercase letter denote the complex number associated with the point labeled by the corresponding uppercase letter. Let M', M and O denote the midpoints of segments $[M_1'M_2']$, $[M_1M_2]$ and $[O_1O_2]$, respectively. Also let $z = \dfrac{m_1 - o_1}{m_1' - o_1} = \dfrac{m_2 - o_2}{m_2' - o_2}$, so that multiplication by z is a rotation about the origin through some angle. Then $m = \dfrac{m_1 + m_2}{2}$ equals

$$\frac{1}{2}(o_1 + z(m_1' - o_1)) + \frac{1}{2}(o_2 + z(m_2' - o_2)) = o + z(m' - o),$$

i.e., the locus of M is the circle centered at O with radius OM'.

(b) We shall use directed angles modulo π. Observe that

$$\widehat{QM_1M_2} = \widehat{QPM_2} = \widehat{QPO_2} = \widehat{QO_1O_2}.$$

Similarly, $\widehat{QM_2M_1} = \widehat{QO_2O_1}$, implying that triangles QM_1M_2 and QO_1O_2 are similar with the same orientations. Hence,

$$\frac{q - o_1}{q - o_2} = \frac{q - m_1}{q - m_2},$$

or equivalently

$$\frac{q - o_1}{q - o_2} = \frac{(q - m_1) - (q - o_1)}{(q - m_2) - (q - o_2)} = \frac{o_1 - m_1}{o_2 - m_2} = \frac{o_1 - m_1'}{o_2 - m_2'}.$$

Because lines O_1M_1' and O_2M_2' meet, $o_1 - m_1' \neq o_2 - m_2'$ and we can solve this equation to find a unique value for q.

Problem 35. Without loss of generality, assume that triangle $A_1A_2A_3$ is oriented counterclockwise (i.e., angle $A_1A_2A_3$ is oriented clockwise). Let P be the reflection of O_1 across T.

We use the complex numbers with origin O_1, where each point denoted by an uppercase letter is represented by the complex number with the corresponding lowercase letter. Let $\zeta_k = a_k/p$ for $k = 1, 2$, so that $z \mapsto \zeta_k(z - z_0)$ is a similarity through angle $\widehat{PO_1A_k}$ with ratio O_1A_3/O_1P about the point corresponding to z_0.

Because O_1 and A_1 lie on opposite sides of line A_2A_3, angles $A_2A_3O_1$ and $A_2A_3A_1$ have opposite orientations, i.e., the former is oriented counterclockwise. Thus, angles PA_3O_1 and $A_2O_3A_1$ are both oriented counterclockwise. Because $\widehat{PA_3O_1} = 2\widehat{A_2A_3O_2} = \widehat{A_2O_3A_1}$, it follows that isosceles triangles PA_3O_1 and $A_2O_3A_1$ are similar and have the same orientation. Hence, $o_3 = a_1 + \zeta_3(a_2 - a_1)$.

Similarly, $o_2 = a_1 + \zeta_2(a_3 - a_1)$. Hence,

$$o_3 - o_2 = (\zeta_2 - \zeta_3)a_1 + \zeta_3 a_2 - \zeta_2 a_3$$

$$= \zeta_2(a_2 - a_3) + \zeta_3(\zeta_2 p) - \zeta_2(\zeta_3 p) = \zeta_2(a_2 - a_3),$$

or (recalling that $o_1 = 0$ and $t = 2p$)

$$\frac{o_3 - o_2}{a_1 - o_1} = \zeta_2 = \frac{a_2 - a_3}{p - o_1} = \frac{1}{2}\frac{a_2 - a_3}{t - o_1}.$$

Thus, the angle between $[O_1A_1]$ and $[O_2O_3]$ equals the angle between $[O_1T]$ and $[A_3A_2]$, which is $\pi/2$. Furthermore, $O_2O_3/O_1A_1 = \frac{1}{2}A_3A_2/O_1T$, or $O_1A_1/O_2O_3 = 2O_1T/A_2A_3$. This completes the proof.

Problem 36. Assume that the origin O of the coordinate system in the complex plane is the center of the circumscribed circle. Then, the vertices A_1, A_2, A_3 are represented by complex numbers w_1, w_2, w_3 such that

$$|w_1| = |w_2| = |w_3| = R.$$

Let $\varepsilon = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then $\varepsilon^2 + \varepsilon + 1 = 0$ and $\varepsilon^3 = 1$. Suppose that P_0 is represented by the complex number z_0. The point P_1 is represented by the complex number

$$z_1 = z_0\varepsilon + (1 - \varepsilon)w_1. \tag{1}$$

The point P_2 is represented by

$$z_2 = z_0\varepsilon^2 + (1 - \varepsilon)w_1\varepsilon + (1 - \varepsilon)w_2,$$

and P_3 by

$$z_3 = z_0\varepsilon^3 + (1 - \varepsilon)w_1\varepsilon^2 + (1 - \varepsilon)w_2\varepsilon + (1 - \varepsilon)w_3$$
$$= z_0 + (1 - \varepsilon)(w_1\varepsilon^2 + w_2\varepsilon + w_3).$$

An easy induction on n shows that after n cycles of three such rotations, we obtain that P_{3n} is represented by

$$z_{3n} = z_0 + n(1 - \varepsilon)(w_1\varepsilon^2 + w_2\varepsilon + w_3).$$

In our case, for $n = 662$ we obtain

$$z_{1996} = z_0 + 662(1 - \varepsilon)(w_1\varepsilon^2 + w_2\varepsilon + w_3) = z_0.$$

Thus, we have the equality

$$w_1\varepsilon^2 + w_2\varepsilon + w_3 = 0. \tag{2}$$

This can be written under the equivalent form

$$w_3 = w_1(1 + \varepsilon) + (-\varepsilon)w_2. \tag{3}$$

Taking into account that $1 + \varepsilon = \cos \frac{\pi}{3} + i \sin \frac{\pi}{3}$, the equality (3) can be translated, using the lemma on p. 218, into the following: the point A_3 is obtained under the rotation of point A_1 about center A_2 through the angle $\frac{\pi}{3}$. This proved that $A_1A_2A_3$ is an equilateral triangle.

Problem 37. Let $B(b, 0), C(c, 0)$ be the centers of the given circles and let $A(0, a)$, $X(0, -a)$ be their intersection points. The complex numbers associated to these point are $z_B = b$, $z_C = c$, $z_A = ia$ and $z_X = -ia$, respectively. After rotating A through angle t about B we obtain a point M and after rotating A about C we obtain the point N. Their corresponding complex numbers are given by formulas:

$$z_M = (ia - b)\omega + b = ia\omega + (1 - \omega)b$$

and

$$z_N = ia\omega + (1 - \omega)c.$$

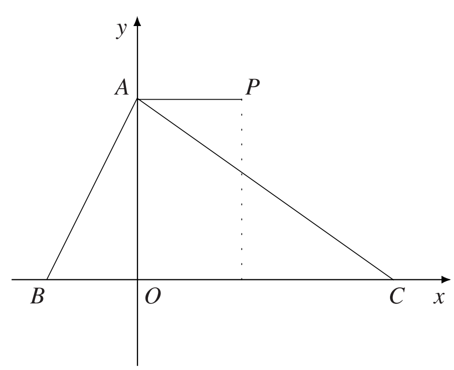

Figure 6.3.

The required result is equivalent to the following: the bisector lines l_{MN} of the segments MN pass through a fixed point $P(x_0, y_0)$. Let R be the midpoint of the segment

MN. Then $z_R = \frac{1}{2}(z_M + z_N)$. A point Z of the plane is a point of l_{MN} if and only if the lines RZ and MN are orthogonal. By using the real product of complex numbers we obtain

$$\left(z - \frac{z_M + z_N}{2}\right) \cdot (z_N - z_M) = 0.$$

This is equivalent to

$$z \cdot (z_N - z_M) = \frac{1}{2}(|z_N|^2 - |z_M|^2).$$

By noting that $z = x + iy$ we obtain

$$x(c-b)(1-\cos t) - y(c-b)\sin t = \frac{1}{2}(|z_N|^2 - |z_M|^2).$$

After an easy computation we obtain

$$|z_M|^2 = 2b^2 + a^2 - 2b^2\cos t - 2ab\sin t$$

and

$$|z_N|^2 = 2c^2 + a^2 - 2c^2\cos t - 2ac\sin t.$$

Thus, the orthogonality condition yields

$$x(1-\cos t) - y\sin t = (b+c) - (b+c)\cos t - a\sin t.$$

This can be written in the form

$$(x - b - c)(1 - \cos t) = (y - a)\sin t.$$

This equation shows that the point $P(x_0, y_0)$ where $x_0 = b + c$, $y_0 = a$ is a fixed point of the family of lines l_{MN}.

The point P belong to the line through A parallel to BC and it is the symmetrical point of X with respect to the midpoint of the segment BC. This follows from the equality

$$z_P + z_X = \frac{b+c}{2}.$$

Problem 38. Let $A(1+i)$, $B(-1+i)$, $C(-1-i)$, $D(1-i)$ be the vertices of the square. Using the symmetry of the configuration of points, with respect to the axes and center O of the square, we will do computations for the points lying in the first quadrant. Then L, M are represented by the complex numbers $L(\sqrt{3}-1)$, $M((\sqrt{3}-1)i)$. The midpoint of the segment LM is $P\left(\frac{\sqrt{3}-1}{2} + i\frac{\sqrt{3}-1}{2}\right)$. Since K is represented by $K(-i(\sqrt{3}-1))$, the midpoint of AK is $Q\left(\frac{1}{2} + i\frac{2-\sqrt{3}}{2}\right)$. In the same way, the

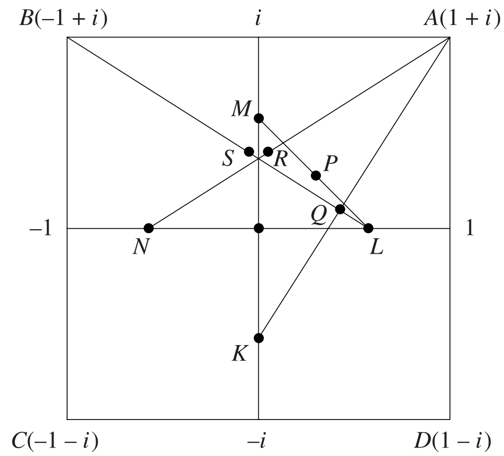

Figure 6.4.

midpoint of AN is $R\Big(\dfrac{2-\sqrt{3}}{2}+\dfrac{i}{2}\Big)$ and the midpoint of BL is $S\Big(\dfrac{-2+\sqrt{3}}{2}+\dfrac{i}{2}\Big)$. It is sufficient to prove that $SR = RP = PQ$ and $\widehat{SRP} = \widehat{RPQ} = \dfrac{5\pi}{6}$. For any point X we denote by Z_X the corresponding complex number. We have

$$RS^2 = |Z_S - Z_R|^2 = (-2+\sqrt{3})^2 = 7 - 4\sqrt{3},$$

$$\begin{aligned} RP^2 = |Z_P - Z_R|^2 &= \left|\frac{\sqrt{3}-1}{2} + i\,\frac{\sqrt{3}-1}{2} - \frac{2-\sqrt{3}}{2} - \frac{i}{2}\right|^2 \\ &= \left|\frac{2\sqrt{3}-3}{2} + i\,\frac{\sqrt{3}-2}{2}\right|^2 = \frac{(2\sqrt{3}-3)^2 + (2\sqrt{3}-2)^2}{4} \\ &= \frac{28-16\sqrt{3}}{4} = 7 - 4\sqrt{3}. \end{aligned}$$

Using reflection in OA, we also have $PQ^2 = RP^2 = 7 - 4\sqrt{3}$.

For angles we have

$$\begin{aligned} \cos\widehat{SRP} &= \frac{\dfrac{3-2\sqrt{3}}{2}(2-\sqrt{3}) + \dfrac{2-2\sqrt{3}}{2}\cdot 0}{7-4\sqrt{3}} \\ &= \frac{(12-7\sqrt{3})(7+4\sqrt{3})}{2(7-4\sqrt{3})(7+4\sqrt{3})} = -\frac{\sqrt{3}}{2}. \end{aligned}$$

This proves that $\widehat{SRP} = \dfrac{5\pi}{6}$. In the same way, $\cos\widehat{RPQ} = -\dfrac{\sqrt{3}}{2}$ and $\widehat{RPQ} = \dfrac{5\pi}{6}$.

Problem 39. Let $1, \varepsilon, \varepsilon^2$, be the coordinates of points A, B, C, M, respectively, where $\varepsilon = \cos 120° + i \sin 120°$.

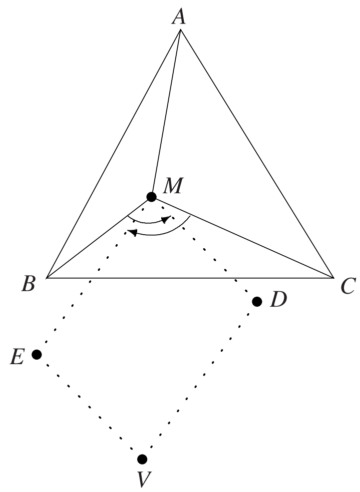

Figure 6.5.

Consider point V such that $MEVD$ is a parallelogram. If d, e, v are the coordinates of points D, E, V, respectively, then

$$v = e + d - m.$$

Using the rotation formula, we obtain

$$d = m + (\varepsilon - m)\varepsilon \quad \text{and} \quad e = m + (\varepsilon^2 - m)\varepsilon^2,$$

hence

$$v = m + \varepsilon^2 - m\varepsilon + m + \varepsilon^4 - m\varepsilon^2 - m$$

$$= m + \varepsilon^2 + \varepsilon - m(\varepsilon^2 + \varepsilon) = m - 1 + m = 2m - 1.$$

This relation shows that M is the midpoint of the segment $[AV]$ and the conclusion follows.

Problem 40. Consider the complex plane with origin at the center of the parallelogram $ABCD$. Let a, b, c, d, m be the coordinates of points A, B, C, D, M, respectively.

It follows that $c = -a$ and $d = -b$.

It suffices to prove that

$$|m - a| \cdot |m + a| + |m - b||m + b| \geq |a - b||a + b|,$$

or

$$|m^2 - a^2| + |m^2 - b^2| \geq |a^2 - b^2|.$$

This follows immediately from the triangle inequality.

Problem 41. Let the coordinates of A, B, C, H and O be a, b, c, h and o, respectively. Consequently, $a\overline{a} = b\overline{b} = c\overline{c} = R^2$ and $h = a + b + c$. Since D is symmetric to A with respect to line BC, the coordinates d and a satisfy

$$\frac{d-b}{c-b} = \overline{\left(\frac{a-b}{c-b}\right)}, \quad \text{or} \quad (\overline{b} - \overline{c})d - (b-c)\overline{a} + (b\overline{c} - \overline{b}c) = 0. \tag{1}$$

Since

$$\overline{b} - \overline{c} = -\frac{R^2(b-c)}{bc} \quad \text{and} \quad b\overline{c} - \overline{b}c = \frac{R^2(b^2-c^2)}{bc},$$

by inserting these expressions in (1), we obtain that

$$d = \frac{-bc + ca + ab}{a} = \frac{k - 2bc}{a},$$

$$\overline{d} = \frac{R^2(-a+b+c)}{bc} = \frac{R^2(h-2a)}{bc},$$

where $k = bc + c + ab$. Similarly, we have

$$e = \frac{k - 2ca}{b}, \quad \overline{e} = \frac{R^2(h-2b)}{ca}, \quad f = \frac{k-2ab}{c} \quad \text{and} \quad \overline{f} = \frac{R^2(h-2c)}{ab}.$$

Since

$$\begin{aligned}
\Delta = \begin{vmatrix} d & \overline{d} & 1 \\ e & \overline{e} & 1 \\ f & \overline{f} & 1 \end{vmatrix} &= \begin{vmatrix} e-d & \overline{e} - \overline{d} \\ f-d & \overline{f} - \overline{d} \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{(b-a)(k-2ab)}{ab} & \dfrac{R^2(a-b)(h-2c)}{abc} \\[2ex] \dfrac{(c-a)(k-2ca)}{ca} & \dfrac{R^2(a-c)(h-2b)}{abc} \end{vmatrix} \\
&= \frac{R^2(c-a)(a-b)}{a^2b^2c^2} \times \begin{vmatrix} -(ck-2abc) & (h-2c) \\ (bk-2abc) & -(h-2b) \end{vmatrix} \\
&= \frac{-R^2(b-c)(c-a)(a-b)(hk-4abc)}{a^2b^2c^2}
\end{aligned}$$

and $\overline{h} = R^2k/abc$, it follows that D, E and F are collinear if and only if $\Delta = 0$. This is equivalent to $hk - 4abc = 0$, i.e., $h\overline{h} = 4R^2$. From the last relation we obtain $OH = 2R$.

Problem 42. Let the coordinates of A, B, C, D and E be a, b, c, d and e, respectively. Then $d = (2b + c)/3$ and $e = 2d - a$. Since $\widehat{ACB} = 2\widehat{ABC}$, the ratio

$$\left(\frac{a-b}{c-b}\right)^2 : \frac{b-c}{a-c}$$

is real and positive. It is equal to $(AB^2 \cdot AC)/BC^3$. On the other hand, a direct computation shows that the ratio

$$\frac{e-c}{b-c} : \left(\frac{c-b}{e-b}\right)^2$$

is equal to

$$\frac{1}{(b-c)^3} \times \left(\frac{(b-a)+2(c-a)}{3}\right)^2 \left(\frac{4(b-a)-(c-a)}{3}\right)$$

$$= \frac{4}{27} + \frac{(b-a)^2(c-a)}{(b-c)^3} = \frac{4}{27} - \frac{AB^2 \cdot AC}{BC^3},$$

which is a real number. Hence the arguments of $(e-c)/(b-c)$ and $(c-b)^2/(e-b)^2$, namely, $\widehat{ECB}$ and $2\widehat{EBC}$, differ by an integer multiple of 180°. We easily infer that either $\widehat{ECB} = 2\widehat{EBC}$ or $\widehat{ECB} = 2\widehat{EBC} - 180°$, according to whether the ratio is positive or negative. To prove that the latter holds, we have to show that $AB^2 \cdot AC/BC^3$ is greater than 4/27. Choose a point F on the ray AC such that $CF = CB$.

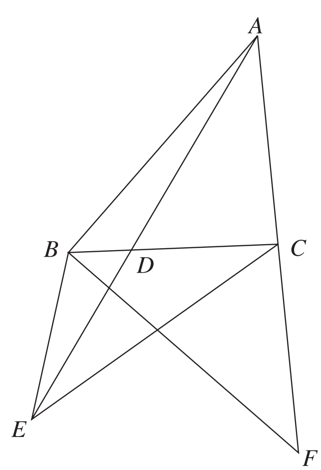

Figure 6.6.

Since $\triangle CBF$ is isosceles and $\widehat{ACB} = 2\widehat{ABC}$, we have $\widehat{CFB} = \widehat{ABC}$. Thus $\triangle ABF$ and $\triangle ACB$ are similar and $AB : AF = AC : AB$. Since $AF = AC + BC$,

$AB^2 = AC(AC + BC)$. Let $AC = u^2$ and $AC + BC = v^2$. Then $AB = uv$ and $BC = v^2 - u^2$. From $AB + AC > BC$, we obtain $u/v > 1/2$. Thus

$$\frac{AB^2 \cdot AC}{BC^3} = \frac{u^4 v^2}{(v^2 - u^2)^3} = \frac{(u/v)^4}{(1 - u^2/v^2)^3} > \frac{(1/2)^4}{(1 - 1/4)^3} = \frac{4}{27},$$

and the conclusion follows.

6.2.5 Solving trigonometric problems (p. 220)

Problem 11. (i) Consider the complex number

$$z = \frac{1}{\cos\theta}(\cos\theta + i\sin\theta).$$

From the identity

$$\sum_{k=0}^{n-1} z^k = \frac{1 - z^n}{1 - z} \tag{1}$$

we derive

$$\sum_{k=0}^{n-1} \frac{1}{\cos^k\theta}(\cos k\theta + i\sin k\theta) = \frac{1 - \dfrac{1}{\cos^n\theta}(\cos n\theta + i\sin n\theta)}{1 - \dfrac{1}{\cos\theta}(\cos\theta + i\sin\theta)}$$

$$= \frac{\cos\theta - \dfrac{1}{\cos^{n-1}\theta}(\cos n\theta + i\sin n\theta)}{-i\sin\theta} = \frac{\sin n\theta}{\sin\theta\cos^{n-1}\theta} + i\frac{\cos^n\theta - \cos n\theta}{\sin\theta\cos^{n-1}\theta}.$$

It follows that

$$\sum_{k=0}^{n-1} \frac{\cos k\theta}{\cos^k\theta} = \frac{\sin n\theta}{\sin\theta\cos^{n-1}\theta}$$

and we just have to substitute $\theta = 30°$.

(ii) We proceed in an analogous way by considering the complex number $z = \cos\theta(\cos\theta + i\sin\theta)$. Using identity (1) we obtain

$$\sum_{k=1}^{n} z^k = \frac{z - z^{n+1}}{1 - z}.$$

Hence

$$\sum_{k=1}^{n} \cos^k \theta (\cos k\theta + i \sin k\theta)$$
$$= \frac{\cos\theta(\cos\theta + i\sin\theta) - \cos^{n+1}\theta(\cos(n+1)\theta + i\sin(n+1)\theta)}{\sin^2\theta - i\cos\theta\sin\theta}$$
$$= i\frac{\cos\theta(\cos\theta + i\sin\theta) - \cos^{n+1}\theta(\cos(n+1)\theta + i\sin(n+1)\theta)}{\sin\theta(\cos\theta + i\sin\theta)}$$
$$= i\Big[\text{cotan}\theta - \frac{\cos^{n+1}\theta(\cos n\theta + i\sin n\theta)}{\sin\theta}\Big]$$
$$= \frac{\sin n\theta\cos^{n+1}\theta}{\sin\theta} + i\Big(\text{cotan}\theta - \frac{\cos^{n+1}\theta\cos n\theta}{\sin\theta}\Big)$$

It follows that

$$\sum_{k=1}^{n} \cos^k\theta\cos k\theta = \frac{\sin n\theta\cos^{n+1}\theta}{\sin\theta}$$

Finally, we let $\theta = 30°$ in the above sum.

Problem 12. Let

$$\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$$

for some integer n. Consider the sum

$$S_n = 4^n + (1+\omega)^{2n} + (1+\omega^2)^{2n} + \cdots + (1+\omega^{n-1})^{2n}.$$

For all $k = 1, \ldots, n-1$, we have

$$1 + \omega^k = 1 + \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} = 2\cos\frac{k\pi}{n}\left(\cos\frac{k\pi}{n} + i\sin\frac{k\pi}{n}\right)$$

and

$$(1+\omega^k)^{2n} = 2^{2n}\cos^{2n}\frac{k\pi}{n}(\cos 2k\pi + i\sin 2k\pi) = 4^n\cos^{2n}\frac{k\pi}{n}.$$

Hence

$$S_n = 4^n + \sum_{k=1}^{n-1}(1+\omega^k)^{2n}$$
$$= 4^n\left[1 + \cos^{2n}\left(\frac{\pi}{n}\right) + \cos^{2n}\left(\frac{2\pi}{n}\right) + \cdots + \cos^{2n}\left(\frac{(n-1)\pi}{n}\right)\right]. \qquad (1)$$

On the other hand, using the binomial expansion, we have

$$S_n = \sum_{k=0}^{n-1}(1+\omega^k)^{2n} = \sum_{k=0}^{n-1}\left(\binom{2n}{0} + \binom{2n}{1}\omega^k + \right.$$

$$+\binom{2n}{2}\omega^{2k}+\cdots+\binom{2n}{n}\omega^{nk}+\binom{2n}{2n-1}\omega^{(2n-1)k}+\binom{2n}{2n}\Bigg)$$

$$=n\binom{2n}{0}+n\binom{2n}{n}+n\binom{2n}{2n}+\sum_{\substack{j=1\\ i\neq n}}^{2n-1}\binom{2n}{j}\cdot\sum_{k=0}^{n-1}\omega^{jk}$$

$$=2n+n\binom{2n}{n}+\sum_{\substack{j=1\\ i\neq n}}^{2n-1}\binom{2n}{j}\cdot\frac{1-\omega^{jn}}{1-\omega^{j}}=2n+n\binom{2n}{n}. \quad (2)$$

The relations (1) and (2) give the desired identity.

Problem 13. For $p=0$, take $a_0=1$. If $p\geq 1$, let $z=\cos\alpha+i\sin\alpha$ and observe that

$$z^{2p}=\cos 2p\alpha+i\sin 2p\alpha,$$

$$z^{-2p}=\cos 2p\alpha-i\sin 2p\alpha$$

and

$$\cos 2p\alpha=\frac{z^{2p}+z^{-2p}}{2}=\frac{1}{2}[(\cos\alpha+i\sin\alpha)^{2p}+(\cos\alpha-i\sin\alpha)^{2p}].$$

Using the binomial expansion we obtain

$$\cos 2p\alpha=\binom{2p}{0}\cos^{2p}\alpha-\binom{2p}{2}\cos^{2p-2}\alpha\sin^2\alpha+\cdots+(-1)^p\binom{2p}{2p}\sin^{2p}\alpha.$$

Hence $\cos 2p\alpha$ is a polynomial of degree p in $\sin^2\alpha$, so there are $a_0,a_1,\ldots,a_p\in\mathbb{R}$ such that

$$\cos 2p\alpha=a_0+a_1\sin^2\alpha+\cdots+a_p\sin^{2p}\alpha \text{ for all } \alpha\in\mathbb{R},$$

with

$$a_p=\binom{2p}{0}-\binom{2p}{2}(-1)^{p-1}+\binom{2p}{4}(-1)^{p-2}+\cdots+\binom{2p}{2p}(-1)^p$$

$$=(-1)^p\left(\binom{2p}{0}+\binom{2p}{2}+\cdots+\binom{2p}{2p}\right)\neq 0.$$

6.2.6 More on the n^{th} roots of unity (pp. 228–229)

Problem 11. Let $p=1,2,\ldots,m$ and let $z\in U_p$. Then $z^p=1$.

Note that $n-m+1, n-m+2,\ldots,n$ are m consecutive integers, and, since $p\leq m$, there is an integer $k\in\{n-m+1,n-m+2,\ldots,n\}$ such that p divides k.

Let $k = k'p$. It follows that $z^k = (z^p)^{k'} = 1$, so $z \in U_k \subset U_{n-m+1} \cup U_{n-m+2} \cup \cdots \cup U_n$, as claimed.

Remark. An alternative solution can be obtained by using the fact that

$$\frac{(a^n - 1)(a^{n-1} - 1) \cdots (a^{n-k+1} - 1)}{(a^k - 1)(a^{k-1} - 1) \cdots (a - 1)}$$

is an integer for all positive integers $a > 1$ and $n > k$.

Problem 12. Rewrite the equation as

$$\left(\frac{bx + a\alpha}{ax + b\overline{\alpha}}\right)^n = \frac{d}{c}.$$

Since $|c| = |d|$, we have $\left|\frac{d}{c}\right| = 1$ and consider

$$\frac{d}{c} = \cos t + i \sin t, \quad t \in [0, 2\pi).$$

It follows that

$$\frac{bx_k + a\alpha}{ax_k + b\overline{\alpha}} = u_k, \tag{1}$$

where

$$u_k = \cos\frac{t + 2k\pi}{n} + i \sin\frac{t + 2k\pi}{n}, \quad k = \overline{0, n-1}.$$

The relation (1) implies that

$$x_k = \frac{b\overline{\alpha}u_k - a\alpha}{b - au_k}, \quad k = \overline{0, n-1}.$$

To prove that the roots x_k, $k = \overline{0, n-1}$ are real numbers, it suffices to show that $x_k = \overline{x_k}$ for all $k = \overline{0, n-1}$.

Denote $|a| = |b| = r$. Then

$$\overline{x_k} = \frac{\overline{b}\alpha\overline{u_k} - \overline{a\alpha}}{\overline{b} - \overline{au_k}} = \frac{\frac{r^2}{b} \cdot \alpha \cdot \frac{1}{u_k} - \frac{r^2}{a} \cdot \overline{\alpha}}{\frac{r^2}{b} - \frac{r^2}{a} \cdot \frac{1}{u_k}}$$

$$= \frac{\alpha a - b\overline{\alpha}u_k}{au_k - b} = x_k, \quad k = \overline{0, n-1},$$

as desired.

Problem 13. Differentiating the familiar identity

$$\sum_{k=0}^{n} z^k = \frac{x^{n+1} - 1}{x - 1}$$

with respect to x, we get

$$\sum_{k=1}^{n} kx^{k-1} = \frac{nx^{n+1} - (n+1)x^n + 1}{(x-1)^2}.$$

Multiplying both sides by x and differentiating again, we arrive at

$$\sum_{k=1}^{n} k^2 x^{k-1} = g(x),$$

where

$$g(x) = \frac{n^2 x^{n+2} - (2n^2 + 2n - 1)x^{n+1} + (n+1)^2 x^n - x - 1}{(x-1)^3}.$$

Taking $x = z$ and using $|z| = 1$ (which we were given), we obtain

$$|g(z)| \le \sum_{k=1}^{n} k^2 |z|^{k-1} = \frac{n(n+1)(2n+1)}{6}. \tag{1}$$

On the other side, taking into account that $z^n = 1$, $z \neq 1$, we get

$$g(z) = \frac{n(nz^2 - 2(n+1)z + n + 2)}{(z-1)^3} = \frac{n(nz - (n+2))}{(z-1)^2}. \tag{2}$$

From (1) and (2) we therefore conclude that

$$|nz - (n+2)| \le \frac{(n+1)(2n+1)}{6} |z-1|^2.$$

Problem 14. Setting $x = y \in M$ yields $1 = \dfrac{x}{y} \in M$. For $x = 1$ and $y \in M$ we obtain $\dfrac{1}{y} = y^{-1} \in M$.

If x and y are arbitrary elements of M, then $x, y^{-1} \in M$ and consequently

$$\frac{x}{y^{-1}} = xy \in M.$$

Let $x_1, x_2, \ldots, x_n$ be the elements of set M and take at random an element $x_k \in M$, $k = \overline{1, n}$. Since $x_k \neq 0$ for all $k = \overline{1, n}$, the numbers $x_k x_1, x_k x_2, \ldots, x_k x_n$ are distinct and belong to the set M, hence

$$\{x_k x_1, x_k x_2, \ldots, x_k x_n\} = \{x_1, x_2, \ldots, x_n\}.$$

Therefore $x_k x_1 \cdot x_k x_2 \cdots x_k x_n = x_1 x_2 \cdots x_n$, hence $x_k^n = 1$, that is, x_k is an n^{th} root of 1.

The number x_k was chosen arbitrary, hence M is the set of the n^{th}-roots of 1, as claimed.

Problem 15. a) We will denote by $S(X)$ the sum of the elements of a finite set X. Suppose $0 \neq z \in A$. Since A is finite, there exists positive integers $m < n$ such that $z^m = z^n$, whence $z^{n-m} = 1$. Let d be the smallest positive integer k such that $z^k \in 1$. Then $1, z, z^2, \dots, z^{d-1}$ are different, and the dth power of each is equal to 1; therefore these numbers are the dth roots of unity. This shows that $A \setminus \{0\} = \bigcup_{k=1}^{m} U_{n_k}$, where $U_p = \{z \in \mathbb{C} |\, z^p = 1\}$. Since $S(U_p) = 0$ for $p \geq 2$, $S(U_1) = 1$ and $U_p \cap U_q = U_{(p,q)}$ we get

$$S(A) = \sum_k S(U_{n_k}) - \sum_{k<l} S(U_{n_k} \cap U_{n_l})$$

$$+ \sum_{k<l<s} S(U_{n_k} \cap U_{n_l} \cap U_{n_s}) + \cdots = \text{ an integer.}$$

b) Suppose that for some integer k there exists $A = \bigcup_{k=1}^{m} U_{n_k}$ such that $S(A) = k$. Let $p_1, p_2, \dots, p_6$ be the distinct primes which are not divisors of any n_k. Then

$$S(A \cup U_{p_1}) = S(A) + S(U_{p_1}) - S(A \cap U_{p_1}) = k - S(U_1) = k - 1.$$

Also

$$\begin{aligned}
&S(A \cup U_{p_1p_2p_3} \cup U_{p_1p_4p_5} \cup U_{p_2p_4p_6} \cup U_{p_3p_5p_6}) \\
&\quad = S(A) + S(U_{p_1p_2p_3}) + S(U_{p_1p_4p_5}) + S(U_{p_2p_4p_6}) + S(U_{p_3p_5p_6}) \\
&\quad\quad - S(A \cap U_{p_1p_2p_3}) - \cdots + S(A \cap U_{p_1p_2p_3} \cap U_{p_1p_4p_5}) \\
&\quad\quad + \cdots - S(A \cap U_{p_1p_2p_3} \cap U_{p_1p_4p_5} \cap U_{p_2p_4p_6} \cap U_{p_3p_5p_6}) \\
&\quad = k + 4 \cdot 0 - 4S(U_1) - \sum_{k=1}^{6} S(U_{p_k}) + 10S(U_1) - 5S(U_1) + S(U_1) \\
&\quad = k - 4 + 10 - 5 + 1 = k + 2.
\end{aligned}$$

Hence, if there exists A such that $S(A) = k$, then there exist B and C such that $S(B) = k - 1$ and $S(C) = k + 2$. The conclusion now follows easily.

6.2.7 Problems involving polygons (p. 237)

Problem 12. Suppose that such a 1990-gon exists and let $A_0A_1 \cdots A_{1989}$ be its vertices. The sides A_kA_{k+1}, $k = 0, 1, \dots, 1989$ define the vectors $\overrightarrow{A_kA_{k+1}}$ which can be represented in the complex plane by the numbers

$$z_k = n_k w^k, \quad k = 0, 1, \dots, 1989$$

where $w = \cos\dfrac{2\pi}{1990} + i\sin\dfrac{2\pi}{1990}$. Here $A_{1990} = A_0$ and $n_0, n_1, \ldots, n_{1989}$ represents a permutation of the numbers $1^2, 2^2, \ldots, 1990^2$.

Because $\sum_{k=0}^{1989} \overrightarrow{A_k A_{k+1}} = 0$, the problem can be restated as follows: find a permutation $(n_0, n_1, \ldots, n_{1989})$ of the numbers $1^2, 2^2, \ldots, 1990^2$ such that

$$\sum_{k=0}^{1989} n_k w^k = 0.$$

Observe that $1990 = 2 \cdot 5 \cdot 199$. The strategy is to add vectors after suitable grouping of 2, 5, 199 vectors such that these partial sums can be directed toward the suitable result.

To begin, let consider the pairing of numbers

$$(1^2, 2^2), (3^2, 4^2), \ldots, (1988^2, 1989^2)$$

and assign these lengths to pairs of opposite vectors respectively:

$$(w_k, w_{k+995}), \quad k = 0, \ldots, 994.$$

By adding the obtained vectors, we obtain 995 vectors of lengths

$$2^2 - 1^2 = 3;\ 4^2 - 3^2 = 7;\ 6^2 - 5^2 = 11; \ldots;\ 1989^2 - 1988^2 = 3979$$

which divide the unit circle of the coordinate plane into 995 equal arcs.

Let $B_0 = 1,\ B_1, \ldots, B_{994}$ be the vertices of the regular 995-gon inscribed in the unit circle. We intend to assign the lengths 3,7,11, … ,3979 to the unit vectors $\overrightarrow{OB}_0, \overrightarrow{OB}_1, \ldots, \overrightarrow{OB}_{994}$ such that the sum of the obtained vectors is zero,

We divide 995 lengths into 199 groups of size 5:

$$(3, 7, 11, 15, 19), (23, 27, 31, 35, 39), \ldots, (3963, 3967, 3971, 3975, 3979).$$

Let $\zeta = \cos\dfrac{2\pi}{5} + i\sin\dfrac{2\pi}{5}$, $\omega = \cos\dfrac{2\pi}{199} + i\sin\dfrac{2\pi}{199}$ be the primitive roots of unity of order 5 and 199, respectively. Let P_1 be the pentagon with vertices $1, \zeta, \zeta^2, \zeta^3, \zeta^4$. Then we rotate P_1 about the origin O with coordinates through angles $\theta_k = \dfrac{2k\pi}{199}$, $k = 1, \ldots, 198$, to obtain new pentagons $P_2, \ldots, P_{198}$, respectively. The vertices of P_{k+1} are $\omega^k, \omega^k\zeta, \omega^k\zeta^2, \omega^k\zeta^3, \omega^k\zeta^4$, $k = 0, \ldots, 198$. We assign to unit vectors defined by the vertices P_k of the respective lengths:

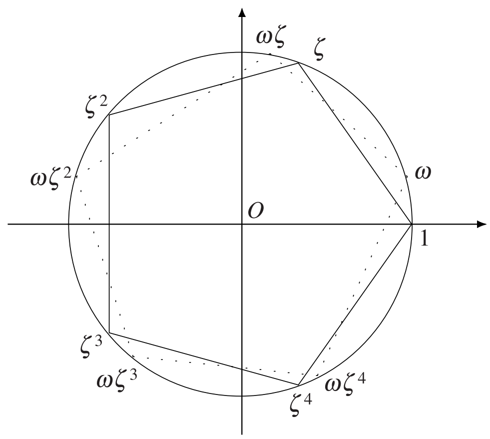

Figure 6.7.

$$2k+3,\ 2k+7,\ 2k+11,\ 2k+15,\ 2k+19 \quad (k=0,\dots,198).$$

Thus, we have to evaluate the sum:

$$\sum_{k=0}^{198}[(2k+3)\omega^k+(2k+7)\omega^k\zeta+(2k+1)\omega^k\zeta^2+(2k+15)\omega^k\zeta^3+(2k+19)\omega^k\zeta^4]$$

$$\sum_{k=0}^{198}2k\omega^k(1+\zeta+\zeta^2+\zeta^3+\zeta^4)+(3+7\zeta+11\zeta^2+15\zeta^3+19\zeta^4)\sum_{k=0}^{198}\omega^k.$$

Since $1+\zeta+\zeta^2+\zeta^3+\zeta^4=0$ and $1+\omega+\omega^2+\cdots+\omega^{198}=0$, it follows that the sum equals zero.

Problem 13. It is convenient to take a regular octagon inscribed in a circle and note its vertices as follows:

$$A=A_0,\ A_1,\ A_2,\ A_3,\ A_4=E,\ A_{-3},\ A_{-2},\ A_{-1}.$$

We imagine a step in the path like a rotation of angle $\frac{2\pi}{8}=\frac{\pi}{4}$ about the center O of the circumscribed circle of the octagon. In this way, a path is a sequence of such rotations, submitted to some conditions. If the rotation is counterclockwise we add the angle $\frac{\pi}{4}$; if the rotation is clockwise we add the angle $-\frac{\pi}{4}$. The starting point is A_0, which is represented by the complex number $z_0=\cos 0+i\sin 0$. Any vertex A_k of the octagon is represented by $z_k=\cos\frac{2k\pi}{8}+i\sin\frac{2k\pi}{8}$. It is convenient to work only with the angles $\frac{2k\pi}{8}$, $-4\le k\le 4$. But these k's are integers considered mod 8, such that $z_4=z_{-4}$ and $A_4=A_{-4}$.

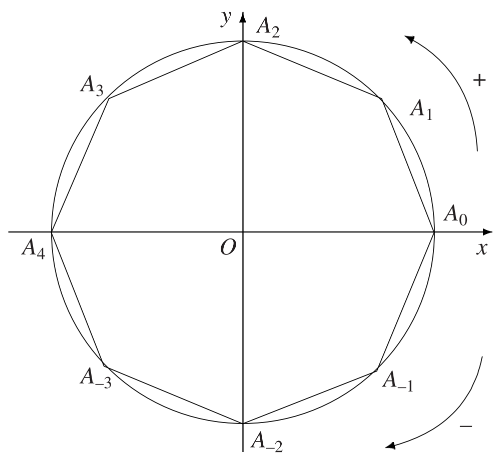

Figure 6.8.

We may associate to a path of length n, say $(P_0P_1\cdots P_n)$, an ordered sequence $(u_{,}u_2,\ldots,u_n)$ of integers which satisfy the following conditions:

a) $u_k=\pm 1$ for any $k=1,2,\ldots,n$; more precisely $u_i=\pm 1$ if the $\text{arc}(P_{k-1}P_k)$ is $\frac{\pi}{4}$ and $u_k=-1$ if the $\text{arc}(P_{k-1}P_k)$ is $-\frac{\pi}{4}$;

b) $u_1+u_2+\cdots+u_k\in\{-3,-2,-1,0,1,2,3\}$ for all $k=1,2,\ldots,n-1$;

c) $u_1+u_2+\cdots+u_n=\pm 4$.

For example, the sequence associated with the path $(A_0, A_{-1}, A_0, A_1, A_2, A_3, A_4)$ is $(-1,1,1,1,1,1)$. From now on we consider only sequences that satisfy a), b), c). It is obvious that conditions a), b), c) define a bijective function between the set of paths and the set of sequences.

For any sequence $u_1,u_2,\ldots,u_n$ and any k, $1\le k\le n$, we call the sum $s_k=u_1+u_2+\cdots+u_k$ a partial sum of the sequence. It is easy to see that for any k, s_k is an even number if and only if k is even. Thus, $a_{2n-1}=0$. Thus we have to prove the formula for even numbers. For small n we have $a_2=0$, $a_4=2$; for example, only sequences $(1,1,1,1)$ and $(-1,-1,-1,-1)$ of length 4 satisfy conditions a)–c).

In the following we will prove a recurrence relation between the numbers a_n, n even. The first step is to observe that if $s_n=\pm 4$, then $s_{n-2}=\pm 2$. Moreover, if $(u_1,u_2,\ldots,u_{n-2})$ is a sequence that satisfies a), b) and $s_{n-2}=\pm 2$ there are only two ways to extend it to a sequence that satisfy c) as well: either the sequence $(u_1,u_2,\ldots,u_{n-2},+1,+1)$ or the sequence $(u_1,u_2,\ldots,u_{n-2},-1,-1)$. So if we denote by x_n the number of sequences that satisfy a), b) and $s_n=\pm 2$, then n is even and $a_n=x_{n-2}$.

Let y_n denote the number of sequences which satisfy a), b) and $s_n = 0$. Then n is even and we have the equality

$$y_n = x_{n-2} + 2y_{n-2}. \tag{1}$$

This equality comes from the following constructions. A sequence $(u_1, \dots, u_{n-2})$ for which $s_{n-2} = \pm 2$ gives rise to a unique sequence of length n with $s_n = 0$ by extending it either to $(u_1, \dots, u_{n-2}, 1, 1)$ or $(u_1, u_2, \dots, u_{n-1}, -1, -1)$. Also, a sequence $(u_1, \dots, u_{n-2})$ with $s_{n-2} = 0$ gives rise either to sequence $(u_1, \dots, u_{n-2}, 1, -1)$ or $(u_1, \dots, u_{n-2}, -1, 1)$. Finally, every sequence of length n with $s_n = 0$ ends in one of the following "terminations": $(-1, -1)$, $(1, 1)$, $(1, -1)$, $(-1, 1)$.

The following equality is also verified:

$$x_n = 2x_{n-2} + 2y_{n-2}. \tag{2}$$

This corresponds to the property that any sequence of length n for which $s_n = \pm 2$ can be obtained either from a similar sequence of length $n-2$ by adding the termination $(1, -1)$ or the termination $(-1, 1)$, or from a sequence of length $n-2$ for which $s_{n-2} = 0$ by adding the termination (1,1) or the termination $(-1, -1)$.

Now, the problem is to derive $a_n = x_{n-2}$, from relations (1) and (2). By subtracting (1) from (2) we obtain $x_{n-2} = x_n - y_n$, for all $n \geq 4$, n even. Thus, $y_{n-2} = x_{n-2} - x_{n-4}$. Substituting the last equality in (2) we obtain the recurrent relation: $x_n = 4x_{n-2} - 2x_{n-4}$, for all $n \geq 4$, n even. Taking into account that $x_n = a_{n+2}$, we obtain the linear recurrent relation

$$a_{n+2} = 4a_n - 2a_{n-2}, \quad n \geq 4, \tag{3}$$

with the initial values $a_2 = 0$, $a_4 = 2$.

The sequence (a_n), $n \geq 2$, n even is uniquely defined by $a_2 = 0$, $a_4 = 2$ and relation (3). Therefore, to answer the question, it is sufficient to prove that the sequence $(c_{2n})_{n\geq 1}$, $c_{2n} = \frac{1}{\sqrt{2}}((2+\sqrt{2})^{n-1} - (2-\sqrt{2})^{n-1})$ obeys the same conditions. This is a straightforward computation.

Problem 14. Consider the complex plane with origin at the center of the polygon. Without loss of generality we may assume that the coordinates of A, B, C are 1, ε, ε^2, respectively, where $\varepsilon = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

Let $z_M = \cos t + i\sin t$, $t \in [0, 2\pi)$ be the coordinate of point M. From the hypothesis we derive that $t > \frac{4\pi}{n}$. Then

$$MA = |z_M - 1| = \sqrt{(\cos t - 1)^2 + \sin^2 t} = \sqrt{2 - 2\cos t} = 2\sin\frac{t}{2};$$

$$MB = |z_M - \varepsilon| = \sqrt{2 - 2\cos\left(t - \frac{2\pi}{n}\right)} = 2\sin\left(\frac{t}{2} - \frac{\pi}{n}\right);$$

$$MC = |z_M - \varepsilon^2| = \sqrt{2 - 2\cos\left(t - \frac{4\pi}{n}\right)} = 2\sin\left(\frac{t}{2} - \frac{2\pi}{n}\right);$$

$$AB = |\varepsilon - 1| = \sqrt{2 - 2\cos\frac{2\pi}{n}} = 2\sin\frac{\pi}{n}.$$

We have

$$MB^2 - AB^2 = 4\sin^2\left(\frac{t}{2} - \frac{\pi}{n}\right) - 4\sin^2\frac{\pi}{n}$$

$$2\left(\cos\frac{2\pi}{n} - \cos\left(t - \frac{2\pi}{n}\right)\right)$$

$$= -2 \cdot 2\sin\frac{\frac{2\pi}{n} - \left(t - \frac{2\pi}{n}\right)}{2}\sin\frac{\frac{2\pi}{n} + \left(t - \frac{2\pi}{n}\right)}{2}$$

$$= 2\sin\frac{t}{2} \cdot 2\sin\left(\frac{t}{2} - \frac{2\pi}{n}\right) = MA \cdot MC,$$

as desired.

Problem 15. Rotate the polygon $A_1A_2 \cdots A_n$ so that the coordinates of its vertices are the complex roots of unity of order n, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$. Let z be the coordinate of point P located on the circumcircle of the polygon and note that $|z| = 1$.

The equality

$$z^n - 1 = \prod_{j=1}^{n}(z - \varepsilon_j)$$

yields

$$|z^n - 1| = \prod_{j=1}^{n}|z - \varepsilon_j| = \prod_{j=1}^{n} PA_j.$$

Since $|z^n - 1| \leq |z|^n + 1 = 2$, it follows that the maximal value of $\prod_{j=1}^{n} PA_j^2$ is 2 and is attained for $z^n = -1$, i.e., for the midpoints of arcs A_jA_{j+1}, $j = 1, \ldots, n$, where $A_{n+1} = A_1$.

Problem 16. Without loss of generality, assume that points A_k have coordinates ε^{k-1} for $k = 1, \ldots, 2n$, where

$$\varepsilon = \cos\frac{\pi}{n} + i\sin\frac{\pi}{n}.$$

Let α be the coordinate of the point P, $|\alpha| = 1$. We have

$$PA_{k+1} = |\alpha - \varepsilon^k|$$

and

$$PA_{n+k+1} = |\alpha - \varepsilon^{n+k}| = |\alpha + \varepsilon^k|,$$

for $k = 0, \dots, n-1$. Then

$$\sum_{k=0}^{n-1} PA_{k+1}^2 \cdot PA_{n+k+1}^2 = \sum_{k=0}^{n-1} |\alpha - \varepsilon^k|^2 \cdot |\alpha + \varepsilon^k|^2$$

$$= \sum_{k=0}^{n-1} [(\alpha - \varepsilon^k)(\overline{\alpha} - \overline{\varepsilon}^k)][(\alpha + \varepsilon^k)(\overline{\alpha} + \overline{\varepsilon}^k)]$$

$$= \sum_{k=0}^{n-1} (2 - \alpha\overline{\varepsilon}^k - \overline{\alpha}\varepsilon^k)(2 + \alpha\overline{\varepsilon}^k + \overline{\alpha}\varepsilon^k)$$

$$= \sum_{k=0}^{n-1} (2 - \alpha^2\overline{\varepsilon}^{2k} - \overline{\alpha}^2\varepsilon^{2k}) = 2n - \alpha^2 \sum_{k=0}^{n-1} \overline{\varepsilon}^{2k} - \overline{\alpha}^2 \cdot \sum_{k=0}^{n-1} \varepsilon^{2k}$$

$$= 2n - \alpha^2 \cdot \frac{\overline{\varepsilon}^{2n} - 1}{\overline{\varepsilon}^2 - 1} - \overline{\alpha}^2 \cdot \frac{\varepsilon^{2n} - 1}{\varepsilon^2 - 1} = 2n,$$

as desired.

6.2.8 Complex numbers and combinatorics (p. 245)

Problem 11. Let us consider the complex number $z = \cos t + i \sin t$ and the sum $t_n = \sum_{k=0}^{n} \binom{n}{k}^2 \sin kt$. Observe that

$$s_n + it_n = \sum_{k=0}^{n} \binom{n}{k}^2 (\cos kt + i \sin kt) = \sum_{k=0}^{n} \binom{n}{k}^2 (\cos t + i \sin t)^k.$$

In the product $(1 + X)^n(1 + zX)^n = (1 + (z + 1)X + zX^2)^n$ we set the coefficient of X^n equal to obtain

$$\sum_{\substack{0 \le k,s \le n \\ k+s=n}} \binom{n}{k}\binom{n}{s} z^s = \sum_{\substack{0 \le k,s,r \le n \\ k+s+r=n \\ s+2r=n}} \frac{n!}{k!s!r!}(z + 1)^s z^r. \tag{1}$$

The above relation is equivalent to

$$\sum_{k=0}^{n} \binom{n}{k}^2 z^k = \sum_{k=0}^{[\frac{n}{2}]} \binom{n}{2k}\binom{2k}{k}(z + 1)^{n-2k} z^k. \tag{2}$$

The trigonometric form of the complex number $1 + z$ is given by

$$1 + \cos t + i \sin t = 2\cos^2 \frac{t}{2} + 2i \sin \frac{t}{2} \cos \frac{t}{2} = 2 \cos \frac{t}{2} \left(\cos \frac{t}{2} + i \sin \frac{t}{2}\right),$$

since $t \in [0, \pi]$. From (2) it follows that

$$s_n + it_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k}\binom{2k}{k}\left(2\cos\frac{t}{2}\right)^{n-2k}\left(\cos\frac{nt}{2} + i\sin\frac{nt}{2}\right),$$

hence

$$s_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k}\binom{2k}{k}\left(2\cos\frac{t}{2}\right)^{n-2k}\cos\frac{nt}{2},$$

$$t_n = \sum_{k=0}^{\left[\frac{n}{2}\right]} \binom{n}{2k}\binom{2k}{k}\left(2\cos\frac{t}{2}\right)^{n-2k}\sin\frac{nt}{2}.$$

Remark. Here we have a few particular cases of (2).

1) If $z = 1$, then

$$\sum_{k=0}^{n}\binom{n}{k}^2 = \sum_{k=0}^{\left[\frac{n}{2}\right]}\binom{n}{2k}\binom{2k}{k}2^{n-2k} = \binom{2n}{n}.$$

2) If $z = -1$, then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 = \begin{cases} 0, & \text{if } n \text{ is odd,} \\ (-1)^{\frac{n}{2}}\binom{n}{n/2}, & \text{if } n \text{ is even.} \end{cases}$$

3) If $z = -\dfrac{1}{2}$, then

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}^2 2^{n-k} = \sum_{k=0}^{\left[\frac{n}{2}\right]}(-1)^k\binom{n}{2k}\binom{2k}{k}2^k.$$

Problem 12. 1) In Problem 4 consider $p = 4$ to obtain

$$\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots = \frac{2^n}{4}\left(1 + 2\left(\cos\frac{\pi}{4}\right)^n\cos\frac{n\pi}{4}\right)$$
$$= \frac{1}{4}\left(2^n + 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}\right).$$

2) Let us consider $p = 5$ in Problem 4. We find that

$$\binom{n}{0} + \binom{n}{4} + \binom{n}{8} + \cdots = \frac{2^n}{5}\left(1 + 2\left(\cos\frac{\pi}{5}\right)^n\cos\frac{n\pi}{5} + 2\left(\cos\frac{2\pi}{5}\right)^n\cos\frac{2n\pi}{5}\right).$$

Using the well-known relations

$$\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4} \quad \text{and} \quad \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$$

the desired identity follows.

Problem 13. 1) Let ε be a cube root of unity different from 1. We have

$$(1-\varepsilon)^n = A_n + B_n\varepsilon + C_n\varepsilon^2, \quad (1-\varepsilon^2)^n = A_n + B_n\varepsilon^2 + C_n\varepsilon$$

hence

$$A_n^2 + B_n^2 - C_n^2 - A_nB_n - B_nC_n - C_nA_n = (A_n + B_n\varepsilon + C_n\varepsilon^2)(A_n + B_n\varepsilon^2 + C_n\varepsilon)$$
$$= (1-\varepsilon)^n(1-\varepsilon^2)^n = (1-\varepsilon-\varepsilon^2+1)^n = 3^n.$$

2) It is obvious that $A_n + B_n + C_n = 0$. Replacing $C_n = -(A_n + B_n)$ in the previous identity we get $A_n^2 + A_nB_n + C_n^2 = 3^{n-1}$.

Problem 14. For any $k \in \{0, 1, \ldots, p-1\}$, consider $x_k = \sum c_1 \cdots c_m$, the sum of all products $c_1 \cdots c_m$ such that $c_i \in \{1, 2, \ldots, n\}$ and $\sum_{i=1}^{m} c_i \equiv k \pmod{p}$.

If $\varepsilon = \cos\dfrac{2\pi}{p} + i\sin\dfrac{2\pi}{p}$, then

$$(\varepsilon + 2\varepsilon^2 + \cdots + n\varepsilon^n)^m = \sum_{c_1,\ldots,c_m \in \{1,2,\ldots,n\}} c_1 \cdots c_m \varepsilon^{c_1+\cdots+c_m} = \sum_{k=0}^{p-1} x_k\varepsilon^k.$$

Taking into account the relation

$$\varepsilon + 2\varepsilon^2 + \cdots + n\varepsilon^n = \frac{n\varepsilon^{n+2} - (n+1)\varepsilon^{n+1} + \varepsilon}{(\varepsilon-1)^2} = \frac{n\varepsilon}{\varepsilon - 1}$$

(see Problem 9 in Section 5.4 or Problem 13 in Section 5.5) it follows that

$$\frac{n^m}{(\varepsilon-1)^m} = \sum_{k=0}^{p-1} x_k\varepsilon^k. \tag{1}$$

On the other hand, from $\varepsilon^{p-1} + \cdots + \varepsilon + 1 = 0$ we obtain that

$$\frac{1}{\varepsilon-1} = -\frac{1}{p}(\varepsilon^{p-2} + 2\varepsilon^{p-3} + \cdots + (p-2)\varepsilon + p - 1),$$

hence

$$\frac{n^m}{(\varepsilon-1)^m} = \left(-\frac{n}{p}\right)^m (\varepsilon^{p-2} + 2\varepsilon^{p-3} + \cdots + (p-2)\varepsilon + p - 1)^m.$$

Put

$$(X^{p-2} + 2X^{p-3} + \cdots + (p-2)X + p - 1)^m = b_0 + b_1X + \cdots + b_{m(p-2)}X^{m(p-2)},$$

and find

$$\frac{n^m}{(\varepsilon-1)^m} = \left(-\frac{n}{p}\right)^m (y_0 + y_1\varepsilon + \cdots + y_{p-1}\varepsilon^{p-1}), \tag{2}$$

where $y_j = \sum_{k\equiv j \pmod p} b_k$.

From (1) and (2) we get

$$x_0 - ry_0 + (x_1 - ry_1)\varepsilon + \cdots + (x_{p-1} - ry_{p-1})\varepsilon^{p-1} = 0,$$

where $r = \left(-\frac{n}{p}\right)^m$. From Proposition 4 in Subsection 2.2.2, it follows that $x_0 - ry_0 = x_1 - ry_1 = \cdots = x_{p-1} - ry_{p-1} = k$. Now it is sufficient to show that $r|k$. But

$$\begin{aligned} pk &= x_0 + \cdots + x_{p-1} - r(y_0 + \cdots + y_{p-1}) \\ &= (1 + 2 + \cdots + n)^m - r(b_0 + \cdots + b_{m(p-2)}) \\ &= (1 + 2 + \cdots + n)^m - r(1 + 2 + \cdots + (p-1))^m, \end{aligned}$$

and we obtain

$$pk = \left(\frac{n(n+1)}{2}\right)^m - r\left(\frac{p(p-1)}{2}\right)^m.$$

Since the right-hand side is divisible by pr, it follows that $r|k$.

Problem 15. Expanding $(1+i\sqrt{a})^n$ by binomial theorem and then separating the even and odd terms we find

$$(1 + i\sqrt{a})^n = s_n + i\sqrt{a}t_n. \tag{1}$$

Passing to conjugates in (1) we get

$$(1 - i\sqrt{a})^n = s_n - i\sqrt{a}t_n. \tag{2}$$

From (1) and (2) it follows that

$$s_n = \frac{1}{2}[(1 + i\sqrt{a})^n + (1 - i\sqrt{a})^n]. \tag{3}$$

The quadratic equation with roots $z_1 = 1+i\sqrt{a}$ and $z_2 = 1-i\sqrt{a}$ is $z^2-2z+(a+1) = 0$. It is easy to see that for any positive integer n the following relation holds:

$$s_{n+2} = 2s_{n+1} - (1 + a)s_n. \tag{4}$$

Now, we proceed by induction by step 2. We have $s_1 = 1$ and $s_2 = 1 - a = 2 - 4k = 2(1 - 2k)$, hence the desired property holds. Assume that $2^{n-1}|s_n$ and $2^n|s_{n+1}$. From (4) it follows that $2^{n+1}|s_{n+2}$, since $1 + a = 4k$ and $2^{n+1}|(1 + a)s_n$.

6.2.9 Miscellaneous problems (p. 252)

Problem 12. Using the triangle inequality, we have

$$2|z|^2 = |x|y| + y|x|| \le |x||y| + |y||x|,$$

so $|z|^2 \le |x| \cdot |y|$. Likewise,

$$|y|^2 \le |x| \cdot |z| \quad \text{and} \quad |z|^2 \le |y||x|.$$

Summing these inequality yields

$$|x|^2 + |y|^2 + |z|^2 \le |x||y| + |y||z| + |z||x|.$$

This implies that

$$|x| = |y| = |z| = a.$$

If $a = 0$, then $x = y = z = 0$ is a solution of the system. Consider $a > 0$. The system may be written as

$$\begin{cases} x + y = \dfrac{2}{a}z^2, \\ y + z = \dfrac{2}{a}x^2, \\ z + x = \dfrac{2}{a}y^2. \end{cases}$$

Subtracting the last two equations gives

$$x - y = \frac{2}{a}(y^2 - x^2), \text{ i.e., } (y - x)\left(y + x + \frac{2}{a}\right) = 0.$$

Case 1. If $x = y$, then $x = y = \dfrac{z^2}{a}$. The last equation implies

$$z + \frac{z^2}{a} = 2\frac{z^4}{a^3}.$$

This is equivalent to

$$2\left(\frac{z}{a}\right)^3 = \frac{z}{a} + 1,$$

hence

$$\frac{z}{a} = 1 \quad \text{or} \quad \frac{z}{a} = \frac{-1 \pm i}{2}.$$

If $z = a$, then $x = y = z = a$ is a solution of the system.

If $\frac{z}{a} = \frac{-1 \pm i}{2}$, then

$$1 = \left|\frac{z}{a}\right| = \left|\frac{-1 \pm i}{2}\right| = \frac{\sqrt{2}}{2},$$

which is a contradiction.

Case 2. If $x + y = -\frac{2}{a}$, then $-\frac{2}{a} = \frac{2}{a}z^2$. We obtain $z = \pm i$ and $a = |z| = 1$. Consider $z = i$; then

$$x = (x + y) - (y + z) + z = 2z^2 - 2x^2 + z = -2 + i - 2x^2$$

or equivalently,

$$2x^2 + x + 2 - i = 0.$$

Then $x = i$ or $x = -\frac{1}{2} - i$. Since $|x| = a = 1$, we have $x = i$. Then $y = 2x^2 - z = -2 - i$ and $|y| = \sqrt{5} \neq a = 1$, so the system has no solution. The case $z = -i$ had the same conclusion.

Therefore, the solutions are $x = y = z = a$, where $a \geq 0$ is a real number.

Problem 13. In any solution (x, y, z) we have $x \neq 0$, $y \neq 0$, $z \neq 0$ and $x \neq y$, $y \neq z$, $z \neq x$. We can divide each equation by others and obtain new equations:

$$\begin{aligned} x^2 + y^2 &= yz + zx, \\ y^2 + z^2 &= xy + zx, \\ z^2 + x^2 &= xy + yz. \end{aligned} \tag{1}$$

By adding them one obtains the equality

$$x^2 + y^2 + z^2 = xy + yz + zx. \tag{2}$$

After subtracting equations (1), the second from the first, one obtains $x + y + z = 0$. By squaring this identity one obtains an improvement of (2):

$$x^2 + y^2 + z^2 = xy + yz + zx = 0. \tag{3}$$

Using (3) in (1) one obtains

$$x^2 = zy, \quad y^2 = zx, \quad z^2 = xy \tag{4}$$

and also

$$x^3 = y^3 = z^3 = xyz.$$

It follows that x, y, z are distinct roots of the same complex number $a = xyz$. From $x^3 = y^3 = z^3 = xyz = a$ we obtain

$$x = \sqrt[3]{a}, \quad t = \varepsilon\sqrt[3]{a}, \quad z = \varepsilon^2\sqrt[3]{a}, \tag{5}$$

where $\varepsilon^2 + \varepsilon + 1 = 0$, $\varepsilon^3 = 1$. When introduce relations (5) in the first equation of the original system, one obtains $a^3(1-\varepsilon)(1-\varepsilon^2) = 3$. Taking into account the computation

$$(1-\varepsilon)(1-\varepsilon^2) = 1 - \varepsilon - \varepsilon^2 + 1 = 3,$$

we have $a^3 = 1$. Hence, we obtain using (5) that (x, y, z) is a permutation of the set $\{1, \varepsilon, \varepsilon^2\}$.

Problem 14. Suppose that the triangles OXY and OZT are counterclockwise oriented, and let x, y, z, t be the coordinates of the points X, Y, Z, T and let m be the coordinate of O. As these are right isosceles triangles we have $x - m = i(y - m)$, $z - m = i(t - m)$. It follows that $m(1-i) = x - iy = z - it$. We deduce $x - z = i(y - t)$.

Reciprocally, if $x - iy = z - it$, the coordinate of O is $m = \dfrac{x - iy}{1 - i}$, and the triangles OXY and OZT are right and isosceles.

Let a, b, c, d, e, f be the coordinates of the given hexagon in that order. We can write $a - ib = c - ie$, $b - id = e - if$. It follows that $a + d = c + f$, i.e., $ACDF$ is a parallelogram.

Multiplying the first equality by i, we obtain $b - ic = e - ia$, i.e., BC and AE are connected.

Problem 15. By standard computations, we find that on the circumscribed circle the sides of the pentagon subtend the following arcs: $\overset{\frown}{AB} = 80°$, $\overset{\frown}{BC} = 40°$, $\overset{\frown}{CD} = 80°$, $\overset{\frown}{DE} = 20°$ and $\overset{\frown}{EA} = 140°$. It is then natural to consider all these measures as multiples of $20°$ that correspond to the primitive 18^{th} roots of unity, say $\omega = \cos\dfrac{2\pi}{18} + i\sin\dfrac{2\pi}{18}$. We thus assign, to each vertex, starting from $A(1)$, the corresponding root of unity: $B(\omega^4)$, $C(\omega^6)$, $D(\omega^{10})$, $E(\omega^{11})$. We shall use the following properties of ω:

$$\omega^{18} = 1, \quad \omega^9 = -1, \quad \overline{\omega}^k = \omega^{18-k}, \quad \omega^6 - \omega^3 + 1 = 0. \tag{A}$$

We need to prove that the coordinate coordinate of the common point of the lines BD and CE is a real number.

The equation of the line BD is

$$\begin{vmatrix} z & \overline{z} & 1 \\ \omega^4 & \overline{\omega}^4 & 1 \\ \omega^{10} & \overline{\omega}^{10} & 1 \end{vmatrix} = 0, \tag{1}$$

and the equation of the line CE is

$$\begin{vmatrix} z & \overline{z} & 1 \\ \omega^6 & \overline{\omega}^6 & 1 \\ \omega^{11} & \overline{\omega}^{11} & 1 \end{vmatrix} = 0. \tag{2}$$

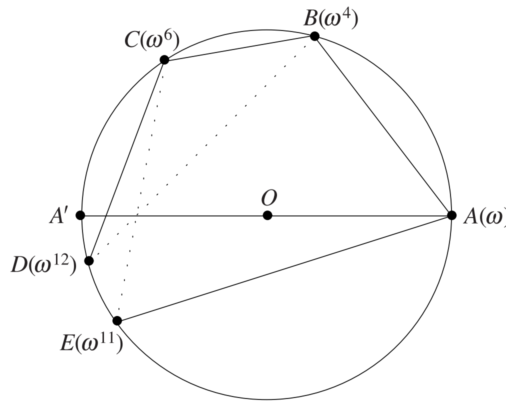

Figure 6.9.

The equation (1) can be written as follows:

$$z(\omega^{14} - \omega^8) - \overline{z}(\omega^4 - \omega^{10}) + (\omega^{12} - \omega^6) = 0$$

or

$$z\omega^8(\omega^6 - 1) + \overline{z}\omega^4(\omega^6 - 1) + \omega^6(\omega^6 - 1) = 0.$$

Using the properties of ω we derive a simplified version of (1):

$$z\omega^4 + \overline{z} + \omega^2 = 0. \qquad (1')$$

In the same way, equation (2) becomes

$$z\omega + \overline{z} - \omega^3(\omega^4 - 1) = 0. \qquad (2')$$

From (1′) and (2′) we obtain the following expression for z:

$$z = \frac{-\omega^7 + \omega^3 - \omega^2}{\omega^4 - \omega} = \frac{-\omega^6 + \omega^2 - \omega}{\omega^6} = -1 + \frac{\omega - 1}{\omega^5}.$$

To prove that z is real, it will suffice to prove that it coincides with its conjugate. It is easy to see that

$$\frac{\omega - 1}{\omega^5} = \frac{\overline{\omega} - 1}{\overline{\omega}^5}$$

is equivalent to

$$\overline{\omega}^4 - \overline{\omega}^5 = \omega^4 - \omega^5,$$

i.e., $\omega^{14} - \omega^{13} = \omega^4 - \omega^5$, which is true by the properties of ω given in (A).

Glossary

Antipedal triangle of point M: The triangle determined by perpendicular lines from vertices A, B, C of triangle ABC to MA, MB, MC, respectively.

Area of a triangle: The area of triangle with vertices with coordinates z_1, z_2, z_3 is the absolute value of the determinant

$$\Delta = \frac{i}{4}\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z_3 & \overline{z_3} & 1 \end{vmatrix}.$$

Area of pedal triangle of point X with respect to the triangle ABC:

$$\text{area}[PQR] = \frac{\text{area}[ABC]}{4R^2}|x\overline{x} - R^2|.$$

Argument of a complex number: If the polar representation of complex number z is $z = r(\cos t^* + i \sin t^*)$, then $arg(z) = t^*$.

Barycenter of set $\{A_1, \dots, A_n\}$ with respect to weights $m_1, \dots, m_n$: The point G with coordinate $z_G = \frac{1}{m}(m_1 z_1 + \cdots + m_n z_n)$, where $m = m_1 + \cdots + m_n$.

Barycentric coordinates: Consider triangle ABC. The unique real number μ_a, μ_b, μ_c such that

$$z_P = \mu_a a + \mu_b b + \mu_c c, \text{ where } \mu_a + \mu_b + \mu_c = 1.$$

Basic invariants of triangle: s, r, R

Binomial equation: An algebraic equation of the form $Z^n + a = 0$, where $a \in \mathbb{C}^*$.

Ceva's theorem: Let AD, BE, CF be three cevians of triangle ABC. Then, lines AD, BE, CF are concurrent if and only if

$$\frac{AF}{FB} \cdot \frac{BD}{DC} \cdot \frac{CE}{EA} = 1.$$

Cevian of a triangle: any segment joining a vertex to a point on the opposite side.

Concyclicity condition: If points $M_k(z_k)$, $k = 1, 2, 3, 4$, are not collinear, then they are concyclic if and only if

$$\frac{z_3 - z_2}{z_1 - z_2} : \frac{z_3 - z_4}{z_1 - z_4} \in \mathbb{R}^*.$$

Collinearity condition: $M_1(z_1), M_2(z_2), M_3(z_3)$ are collinear if and only if $\frac{z_3 - z_1}{z_2 - z_1} \in \mathbb{R}^*$.

Complex coordinate of point A of cartesian coordinates (x, y): The complex number $z = x + yi$. We use the notation $A(z)$.

Complex coordinate of the midpoint of segment $[AB]$: $z_M = \frac{a+b}{2}$, where $A(a)$ and $B(b)$.

Complex coordinates of important centers of a triangle: Consider the triangle ABC with vertices with coordinates a, b, c. If the origin of complex plane is in the circumcenter of triangle ABC, then:

- the centroid G has coordinate $z_G = \frac{1}{3}(a + b + c)$;
- the incenter I has coordinate $z_I = \frac{\alpha a + \beta b + \gamma c}{\alpha + \beta + \gamma}$, where α, β, γ are the sides length of triangle ABC;
- the orthocenter H has coordinate $z_H = a + b + c$;
- the Gergonne point J has coordinate $z_J = \frac{r_\alpha a + r_\beta b + r_\gamma c}{r_\alpha + r_\beta + r_\gamma}$, where $r_\alpha, r_\beta, r_\gamma$ are the radii of the three excircles of triangle;
- the Lemoine point K has coordinate $z_K = \frac{\alpha^2 a + \beta^2 b + \gamma^2 c}{\alpha^2 + \beta^2 + \gamma^2}$;
- the Nagel point N has coordinate $z_N = \left(1 - \frac{\alpha}{s}\right) a + \left(1 - \frac{\beta}{s}\right) b + \left(1 - \frac{\gamma}{s}\right) c$;
- the center O_9 of point circle has coordinate $z_{O_9} = \frac{1}{2}(a + b + c)$.

Complex number: A number z of the form $z = a + bi$, where a, b are real numbers and $i = \sqrt{-1}$.

Complex product of complex numbers *a* and *b*: $a \times b = \frac{1}{2}(\overline{a}b - a\overline{b})$.

Conjugate of a complex number: The complex number $\overline{z} = a - bi$, where $z = a + bi$.

Cyclic sum: Let n be a positive integer. Given a function f of n variables, define the cyclic sum of variables $(x_1, x_2, \ldots, x_n)$ as

$$\sum_{\text{cyc}} f(x_1, x_2, \ldots, x_n) = f(x_1, x_2, \ldots, x_n) + f(x_2, x_3, \ldots, x_n, x_1)$$
$$+ \cdots + f(x_n, x_1, x_2, \ldots, x_{n-1})$$

De Moivre's formula: For any angle α and for any integer n,

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha.$$

Distance between points $M_1(z_1)$ and $M_2(z_2)$: $M_1M_2 = |z_2 - z_1|$.

Equation of a circle: $z \cdot \overline{z} + \alpha \cdot z + \overline{\alpha} \cdot \overline{z} + \beta = 0$, where $\alpha \in \mathbb{C}$ and $\beta \in \mathbb{R}$.

Equation of a line: $\overline{\alpha} \cdot \overline{z} + \alpha z + \beta = 0$, where $\alpha \in \mathbb{C}^*$, $\beta \in \mathbb{R}$ and $z = x + iy \in \mathbb{C}$.

Equation of a line determined by two points: If $P_1(z_1)$ and $P_2(z_2)$ are distinct points, then the equation of line P_1P_2 is

$$\begin{vmatrix} z_1 & \overline{z_1} & 1 \\ z_2 & \overline{z_2} & 1 \\ z & \overline{z} & 1 \end{vmatrix} = 0.$$

Euler's formula: Let O and I be the circumcenter and incenter, respectively, of a triangle with circumradius R and inradius r. Then

$$OI^2 = R^2 - 2Rr.$$

Euler line of triangle: The line determined by the circumcenter O, the centroid G, and the orthocenter H.

Extend law of sines: In a triangle ABC with circumradius R and sides α, β, γ the following relations hold:

$$\frac{\alpha}{\sin A} = \frac{\beta}{\sin B} = \frac{\gamma}{\sin C} = 2R.$$

Heron's formula: The area of triangle ABC with sides α, β, γ is equal to

$$\text{area}[ABC] = \sqrt{s(s-\alpha)(s-\beta)(s-\gamma)},$$

where $s = \frac{1}{2}(\alpha + \beta + \gamma)$ is the semiperimeter of the triangle.

Isometric transformation: A mapping $f : \mathbb{C} \to \mathbb{C}$ preserving the distance.

Lagrange's theorem: Consider the points $A_1, \ldots, A_n$ and the nonzero real numbers $m_1, \ldots, m_n$ such that $m = m_1 + \cdots + m_n \neq 0$. For any point M in the plane the following relation holds:

$$\sum_{j=1}^{n} m_j MA_j^2 = mMG^2 + \sum_{j=1}^{n} m_j GA_j^2,$$

where G is the barycenter of set $\{A_1, \ldots, A_n\}$ with respect to weights $m_1, \ldots, m_n$.

Modulus of a complex number: The real number $|z| = \sqrt{a^2 + b^2}$, where $z = a + bi$.

Morley's theorem: The three points of adjacent trisectors of angles form an equilateral triangle.

Nagel line of triangle: The line I, G, N.

n

n^{th} roots of complex number z_0: Any solution Z of the equation $Z^n - z_0 = 0$.

n^{th} roots of unity: The complex numbers

$$\varepsilon_k = \cos\frac{2k\pi}{n} + i \sin\frac{2k\pi}{n}, \ k \in \{0, 1, \ldots, n-1\}.$$

The set of all these complex numbers is denoted by U_n.

Orthogonality condition: If $M_k(z_k)$, $k = 1, 2, 3, 4$, then lines M_1M_2 and M_3M_4 are orthogonal if and only if

$$\frac{z_1 - z_2}{z_3 - z_4} \in i\mathbb{R}^*.$$

Orthopolar triangles: Consider triangle ABC and points X, Y, Z situated on its circumcircle. Triangles ABC and XYZ are orthopolar (or S-triangles) if the Simson–Wallance line of point X with respect to triangle ABC is orthogonal to line YZ.

Pedal triangle of point X: The triangle determined by projections of X on sides of triangle ABC.

Polar representation of complex number $z = x + yi$: The representation $z = r(\cos t^* + i \sin t^*)$, where $r \in [0, \infty)$ and $t^* \in [0, 2\pi)$.

Primitive n^{th} root of unity: An n^{th} root $\varepsilon \in U_n$ such that $\varepsilon^m \neq 1$ for all positive integers $m < n$.

Quadratic equation: The algebraic equation $ax^2 + bx + c = 0$, $a, b, c \in \mathbb{C}$, $a \neq 0$.

Real product of complex numbers a and b: $a \cdot b = \frac{1}{2}(\overline{a}b + a\overline{b})$.

Reflection across a point: The mapping $s_{z_0} : \mathbb{C} \to \mathbb{C}$, $s_{z_0}(z) = 2z_0 - z$.

Reflection across the real axis: The mapping $s : \mathbb{C} \to \mathbb{C}$, $s(z) = \overline{z}$.

Rotation: The mapping $r_a : \mathbb{C} \to \mathbb{C}$, $r_a(z) = az$, where a is a given complex number.

Rotation formula: Suppose that $A(a)$, $B(b)$, $C(c)$ and C is the rotation of B with respect to A by the angle α. Then $c = a + (b - a)\varepsilon$, where $\varepsilon = \cos\alpha + i\sin\alpha$.

Similar triangles: Triangles $A_1A_2A_3$ and $B_1B_2B_3$ of the same orientation are similar if and only if

$$\frac{a_2 - a_1}{a_3 - a_1} = \frac{b_2 - b_1}{b_3 - b_1}.$$

Simson Line: For any point M on the circumcircle of triangle ABC, the projections of M on lines BC, CA, AB are collinear.

Translation: The mapping $t_{z_0} : \mathbb{C} \to \mathbb{C}$, $t_{z_0}(z) = z + z_0$.

Trigonometric identities

$$\sin^2 x + \cos^2 x = 1,$$
$$1 + \cot^2 x = \csc^2 x,$$
$$\tan^2 x + 1 = \sec^2 x;$$

addition and subtraction formulas:

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b,$$
$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b,$$
$$\tan(a \pm b) = \frac{\tan a \pm \tan b}{1 \mp \tan a \tan b},$$
$$\cot(a \pm b) = \frac{\cot a \cot b \mp 1}{\cot a \pm \cot b};$$

double-angle formulas:

$$\sin 2a = 2\sin a \cos a = \frac{2\tan a}{1 + \tan^2 a},$$
$$\cos 2a = 2\cos^2 a - 1 = 1 - 2\sin^2 a = \frac{1 - \tan^2 a}{1 + \tan^2 a},$$
$$\tan 2a = \frac{2\tan a}{1 - \tan^2 a};$$

triple-angle formulas:

$$\sin 3a = 3\sin a - 4\sin^3 a,$$
$$\cos 3a = 4\cos^3 a - 3\cos a,$$
$$\tan 3a = \frac{3\tan a - \tan^3 a}{1 - 3\tan^2} a;$$

half-angle formulas:

$$\sin^2\frac{a}{2}=\frac{1-\cos a}{2},$$
$$\cos^2\frac{a}{2}=\frac{1+\cos a}{2},$$
$$\tan\frac{a}{2}=\frac{1-\cos a}{\sin a}=\frac{\sin a}{1+\cos a};$$

sum-to-product formulas:

$$\sin a+\sin b=2\sin\frac{a+b}{2}\cos\frac{a-b}{2},$$
$$\cos a+\cos b=2\cos\frac{a+b}{2}\cos\frac{a-b}{2},$$
$$\tan a+\tan b=\frac{\sin(a+b)}{\cos a\cos b};$$

difference-to-product formulas:

$$\sin a-\sin b=2\sin\frac{a-b}{2}\cos\frac{a+b}{2},$$
$$\cos a-\cos b=-2\sin\frac{a-b}{2}\sin\frac{a+b}{2},$$
$$\tan a-\tan b=\frac{\sin(a-b)}{\cos a\cos b};$$

product-to-sum formulas:

$$2\sin a\cos b=\sin(a+b)+\sin(a-b),$$
$$2\cos a\cos b=\cos(a+b)+\cos(a-b),$$
$$2\sin a\sin b=-\cos(a+b)+\cos(a-b).$$

Vieta's theorem: Let $x_1, x_2, \dots, x_n$ be the roots of polynomial

$$P(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0,$$

where $a_n\neq 0$ and $a_0, a_1, \dots, a_n \in \mathbb{C}$. Let s_k be the sum of the products of the x_i taken k at a time. Then

$$s_k=(-1)^k\frac{a_{n-k}}{a_n},$$

that is,

$$x_1+x_2+\cdots+x_n=\frac{a_{n-1}}{a_n},$$
$$x_1x_2+\cdots+x_ix_j+x_{n-1}x_n=\frac{a_{n-2}}{a_n},$$
$$\cdots$$
$$x_1x_2\cdots x_n=(-1)^n\frac{a_0}{a_n}.$$

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